Game-Based Redistricting for Small-Population States

Category: Mathematics and Computer Science

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Abstract

Congressional districts are often perceived as being biased towards specific parties or groups. Recently, due to major court cases the problem of gerrymandering has received much attention from mathematicians and computer scientists. One proposal suggests a game-based districting method where two players alternate creating districts, but it does not work for small-population states. This project proposes and evaluates an alternate game for two-district states like Maine, where two districts are created by players who alternate adding precincts. To be politically viable, such a game must:

- follow simple rules,
- create compact and continuous districts, and
- not admit a dominant strategy for either player.

The effectiveness of this game is analyzed using simulated state maps and players programmed to follow simple strategies. It is shown that the game creates compact and continuous districts, and does not admit a dominant strategy unless one party has a significant electoral advantage in the state.

Contents

1 Introduction

Congressional districts are often perceived as being biased towards specific parties or groups, decreasing people's trust in the United States' political system. This issue has been long present in the minds of politicians and, due to recent major court cases [7], has received an increasing amount of attention from mathematicians and computer scientists, many of whom have proposed their own methods for fairly creating congressional districts [1] [4].

One redistricting method was proposed by Wesley Pegden and collaborators [3]. It suggests a game-based approach for creating districts. In it, players alternate drawing and choosing districts. After one player draws a district map, the next player chooses, or freezes, one of those districts. That district then becomes a district in the final districting plan. The players then switch, and this process repeats until the whole state has been fully divided into districts.

This method is effective in larger states where there are many districts to be chosen. However, in smaller states such as Maine where there are only two districts, this approach is completely ineffective as one player has the power to determine both of the final districts.

This project proposes and evaluates an alternate game specifically designed for two-district states, where the two districts are created by players who alternate adding precincts. To be politically viable, such a game must: follow simple rules; create compact, that is regularly shaped, and continuous districts; and not admit a winning or dominant strategy for either player.

The effectiveness of this game is analyzed using simulated state maps and players programmed to follow simple strategies. This paper illustrates that such a game creates compact and continuous districts, and does not admit a winning or dominant strategy unless one party has a significant electoral advantage in the state.

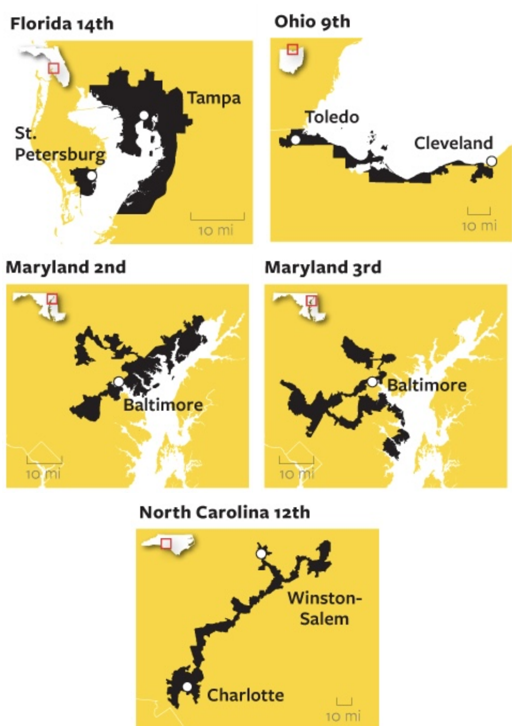


Figure 1. Examples of districts that are suspected to have been gerrymandered. Source: Peter Bell

2 Acknowledgements

This project was completed at Brunswick High School in Brunswick, Maine during the past year. The work for this project was done under the supervision of Susan Perkins of Brunswick High School, Brunswick Maine. This paper was inspired by a lecture given by Moon Duchin from Tufts University online covering gerrymandering. The idea of using a game-based redistricting method came from work by Wesley Pegden, Ariel D. Procaccia, and Dingli Yu. Thomas and Jennifer Pietraho assisted with the programming for this project, and editing this paper. Alden Walker from the Center for Communications Research gave useful feedback relating to this project.

3 A Game-Based Redistricting Method

Wesley Pegden of Carnegie Mellon University and collaborators proposed a potential solution to the problem of fair districting [3]. Their idea was a multi-player game called I-Cut-You-Freeze. Below is the description of a two-player version.

I-Cut-You-Freeze State Redistricting

In this game, there are two players who are drawing a redistricting plan for a state.

1. One player draws a full set of voting districts for the state.
2. The other player chooses one district from the other's plan to keep, and redraws the rest of the districting map.
3. The two players alternate drawing and freezing districts until the state has been fully divided.

Example. An example of the I-Cut-You-Freeze state redistricting method is shown in Figure 2.

The first image of Figure 2 shows the initial set of districts created by Player 1. The second image shows the district that Player 2 chose to keep

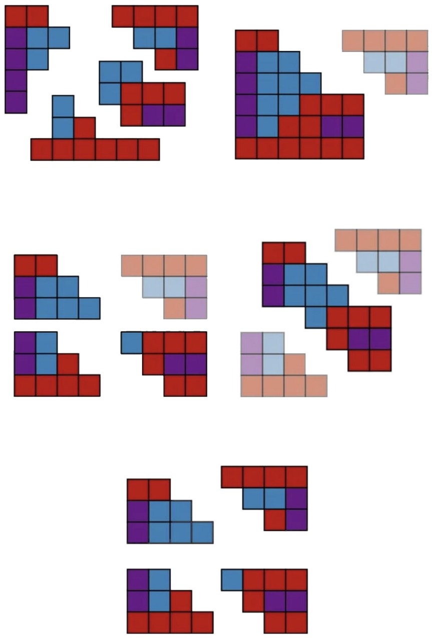


Figure 2. The process of redistricting a state using the I-Cut-You-Freeze method. The color of each square represents the political leaning of each precinct. Source: Slack.com

from that initial plan, and the third shows the new districts that Player 2 drew from the remaining area in the state. The fourth image shows the district selected by Player 1 to freeze, and in the fifth image the final redistricting plan is shown.

The I-Cut-You-Freeze method is effective for dividing larger states into many districts. However, for smaller states such as Maine where there are only two districts, this approach would be completely ineffective. When using this method to divide two-district states, after the first player draws both districts and the other player chooses one to freeze, the second district from the initial plan is also frozen, as the second player cannot redraw the single remaining district. This effectively gives one player the power to determine both of the final districts, making the game ineffective in redistricting small states with only two districts. For similar reasons, this game is also relatively ineffective for redistricting slightly larger states, perhaps with three or four districts.

4 Redistricting for Small States

This section proposes a game which adapts the I-Cut-You-Freeze method to two-district states. Section 4.1 describes the outline of the game. This paper evaluates the game for a simple model of precincts and states. Section 4.2 describes this model, and Section 4.3 defines the notions of connected and compact districts using this model.

4.1 How to Play the Game

Voting districts are made up of small units of population. This paper calls these units precincts. In this proposed game, there are two players, each tasked with selecting precincts that will form a voting district.

Game for Two-District States

1. The two players alternate adding precincts to their district until every precinct has been selected.
2. After every move, both districts must be continuous and, except for the first ten moves, above a specified compactness threshold.
3. If no precinct can be selected and keep the compactness above the specified threshold, a player must add the precinct that yields the highest compactness score.
4. After every move, all unchosen precincts must be able to be added to either district. That is, no unchosen precinct may be fully surrounded by a single district.

As desired, the rules are simple and easy to understand. Before this game can be played, definitions for connected and compact district must be established, and a threshold for compactness set. The first ten moves of the game disregard this compactness requirement, allowing the players to begin the game with some flexibility in terms of the shape of their district.

4.2 Creating Simulated States

To model the proposed game, this project creates simulated states, which allows for multiple examples to be tested, as simulated states can be generated an arbitrary number of times. It also allows for the game to be run under specific scenarios, as these models can be designed with different parameters.

Simple models are used for these simulated states, where all precincts are squares of equal size. Equivalently, these can be thought of as mathematical graphs [2], where each precinct is represented by a node and edges connect adjacent precincts (Figure 3).

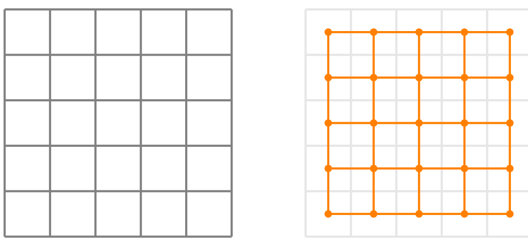


Figure 3. A model state and its precincts on the left, and its representation as a graph on the right. Adjacent precincts are connected by edges in the graph. Image by the author.

In this analysis, each simulated state is square and contains 900 precincts in total. This size was chosen as it was large enough to capture complex behaviors but not large enough to be overly computationally expensive. Two-district states, like Maine, also tend to have roughly this many towns.

The model also accounts for the distribution of voters in each precinct. States contain two types of voters: those who vote for the Purple Party, and those who vote for the Orange Party. All nodes have equal populations, and are assigned a number which represents the percentage of the population that votes with the Purple Party.

This percentage of Purple voters is randomly selected for each precinct according to a bell curve with a standard deviation of 10% and centered between 50% and 55% Purple voters depending on the simulation. This standard deviation was selected based on data from real states. Wisconsin's distribution can be seen in Figure 4 [5].

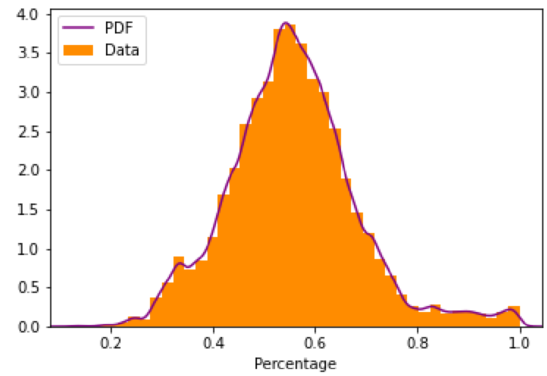


Figure 4. The voter distribution in Wisconsin. Image and probability density function computed by the author.

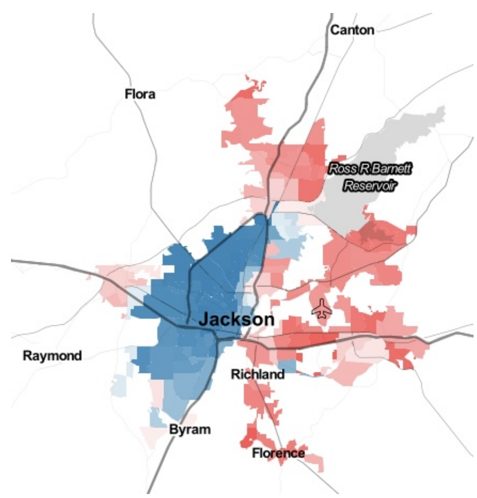


Figure 5. The clumping of voters by political party in Jackson, Mississippi. Source: FiveThirtyEight.

The model makes an additional adjustment to the distribution of voters throughout the state. In real life, voters with similar political views often settle near each other, forming clumps. This can be seen in Figure 5 using a map of Jackson, Mississippi. In order to simulate this phenomenon, this project uses the migration algorithm below.

Migration algorithm.

Repeat the following 10,000 times:

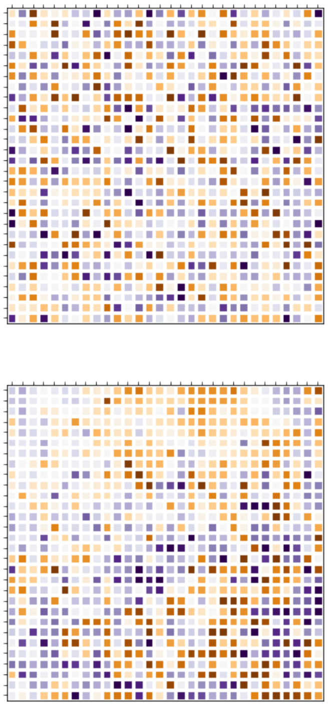


Figure 6. A 20×20 state with randomly selected voter distribution before and after migration. Darker purple nodes represent precincts with higher percentages of Purple voters. Image by the author.

1. Randomly select one node $\mathcal{A}$ and let a be the numerical value of the node associated with the percentage of Purple voters in the precinct it represents.

2. Average the values of each of the nodes connected to $\mathcal{A}$. Call that value b and compute $|a - b|$.

3. Randomly select one of the nodes connected to $\mathcal{A}$. Call this node $\mathcal{B}$.

4. Average the values of each of the nodes connected to $\mathcal{B}$. Call that value c and compute $|a - c|$.

5. If $|a - b| > |a - c|$, then switch the values of $\mathcal{A}$ and $\mathcal{B}$.

In this migration algorithm, a node will switch positions with an adjacent node if, by doing so, its numerical value will be closer to the value of its new neighbors than it was in its original neighbouring nodes. Repeating this process has the effect of clumping nodes with similar values. Figure 6 shows the outcome of this process. In the size of states considered, 10,000 repeats of this algorithm allows for precincts to clump together without fully segregating the two parties.

4.3 Requirements for a District

There are a few requirements that each voting district must satisfy. These requirements include being compact and continuous, as well as having equal populations. In the case of the graph-based districts used in this project, any district is continuous if any two of its nodes can be connected using edges in the district. Districts have equal populations if each district contains the same number of nodes. Compactness is more difficult to describe, so specifics for how it is measured in this project are explained in Section 4.3.1.

4.3.1 Compactness

One of the requirements for forming a district, both in this proposed districting game and in real life, is being sufficiently compact. This means that districts must cover an area that is regularly formed, rather than long and winding throughout a state.

A district's compactness is a very important aspect of its creation, as it can play into both the public acceptance of the district and its ability to accurately represent a state's population.

Gerrymandering is a practice that involves manipulating the boundaries of voting districts in order to skew the political leaning of the district towards one party or another, generally leaving the districts as a misrepresentation of the overall population of the state. By creating compact districts, the ability for someone to gerrymander a state is greatly reduced, as well as the public's perception of gerrymandering.

Computing the compactness of real districts can be done in a variety of ways. One method used by the MGGG Redistricting Lab, led by Moon

Duchin, computes compactness as a measure of the perimeter of a district over its area [6].

Definition 4.1. The compactness score of a district is defined as the ratio:

$$\mathcal{C} = \frac{\text{perimeter}}{\text{area}}.$$

Example. Pennsylvania's Seventh Congressional District (PA-07) in 2011 had an extensive perimeter, giving it a large compactness score. When the district was redrawn in 2018 due to suspected gerrymandering, its perimeter was shortened, giving it a lower compactness score and a more regular shape. In this formulation, low compactness scores are desirable.

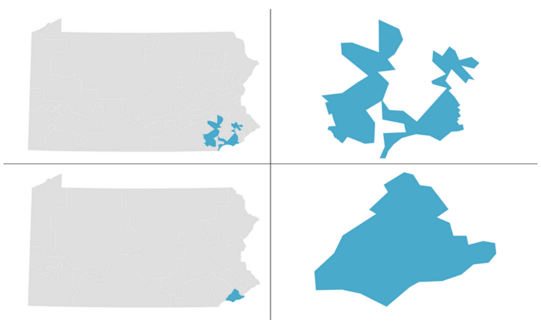


Figure 7. The top district is Pennsylvania's Seventh Congressional District (PA-07) in 2011, and the bottom is PA-07 in 2018. Source: Brennan Center for Justice

This project uses graphs to depict districts, so this formula must be adapted to districts made of edges and nodes.

Computing the perimeter of a district drawn as a graph can be done by counting the number of edges that lie on its perimeter, and its area can be represented by the number of nodes in the district. A first attempt at a compactness score for graphs is as follows:

Definition 4.2. The compactness score $\mathcal{C}$ for a graph-based district is:

$$\mathcal{C} = \frac{\#\ \text{perimeter edges}}{\#\ \text{nodes in district}}$$

Example. The perimeter of the district in Figure 8 is 16 edges, and its area is 21 nodes. This makes $\mathcal{C} = \frac{16}{21} \approx 0.761$

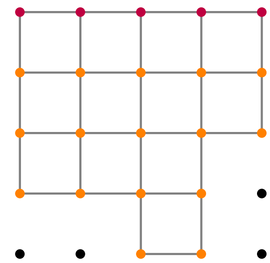


Figure 8. An example of a graph-based district. Image by the author.

This would be an ideal formula to apply to the simulated districts in this project. However, counting the number of perimeter edges in a graph-based district can be time-consuming for a computer in large, complex districts. To make this computation simpler, consider the following definition:

Definition 4.3. The compactness score $\mathcal{D}$ for a graph-based district is:

$$\mathcal{D} = \frac{2 \cdot \#\ \text{edges in district}}{\#\ \text{nodes in district}}$$

This computes the average number of edges connected to each node. Nodes in the center of a district will have four edges coming out of them, and nodes on the boundary of the district will have fewer. The closer $\mathcal{D}$ is to 4, the less perimeter the district has, and the more compact the district will be. One advantage of this statistic is that it is easy to compute.

Example. The compactness score $\mathcal{D}$ for the district in Figure 8 is:

$$\mathcal{D} = \frac{2 \cdot 32}{21} \approx 3.048$$

However, one additional modification to the definition of the compactness score must be made. Not every node in a state or district can be connected to four edges. Nodes on the boundary of the state will have a maximum of three edges to connect to other nodes. The score $\mathcal{D}$ relies on the total number of edges contained within a district, so these state edge nodes must be accounted for as not to lower the district's compactness score. A more accurate definition of a compactness score is as follows:

Definition 4.4. The compactness score $\mathcal{E}$ for a graph-based district is:

$$\mathcal{E} = \frac{2 \cdot \#\ \text{edges in district} + \#\ \text{edge nodes}}{\#\ \text{nodes in district}}$$

Example. For the district in Figure 8, assuming that the top edge of the district lies on the boundary of the state:

$$\mathcal{E} = \frac{2 \cdot 32 + 5}{21} \approx 3.286$$

This definition of compactness should also align with the intuitive notion of compactness: districts with higher compactness scores should look more regular. This is examined in the following example.

Example. Consider District A and District B from Figure 9. District A appears to be more compact than District B, as District B has a more irregular shape with several protrusions whereas District A is very rectangular. These qualitative observations can be compared with the compactness scores $\mathcal{E}$ for each district. For this example, assume that

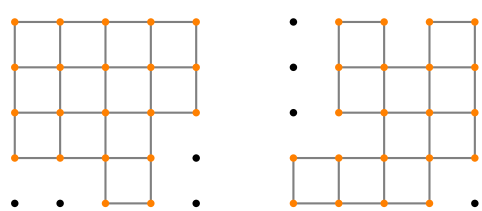


Figure 9. District A and District B. Image by the author

neither District A nor District B from Figure 9 lie on the edge of a state. For District A:

$$\mathcal{E} = \frac{2 \cdot 32 + 0}{21} = \frac{64}{21} \approx 3.048$$

And for District B:

$$\mathcal{E} = \frac{2 \cdot 30 + 0}{21} = \frac{60}{21} = 2.857$$

These compactness scores show that District A is more compact than District B.

After establishing the method for evaluating the compactness of a district, a threshold must be established for determining which districts are compact enough. This was accomplished by generating many simulated districts on a 30 by 30 node state, sorting them into two categories, compact-looking, and non compact-looking, and then looking at the compactness scores from the first category to determine a cutoff point. This cutoff point was determined to be a 3.7 compactness score. See Figure 10 for a comparison between different values of the compactness score and the resulting districts.

5 Game Strategies

In a two-district state, any redistricting plan will have the majority party win at least one of the districts, and it is possible for that party to win both if the districts are drawn in the right way. So for this proposed game, it will be said that the majority party *wins* the game if they win the majority of the vote in both districts and the minority part wins if they are able to secure the vote in at least one of the two districts.

Winning strategies are strategies that, if used by one of the players, can guarantee that no matter what strategy their opponent uses, they will always win.

Goal: The primary goal of this paper is to analyze the previously proposed game to determine if there are any simple winning strategies that can be found when the majority party represents 50% to 55% of the state.

Having a winning strategy for either party would render this game ineffective at redistricting states, especially when the difference in voter distribution is small. However, as the percentage gap between the majority party and the minority party grow, it becomes more likely that the majority party will win both districts regardless of the

(a) Compactness score is approximately 3.3.

(b) Compactness score is approximately 3.4.

(c) Compactness score is approximately 3.5.

(d) Compactness score is approximately 3.6.

(e) Compactness score is approximately 3.7.

(f) Compactness score is approximately 3.8.

(g) Compactness score is approximately 3.9.

Figure 10. The effect of the compactness threshold on district states. Images by the author.

strategies used.

5.1 Two Real-Life Strategies

Two strategies that are used in real life to gerrymander districts are *packing* and *cracking*. This paper will used them as models for strategies used to analyze the proposed game.

Packing is a strategy that involves taking as many people as possible from a specific group - whether that be based on race, political alignment, etc. - and packing them together into a single or small number of districts. This minimizes the voice of that group by only representing them in those district(s) rather than throughout the state. Usually this strategy is used with negative intentions. However, packing is also used purposefully to guarantee certain minorities a voice in government, though this use is beyond the scope of this project.

The second strategy is cracking, where a group of people is broken up throughout many districts. This minimizes their input in government, as they do not have a majority in any districts.

Example. In Figure 11, a single state is divided into three districts in two ways. The initial state, as shown on the far left, has 60% of its precincts with a majority Purple Party and 40% with an Orange Party majority. Given this distribution, it would be expected that the Purple Party wins

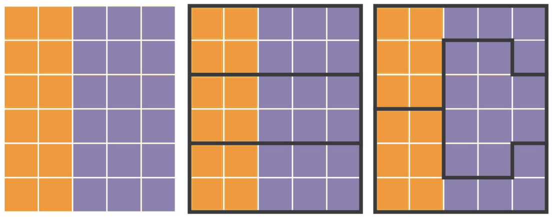


Figure 11. A district, shown on the far left, divided into three districts in two different ways. Source: Adapted from Steven Nass

majority in two of the districts, and the Orange Party wins one.

In the center image, a districting plan is shown where the Purple Party wins all three of the districts. These districts show an example of the cracking strategy, as the Orange-majority precincts have been split throughout the three districts so that they have no majority in any of them.

The rightmost image shows a different set of districts where the Orange Party wins two of the districts. This image shows an example of packing, as the purple party is shoved largely into a singular district, ensuring them the majority vote in that district but leaving the other two districts to be majority Orange Party.

5.2 Strategies Evaluated

This project evaluates three strategies for playing the proposed game. The Random Strategy acts as a baseline strategy, the Greedy Strategy adapts the real-life strategy of packing, and the Barely-Winning Strategy adapts the strategy of cracking.

Before describing these strategies in detail, the notion of a *valid node* must be defined.

It is an important problem to make sure that, in a two-player game like this one, each player always has a valid move throughout the game. To ensure that this is always true, this project uses an idea from graph theory.

Definition 5.1. If the graph G has the property that for any two vertices x and y, one can find a path from x to y, then we say that G is a connected graph. [2]

At all times in the game:

1. The precincts in the Player 1's district and the unclaimed precincts must create one connected graph, and

2. the precincts in the Player 2's district and all the unclaimed precincts must create one connected graph.

Definition 5.2. A valid node is an unchosen node that, when added to a district, that district will satisfy conditions 1 and 2 above.

These two conditions are important as they ensure that every unchosen node can be added to either district. That is, no unchosen precinct will be fully surrounded by a single district. If any precincts were fully surrounded by a singular district but remained unchosen until the very end, the game could end in a stalemate as the other district could not add anymore precincts to their district but would be required to in order to form evenly sized districts.

The first strategy is the Random Strategy, where valid nodes are added to the district randomly. This strategy acts as a baseline to compare the other strategies to.

Random Strategy.

On each move:

1. Randomly select one node from the set of all unchosen nodes. Call that node $\mathcal{A}$.

2. Add $\mathcal{A}$ to the district and check that the district meets all of the requirements specified in Section 4.3 and that $\mathcal{A}$ is a valid node.

3. If all of the requirements are met, keep $\mathcal{A}$ as part of the district. If not, remove it and try again until a node is found that will satisfy the requirements.

4. If adding any node will lower the compactness score of the district below the compactness threshold, add the node that will result in a district with the highest compactness score.

The second strategy is the Greedy Strategy, where each player is trying to pack as many voters from their party into their district.

Greedy Strategy for the Majority Party.

On each move:

1. Select a node $\mathcal{A}$ with the highest numerical value from the set of all unchosen nodes.
2. Add $\mathcal{A}$ to the district and check that the district meets all of the requirements specified in Section 3.3 and that $\mathcal{A}$ is a valid node.
3. If all of the requirements are met, keep $\mathcal{A}$ as part of the district. If not, remove it and try the node with the next highest value until a node is found that will satisfy the requirements.
4. If adding any node will lower the compactness score of the district below the compactness threshold, add the node that will result in a district with the highest compactness score.

The Greedy Strategy for the minority party is identical to the one for the majority party except that instead of choosing the highest value nodes to add to the district, the lowest value nodes are added.

In the final strategy, players try to keep the voter distribution of their district as close to evenly split as possible. For this strategy, another statistic must be known about a district. The *district score* $\mathcal{S}$ for a district is the distribution of voters throughout the whole district, that is, the percentages of voters who vote for the Purple Party. The district score is also used to collect data on the final districts created no matter the strategy used.

Definition 5.3. The district score $\mathcal{S}$ is defined as the ratio:

$$\mathcal{S} = \frac{\text{sum of all nodes}}{\text{total \# of nodes}}$$

There will also be a *cutoff score* $\mathcal{K}$, which determines the lowest desirable value of $\mathcal{S}$ for a district. In this project, $\mathcal{K} = 51$ for the majority party, and $\mathcal{K} = 49$ for the minority party.

Definition 5.4. The minimum value $\mathcal{X}$ of a node that can be added to a district and keep its district score $\mathcal{S}$ above the cutoff score $\mathcal{K}$ is defined as:

$$\mathcal{X} = \mathcal{K} \cdot (\text{\# of nodes} + 1) - \mathcal{S} \cdot (\text{\# of nodes})$$

By adding nodes to the district which have a numerical value close to $\mathcal{X}$ allows for the party to keep a majority in their district while maximizing the number of voters for their party which will be in the other districts. This Barely-Winning strategy acts almost as a version of the real-life cracking strategy, though applied to one's self rather than their opponent.

Barely-Winning Strategy for the Majority Party.

On each move:

1. Sort all of the unchosen nodes according to the difference between their value, a and $\mathcal{X}$. First list those where $a > \mathcal{X}$ in order from least to greatest, and then list those where the value of $a \leq \mathcal{X}$ in order from greatest to least. Select the first node, $\mathcal{A}$, on that list.
2. Add $\mathcal{A}$ to the district and check that the district meets all of the requirements specified in Section 3.3 and that $\mathcal{A}$ is a valid node.
3. If all of the requirements are met, keep $\mathcal{A}$ as part of the district. If not, remove it and try the next node on the list until a node is found that will satisfy the requirements.

4. If adding any node will lower the compactness score of the district below the compactness threshold, add the node that will result in a district with the highest compactness score.

The Barely-Winning Strategy for the minority party is identical to the one used for the majority party except for the first step. The unchosen nodes are instead sorted as follows: sort all of the unchosen nodes given the difference between their value, a and $\mathcal{X}$. First list those where $a < \mathcal{X}$ in order from greatest to least, and then list those where the value of $a \geq \mathcal{X}$ in order from least to greatest.

6 Simulations

One of the overarching goals of this project is to determine whether the proposed game admits a simple dominant strategy. This section examines this question by applying the three strategies detailed in Section 5 to the game and simulating the outcomes.

6.1 Simulation Process

This section details the process in which the full game is simulated and data is collected about the number of districts won by the majority party.

Full Simulation Process.

1. Choose a percentage to use as the mean of the voter-distribution bell curve and the strategies to be used by each player.

2. Using the process in Section 4.2, generate a state and its precincts.

3. Let the players play through the game until all precincts belong to a district.

4. Calculate the district scores $\mathcal{S}$ for both districts using Definition 5.3 and determine if the majority party or the minority party won.

In this project, every state that was generated was run through this full simulation process nine times, each to test a different combination of strategies against each other. In total, 60 states were generated, 10 for each bell curve used, to produce a total of 540 redistricting plans.

6.2 Districts Created

In this section, the districts produced through many simulations of the game are examined to:

- verify that the voting districts are below the compactness threshold and appear compact and connected;
- and examine whether different strategies form qualitatively different districts.

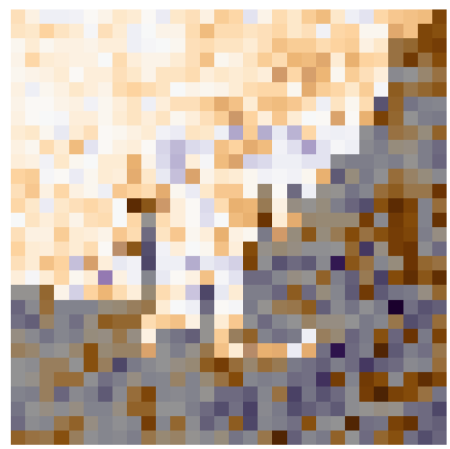


Figure 12. One example result of the full simulation process, run with a voter distribution of 51% Purple and pitting the Greedy Strategy against the Barely-Winning Strategy. Image by the author.

Example. In Figure 12, a state with a voter distribution of 51% Purple and 49% Orange was divided into two districts. The background graph provides a visual representation of the precincts in the state, and the color of each square represents the percentage of voters within the precinct that vote with the Purple Party (the darker purple the square is, the higher percentage of Purple voters, and the darker orange, the higher the percentage of Orange voters).

Player 1 was playing with the Greedy Strategy, trying to pack as many Purple voters into its district. That district is shown in Figure 12 as the shaded district. Player 2 was playing with the Barely-Winning Strategy, and created the other district.

It can be seen that Player 1 was able to capture many of the high percentage Purple precincts and Player 2 was able to keep their district with relatively split precincts, as indicated by the overall lighter and less saturated precincts. The lack of color represents this positioning. An interesting effect of Player 2's strategy is that, since they maintained a fairly evenly-split district, Player 1's district not only picked up the high value Purple precincts but also the high value Orange Precincts. This can been seen in the more saturated colors of the shaded districts in 12 and the more uniform coloration of the other district.

7 Conclusions

Throughout all of the simulations of the game, both players always had a valid move, allowing them to fully form their respective districts, and the districts created were looked compact as well as exceed the numerical compactness threshold, and were continuous. More nuanced aspects of the game are also investigated:

- Section 7.1 examines whether this proposed game admits any simple winning or dominant strategies.
- In Section 7.2, the effects of the compactness requirements on the fairness of the game are analysed.

7.1 No Dominant Strategies

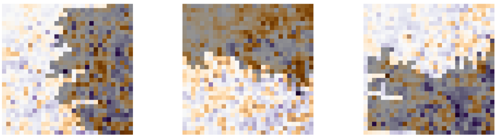
It is important that this game does not admit simple winning or dominant strategies. *Winning strategies* are strategies that can be used by a player to ensure that they win the game, gain majority in both districts for Purple and gain majority in one district for Orange, no matter how their opponent plays. *Dominant strategies* are strategies that give a player the best overall outcomes against any strategy used by their opponent, though they do not necessarily always win. Having a dominant strategy eliminates the need for a player to participate in the game, as the results of their district could be computed algorithmically. Winning strategies make playing the game almost entirely pointless, as one player will win no matter what the other player does.

From all of the simulations of this game, the number of districts won by each party can be used to determine if any of the three strategies tested are dominant or winning strategies. Table 1 show the average number of districts won per state given the strategies pitted against each other and the overall voter distribution of the state.

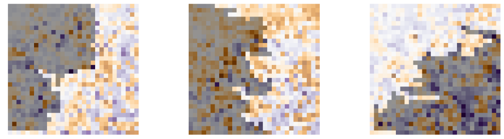
Example. Table 1a shows which the number of districts won by each party given the strategies used to create District A and District B. This data can be analyzed to determine which strategies would give each party the highest winning rate when matched with each strategy used by their opponent.

If Purple plays with the Random Strategy or the Greedy Strategy, than Orange wants to play with the Greedy Strategy to win the most districts, but if Purple plays with the Barely-Winning Strategy, than Orange would do best with either the Greedy or the Barely-Winning Strategy. On the other hand, if Orange plays with the Barely-Winning Strategy, than Purple would do best with the Greedy Strategy but otherwise would do best with the Barely-Winning Strategy.

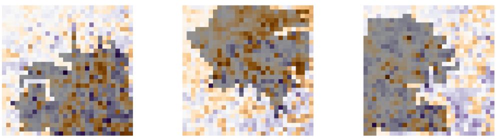
Since neither side had one strategy that always allowed them the most number of district won, it can be concluded that there is no dominant strategy for either player.



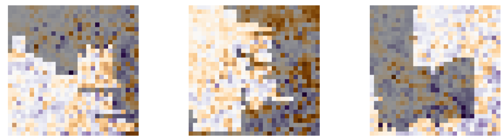
(a) Random vs Random.



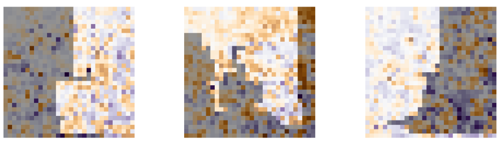
(b) Random vs Greedy.



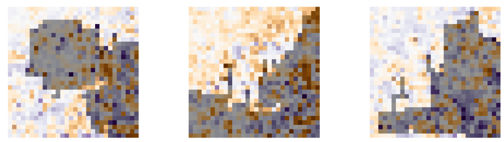
(c) Random vs Barely-Winning.



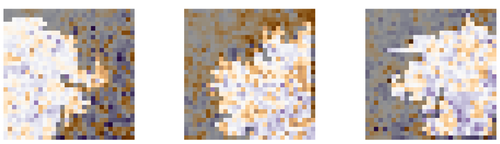
(d) Greedy vs Random.



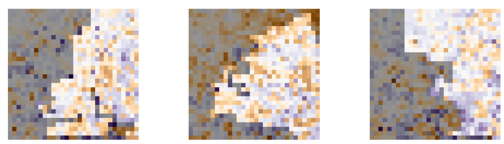
(e) Greedy vs Greedy.



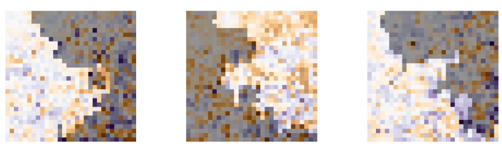
(f) Greedy vs Barely-Winning.



(g) Barely-Winning vs Random.



(h) Barely-Winning vs Greedy.



(i) Barely-Winning vs Barely-Winning.

Figure 13. These images present the different districts that are formed when different strategies are paired against each other in simulations of the game. In each triplet of images, the strategies pitted against each other remain the same but the voter distribution of the state varies. The leftmost state has a 50-50 distribution, the center state has a 51-49 distribution, and the rightmost has a 52-47 distribution. Also, all of the states with the same state voter distribution shown in this figure are the same, just divided into different districts depending on the strategies used by either player. The darker and lighter portions of the state represent the two different districts created during the game, with the darker districts representing the district formed according to the first strategy listed in the captions, and the lighter district corresponding to the second. Images by the author.

7.2 The Compactness Trade-off

As the simulations took place, an unexpected phenomena could be seen. Once the majority party controlled 53% of the state and the minority party only 47%, the minority Orange party could no longer win any districts.

50-50			
	Random	**Greedy**	**Barely**
Random	1.07, 0.93	1.10, 0.90	1.30, 0.70
Greedy	1.00, 1.00	1.00, 1.00	1.03, 0.97
Barely	1.07, 0.93	1.13, 0.86	1.03, 0.97

(a) Results for states with a 50-50 voter split.

51-49			
	Random	**Greedy**	**Barely**
Random	1.77, 0.23	1.73, 0.27	1.97, 0.03
Greedy	1.70, 0.30	1.30, 0.70	1.87, 0.13
Barely	1.70, 0.30	1.44, 0.56	1.77, 0.23

(b) Results for states with a 51-49 voter split.

52-48			
	Random	**Greedy**	**Barely**
Random	2.00, 0.00	2.00, 0.00	2.00, 0.00
Greedy	2.00, 0.00	1.86, 0.13	1.97, 0.03
Barely	2.00, 0.00	1.87, 0.13	2.00, 0.00

(c) Results for states with a 52-48 voter split.

53-47			
	Random	**Greedy**	**Barely**
Random	2.00, 0.00	1.90, 0.10	2.00, 0.00
Greedy	2.00, 0.00	2.00, 0.00	2.00, 0.00
Barely	2.00, 0.00	2.00, 0.00	2.00, 0.00

(d) Results for states with a 53-47 voter split.

Table 1. The average number of districts won by either party given the voter distribution and strategies used.

There will inevitably be a point with any initial distribution of voters where the minority party can no longer win majority in either district. When the voter distribution reaches 60-40, over two-thirds of the state will be in the majority party for this particular game, given its standard deviation of 10%, making it virtually impossible for the minority part to win either district. But though there is a certain cutoff point for when the minority party can win any districts, one would not expect it to be as low as a 53-47 split.

One possible explanation for such a low cutoff level could be the condition of compactness on each district, as the constant need to have a compact district could limit a player's ability to add the precincts they may really want.

After noticing this phenomena in the initial simulations of the game, a few more simulations were run on states with a 53-47 split. Some of these districts were required to remain above the compactness threshold, and some were given no compactness requirement. The final percentage of Purple voters for each district created in this experiment are shown in the tables below:

As can be seen in Table 2, Purple wins every district. However, once the compactness requirement was removed, the minority Orange party was able to win one district approximately half of the time, as shown in Table 3.

To determine the point at which the minority

53-47 with Compactness Requirement		
	District A	District B
Trial 1	54.74	51.10
Trial 2	53.82	51.34
Trial 3	55.23	51.10
Trial 4	53.66	51.84
Trial 5	55.56	50.98

Table 2. Percentage of Purple voters in each of the final districts for five simulations of the game. These simulations required each district to stay above the compactness threshold.

53-47 with No Compactness Requirement		
	District A	District B
Trial 1	56.28	50.31
Trial 2	55.96	49.95
Trial 3	54.60	50.28
Trial 4	57.64	49.38
Trial 5	56.74	49.78

Table 3. Percentage of Purple voters in each of the final districts for five simulations of the game. These simulations did not have a compactness requirement.

party could no longer win any districts when there was no compactness threshold requirement, a few more simulations were run. Tables 4 and 5 illustrate these runs and the final percentage of Purple voters in each district. From these two tables, it can be seen that the minority party is able to win districts until the voter distribution reaches 55-45 when these is no compactness threshold requirement.

This data shows that there is a trade-off between having compact, and thus nicer-looking, districts and fairness is representation of the voter distribution within those districts.

54-46 with No Compactness Requirement		
	District A	District B
Trial 1	57.62	51.00
Trial 2	58.40	50.15
Trial 3	58.76	49.48
Trial 4	57.54	49.98
Trial 5	58.41	50.53

Table 4. Percentage of Purple voters in each of the final districts for five simulations of the game. These simulations did not have a compactness requirement.

55-45 with No Compactness Requirement		
	District A	District B
Trial 1	58.80	50.57
Trial 2	59.34	51.58
Trial 3	58.42	51.04
Trial 4	57.53	53.22
Trial 5	58.48	51.12

Table 5. Percentage of Purple voters in each of the final districts for five simulations of the game. These simulations did not have a compactness requirement.

8 Future Work

This paper sets the groundwork for a redistricting game that can be applied for two-district states. There are a number of natural ways that this work can be expanded.

- This project only considers the proposed game in terms of simulated data and very abstract representations of states. This is done to limit the preconceived notions of good or bad that are associated with real-life states and or political parties. However, it is important to assess the political viability of this game, and so applying real state shapes and voter data is a natural next step to take with this project.

- In real-life, people do not vote for candidates from only two parties, and they do not always vote with or for the same party. This project vastly simplifies voters into voting for only two different parties and always sticking with the same party. Adding in these elements that were simplified in this project would make the simulations and analysis of the game more realistic and interesting.

- The strategies used to test the game in this project are simple, always sticking to the same criterion for adding precincts to their district and not planning ahead for future moves. Adding in more complex strategies could allow for further testing of the game and possibly some interesting data to analyze.

- Through the analysis of this project, the trade-off between the compactness of a district and its ability to represent the voter make-up of the state came as an unexpected but intriguing observation. Extensions of the project relating to this trade-off would make for compelling future work, including:
  - examining if it is inevitable that the minority party cannot win any districts even when there is not a great difference between the majority and minorities parties if there is a compactness threshold requirement;
  - adapting either the game or the method for computing the compactness of a district to minimize this compactness trade-off;
  - and analysing whether this trade-off can be seen in real-life, or if it is unique to graph based states, this method of computing compactness, etc.

9 Implementation details

Programming for this project was done by the author using Python. The code used can be found at: `https://github.com/maia-pi`. Graphs for the simulated states were made using the `NextworkX` package. This package also was used to compute the connectivity of districts and compactness-related statistics. Images were created using the `matplotlib` Python Package. Win distribution statistics for all of the simulations were computed by hand by the author.

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References

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