Cauchy-Riemann Equations

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$z = x + iy \qquad w = f(z)$

$w = f(z) = u(x, y) + iv(x, y)$

use iC

For f(z) to be differentiable

- $U_x = V_y$
- $U_y = -V_x$

Analytic at z_0

if differentiable in a **neighbourhood** of z_0

Holomorphic

if analytical for $z \in \mathbb{C}$

Singularity

analytic everywhere expect for one value of Z

This value is the singularity

Properties of Analytic Functions

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$U_{xx} + U_{yy} = 0$
$V_{xx} + V_{yy} = 0$
U and V are said to be harmonic conjugates from Laplace's equations

Using this, find f(z) from u(x,y):

- check $U_{xx} + U_{yy} = 0$
- Get V_x and V_y from CR equations
- Partial integration to find V(x,y)
- now $f(z) = U + iV$
- **use iC**

Orthogonality

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Family of curves where $\boldsymbol{u = const \quad and \quad v = const}$

$$\left.\frac{dy}{dx}\right|_{u=const} * \left.\frac{dy}{dx}\right|_{v=const} = -1$$

So curves of U const and V const are **orthogonal**

Conformal Mappings

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<u>Conformal Mapping</u>:

- **preserves angles** in magnitude + sense

Theorem

the mapping by analytical $f(z)$ is conformal everywhere except at points where $f'(z) = 0$

The map 1/z

ALWAYS GET U and V in terms of X,Y FIRST

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Z-plane general equation for circles and straight lines:

$$\boldsymbol{\alpha\left(x^2+y^2\right)+\beta x+\gamma y+\Delta=0} \quad \text{in Z-plane}$$

$$\boldsymbol{\alpha+\beta U-\gamma V+\Delta\left(U^2+V^2\right)=0} \quad \text{in W-plane}$$

α	Δ	Z-plane	W-plane
!=0	!=0	Circle	Circle
!=0	==0	Circle	Line
==0	==0	Line	Line
==0	!=0	Line	Circle

$$\alpha == 0 \to Line\ in\ Z$$
$$\alpha\ != 0 \to Circle\ in\ Z$$

$$\Delta == 0 \to Line\ in\ W$$
$$\Delta\ != 0 \to Circle\ in\ W$$

<u>Mobius Transformations</u>

The rules on the left extend to any map of the type:

$$f(z)=\frac{az+b}{cz+d}$$

fixed point
w = z
find roots

Cauchy's Theorem

21 December 2017 14:25

if F(z) is analytic everywhere within and on a contour C, then

$$\oint_C F(z)dz = \mathbf{0}$$

So integral only has value !=0 when Contour contains parts which are not analytical,

like singularities

Poles and Residues

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Simple pole z=a if

$$F(z) = \frac{g(z)}{z-a}$$

g(z) analytical at z=a, but F(z) is not

Pole of multiplicity m

$$F(z) = \frac{g(z)}{(z-a)^m}$$

e.g. m=2, -> double pole

REWRITE F(Z) as separate factors first

Residue:

Simple Pole:

Residue = $\lim_{z\to a}(z-a)F(z) = \lim_{z\to a}\frac{(z-a)g(z)}{(z-a)} = \boldsymbol{g(a)}$

Pole of multiplicity m: (formula given)

Residue = $\lim_{z\to a}[\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}((z-a)^m F(z))] = \frac{1}{(m-1)!}\frac{d^{m-1}g}{dz^{m-1}}\bigg|_{z=a}$

Residue with Simple Zero

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<u>Definition:</u>

g(z) has a simple zero if at z=a:

g(a) = 0;

g'(a) != 0;

Residue of $F(z) = \frac{h(z)}{g(z)}$ $at\ z = a, where\ g(z) has\ simple\ zero\ at\ z = a$

$$\text{Res} = \frac{h(a)}{g\prime(a)}$$

for zeros of multiplicity m:

$$\text{Res} = \frac{h(a)}{\frac{g^{m}(a)}{m!}}$$

Residue Theorem

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If the only singularities within a contour C, are poles then:

$$\oint_C F(z)dz = 2\pi i\ (\boldsymbol{Sum\ of\ residues}\ of\ F(z)\ at\ its\ \boldsymbol{poles}\ within\ C)$$

Improper Integrals and Jordan's Lemma

22 December 2017 13:15

MUST STATE IF SATISFY COND. IF USING JL

Used to evaluate improper integrals of type $\int_{-R}^{R} e^{imx} f(x)dx$

Step 1: (H_R^+ indicates upper semicircle)

$$\oint_C e^{imz} f(z)dz = \int_{-R}^{R} e^{imx} f(x)dx + \int_{H_R^+} e^{imz} f(z)dz$$

residue theorem = target integral + extra that need to show =0
then target integral = value from residue theorem

the only singularities are poles
if m>0, tend
if m=0 other tend

Step 2:
Need

$$\lim_{R\to\infty} \int_{H_R} e^{imz} f(z)dz = 0$$

- $m > 0, then\ |F(z)| \to 0\ as\ R \to \infty$
- $m = 0, then\ |F(z)| \to 0\ faster\ then\ \left|\frac{1}{z}\right| \to 0\ as\ R \to \infty$
- m < 0 - use lower half of plane - pi to 2pi

Step 3:
If F(z) satisfies conditions for Jordan Lemma then

$$\int_{-\infty}^{\infty} e^{imx} f(x)dx$$
$$= 2\pi i\ (sum\ of\ residues\ of\ poles\ of\ e^{imz}F(z)\ in\ \boldsymbol{upper\ half\ of\ the\ plane})$$

Poles on the real axis

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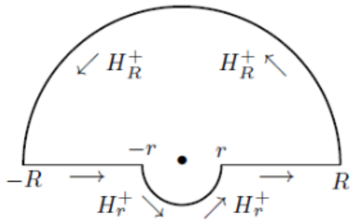
Adjustment to Jordan Lemma when pole lies on the real axis:



$$\oint_C \frac{f(z)}{z}dz = \int_{-R}^{-r}\frac{f(x)}{x}dx + \int_{H_r}\frac{f(z)}{z}dz + \int_{r}^{R}\frac{f(x)}{x}dx + \int_{H_R}\frac{f(z)}{z}dz$$

limits taken: $\boldsymbol{R \to \infty \ \ and \ \ r \to 0}$

$$2\pi i\begin{bmatrix} sum\ of \\ residues \\ on\ axis \\ and\ within\ C\end{bmatrix} = \int_{-\infty}^{\infty}\frac{f(x)}{x}dx + \lim_{r\to 0}\int_{H_r}\frac{f(z)}{z}dz \ + \lim_{R\to\infty}\int_{H_R}\frac{f(z)}{z}dz$$

Want last part along $to = 0$

To solve second last integral:

- $z = re^{i\theta} \quad \theta: \pi \dots 2\pi \ \left(for\ H_r\ going\ anti-clockwise\right)$
- $dz = izd\theta$
- $sub\ into\ \lim_{r\to 0}\int_{H_r}\frac{f(z)}{z}dz\ and\ get\ value$
    - **make use of r -> 0**

Integrals around the unit circle

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Integrals of the type $\int_0^{2\pi} f(cos\theta, sin\theta)d\theta = \oint_C f(z)dz$

Step 1:

$$z = e^{i\theta}$$
$$d\theta = \frac{dz}{iz}$$
$$cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$
$$sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

Step 2:

calculate residues within C

Step 3:

$$\int_0^{2\pi} f(cos\theta, sin\theta)d\theta = 2\pi i\ [sum\ Step\ 2]$$

Introduction

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$$\mathcal{L}[f(t)] = \bar{f}(s)$$

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

Make use of **Re(s)** > 0 in show that's

Library of LT's

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Funciton	$f(t)$	$\bar{f}(s)$	Region of Convergence
Exponential	e^{at}	$\frac{1}{s-a}$	$Re\ (s-a) > 0$
Constant	1	$\frac{1}{s}$	$Re\ s > 0$
Sine(im e^{iat})	$\sin(at)$	$\frac{a}{s^2+a^2}$	$Re\ s > 0$
Cosine(re e^{iat})	$\cos(at)$	$\frac{s}{s^2+a^2}$	$Re\ s > 0$
Polynomial	t^n	$\frac{n!}{s^{n+1}}$	$(n \geq 0)\ \ Re\ s > 0$
Heaviside	$H(t-a)$	$\frac{e^{-sa}}{s}$	$(a > 0)\ \ Re\ s > 0$
Dirac	$\delta(t-a)$	e^{-sa}	$a \geq 0$
Shift in s	$e^{at}f(t)$	$\bar{f}(s-a)$	$Re\ (s-a) > 0$
Shift in t	$H(t-a)f(t-a)$	$e^{-sa}\bar{f}(s)$	$a > 0$
Convolution(prove via double integral)	$f * g$	$\bar{f}(s)\bar{g}(s)$	$\int_0^t f(u)g(t-u)du$
Integral	$\int_0^t f(u)du$	$\frac{\bar{f}(s)}{s}$	Convolution of $f(t)\ and\ 1$
Derivative (prove via integration by parts)	$f'(t)$	$s\bar{f}(s) - f(o)$	use $g(t) = f'(t)$ to get higher order derivates - sub in recursively

roof via recurrence

start integral from a

$\tau = t - a$

Convolution proof:
- u=t
- $\tau = t - u$

Need to memorise

Nonstandard inverse LT's

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When no direct inverse can be used from library

- Partial Fractions
- Shift in s
- Shift in t
- Convolution
- Differentiate w.r.t. a

Solving ODE's with LT

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can plug in IC to DE to find missing IC

Use:

- $\ddot{x}(t) = s^2\bar{x}(s) - sx(0) - \dot{x}(0)$
- $\dot{x}(t) = s\bar{x}(s) - x(0)$

to transform ODE into the form below:

$$\bar{x}(s) = \frac{\bar{f}(s)}{s^2 + \alpha s + {\omega_0}^2} + \frac{(s+\alpha)x(0) + \dot{x}(0)}{s^2 + \alpha s + {\omega_0}^2}$$

- **Denominator is same auxiliary equation, but with parameter s**
- can use above straight away as long as LHS of ODE is of form $\ddot{x} + \alpha\dot{x} + {\omega_0}^2 = f(t)$

- PI(from forcing function f(t)) + CF(factors in initial conditions)
- both $x(0)\ \ and\ \ \dot{x}(0)$ have to at same point in time
- break up RHS into simple functions whose inverse LT can be found form library

- can find general version for when f(t) not given
  - use convolution property of LTs to express PI in time domain