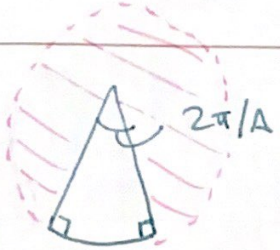
Cone point



↑ "unfolded cone"

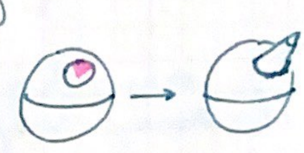
$$(2-3)\pi + \sum_{i=1}^{3} \nu_i = \int K\,dA$$

$$\underbrace{-\pi} + \underbrace{\left(\frac{\pi}{2}+\frac{\pi}{2}+\frac{2\pi}{A}\right)} =$$

$$\frac{2\pi}{A} = \int K\,dA = 2\pi\chi$$

$$\frac{1}{A} = \chi_{cone}$$

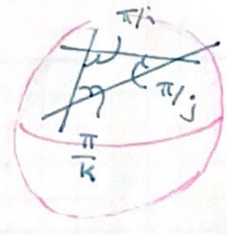
Before we "pushed out" the cone from the surface the apex angle was 2π $(A=1)$.
So the change is (we cut out the pink part)



how much we cut out $\rightarrow \frac{1}{1} - \frac{1}{A} = 1 - \frac{1}{A}$

Corner Point

Consider a polygonal face generated by intersecting mirrors.
$*ijk\ldots$ has a total of p characters



$$(2-p)\pi + \sum_{i=1}^{p} \nu_i$$

$$= 2\pi - \pi p - \left(-\sum_{i=1}^{p}\nu_i\right)$$

$$= 2\pi - \left[\sum_{i=1}^{p}(\pi - \nu_i)\right]$$

$$= 2\pi - \left(\pi - \frac{\pi}{i} + \pi - \frac{\pi}{j} + \pi - \frac{\pi}{k} + \cdots\right)$$

$$= \int K\,dA = 2\pi\chi$$

$$\chi_{face} = 1 - \left(\frac{i-1}{2i} + \frac{j-1}{2j} + \frac{k-1}{2k} + \cdots\right)$$

← this is what we want so the amount to "cut out" is $\chi_{sphere} - \chi_{face} = 2 - [1-(\,)] = 1 + \left(\frac{i-1}{2i}\right) + \ldots$

If we had only one boundary (*) w/ no additional characters we are excising a "circle" ie $p \to \infty$

$$\sum \nu_i = \pi p \quad \text{(geodesic)}$$

$$\int K\,dA = 2\pi\chi = 2\pi \rightarrow \chi = 1$$

Indeed, it is a standard result in topology

Handles = make 2 boundaries (holes) and glue them together

so $\chi_{handle} = 2(1) = 2$

Cross-caps are harder to see, but are basically like möbius strips (in fact, möbius band is $*\times$) and has only one boundary after gluing together so

$\chi_{cross} = 1$

Thus, if we start with a sphere ($\chi = 2$) and create surfaces by punching holes, etc. (different surgical operations)

$$\chi = \underset{\uparrow \atop \text{sphere}}{2} - \left\{ \begin{array}{ll} 2\alpha & (\alpha \text{ handles}) \\ + \beta & (\beta \text{ cross-caps}) \\ + \frac{A-1}{A} + \cdots & (\text{all cones}) \\ + 1 + \frac{i-1}{2i} + \cdots & \\ + 1 + \frac{m-1}{2m} + \cdots & \end{array} \right.$$

associated w/ * even if not intersecting

associated w/ characters after *

all kaleidoscopes

All values in { } are positive

$$o^{\alpha} \, ABC \cdots * ijk * mpq \cdots \times^{\beta}$$

if $\chi > 0$ $\longrightarrow$ isometries acting on sphere (S^2)
(point groups)

$\chi = 0$ $\longrightarrow$ isometries acting on Euclidean plane (E^2)
(wallpaper groups)

$\chi < 0$ $\longrightarrow$ isometries acting on Hyperbolic plane (H^2)

- Many $\chi > 0$ are possible but have some restrictions
(if AB then A = B, if $*ij$ then $i = j$)
- Only a few (17) have $\chi = 0$ $\longrightarrow$ these are wallpaper groups.

These are the cases where we "cut" out enough curvature so that what remains of the sphere is a <u>flat</u> surface

- Most have $\chi < 0$

$|\chi|$ = area of the asymmetric unit (fundamental domain) in its universal covering space

$\hookrightarrow$ for E^2 area is independent of $|\chi|$