Intro

15 January 2018 20:33

$$SNR = \frac{signal\ power}{noise\ power}$$

External noise - interference
Internal noise - channel properties

Efficiency:
number of bits transmitted of unit: power, time, and bandwidth

Reliability:
expressed in terms of SNR or probability of error

Information Theory:

- Shannon capacity formula
  - $C = Wlog(1 + SNR)bps$
    - $W(Hz) - channel\ bandwidth$
    - C - channel capacity

Probability

Normal Distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} \quad \text{for } -\infty < x < \infty$$

$$F_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-\frac{(y-m)^2}{2\sigma^2}} dy$$

Uniform Distribution

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

$$E[X] = \frac{a+b}{2}$$

$$\sigma_X{}^2 = \frac{(b-a)^2}{12}$$

Joint Distribution

Joint distribution function for two random variables X and Y

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

Joint probability density function

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

Properties

1) $F_{XY}(\infty,\infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(u,v)\,du\,dv = 1$
2) $f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy$
3) $f_Y(x) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx$
4) X, Y are **independent** $\Leftrightarrow f_{XY}(x,y) = f_X(x) f_Y(y)$
5) X, Y are **uncorrelated** $\Leftrightarrow E[XY] = E[X]E[Y]$

If normal - uncorrelated -> independent

Random Process

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Time-varying function which assigns to each outcome, s, a function of time

$X(t, s_j)$

Fixed S_j - sample function - signal

Fixed t - random variable
- $\mu_X(t) = E[X(t)]$
  - **mean varies with time**

Autocorrelation - measures correlation between two samples

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x, y; t_1, t_2)dxdy$$

$R_X(constant) - difference\ between\ t_1 and\ t_2 is\ a\ constant$

$AutoCorr = R_{ff}(t_1, t_2) - \mu_f(t_1)(t_2)$

For gaussian
- uncorrelated -> independent

Stationary - homogenous

-check these as well for t_1-t_2

$\boldsymbol{R_X(0) = E[X^2(t)]}$
$\boldsymbol{R_X(t) = R_X(-t)}$
$\boldsymbol{|R_X(t)| \le R_X(0)}$

Strict-Sense Stationary
- all underlying **n** RVs time **shifted by a constant**
- previous joint PDF = new PDF
- *implies WSS*

1st order

$f_X(x;t)$ is invariant over time

2nd order

$R_X(t_1, t_2) = R_X(t_2 - t_1)$

Wide-sense Stationary
both mean and autocorrelation are time invariant

$\mu_X(t) = \mu_X$
$R_X(t, t + \tau) = R_X(\tau)$

Asked about $F\left(X(t_1), X(t_2)\right)$

$$\boldsymbol{E\left[F\left(X(t_1), X(t_2)\right)^2\right] = G(R_X(0), R_x(\tau)), find\ tau}$$

PSD - $\boldsymbol{S_X(f)} = \int_{-\infty}^{\infty} \boldsymbol{R_X(\tau)e^{-j2\pi f\tau}d}\tau \ge 0$

Passing through an LTI filter

$y = convolution\ of\ x\ and\ impule\ response\ h$

Mean

$\mu_Y(t) = \mu_X(t) * h(t)$
$\mu_Y(t) = \mu_X * h(0) \quad (WSS)$

Autocorrelation

$R_Y(t, u) = h(t) * [h(u) * R_X(t, u)]$
$R_Y(\tau) = h(\tau) * [h(-\tau) * R_X(\tau)] \quad (WSS)$
$R_Y(\tau) = h(\tau) * [h^*(-\tau) * R_X(\tau)] \quad (WSS\ - complex\ X(t))$

Autocorrelation

$S_Y(f) = |H(f)|^2 S_X(f)$

Baseband and Passband

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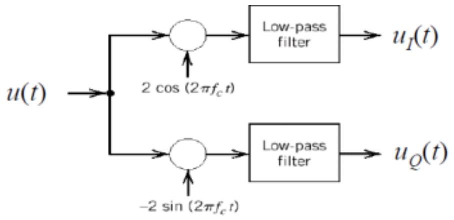
$$E = \int_{-\infty}^{+\infty} |s(t)|^2 \, dt = \int_{-\infty}^{+\infty} |S(f)|^2 \, df$$

$$P = \frac{1}{T} \int_{-T/2}^{+T/2} |s(t)|^2 \, dt$$

<u>Baseband to passband</u> - send two baseband in parallel

$$u(t) = u_I(t)\cos(2\pi f_c t) - u_Q(t)\sin(2\pi f_c t)$$

Demodulation



physical baseband channel - real baseband signals
physical passband channel - complex baseband signals

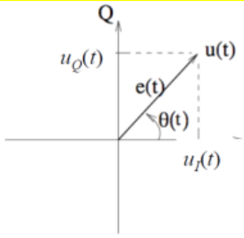
Hilbert transform - if FT of x(t) is X(f)

FT of $\hat{x}(t) = -jsgn(f)X(f)$

<u>Pre-envelope</u>

$$u_+(t) = u(t) + j\hat{u}(t)$$
$$u_-(t) = u(t) - j\hat{u}(t)$$

<u>Passband - complex envelope</u>



$$\tilde{u}(t) = u_I(t) + ju_Q(t) = e(t)e^{j\theta(t)}$$ **Complex envelope**

$$e(t) = \sqrt{u_I^2(t) + u_Q^2(t)} \qquad \theta(t) = \tan^{-1}\frac{u_Q(t)}{u_I(t)}$$

$$u_I(t) = e(t)\cos\theta(t), \; u_Q(t) = e(t)\sin\theta(t)$$

$$\boxed{u(t) = u_I(t)\cos(2\pi f_c t) - u_Q(t)\sin(2\pi f_c t)}$$ In terms of I and Q components

$$\boxed{u(t) = e(t)\cos(2\pi f_c t + \theta(t))}$$ In terms of envelope and phase

$$\boxed{u(t) = \mathrm{Re}\{\tilde{u}(t)e^{j2\pi f_c t}\}}$$ In terms of complex envelope

Noise in communications systems

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<u>White Noise</u>

$$S_N(f) = \frac{N_0}{2} \qquad -\infty < f < \infty \; - \boldsymbol{flat\ PSD\ - is\ a\ random\ process}$$

$$P_N = \int_{f_1}^{f_2} S_N(f)\,df$$

$$\boldsymbol{R_N(\tau) = \frac{N_0}{2}\delta(\tau)}$$

Power of zero mean Gaussian:
= Variance

White - samples are uncorrelated
- since Gaussian -> independent

process is Gaussian if pdf of samples follows Gaussian distribution
-CLT

N Power in LP Filter = N_0B

N Power in BP Filter = $2N_oB$

n(t) in BP can be split into n_I(t) and n_Q(t) - (both base-band random processes)

- If noise $n(t)$ has zero mean, then so do $n_I(t)$ and $n_Q(t)$.
- If noise $n(t)$ is Gaussian, then so are $n_I(t)$ and $n_Q(t)$.
- If noise $n(t)$ is stationary, then so are $n_I(t)$ and $n_Q(t)$.
- If noise $n(t)$ is Gaussian and its power spectral density $S(f)$ is symmetric with respect to central frequency f_c, then $n_I(t)$ and $n_Q(t)$ are statistically independent.
- $n_I(t)$ and $n_Q(t)$ have the same variance (i.e., same power) as $n(t)$.

$$S_{N_I}(f) = S_{N_Q}(f) = N_0 \qquad |f| \le B$$
$$P_{N_I} = P_{N_Q} = 2N_0B$$

Noise performance of DSB

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$$SNR_0 = \frac{avg\ power\ of\ message\ signal\ at\ receiver}{avg\ power\ of\ noise}$$

$P = E\{m^2(t)\} = P\ of\ message$

$SNR\ (dB) = 10\log_{10}(SNR)$

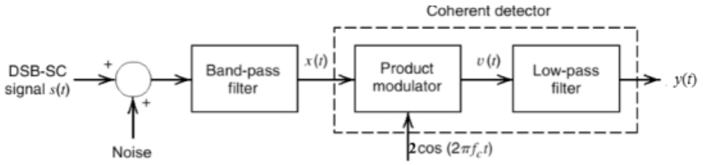
increase P_T - P_S increases
- uses same P_T to compare schemes
- use $SNR_{baseband}$ as ref.

Baseband SNR

- $P_S = P_T = P$
- $P_N = 2W * \frac{N_0}{2} = \boldsymbol{WN_0}$
- $SNR_{baseband} = \frac{P_T}{N_0 W}$

DSB-SC SNR

- received signal
  - $x(t) = Am(t)\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$



- multiply with $2\cos(2\pi f_c t)$ and pass through LPF
  - $\boldsymbol{y(t) = Am(t) + n_I(t)}$
    - $n_Q(t)$ removed!

- $P_S = E\{A^2 m^2(t)\} = A^2 P$
- $P_N = 2W * N_0 = 2N_0 W$

- $SNR = \frac{A^2 P}{2N_o W}$
- $P_T = E\{A^2 m^2(t)\cos(2\pi f_c t)\} = \frac{A^2 P}{2}$

- $SNR_{DSB-SC} = \frac{P_T}{N_0 W} = SNR_{baseband}$

- DSB-SC has same SNR, but less interference as at higher freq. transmission

Noise performance of SSB and AM

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SSB SNR

- moduated signal: $\boldsymbol{s(t)} = \frac{\boldsymbol{A}}{\mathbf{2}}\boldsymbol{m(t)}\cos(\mathbf{2}\boldsymbol{\pi f_c t}) - \frac{\boldsymbol{A}}{\mathbf{2}}\boldsymbol{\widehat{m}(t)}\sin(\mathbf{2}\boldsymbol{\pi f_c t})$
  - m(t) and ^m(t) have power P
- $P_T = \frac{A^2 P}{4}$
- received signal: $y(t) = \frac{A}{2}m(t) + n_I(t)$
- $P_S = \frac{A^2 P}{4}$
- $P_N = (W + W) * \frac{N_0}{2} = N_0 W$
- $SNR_{SSB} = \frac{A^2 P}{4N_0 W} = \frac{P_T}{N_0 W} = SNR_{baseband}$
- same SNR as baseband - less interference
- same SNR as DSB-SC - **req. half the bandwidth**

$$\boldsymbol{SSB = m(t)\cos\omega_c t \mp m_h(t)\sin\omega_c t}$$

USB MINUS LSB PLUS

USB - real M for real S, Im M for S as well

Full AM SNR

$\mu = \frac{m_p}{A}$

- $P_T = \frac{A^2 + P}{2}$
- received signal after LPF: $y(t) = A + m(t) + n_I(t)$
  - $P_S = P$
- $P_N = 4W * \frac{N_0}{2}$
- $SNR_{AM} = \frac{P}{2N_0 W}$
- $SNR_{baseband} = \frac{P_T}{N_0 W} = \frac{A^2 + P}{2N_0 W}$
- $SNR_{AM} = \frac{P}{A^2 + P} SNR_{baseband}$
- $\frac{P}{A^2 + P} < 1$
  - lower SNR than baseband for coherent AM

Envelope detection:

- RC circuit
- if small noise, same as above
- large noise - signal is lost

Summary of AM SNR

24 May 2018 00:06

(De-) Modulation Format	Output SNR	Transmitted Power	Baseband Reference SNR	Figure of Merit (= Output SNR / Reference SNR)
AM Coherent Detection	$\frac{P}{2N_0W}$	$\frac{A^2+P}{2}$	$\frac{A^2+P}{2N_0W}$	$\frac{P}{A^2+P}<1$
DSB-SC Coherent Detection	$\frac{A^2P}{2N_0W}$	$\frac{A^2P}{2}$	$\frac{A^2P}{2N_0W}$	1
SSB Coherent Detection	$\frac{A^2P}{4N_0W}$	$\frac{A^2P}{4}$	$\frac{A^2P}{4N_0W}$	1
AM Envelope Detection (Small Noise)	$\frac{P}{2N_0W}$	$\frac{A^2+P}{2}$	$\frac{A^2+P}{2N_0W}$	$\frac{P}{A^2+P}<1$
AM Envelope Detection (Large Noise)	Poor	$\frac{A^2+P}{2}$	$\frac{A^2+P}{2N_0W}$	Poor

A: carrier amplitude, P: power of message signal, N_0: single-sided PSD of noise, W: message bandwidth.

Noise performance of FM

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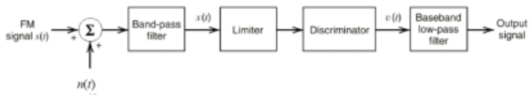
$$f_i(t) = f_c + k_f m(t)$$

$$s(t) = A\cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right]$$

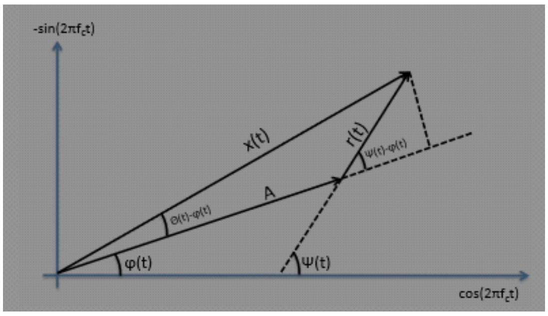
$$\Delta f = k_f m_p$$

$$\beta = \frac{\Delta f}{W} \quad W = message\ bandwidth$$

Transmission Bandwidth
$B_T = 2(\Delta f + W) = 2W(\beta + 1)$



BP Filter: $\pm f_c \pm \frac{B_T}{2}$
Limiter - remove amplitude variations



$$x(t) = s(t) + n(t),$$
$$= A\cos[2\pi f_c t + \phi(t)] + r(t)\cos[2\pi f_c t + \psi(t)].$$

$$v(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} \simeq k_f m(t) + n_d(t).$$

$$n_d(t) = \frac{1}{2\pi A}\frac{dn_Q(t)}{dt}$$

$$S_{N_o}(f) = \frac{f^2}{A^2}N_0 \qquad |f| \le W$$

FM SNR

- $P_S = k_f^2 P$
- $P_N = \frac{2N_0W^3}{3A^2}$
- $SNR_O = \frac{3A^2k_f^2P}{2N_0W^3}$
- $P_T = \frac{A^2}{2}$
- $SNR_{baseband} = \frac{P_T}{N_0W} = \frac{A^2}{2N_0W}$
- $SNR_{FM} = \frac{3k_f^2P}{W^2}SNR_{baseband}$
- since $\frac{k_f}{W} = \frac{\beta}{m_p}$
- $SNR_{FM} = 3\beta^2\frac{P}{m_p^2}SNR_{baseband} \propto \beta^2 SNR_{baseband}$

- doubling bandwidth, B_T , gives quadratic increase in SNR

Pre/de-emphasis for FM

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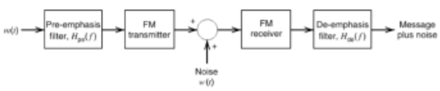
Threshold:

- carrier-to-noise ratio
  - $\rho = \dfrac{A^2}{2N_0 B_T}$
  - below 10, FM receiver breaks
  - small changes to phase can result in changes of 2pi

Improve Output SNR:

- noise is proportional to square of frequency at LPF output
- -increase message amplitude at higher frequency
  - increase SNR at high freq.

Pre-emphasis and De-emphasis



- $H_{pe}(f)$: artificially emphasizes high frequency components of the message prior to modulation (before noise is introduced).
- $H_{de}(f)$: de-emphasizes high frequency components at the receiver, and restore the original PSD of the message.
- In theory, $H_{pe}(f) \propto f$, $H_{de}(f) \propto 1/f$
- This can improve output SNR by around 13 dB.

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Improvement Factor:

$$I = \frac{P_N old}{P_N new} = \frac{2W^3}{3\int_{-W}^{W} f^2 |\mathrm{H_{de}}(f)|^2 df}$$

Example

- (a) Pre-emphasis filter

$$H_{pe}(f) \approx 1 + j\frac{f}{f_0}$$

$$f_0 = 1/(2\pi rC), \quad R \ll r, \quad 2\pi frC \ll 1$$

- (b) De-emphasis filter

$$H_{de}(f) = \frac{1}{1 + jf/f_0}$$

- Improvement

$$I = \frac{2W^3}{3\int_{-W}^{W} f^2/(1+f^2/f_0^2)df} = \frac{(W/f_0)^3}{3[(W/f_0) - \tan^{-1}(W/f_0)]}$$

- (Full) AM: SNR performance is 4.8 dB worse than a baseband system, transmission bandwidth is $B_T = 2W$
- DSB: SNR performance is identical to a baseband system, transmission bandwidth is $B_T = 2W$
- SSB: SNR performance is again identical, transmission bandwidth is only $B_T = W$
- FM: SNR performance is 15.7 dB better than a baseband system, transmission bandwidth is $B_T = 2(\beta + 1)W = 12W$
  (with pre- and de-emphasis SNR performance is increased by about 13 dB with the same transmission bandwidth).

Digital representation of signals

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sample at $f_s = 2W$

Quantisation: Uniform

- difference between actual and taken = q

$$f_Q(q) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} \le q \le \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases}$$

has variance $P_N = \frac{\Delta^2}{12}$

n bits

2^n levels

$2^n\Delta$ dynamic range

max error = $\frac{\Delta}{2}$

$$n = \log_2 \frac{1}{\max\_error}$$

$$m_p = 2^{n-1}\Delta$$

$$SNR_0 = \frac{P_S}{P_N} = \frac{12P}{\Delta^2} = \frac{12P2^{(n-1)2}}{m_p^2} = \frac{3P}{m_p^2}2^{2n}$$

$$\boldsymbol{SNR_0(dB) = 6n + 10\log_{10}\left(\frac{3P}{m_p^2}\right)}$$

Companding:

- compress signal
    - tighter grouping
    - reduces peak to average power ratio
- quantise using **uniform quantizer**
- expand signal
    - ideally - inverse of compression
- can increase SNR
    - **make output SNR insensitive to peak:avg power ratio**

Baseband digital transmission

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Analog Communication - quality via SNR

Digital Communication - **quality via BER**
- less susceptible to noise

Matched filter

- filter on receiver that makes instantaneous power at sampling time as large as possible compared to noise component

impulse response of filter:

$$\boldsymbol{h_{opt}(t) = kg(T_b - t)}$$

T_B - symbol period
g(t) - transmitter pulse shape
k -gain

if $h_{opt}(t)$

$$\boldsymbol{\eta_{max} = \frac{2E}{N_0} = SNR}$$

- proportional to signal energy

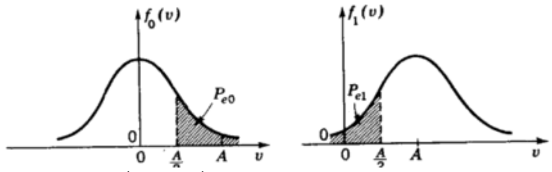
for Rectangular pulse:
match with same pulse but scaled by k
equivalent: integrate

n-ary PAM:

$$\frac{\log n \text{ bits per symbol}}{T \text{ secs per symbol}} = \boldsymbol{\frac{\log n}{T} \, bps}$$

Error:

- binary PCM on-off
- noise is AWGN $N \sim N(0, \sigma^2)$
- if 0, signal $Y = N < \lambda$
  - $Y \sim N(0, \sigma^2)$
- if 1, signal $Y = A + N > \lambda$
  - $Y \sim N(A, \sigma^2)$



$$P_e(\lambda) = p_1 P_{e1} + (1 - p_1) P_{e0}$$

Choose λ so that $\frac{dP_e(\lambda)}{d\lambda} = 0$

$$\rightarrow \text{minimise } P_e(\lambda) \rightarrow -\frac{\sigma^2}{A} \ln \frac{p_1}{1 - p_1} \rightarrow \boldsymbol{\lambda = \frac{A}{2}} \text{ for equi probable}$$

$$P_e = P_0 \int_{A(T)} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx + P_1 \int^{B(T)} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx.$$

$$\frac{dP_e}{dT} = -P_0 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(A(T))^2}{2\sigma^2}} + P_1 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(B(T))^2}{2\sigma^2}} = 0.$$

$$\boldsymbol{P_e = Q\left(\frac{A}{2} * \frac{1}{\sigma}\right)}$$

$$E_{pulse} = \frac{A^2}{T_b} \quad \rightarrow \quad \text{avg energy ber bit}, E_b = \frac{A^2}{2T_b}$$

$$P_N = \sigma^2 = \frac{N_0}{2T_b}$$

$$\boldsymbol{P_e = Q\left(\sqrt{\frac{E_b}{N_o}}\right)}$$

- small increase in SNR - HUGE increase in reliability

Digital modulation - coherent

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<u>ASK - A = 0,A</u>

$\underline{x(t) = A(t)cos(2\pi f_c t) - A\epsilon\{0, A\}}$
$\underline{y_o(t) = A(t) + n_i(t)}$

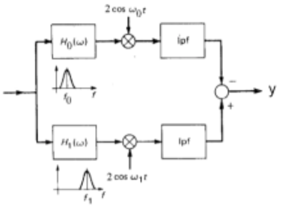
$$E_{pulse} = \frac{A^2}{2T_b}\ (half\ from\ cosine)$$

$$E_b = \frac{A^2}{4T_b}, \sigma^2 = \frac{N_0}{T_b}$$

$$\boldsymbol{P_{e,ASK} = Q\left(\frac{A}{2\sigma}\right) = Q\left(\sqrt{\frac{E_b}{N_o}}\right) - same\ as\ baseband}$$

<u>FSK</u>

$f = f_c + (0,1)$



$$\text{LPF output on each branch} = \begin{cases} A + \text{noise} & \text{if symbol present} \\ \text{noise} & \text{if symbol not present} \end{cases}$$

$$\boldsymbol{y(t) = \pm A + [n_1(t) - n_0(t)]}\ \{+= 1, -= 0, n_i(t) = n_1\ and\ n_o\}$$

-> noise variance doubles

$$\boldsymbol{P_{e,FSK} = Q\left(\frac{A}{\sqrt{2}\sigma}\right) = Q\left(\sqrt{\frac{E_b}{N_o}}\right)}$$

<u>PSK</u>

<u>BPSK</u> - change phase of cosine by pi
- time by 1 or -1
- -1 - angle increases by pi

$\underline{x(t) = A(t)cos(2\pi f_c t) - A\epsilon\{-A, A\}}$
$\underline{y_o(t) = A(t) + n_i(t)}$

$$\boldsymbol{P_{e,PSK} = Q\left(\frac{A}{\sigma}\right) = Q\left(\sqrt{\frac{2E_b}{N_o}}\right) < better\ than\ baseband}$$

non-coherent demodulation

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$$P_{e,ASK,noncoherent} \approx \mathbf{\frac{1}{2}e^{-\frac{A^2}{8\sigma^2}}} \geq Q\left(\frac{A}{2\sigma}\right)\ (coherent)$$

- lose some performance, bust for large SNR, lose less

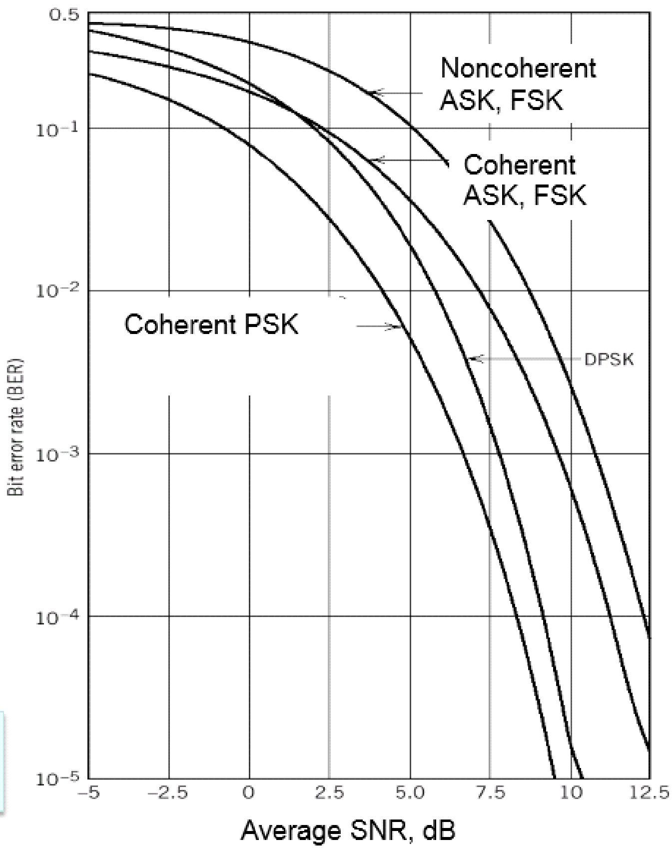
$$P_{e,FSK,noncoherent} \approx \mathbf{\frac{1}{2}e^{-\frac{A^2}{4\sigma^2}}} \geq Q\left(\frac{A}{\sqrt{2}\sigma}\right)\ (coherent)$$

$$P_{e,DPSK,noncoherent} \approx \mathbf{\frac{1}{2}e^{-\frac{A^2}{2\sigma^2}}} \geq P_{e,PSK} = Q\left(\frac{A}{\sqrt{2}\sigma}\right)\ (coherent)$$

- cannot do PSK using envelope
- DPSK: $b_n = b_{n-1} * a_n$

Scheme	Bit Error Rate
Coherent ASK	$Q(A/2\sigma)$
Coherent FSK	$Q(A/\sqrt{2}\sigma)$
Coherent PSK	$Q(A/\sigma)$
Noncoherent ASK	½ $\exp(-A^2/8\sigma^2)$
Noncoherent FSK	½ $\exp(-A^2/4\sigma^2)$
DPSK	½ $\exp(-A^2/2\sigma^2)$

Caution: ASK and FSK have the same bit error rate with respect to average SNR.



Entropy and source Coding

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$$\boldsymbol{I(s) = \log \frac{1}{p}}$$

$Information\ in\ a\ sybmol, s$
$a\ unique\ symbol\ s, has\ probaility, p$

Discrete Memoryless Source

- sequence of symbols from a finite alphabet
- successive symbols **are independent and identically distributed**

Source Entropy

$$H(S) = E[I(s_k)] = \sum_{k=1}^{K} p_k I(s_k) = \boldsymbol{-\sum_{k=1}^{K} p_k \log_2 p_k}$$

- units: bits/symbol

Average Codeword Length

$l_k = number\ of\ bits\ to\ encode\ symbol\ k$

$$\boldsymbol{\bar{L} = \sum_{k=1}^{K} p_k l_l}$$

Typical sequence of length N

$$\boldsymbol{p(S_N)} = p_1^{Np_1} \times p_2^{Np_2} \times \cdots \times p_K{}^{Np_K}$$

$$\boldsymbol{L_N} = \log_2 \frac{1}{p(S_N)} = -\log(p_1^{Np_1} \times p_2^{Np_2} \times \cdots \times p_K{}^{Np_K})$$

$$\rightarrow \boldsymbol{L_N} = -N \sum_{k=1}^{K} p_k \log_2 p_k = \boldsymbol{NH(S)}$$

L_N is average number of symbols to code a sequence of length N, so

$$\bar{L} = \frac{L_N}{N} = H(S) \qquad \rightarrow \boldsymbol{\bar{L} \geq H(S)}$$

Huffman Coding:

- **assign each of the two smallest probabilities a bit, then merge**
  - **repeat until only one symbol left**
- (read backwards for codeword of a symbol)
- not unique - can reorder probabilities
- the most efficient **prefix code** - **any codeword does not contain, at its start, the code of any other codeword**
- drawback - need to know probabilities

Channel capacity

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Conditional Entropy

$$H(X|Y=y_k)=\sum_{j=0}^{J-1}p(x_j|y_k)\log_2\left[\frac{1}{p(x_j|y_k)}\right]$$

Mean Entropy:

$$H(X|Y)=\sum_{k=0}^{K-1}\sum_{j=0}^{J-1}p\left(x_j,y_k\right)\log_2\left[\frac{1}{p(x_j|y_k)}\right]$$

- amount of uncertainty after observing output

Mutual Information:

$$\boldsymbol{I(X;Y)}=H(X)-H(X|Y)=\sum_{k=0}^{K-1}\sum_{j=0}^{J-1}p\left(x_j,y_k\right)\log_2\left[\frac{p(x_j|y_k)}{p(y_k)}\right]$$

- $I(X;Y)=I(Y;X)\geq 0$
- at worst, Y is independent of X, and I(X;Y) = 0

Channel Capacity

$$\boldsymbol{C=\max_{p(x_j)} I(X;Y)}$$

AWGN Channel

$$\boldsymbol{C=Blog_2(1+SNR)}=Blog_2\left(1+\frac{P}{N_0B}\right)\ \boldsymbol{bps}$$

- B - channel bandwidth

Simple Error Checks - offers no correction

Repetition, halves bit rate

Parity bit, detect odd amount of error

Block Codes:

- encode source block into longer codeword
- detect - not valid code word
- correct - match corrupted block to a valid one via its proximity

- (n,k) - binary linear block code
  - k bits of source
  - n bit codeword
  - code rate, $R=\frac{k}{n}$
- generator matrix, G
  n by k matrix
- **Hamming weight**
  - $w_H(a)=\boldsymbol{number\ of\ non-zero\ elements\ in\ the\ vector}$
- **Hamming Distance**
  - $d_H(a,b)=\boldsymbol{w_H(a+b)}$

- error detection
  $d_{min}=\min\{w_H(codeword),c\epsilon C,c\neq 0\}$
  number of errors that can be detected = $\boldsymbol{d_{min}-1}$
- error correction
  - number of errors that can be corrected = $\boldsymbol{\frac{d_{min}-1}{2}}$