LU Decomposition

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$ax + by = c$
$dx + ey = f$

Row Formulation:

- $\begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$

Column Formulation:

- $x\begin{bmatrix} a \\ d \end{bmatrix} + y\begin{bmatrix} b \\ e \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$
- More generally
  - $x_1 v_1 + x_2 v_2 + \cdots + x_N v_N = b$
  - $v_1 \ldots v_N$ need to be linearly independent for a unique solution to always exist
    - Can then fully span N-dimensions to reach any point

Gaussian Elimination:

- $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \rightarrow \begin{bmatrix} p_1 & u & u & b_1 \\ 0 & p_2 & u & b_2 \\ 0 & 0 & p_3 & b_3 \end{bmatrix}$
  - Augmented matrix $[A|b]$
- Solve original equation using 'back substitution', i.e. $p_! = b_3$
- Viewing as matrix multiplication
  - $[i] - c[j]$ reads as substract c * j from row i
  - $E_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $, replace\ i, j\ elememts\ with\ c$
    - Then multiply original matrix A with E_{ij}
    - $E_{ij}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \boldsymbol{-c} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Last N(null space dim) rows of ΠE
  - give a basis for LEFT null space of A

LU Decomposition:

- In general, 3 operations for a 3x3
  - $E_{32}E_{31}E_{21}A = u$
  - $A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}u$
- Lower Triangular matrix
  - $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$
- $A = Lu$
  - Pivots in L must be 1
    - 0 values in upper half
  - Pivots in u can be any value
    - 0 values in lower half
  - $\mathbf{det}(\boldsymbol{A}) = \mathbf{det}(\boldsymbol{L})\,\mathbf{det}(\mathbf{U})$
  - $\mathbf{det}(\boldsymbol{L}) = \mathbf{1}\ (\boldsymbol{product\ of\ diagonals})$
  - $\mathbf{det}(\boldsymbol{U}) = \boldsymbol{product\ of\ diagonals}$

Vector Spaces

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- **0 vector is always part** of the vector space
- Inner Product
  - for column vectors
    - $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^H \boldsymbol{x}$
      - H - Hermitian transpose
        - when **y** is complex, use complex conjugate of each element
      - Otherwise, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^T \boldsymbol{x}$
- Norm (Induced)
  - $||\boldsymbol{x}||_2 = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}\ ('l2\ norm')$
    - $'lp\ norm': ||\boldsymbol{x}||_p = \sqrt[p]{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}, (p \geq 1)$
- two matrices
  - inner product - '*element by element*' product
    - $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \sum_{i,j} a_{i,j} b_{i,j}^*$
  - induced norm
    - $||\boldsymbol{A}|| = \sqrt{\sum |a_{i,j}|^2}$

- Orthogonal Matrices
  - If the vectors in the matrix A form an orthogonal basis then
    - $AA^H = A^H A = I$
      - $\boldsymbol{A^H = A^{-1}}$
  - preserves induced norm: if $y = Ax$
    - $||y||^2 = y^H y = (Ax)^H Ax = x^H A^H Ax$ $= x^H x = ||x||^2$
- **Orthogonal Subspaces**
  - E is a vector space, with V and W subspaces
  - <V,W>=0 if every vector in V is orthogonal to every vector in W
    - **Check against basis of subspace**
  - Orthogonal complement
    - subspace of all vectors orthogonal to V:
      - $V^\perp\ (E = V \oplus V^\perp)$
- Linear Mapping
  - domain X - n dimension
  - range Y - m dimension
  - $A\ - m\ x\ n$
  - consider basis elements of X

for n-dimension rotate - do n 2D rotates

- Cauchy-Schwartz inequality
  - $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leq ||\boldsymbol{x}||\,||\boldsymbol{y}||$
- Angle: $\cos\theta = \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{||\boldsymbol{x}||\,||\boldsymbol{y}||}$
  - **minimally similar when**
    - $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$ -> **orthogonal**
      - $|\boldsymbol{x}| = |\boldsymbol{y}| = \boldsymbol{1}$ -> **orthonormal**
  - **maximally similar when one is the rescaled version of the other**

- Linear independence
  - $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 + \cdots + c_n\boldsymbol{v}_n \neq 0$
    - for all $c_i \neq 0$

- Span
  - a set of vectors T in vector space E
  - take all **linear combinations of T**
    - gives $\boldsymbol{V = span\{T\}}$
    - V is a 'sub-space; of E
- Basis and Dimension
  - if $span\{T\} = E$
    - T forms a **basis** for E
    - Cardinality of T = dimension of E

- Rank A = number of pivots of A in echelon form
- **Range Space**
  - set of values in Y that are reach from X
  - $R(A) = span\{column\ space\ of\ A\}$
    - dimension = **rank A**
- **Null Space**
  - set of values in X that are transformed to **0** in Y
  - **dimension = n - rank A**
    - if >0, mapping is **not invertible**
    - n= dimension of A
  - **Computing Nullspace**
    - perform Gaussian elimination of **Ax = 0**
      - pivot columns
      - free columns
    - find simultaneous equations
      - **find pivot variables in terms of free variables**
    - **special solution**
      - assign a free variable to 1, others to 0
        - get one special solution
    - Null space
      - **linear combination** of the special solutions

Discrete-Time

12 November 2018 23:07

Linear Convolution:

- A Linear Mapping from $\mathcal{R}^n \ to \ \mathcal{R}^m$
  - $y_n = \sum_k h_k x_{n-k}$
    - $x_i = 0 \ for \ i \notin \{0, \dots, n-1\}$
  - Model this as y = Ax
  - $m = n + k \ - 1$

- **Toeplitz** matrix
  - Same entry along every left->right diagonal
  - Example with k = 3
    - n=3, m = 3+3-1 = 5
  - $A = \begin{matrix} h_0 & 0 & 0 \\ h_1 & h_0 & 0 \\ h_2 & h_1 & h_0 \\ 0 & h_2 & h_1 \\ 0 & 0 & h_2 \end{matrix}$

Circulant Convolution:

Circulant Matrices:

- A is nxn square
- X is **periodic** outside the interval (n)

Determinants

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Invertible if and only if det(A) != 0

$$\det(A) = \sum (-1)^{i+j} a_{ij} M_{ij} \quad , for\ fixed\ i\ or\ j$$

Cofactor Matrix - don't multiply by the element

Properties:

- Exchange two rows -> sign of det changes
  - Two equal rows -> det = 0
- det same after row reduction
- Scaling a row by T -> scales det by T
- det is linear for a **single** row
- For an upper triangle matrix
  - det = product of diagonals
- If A is singular, det(A) = 0
- For square matrices A,B
  - det(AB) = det(A)det(B)
- $\det(A^{-1}) = \frac{1}{\det(A)}$
- $\det(cA) = c^n \det(A)\, , where\ A\ is\ nxn$
- $\det(A^T) = \det(A)$

Inverse using cofactors:

- $A^{-1} = \frac{1}{\det(A)} C^T$
  - $C_{ij} = (-1)^{i+j} M_{ij}$

Rank:

- For square matrix A, nxn
  - rank(A) = n if and only if $\det(A) \neq 0$
- For matrix A, mxn
  - $rank(A) \leq \min(n, m)$
  - $rank(AB) \leq \min(rank(A), rank(B))$

Trace:

- Sum of elements along the main diagonal
- Also, $trace(ABC) = trace(BCA) = trace(CAB)$

Eigen / Diagonalization

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Eigenvalues:

- A nxn matrix will have n eigenvalues
- Same eigen values for $A\ and\ A^T$
- $\sum \lambda = trace(A)$
- $\prod \lambda = \det(A)$

Solving:

- $(A - \lambda I)\boldsymbol{x} = 0$
  - $\det(A - \lambda I) = 0$
- Plug values of $\lambda\ into\ (A - \lambda I)\boldsymbol{x} = 0$
  - To find eigenvector
    - **Families** of eigenvectors

- Consider $A = B + cI$
  - A has B's eigenvalues + c
    - **if $-B, reverse\ sign + C$**
  - A has same eigenvectors as B
- (upper) **triangular matrix**
  - eigenvalues are the diagonal values

Matrix Diagonalization:

- place independent eigenvectors of A into S
- place eigenvalues of A into diagonal of a matrix, λ

Can diagonalize when:

- eigenvalues are diff
- values same, but eigenvectors independent

- $AS = S\lambda$
  - $\boldsymbol{A = S\lambda S^{-1}}$
  - $S^{-1}AS = \lambda$

- if A and B are diagonalizable, they share same S
  - iff - AB = BA

- $A^k = S\lambda^k S^{-1}$
  - $\lambda^k = \begin{matrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{matrix}$

- assuming square and invertible
  - $\boldsymbol{A^{-1}} = \left(S\lambda S^{-1}\right)^{-1} = \left(s^{-1}\right)^{-1}\lambda^{-1}S^{-1} = S\boldsymbol{\lambda^{-1}}S^{-1}$

- Markov Process
  - difference equation
  - get A - find eigen
    - can compute S, λ^k, S^{-1} - solution is easier
  - properties of A:
    - all entries >= 0
    - each column sums to 1
    - $\lambda_1 = 1\ \left(u_1 is\ steady\ state\right)$
    - others: $\left|\lambda_i\right| \leq 1$

- Circulant Matrices

Symmetric

29 October 2018 08:23

Real: Symmetry: $A = A^T$
Complex: $A^{*T} = A^H = A$

- complex conjugate then transpose
- said to be Hermitian

Eigenvalues are ALWAYS real
Eigenvectors are Orthogonal

- $A = Q\lambda Q^{-1} = Q\lambda Q^T \ (Q^H for\ complex)$

Proof for orthogonal:

- $Ax = \lambda_1 x \ , Ay = \lambda_2 y$
- $(\lambda_1 x)^H y = (Ax)^H y$
- $x^H \lambda_1 y = x^H A^H y$
  $x^H \lambda_1 y = x^H A y = x^H \lambda_2 y$
- $\lambda_1 \neq \lambda_2$
  - $x^H y == 0$
    - x and y are perpendicular

<u>Positive Definite Hermitian Matrices:</u>

- $x^H A x > 0$
- **all $\lambda_i > 0$**
- all pivots satisfy $d_i > 0$
  - (without row exchanges)
- semi-positive (≥)

Proof (Real):

- Assume $A = A^T$
- $Ax = \lambda x$ (1)
- take complex conjugate
- $A^* x^* = \lambda^* x^*$
- A is real (***)
- $A x^* = \lambda^* x^*$
- transpose both sides
- $(Ax^*)^T = (\lambda^* x^*)^T$
  - $x^{*T} A^T = \lambda^* x^{*T} \Rightarrow x^{*T} A = \lambda^* x^{*T}$
- multiply from RHS by x, to get norm
- $x^{*T} A x = \lambda^* x^{*T} x$ (2)
- from (1), multiply form LHS by x^{*T}
  - $x^{*T} A x = x^{*T} \lambda x = \lambda x^{*T} x$ (3)
- combine (2) and (3)
  - $\lambda^* x^{*T} x = \lambda x^{*T} x$
  - $\lambda^* ||x||^2 = \lambda ||x||^2$
- since $||x||^2 > 0 \ (unless\ \boldsymbol{x} = 0)$
  - $\lambda^* == \lambda$
  - eigenvalues are always real

For Complex case:

- (***) is replaced with
- $A^{*T} = A^H = A$

Ax=b

$A: mxn, b: m$

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m=n: (square)

m>n: (tall - more rows)

m<n: (fat - more columns)

A maps from C^n to C^m

- b is in C^m
- existence - rows
  - if **range(A) = C^m**
    - *-> full row rank - rank(A)=m*
    - a solutions always exists
    - $m \leq n$ (fat/square)
  - else
    - if b not in range(A)
      - no solutions exists
- uniqueness - columns
  - if solution exists, it is unique
    - if null space of A is trivial
    - -> full column rank - rank(A)=n
    - $m \geq n$ (tall/square)
  - solution is not unique if
    - null space of A is non-trivial

Four Fundamental Subspaces:

- Range and Null of A and A^T
- C^m covered by range(A) and null(A^T)
  - these two are orthogonal complements
- C^N covered by range(A^T) and null(A)
  - these two are orthogonal complements

Inverse:

- Left Inverse: BA=I - tall (full column rank)
- Right Inverse: AC = I - fat (full row rank)

Projections: (Orthogonal)

- view in lower dimension subspace $V \in H$
- $\hat{x} \in V, s.t. ||x - \hat{x}|| \leq ||x - v||$ $for\ all\ v \in V$
  - $x - \hat{x} \perp V$
  - difference is orthogonal to the subspace
  - $\hat{x}$ is unique
- $\hat{x} = Px$, $where\ P\ depends\ on\ V, not\ x$
- **P is:**
  - *Idempotent* - $P(Px) = Px$ $for\ all\ x \in H$
  - *Self-Adjoint* - $P = P^H$
- Finding P
  - N < M
  - Project x from C^m onto subspace C^n
  - Vectors $a_1 \ldots a_n$ span C^n - (the subspace V)
    - Each a_i has m entries
    - Stack together to form A - C^{mxn} - rank(A)=n
  - $\boldsymbol{P = A(A^H A)^{-1} A^H}$
  - $P - C^{mxm} - rank(P) = n - \det(P) = 0$
    - If n=m, det(P)=1

Gram-Schmidt Process

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Make a matrix A orthogonal - so that $A^H = A^{-1}$

- Start with n independent vectors
- Get n orthogonal vectors
- Finally get n orthonormal vectors

1. Start with three independent vectors, **a,b,c**
2. Choose **A = a**
   a. $q_1 = \frac{A}{||A||}$
3. $\boldsymbol{B = b - \hat{b} = b - Pb = b - A\left(A^H A\right)^{-1} A^H b}$

   $\boldsymbol{B = b - \frac{AA^T}{A^T A} b}$
4. $\boldsymbol{C = c - \frac{AA^T}{A^T A} c - \frac{BB^T}{B^T B} c}$
5. Can continue for d, e, etc.

Full Column Rank Case

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<u>Full column rank case:</u>

- tall (or square)
- if solution exits it is unique
- either find unique solution
- or **least-squares approximation**
  - minimise L2 norm
- $\left|\left|Ax_{LS} - b\right|\right|_2 \leq \left|\left|Ax - b\right|\right|_2$
  - If solution exists, x_{LS} is that solution

<u>Least Squares Minimization:</u>

- $\hat{x}\ minimizes\ \left|\left|Ax - b\right|\right|$
  - $iff\ A^H A\hat{x} = A^H b$
- $A^H A\hat{x} = A^H b\ \Rightarrow A^H(b - A\hat{x}) = 0$
  - $\hat{x} = Px$

Full Row Rank Case

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<u>Full row rank case:</u>

- fat(or square)
- At least one solution exists
- If solution not unique
  - Let x_p be one solution to $Ax = b$
  - Any other solution
    - $x = x_p + z \, with \, z \in \mathcal{N}(A)$

<u>Minimum Norm Solution:</u>

- $minimize \, ||x|| \, s.t. \, Ax = b$
- $x_{MN} = A^H (AA^H)^{-1} b$

Singular Value Decomposition

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neither full column or row rank
- find Minimum Norn Least Squares Solution to Ax=b

SVD aim - 'diagonalize' and matrix - not just square and/or full rank ones

Every Matrix $A \in \mathbb{C}^{mxn}$ can be factored as:

- $\boldsymbol{A = U\Sigma V^H}$
- $U \; - U^H U = I, columns\ u_i, dimensions\ m\ x\ m$
- $\Sigma \; - non - negative\ diagonal\ entires\ \sigma_i, dimensions\ m\ x\ n$
- $V \; - V^H V = I, columns\ v_i, dimensions\ n\ x\ n$

$A^H A = V\Sigma^H U^H U \Sigma V^H = V\Sigma^H \Sigma V^H = V\Lambda V^{-1}$

- $\Sigma^H \Sigma \; - dimension\ n\ x\ n$
- Singlaur values squared, σ_i^2, are the eigenvalues of $A^H A$
  - and v_i are the eigenvectors

Similarly, $AA^H = U\Sigma V^H V \Sigma^H U^H = U\Sigma\Sigma^H U^H = U\Lambda U^{-1}$

- $\Sigma\Sigma^H \; - dimension\ m\ x\ m$
- Singlaur values squared, σ_i^2, are the eigenvalues of AA^H
  - and u_i are the eigenvectors

Eigenvector Decomposition of $\boldsymbol{A^H A\ and\ AA^H}$ will give us all the information for the SVD of A

Reduced SVD:

- truncate to get rid of zero valued eigen values/vectors
- so for A with rank r
  - $A = \sum_{i=1}^{r} \sigma_i \boldsymbol{u_i v_i^H}$

Pseudoinverses and SVD:

- the minimum norm least squares solution to $Ax = b$
  - $x = A^+ b$
  - $\hat{x} = V\Sigma^+ U^H b$
  - where A^+ is the **pseudoinverse** of A
    - $A = U\Sigma V^H$
    - $A^+ = V\Sigma^+ U^H$
      - $\Sigma^+ = \begin{pmatrix} \frac{1}{\sigma_1} & \\ & \ddots \end{pmatrix}$

Matrices U and V:

- $A, A^H A\ and\ AA^H have\ same\ rank\ r$
- All have r **non-zero** eigen values
- $A^H A\ has\ n - r\ zero\ eigen\ values$
- $AA^H\ has\ m - r\ zero\ eigen\ values$

Filling U and V:

- sort by descending magnitude of corresponding eigen values

$AV = U\Sigma$

- $A\boldsymbol{v_i} = \sigma_i \boldsymbol{u_i}$
- $v_i' s$ are in the row space of A $range(A^H)$
  - for i corresponding to non-zero eigen values
  - form a basis for the row space of A
    - since are orthonormal
- the $v_i' s\ corresponding\ to\ zero\ eigen\ values$
  - from a basis for $null(A)$
- $u_i' s$ are in the column space of A $range(A)$
  - for i corresponding to non-zero eigen values
  - form a basis for the column space of A
    - since are orthonormal
- the $u_i' s\ corresponding\ to\ zero\ eigen\ values$
  - from a basis for $null(A^H)$

Summarize:

- $\mathcal{R}(A) \quad = span\{U_1\}$
- $\mathcal{N}(A) \quad = span\{V_2\}$
- $\mathcal{R}(A^H) = span\{V_1\}$
- $\mathcal{N}(A^H) = span\{U_2\}$