Basics of Interest

17 October 2019 19:00

Compounding Intervals

- $period = 1/m$ (years
- $period_{rate} = \frac{r}{m}$
- *Growth over* k *periods*
  - $\left[1 + \frac{r}{m}\right]^k$
- continuous: growth = e^{rt}

Discount Factor

- $d_k = \frac{1}{\left(1 + \frac{r}{m}\right)^k} < 1$
- $d_k A$ *is the present value of* A
- continuous: $d_k = e^{-rt}$

Ideal Bank

- same interest rate to deposits and loans
- no costs
- same rate for any size of principal
- **Constant** ideal bank
  - rate is independent of length of time

Future and Present Values

- cash flow stream: $x_0, x_1, x_2, \dots, x_n$
- $FV = \sum_{k=0}^{n} x_k \left(1 + \frac{r}{m}\right)^{n-k} \quad (compounding)$
- $PV = \sum_{k=0}^{n} \frac{x_k}{\left(1 + \frac{r}{m}\right)^k} \quad (discounting)$

Two CF streams are equivalent if they can be transformed into each other by an ideal bank

Two CF stream are equivalent for a constant ideal bank with interest rate r **i.f.f their PVs are equal**

Net Present Value:

- using both +ve and -ve cash flows
- higher NPV -> more desirable investment
- **Repeatable activities** must be compared over the **same time horizon**

Running Present Value:

- for any $i < j < k$
  - $d_{i,k} = d_{i,j} d_{j,k}$
- $PV(0) = x_0 + d_1 x_1 + \cdots + d_n x_n$
- $= x_0 + d_{0,1}(x_1 + \cdots + d_{1,n} x_n)$
- $\rightarrow PV(0) = x_0 + d_{0,1} PV(1)$
- $PV(k) = x_k + d_{k,k+1} PV(k+1)$

Spot Rate:

- $\boldsymbol{s_t}$: *annualised interest rate* charged for money held from ***present time until time t***
- *compounding*: $\left(1 + \frac{s_t}{m}\right)^{mt}$
- *discount* $\frac{1}{(1+s_t)^t}$

Forward Rate:

- f_{t_1,t_2}
  - *borrow at* t_1, *repaid at* t_2 $(t_1 < t_2)$
  - rate agreed at $t = 0$
- *implied* forward rate $f_{i,j}$
  - $\boldsymbol{(1 + s_j)^j = (1 + s_i)^i (1 + f_{i,j})^{j-i}}$
  - can differ to *market* forward rate
- discount factors
  - $d_i, j = \frac{1}{\left(1 + \frac{f_{i,j}}{m}\right)^{j-i}}$
  - continuous: $e^{-f_{i,j}(j-i)}$

Fixed-Income Securities

17 October 2019 19:00

Geometric Series:

- $S_n = \sum_{k-1}^{n} x_k$
- $S_n = \frac{1-x^{n+1}}{1-x}$
- $for\ x < 1, S_\infty = \frac{1}{1-x}$

Annuity:

- pays fixed amount A periodically
- $PV = \sum_{k=1}^{n} d_k A$
- if spot rate curve is constant
  - $A = \frac{r(1+r)^n PV}{(1+r)^n - 1}$
  - use for **amortization**

Yield:

- interest rate required to go from
  - price to PV of stream of payments
- price P, face value F
- m coupon payments of C/m per year
- n remaining periods
- **yield to maturity (YTM)** λ
  - $P = \frac{F}{\left(1+\frac{\lambda}{\mathrm{m}}\right)^n} + \sum_{k=1}^{n} \frac{C/m}{\left(1+\frac{\lambda}{m}\right)^k}$
  - $P = \frac{F}{\left(1+\frac{\lambda}{\mathrm{m}}\right)^n} + \frac{C}{\lambda}\left\{1 - \frac{1}{\left(1+\frac{\lambda}{m}\right)^n}\right\}$

Macaulay Duration:

- receive n cash flows at $t_0 \dots t_n$
- $D = \sum_{k=1}^{n} w_k t_k\ (years)$
  - $w_k = \frac{PV(t_k)}{PV_{tot}}$
    $(computed\ w.r.t\ the\ yield)$
- $D = \frac{1+\frac{\lambda}{m}}{\lambda} - \frac{1+\frac{\lambda}{m}+n\left(\frac{c}{m}-\frac{\lambda}{m}\right)}{c\left[\left(1+\frac{\lambda}{m}\right)^n - 1\right]+\lambda}$

Modified Duration:

- $P(\lambda)\ price\ as\ a\ function\ of\ yield$
- $D_M = -\frac{1}{P(\lambda_0)}\frac{dP(\lambda)}{d\lambda}\Big|_{\lambda=\lambda_0}$
  - sensitivity = derivative
  - negative -> duration >= 0
  - divide -> relative
- $D_M \approx -\frac{1}{P}\frac{\Delta P}{\Delta\lambda} \rightarrow \boldsymbol{\Delta P \approx -D_M P \Delta\lambda}$
  - longer duration -> larger sensitivity

Relationship:

- $\boldsymbol{D_M = \frac{D}{1+\frac{\lambda}{m}}}$

Duration of a portfolio:

- weighted average of assets durations

Immunization:

- need to meet a price P in time D
  - $P = x_1 P_1 + x_2 P_2$
  - $D = \frac{x_1 P_1}{P} D_1 + \frac{x_2 P_2}{P} D_2$
- only works for two assets (only have 2 eqns)

Taylor expansion of $P(\lambda)$ provides more terms
$P(\lambda) = P(\lambda_0) + P'(\lambda_0)(\lambda-\lambda_0) + \frac{1}{2}P''(\lambda_0)(\lambda-\lambda_0)^2 + \cdots$

Convexity:

- 2nd term - Convexity
  - $C = \frac{1}{P(\lambda_0)}\frac{d^2P(\lambda)}{d\lambda^2}\Big|_{\lambda=\lambda_0}$
- Revised approximation (second-order)
  - $\Delta P \approx -D_M P \Delta\lambda + \frac{PC}{2}(\Delta\lambda)^2$

$$\left(1+\frac{\lambda}{\mathrm{m}}\right)^{n} \quad \lambda\left\{ \quad \left(1+\frac{\lambda}{m}\right)^{n}\right\}$$

- **Inverse dependence** between price and yield
- the longer the time to maturity -> the more sensitive the price is to the yield

- $\boldsymbol{D_M = \dfrac{D}{1+\frac{\lambda}{m}}}$

Mean-Variance Portfolio Theory

26 October 2019 01:02

$R = 1 + r$
$X_1 = (1+r)X_0$

Portfolio Returns:

- $\sum_{i=1}^{n} X_{0i} = X_o$, $\sum_{i=1}^{n} w_i = 1$
- $R = \sum_{i=1}^{n} R_i w_i$

Variance as a risk measure

- Assets
    - $E(r_i) = \bar{r_i}$
    - $var(r_i) = \sigma_i^2$
    - $cov(r_i, r_j) = \sigma_{ij}$

- Portfolio expected return
    - $\bar{r} = \sum_{i=1}^{n} w_i \bar{r_i}$
- Portfolio return variance
    - $\boldsymbol{\sigma^2 = \sum_{i,j=1}^{n} w_i \sigma_{ij} w_j}$

Mean-variance diagrams

- Feasible set / region
    - matches the constraints
- Minimum-variance set
    - left boundary of FS
    - min var point
- **Risk-averse**
    - same mean return
    - prefer lower variance
- **Greedy-investor**
    - same variance
    - prefer higher mean return
- **Efficient frontier**
    - **upper** half of min var set

Parameter estimation

- non overlapping periods
    - Don't get biased estimator

Estimate mean

(average the samples)

- $\hat{\bar{r}} = \frac{1}{n} \sum_{i=1}^{n} r_i$
    - $E(\hat{\bar{r}}) = \bar{r}$
    - $var(\hat{\bar{r}}) = \frac{1}{n} \sigma^2$

Estimate variance (and covariances)

(sample variance)

- $\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (r_i - \hat{\bar{r}})^2$
    - $E(\hat{\sigma}^2) = \sigma^2$

Markowitz Model:

- $\min \frac{1}{2} \boldsymbol{w}^T \Sigma \boldsymbol{w} = \min \frac{1}{2} \sigma_p^2$
- $s.t.$
    - $\boldsymbol{w}^T \bar{\boldsymbol{r}} - \bar{r_p} = 0$
    - $\boldsymbol{w}^T \boldsymbol{1} - 1 = 0$
    - *to not allow short selling*
        - $w_i \geq 0$
        - leads to a quadratic program
- finding weights for min variance, to reach $\bar{r_p}$

Solving:

- Lagrangian function
- $L(\boldsymbol{w}, \lambda, \mu) = \frac{\boldsymbol{1}}{\boldsymbol{2}} \boldsymbol{w}^T \Sigma \boldsymbol{w} - \lambda(\boldsymbol{w}^T \bar{\boldsymbol{r}} - \bar{r_p}) - \mu(\boldsymbol{w}^T \boldsymbol{1} - 1)$
- Optimality conditions
    - $\frac{\partial L}{\partial \boldsymbol{w}} = \Sigma \boldsymbol{w} - \lambda \bar{\boldsymbol{r}} - \mu \boldsymbol{1} = 0$ $(n \ equations)$
    - $\frac{\partial L}{\partial \lambda} = -\boldsymbol{w}^T \bar{\boldsymbol{r}} + \bar{r_p} = 0 \rightarrow \bar{\boldsymbol{r}}^T \boldsymbol{w} = \bar{r_p}$
    - $\frac{\partial L}{\partial \mu} = -\boldsymbol{w}^T \boldsymbol{1} + 1 = 0 \rightarrow \boldsymbol{1}^T \boldsymbol{w} = 1$
- **solvable if** Σ has full-rank and $\bar{\boldsymbol{r}}$ is not a multiple of $\boldsymbol{1}$

$$\begin{pmatrix} \boldsymbol{w} \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \Sigma & -\bar{\boldsymbol{r}} & -\boldsymbol{e} \\ -\bar{\boldsymbol{r}}^\top & 0 & 0 \\ -\boldsymbol{e}^\top & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{0} \\ -\bar{r}_\mathrm{P} \\ -1 \end{pmatrix} = f(\bar{r}_\mathrm{P}) .$$

- $var(\hat{\sigma}^2) = \dfrac{2\sigma^4}{n-1}$

CAPM

29 October 2019 09:36

Two-Fund Theorem:
- $\bar{r}_p^3 = \alpha\bar{r}_p^1 + (1-\alpha)\bar{r}_p^2$
- $(\boldsymbol{w}_3, \lambda_3, \mu_3) = \alpha(\boldsymbol{w}_1, \lambda_1, \mu_1) + (1-\alpha)(\boldsymbol{w}_2, \lambda_2, \mu_2)$
- importance
  - efficient portfolios can be made from two efficient funds
  - no need for anyone to buy individual stocks

Inclusion of Risk-Free Asset:
- $r_f = E(r_f)\ , \qquad \sigma = 0$
- $cov(r_i, r_f) = 0$
- Portfolio with r_f
  - $\bar{r}_p = \alpha r_f + (1-\alpha)\bar{r}$
  - $\sigma_p^2 = (1-\alpha)^2\sigma^2$
  - straight line on plot $(\sigma, \bar{r})$
    - y-intercept r_f

One-Fund Theorem:
- when risk-free borrowing and lending are available
- exists a single fund F of risky assests
- such that any efficient portfolio can be constructed from
  - $F\ (\boldsymbol{market\ portfolio} - market\ cap\ wieghted)$
  - r_f
- Markowitz problem:
  - $\min \frac{1}{2}\boldsymbol{w}^T\Sigma\boldsymbol{w}$
  - $s.t.$
    - $w_0 r_f + \boldsymbol{w}^T\bar{\boldsymbol{r}} = \bar{r}_p$
    - $w_0 + \boldsymbol{w}^T\boldsymbol{1} = 1$

Capital Market Line:
- In presence of the r_f
- $effecient\ frontier \to \boldsymbol{CML}$
  - $\bar{r} = r_f + \left(\frac{\bar{r}_M - r_f}{\sigma_M}\right)\sigma$

CAPM:
- $\bar{r}_i = r_f + \beta_i(\bar{r}_M - r_f)$ (SML)
  - $\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$
  - correlation with the market
  - expected excess rate of return w.r.t to r_f
  - $\beta_p = \sum w_i\beta_i$

Risk:
- actual return
- $r_i = r_f + \beta_i(r_M - r_f) + \epsilon_i$
  - CAPM implies
    - $E(\epsilon_i) = 0$
    - $cov(\epsilon_i, r_M) = 0$
- $\sigma_i^2 = \beta^2\sigma_M^2 + var(\epsilon_i)$
  - systematic risk
    - cannot be diversified
  - non-systematic
    - uncorrelated with market
    - can be reduced by diversification

CAPM Summary:
- only rewarded for risk that cannot be diversified away
- Risk is measured by β
- r_i is determind by how it fits into the market portfolio

CAPM as a pricing model:
- $r = \frac{Q-p}{P} \to \frac{\bar{Q}-P}{P} = r_f + \beta(\bar{r}_M - r_f)$
- $P = \frac{\bar{Q}}{1 + r_f + \beta(\bar{r}_M - r_f)}$

- Certainty Equivalent Form
  - $P = \frac{1}{1+r_f}\left(\bar{Q} - \frac{cov(Q, r_M)}{\sigma_M^2}(\bar{r}_M - r_f)\right)$
  - discount the 'risk-adjusted payoff'
  - at risk free rate r_f

General Principles of Risk

08 November 2019 14:49

<u>Utility functions</u>

- Ranking random wealth levels
- Alternative to weighted average
- $U(w) \to \mathcal{R}$
  - Depends on risk tolerance
  - Individual financial environment
  - Modify without changing ranking
    - Adding a constant
    - Multiply by a constant

<u>Risk Aversion</u>

- Concave functions
- $U[\alpha x + (1-\alpha)y] \geq \alpha U[x] + (1-\alpha)U[y]$
- $U'(x) > 0\,, U''(x) < 0$
- Arrow-Pratt
  - How risk aversion changes with wealth
  - $a(x) = -\dfrac{U''(x)}{U'(x)}$

<u>Certainty equivalent</u>

- $U(C) = E[U(x)]$

<u>Measuring Utility Functions:</u>

- Method 1
  - $A\ has\ p, B\ has\ 1-p$
  - For $p \in [0,1]$, ask for C
  - $C(e), where\ c = pA + (1-p)B$
  - $U(x) = C^{-1}(x)$
- Method 2
  - Select a parameterized function
    - e.g. $-e^{-ax}$
  - Ask a single 50/50
  - $-e^{-Ca} = -0.5e^{-Aa} - 0.5e^{-Ba}$

Connection to mean-var criterion:

- Quadratic utility: $U(x) = ax^2 - \frac{b}{2}x$
- Portfolio with random wealth level y
- $E[U(y)] = aE(y) - \dfrac{b}{2}E(y)^2 - \dfrac{b}{2}var(y)$

<u>Security:</u>

- Random Payoff d
- Price P

Type A Arbitrage:

- $P < 0, d = 0$

Type B Arbitrage:

- $P \leq 0, d \geq 0, \qquad \Pr(d > 0))0$

Ideal Market:

- Short sales
- No transaction costs
- Securities can be arbitrarily divided

<u>Linearity of Pricing:</u>

- $P = \sum_{i=1}^{n} \theta_i P_i$
- $\left(d = \sum_{i=1}^{n} \theta_i d_i\right)$

- In Ideal market, Absence of Type A -> LP

Portfolio Problem:

- $\max E\left[U\left(\sum_{i=1}^{n} \theta_i d_i\right)\right]$
- $s.t. \sum_{i=1}^{n} \theta_i P_i = W$
- If a solution exists:
  - Type A or B cannot exist

- $x^* = \sum_{i=1}^{n} \theta_i^* d_i$

- $\dfrac{\partial L}{\partial \theta_i} = E[U'(x^*)d_i] = \lambda P_i$

- $P_i = \dfrac{E[U'(x^*)d_i]}{RE[U'(x^*)]}$

Asset Price Dynamics

23 November 2019 12:29

Additive:

- $S(k+1) = aS(k) + u(k)$
- u(k) Normal , $a \geq 1$
- $E[s(k)] = a^k S(0)$
- **variance not proportional to price**
- **price shock can cause neg price**

Multiplicate:

- $S(k+1) = u(k)S(k)$
- $\ln S(k+1) = \ln S(k) + \ln\ u(k)$

- $\mathrm{w(k)} = \ln u(k)$ Normal
- $u(k) = e^{w(k)}$ - log normal RV

- $w(k) \sim \mathcal{N}(v, \sigma^2)$
- $X(k) \sim \mathcal{N}(X(0) + kv, k\sigma^2)$
- $S(k) = e^{X(k)}\ - \log normal$

- $\boldsymbol{E[u] = e^{v+\frac{\sigma^2}{2}}}$
- $\boldsymbol{var(u) = e^{(2v+\sigma^2)}(e^{\sigma^2} - 1)}$

Binomial Lattice and Multiplicative

- BL: multiple by $u\ with\ pr\ p, d\ with\ pr\ 1-p$
- MM: multiply by e^w, $w \sim \mathcal{N}(v\Delta t, \sigma^2 \Delta t)$

- match the expectation and the variance of $\boldsymbol{w}\ and\ \boldsymbol{BL}$

- $U = \ln u\,, \qquad D = \ln d\,, \qquad set\ D = -U$
- $(2p-1)U = v\Delta t$
- $4p(1-p)U^2 = \sigma^2 \Delta t$

- $\boldsymbol{p \approx \frac{1}{2} + \frac{1}{2}\frac{v}{\sigma}\sqrt{\Delta t}}$
- $\boldsymbol{\ln u \approx \sigma\sqrt{\Delta t}}$
- $\boldsymbol{\ln d \approx -\sigma\sqrt{\Delta t}}$

Basic Options Theory

23 November 2019 12:29

Call:

- Holder can purchase from writer at strike time at strike price
- $C = \max\{0, S(T) - K\}$

Put:

- Holder can sells to writer at strike time at strike price
- $P = \max\{0, K - S(T)\}$

American - Exercise any time before strike time

Inner Value:

- value of option at time $t \leq T$

Combinations:

- Butterfly Spread
  - Buy **two** calls with $K_1 and\ K_3$
  - Sell **two** calls with K_2 each
  - $K_1 < K_2 < K_3$

$\underline{Put - Call\ Parity}$

- $C - P + d_T K = S$
- buy one call, sell one put, lend $d_T K$
- payoff identical to the underlying stock

$\underline{Binomial\ Options\ Theory}$:

- In one time step
  - stock price S multiplied by $u\ or\ d$
  - risk free asset $d < R < u$
    - avoid Type B arb
- Match payoff of option to
  - a portfolio that invests **x** in stock and **b** in risk free
  - $C_u = \max\{0, uS - K\} = ux + Rb$
  - $C_d = \max\{0, dS - K\} = dx + Rb$
  - $x = \dfrac{C_u - C_d}{u - d}, b = \dfrac{uC_d - dC_u}{R(u - d)}$
  - payoff of replicating portfolio
  - $x + b = \boldsymbol{\dfrac{1}{R}\left(\dfrac{R-d}{u-d}C_u + \dfrac{u-R}{u-d}C_d\right) = C}$
  - option has same payoff, otherwise Type A arb
  - $C = \dfrac{1}{R}[qC_u + (1-q)C_d]\ , where\ q = \dfrac{R-d}{u-d}$

Real Options

- End of term/lease value is zero
- Value at node $\frac{1}{R}[cashflow + risk\ neutral\ value\ next\ node]$