ASSIGNMENT - 1

Q1. Let $f(n)$ and $g(n)$ be asymptotically non-negative functions. Using the basic defination of Θ-notation, prove that

$$\max(f(n), g(n)) = \Theta(f(n)+g(n))$$

A1. To prove the above equation, we need to prove that

$$\max(f(n), g(n)) = \begin{cases} O(f(n)+g(n)) \text{ and} \\ \Omega(f(n)+g(n)) \end{cases}$$

Let, $\max(f(n), g(n)) = O(f(n)+g(n))$ be eq^n ①
and $\max(f(n), g(n)) = \Omega(f(n)+g(n))$ be eq^n ②

<u>Proving eq^n ①</u>

If $\max(f(n), g(n))$ is $f(n)$ then,

$$\max(f(n), g(n)) \geq g(n) \quad — ③$$

and if $\max(f(n), g(n))$ is $g(n)$ then,

$$\max(f(n), g(n)) \geq f(n) \quad — ④$$

<u>Adding eq^n ③ and ④</u>

$$2\max(f(n), g(n)) \geq f(n) + g(n)$$

<u>Moving 2 to RHS</u>

$$\max(f(n), g(n)) \geq \frac{1}{2}(f(n)+g(n))$$

$\because \frac{1}{2} > 0$, let $1/2$ be a constant 'c'

$$\max(f(n), g(n)) \geq c\,(f(n) + g(n)) \quad \text{— (5)}$$

Let, $\max(f(n), g(n)) = K(n)$
and $f(n) + g(n) = h(n)$
~~Hence~~ Now, eq^n (5) becomes

$$K(n) \geq c\,h(n) \geq 0$$

So, $$K(n) = \Omega\, h(n)$$

Therefore, $\max(f(n), g(n)) = \Omega(f(n) + g(n))$ — (6)

Proving eq^n (2)

$$f(n) \leq f(n) + g(n)$$
$$g(n) \leq f(n) + g(n)$$

So, $$\max(f(n), g(n)) \leq (f(n) + g(n))$$

Multiply RHS by 1

$$\max(f(n), g(n)) \leq 1\,(f(n) + g(n)) \quad \text{— (7)}$$

Let, $\max(f(n), g(n)) = K(n)$
and, $(f(n) + g(n)) = h(n)$
Now, eq^n (7) becomes

$$K(n) \leq 1\,h(n)$$

Let 1 be the constant c

$$K(n) \leq c\,h(n)$$
$$K(n) = c\,h(n)$$

Therefore, $\max(f(n), g(n)) = O(f(n) + g(n))$ — (8)

From eq^n ⑥ and ⑧,

$$\boxed{max(f(n) \cdot g(n)) = \Theta(f(n) + g(n))}$$

Hence Proved.

Q2. Rank the following functions in terms of asymtotic growth. In other words find an arrangement of the function f1, f2, f3 ..... such that, for all i, $fi = \Omega(fi+1)$.

A2. Let us consider the value of $n = 20$ to solve the order of functions and log as natural log n (ln)....

$f1(n) = \sqrt{n} \log n$

$f1(20) = \sqrt{20} \log 20$ — ④

$= \sqrt{20} \ln 20$

$= 7.7404$

$f2(n) = \log(\log n^2)$

$f2(20) = \ln(\ln 20^2)$ — ②

$= \ln(\ln 400)$

$= 1.7903$

$f_3(n) = 2^{(\log n)^2}$

$f_3(20) = 2^{(\ln 20)^2}$ — ⑥

$= 2^{(8.974)}$

$\neq \cancel{631557} = 491.14$

$f_4(n) = n!$ — ⑦

$f_4(20) = 20!$

$= 2432902008176640000$

$f_5(n) = n^{0.0001}$

$f_5(20) = 20^{0.0001}$ — ①

$= 1.00029$

$f_6(n) = 2^{(2\log n)}$

$f_6(20) = 2^{(2\ln 20)}$ — ⑤

$= 2^{(2\times 2.99)}$

$= 2^{(5.98)}$

$= 63.1188$

$f_7(n) = (\log n)!$ — ③

$f_7(20) = (\log 20)!$

$= 5.9679$

- Functions in ascending order of their growth rate

$f_5 < f_2 < f_7 < f_1 < f_6 < f_3 < f_4$