CW

April 8, 2020

[110]:
```
import sys

if not sys.warnoptions:
    import warnings
    warnings.simplefilter("ignore")
```

1 Q1 Regression Methods

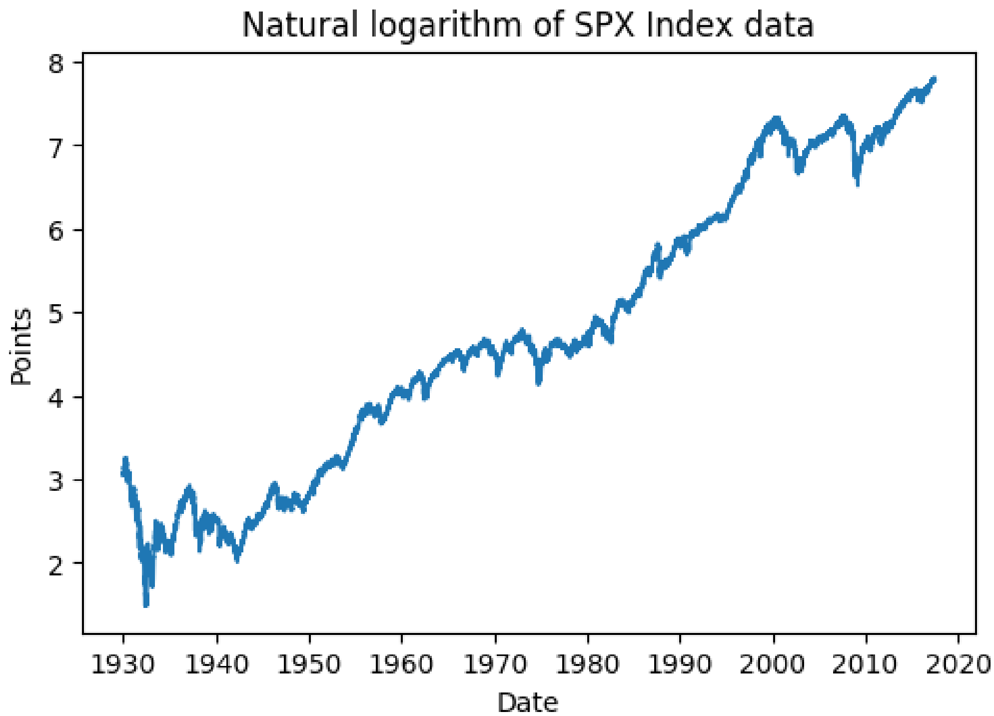
[111]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
%matplotlib inline

from tabulate import tabulate
import datetime as dt
```

1.1 Q1.1 Processing stock pirce data in Python

1.1.1 Q1.1.1

[3]:
```
px = pd.read_csv("./../data/priceData.csv").set_index('date').dropna()
dates = px.index
date_axis = [dt.datetime.strptime(d,'%d/%m/%Y').date() for d in dates]
logpx = np.log(px)
```

[4]:
```
plt.figure(dpi=100)
plt.plot(date_axis,logpx)
plt.title('Natural logarithm of SPX Index data')
plt.xlabel('Date')
plt.ylabel('Points')
plt.show()
```

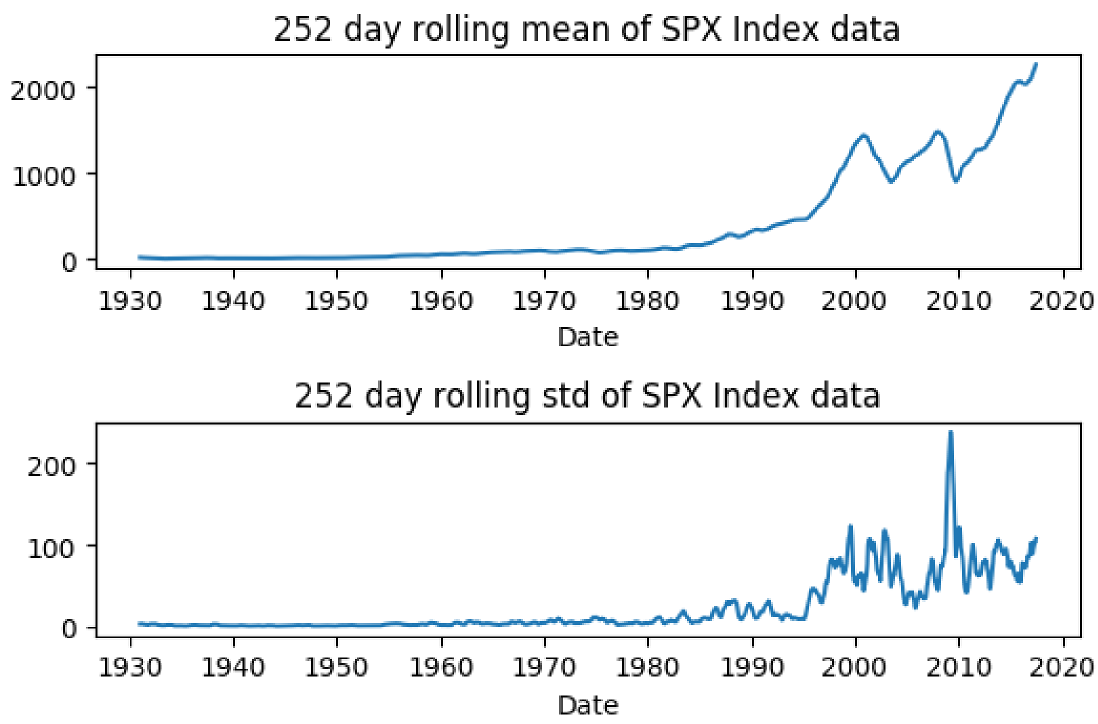


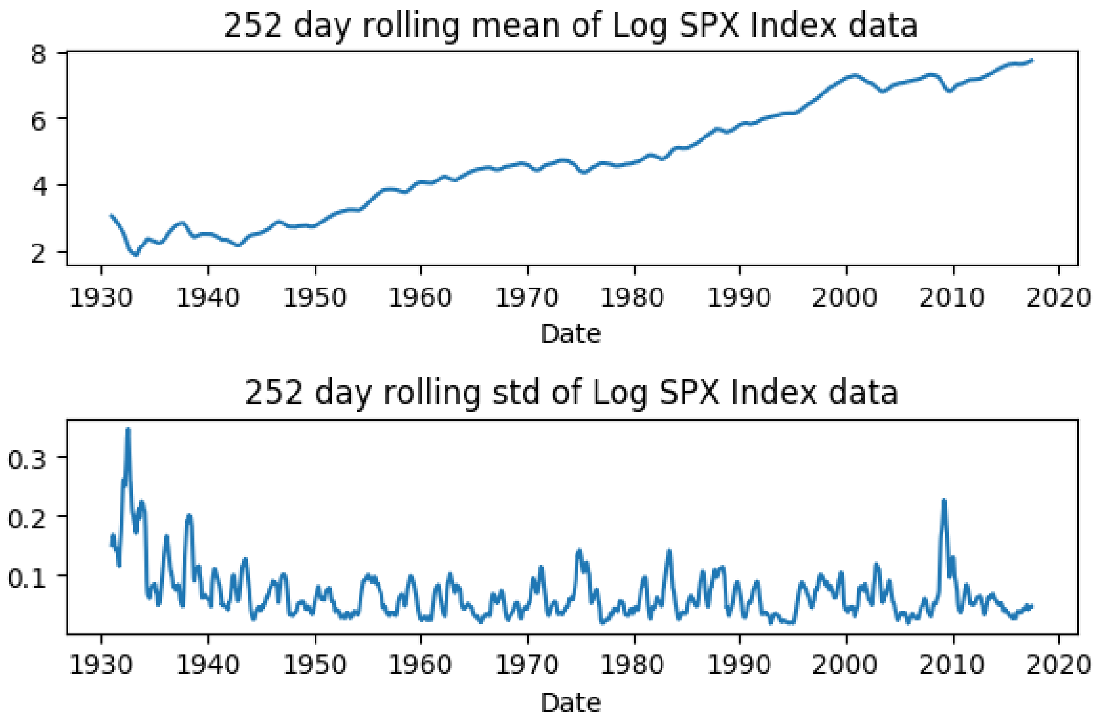
1.1.2 Q1.1.2

[5]:
```
WINDOW_SZ = 252
```

[6]:
```
plt.figure(dpi=100)
plt.subplot(211)
plt.title('252 day rolling mean of SPX Index data')
plt.xlabel('Date')
plt.plot(date_axis,px.rolling(WINDOW_SZ).mean())
# Sliding standard deviation
plt.subplot(212)
plt.plot(date_axis,px.rolling(WINDOW_SZ).std())
plt.title('252 day rolling std of SPX Index data')
plt.xlabel('Date')
plt.tight_layout()
plt.show()
# Log Sliding mean
plt.figure(dpi=100)
plt.subplot(211)
plt.plot(date_axis,logpx.rolling(WINDOW_SZ).mean())
plt.title('252 day rolling mean of Log SPX Index data')
plt.xlabel('Date')
```

```
# Log Sliding standard deviation
plt.subplot(212)
plt.plot(date_axis,logpx.rolling(WINDOW_SZ).std())
plt.title('252 day rolling std of Log SPX Index data')
plt.xlabel('Date')
plt.tight_layout()
plt.show()
```





Stationarity of price time-series The rolling mean for both price and log price shows that the mean price trends upwards. Neither of these time-series are stationairy.

The rolling std for price also trends upwards, but the log price std does not exhibit a clear, positive trend. The simple price rolling std is susceptible to the exponential growth of the price. The same percentge change in 2019, would produce a larger variance value than that same percentage change in 1940. So the rolling std of the price is not stationary.

The rolling std of the log price is obtained from the linearly increasing log price. This makes is less susceptible to the trend of the log price time-series, and is stationary.

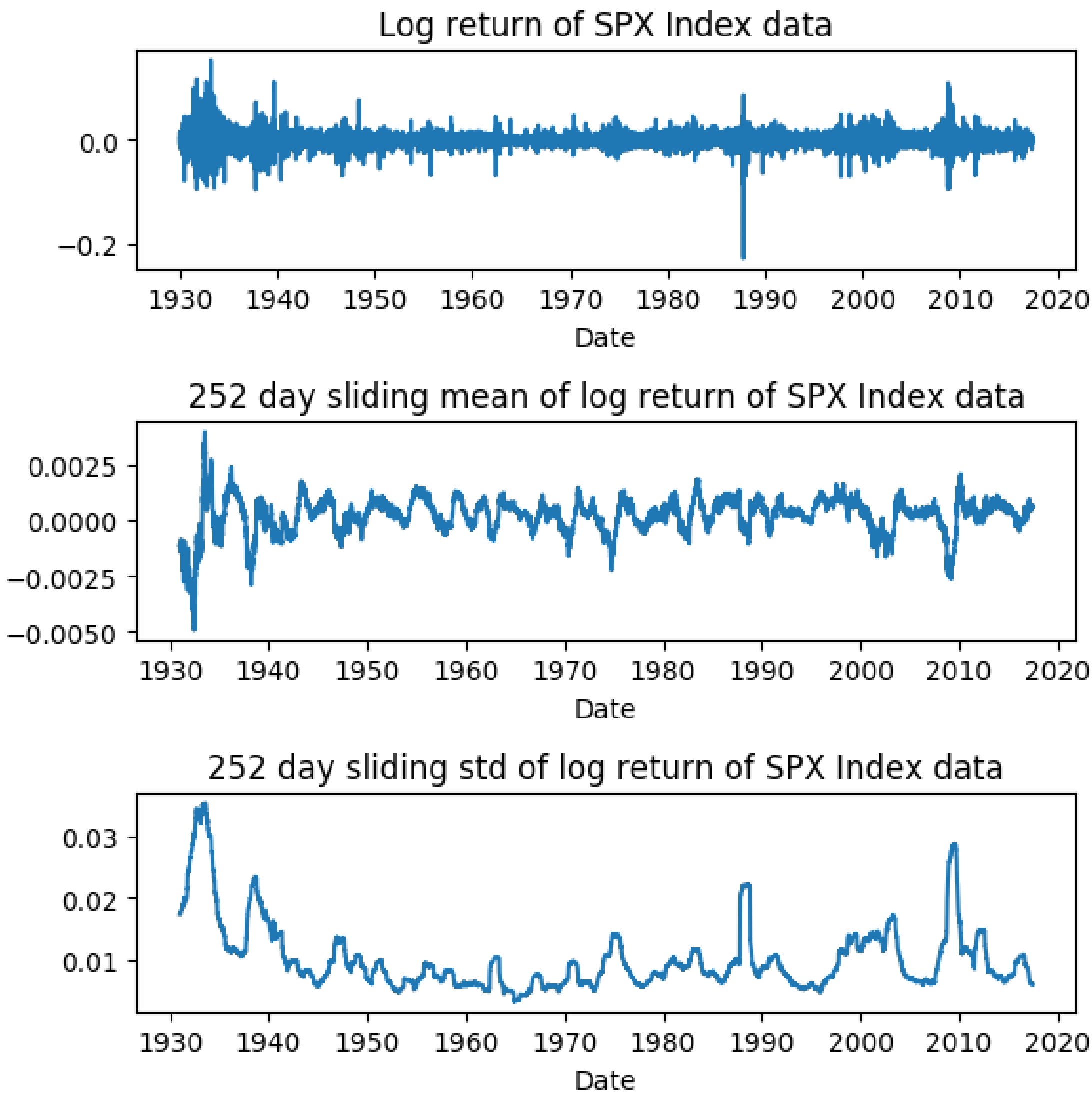
1.1.3 Q1.1.3

[7]:
```
# log return
logret = logpx.diff()

# simple return
simpret = px.pct_change()
```

Sliding statistics of log returns

[8]:

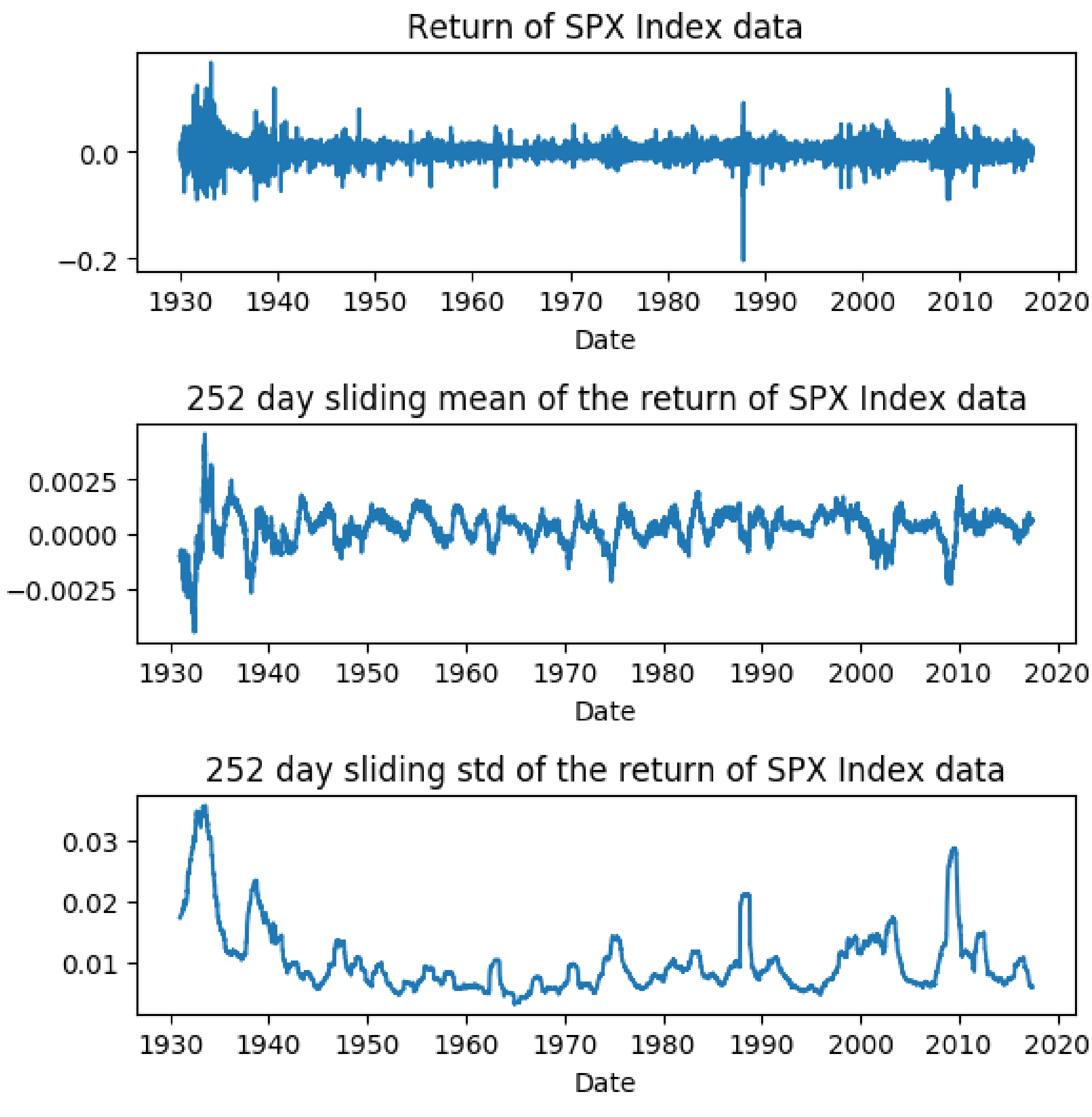
```
# Log returns
plt.figure(dpi=100, figsize=(6,6))
plt.subplot(311)
plt.plot(date_axis,logret)
plt.title('Log return of SPX Index data')
plt.xlabel('Date')
plt.tight_layout()
# Sliding mean Log return
plt.subplot(312)
logret_roll_mean = logret.rolling(WINDOW_SZ).mean()
plt.plot(date_axis,logret_roll_mean)
plt.title('252 day sliding mean of log return of SPX Index data')
plt.xlabel('Date')
5
plt.tight_layout()
# Sliding var Log return
plt.subplot(313)
logret_roll_var = logret.rolling(WINDOW_SZ).std()
plt.plot(date_axis,logret_roll_var)
plt.title('252 day sliding std of log return of SPX Index data')
plt.xlabel('Date')
plt.tight_layout()
plt.show()
```



Sliding statistics of simple returns

[9]:
```
# Simple return
plt.figure(dpi=100, figsize=(6,6))
plt.subplot(311)
plt.title('Return of SPX Index data')
plt.xlabel('Date')
plt.plot(date_axis,simpret)
plt.tight_layout()
# Sliding mean simple return
plt.subplot(312)
simpret_roll_mean = simpret.rolling(WINDOW_SZ).mean()
plt.title('252 day sliding mean of the return of SPX Index data')
```

```
plt.xlabel('Date')
plt.plot(date_axis,simpret_roll_mean)
plt.tight_layout()
# Sliding variance simple return
plt.subplot(313)
simpret_roll_var = simpret.rolling(WINDOW_SZ).std()
plt.title('252 day sliding std of the return of SPX Index data')
plt.xlabel('Date')
plt.plot(date_axis,simpret_roll_var)
plt.tight_layout()
```



The rolling mean of both the log return and simple return are stationary, unlike the rolling means of the log price and price.

This makes the time-series more suitable for analysis as it removes trends and allows for better comparison to other time-series.

1.1.4 Q1.1.4

Suitability of log returns over simple returns for SP purposes: If we assume that prices are log normally distributed, then the log returns would be normally distributed. This property is useful for when normality is assumed in statisics.

When returns are very small, the log returns are very close in value to the simple returns. Small returns are common for trades holding the asset for a short duration, e.g. one day / daily returns.

Log returns can be decomposed into the difference between two logs. So the compound return over n time periods can be done in O(1) time. The sum of normal values is normal, but the product is not. So the compound return obtained via log returns is also normally distributed.

[10]:
```
# Jarque-Bera test for Gaussianity
from scipy import stats

m=np.array([ ["log price",stats.jarque_bera(logpx)[1]],
             ["log returns",stats.jarque_bera(logret[1:])[1]],
             ["simple returns",stats.jarque_bera(simpret[1:])[1]] ])
headers = ["Data", "JB p-value"]
table = tabulate(m, headers, tablefmt="fancy_grid")
print(table)
```

Data	JB p-value
log price	0
log returns	0
simple returns	0

The Jarque-Bera test returned 0 for the log returns, thus rejected the null hypothesis. This implies that the original price data was not log normally distributed. However, over shorter period of time, the prices should be log normally distributed. So the theoretical advantages of using simple returns would not apply to this SPX time-series.

1.1.5 Q1.1.5

[11]:
```
data = {'prices': [1, 2, 1]}
price_df = pd.DataFrame(data, columns=['prices'])
simple_ret = price_df.pct_change()
log_ret = np.log(price_df).diff()
# correct for missing first return
```

```
simple_ret.prices[0] = 0
log_ret.prices[0] = 0

m=np.array([ [1,simple_ret.prices[0]], [2,simple_ret.prices[1]], [3,simple_ret.
↪prices[2]] ])
headers = ["Day", "Simple Return"]
table = tabulate(m, headers, tablefmt="fancy_grid")
print(table)
print("The simple returns give a total return of: {}".format(sum(simple_ret.
↪prices)))

m=np.array([ [1,log_ret.prices[0]], [2,log_ret.prices[1]], [3,log_ret.
↪prices[2]] ])
headers = ["Day", "Simple Return"]
table = tabulate(m, headers, tablefmt="fancy_grid", floatfmt=".3f")
print(table)
print("The log returns give a total return of: {}".format(sum(log_ret.prices)))
```

Day	Simple Return
1	0
2	1
3	-0.5

```
The simple returns give a total return of: 0.5
```

Day	Simple Return
1.000	0.000
2.000	0.693
3.000	-0.693

```
The log returns give a total return of: 0.0
```

This example shows that log returns are symmetric, and so the compound log returns can be calculated with just the initial and final prices.

1.1.6 Q1.1.6

Log returns should not be used over long periods of times, as the assumption of log-normallity is unrealistic.

Log returns are not linearly additive across assets, and so should not be used when dealing with multi-asset portfolios.

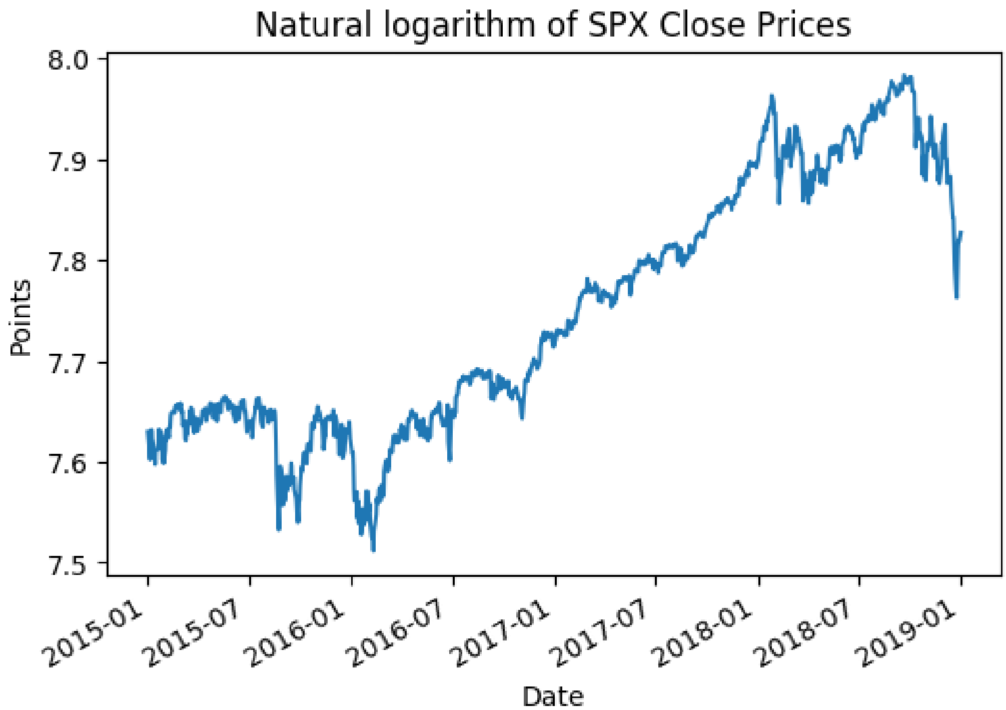
1.2 Q1.2 ARMA vs ARIMA Models for Financial Applications

1.2.1 Q1.2.1

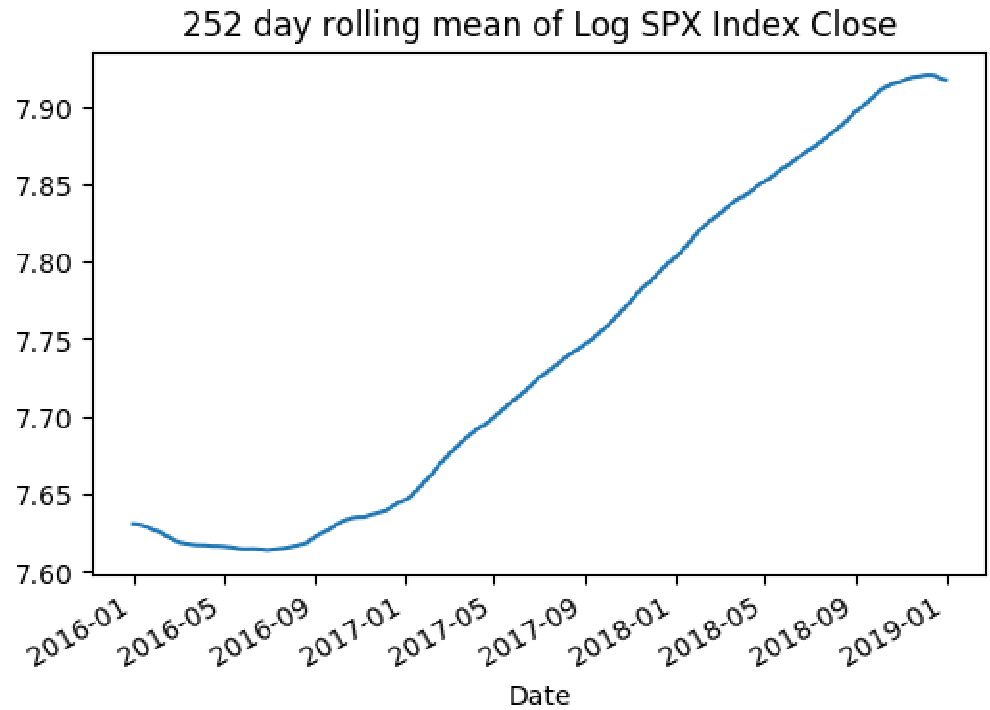
[12]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
%matplotlib inline

from statsmodels.tsa.ar_model import AR
from statsmodels.tsa.arima_model import ARIMA
import copy
import datetime as dt
```

[13]:
```
snp = pd.read_csv("./../data/snp_500_2015_2019.csv").set_index(['Date'])
snp_close = snp['Close'].to_frame().apply(np.log)
#snp_close.head()

dates = snp.index
date_axis = [dt.datetime.strptime(d,'%Y-%m-%d').date() for d in dates]
```

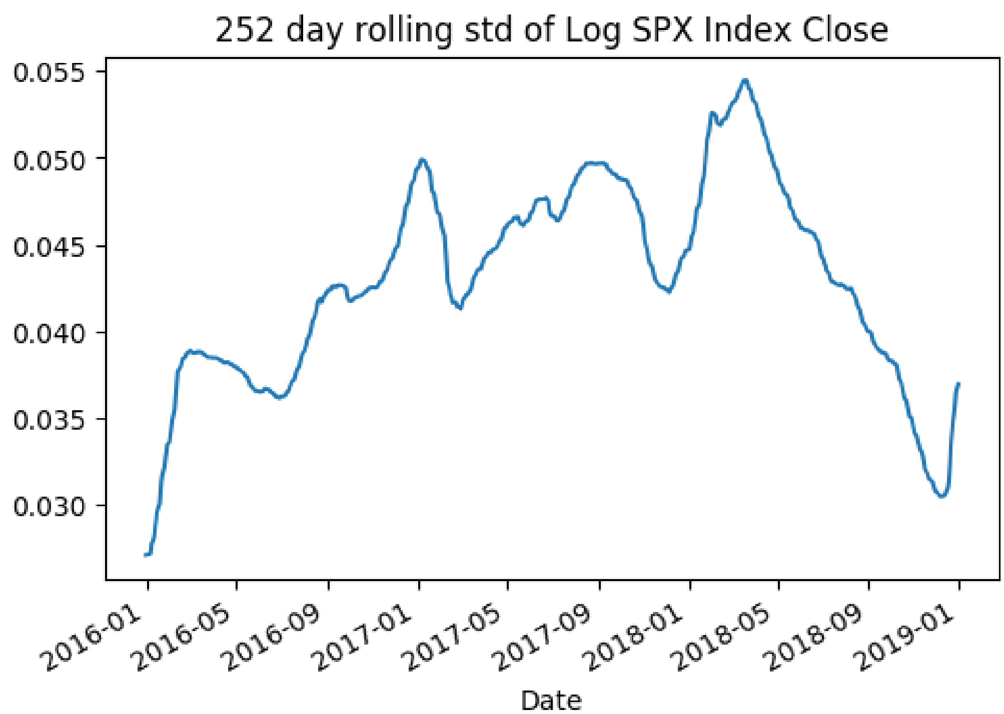
[14]:
```
fig, ax = plt.subplots(dpi=100)
ax.plot(date_axis, snp_close)
# rotate and align the tick labels so they look better
fig.autofmt_xdate()

plt.title('Natural logarithm of SPX Close Prices')
plt.xlabel('Date')
plt.ylabel('Points')
plt.show()
```



[15]:
```
WINDOW_SZ = 252
# Log Sliding mean
fig, ax = plt.subplots(dpi=100)
ax.plot(date_axis, snp_close.rolling(WINDOW_SZ).mean())
# rotate and align the tick labels so they look better
fig.autofmt_xdate()
plt.title('252 day rolling mean of Log SPX Index Close')
plt.xlabel('Date')
plt.show()

# Log Sliding standard deviation
fig, ax = plt.subplots(dpi=100)
ax.plot(date_axis, snp_close.rolling(WINDOW_SZ).std())
# rotate and align the tick labels so they look better
fig.autofmt_xdate()
plt.title('252 day rolling std of Log SPX Index Close')
plt.xlabel('Date')
plt.show()
```



252 day rolling mean of Log SPX Index Close
7.90
7.85
7.80
7.75
7.70
7.65
7.60
2016-01
2016-05
2016-09
2017-01
2017-05
2017-09
2018-01
2018-05
2018-09
2019-01
Date



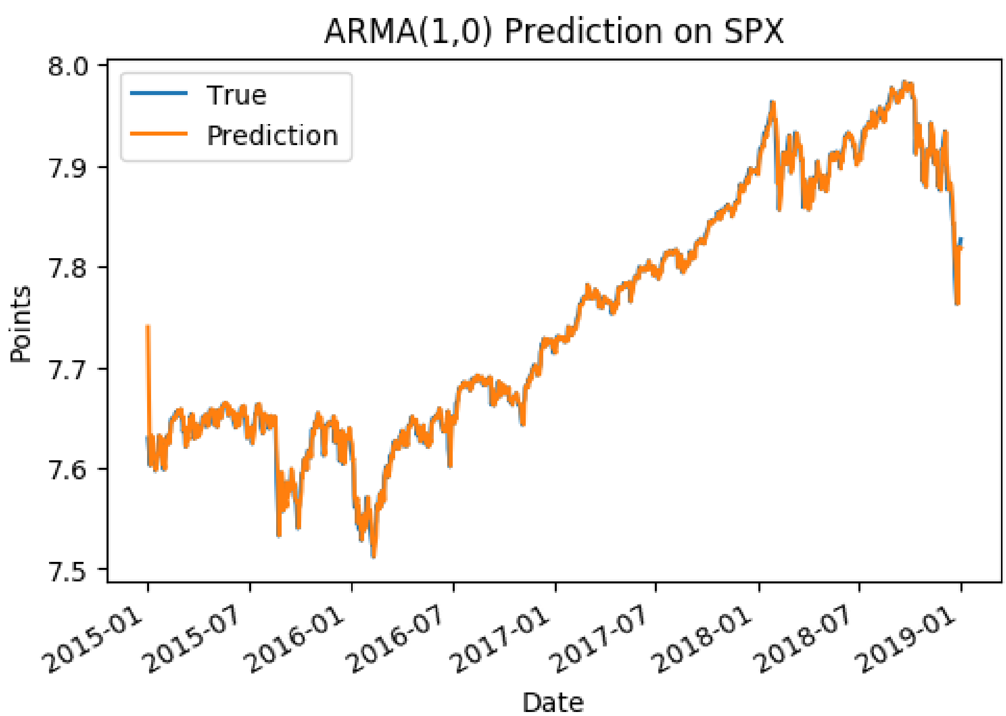
252 day rolling std of Log SPX Index Close
0.055
0.050
0.045
0.040
0.035
0.030
2016-01
2016-05
2016-09
2017-01
2017-05
2017-09
2018-01
2018-05
2018-09
2019-01
Date

The rolling mean shows that the time-series is not stationary, so ARMA not suitable. and so ARIMA is more appropriate.

1.2.2 Q1.2.2

[19]:
```
snp_arma = copy.deepcopy(snp_close)
snp_arma.columns = ['True']
snp_arma['Res'] = ARIMA(snp_arma, order=(1,0,1)).fit().resid
snp_arma['Prediction'] = snp_arma['True'] - snp_arma['Res']
```

[20]:
```
fig, ax = plt.subplots(dpi=100)
ax.plot(date_axis, snp_arma['True'], label="True")
ax.plot(date_axis, snp_arma['Prediction'], label="Prediction")
# rotate and align the tick labels so they look better
fig.autofmt_xdate()
plt.title('ARMA(1,0) Prediction on SPX')
plt.xlabel('Date')
plt.ylabel("Points")
plt.legend()
plt.show()
```



[21]:
```
arma_1_0_sse = np.sum((snp_arma['True']-snp_arma['Prediction'])**2)
print("SSE of ARMA(1,0) Predictions: {}".format(arma_1_0_sse))
```

```
SSE of ARMA(1,0) Predictions: 0.08675908609244501
```

The ARMA(1,0) model appears to perform well at predicting this time-series.

The model predicts the next days prices as the current days price, with an adiional amount of small, random noise.

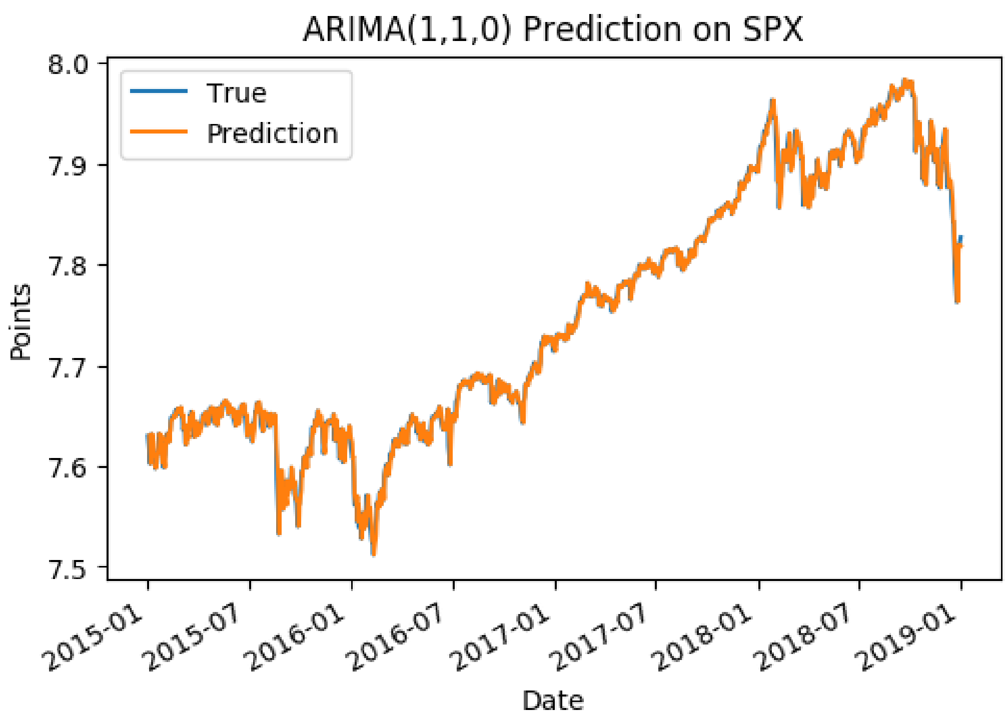
This is effectively a random walk, which can be viable for very short term predictions, shown by the low SSE of the model.

In practice, this is not that useful unless the stock is only held for a day. Since the model heavily relies on the trend, predictions for prices further than one day away will have this trend bias along with compounding errors.

1.2.3 Q1.2.3

[22]:
```
snp_arima = copy.deepcopy(snp_close)
snp_arima.columns = ['True']
snp_arima['Res'] = ARIMA(snp_arima, order=(1,1,0)).fit().resid
snp_arima['Prediction'] = snp_arima['True'] - snp_arima['Res']
```

[23]:
```
fig, ax = plt.subplots(dpi=100)
ax.plot(date_axis, snp_arima['True'], label="True")
ax.plot(date_axis, snp_arima['Prediction'], label="Prediction")
# rotate and align the tick labels so they look better
fig.autofmt_xdate()
plt.title('ARIMA(1,1,0) Prediction on SPX')
plt.xlabel('Date')
plt.ylabel("Points")
plt.legend()
plt.show()
```



[24]:
```
arima_1_1_0_sse = np.sum((snp_arima['True']-snp_arima['Prediction'])**2)
print("SSE of ARIMA(1,1,0) Predictions: {}".format(arima_1_1_0_sse))
```

```
SSE of ARIMA(1,1,0) Predictions: 0.0746545241081268
```

The ARIMA(1,1,0) model produced predictions with a lower SSE.

ARIMA 'detrends' the data, and the reliance on trends is minimal compared to ARMA(1,0). This makes the ARIMA more suitable for predicting returns further than a day out.

1.2.4 Q1.2.4

The log prices have the symmetric property. This is important for the initial differencing step in ARIMA to correctly remove the trends.

1.3 Q1.3

1.3.1 Q1.3.1

VAR can be represented as

$$\mathbf{Y} = \mathbf{B}\mathbf{Z} + \mathbf{U} \quad (1.3.11)$$

where:

(1.3.12)

$$\mathbf{Y} = [\mathbf{y}[p][\mathbf{y}[p+1]...[\mathbf{y}[T]]$$

$$\mathbf{B} = [\mathbf{c}\mathbf{A}_1\mathbf{A}_2...\mathbf{A}_p]$$

$$\mathbf{Z} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{y}[p-1] & \mathbf{y}[p] & \cdots & \mathbf{y}[T-1] \\ \mathbf{y}[p-2] & \mathbf{y}[p-1] & \cdots & \mathbf{y}[T-2] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}[0] & \mathbf{y}[1] & \cdots & \mathbf{y}[T-p] \end{bmatrix}$$

$$\mathbf{U} = [\mathbf{e}[p][\mathbf{e}[p+1]...[\mathbf{e}[T]]$$

1.3.2 Q1.3.2

Using the LSE estimator on the model, the cost function is given by:

$$J(B) = (Y - BZ)^T(Y - BZ) \quad (1.3.21) \tag{1}$$

when expanded, it turns into

$$J(B) = YY^T - 2Y^TBZ + Z^TB^TBZ \quad (1.3.22) \tag{2}$$

The aim is to minimize the cost function:

$$\frac{J(B)}{dB} = 0 \quad (1.3.23) \tag{3}$$

This results in:

$$-2YZ^T + B_{opt}(ZZ^T + ZZ^T) = 0 \quad (1.3.24) \tag{4}$$

$$-2YZ^T + 2B_{opt}ZZ^T = 0 \quad (1.3.25) \tag{5}$$

Finally, the optimal parameters are given by:

$$B_{opt} = YZ^T(ZZ^T)^{-1} \quad (1.3.26) \tag{6}$$

1.3.3 Q1.3.3

For the system to be stable the eigenvalues of the matrix A must be smaller than one in absolute value. This is proven by taking the Z-transform:

$$Y(z) = AY(z)z^{-1} + E(z) \quad (1.3.31) \tag{7}$$

$$Y(z)(I - Az^{-1}) = E(z) \tag{8}$$

$$H(z) = (I - Az^{-1})^{-1} \tag{9}$$

$$A = QDQ^{-1} \Rightarrow H(z) = (I - QDQ^{-1}z^{-1})^{-1} \quad (1.3.32) \tag{10}$$

$$\text{where } A \text{ has been diagonalized.} \tag{11}$$

$$\text{For the system to be stable, there must be no poles. This happens when} \tag{12}$$

$$|\lambda| < 1 \tag{13}$$

1.3.4 Q1.3.4

[25]:
```
import pandas as pd
import matplotlib.pyplot as plt
import numpy as np
from statsmodels.tsa.api import VAR
import datetime as dt
```

[26]:
```
df = pd.read_csv(r'./../data/snp_allstocks_2015_2019.csv')
dates = df['Date']
date_axis = [dt.datetime.strptime(d,'%Y-%m-%d').date() for d in dates]
df = df.set_index('Date')
info = pd.read_csv(r'./../data/snp_info.csv')
info.drop(columns=info.columns[0],inplace=True)
```

[27]:
```
tickers = ['CAG', 'MAR', 'LIN', 'HCP', 'MAT']
stocks = df[tickers]
stocks_ma = stocks.rolling(window=66).mean()
stocks_detrended = stocks.sub(stocks_ma).dropna()
```

[28]:
```
model = VAR(stocks_detrended)
results = model.fit(1)
A = results.params[1:].values
eigA, _ = np.linalg.eig(A)
```

[29]:
```
headers = ["eigenvalue magnitude"]
# tabulate data
table = tabulate(np.array([np.absolute(eigA)]).T, headers,␣
 ↪tablefmt="fancy_grid", floatfmt=".3f")
```

```python
# output
print(table)
```

```
eigenvalue magnitude

                0.726

                0.726

                1.006

                0.861

                0.911
```

It would not make sense to construct a portfolio using these stocks as the third eigenvlue has a magnitude greater than 1. This means that the VAR(1) process is not stable here.

1.3.5 Q1.3.5

[30]:
```python
def eig_regression(stocks):
    stocks_ma = stocks.rolling(window=66).mean()
    stocks_detrended = stocks.sub(stocks_ma).dropna()
    model = VAR(stocks_detrended)
    results = model.fit(1)
    A = results.params[1:].values
    eigA, _ = np.linalg.eig(A)
    return eigA
```

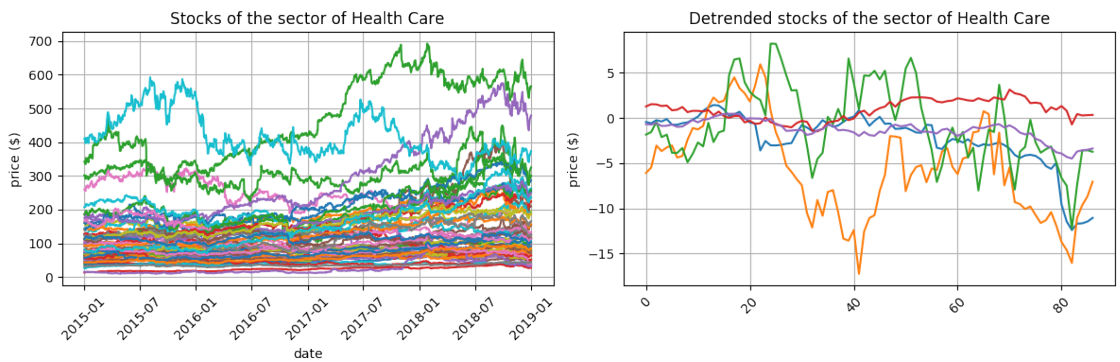
[31]:
```python
sector_stocks = {}
print("Sectors:")
for sector in info['GICS Sector'].unique():
    tickers = info.loc[info['GICS Sector']== sector]['Symbol'].tolist()
    stocks = df[tickers]
    sector_stocks[sector] = stocks
```

```
Sectors:
```

[32]:
```python
chosen_sector = 'Health Care'
```

[33]:
```python
stocks = sector_stocks[chosen_sector]
eigA = eig_regression(stocks)

plt.figure(dpi=100, figsize=(12,4))
```

```
plt.subplot(121)
plt.plot(date_axis,stocks)
plt.title('Stocks of the sector of ' + str(chosen_sector))
plt.ylabel('price ($)')
plt.xlabel('date')
plt.xticks(rotation=45)
plt.tight_layout()
plt.grid()

plt.subplot(122)
plt.plot(np.array(stocks_detrended))
plt.title('Detrended stocks of the sector of ' + str(chosen_sector))
plt.ylabel('price ($)')
plt.xticks(rotation=45)
plt.tight_layout()
plt.grid()
plt.show()
```



[34]:
```
print("Largest eigenvalue Magnitude of VAR(1) in health sector:", np.
 ↪absolute(max(eigA)))
```

```
Largest eigenvalue Magnitude of VAR(1) in health sector: 0.9941531205433555
```

[35]:
```
sector_lem = []
for sector, stocks in sector_stocks.items():
    eigA = eig_regression(stocks)
    lem = np.absolute(max(eigA))
    sector_lem.append( (sector, lem) )

headers = ["sector", "largest eigenvalue magnitude"]
# tabulate data
table = tabulate(sector_lem, headers, tablefmt="fancy_grid", floatfmt=".3f")
print(table)
```

sector	largest eigenvalue magnitude
Industrials	0.992
Health Care	0.994
Information Technology	0.993
Communication Services	0.982
Consumer Discretionary	0.991
Utilities	0.986
Financials	1.004
Materials	0.992
Real Estate	0.983
Consumer Staples	0.992
Energy	0.986

Constructing a portfolio that consists of stocks from just one sector isn't advisable unless the investor is OK with the additional risk carried. Any negative effects to the industry would cause the majoirty of stocks to lose value. However, the advantange of investing in the sector instead of a single company is that is is less risky. A single company can lose value whilst the overall sector retains a positive or neutral outlook.

It is usefull to analyse sectors as it can show stocks that outperform the rest of the industry. Also the sector wide analysis is less sensistive to individual stock fluctuations, which can make analysis more insightful.

Additionally, the largest eigenvalue magnitude for the Health sector was under 1. All sectors, apart from the financials sector also had a largest eigenvalue magnitude of under 1. This supports that a portfolio could constructed from the sector as the VAR(1) of that portfolio would be stable.

2 Q2 Bond Pricing

2.1 2.1 Examples of bond pricing

2.1.1 Q2.1.1

An investor receives USD 1,100 in one year in return for an investment of USD 1,000 now. Calculate the percentage return per annum with: a) Annual compounding, b) Semiannual compounding, c)

Monthly compounding, d) Continuous compounding

[36]:
```
import math

ret_ann = 1100/1000 - 1
ret_sann = ((1100/1000)**(1/2) -1) * 2
ret_monthly = ((1100/1000)**(1/12) -1) * 12
ret_cont = math.log(1100/1000)

m = [ ["annual", ret_ann*100], ["semiannual", ret_sann*100], ["monthly",␣
↪ret_monthly*100], ["continuous", ret_cont*100]]

headers = ["compounding type", "return per annum (%)"]
# tabulate data
table = tabulate(m, headers, tablefmt="fancy_grid", floatfmt=".3f")
print(table)
```

compounding type	return per annum (%)
annual	10.000
semiannual	9.762
monthly	9.569
continuous	9.531

2.1.2 Q2.1.2

What rate of interest with continuous compounding is equivalent to 15% per annum with monthly compounding?

[37]:
```
equiv_cont = math.log((1+0.15/12)**12)
print("{:.2f}% continuous compounding is equivalent to 15% per annum with␣
↪monthly compounding".format(equiv_cont*100))
```

```
14.91% continuous compounding is equivalent to 15% per annum with monthly
compounding
```

2.1.3 Q2.1.3

A deposit account pays 12% per annum with continuous compounding, but interest is actually paid quarterly. How much interest will be paid each quarter on a USD 10,000 deposit?

We calculate the continuous compounding interest rate:

[38]:
```python
cont_rate = math.log(1.12)
print("continuous compounding rate : {:.3f}%".format(cont_rate*100))
```

```
continuous compounding rate : 11.333%
```

[39]:
```python
init_deposit = 10000
# calc account value after each quarter
first_quarter  = init_deposit   * math.exp(cont_rate/4)
second_quarter = first_quarter  * math.exp(cont_rate/4)
third_quarter  = second_quarter * math.exp(cont_rate/4)
fourth_quarter = third_quarter  * math.exp(cont_rate/4)

m = [["first", first_quarter-init_deposit],
     ["second", second_quarter-first_quarter],
     ["third", third_quarter-second_quarter],
     ["fourth", fourth_quarter-third_quarter]]

headers = ["quarter", "interest paid ($)"]
table = tabulate(m, headers, tablefmt="fancy_grid", floatfmt=".2f")
print(table)
```

quarter	interest paid ($)
first	287.37
second	295.63
third	304.13
fourth	312.87

2.2 Q2.2 Forward rates

Suppose that the one–year interest rate, r1 is 5%, and the two–year interest rate, r2 is 7%. If you invest USD 100 for one year, your investment grows to $100 \times 1.05 =$ USD105; if you invest for two years, it grows to $100 \times 1.072 =$ USD114.49. The extra return that you earn for that second year is $1.072/1.05-1 = 0.090$, or 9.0 %

- The decision to invest for two years instead of one depends on how risk-averse the investor is and how long they can comfortably invest for. If a two year period is suitable and the bond has a low risk (i.e. developed country Goverment bond), then the two year investment seems suitable, especially if it beats inflation.
- The 5% strategy pays out the quickest, allowing the investor to re-evaluate after one year. This is suitable if the investor would like to invest in other assets after one year. The 7% strategy has the largest return over the two year period, but requires capital to be locked

away the longest. If the investor has no desire to hold other assets over the period, this is the ideal strategy, assuming the bond has very low risk. The 9% strategy, like the 5%, only requires capital to be tied down for one year, but it is tied down in the second year.

- Whilst providing a higher return compared to 5%, the investor must have the agreed initial captial available after one year to loan. So while free to invest elsewhere in the first year, there is an additional risk of not having the funds when required. This risk is created when investing in other assets in the first year, with the aim of beating 5% for the year.
- If investing for one year at 5%, then a further 9% is required if reinvesting over the second year, to match the two year investment. The forward rate implies that after one year, the new one year rate would be 9%. However this is unlikely to hold after one year as the one-year and two-year rates are likely to change before then.

2.3 Q2.3 Duration of a coupon-bearing bond

2.3.1 Q2.3.1 a)

The duration of the 1% bond in Table is found by summing the (Year x Fraction of PV) terms

[40]:
```
year_x_frac_pv = [0.0124, 0.0236, 0.0337, 0.0428, 0.0510, 0.0583, 6.5377]

print("The duration is {:.2f} years".format(sum(year_x_frac_pv)))
```

```
The duration is 6.76 years
```

2.3.2 Q2.3.1 b)

[41]:
```
modified_duration = sum(year_x_frac_pv)/(1+0.05)
print("The modified duration is {:.2f}".format(modified_duration))
```

```
The modified duration is 6.44
```

This measures how senstitive the bond price is with respect to the yield, while the duration is the weighted average of the times to each of the cash payments.

2.3.3 Q2.1.3 c)

Using the duration allows for immunization (sensitivity analyses). A first order immunization using two bonds can allow a pension plan to meet a price P in time D using the following:

$$P = x_1 P_1 + x_2 P_2 \quad (2.1.31)$$

$$D = \frac{x_1 P_1}{D_1} + \frac{x_2 P_2}{D_2} \quad (2.1.32)$$

Where $x_Y, P_Y and D_Y$ are the portfolio weighting, bond price and bond duration respectively, for bond Y.

This helps to reduce the impact on the pension portfolio against unexpected changes in interest rates (which would affect the prices of the two individual bonds).

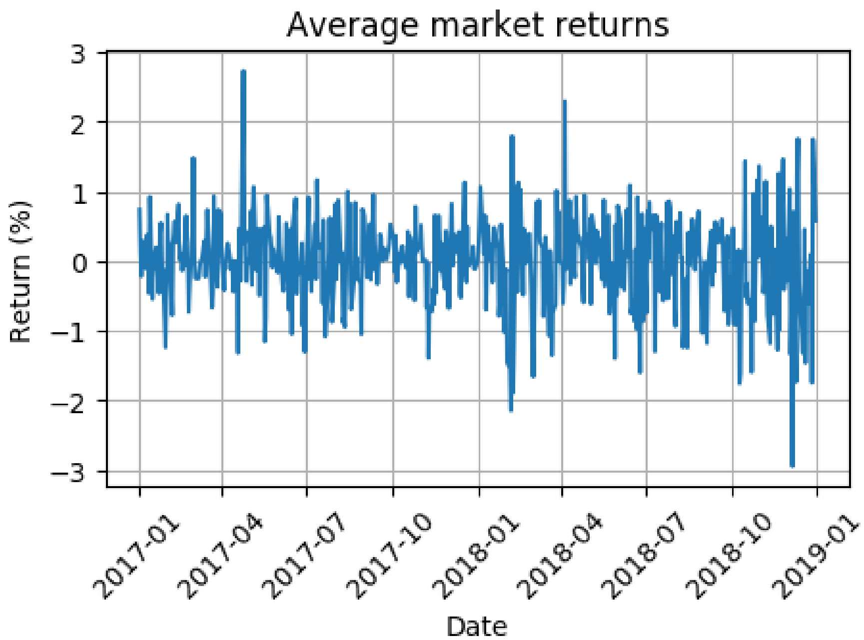
2.4 Q2.4 Capital Assest Pricing Model (CAPM) and Arbitrage Pricing Theory (APT)

[42]:
```
import pandas as pd
import numpy as np
import datetime as dt
import matplotlib.pyplot as plt
df = pd.read_csv(r'./../data/fsp_case_31_BSD.csv', index_col=0, header=[0,1])
dates = df['ret'].index
date_axis = [dt.datetime.strptime(d,'%Y-%m-%d').date() for d in dates]
```

2.4.1 Q2.4.1 Market returns per day

[43]:
```
market_returns = df['ret'].mean(axis=1)
```

[44]:
```
plt.figure(dpi=100, figsize=(5,3))
plt.plot(date_axis, market_returns*100)
plt.title('Average market returns')
plt.xticks(rotation=45)
plt.xlabel('Date')
plt.ylabel('Return (%)')
plt.grid()
plt.show()
```



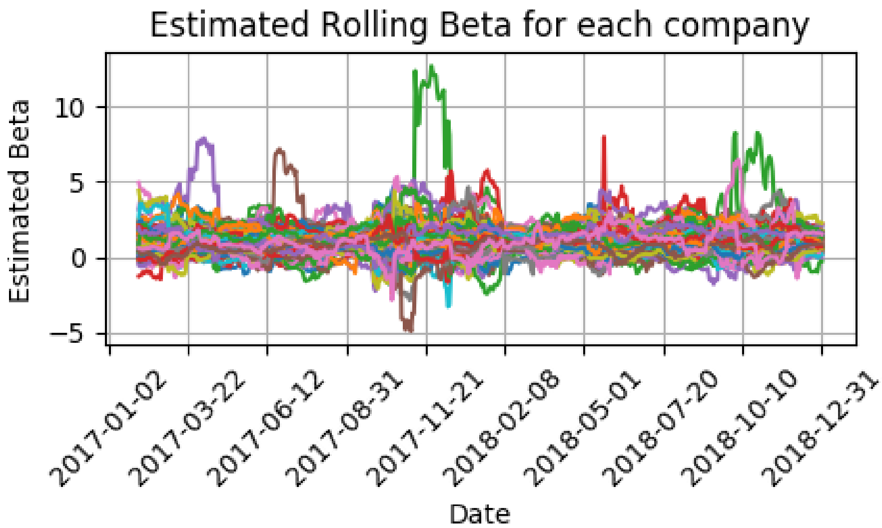
[45]:
```
print("The average market return was {:.7f}% with a std of {:.5f}%".format(np.
↪mean(market_returns), np.std(market_returns)))
```

```
The average market return was 0.0000460% with a std of 0.00658%
```

[46]:
```
betas = (df['ret'].rolling(22).cov(df['ret'].mean(axis=1).rolling(22)).T /␣
↪df['ret'].mean(axis=1).rolling(22).var()).T
```

2.4.2 Q2.4.2 Rolling beta

[47]:
```
WINDOW = 22 # days
# beta_i = cov_iM / var_MM
#betas = df['ret'].rolling(22).cov(market_returns)/np.var(market_returns)

plt.figure(dpi=100, figsize=(5,3))
plt.plot(betas.index, betas)
plt.title('Estimated Rolling Beta for each company')
plt.xticks([betas.index[i] for i in np.linspace(0,len(betas.index)-1, 10).
↪astype(int)],rotation=45)
plt.xlabel('Date')
plt.ylabel('Estimated Beta')
plt.grid()
plt.tight_layout()
plt.show()
```



[48]:
```
print("The average Beta for all companies was {:.3f} with a an average std of {:
↪.3f}".format(np.mean(np.mean(betas)), np.mean(np.std(betas))))
```

```
The average Beta for all companies was 0.983 with a an average std of 0.550
```

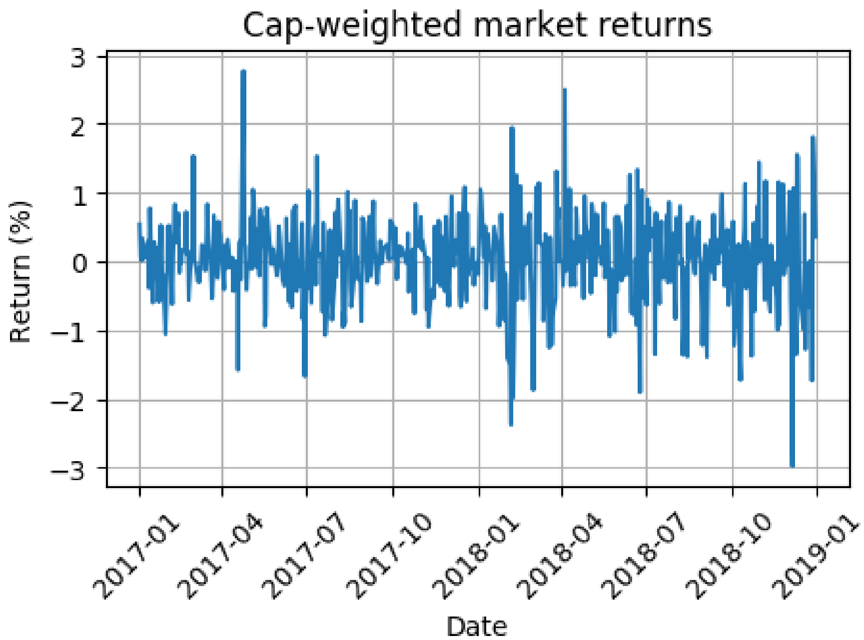
Volatility of Rolling Beta Most rolling beta can be seen above to be volatile, and so cannot be assumed to be constant. This is supported by the std of the mean Beta per company, being close in value to the mean of mean Beta per company.

2.4.3 Q2.4.3 Market-cap weighted returns

[49]:
```
market_caps = df['mcap'].sum(axis=1)
mcapw_returns = (df['mcap']*df['ret']).sum(axis=1) / market_caps
betas = (df['ret'].rolling(22).cov(mcapw_returns.rolling(22)).T / mcapw_returns.
↪rolling(22).var()).T
```

[50]:
```
market_caps = df['mcap'].sum(axis=1)
mcapw_returns = (df['mcap']*df['ret']).sum(axis=1) / market_caps
```

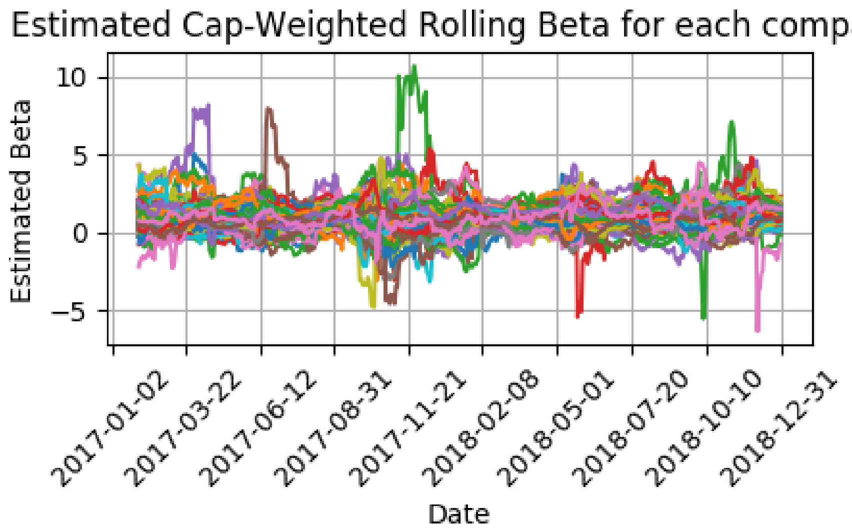
[51]:
```
plt.figure(dpi=100, figsize=(5,3))
plt.plot(date_axis, mcapw_returns*100)
plt.title('Cap-weighted market returns')
plt.xticks(rotation=45)
plt.xlabel('Date')
plt.ylabel('Return (%)')
plt.grid()
plt.show()
```



Weighting coefficient This coeffecient is the total market capitalization during that day, assuming the stocks used are all stocks in the market being studied.

2.4.4 Q2.4.4 Cap-weighted Beta

[52]:
```
WINDOW = 22 # days
# beta_i = cov_iM / var_MM
betas = (df['ret'].rolling(22).cov(mcapw_returns.rolling(22)).T / mcapw_returns.
 ↪rolling(22).var()).T

plt.figure(dpi=100, figsize=(5,3))
plt.plot(betas.index, betas)
plt.title('Estimated Cap-Weighted Rolling Beta for each company')
plt.xticks([betas.index[i] for i in np.linspace(0,len(betas.index)-1, 10).
 ↪astype(int)],rotation=45)
plt.xlabel('Date')
plt.ylabel('Estimated Beta')
plt.grid()
plt.tight_layout()
plt.show()
```



[53]:
```
print("The average cap-weighted Beta for all companies was {:.3f} with a an␣
↪average std of {:.3f}".format(np.mean(np.mean(betas)), np.mean(np.
↪std(betas))))
```

```
The average cap-weighted Beta for all companies was 0.938 with a an average std
of 0.562
```

Comparison to equally-weighted Betas The cap-weighted Betas are also volatile, but have a lower magnitude compared to the equally-weighted, (0.938 mean vs 0.983)

High beta values have decreased in magnitude, which can be observed from the peaks of both plots. This suggests the corresponding companies were being over weighted in the equal wighting. Large Beta suggests that company drives/correlates with the market. If the Beta decreases then that company was contributing more to the market return than it's market capitlization would allow in cap-weighted. This highlights the importance of using cap-weighted Betas when assessing individual companies Betas for investment decisions.

2.4.5 Q2.4.5 Arbitrage Pricing Theory (APT) for a two factor model

For this exercise these factors are the cap-weighted market returns and the small minus big (SMB). The sensitivity to SMB is taken to be the natural logarithm of the market value of the asset. The sensitivity to the market returns are the cap-weighted Betas.

Model: $r_i = a + b_{m_i}R_m + b_{s_i}R_s + \epsilon_i$

[54]:
```
from sklearn.linear_model import LinearRegression
```

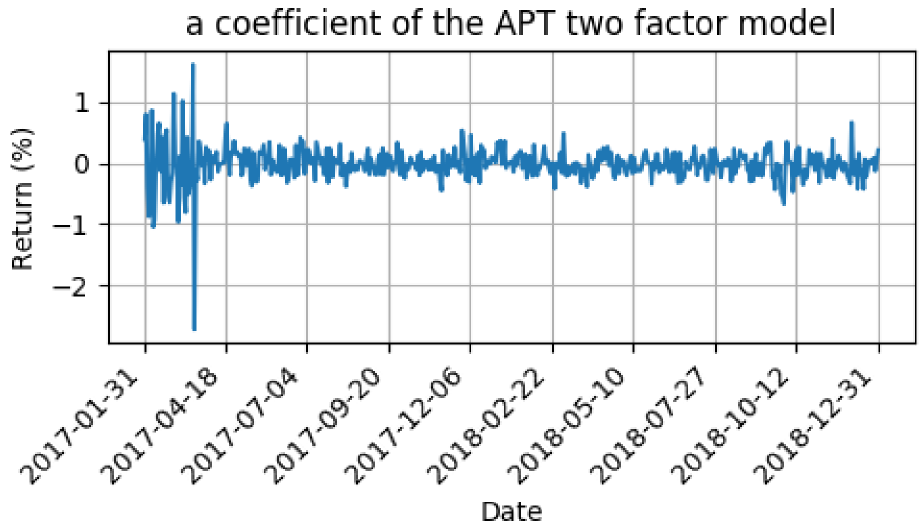
Q2.4.5 a)

[55]:
```
num_days = len(betas) - WINDOW + 1
num_stocks = 157

# specific return
e  = np.zeros([num_days, num_stocks])
# coeffecients - per day
rs = np.zeros([num_days])
rm = np.zeros([num_days])

# sensitivities
bs = np.zeros([num_stocks, num_days])
bm = np.zeros([num_stocks, num_days])

# independent term
a  = np.zeros([num_days])

# Y
ri = np.zeros([num_stocks, num_days])

# Fill above arrays
df1 = df.replace([np.inf, -np.inf], np.nan)
df1 = df.fillna(0)
i = 0
for date, stock in df1['mcap'].iteritems():
    stock_ = stock.replace([0], 1) # log(1) = 0
    bs[i] = np.log(stock_)[21:]
    #bs[i][bs[i] == -np.inf] = 0
    i += 1
i = 0
betas_ = betas.replace([np.inf, -np.inf], np.nan)
betas_ = betas_.fillna(0)
for date, stock in betas_.iteritems():
    bm[i] = stock[21:]
    i += 1
i = 0
for date, stock in df1['ret'].iteritems():
    ri[i] = stock[21:]
    i += 1

# Prep for Linear Regression
X = np.zeros([num_days, num_stocks, 2])
for t in range(num_days):
    for i in range(num_stocks):
        X[t,i,0] = bm[i,t]
        X[t,i,1] = bs[i,t]
Y = ri.T.reshape([num_days, num_stocks])
```

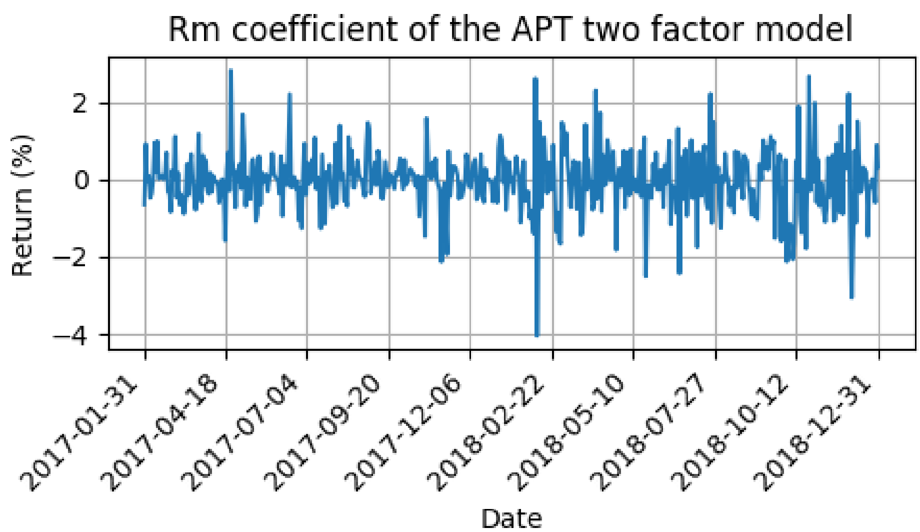
```
for t in range(num_days):
    reg = LinearRegression().fit(X[t,:,:],Y[t,:])
    y_pred = reg.predict(X[t,:,:])
    e[t,:] = Y[t,:] - y_pred
    a[t] = reg.intercept_
    rm[t] = reg.coef_[0]
    rs[t] = reg.coef_[1]
```

[56]:
```
plt_dates = betas.index[21:]

fig, ax = plt.subplots(dpi=100, figsize=(5,3))
ax.plot(plt_dates, a*100)
# rotate and align the tick labels so they look better
fig.autofmt_xdate()
plt.title('a coefficient of the APT two factor model')
plt.xlabel('Date')
plt.ylabel('Return (%)')
plt.grid()
plt.xticks([plt_dates[i] for i in np.linspace(0,len(plt_dates)-1, 10).
 ↪astype(int)],rotation=45)
plt.tight_layout()
plt.show()

fig, ax = plt.subplots(dpi=100, figsize=(5,3))
ax.plot(plt_dates, rm*100, label='Rm')
# rotate and align the tick labels so they look better
fig.autofmt_xdate()
plt.title('Rm coefficient of the APT two factor model')
plt.xlabel('Date')
plt.ylabel('Return (%)')
plt.grid()
plt.xticks([plt_dates[i] for i in np.linspace(0,len(plt_dates)-1, 10).
 ↪astype(int)],rotation=45)
plt.tight_layout()
plt.show()

fig, ax = plt.subplots(dpi=100, figsize=(5,3))
ax.plot(plt_dates, rs*100, label='Rs')
# rotate and align the tick labels so they look better
fig.autofmt_xdate()
plt.title('Rs coefficient of the APT two factor model')
plt.xlabel('Date')
plt.ylabel('Return (%)')
plt.grid()
plt.xticks([plt_dates[i] for i in np.linspace(0,len(plt_dates)-1, 10).
 ↪astype(int)],rotation=45)
plt.tight_layout()
```

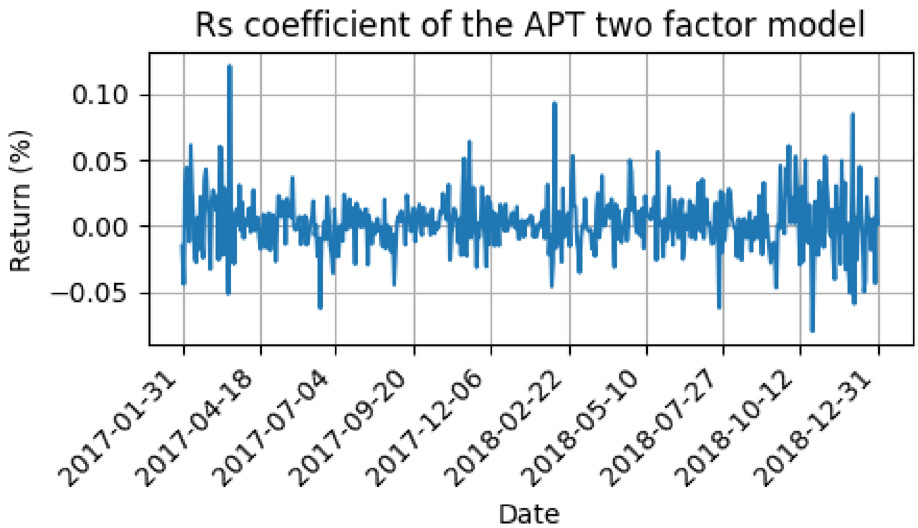
```
plt.show()
```



a coefficient of the APT two factor model
Return (%)
1
0
−1
−2
2017-01-31
2017-04-18
2017-07-04
2017-09-20
2017-12-06
2018-02-22
2018-05-10
2018-07-27
2018-10-12
2018-12-31
Date



Rm coefficient of the APT two factor model
Return (%)
2
0
−2
−4
2017-01-31
2017-04-18
2017-07-04
2017-09-20
2017-12-06
2018-02-22
2018-05-10
2018-07-27
2018-10-12
2018-12-31
Date



Q2.4.5 b)

[57]:
```
a_mean, a_std   = np.mean(a), np.std(a)
rm_mean, rm_std = np.mean(rm), np.std(rm)
rs_mean, rs_std = np.mean(rs), np.std(rs)

ma_mean, ma_std   = np.mean(np.absolute(a)), np.std(np.absolute(a))
mrm_mean, mrm_std = np.mean(np.absolute(rm)), np.std(np.absolute(rm))
mrs_mean, mrs_std = np.mean(np.absolute(rs)), np.std(np.absolute(rs))

m = [["a", a_mean*100, a_std*100],
     ["Rm", rm_mean*100, rm_std*100],
     ["Rs", rs_mean*100, rs_std*100],
     ["|a|", ma_mean*100, ma_std*100],
     ["|Rm|", mrm_mean*100, mrm_std*100],
     ["|Rs|", mrs_mean*100, mrs_std*100]]

headers = ["parameter", "mean (%)", "stddev"]
table = tabulate(m, headers, tablefmt="fancy_grid", floatfmt=".5f")
print(table)
```

parameter	mean (%)	stddev
a	-0.02311	0.27715
Rm	-0.03306	0.78403
Rs	0.00254	0.02126

```
|a|              0.17948    0.21245

|Rm|             0.56310    0.54655

|Rs|             0.01517    0.01511
```

The mean magnitdue of Rm is about 0.6%, whereas the mean magnitude of Rs is around 0.015%, or over 30 times smaller. This means that the market return is more significant than the exposure to size for daily returns. A also has a significantly larger mean magnitude compared to Rs, just over 10 times larger.

Rm also has a very large variance. This can be explained by there being days where prices change significantly and there being many days where prices remain at similar levels. Essentially, the market is volatile inter-day.

Rs has a vary low variance. By taking the log of the market cap, this factor has been made less sensitive to changes and so the inter-day variation is low.

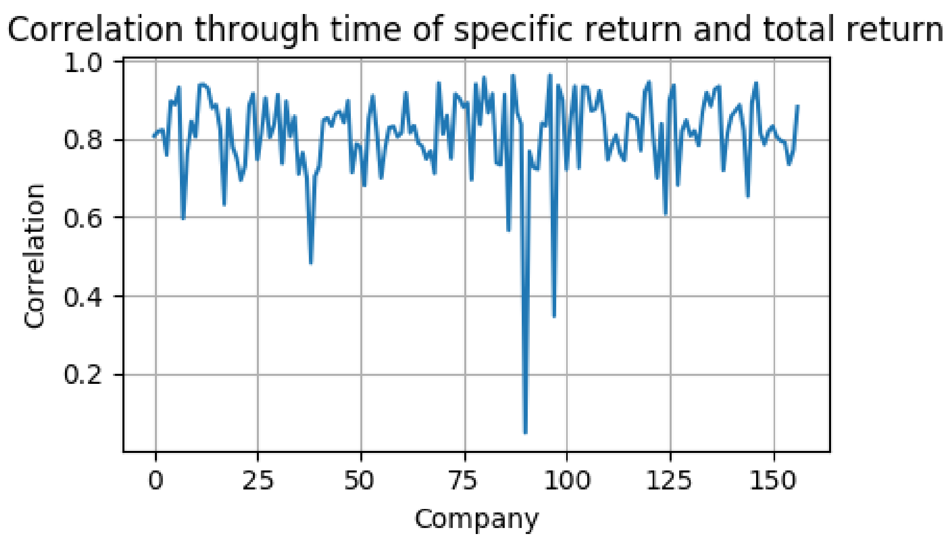
The value of 'a' mainly varies between 1% and -1%, with a mean of approximately 0 and a stddev of 0.277. This term can be considered as the risk-free rate for that day. The negative values can be viewed as the market pricing the rate as negative, as seen with German goverment bonds. If analysing data from the USA, over the same period, 'a' should have a larger positive mean, due to US goverment bonds having a higher yield.

Q2.4.5 c)

[58]:
```
e_bystock = e.T
```

[59]:
```
e_corr = np.zeros([num_stocks])
for i in range(num_stocks):
    e_corr[i] = np.corrcoef(e_bystock[i], ri[i])[1][0]
```

[61]:
```
plt.figure(dpi=100, figsize=(5,3))
plt.plot(e_corr)
plt.title('Correlation through time of specific return and total return')
plt.xlabel('Company')
plt.ylabel('Correlation')
plt.grid()
plt.tight_layout()
plt.show()
```



Q2.4.5 d)

[62]:
```
R = np.array([rm, rs]).T
R_cov = np.cov(R, rowvar=False)
```

[63]:
```
print("The covariance matrix of R is")
print(R_cov)
```

```
The covariance matrix of R is
[[ 6.15936602e-05 -8.05737998e-07]
 [-8.05737998e-07  4.52932279e-08]]
```

[64]:
```
R_eig, _ = np.linalg.eig(R_cov)
print('The eigen values of the covariance matrix are')
print(R_eig)
```

```
The eigen values of the covariance matrix are
[6.16042064e-05 3.47470098e-08]
```

The covariance between Rm and Rs is $-8.06e^{-7}$.

This is small, approximately zero. This could be due to the small magnitude of Rs, but Rs can be said to have no significant impact on Rm.

This matrix is also stable as the magnitude of the eigen values are less than 1.

Q2.4.5 e)

[65]:
```
E_cov = np.cov(e, rowvar=False)
```

[66]:
```python
from sklearn.decomposition import PCA
```

[67]:
```python
pca_e = PCA()
pca_e.fit(E_cov)

m = [(i+1, pca_e.explained_variance_ratio_[i]*100) for i in range(10)]
headers = ["component", "perc. variance explained (%)"]
table = tabulate(m, headers, tablefmt="fancy_grid", floatfmt=".3f")
print("Percentage of the variance explained by the  rst ten principal␣
↪components - Specific Returns Covariance")
print(table)
```

```
Percentage of the variance explained by the  rst ten principal components -
Specific Returns Covariance
```

component	perc. variance explained (%)
1	25.744
2	11.505
3	9.949
4	5.978
5	4.391
6	3.799
7	3.148
8	2.709
9	2.498
10	2.115

The first 10 components, out of 157, account for explaining 71.84% of the variance.

The specific return used, deducts the market return and market size factors already. Therefore, the covariance between these returns can show if there exists other factors not alreasy accounted for. By performing PCA, these factors can be extracted out. The first compoment represents a potential factor that could explain about 26% of the specific returns. This is a significant amount, and so a third factor derived from this, by assessing the covariance has the potential to increase the prediction accuracy of a new, 3 factor APT.

[68]:
```
DR_cov = np.cov(ri)
pca_dr = PCA()
pca_dr.fit(DR_cov)

m = [(i+1, pca_dr.explained_variance_ratio_[i]*100) for i in range(10)]
headers = ["component", "perc. variance explained (%)"]
table = tabulate(m, headers, tablefmt="fancy_grid", floatfmt=".3f")
print("Percentage of the variance explained by the  rst ten principal␣
 ↪components - Daily Returns Covariance")
print(table)
```

```
Percentage of the variance explained by the  rst ten principal components -
Daily Returns Covariance
```

component	perc. variance explained (%)
1	35.439
2	19.105
3	7.855
4	6.900
5	3.509
6	2.889
7	2.301
8	1.732
9	1.709
10	1.462

The table above looks at the first few components, when performing PCA on the covariance of the original daily returns. The first two components explain a higher percentage of variance. This supports the idea that the factors used in the APT model account for taking away variance in the specific returns. Namely, the market return and market size factors reduced the variance of the specific returns, leading to the first component of PCA explaining a lower ratio of the variance.

3 Q3 Portfolio Optimization

3.1 Q3.1 Adaptive minimum-variance portfolio optimization

3.1.1 Q3.1.1 Minimum Variance Portfolio Optimization

By allowing negative weights on assets, or, allowing short selling, the minimum-variance portfolio is given by:

$$min \quad \frac{1}{2}\mathbf{w}^T\mathbf{C}\mathbf{w} \qquad (3.1.11)$$

$$subjectto \quad \mathbf{w}^T\mathbf{1} = 1$$

The minimum-vaariance portfolio optimization problem can be solved by finding the optimal conditions for the associated Lagrangian function:

$$L(\mathbf{w}, \lambda, \mu) = \frac{1}{2}\mathbf{w}^T\mathbf{C}\mathbf{w} - \lambda(\mathbf{w1} - 1) \quad (3.1.12)$$

By differentiating w.r.t. $\mathbf{w}$ and λ and assuming that $\mathbf{C}$ is invertible, the following optimality conditions are obtained:

$$\frac{dL}{d\mathbf{w}} = \mathbf{Cw} - \lambda\mathbf{1} = 0 \text{ -> } \mathbf{w} = \lambda\mathbf{C}^{-1}\mathbf{1} \quad (3.1.13)$$

$$\frac{dL}{d\lambda} = \mathbf{w}^T\mathbf{1} - 1 = 0 \text{ -> } \mathbf{w}^T\mathbf{1} = 1 \quad (3.1.14)$$

The set of optimal weights $\mathbf{w}_{opt}$ can be found by solving for λ and substituting:

$$\mathbf{w}^T\mathbf{1} = (\lambda\mathbf{C}^{-1}\mathbf{1})^T\mathbf{1} = \lambda\mathbf{1}^T\mathbf{C}^{-1^T}\mathbf{1} = 1 \quad (3.1.15)$$

$$\text{-> } \lambda = \frac{1}{\mathbf{1}^T\mathbf{C}^{-1^T}\mathbf{1}} \quad (3.1.16)$$

$$\mathbf{w}_{opt} = \lambda\mathbf{C}^{-1}\mathbf{1} = \frac{1}{\mathbf{1}^T\mathbf{C}^{-1^T}\mathbf{1}}\mathbf{C}^{-1}\mathbf{1} \quad (3.1.17)$$

The theoretical variance of the returns using $\mathbf{w}_{opt}$ is given by:

$$\sigma_p^2 = \mathbf{w}_{opt}^T\mathbf{C}\mathbf{w}_{opt} = \frac{1}{\mathbf{1}^T\mathbf{C}^{-1^T}\mathbf{1}}(\mathbf{C}^{-1}\mathbf{1})^T\mathbf{C}\frac{1}{\mathbf{1}^T\mathbf{C}^{-1^T}\mathbf{1}}\mathbf{C}^{-1}\mathbf{1} \quad (3.1.18)$$

$$\sigma_p^2 = \frac{1}{(\mathbf{1}^T\mathbf{C}^{-1^T}\mathbf{1})^2}(\mathbf{C}^{-1}\mathbf{1})^T\mathbf{C}(\mathbf{C}^{-1}\mathbf{1}) = \frac{1}{(\mathbf{1}^T\mathbf{C}^{-1^T}\mathbf{1})^2}\mathbf{1}^T\mathbf{C}^{-1^T}\mathbf{1} = \frac{1}{\mathbf{1}^T\mathbf{C}^{-1^T}\mathbf{1}} \quad (3.1.19)$$

3.1.2 Q3.1.2 Static Mimimum Variance Portfolio

[90]:
```
import pandas as pd
import numpy as np
import datetime as dt
import matplotlib.pyplot as plt
from tabulate import tabulate
```

[91]:
```
df = pd.read_csv(r'./../data/fsp_case_31_BSD.csv', index_col=0, header=[0,1])
dates = df['ret'].index
date_axis = [dt.datetime.strptime(d,'%Y-%m-%d').date() for d in dates]
df = df['ret'].T[-10:].T
```

[92]:
```
df_train = df[:260]
df_test  = df[260:]
```

[93]:
```
def optimal_weights(returns):
    C = np.cov(returns, rowvar=False)
    C_inv = np.linalg.inv(C)
    ones = np.ones(shape=[C.shape[0]])

    w_opt = ( np.dot(C_inv,ones) ) / ( np.dot(np.dot(ones.T,C_inv.T), ones) )

    theo_var = 1 / ( np.dot(np.dot(ones.T,C_inv.T), ones) )

    return w_opt, theo_var
```

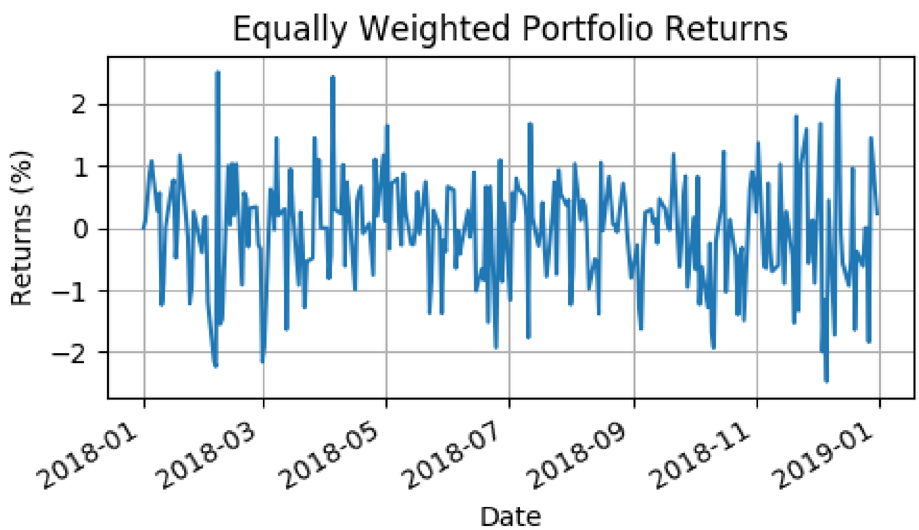
[94]:
```
def test_portfolio(weights, returns):
    port_returns = (returns*weights).sum(axis=1)

    var = np.var(port_returns)

    cum_returns = np.log1p(port_returns)
    cum_returns = np.expm1(cum_returns.cumsum())
    return port_returns, var, cum_returns
```

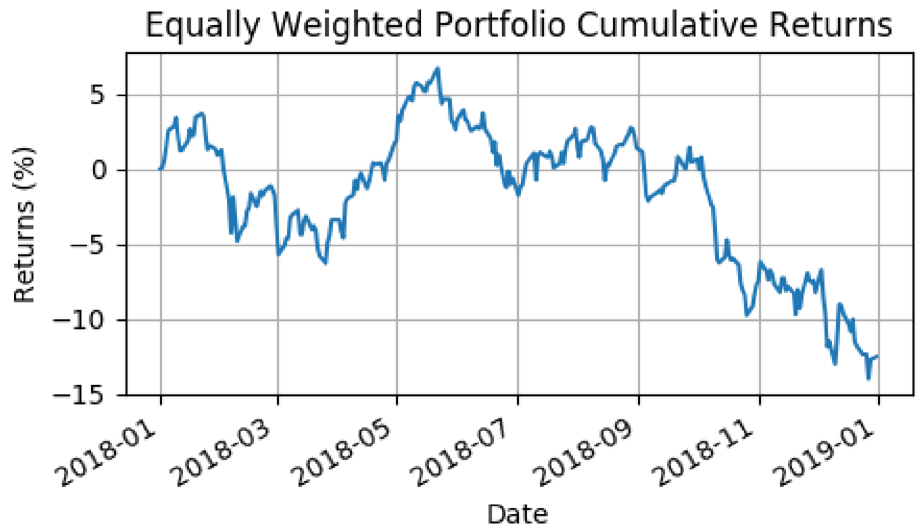
Equally Weighted Portfolio

[95]:
```
weights = [1/10 for _ in range(10)]
ret, var_ew, cum_ret = test_portfolio(weights, df_test)
```

[96]:
```
fig, ax = plt.subplots(dpi=100, figsize=(5,3))
ax.plot(date_axis[260:], ret*100)
# rotate and align the tick labels so they look better
fig.autofmt_xdate()
plt.title('Equally Weighted Portfolio Returns')
```

```
plt.xlabel('Date')
plt.ylabel('Returns (%)')
plt.grid()
plt.tight_layout()
plt.show()

fig, ax = plt.subplots(dpi=100, figsize=(5,3))
ax.plot(date_axis[260:], cum_ret*100)
# rotate and align the tick labels so they look better
fig.autofmt_xdate()
plt.title('Equally Weighted Portfolio Cumulative Returns')
plt.xlabel('Date')
plt.ylabel('Returns (%)')
plt.grid()
plt.tight_layout()
plt.show()
```





Min-Var Portfolio

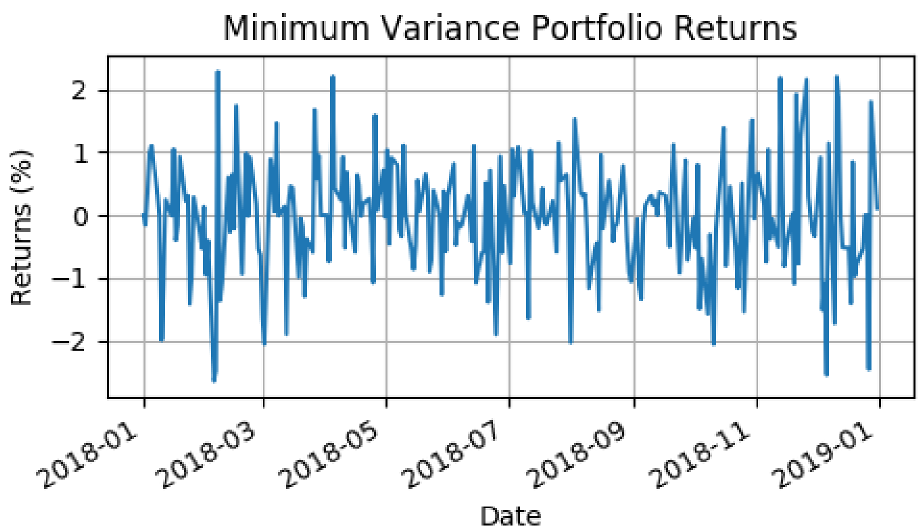
[97]:
```
weights, theo_var = optimal_weights(df_train)
ret_mv, var_mv, cum_ret_mv = test_portfolio(weights, df_test)
```

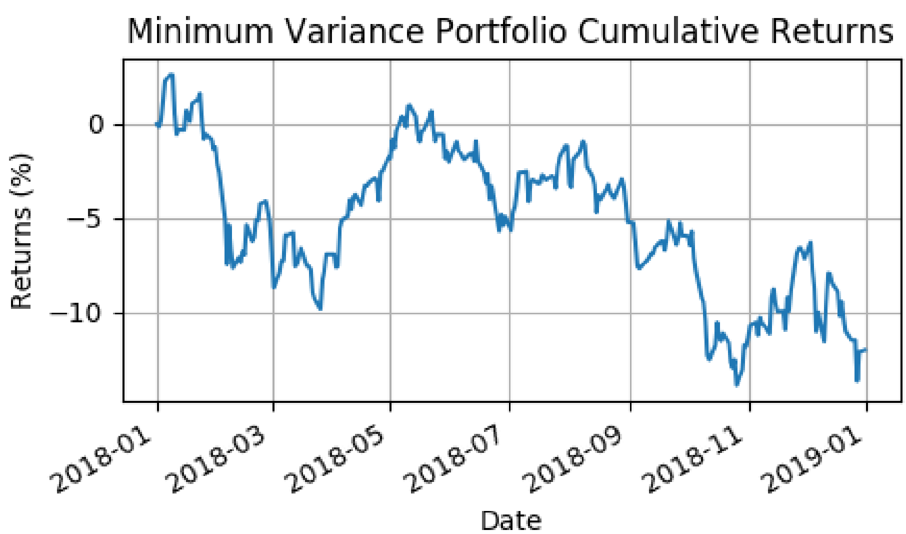
[98]:
```
print("Theoretical variance of this portfolio: {:.7f}".format(theo_var))
```

```
Theoretical variance of this portfolio: 0.0000286
```

[99]:
```
fig, ax = plt.subplots(dpi=100, figsize=(5,3))
ax.plot(date_axis[260:], ret_mv*100)
# rotate and align the tick labels so they look better
fig.autofmt_xdate()
plt.title('Minimum Variance Portfolio Returns')
plt.xlabel('Date')
plt.ylabel('Returns (%)')
plt.grid()
plt.tight_layout()
plt.show()

fig, ax = plt.subplots(dpi=100, figsize=(5,3))
ax.plot(date_axis[260:], cum_ret_mv*100)
# rotate and align the tick labels so they look better
fig.autofmt_xdate()
plt.title('Minimum Variance Portfolio Cumulative Returns')
plt.xlabel('Date')
plt.ylabel('Returns (%)')
plt.grid()
plt.tight_layout()
```

```
plt.show()
```





[101]:
```
m = [["Equal Weights", ret.mean()*100, cum_ret[-1]*100, var_ew],
     ["Min-Var", ret_mv.mean()*100, cum_ret_mv[-1]*100, var_mv]]
headers = ["Strategy", "Mean Return (%)", "Final Cumulative Ret (%)",␣
 ↪"Variance"]
table = tabulate(m, headers, tablefmt="fancy_grid", floatfmt=".7f")
print(table)
```

Strategy	Mean Return (%)	Final Cumulative Ret (%)	Variance
Equal Weights	-0.0473190	-12.5290991	0.0000790
Min-Var	-0.0450505	-12.0391415	0.0000816

The min-var portfolio had a slighly lower mean daily return than the equally weighted portfolio, but had a better final cumalative return.

The variance of the portfolio returns, $8.16e-5$ was greater than the theoretical variance of $2.86e-5$. It was also greater than the variance of the equally weighted portfolio ($7.90e-5$). This shows that a minimum variance portolio over one time period, is not guaranteed to also be a minimum variance portfolio over a different time period.

3.1.3 Q3.1.3 Adaptive Mimimum Variance Portfolio

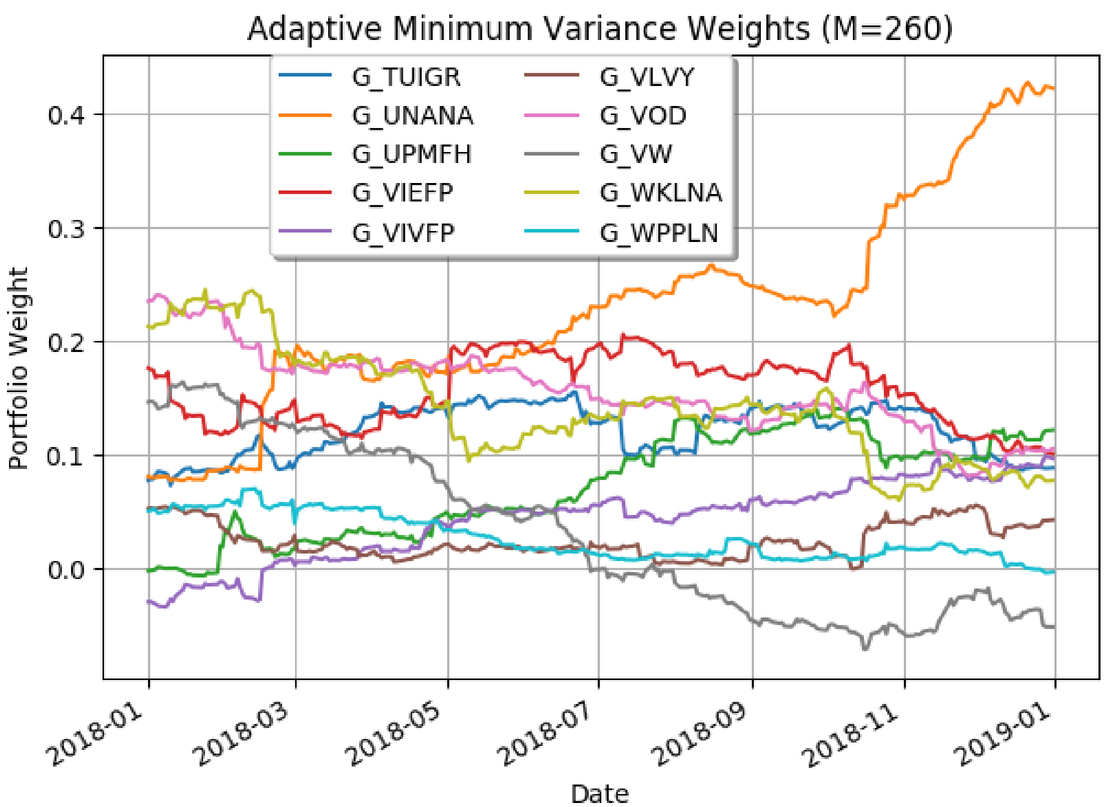
[102]:
```
def optimal_weights_rolling(returns, M):
    Cs = df_t.rolling(M).cov().to_numpy()[M*10:,:].reshape([261,10,10])
    C_invs = np.linalg.inv(Cs)
    ones = np.ones(shape=[10])

    num = np.dot(C_invs,ones)
    div = np.dot(np.dot(ones.T,C_invs.transpose((0,2,1))), ones)

    w_opt = num
    for i, d in enumerate(div):
        w_opt[i] /= d

    return w_opt
```

[103]:
```
def test_portfolio_rolling(returns, weights, M):

    port_returns = (returns[M:]*weights).sum(axis=1)

    var = port_returns.var()

    cum_returns = np.log1p(port_returns)
    cum_returns = np.expm1(cum_returns.cumsum())
    return port_returns, var, cum_returns
```

[104]:
```
M = 260
df_t = df[260-M:]

w = optimal_weights_rolling(df_t, M)

rt, var, cum_rt = test_portfolio_rolling(df_t, w, M)
```

[105]:

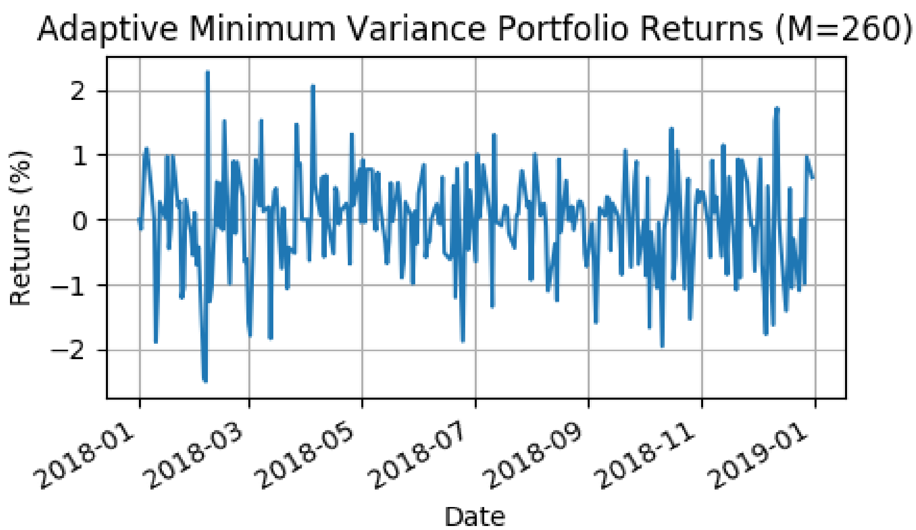
```
fig, ax = plt.subplots(dpi=100, figsize=(7,5))
i=0
for stock in df:
    ax.plot(date_axis[260:], w.T[i], label=stock)
    i += 1
# rotate and align the tick labels so they look better
fig.autofmt_xdate()
plt.title('Adaptive Minimum Variance Weights (M=260)')
plt.xlabel('Date')
plt.ylabel('Portfolio Weight')
plt.grid()
#plt.tight_layout()
ax.legend(loc='upper center', bbox_to_anchor=(0.4, 1.02),
          ncol=2, fancybox=True, shadow=True)
plt.show()
```

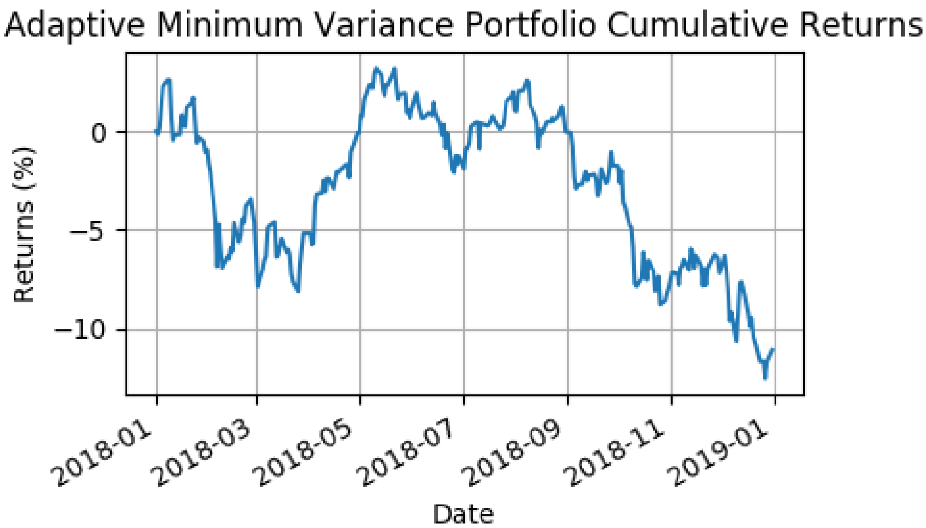


[106]:

```
fig, ax = plt.subplots(dpi=100, figsize=(5,3))
ax.plot(date_axis[260:], rt*100)
# rotate and align the tick labels so they look better
fig.autofmt_xdate()
plt.title('Adaptive Minimum Variance Portfolio Returns (M=260)')
```

```
plt.xlabel('Date')
plt.ylabel('Returns (%)')
plt.grid()
plt.tight_layout()
plt.show()

fig, ax = plt.subplots(dpi=100, figsize=(5,3))
ax.plot(date_axis[260:], cum_rt*100)
# rotate and align the tick labels so they look better
fig.autofmt_xdate()
plt.title('Adaptive Minimum Variance Portfolio Cumulative Returns')
plt.xlabel('Date')
plt.ylabel('Returns (%)')
plt.grid()
plt.tight_layout()
plt.show()
```





[107]:
```python
m = [["Equal Weights", ret.mean()*100, cum_ret[-1]*100, var_ew],
     ["Static Min-Var", ret_mv.mean()*100, cum_ret_mv[-1]*100, var_mv],
     ["Adaptive Min-Var", rt.mean()*100, cum_rt[-1]*100, var]]
headers = ["Strategy", "Mean Return (%)", "Final Cumulative Ret (%)",␣
 ↪"Variance"]
table = tabulate(m, headers, tablefmt="fancy_grid", floatfmt=".7f")
print(table)
```

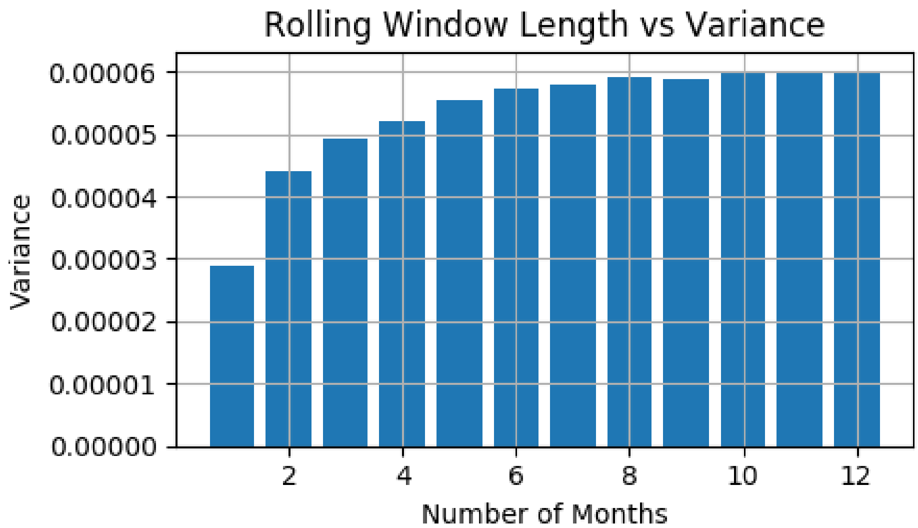
Strategy	Mean Return (%)	Final Cumulative Ret (%)	Variance
Equal Weights	-0.0473190	-12.5290991	0.0000790
Static Min-Var	-0.0450505	-12.0391415	0.0000816
Adaptive Min-Var	-0.0419880	-11.0838185	0.0000603

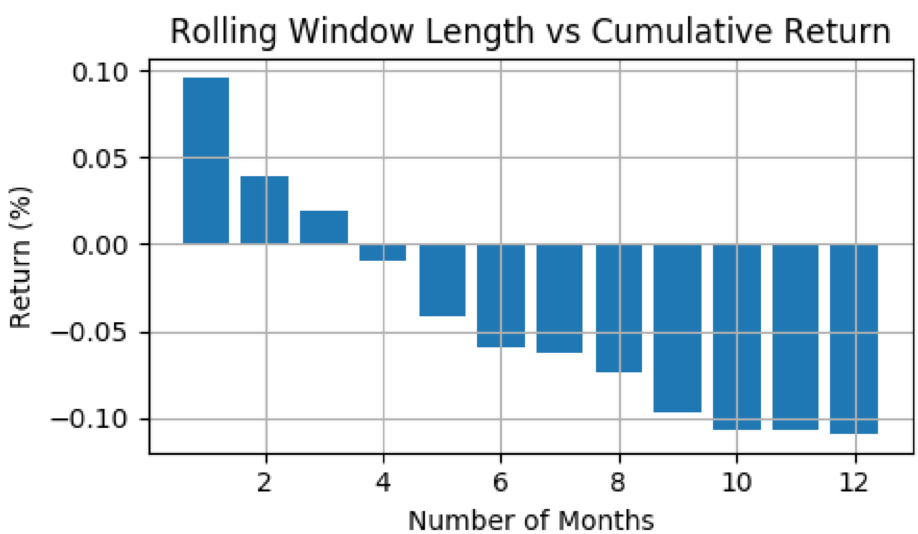
The adaptive portfolio showed better performance vs the previous two strategies in terms of variance, when using $M = 260$ having the lowest at $6.03e-5$. It also resulted in a better final cumulative return, but still lost value. However, in practice, updating portfolio weights daily substantially increases transactions costs, and would resslt in lower returns than shown.

Varying Roliing Window, M

[108]:
```
base_M = 21

variances = []
cum_returns = []

for i in range(1, 13):
    M = i*base_M
    df_t = df[260-M:]

    w = optimal_weights_rolling(df_t, M)
    rt, var, cum_rt = test_portfolio_rolling(df_t, w, M)

    variances.append(var)
    cum_returns.append(cum_rt[-1])
```

[109]:
```
fig, ax = plt.subplots(dpi=100, figsize=(5,3))
ax.bar(list(range(1,13)), variances)
plt.title('Rolling Window Length vs Variance')
plt.xlabel('Number of Months')
plt.ylabel('Variance')
plt.grid()
plt.tight_layout()
plt.show()

fig, ax = plt.subplots(dpi=100, figsize=(5,3))
ax.bar(list(range(1,13)), cum_returns)
plt.title('Rolling Window Length vs Cumulative Return')
plt.xlabel('Number of Months')
plt.ylabel('Return (%)')
plt.grid()
plt.tight_layout()
plt.show()
```





Taking 21 days as one trading month, the number of months the minimum-variance portfolio weights was trained on was varied from 1 to 12. As the number of months increased, so did the variance of the returns, and the cumaltive return decreased. Using shorter rolling windows of 1 to 3 months, resulted in positive cumulative returns and significantly smaller variances. This shows that the market was changing enough at least every quarter, that a rebalancing would be required to produce good performance. It is important to note that transactions costs would be independent of the rolling window length, as the portolfio weights are updated daily for each M tested.

The covariance matrix used equal weightings to each daily return in the rolling window. However, there are other methods to assign the weightings. For example, it is possible to place a greater reliance on more recent daily returns, by using an exponential weight with the daily returns when calculating the rolling covariance matrices. So, more recent returns have a larger weighting than older returns. This introduces additional parameters to investigate, such as the decay rate of the weightings. This would be more useful when looking at windows of one quarter or greater, as it allows historical data to have some influence whilst also heavily capturing the more recent trends.

4 Q4 Robust Statistics and Non Linear Methods

4.1 Q4.1 Data Import and Non Linear Models

4.1.1 Q4.1.1 Key Descriptive Statistics

For each of AAPL, IBM, JPM and DJI, the tables below show the mean, median, standard deviation, median absolute deviation, interquartile range, skew and kurtosis.

For all assets, the mean and median are quite similar apart from the returns, where the median is higher. The measure of dispersion differ more, namely the std.dev and MAD.

[8]:
```
aapl = pd.read_csv('./../data/AAPL.csv').set_index('Date')
ibm = pd.read_csv('./../data/IBM.csv').set_index('Date')
jpm = pd.read_csv('./../data/JPM.csv').set_index('Date')
dji = pd.read_csv('./../data/DJI.csv').set_index('Date')
stocks = [aapl, ibm, jpm, dji]
dates = aapl['Open'].index
date_axis = [dt.datetime.strptime(d,'%Y-%m-%d').date() for d in dates]
```

[9]:
```
# add 1-day returns
for stock in stocks:
    stock['ret'] = stock['Adj Close'].pct_change()
```

[10]:
```
names = ["AAPL", "IBM", "JPM", "DJI"]
data = []
for stock in stocks:
    _data = []
    _data.append(["Mean"]+list(stock.mean().to_numpy()))
    _data.append(["Median"]+list(stock.median().to_numpy()))
    _data.append(["StdDev"]+list(stock.std().to_numpy()))
    _data.append(["MAD"]+list(stock.mad().to_numpy()))
    _data.append(["IQR"]+list(stock.quantile(0.75).to_numpy()-stock.quantile(0.
↪25).to_numpy()))
    _data.append(["Skew"]+list(stock.skew().to_numpy()))
    _data.append(["Kurtosis"]+list(stock.kurtosis().to_numpy()))
    data.append(_data)

headers = ["Statistic", "Open", "High", "Low", "Close", "Adj. Close", "Volume",␣
↪"Return"]
```

```python
for i, name in enumerate(names):
    print(name)
    table = table = tabulate(data[i], headers, tablefmt="fancy_grid")
    print(table)
```

AAPL

Statistic	Open	High	Low	Close	Adj. Close	Volume	Return
Mean	187.687	189.562	185.824	187.712	186.174	3.27048e+07	0.000425548
Median	186.29	187.4	184.94	186.12	184.352	2.9184e+07	0.00161136
StdDev	22.1456	22.2816	22.0088	22.1607	21.9047	1.41797e+07	0.019323
MAD	18.1422	18.2413	18.0637	18.1758	17.9117	1.03785e+07	0.0133526
IQR	36	36.34	36.06	36.755	35.6854	1.63117e+07	0.0181045
Skew	0.259917	0.300385	0.220489	0.263849	0.29077	1.74332	-0.41564
Kurtosis	-0.912594	-0.924602	-0.917632	-0.932425	-0.928017	4.35318	4.24306

IBM

Statistic	Open	High	Low	Close	Adj. Close	Volume	Return

Mean	138.454	139.492	137.329	138.363	134.903	5.19894e+06	-0.000251616
Median	142.81	143.99	142.06	142.71	138.566	4.2379e+06	0.000409482
StdDev	12.1143	11.9131	12.2046	12.0281	10.6716	3.32896e+06	0.0155616
MAD	9.99034	9.81456	10.0753	9.9129	8.68543	2.02542e+06	0.010278
IQR	15.38	14.72	16.34	15.505	14.1039	1.95295e+06	0.0132241
Skew	-0.676024	-0.622707	-0.713446	-0.682246	-0.811222	3.1929	-0.309538
Kurtosis	-0.585272	-0.623607	-0.561975	-0.584037	-0.420852	11.7969	7.23577

JPM

Statistic	Open	High	Low	Close	Adj. Close	Volume	Return
Mean	108.708	109.652	107.683	108.607	107.263	1.47007e+07	-0.00013302
Median	109.18	110.53	107.79	109.02	107.219	1.3633e+07	-0.000602616
StdDev	5.35908	5.20287	5.43254	5.30048	4.83332	5.34977e+06	0.0130878
MAD	4.41151	4.33669	4.43555	4.37808	3.92939		

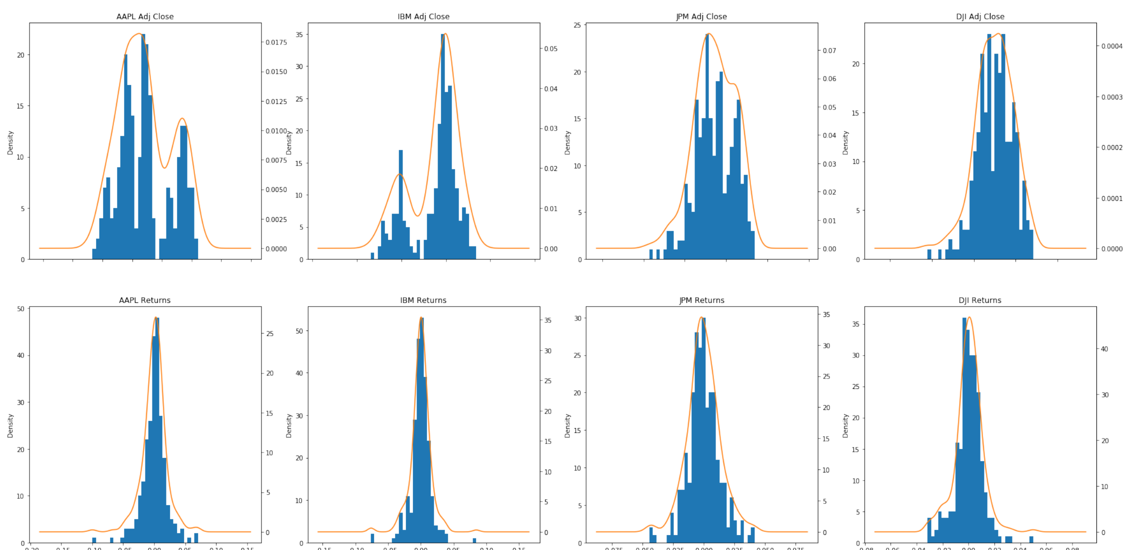
Statistic	Open	High	Low	Close	Adj. Close	Volume	Return
						3.945e+06	0.00975647
IQR	8.81001	8.845	8.845	8.835	7.22244	6.2336e+06	0.01497
Skew	-0.420811	-0.376221	-0.377517	-0.374853	-0.344491	1.69346	0.0228698
Kurtosis	-0.322536	-0.544163	-0.2657	-0.396579	-0.105437	4.4302	1.31859

DJI

Statistic	Open	High	Low	Close	Adj. Close	Volume	Return
Mean	25001.3	25142	24846	24999.2	24999.2	3.32889e+08	0.00019681
Median	25025.6	25124.1	24883	25044.3	25044.3	3.1379e+08	0.000374537
StdDev	858.835	815.204	903.302	859.132	859.132	9.4078e+07	0.0104764
MAD	682.034	658.96	712.026	686.324	686.324	6.87602e+07	0.0074519
IQR	1109.43	1077.82	1204.42	1158.16	1158.16	1.0893e+08	0.00988984
Skew	-0.372127	-0.239367	-0.456447	-0.380147	-0.380147	1.73956	-0.015718
Kurtosis	0.485736	0.118153	0.557592	0.400668	0.400668	5.85758	2.77395

4.1.2 Q4.1.2 Histograms and Probability Density Functions

[11]:
```
fig, axis = plt.subplots(2, 4, figsize=(30,15))
# adj_close
for i, ax in enumerate(axis[0]):
    ax.set_title("{} Adj Close".format(names[i]))
    ax.hist(stocks[i]['Adj Close'], bins=30)
    stocks[i]['Adj Close'].plot.kde(ax=ax, secondary_y=True)
# returns
for i, ax in enumerate(axis[1]):
    ax.set_title("{} Returns".format(names[i]))
    ax.hist(stocks[i]['ret'].dropna(), bins=30)
    stocks[i]['ret'].dropna().plot.kde(ax=ax, secondary_y=True)
```



The pdfs of the returns appear to be a normal distribution, making the use of mean and standard deviation viable for them. However, the pdfs of the prices do not to appear to be normal distributions. So the classical statistics of mean and std should not be used with classic methods that assume normallity of the underlying distribution, when using the price data.

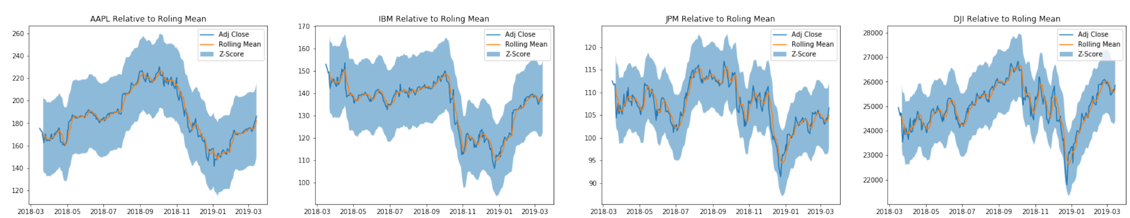
4.1.3 Q4.1.3 Mean vs Median Outlier Detection

[12]:
```
outliers = [ ["AAPL", 0, 0, 0, 0],
             ["IBM", 0, 0, 0, 0],
             ["JPM", 0, 0, 0, 0],
```

```
            ["DJI", 0, 0, 0, 0] ]
def count_outliers(close, low, high):
    l = low-close
    count = l[l>0].count()
    h = high-close
    count += h[h<0].count()
    return count
```

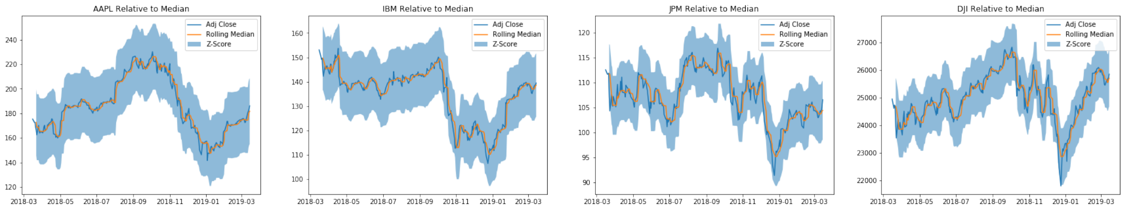
Using the static dispersion measurements:

```
[13]: fig, axis = plt.subplots(1, 4, figsize=(30,5))
# adj_close
for i, ax in enumerate(axis):
    std = stocks[i]['Adj Close'].std()
    ax.set_title("{} Relative to Roling Mean".format(names[i]))
    ax.plot(date_axis, stocks[i]['Adj Close'], label="Adj Close")
    ax.plot(date_axis, stocks[i]['Adj Close'].rolling(5).mean(), label='Rolling␣
↪Mean')
    minus = stocks[i]['Adj Close'].rolling(5).mean() - 1.5*std
    plus = stocks[i]['Adj Close'].rolling(5).mean() + 1.5*std
    outliers[i][1] = count_outliers(stocks[i]['Adj Close'], minus, plus)
    ax.fill_between(date_axis,minus,plus, label="Z-Score", alpha=0.5)
    ax.legend()
plt.show()
```



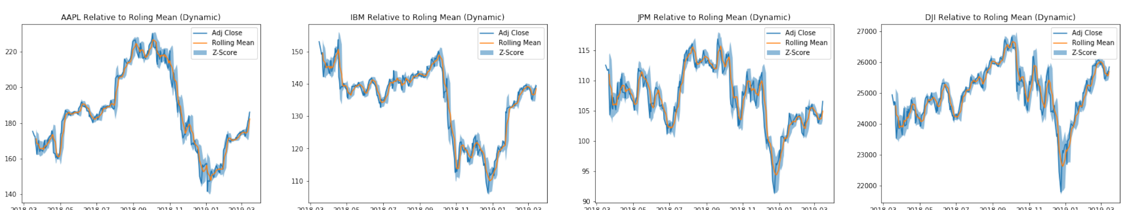

```
[14]: fig, axis = plt.subplots(1, 4, figsize=(30,5))
# adj_close
for i, ax in enumerate(axis):
    mad = stocks[i]['Adj Close'].mad()
    ax.set_title("{} Relative to Median".format(names[i]))
    ax.plot(date_axis, stocks[i]['Adj Close'], label="Adj Close")
    ax.plot(date_axis, stocks[i]['Adj Close'].rolling(5).median(),␣
↪label='Rolling Median')
    minus = stocks[i]['Adj Close'].rolling(5).median() - 1.5*mad
    plus = stocks[i]['Adj Close'].rolling(5).median() + 1.5*mad
    outliers[i][2] = count_outliers(stocks[i]['Adj Close'], minus, plus)
    ax.fill_between(date_axis,minus,plus, label="Z-Score", alpha=0.5)
    ax.legend()
```

```
plt.show()
```



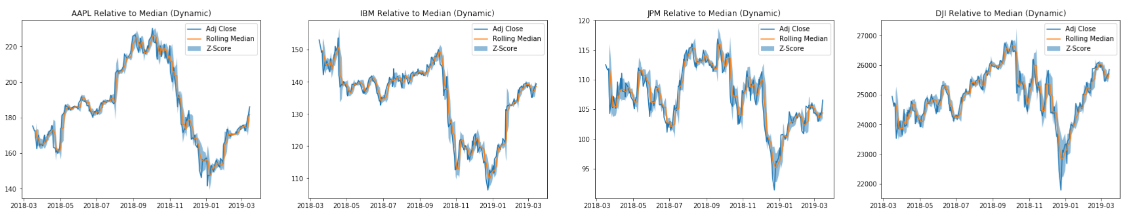
Using dynamic dispersion measurements (rolling):

[15]:
```
fig, axis = plt.subplots(1, 4, figsize=(30,5))
# adj_close
for i, ax in enumerate(axis):
    ax.set_title("{} Relative to Roling Mean (Dynamic)".format(names[i]))
    ax.plot(date_axis, stocks[i]['Adj Close'], label="Adj Close")
    ax.plot(date_axis, stocks[i]['Adj Close'].rolling(5).mean(), label='Rolling␣
↪Mean')
    std = stocks[i]['Adj Close'].rolling(5).std()
    minus = stocks[i]['Adj Close'].rolling(5).mean() - 1.5*std
    plus = stocks[i]['Adj Close'].rolling(5).mean() + 1.5*std
    outliers[i][3] = count_outliers(stocks[i]['Adj Close'], minus, plus)
    ax.fill_between(date_axis,minus,plus, label="Z-Score", alpha=0.5)
    ax.legend()
plt.show()
```



[16]:
```
fig, axis = plt.subplots(1, 4, figsize=(30,5))
# adj_close
for i, ax in enumerate(axis):
    ax.set_title("{} Relative to Median (Dynamic)".format(names[i]))
    ax.plot(date_axis, stocks[i]['Adj Close'], label="Adj Close")
    ax.plot(date_axis, stocks[i]['Adj Close'].rolling(5).median(),␣
↪label='Rolling Median')

    mad = lambda x: np.fabs(x - x.mean()).mean()
    mad = stocks[i]['Adj Close'].rolling(5).apply(mad, raw=True)
```

```
    minus = stocks[i]['Adj Close'].rolling(5).median() - 1.5*mad
    plus = stocks[i]['Adj Close'].rolling(5).median() + 1.5*mad
    outliers[i][4] = count_outliers(stocks[i]['Adj Close'], minus, plus)
    ax.fill_between(date_axis,minus,plus, label="Z-Score", alpha=0.5)
    ax.legend()
plt.show()
```



[17]:
```
headers = ["Stock", "Mean, Static Std", "Median, Static MAD", "Mean, Dynamic␣
 ↪Std", "Median, Dynamic MAD"]
table = tabulate(outliers, headers, tablefmt="fancy_grid")
print("Outliers from each Detection Method")
print(table)
```

```
Outliers from each Detection Method
```

Stock	Mean, Static Std	Median, Static MAD	Mean, Dynamic Std	Median, Dynamic MAD
AAPL	0	0	30	98
IBM	0	0	31	98
JPM	0	2	33	97
DJI	0	3	30	95

In practice, the static dispersion measures cannot be used as they require future data. The dynamic approach, calculates the std. and MAD from the same rolling window used for the mean and median.

The mean method results in less outliers being detected. This is due to an outlier effecting the std. greatly, whereas it has minimal impact on the MAD. As the std. increases more with outliers, they are less likely to be considered out of the range compared to the mean.

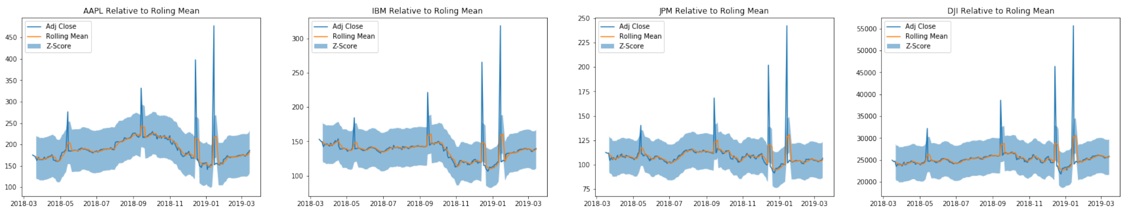
Overall, the dynamic method detects more outliers than the static method, which is expected as the price range has a tighter bound / window.

4.1.4 Q4.1.4 Impact of Artificial Outliers

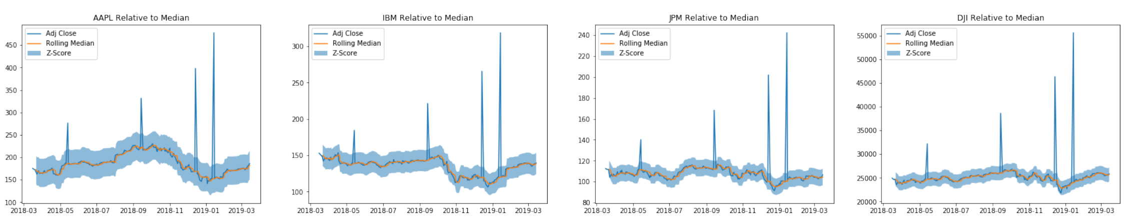
[18]:
```
stocks_o = []
for stock in stocks:
    s = pd.DataFrame.copy(stock)
    s.loc['2018-05-14', 'Adj Close'] = 1.2*s['Adj Close'].max()
    s.loc['2018-09-14', 'Adj Close'] = 1.2*s['Adj Close'].max()
    s.loc['2018-12-14', 'Adj Close'] = 1.2*s['Adj Close'].max()
    s.loc['2019-01-14', 'Adj Close'] = 1.2*s['Adj Close'].max()
    stocks_o.append(s)
```

[19]:
```
outliers = [ ["AAPL", 0, 0, 0, 0],
             ["IBM", 0, 0, 0, 0],
             ["JPM", 0, 0, 0, 0],
             ["DJI", 0, 0, 0, 0] ]
def count_outliers(close, low, high):
    l = low-close
    count = l[l>0].count()
    h = high-close
    count += h[h<0].count()
    return count
```

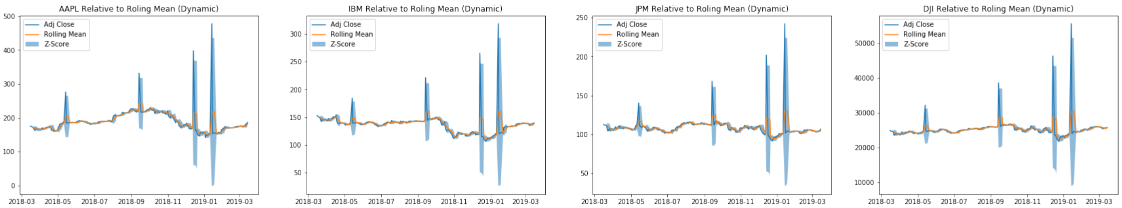
[20]:
```
fig, axis = plt.subplots(1, 4, figsize=(30,5))
# adj_close
for i, ax in enumerate(axis):
    std = stocks_o[i]['Adj Close'].std()
    ax.set_title("{} Relative to Roling Mean".format(names[i]))
    ax.plot(date_axis, stocks_o[i]['Adj Close'], label="Adj Close")
    ax.plot(date_axis, stocks_o[i]['Adj Close'].rolling(5).mean(),␣
 ↪label='Rolling Mean')
    minus = stocks_o[i]['Adj Close'].rolling(5).mean() - 1.5*std
    plus = stocks_o[i]['Adj Close'].rolling(5).mean() + 1.5*std
    outliers[i][1] = count_outliers(stocks_o[i]['Adj Close'], minus, plus)
    ax.fill_between(date_axis,minus,plus, label="Z-Score", alpha=0.5)
    ax.legend()
plt.show()
```



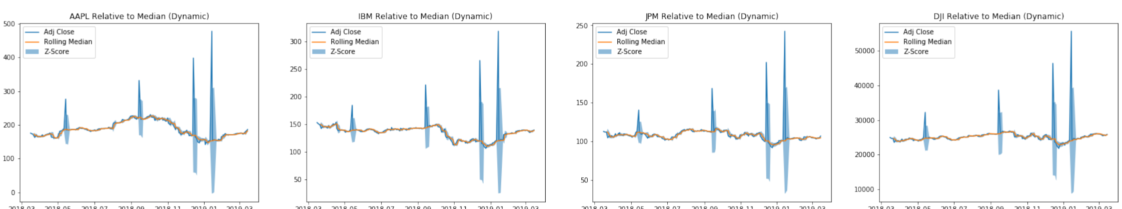
[21]:
```
fig, axis = plt.subplots(1, 4, figsize=(30,5))
# adj_close
for i, ax in enumerate(axis):
    mad = stocks_o[i]['Adj Close'].mad()
    ax.set_title("{} Relative to Median".format(names[i]))
    ax.plot(date_axis, stocks_o[i]['Adj Close'], label="Adj Close")
    ax.plot(date_axis, stocks_o[i]['Adj Close'].rolling(5).median(),␣
↪label='Rolling Median')
    minus = stocks_o[i]['Adj Close'].rolling(5).median() - 1.5*mad
    plus = stocks_o[i]['Adj Close'].rolling(5).median() + 1.5*mad
    outliers[i][2] = count_outliers(stocks_o[i]['Adj Close'], minus, plus)
    ax.fill_between(date_axis,minus,plus, label="Z-Score", alpha=0.5)
    ax.legend()
plt.show()
```



[22]:
```
fig, axis = plt.subplots(1, 4, figsize=(30,5))
# adj_close
for i, ax in enumerate(axis):
    ax.set_title("{} Relative to Roling Mean (Dynamic)".format(names[i]))
    ax.plot(date_axis, stocks_o[i]['Adj Close'], label="Adj Close")
    ax.plot(date_axis, stocks_o[i]['Adj Close'].rolling(5).mean(),␣
↪label='Rolling Mean')
    std = stocks_o[i]['Adj Close'].rolling(5).std()
    minus = stocks_o[i]['Adj Close'].rolling(5).mean() - 1.5*std
    plus = stocks_o[i]['Adj Close'].rolling(5).mean() + 1.5*std
    outliers[i][3] = count_outliers(stocks_o[i]['Adj Close'], minus, plus)
    ax.fill_between(date_axis,minus,plus, label="Z-Score", alpha=0.5)
    ax.legend()
plt.show()
```



[23]:
```
fig, axis = plt.subplots(1, 4, figsize=(30,5))
# adj_close
for i, ax in enumerate(axis):
    ax.set_title("{} Relative to Median (Dynamic)".format(names[i]))
    ax.plot(date_axis, stocks_o[i]['Adj Close'], label="Adj Close")
    ax.plot(date_axis, stocks_o[i]['Adj Close'].rolling(5).median(),␣
 ↪label='Rolling Median')

    mad = lambda x: np.fabs(x - x.mean()).mean()
    mad = stocks_o[i]['Adj Close'].rolling(5).apply(mad, raw=True)
    minus = stocks_o[i]['Adj Close'].rolling(5).median() - 1.5*mad
    plus = stocks_o[i]['Adj Close'].rolling(5).median() + 1.5*mad
    outliers[i][4] = count_outliers(stocks_o[i]['Adj Close'], minus, plus)
    ax.fill_between(date_axis,minus,plus, label="Z-Score", alpha=0.5)
    ax.legend()
plt.show()
```



[24]:
```
headers = ["Stock", "Mean, Static Std", "Median, Static MAD", "Mean, Dynamic␣
 ↪Std", "Median, Dynamic MAD"]
table = tabulate(outliers, headers, tablefmt="fancy_grid")
print("Outliers from each Detection Method")
print(table)
```

```
Outliers from each Detection Method


  Stock       Mean, Static Std     Median, Static MAD     Mean, Dynamic Std
Median, Dynamic MAD
```

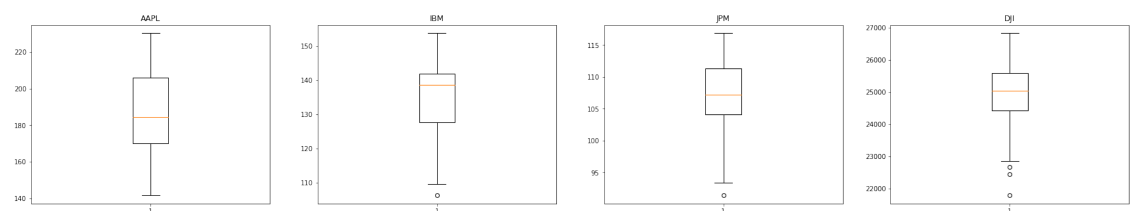
AAPL	11	4	32	93
IBM	12	4	31	94
JPM	12	4	33	88
DJI	12	4	29	90

As expected, the dynamic methods were not impacted that much by the artificial outliers.

The introduced outliers impacted the mean based, static std. detection method the most, making the average number of detected outliers 12, instead of the previous total of 0. The outliers increased the mean, and skewed the data to be short-tailed. This lead to the lower values being less likely to fall within the window around the rolling mean, and hence be detected as outliers.

4.1.5 Q4.1.5 Box Plots

[25]:
```python
fig, axis = plt.subplots(1, 4, figsize=(30,5))
for i, ax in enumerate(axis):
    ax.set_title(names[i])
    ax.boxplot(stocks[i]['Adj Close'])
plt.show()
```



The box plot provideS: - median - the orange line - interquartile rang (IQR) - length of the box - 25% quartile - bottom line of box - 75% quartile - top line of box - min value excl. outliers - bottom whisker - max value excl. outliers - top whisker - points outside 1.5x the IQR - points on the plot

Each of the plots either have one or more of: - asymmetric quartiles - asymmetric distance to median from min vs max - outlier points

This suggests that normal / gassian distribution of the data cannot be assumed.

4.2 Q4.2 Robust Estimators

4.2.1 Q4.2.1 Implementation

[29]:
```
def median(s):
    _sorted = s.sort_values()
    return _sorted[int(len(_sorted)/2)]

def IQR(s):
    _sorted = s.sort_values()
    quarter = len(_sorted)/4
    lo = _sorted[int(quarter)]
    hi = _sorted[int(3*quarter)]
    return hi-lo

def MAD(s):
    med = median(s)
    devs = abs(s-med)
    return median(devs)
```

4.2.2 Q4.2.2 Complexity Analysis

For series length N.

Mean: - Requires a sort - $O(nlogn)$ - Array lookup for middle term - $O(1)$ - Overall - $O(nlogn)$

IQR: - Requires a sort - $O(nlogn)$ - Index calculation, array lookups, difference - $O(1)$ - Overall - $O(nlogn)$ - Same complexity as median, 1 more lookup, 3 extra calculations - For large N, same performance as median

MAD: - Two median calculations - $O(nlogn)$ - Absolute difference calculation - $O(n)$ - Overall - $O(nlogn)$ - Same complexity as both median and IQR, but over twice the amount of operations

All three estimators are less computationally effecient than both mean and stddev ($O(n)$).

4.2.3 Q4