MPRI 2.18.1 (2019/20): Distributed algorithms for networks, 2nd part

Lecture 3: Rumor Spreading in Realistic Networks

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<u>Outline:</u>

Homeworks:

Hypercubes = nice networks

Random geometric graphs = wireless sensor network (next lecture)

Preferential attachment graphs = social networks (next lecture)

Contents of This and Next Lecture

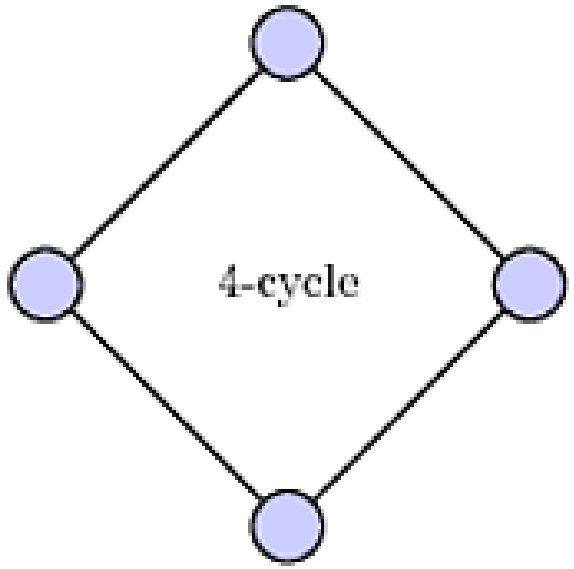
- 3 important networks classes
  - hypercubes: regular, logarithmic degree, logarithmic diameter, many paths between any two vertices
  - random geometric graphs: model for radio networks, connectivity threshold, giant component threshold
  - preferential attachment graphs: model for real networks, low diameter, power-law degree distribution

- Rumor spreading in such networks
  - hypercubes: RRS takes logarithmic time (=diameter)
    - cool proof: backward spreading of non-information
  - random geometric graphs: RRS takes time approx. diameter. Argument: Take a sufficiently sparse grid, then RRS is roughly like in the grid graph
  - preferential attachment: push takes long, push-pull takes logarithmic time, push-pull without double-calls takes approx. diameter

Homework 2.1: Rumor Spreading in the 4-Cycle

- Let G be the graph that forms a cycle of 4 vertices. Assume that you run the randomized rumor spreading protocol in this graph: the rumor starts in an arbitrary node and then in each round, each node calls a random neighbor (not including itself). Compute precisely the expected time it takes until all vertices are informed.

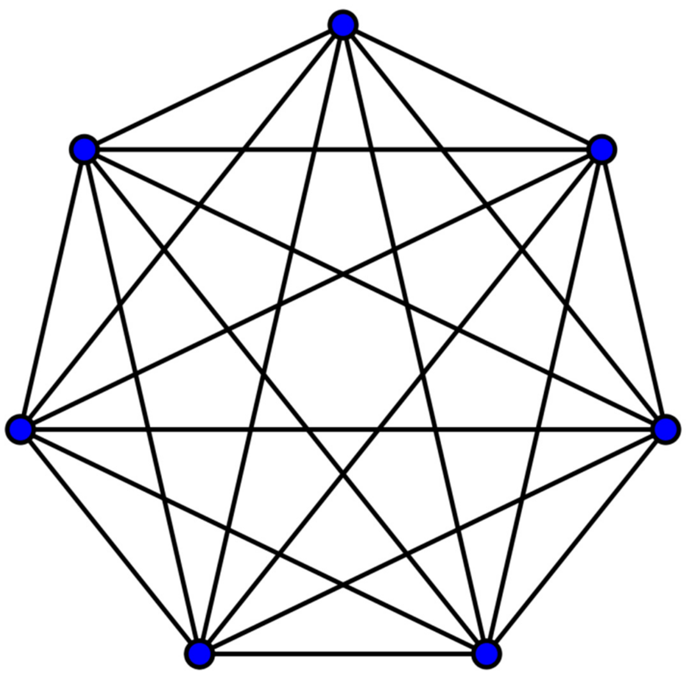
- Solution "Markov chain thinking": Let T_i be the expected time to inform all nodes when starting with i informed nodes (forming a connected piece on the cycle)
  - $T_4 = 0$
  - $T_3 = 1 + \frac{1}{4}T_3 + \frac{3}{4}T_4$ "after one round, w.p. $\frac{1}{4}$ you still have 3 informed vertices and w.p. $\frac{3}{4}$ you have 4 informed vertices" → $T_3 = \frac{4}{3}$.
  - $T_2 = 1 + \frac{1}{4}T_2 + \frac{1}{2}T_3 + \frac{1}{4}T_4 = \frac{20}{9}$.
  - $T_1 = 1 + T_2 = \frac{29}{9}$.



Homework 2.2: Path Lemmas Can Overestimate Runtimes

- Give an example showing the path lemmas can greatly over-estimate the time it takes to inform a vertex. Why?

- Example: Complete graph on n vertices.
  - Rumor spreading time: $\Theta(\log n)$
  - Path lemmas:
    - Path_lenght $\times$ degree $= 1 \times (n-1)$

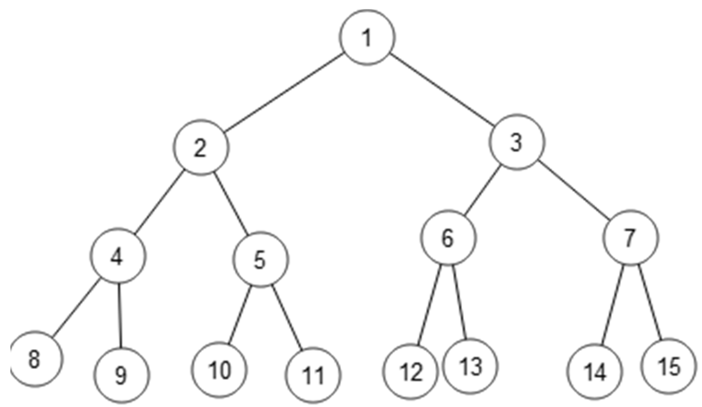


- Reason: There are many paths from the source to a give target, hence the time to traverse a particular path is much higher than the time to traverse any path.

Homework 2.4: Lower Bound Tree

- Prove a lower bound of $\Omega(hk \log k) = \Omega(k \log n)$ for the rumor spreading time in a k-regular tree of height h.

- **Proof 1 (adaptive path argument):** Assume the rumor starts in the root x_0.
- Given that some x_i is defined, let x_{i+1} be the child of x_i which is called last by x_i. This defines inductively a path from the root to a leaf.
- By a coupon collector argument, the expected time it takes until a node has called all its children (and hence the last child) is $\Omega(k \log k)$.
- Hence the expected time for the rumor to reach the end of the path is $\Omega(hk \log k)$.



Homework 2.4: Lower Bound Tree (2)

$$\left(1-\frac{1}{k+1}\right)^k \geq \frac{1}{e}$$

- **Proof 2 (uninformed leaves):** Assume that all nodes except the leaves are informed.
- The probability that a particular leaf is uninformed after $T = 0.2\, k \ln n$ rounds is $\left(1-\frac{1}{k+1}\right)^T \geq \exp(-0.2 \ln n) = n^{-0.2}$, hence the expected number of uninformed nodes is at least $\frac{1}{2} n^{0.8}$ – note that at least half the nodes are leaves.

simple adhoc arguments work as well

- We now use the method of bounded differences to argue that with high probability, not all nodes are informed after T rounds:
  - Let X be the vector of random decisions in these T rounds. Then X consists of at most $nT = \Theta(n \log n)$ independent random decisions.
  - Denote by $f(X)$ the number of uninformed leaves after T rounds. Note that changing a single random decision in X changes $f(X)$ by at most 1.
  - Hence $\Pr\left[f(X) \leq E[f(x)] - \frac{1}{4} n^{0.8}\right] \leq \exp\left(2\left(\frac{1}{4} n^{0.8}\right)^2 \left(\Theta(n \log n)\right)^{-1}\right)$

Homework 2.5: Rumor Spreading in Arbitrary Graphs

- Theorem: For any connected graph $G = (V, E)$, a rumor starting in an arbitrary vertex s with high probability reaches all vertices within $O(n \log n)$ rounds.

- Definition: Consider a path $P: x_0, x_1, \dots, x_k$ in G. Assume that x_0 is informed at the start of round T_1. We say that the *rumor traverses the path* P in the time interval $[T_1, T_2]$ if there are $T_1 \leq t_0 < t_1 < \cdots < t_{k-1} \leq T_2$ such that for all $i \in [0..k-1]$ vertex x_i calls x_{i+1} in round t_i.
  - Note: This implies that all nodes on the path are informed after round T_2

- Degree-Lemma: If $\boldsymbol{P}$ is a shortest path, then $\sum_{i=0}^{k-1} \mathbf{deg}\, \boldsymbol{x_i} \leq \mathbf{3}\boldsymbol{n}$
- Proof: Let z be a vertex not on P. Let $N_P(z)$ be the set of neighbors of z that lie on P. Then $N_P(z) \subseteq \{x_{i-1}, x_i, x_{i+1}\}$ for some $i \in [1..k-1]$ – otherwise there would be a shorter path from x to y.
  - Hence $\sum_{i=0}^{k-1} \deg x_i = \sum_z |N_P(z)| + 2(k-1) \leq 3(n-k) + 2k \leq 3n$.

Homework 2.5: Rumor Spreading in Arbitrary Graphs (2)

- Path lemma & degree lemma: If P is a shortest path, then the first T_2 such that the rumor traverses P in the interval $[T_1, T_2]$ satisfies $E[T_2] \leq T_1 + 3n - 1$
- Markov's inequality: The probability that the rumor does not traverse the shortest path P in the interval $[T_1, T_1 + 6n - 1]$, is at most $1/2$.
- Restart argument: the probability that the rumor does not traverse the shortest path in $[T_1, T_1 + 6kn - 1]$, is at most 2^{-k}
  - proof: consider the intervals $[T_1 + 6(i-1)n, T_1 + 6in - 1]$, $i \in [1..k]$
  - for each interval, the probability that the rumor does not traverse the path in this interval, is at most $1/2$
  - by the Markov property of rumor spreading, the probability is at most 2^{-k} that all these k intervals fail to let the rumor traverse the path
  - hence 2^{-k} is an upper bound for the probability that the rumor does not traverse the path in $[T_1, T_1 + 6kn - 1]$

Homework 2.5: Rumor Spreading in Arbitrary Graphs (3)

- Let $v \in V \setminus \{s\}$. Then the probability that v is not informed after $12\, n \log_2(n)$ rounds, is at most $1/n^2$
  - Proof: choose a shortest path from s to v.The probability that the rumor does not this path in the time interval $[0,\ 12\, n \log_2(n) - 1]$ is at most $1/n^2$.

- Union bound: With probability at most $1/n$, there is a vertex that is not informed after $12\, n \log_2(n)$ rounds.

Homework 2.5: Rumor Spreading in Arbitrary Graphs (4)

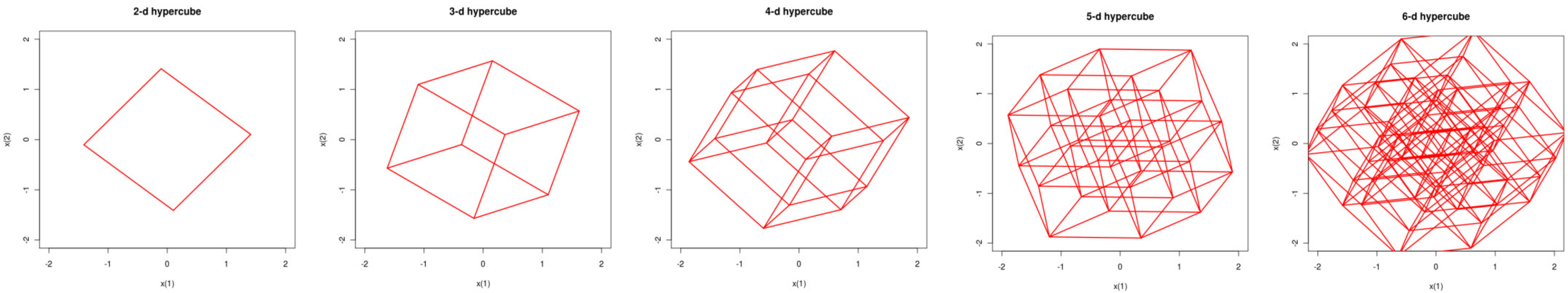
- <u>Alternative proof:</u>
- The probability that a node x has not called its neighbor y within $T_x = 3\deg(x)\ln(n)$ rounds after becoming informed is $(1 - 1/\deg(x))^T \leq \exp(-3\ln(n)) = n^{-3}$.
- Union bound: The probability that x has not called all neighbors within that time is n^{-2}.
- Union bound: The probability that there is such an x is n^{-1}.
- Hence with probability at least $1 - n^{-1}$, all nodes x call all their neighbors within T_x rounds after becoming informed.
- Assuming this, the rumor traverses any path in time at most $3\ln(n)$ times its degree sum. Hence the claim follows from the fact that all shortest paths have a degree sum of at most $3n$

Today's (and Next Week's) Topics

- Rumor spreading in "relevant" networks:
  - nice man-made communication networks = hypercube,
  - wireless sensor networks = random geometric graphs,
  - social networks = preferential attachment graphs

Part 1: Rumor Spreading in Hypercubes H_d

- **Definition:** The d-dimensional *hypercube* is a graph H_d having
  - $V = \{0,1\}^d$ as vertex set (hence $n := |V| = 2^d$), and
  - two vertices are adjacent if they differ in exactly one position.



- Maximum degree: $\Delta(G) = d = \log n$
- Distances in H_d: $d(u, v) =$ "number of positions u and v differ in".
- Diameter (max. distance between vertices): $\mathrm{diam}(G) = d = \log n$,

- Good communication network: Small diameter, relatively few edges, high connectivity (d disjoint paths between any two vertices)

Rumor Spreading in Hypercubes

- The degree-diameter bound gives a rumor spreading time of
$$O(\Delta \max\{\mathrm{diam}(H_d), \log n\}) = O(\log^2 n)$$

- Might be overly pessimistic, because there are many path between any pair of vertices:
  - $d!$ different shortest paths between $(0, \dots, 0)$ and $(1, \dots, 1)$
  - so there might be one path where we are much more lucky than what the expectation tell us.

- **Theorem:** With probability $1 - 1/n$, a rumor started in an arbitrary node of the hypercube has reached all nodes after $O(\log n)$ rounds.
  - beautiful proof (next couple of slides)
  - major open problem to determine the leading constant

Proof: Preparations

- We assume that the rumor starts in $s = (0, \dots, 0)$. [symmetry]

- We show that for any $\beta > 0$ there is a $K > 0$ such that after $K \log n$ rounds, the vertex $t = (1, \dots, 1)$ is informed with probability $1 - n^{-\beta}$
  - similar arguments work for any target t
  - a union bound shows that all vertices are informed w.p. $1 - n^{-\beta+1}$

- Two technical assumptions that do not change how the rumor spreads, but help in the proof
  - all-work assumption: We assume that in each round *every* node calls a random neighbor – if the caller is not informed, nothing happens
  - everything-predefined assumption: We assume that before the process starts, each node has already fixed whom to call in which round

Expansion Phase

- Observation: The rumor quickly moves away from $s = (0, \dots, 0)$, but it is increasingly difficult to argue that the rumor truly approaches the target.

- Plan: Show that you get at least close to the target!
  - for reasons that will become clear later, we show that we get close to any target we want.

- Expansion Lemma: Let $\alpha > 0$. Let $v \in V$. Let $C \geq 2$. After Cd/α rounds, with probability at least $1 - \exp(-\ Cd/8)$ there is an informed vertex w such that $d(v, w) \leq \alpha d$.
  - "in $\Theta(d)$ rounds the rumor reaches v apart from at most the last αd steps (and apart from an $O(n^{-\Omega(1)})$ failure probability"

Proof: Expansion Lemma

- Similar to the analysis how rumors traverse a path.
- Let d_t denote the distance of v to the closest informed vertex after round t.
- Define binary random variables X_t (counting true/artificial progress) as follows
  - if $d_{t-1} > \alpha d$, then $X_t = 1$ if and only if $d_{t-1} > d_t$
  - if $d_{t-1} \leq \alpha d$, then $X_t = 1$ with probability α (independent of everything)
- $\Pr[X_t = 1] \geq \alpha$ for all t
- Note: $X^T := \sum_{t=1}^{T} X_t \geq d(s,v) - \alpha d$ if and only if $d_T \leq \alpha d$ (our aim)
- The X_t are not independent, but we have $\Pr[X_t = 1 | X_1 = x_1, \dots, X_{t-1} = x_{t-1}] \geq \alpha$ for all $x_1, \dots, x_{t-1} \in \{0,1\}$. Hence X^T dominates a sum Y^T of T independent random variables that are 1 with probability exactly α (Lemma 1.18 in book chapter).
- For $T = Cd/\alpha$ we have
$$\Pr[X^T \leq d] \leq \Pr[Y^T \leq d] \leq \Pr\left[Y^T \leq \frac{1}{2} E[Y^T]\right] \leq \exp\left(-\frac{E[Y^T]}{8}\right) \leq \exp\left(-\frac{Cd}{8}\right)$$
by the multiplicative Chernoff bound.

Backward Phase

- Plan: Do something “dual”: starting in t and going backward in time, spread “uninformedness”

- Recall that we assumed that all nodes call in each round.

- Assume that our target node t is uninformed after some round T.
  - if some node x calls t in round T, then x must be uninformed after round $T-1$
  - iterate this argument to construct a path ending in t such that if the start of the path was informed at some time $T-i$ then t would be informed at time T

- Here we use the all-work and all-predetermined assumptions!

Backward Phase – Some Details

- Lemma: Let T be large. Let $\alpha > 0$. Let $v \in V$. Then with probability at least $1 - \exp(-Cd/8)$ there is a $w \in V$ such that $d(w, v) \leq \alpha d$ and if w is informed after round $T - Cd/(1 - \exp(-\alpha))$, then t is informed after round T.

- Proof:
  - For $i = 0,1,2 \ldots$ let d_i be the smallest $d(v, x)$ of a node x having the property that if x is informed at the end of round $T - i$, then t is informed after round T.
    - $d_0 = d(v, t) \leq d$
    - if $d_i > \alpha d$, then $\Pr[d_{i+1} = d_i - 1] \geq 1 - (1 - 1/d)^{\alpha d} \geq 1 - \exp(-\alpha)$
  - Use an analogous "artificial progress counting" argument as before
    - $X_i = 1$ if $d_i < d_{i-1}$ and $d_{i-1} > \alpha d$, otherwise independent random bit that is 1 with prob. $1 - \exp(-\alpha)$
    - …

Coupling Phase

- So far: For any $v \in V$, with probability $1 - \exp(\Omega(d))$ [as large as we want]
  - there is an $A_v \in V$ such that $d(v, A_v) \leq \alpha d$ and s informs A_v within $O(d)$ rounds
  - there is a $B_v \in V$ such that $d(v, B_v) \leq \alpha d$ and “the rumor would go from B_v to t in $O(d)$ rounds”

- Remains to do: Get the rumor from A_v to B_v!
  - Problem: Very hard to get the rumor exactly somewhere (we need already d rounds to call a particular neighbor)
  - Solution: Take many v as above, sufficiently far apart, and play this game many times in parallel – once we will be lucky ☺

Quickly From A_v to B_v With Small (but Large Enough) Probability

- Let $B := B(v, 2\alpha d) := \{u \in V \mid d(u, v) \leq 2\alpha d\}$ "$2\alpha d$ ball around v"
- Target: Get the rumor from A_v to B_v, but only using nodes in B
  - needed later to ensure that processes for different v don't interact

- Lemma: The probability that the rumor moves inside B from A_v to B_v in time at most $2\alpha d$, is at least $(2\alpha/e)^{2\alpha d}$.

- Proof: Send the rumor along a direct path with speed one! ($d' := d(A_v, B_v)$)
  - Probability that the rumor moves closer to B_v in every round:

$$\prod_{i=1}^{d'} \frac{i}{d} \geq \prod_{i=1}^{2\alpha d} \frac{i}{d} = \frac{(2\alpha d)!}{d^{2\alpha d}} \geq \frac{(2\alpha d/e)^{2\alpha d}}{d^{2\alpha d}} = \left(\frac{2\alpha}{e}\right)^{2\alpha d}$$

  - Small exercise: Any such path remains in B

Many v's that are far apart

- Target: Find a large set of v's such that the distance of any two is more than $4\alpha d$ – so the $2\alpha d$-balls around them do not intersect.

- **Lemma**: There is a set $S \subseteq V$ such that $|S| = \exp(d/32) =: m$ and for all $x, y \in S$ with $x \neq y$ we have $d(x, y) \geq d/4$.

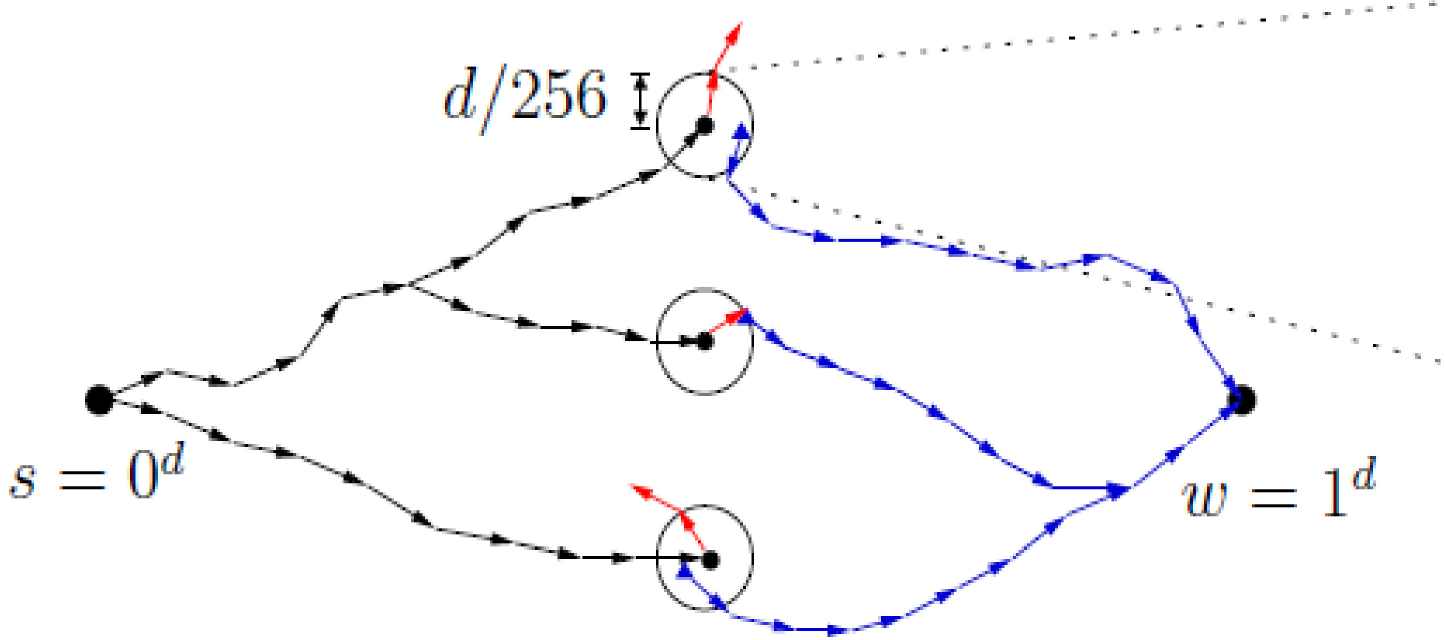
- Proof: Take a random set!
  - Let $x_1, \dots, x_m$ be random vertices.
  - For $i \neq j$, we have $E\left[d\left(x_i, x_j\right)\right] = d/2$
  - $d\left(x_i, x_j\right)$ is a sum of d independent $\{0,1\}$ random variables
  - Chernoff bound: $p := \Pr\left[d\left(x_i, x_j\right) \leq d/4\right] \leq \exp(-d/16)$
  - Union bound: $\Pr[S \text{ bad}] \leq \sum_{i,j} \Pr\left[d\left(x_i, x_j\right) \leq d/4\right] < m^2 p \leq 1$
    - Consequently, there is such a set S

Proof: Putting Everything Together

- Choose α small enough so that $(2\alpha/e)^{2\alpha} > \exp(-1/32)$ and $\alpha d < d/4$
  - note that $(2\alpha/e)^{2\alpha}$ tends to one for $\alpha \to 0$
- Choose the set S as on the previous slide.
- Apply expansion lemma with C large enough and union bound to show that with probability $1 - n^{-\beta}$ for all $v \in S$ there is an $A_v \in B(v, \alpha d)$ that is informed after $T_1 = O(d)$ rounds
- Apply backward lemma with C large enough and union bound to show that with probability $1 - n^{-\beta}$ for all $v \in S$ there is a $B_v \in B(v, \alpha d)$ such that if B_v is informed after $T_2 := T_1 + 2\alpha d$ rounds, then t is informed after $T_2 + O(d)$ rounds
- Coupling phase: The probability that for no $v \in S$ the rumor goes (inside $B(v, 2\alpha d)$ from A_v to B_v is at most $\left(1 - (2\alpha/e)^{2\alpha d}\right)^{\exp(d/32)} \leq \exp\left(-\left((2\alpha/e)^{2\alpha} \exp(1/32)\right)^d\right) = \exp\left(-n^{\Theta(1)}\right)$
- Hence apart from a failure prob. of $2n^{-\beta} + \exp(-n^{\Theta(1)})$, t is informed….

Summary Part 1

- Result: A rumor spreads in a hypercube in time $O(\log n)$ w.p. $1 - 1/n$
- Proof in a picture (but with different constants, so ignore these)



Homework

- Carefully study the analysis of randomized rumor spreading in hypercubes. Then answer the following two questions:

- Homework 3.1: Consider the following rumor spreading process in the hypercube. The rumor starts in $s = (0, \dots, 0)$ and our only target is to get the rumor to the opposite node $t = (1, \dots, 1)$.

- We run the classic randomized rumor spreading protocol with the following modification: In each round, each informed node selects a random neighbor. *If this neighbor is closer to* t *than itself*, then the node actually calls this neighbor. If the selected neighbor is further away from t, then the node does nothing in this round.

- How long does this process need to get t informed? Why?

Homework 3.2

- Consider the classic randomized rumor spreading process in the hypercube started at vertex $s = (0, \dots, 0)$ as regarded in the lecture. Together with the insight from the previous problem, we have shown that with probability at least $1 - n^{-\Omega(1)}$ the rumor traverses a shortest path from s to $t = (1, \dots, 1)$ in $O(d)$ time.

- Discuss whether the same property is true for all target vertices t.