Books

14 January 2018 20:43

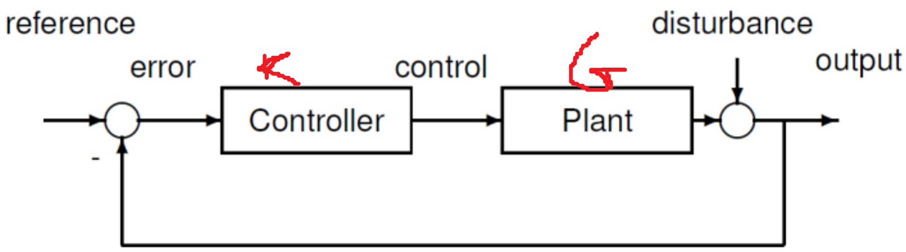
(Use latest editions)

- *Modern Control System Theory and Design*, S.M. Shinners, Wiley. Good for the state-space approach.
- *Modern Control Systems*, R.C. Dorf and R.H. Bishop, Addison-Wesley. Good for the transfer function approach.
- *Feedback Control of Dynamic Systems*, G.F. Franklin, J.D. Powell and A. Emami-Naeini, Addison-Wesley. Good for practical design issues.
- *Feedback Control Systems*, C.L. Phillips and R.D Harbor, Prentice-Hall. Follows the lecture course closely, and is useful for third year course too.

Aims of Control

14 January 2018 20:47

Use of feedback to force a given plant to exhibit desired characteristics



$error = reference - output$

keep it small - output close to reference

Base Model around existing G's (plant) frequency response

Design Controller

Feedback For Static Systems

14 January 2018 21:16

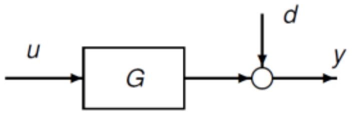
<u>Block Diagrams:</u>

- A way of representing equations involving signals

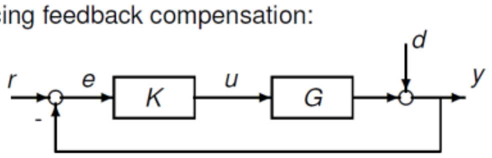
Problem with open loop: Can't make y=u
Don't know disturbance before hand

Consider the open loop static system:

$$y = Gu + d.$$



Introducing feedback compensation:



R and D - external signals
E, U, Y - internal signals

Start with E: solve for

$$e = F(r, d, K, G)$$

Then solve for U and Y , eliminating e with above equation

$$\rightarrow Y = \frac{GK}{1+GK} r + \frac{1}{1+GK} d \quad , |GK| is \gg 1$$

$$y \rightarrow r$$

Good tracking and **disturbance rejection require high loop gain**

Sensitivity:

- $S = \dfrac{1}{1+GK} (system\ to\ the\ left)$
  - reduce sensitivity - high loop gain

Use of Laplace Transforms in Circuit

14 January 2018 20:48

convert ODE into simple quadratic

Using LT tables convert back into time

Impedance

$$Z(s) = \frac{V(s)}{I(s)}$$

- Resistor, $Z(s) = R$
- Inductor, $Z(s) = sL$
- Capacitor, $Z(s) = \frac{1}{sC}$

Transfer Function

$$G(s) = \frac{V_2(s)}{V_1(s)}$$

- ratio of LT of output / input
- for a linear system
- all initial conditions assumed to be zero

Standard Format

14 January 2018 20:48

Decompose an nth order DE into a set of n 1st order DE's
written in matrix-vector form

$$\dot{x}(t) = Ax(t) + Bu(t)\ (\textbf{state equation}),\ x(0) = x_0\ (\textbf{initial condition})$$
$$y(t) = Cx(t) + Du(t)\ (\textbf{output equation})$$

$$\begin{array}{llll} A : n \times n & (\textbf{system matrix}); & B : n \times n_u & (\textbf{input matrix}) \\ C : n_y \times n & (\textbf{output matrix}); & D : n_y \times n_u & (\textbf{direct feedthrough}) \\ x(t) : n \times 1 & (\textbf{state vector}); & u(t) : n_u \times 1 & (\textbf{input vector}) \\ y(t) : n_y \times 1 & (\textbf{output vector}) & & \end{array}$$

Example: 2-input 2-output state-variable model

Suppose $\ddot{y}_1 + k_1\dot{y}_1 + k_2 y_1 = u_1 + k_3 u_2$, $\dot{y}_2 + k_4 y_2 + k_5 \dot{y}_1 = k_6 u_1$.
Define $x_1 = y_1$, $x_2 = \dot{y}_1$, $x_3 = y_2$. Then

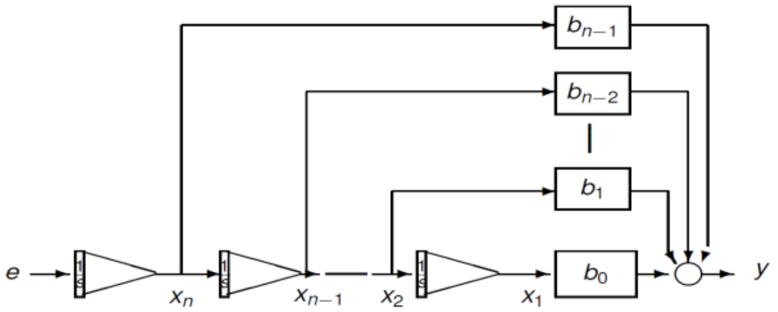
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_2 x_1 - k_1 x_2 + u_1 + k_3 u_2 \\ \dot{x}_3 &= -k_5 x_2 - k_4 x_3 + k_6 u_1 \\ y_1 &= x_1, \qquad y_2 = x_3 \end{aligned}$$

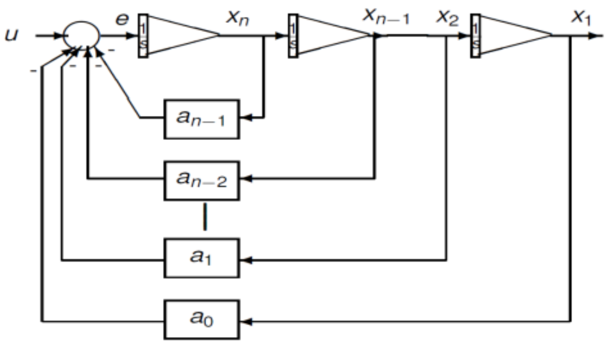
$$\overbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}}^{\dot{x}} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ -k_2 & -k_1 0 & \\ 0 & -k_5 & -k_4 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}^{x} + \overbrace{\begin{bmatrix} 0 & 0 \\ 1 & k_3 \\ k_6 & 0 \end{bmatrix}}^{B} \overbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}^{u}$$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x}.$$

Simulation Diagrams

05 February 2018 22:45





<u>Three Elements:</u>

- **integrator - transfer function $\frac{1}{s}$**
- **pure gain**
- **summer**

Models from transfer functions:

- Divide numerator and denominator by s^n:

$$y(s) = \frac{b_{n-1}s^{-1} + b_{n-2}s^{-2} + \cdots + b_0 s^{-n}}{1 + a_{n-1}s^{-1} + a_{n-2}s^{-2} + \cdots + a_0 s^{-n}}\, u(s).$$

- Set

$$e(s) := \frac{u(s)}{1 + a_{n-1}s^{-1} + a_{n-2}s^{-2} + \cdots + a_0 s^{-n}}$$

  so that

$$y(s) = [b_{n-1}s^{-1} + b_{n-2}s^{-2} + \cdots + b_0 s^{-n}]\, e(s).$$

- This gives

$$e(s) = u(s) - [a_{n-1}s^{-1} + a_{n-2}s^{-2} + \cdots + a_0 s^{-n}]\, e(s).$$

Matrix Exponential function

21 January 2018 23:41

- In matrix form this yields the state-variable model:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-2} & b_{n-1} \end{bmatrix} x.$$

- Note the direct connection with the coefficients of the transfer function:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0}.$$

For a Square Matrix A - **Matrix Exponential Function**

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots + \frac{A^kt^k}{k!} + \cdots$$

Useful Properties:

- P1: $e^0 = I$
- P2: $e^{T\Lambda T^{-1}} = Te^{\Lambda}T^{-1}$ for any nonsingular T
- P3: $e^{(\alpha+\beta)A} = e^{\alpha A}e^{\beta A}$
- P4: $e^{-A} = (e^A)^{-1}$
- P5: $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$
- P6: $\mathcal{L}(e^{At}) = (sI - A)^{-1}$

Test signals

02 June 2018 12:27

Unit impulse:

- $\delta(t) -> 1$

Unit Step:

- $u(t) = 1, t \geq 0 \rightarrow \frac{1}{s}$

Unit Ramp

- $u(t) = t, t \geq 0 \rightarrow \frac{1}{s^2}$

$$cos\omega t \rightarrow \frac{s}{s^2 + \omega^2}$$

$$sin\omega t \rightarrow \frac{\omega}{s^2 + \omega^2}$$

First-order

14 January 2018 20:48

Steady-state - not affected by T
Transient - affected by T

Steady-State

02 June 2018 12:23

Suppose that

$$y(s) = G(s)u(s),$$

where $G(s)$ is a given transfer function.
Suppose that the limit

$$\lim_{t \to \infty} y(t)$$

exists.
Using the **final value theorem**:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sy(s) = \lim_{s \to 0} sG(s)u(s).$$

If $u(t)$ is a unit step, this becomes:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sG(s)\frac{1}{s} = \lim_{s \to 0} G(s) = G(0).$$

$G(0)$ is often called the **d.c. gain** of the system.

Second-order

02 June 2018 12:23

for $ky(t) + B\dot{y}(t) + M\ddot{y}(t) = u(t)$

$$G(s) = K\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$K = \frac{1}{k} \quad , \qquad \omega_n = \sqrt{\frac{k}{M}} \quad , \qquad \zeta = \frac{B}{2\sqrt{kM}}$$

$G(s)$ *has poles at*:

- $\boldsymbol{p_1, p_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}}$

ζ	Case	Description
>1	Overdamped	p_1 and p_2 are real, negative and unequal
=1	Critically Damped	p_1 and p_2 are real, negative and equal
$0 \leq 1$	Underdamped	p_1 and p_2 are complex conjugates, negative real parts
< 0	Unstable	p_1 and p_2 positive real parts

Unit Step response:

- underdamped system
- *damped natrual frequency*: $\boldsymbol{\omega_d = \omega_n\sqrt{1 - \zeta^2}}$

$$y(t) = 1 - \frac{1}{\sqrt{1 - \zeta^2}}e^{-\zeta\omega_n t}\sin(\boldsymbol{\omega_d t} + \alpha)$$

- exponential - envelope
  - $time\ constant\ \tau = \dfrac{1}{\zeta\omega_n}$
- sine - Im part of the poles

Specifications: underdamped 2nd order

- $T_r - rise\ time\ (10\%\ to\ 90\%)$
- $M_p - peak\ value = \boldsymbol{1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}$
- $T_p - time\ to\ first\ peak = \boldsymbol{\dfrac{\pi}{\sqrt{1 - \zeta^2}\omega_n}}$
- $y_{ss} - steady\ state\ value = 1\ - unit\ step$
- $\%overshoot = \dfrac{M_p - y_{ss}}{y_{ss}} \times 100 = \boldsymbol{100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}$
- $T_s - settling\ time$
  - time to enter range of $y_{ss} \pm d\ tube$
  - $\boldsymbol{T_s \approx 4\tau = \dfrac{4}{\zeta\omega_n}}\ for\ 2\%$

5% overshoot, $\boldsymbol{\zeta = 0.7 \approx \frac{1}{\sqrt{2}}}$

As ζ *increases from* 0 *to* 1

- **tracking properties, $\boldsymbol{T_s}$ *and* $\boldsymbol{\%overshoot}$ *decrease***
- **speed of response, $\boldsymbol{T_p}$ *and* $\boldsymbol{T_r}$ *increase***

Overdamped: T_r, T_s *and* y_{ss} *well defined*, *but not peak values*, M_p *and* T_p

Interpret Pole locations

02 June 2018 12:23

Each pole is either real - contributes a pure exponential
Or occur in complex conjugate pairs - contributing an exponentially decaying sinusoid

Can approximate a system a system if G(s) has **dominant poles of 2nd order**,
i.e. other poles are located far into left half plane

Steady-state accuracy

02 June 2018 12:23

System type N: number of free integrators in the loop (s^{-1}'s)

Table:

The following table gives e_{ss} for type 0, 1, and 2 systems for step, ramp and parabolic inputs.

Type	$r(s) = \frac{1}{s}$	$r(s) = \frac{1}{s^2}$	$r(s) = \frac{1}{s^3}$	Error constants
0	$\frac{1}{1+K_p}$	∞	∞	$K_p = \lim_{s\to 0} Q(s)$
1	0	$\frac{1}{K_v}$	∞	$K_v = \lim_{s\to 0} sQ(s)$
2	0	0	$\frac{1}{K_a}$	$K_a = \lim_{s\to 0} s^2Q(s)$

Steady-state error constants

Bode to Nyquist

05 June 2018 03:17

If phase decreasing - moving clockwise

reducing gain - moving towards origin

BIBO stability

02 June 2018 14:51

System is **unstable when one or more poles in the RHS plane**

BIBO stability when poles all poles in **open LHS plane (not including Im axis)**

Marginally stable - when all poles lie in **closed LHS** (plane) - poles on **Im axis must also be simple poles**

Criteria

14 January 2018 20:49

For 1st and 2nd order, sufficient to have all coefficients have same sign

Fir higher order, this is necessary but not sufficient

Routh Array

Degree N, -> **N+1 rows**

First two rows:

Nth Coefficient, skipping one until end
Nth-1 Coefficient skipping one until end

Next rows:

$$\boldsymbol{i^{th} entry} = -\frac{1}{row\ above, first\ element}\begin{vmatrix} two\ rows\ above, 1st\ el & two\ rows\ above, i+1\ entry \\ one\ row\ above, 1st\ el & one\ row\ above, i+1\ entry \end{vmatrix}$$

first element in a row = 0, and other elements will be non-zero in same row

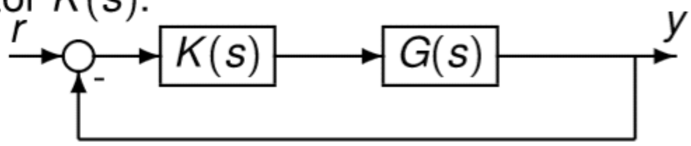
replace 0 with ϵ

 # unstable roots = number of sign changes in first column

Application

02 June 2018 14:52

- Consider the feedback system involving a plant $G(s)$ and compensator $K(s)$.



- The closed-loop transfer function $H(s)$ is

$$H(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}.$$

Can use Routh Array to see if can find a **constant K, if $\boldsymbol{K(s) = K}$**,

can make feedback loop stable, if G(s) by itself gives an unstable closed-loop

If not, $\boldsymbol{K(s)\ needs\ to\ be\ dynamic}$

Frequency Response

14 January 2018 20:49

A transfer function G(s):

- **frequency-response function** $G(j\omega) = \boldsymbol{|G(j\omega)|e^{\angle G(j\omega)}}$

- **Nyquist Diagram** characterisation
  - $Re\{G(j\omega)\}$
  - $Im\{G(j\omega)\}$

- Real and Imaginary axis
- $\pi\ to\ -\pi\ clockwise$
- top half is for $-\omega$, bottom half for $+\omega$

Nyquist Analysis

02 June 2018 18:42

$F(s) = 1 + G(s) = 0 \ \rightarrow characterstic\ equation\ of\ closed - loop, H(s) = \frac{G(s)}{1 + G(s)}$

$Z - \ zeros\ of\ F(s) - closed\ loop\ ploes\ - unkown$
$P - poles\ of\ F(s) - open\ loop\ poles\ - known$

$Nyquist\ Contour$ - covers entire RHP
As s traverses Nyquist contour, let plot of F be value of F(s) as s traverses

$N\ - number\ of\ clockwise\ encirclement\ of\ origin\ by\ F(s)$
$\boldsymbol{N = Z_{RHP} - P_{RHP}}$

If we have plot of G(s)

- can just **number of encirclements of -1 +0j for G(s) plot**

Nyquist Stability Criterion

02 June 2018 18:42

- Consider the feedback loop shown in the figure.
  1. Assume that $G(s)$ has no imaginary axis poles.
  2. Let P be the number of unstable poles of $G(s)$.
  3. Let Z denote the number of unstable poles of the closed-loop system $H(s) = G(s)/(1 + G(s))$
  4. Let Γ denote the Nyquist contour.
  5. Let Γ_G, called the **Nyquist diagram** of $G(s)$, denote the closed curve defined by the mapping $s \to G(s)$ as s traverses Γ clockwise.
  6. Let N be the number of clockwise encirclement of the $-1+j0$ point by Γ_G.
- Then $N = Z - P$.

Only stable when $\boldsymbol{Z = N + P = 0}$

<u>Nyquist Contour has 4 regions:</u>

- the point s=0, G(0)-> D.C. gain
- jw-axis -> G(jw)
  - use fact that $j\omega \to \infty$
  - look what happens to magnitude and phase of poles
  - phase -> **sum of effect on individual poles**
    - **if net of poles is increase, angle of G decreases**
    - **usual poles:**
      - **0 to 90**
      - **180 to 90**
- Infinite arc, generally maps to the origin
  - $\lim_{s\to\infty} G(s)$
- lower half of jw-axis reflect upper half (opposite direction)

find intersection with real axis:

- **Routh Array for 1+KG(s) = 0 , gives G(jw$_1$)**
  - for K making system marginally stable
- **to find $\omega\ for\ G(j\omega_1)$ form aux equation from first row involving K**

alternatively, set Im part of G(jw) to 0

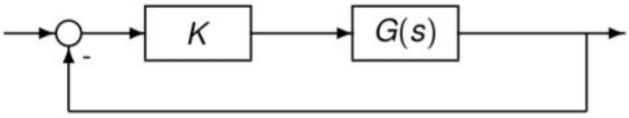
Then do Z=N+P

Gain Compensation

02 June 2018 18:43

$\boldsymbol{clockwise\ encirclement\ of}\ -\frac{1}{K} + j0$

- It follows that the Nyquist diagram of $KG(s)$ encircles the $-1 + j0$ point if and only if the Nyquist diagram of $G(s)$ encircles the $-\frac{1}{K} + j0$ point, and we have the following modified Nyquist criterion.
- Consider the feedback loop in the figure. Let
  1. P be the number of unstable poles of $G(s)$.
  2. Z be the number of unstable closed-loop poles.
  3. N be the number of clockwise encirclement of the $-\frac{1}{K} + j0$ point by the Nyquist diagram of $G(s)$.
- Then $N = Z - P$.
- That is, the closed-loop is stable if and only if $-N = P$.



Relative Stability

02 June 2018 18:43

gain margin - factor to make a stable closed-loop gain become marginally stable

at least 2

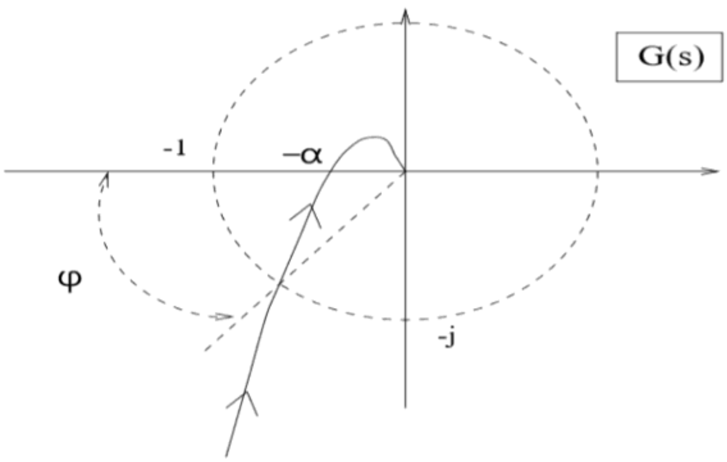
phase margin - min angle the Nyquist diagram must be rotated to intersect with $-1 + j0$

without changing the gain

at least 45 degree

$\boldsymbol{set\ |G(j\omega)| = 1}$

$\boldsymbol{solve\ for\ \omega_1, sub\ into\ \angle G(jw)}$

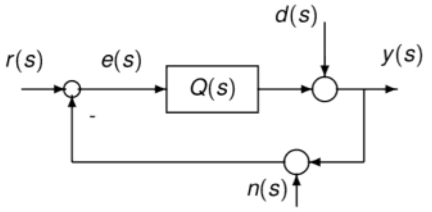


Gain Margin = $1/\alpha$, Phase Margin = ϕ

Prototype feedback system

14 January 2018 20:49

$Q(s) = G(s)K_c(s)$: open − loop gain
$r(s)$: reference signal
$y(s)$: output signal
$e(s)$: error signal
$d(s)$: output disturbance
$n(s)$: sensor noise



<u>Error signal:</u>

$$e(s) = \frac{1}{1+Q(s)}[r(s) - d(s) - n(s)]$$

<u>Output signal:</u>

$$y(s) = \frac{1}{1+Q(s)}d(s) + \frac{Q(s)}{1+Q(s)}[r(s) - n(s)]$$

<u>Closed-loop TF:</u>

$$\boldsymbol{H(s) = \frac{Q(s)}{1+Q(s)}}$$

<u>Sensitivity TF:</u>

$$\boldsymbol{S(s) = \frac{1}{1+Q(s)}}$$

$$\boldsymbol{H(s) + S(s) = 1}$$

Design Objectives

02 June 2018 22:52

Steady-state accuracy:

- need y to track r, so want
- $\boldsymbol{e_{ss}}$ ***to be small***

- if input $r(t) = e^{j\omega_0 t}$
  - $e_{ss}(t) = \boldsymbol{S(j_{\omega 0})} e^{j\omega_0 t}$
  - need $|\boldsymbol{S(j\omega_0)}|$, the steadt-state gain of error at frequency ω_0 , $to\ be \ll \mathbf{1}$
  - hence, $\boldsymbol{Q(j\omega_0) \gg 1}$

- good steady-state accuracy requires **large open-loop gain** over wide range of frequencies
  - **large closed-loop bandwidth**

Output disturbance rejection:

- **also requires large open-loop gain**
  - same as large closed-loop bandwidth

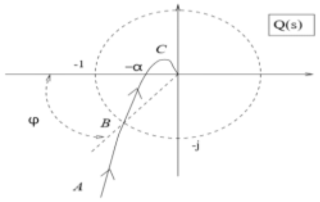
Sensor noise attenuation

- Need $Q(j\omega_0) \ll 1$
- requires small open-loop gain
  - same as small closed-loop bandwidth

Trade-off:

- SSA and ODR conflict with SNA
- usually noise signal has much higher frequency
- so:
  - **Large gain at low frequencies**
  - **Small gain at high frequencies**
  - **usually aim for 40dB decrease per decade after, i.e.** $1/s^2$

Transient Response and Stability margins:



The diagram is divided into three regions:

- A is the low frequency region ($|Q(j\omega)| \gg 1$).
- B is the **crossover** region ($|Q(j\omega)| \approx 1$).
- C is the high frequency region ($|Q(j\omega)| \ll 1$).

Also want low open-loop gain at high frequency (s~0)
to have good gain and phase margin

Shape transient response by regions B and C
Steady-state response from region A

Compensation

02 June 2018 23:14

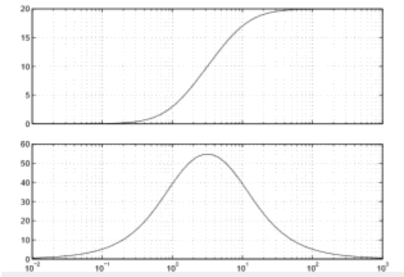
Constant, $\boldsymbol{K_C(s) = K}$

- multiplies gain uniformly for all frequencies
- **if K<1**
  - **increase gain and phase margins**
  - **reduce overshoot** and closed-loop bandwidth
- but
  - **slower response time** - increase in rise time

Phase-lead controller: - hard to balance

- $K_C(s) = K\dfrac{1+s/\omega_0}{1+s/\omega_p} = \boldsymbol{K\dfrac{\omega_p\, s+\omega_0}{\omega_0\, s+\omega_p}}$, $\boldsymbol{\omega_0 < \omega_p}$

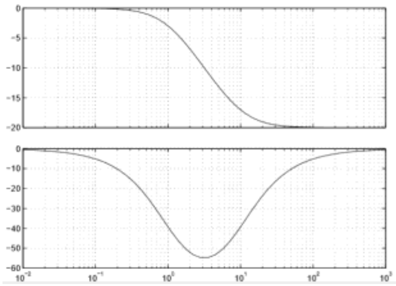
- increasing gain above ω_p
  - bad for margins
- adds **phase-lead (+)** between ω_0 *and* ω_p
  - so choose $\boldsymbol{\omega_0}$ ***and*** $\boldsymbol{\omega_p}$ **to be in the crossover region B**
  - **Crossover freq = geometric mean between $\boldsymbol{\omega_0}$ *and* $\boldsymbol{\omega_p}$**



Phase-lag controller: - choose K to increase low freq. gain

- $K_C(s) = K\dfrac{1+s/\omega_0}{1+s/\omega_p} = \boldsymbol{K\dfrac{\omega_p\, s+\omega_0}{\omega_0\, s+\omega_p}}$, $\boldsymbol{\omega_p < \omega_0}$

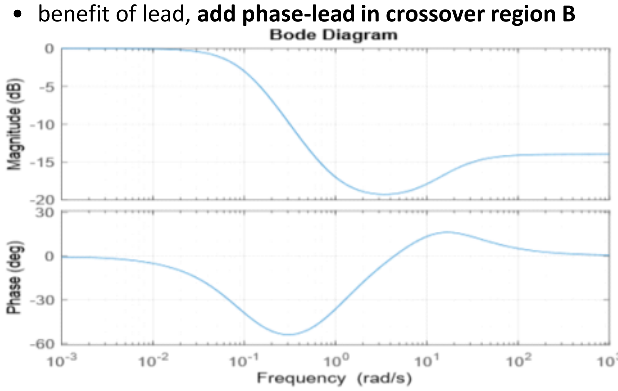
- decreasing gain above ω_0
  - good for margins
- adds **phase-lag (-)** between ω_p *and* ω_0
  - so choose $\boldsymbol{\omega_p}$ ***and*** $\boldsymbol{\omega_0}$ **to be in the lower frequency range A**



Lag-lead controller:

- $K_C(s) = K\dfrac{1+s/\omega_{0_1}}{1+s/\omega_{p_1}}\dfrac{1+s/\omega_{0_2}}{1+s/\omega_{p_2}}$, $\boldsymbol{\omega_{0_1} > \omega_{p_1}\ ,\omega_{0_2} < \omega_{p_2}}$

- Main benefits of Lag, **decrease gain, add lag in region A**
- benefit of lead, **add phase-lead in crossover region B**



Ziegler-Nichols tuning rules

02 June 2018 23:48

Special form of	Name	K(s)
lag	PI - improve steady state accuracy up to a point much lower than crossover frequency	$K_p + \frac{K_i}{s}$
lead	PD - add phase lead at crossover, but can cause instability via high freq. gain	$K_p + K_d s$
lag-lead	PID - best of both	$K_p + \frac{K_i}{s} + K_d s$

By experimenting with a constant gain compensator, get K_{po} and T_o

- when closed-loop becomes marginally stable
  - K_{po} value of the constant
  - T_o period of oscillations

For a G(s) which is

- stable
- Type 0 - rules out PD
- Overdamped (real poles and zeros)

Compensator is defined either by:

- **P:** $\boldsymbol{K(s) = 0.5K_{po}}$
- **PI:** $\boldsymbol{K(s) = 0.45K_{po} + \frac{0.54K_{po}/T_0}{s}}$
- **PID:** $\boldsymbol{K(s) = 0.6K_{po} + \frac{1.2K_{po}/T_o}{s} + 0.075K_{po}T_0 s}$