DISCOVERY THROUGH GOSSIP

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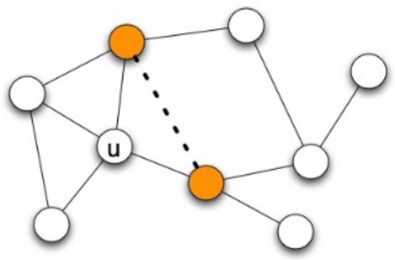
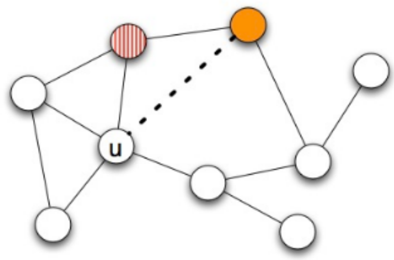


(a) Triangulation



(b) Two-hop walk

Figure 1: Discovery Methods

1 INTRODUCTION

We want to study Gossip based discovery processes in both directed and undirected graphs using push discovery (triangulation) and pull discovery (two-hop walk process).

We are interested in studying the time taken by process to converge to the transitive closure of the graph.

1.1 Notation

Table 1: Table of Notations

Notation	Description
δ_t	Minimum degree of G_t
$N_t^i(u)$	Set of nodes at distance i from u in G_t
$d_t(u)$	Degree of u in G_t
$d_t(u, S)$	Degree induced on S

1.2 Useful lemmas

***Lemma* 1.** $|\cup_{i=1}^{4} N_t^i(u)| \geqslant \min\{2\delta_t, n-1\}$ for all $u \in G_t$.

Proof. If $N_t^3 \neq \emptyset$, then $|\cup_{i=2}^{4} N_t^i(u)| \geqslant \delta_t$ and $|N_t^1(u)| \geqslant \delta_t$. So $|\cup_{i=1}^{4} N_t^i(u)| \geqslant 2\delta_t$ since the two sets are disjoint.
If $N_t^3 = \emptyset$, $N_t^1(u) \cup N_t^2(u) = n-1$ since G_t is connected. □

***Lemma* 2.** Consider k Bernoulli experiments in which the success probability of the ith experiment is at least i/m where $m \geqslant k$. If X_i denotes the number of trials needed for experiment i to output a success and $X = \sum_{i=1}^{k} X_i$, then $\Pr[X > (c+1)m \ln m] < \frac{1}{m^c}$.

Proof. w.l.o.g assume that $k = m$. The problem can be seen as *coupon collector problem* where X_{m+i-1} is the number of steps to collect ith coupon. Consider the probability of not obtaining the ith coupon after $(c+1)m \ln m$ steps, we have: $(1 - \frac{1}{m})^{(c+1)m \ln m} < e^{-(c+1)\ln m} = \frac{1}{m^{c+1}}$ By union bound, the probability that some coupon has not been collected after $(c+1)m \ln m$ steps is less than $\frac{1}{m^c}$. □

2 TRIANGULATION RESULTS

2.1 Upper Bound

Theorem 3 (Upper bound for triangulation process)**.** *For any connected undirected graph, the triangulation process converges to a complete graph in* $O(n \log^2 n)$ *rounds with high probability.*

In order to prove Theorem 3, we prove that the minimum degree of the graph increases by a constant factor (or equals to $n-1$) in $O(n \log n)$ steps. We say that a node v is **weakly tied** to a set of nodes S if $d_t(v, S) < \delta_0/2$, and **strongly tied** to a set of nodes S if $d_t(v, S) \geqslant \delta_0/2$.

***Lemma** 4.* If $d_t(u) < \min\{n-1, (1+\frac{1}{4}\delta_0)\}$ and $w \in N_t^1(u)$ has at least $\frac{\delta_0}{4}$ edges to $N_t^2(u)$, then the probability that u connects to a node in $N_t^2(u)$ through w in round t is at least $\frac{1}{6n}$.

Proof. The probability that u connects to a node in $N_t^2(u)$ through w in round t is:

$$\frac{d_t(w, N_t^2(u))}{D_t(w)} \times \frac{1}{d_t(w)} \geqslant \frac{d_t(w, N_t^2(u))}{D_t(w)} \times \frac{1}{n} \geqslant \frac{d_t(w, N_t^2(u))}{|N_t^1(u)| + d_t(w, N_t^2(u))} \times \frac{1}{n} \geqslant$$
$$\frac{d_t(w, N_t^2(u))}{(1+\frac{1}{4})\delta_0 + d_t(w, N_t^2(u))} \times \frac{1}{n} \geqslant \frac{\frac{\delta_0}{4}}{(1+\frac{1}{4})\delta_0 + \frac{\delta_0}{4}} \times \frac{1}{n} = \frac{1}{6n}$$

□

***Lemma** 5.* If $d_t(u) < \min\{n-1, (1+\frac{1}{4}\delta_0)\}$ and $w \in N_t^1(u)$ is weakly tied to $N_t^2(u)$, and $v \in N_0^2(u) \cap N_0^1(w)$, then u connects to v through w in round t with probability at least $\frac{1}{4\delta_0^2}$.

Proof. Since w is weakly tied to $N_t^2(u)$ and $d_t(w)$ is at most $|N_t^1(u)| + d_t(w, N_t^2(u))$, we obtain that $d_t(w)$ is at most $(1+\frac{1}{4})\delta_0 + \frac{\delta_0}{2}$. Therefore, the probability that u connects to v through w in round t equals:

$$\frac{1}{d_t(w)^2} \geqslant \frac{1}{((1+\frac{1}{4})\delta_0 + \frac{\delta_0}{2})^2} \geqslant \frac{1}{\frac{7\delta_0}{4}^2} \geqslant \frac{1}{4\delta_0^2}.$$

□

To analyze the growth in the degree of a node u, we consider two overlapping cases. The first case is when more than $\delta_0/4$ nodes of $N_t^1(u)$ are strongly tied to $N_t^2(u)$, and the second is when less than $\delta_0/3$ nodes of $N_t^1(u)$ are strongly tied to $N_t^2(u)$.

***Lemma** 6* (Several nodes are strongly tied to two-hop neighbors)*.* There exists a $T = O(n \log n)$ such that if more than $\frac{\delta_0}{4}$ nodes in $N_t^1(u)$ each have at least $\frac{\delta_0}{4}$ edges to $N_t^2(u)$ for all $t < T$, then with probability at least $1 - \frac{1}{n^2}$ it holds that $d_T(u) \geqslant \min\{n-1, (1+\frac{1}{4})\delta_0\}$.

Proof. We assume that $d_t(u) < \min\{n-1, (1+\frac{1}{4})\delta_0\}$ for all $t < T$. (Otherwise there is nothing to prove). Let $w \in N_t^1(u)$ be a node that has at least $\frac{\delta_0}{4}$ edges to $N_t^2(u)$. By Lemma 4 we know that:

$$\Pr[u \text{ connects to a node in } N_t^2(u) \text{ through } w \text{ in round } t] \geqslant \frac{1}{6n}$$

There are more than $\frac{\delta_0}{4}$ such w's in $N_t^1(u)$, each of which independently executes a triangulation step in any given round. Consider $T = \frac{72 n \ln n}{\delta_0}$. We have $18 n \ln n$ chances to add an edge between u and a node in $N_t^2(u)$. Thus,

$$\Pr[u \text{ connects to a node in } N_t^2(u) \text{ after } T \text{ rounds}] \geqslant 1 - (1 - \tfrac{1}{6n})^{18 n \ln n}$$
$$\geqslant 1 - e^{-3 \ln n} = 1 - \frac{1}{n^3}.$$

If a node that is two hops away from u becomes a neighbor of u by round t, it is no longer in $N_t^2(u)$. Therefore, in $T = T_1 \frac{\delta_0}{4} = O(n \log n)$ rounds, u will connect to at least $\frac{\delta_0}{4}$ new nodes with probability at least $1 - \frac{1}{n^2}$ and $d_T(u) \geqslant (1 + \frac{1}{4})\delta_0$ □

Lemma 7 (Few neighbors are strongly tied to two-hop neighbors). There exists $T = O(n \log n)$ such that if less than $\frac{\delta_0}{3}$ nodes in $N_t^1(u)$ are strongly tied to $N_t^2(u)$ for all $t < T$, and there exists a node $v_0 \in N_0^1(u)$ that is strongly tied to $N_0^2(u)$, then $d_T(u) \geqslant \min\{n-1, (1+\frac{1}{8})\delta_0\}$ with probability at least $1 - \frac{1}{n^2}$.

Proof. Let $T, T_1, T_2 = O(n \log n)$. We assume $d_t(u) < \min\{n-1, (1+\frac{1}{8})\delta_0\}$ for all $t < T$. Let S_t^0 denote the set of v_0's neighbors in $N_t^2(u)$ which are strongly tied to $N_t^1(u)$ at round t and W_t^0 denote the set of v_0's neighbors in $N_t^2(u)$ which are weakly tied to $N_t^1(u)$ at round t.
Consider $v \in S_t^0$. Less than $\frac{\delta_0}{3}$ nodes in $N_t^1(u)$ are strongly tied to $N_t^2(u)$ (hype), thus more than $\frac{\delta_0}{2} - \frac{\delta_0}{3} = \frac{\delta_0}{6}$ neighbors of v in $N_t^1(u)$ are weakly tied to $N_t^2(u)$. Let w be such node. By Lemma 5, the probability that u connects to v through w in round t is at least $\frac{1}{4\delta_0^2}$. We have at least $\frac{\delta_0}{6}$ choices for w, each of which executes a triangulation step each round. Consider $T_1 = 72\delta_0 \ln n$ rounds of the process. Then the probability that u connects to v in T_1 rounds is at least:

$$1 - (1 - \frac{1}{4\delta_0^2})^{12\delta_0^2 \ln n} \geqslant 1 - e^{-3\ln n} = 1 - \frac{1}{n^3}.$$

Thus if there exists $t < T_2$ such that $|S_t^0| \geqslant \frac{\delta_0}{8}$, then in $T_1 + T_2$ rounds, $d_{T_1+T_2}(u) \geqslant (1+\frac{1}{8})\delta_0$ with probability at least $1 - \frac{1}{n^2}$. This implies the claim of the lemma if we set $T = T_1 + T_2$.
Therefore, let $S_t^0 < \frac{\delta_0}{8}$ for all $t \leqslant T_2$. Define $R_t^0 = R_{t-1}^0 \cup W_t^0, R_0^0 = W_0^0$. If at least $\frac{\delta_0}{8}$ nodes in R_t^0 are connected to u at any time $t \leqslant T_2$, then the claim of the lemma holds. Consider $|R_t^0 \cap N_t^1(u)| < \delta_0/8$ for all $t \leqslant T_2$. Consider any round $t \leqslant T_2$. From the definition of R_t^0, we have:

$$|R_t^0| \geqslant |W_t^0| = d_t(v_0, N_t^2(u)) - |S_t^0| \geqslant d_t(v_0, N_t^2(u)) - \frac{\delta_0}{8}.$$

At round 0, v_0 is strongly tied to $N_0^2(u)$, i.e., $d_0(v_0, N_0^2(u)) \geqslant \frac{\delta_0}{2}$. Since $\delta_0 \leqslant d_t(u) < (1+1/8)\delta_0$, we have:

$$d_t(v_0, N_t^2(u)) \geqslant d_t(v_0, N_0^2(u)) - \delta_0/8 \geqslant 3\delta_0/8.$$

let e_1 denote the event [u connect to a node in $R_t^0 \backslash N_t^1(u)$ through v_0 in round t].

$$\Pr[e_1] = \frac{|R_t^0 \backslash N_t^1(u)|}{d_t(v_0)} \times \frac{1}{d_t(v_0)} = \frac{|R_t^0| - |R_t^0 \cap C_t^1(u)|}{d_t(v_0)} \times \frac{1}{d_t(v_0)} \geqslant \frac{|R_t^0| - |R_t^0 \cap C_t^1(u)|}{d_t(v_0)|} \times \frac{1}{n} \geqslant \frac{|R_t^0| - |R_t^0 \cap C_t^1(u)|}{|N_t^1(u)| + d_t(v_0, N_t^2(u))} \times \frac{1}{n} \geqslant \frac{|R_t^0| - \delta_0/8}{|N_t^1(u)| + d_t(v_0, N_t^2(u))} \times \frac{1}{n} \geqslant \frac{d_t(v_0, N_t^2(u)) - \delta_0/8 - \delta_0/8}{|N_t^1(u)| + d_t(v_0, N_t^2(u))} \times \frac{1}{n} \geqslant \frac{3\delta_0/8 - \delta_0/8}{|N_t^1(u)| + 3\delta_0/8} \times \frac{1}{n} \geqslant \frac{3\delta_0/8 - \delta_0/8}{(1+1/8)\delta_0 + 3\delta_0/8} \times \frac{1}{n} = \frac{1}{12n}.$$

Let X_1 be the number of rounds it takes for e_1 to occur and let v_1 denote a witness for e_1. Since v_1 is in $R_{X_1}^0$, it is also in $W_{t_1}^0$ for some $t_1 \leqslant X_1$. Therefore, v_1 is weakly tied to $N_{t_1}^1(u)$ and strongly tied to $N_{t_1}^2(u) \cup N_{t_1}^3(u)$ at the start of round t_1. Thus, at the start of round t_1, v_1 has at least $\delta_0/2$ neighbors that are not neighbors of u. If $d_t(v_1, N_t^2(u)) < 3\delta_0/8$ for any round t $X_1 \leqslant t \leqslant T_2$, then at least $\delta_0/2 - 3\delta_0/8 = \delta_0/8$ neighbors of v_1 that were not neighbors of u at the start of round t_1 became neighbors of u by the start of round t. This would imply that $d_t(u) \geqslant (1+1/8)\delta_0$ which violates the assumption at the start of the proof. So consider the case where $d_t(v_1, N_t^2(u)) \geqslant 3\delta_0/8$ for all $X_1 \leqslant t \leqslant T_2$. Let $S_t^1(W_t^1)$ denote the set of v_1's neighbors in $N_t^2(u)$ that are strongly(weakly) tied to $N_t^1(u)$. If $|S_t^1| \geqslant \delta_0/8$ for any $t \leqslant T_2$, then as for the case $|S_t^0| \geqslant \delta_0/8$, in at most $T_1 + T_2$ rounds, the degree of u is at least $(1+1/8)\delta_0$ with probability at least $1 - \frac{1}{n^2}$.
So assume that $|S_t^1| < \delta_0/8$ for all $t \leqslant T_2$. Consider a round t such that $X_1 \leqslant t \leqslant T_2$. Define $R_t^1 = R_{t-1}^1 \cup W_t^1, R_{t_1}^1 = W_{t_1}^1$. Let e_2 denote the event [u connect to a node

in $R_t^0 \backslash N_t^1(u)$ or $R_t^1 \backslash N_t^1(u)$through v_0 or v_1 in round t]. By the same calculation as for v_0, we have $\Pr[e_2] \geqslant 2/12n$, as long as $T_2 \geqslant X_1 + X_2$. Similarly, we define $e_3, \ldots, e_{\lceil \delta_0/8 \rceil}$ and $X_3, \ldots, X_{\lceil \delta_0/8 \rceil}$, and obtain that $\Pr[i_2] \geqslant i/12n$ for $1 \leqslant i \leqslant \lceil \delta_0/8 \rceil$, as long as $T_2 \geqslant \sum_{j=1}^{i} X_j$. The total number of rounds for u to gain at least $\delta_0/8$ nodes as neighbors is given by $\sum_{i=1}^{\lceil \delta_0/8 \rceil} X_i$, which, according to Lemma 2, is bounded by $36n \ln n$ with probability at least $1 - \frac{1}{n^2}$. So setting $T_2 = 36n \ln n$ and $T = T_1 + T_2$ completes the proof. □

***Lemma* 8** (All neighbors are weakly tied to two-hop neighbors). There exists a $T = O(n \log n)$ such that if all nodes in $N_t^1(u)$ are weakly tied to $N_t^2(u)$ for all $t < T$, then with probability at least $1 - \frac{1}{n^2}$ it holds that $d_T(u) \geqslant \min\{(1 + 1/8)\delta_0, n - 1\}$.

Proof. Assume $d_t(u) < \min\{(1 + 1/8)\delta_0, n - 1\}$ for all $t < T$. We first show that any node $v \in N_0^2(u)$ will have at least $\delta_0/4$ edges to $N_{T_1}^1(u)$, where $T_1 = O(n \log n)$. After that, v will connect to u in $T_2 = O(n \log n)$ rounds. Therefore, the total number of rounds for v to connect to u is $T_3 = T_1 + T_2 = O(n \log n)$.
Node v at least connects to one node in $N_0^1(u)$. Call it w_1. Because all nodes in $N_t^1(u)$ are weakly tied to $N_t^2(u)$, we ave $d_t(w_1, N_t^1(u)) \geqslant \delta_0 - \delta_0/2 = \delta_0/2$. If $d_t(w_1, N_t^1(u)) \backslash N_t^1(v) < \delta_0/4$, then v already has $\delta_0/4$ edges to $N_t^1(u)$. So consider the case where $d_t(w_1, N_t^1(u) \backslash N_t^1(v)) \geqslant \delta_0/4$. Consider the event $e_1 =$ [v connects to a node in $N_t^1(u) \backslash N_t^1(v)$ through w_1] and obtain for its probability:

$$\Pr[e_1] = \frac{d_t(w_1, N_t^1(u) \backslash N_t^1(v))}{d_t(w_1)} \times \frac{1}{d_t(w_1)} \geqslant \frac{d_t(w_1, N_t^1(u) \backslash N_t^1(v))}{|N_t^1(u)| + d_t(w_1, N_t^2(u))} \times \frac{1}{d_t(w_1)} \geqslant \frac{\delta_0/4}{(1+1/8)\delta_0 + \delta_0/2} \times \frac{1}{d_t(w_1)} \geqslant \frac{2}{13} \times \frac{1}{n} > \frac{1}{7n}.$$

Let X_1 be the number of rounds needed for e_1 to happen and w_2 be the witness for it. By our choice, w_2 is also weakly tied to $N_t^2(u)$. With the same argument as above, we have $d_t(w_2, N_t^1(u) \backslash N_t^1(v)) \geqslant \delta_0/4$. Let e_2 denote the event [v connects to a node in $N_t^1(u)$ through w_1 or w_2]. We have $\Pr[e_2] \geqslant 2/7n$. Let X_2 be the number of rounds needed for e_2 to occur. Similarly, define $e_3, X_3, \ldots, e_{\delta_0/4}, X_{\delta_0/4}$ and show that $\Pr[e_i] \geqslant i/7n$. Let $T_1 = \sum_i X_i$, which is the bound on the number of rounds needed for v to have at least $\delta_0/4$ neighbors in $N_t^1(u)$. By Lemma 5, the probability that u connects to v through w_i in round t is at least $\frac{1}{4\delta_0^2}$. There are $\delta_0/4$ such w_i's independently running a triangulation step each round. Consider $T_2 = 48\delta_0 \ln n$ rounds of the process, Then:

$$\Pr[u \text{ connects to } v \text{ in } T_2 \text{ rounds}] \geqslant 1 - \left(1 - \frac{1}{4\delta_0^2}\right)^{12\delta_0^2 \ln n} \geqslant 1 - \frac{1}{n^3}.$$

So each $v \in N_0^2(u)$ will connect to u in round $T_3 = T_1 + T_2$ with probability at least $1 - \frac{1}{n^3}$. So in round T_3, u will connect to all nodes in $N_0^2(u)$ with probability at least $1 - \frac{|N_0^2(u)|}{n^3}$. Then, $N_0^2(u) \subseteq N_{T_3}^1(u), N_0^3(u) \subseteq N_{T_3}^1(u) \cup N_{T_3}^2(u), N_0^4(u) \subseteq N_{T_3}^1(u) \cup N_{T_3}^2(u) \cup N_{T_3}^3(u)$. We apply the above analysis twice, and obtain that in round $T = 3T_3 = O(n \log n), N_0^2(u) \cup N_0^3(u) \cup N_0^4(u) \subseteq N_T^1(u)$ with probability at least $1 - \frac{1}{n^2}$. By Lemma 1, $|N_0^2(u) \cup N_0^3(u) \cup N_0^4(u)| \geqslant \min\{2\delta_0, n - 1\}$. □

Now we can prove Theorem 3.

Proof of Theorem 3. We show that in $O(n \log n)$ rounds, either the graph becomes complete or its minimum degree increases by a factor of at least 1/8. Then we apply this argument $O(\log n)$ times to complete the proof.
For each u where $d_0(u) < \min\{(1 + 1/8)\delta_0, n - 1\}$, we consider two cases. Let S be the set of nodes in $N_0^1(u)$ that are strongly tied to $N_0^2(u)$. The first case is when $|S| > \delta_0/3$. By Lemma 6, there exists $T = O(n \log n)$ such that if at least $\delta_0/4$ nodes in $N_t^1(u)$ have each at least $\delta_0/4$ edges to $N_t^2(u)$ for all $t < T$, then $d_T(u) \geqslant (1 + 1/8)\delta_0$ with probability at least $1 - \frac{1}{n^2}$. So if every node is S has at least $\delta_0/4$ edges to $N_t^2(u)$ for all $t < T$, the desired claim of lemma holds. On the

other hand, if there is a node in S that has fewer than $\delta_0/4$ edges to $N_t^2(u)$ for some $t < T$, then u has gained at least $\delta_0/2 - \delta_0/4 = \delta_0/4$ new neighbors, giving the desired claim.
If $S < \delta_0/3$, we have two cases. If we remain in this case for $T = O(n \log n)$ rounds and fine a node $v_0 \in N_0^1(u)$ that is strongly tied to $N_T^2(u)$, then by Lemma 7, we have $d_T(u) \geqslant \min\{(1 + 1/8)\delta_0, n - 1\}$ with probability at least $1 - \frac{1}{n^2}$. Otherwise, by Lemma 8, $T = O(n \log n)$ exists such that if we remain in this case for T rounds, then $d_T(u) \geqslant \min\{(1 + 1/8)\delta_0, n - 1\}$ with probability at least $1 - \frac{1}{n^2}$. Using the union bound, we obtain $\delta_T \geqslant \min\{(1 + 1/8)\delta_0, n - 1\}$ in $T = O(n \log n)$ rounds with probability at least $1 - 1/n$. Apply the above argument $O(\log n)$ times to obtain the desired upper bound. □

2.2 Lower Bound

Theorem 9 (Lower bound for triangulation process)**.** *For any connected undirected graph G that has $k \geqslant 1$ edges less than the complete graph, the triangulation process takes $\Omega(n \log k)$ steps to complete with probability at least $1 - O(e^{-k^{1/4}})$.*

Proof. During the triangulation process, there is a time t when the number of missing edges is at least $m = \Omega(\sqrt{k})$ and the minimum degree is at least $n/3$. If $K < 2n/3$, then this is true initially and for larger k, this is true at the first time t the minimum degree is large enough. The second case follows since the degree of a node can at most double in each step guaranteeing that the minimum degree is not larger than $2n/3$ at time t also implying that at least $n/3 = \Omega(k)$ edges are still missing.
Given the bound on the minimum degree, any missing edge $\{u, v\}$ is added by a fixed node w with probability at most $\frac{18}{n^2}$. Since there are $n - 2$ such nodes, the probability that a missing edges get added is at most $18/n$. Let X_i be the random variable counting the number of steps needed until the ith of m missing edges is added. We would like to analyze $\Pr[X_1 \leqslant T, \ldots, X_m \leqslant T]$ for an appropriate T. We have $\Pr[X_1 \leqslant T] \leqslant 1 - (1 - 18/n)^T \leqslant 1 - \frac{1}{\sqrt{m}}$ for $T = \Omega(n \log m)$. Thus:

$$\Pr[X_1 \leqslant T, \ldots, X_m \leqslant T] = \Pr[X_1 \leqslant T | X_2 \leqslant T, \ldots, X_m \leqslant T] \times \ldots \times \Pr[X_m \leqslant T] \leqslant (1 - \frac{1}{\sqrt{m}})^m = O(e^{-\sqrt{m}}) = O(e^{-k^{1/4}}).$$

This shows that the triangulation process takes with probability at least $1 - O(e^{-k^{1/4}})$ at least $\Omega(n \log m) = \Omega(n \log k)$ steps to complete. □

3 THE TWO-HOP WALK RESULTS

3.1 Upper Bound

***Lemma* 10** (When two-hop neighborhood is not too large). If $N_t^2(u) < \delta_0/3$, there exists $T = O(n \log n)$ such that with probability at least $1 - 1/n^2$, we have $|N_T^2(u)| \geqslant \delta_0/3$ or $d_T(u) \geqslant \min\{2\delta_0, n - 1\}$.

Proof. Omitted. □

***Lemma* 11** (When two-hop neighborhood is not too small). If $N_t^2(u) \geqslant \delta_0/4$, there exists $T = O(n \log n)$ such that with probability at least $1 - 1/n^2$, we have $|N_T^2(u)| < \delta_0/4$ or $d_T(u) \geqslant \min\{(1 + 1/8)\delta_0, n - 1\}$.

Proof. Omitted. □

Theorem 12 (Upper bound for two-hop walk process)**.** *For connected undirected graphs, the two-hop walk process completes in $O(n \log^2 n)$ rounds with high probability.*

Proof. In time $T = O(n \log n)$, the minimum degree of the graph increases by a factor of $1/8$. For each u that $d_0(u) < \min\{(1+1/8)\delta_0, n-1\}$, we analyze by the two following cases. First, if $N_0^2(u) \geqslant \delta_0/2$, by Lemma 11 we know as long as $|N_t^2(u)| \geqslant \delta_0/4$ for all $t \geqslant 0$, $d_T(u) \geqslant \min\{(1+1/8)\delta_0, n-1\}$ with probability $1 - 1/n^2$ where $T = O(n \log n)$. If the condition is not satisfied, we know at least $\delta_0/4$ nodes in $N_0^2(u)$ have been moved to $N_T^1(u)$, which means that $d_T(u) \geqslant \min\{(1+1/8)\delta_0, n-1\}$.
Second, If $N_0^2(u) < \delta_0/2$, by Lemma 10 we know as long as $|N_t^2(u)| < \delta_0/2$ for all $t \geqslant 0$, $d_T(u) \geqslant \min\{(1+1/8)\delta_0, n-1\}$ with probability $1 - 1/n^2$ where $T = O(n \log n)$. If the condition is not satisfied, we are back to analysis in the first case. So with probability $1 - 1/n$ the minimum degree of G will become at least $\min\{(1+1/8)\delta_0, n-1\}$. Now we apply the argument $O(\log n)$ times to show that the two-hop process completes in $O(n \log^2 n)$ steps with high probability. □

3.2 Lower Bound

Theorem 13 (Lower bound for two-hop walk process)**.** *For any connected undirected graph G that has $k \geqslant 1$ edges less than the complete graph, the two-hop process takes $\Omega(n \log k)$ steps to complete with probability at least $1 - O(e^{-k^{1/4}})$.*

Proof. Omitted. Same as the proof of Theorem 9. □

REFERENCES