ASSIGNMENT - 2

Q1. Use the master method to give tight asymptotic bounds for the following recurrences

a. $T(n) = 2T(n/4) + 1$

b. $T(n) = 2T(n/4) + \sqrt{n}$

c. $T(n) = 2T(n/4) + n$

d. $T(n) = 2T(n/4) + n^2$

A1.

a] $T(n) = 2T(n/4) + 1$

$$a = 2 \; ; \; b = 4 \; ; \; f(n) = 1$$

$$\log_b a = \log_4 2 = 0.5$$

MASTER THEOREM (Case 1)

If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$

then $T(n) = \Theta(n^{\log_b a})$

Now, $f(n) = O(n^{\log_b a - \epsilon})$

$f(n) = O(n^{0.5 - \epsilon})$

Let $\epsilon = 0.5$

$\underline{1} = O(n^{0})$

$\boxed{a^0 = 1}$

Case 1 applies.

$$1 = O(1)$$

Therefore, $T(n) = \Theta(n^{1/2})$

$$\boxed{T(n) = \Theta(\sqrt{n})}$$

$$\boxed{n^{1/2} = \sqrt{n}}$$

b] $T(n) = 2T(n/4) + \sqrt{n}$.

Let us consider for CASE 1 of MASTER THEOREM.

Case 1: If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$.

then $T(n) = \Theta(n^{\log_b a})$

$$T(n) = 2T(n/4) + \sqrt{n}$$

$a = 2$; $b = 4$; $f(n) = \sqrt{n}$.

$$n^{\log_b a} = n^{\log_4 2} = n^{1/2}.$$

$$\sqrt{n} = O(n^{1/2 - \epsilon})$$

Let $\epsilon = 1/2$

$$\sqrt{n} \neq O(n^0)$$

Case 1 does not apply

Let us consider CASE 2 of MASTER THEOREM.

Case 2: If $f(n) = \Theta(n^{\log_b a})$

then $T(n) = \Theta(n^{\log_b a} \lg n)$

$T(n) = 2T(n/4) + \sqrt{n}$.

$a = 2$; $b = 4$; $f(n) = \sqrt{n}$.

$n^{\log_b a} = n^{\log_4 2} = n^{1/2}$.

$f(n) = \Theta(n^{\log_b a})$

$\sqrt{n} = \Theta(n^{1/2})$

$\sqrt{n} = \Theta(\sqrt{n})$

Case 2 applies

Therefore, $T(n) = \Theta(\sqrt{n} \lg n)$

c] $T(n) = 2T(n/4) + n$.

Applying Case 1: $f(n) = O(n^{\log_b a - \epsilon})$ for $\epsilon > 0$.

$n \neq O(n^{1/2 - \epsilon})$

Case 1 does not apply.

Applying Case 2:

$$f(n) = \Theta(n^{\log_b a})$$

$$n = \Theta(n^{1/2})$$

$$n \neq \Theta(\sqrt{n})$$

Case 2 does not apply.

Applying Case 3:

If $f(n) = \Omega(n^{\log_b a + \epsilon})$ where $\epsilon > 0$.

and $a \cdot f(n) \leq c \cdot f(n)$ for some $c < 1$ and $\forall$ n.

then $T(n) = \Theta(f(n))$.

$a = 2$; $b = 4$; $f(n) = n$; $n^{\log_b a} = n^{1/2}$.

$$f(n) = \Omega(n^{\log_b a + \epsilon})$$

Let $\epsilon = 1/2$

$$f(n) = \Omega(n^{1/2 + 1/2})$$

$$n = \Omega(n^1)$$

$$a \cdot f(n/b) \leq c \cdot f(n)$$

$$2(n/4) \leq c \cdot n$$

Let $c = 1/2$

$$2(n/4) \leq \frac{1}{2} n$$

$$1/2\, n \leq 1/2\, n.$$

Case 3 applies

Therefore, $\boxed{T(n) = \Theta(f(n) = \Theta(n)}$

d] $T(n) = 2T(n/4) + n^2$.

$a = 2$; $b = 4$; $f(n) = n^2$; $n^{\log_b a} = n^{1/2}$.

Case 1:

$$f(n) = O(n^{\log_b a - \epsilon})$$

$$n^2 = O(n^{1/2 - \epsilon})$$

Let $\epsilon = 1/2$

$$n^2 \neq O(n^0)$$

Case 1 does not apply.

Applying Case 2 :

$$f(n) = \Theta(n^{\log_b a} \lg n)$$

$$n^2 \neq \Theta(n^{1/2} \lg n)$$

Case 2 does not apply.

Applying Case 3 :

$$f(n) = \Omega(n^{\log_b a + \epsilon})$$

$$n^2 = \Omega(n^{1/2 + \epsilon})$$ Let $\epsilon = 1.5$

$$n^2 = \Omega(n^2)$$

Now,

$$a \cdot f(n/b) \leq c \cdot f(n)$$

$$2 \cdot \frac{n^2}{b} \leq c \cdot n^2$$

$$\cancel{2} \cdot \frac{n^2}{\cancel{4}2} \leq c \cdot n^2$$

$$\frac{n^2}{2} \leq c \cdot n^2$$

Let $c = 1/2$

$$\frac{n^2}{2} \leq \frac{n^2}{2}$$

Case 3 applies.

Therefore, $T(n) = \Theta(f(n))$

$$\boxed{T(n) = \Theta(n^2)}$$