MPRI 2.18.1 (2019/20), Homework 2.3

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The aim of this homework is to give a guided tour through the proof of the degree-diameter bound, including showing where the problems are. Most questions can be solved independently, so if you cannot solve one, continue with the next.

Problem 2.3: Let $G = (V, E)$ be an undirected graph with n vertices, maximum degree at most Δ (that is, all vertices have at most Δ neighbors) and diameter at most D (that is, the distance between any two vertices is at most D). We shall show that randomized rumor spreading started in an arbitrary node s informs the whole graph in $O(\Delta \max\{D, \ln n\})$ rounds with high probability. Let $s = x_0, x_1, \ldots, x_\ell$ be a path (of length ℓ) in G. For $i \in [\ell] := \{1, \ldots, \ell\}$, let T_i denote the round in which x_i becomes informed (set $T_0 := 0$).

(a) Give an upper bound for $E[T_1]$ (a bound which is sharp in some graphs).
(b) Denote by $X_i := T_i - T_{i-1}$ the time between x_{i-1} and x_i becoming informed. Give an upper bound for $E[X_i]$.
(c) Are the X_i independent?

Since the previous approach was not good enough, let us be more clever. Define inductively random variables $\bar{T}_i$ as follows. Let $\bar{T}_0 = 0$. For $i > 0$, let $\bar{T}_i$ be the first round after $\bar{T}_{i-1}$ in which x_{i-1} calls x_i.

(d) Show that $T_i \le \bar{T}_i$ for all i.
(e) Let $\bar{X}_i := \bar{T}_i - \bar{T}_{i-1}$. Observe that $T_\ell \le \sum_{i=1}^{\ell} \bar{X}_i$.
(f) What can you say about the value of $E[\bar{X}_i]$?
(g) Are the $\bar{X}_i$ independent?
(h) Why is it difficult to apply Chernoff bounds to $\bar{T}_\ell$?

We overcome this difficulty with another simple trick. We define random variables Y_t, for $t \in \mathbb{N}$, as follows. For each $t \in [\bar{T}_\ell]$, let Y_t be the indicator random variable for the event that there is a $k \in [\ell]$ such that $t = \bar{T}_k$ ("true progress in round t"). For each $t > \bar{T}_\ell$, let Y_t be an independent binary random variable that is one with probability $1/\Delta$ ("fake progress in round t").

(i) Observe that for all $k \in [\ell]$ and $T \in \mathbb{N}$, we have $\bar{T}_k \le T$ if and only if $\sum_{t=1}^{T} Y_t \ge k$.
(j) Show that $E[Y_t] \ge 1/\Delta$ for all $t \in \mathbb{N}$.
(k) Are the Y_t are independent?
(l) Fix $\ell' \ge \ell$ arbitrarily. Use a Chernoff bound to show that for $T = 2\Delta\ell'$ and $Y := \sum_{t=1}^{T} Y_t$, $\Pr[Y < \ell] \le e^{-\ell'/4}$.
(m) Show that for $T = 2\Delta \max\{\ell, 8 \ln n\}$ and Y defined as above in (l), $\Pr[Y < \ell] \le 1/n^2$.
(n) What does the latter imply for the rumor spreading time?