THEORY

- Linear scalar differential eq: $x^{(m)} + a_1(t)x^{(m-1)} + \dots + a_{m-1}(t)x' + a_m(t)x = f(t)$ (1)

$C^m(I)$: the set of m times continuously differentiable functions on I

If $f \equiv 0$, then (1) is homogeneous (i.e. f is the non-homogeneous part of (1))

- Terminology:
  1. Linear: an eq. is linear if x appears only as a derivative
  2. Homogeneous: all terms are somehow related to x
  3. (Non)Constant Coefficients: self explanatory

Thm 1.2: the IVP $\begin{cases} x^{(m)} + \dots + a_m(t)x = f(t) \\ x(t_0) = \eta_1 \\ \vdots \\ x^{(m-1)}(t_0) = \eta_m \end{cases}$ has a unique solution

Prop. 1.3: $\mathcal{L}: C^m(I) \to C(I)$, $\mathcal{L}(x) = x^{(m)} + a_1 x^{(m-1)} + \dots + a_m x$. Then $\mathcal{L}$ is linear

Lemma 1.4: (3) $\Leftrightarrow \mathcal{L}x = 0 \Rightarrow$ the set of solutions of (3) is $\operatorname{Ker}\mathcal{L}$

Prop 1.5 (~~Linearity principle~~) (Superposition principle)

Let $f = \alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_m f_m$ and x_{pk} be a particular solution of f_k. Then

$$\boxed{x_p = \sum_{i=1}^{m} \alpha_i x_{p_i} \text{ is a particular solution of (1)}}, \forall \alpha_i \in \mathbb{R}$$

~~Thm 1.7~~

Prop 1.6 (Linearity principle): If $x_1, \dots, x_m$ are m solutions of (3), then $\sum \alpha_i x_i$ is also a solution of (3), $\forall \alpha_i \in \mathbb{R}$

Thm 1.7 (The fundamental theorem ~~of~~ for LHDEs)

~~Let $x_1, x_2, \dots, x_m$ be~~

The set of solutions of (3) is a linear space of dimension m. Then $\exists\, x_1, \dots, x_m$ lin. independent solutions of (3). The general solution is:

$$\boxed{x = c_1 x_1 + \dots + c_m x_m}, \quad c_1, \dots, c_m \in \mathbb{R}$$

Thm 1.10 (Fundamental thm. for LnHDEs): $x = x_h + x_p$

1st order LDEs

Let (4) $\underbrace{x' + a(t)x}_{(5)=0} = f(t)$ and $A(t) = \int_{t_0}^{t} a(s)\,ds$

Prop. 1.14 $x = ce^{-A(t)}$ is the general solution of (5)

Prop 1.15 Let φ be a solution of (5). Then

i) $\varphi(t) = 0, \forall t$ or $\varphi(t) \neq 0, \forall t$

ii) if $a(t)$ ~~is con~~ $\neq 0, \forall t$, then φ is strictly monotone (for $\varphi \neq 0$)

THEORY

= Separation of variables method = → gen. sol of hom. eq.

$$x'(t) + a(t)x(t) = 0 \overset{x\neq 0}{\Leftrightarrow} \frac{x'(t)}{x(t)} = -a(t) \Leftrightarrow \ln|x(t)| = -a(t) \Leftrightarrow \ln|x(t)| = -\int a(t)\,dt + c$$

$$\Leftrightarrow x(t) = ce^{-at}, \quad c \in \mathbb{R}$$

= Lagrange method = → particular sol of non hom. eq.

Look for φ s.t. $x_p = \varphi(t)e^{-A(t)}$ is a solution of (4)

$$x_p = \int_{t_0}^{t} e^{-A(t)+A(s)} f(s)\,ds$$

= Integrating factor =

$\mu(t) = e^{A(t)}$ is an integrating factor for (4)

2nd order LDEs

= Lagrange method =

Look for φ_1, φ_2 s.t. $\begin{cases} \varphi_1'(t)x_1(t) + \varphi_2'(t)x_2(t) = 0 \\ \varphi_1'(t)x_1'(t) + \varphi_2'(t)x_2'(t) = 0 \end{cases}$, where $x_{1,2}$ sols. of the hom eq.

$$\rightarrow x_p = \varphi_1 x_1 + \varphi_2 x_2$$

LDEs w C.C.

Complex exponential: $\boxed{e^{x+iy} = e^x\cos y + ie^x \sin y = e^x(\cos y + i\sin y)}$

Characteristic eq. method: $x^{(n)} + a_1(t)x^{(n-1)} + \dots + a_{n-1}(t)x' + a_n(t)x = 0 \longrightarrow r^n + a_1(t)r^{n-1} + \dots + a_n(t) = 0$ characteristic eq.

1. Find the n roots of the
2. Let r be a root of multiplicity "m"

   I $r \in \mathbb{R} \longmapsto e^{r}, te^{r}, t^2e^{r}, \dots, t^{m-1}e^{r}$ (m functions)

   II $r = \alpha + i\beta \longmapsto e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t, te^{\alpha t}\cos\beta t, te^{\alpha t}\sin\beta t, \dots, t^{m-1}e^{\alpha t}\cos\beta t, t^{m-1}e^{\alpha t}\sin\beta t$

   (2 m functions, but if $r = \alpha + i\beta$ is a sol, then so is $\alpha - i\beta$)

=> sol. of hom. eq.

sol of non hom eq.

Undetermined coefficients method

- Assume $f = P_k(t)e^{\alpha t}$ – P_k polynomial of order k

  I α is NOT a root of the char. poly. => $x_p = Q_k e^{\alpha t}$, Q_k poly to be determined

  II α root of multiplicity m => $x_p = t^m Q_k(t)e^{\alpha t}$, — " —

- Assume $f = P_k(t)e^{\alpha t}\cos\beta t + \tilde{P}_k(t)e^{\alpha t}\sin\beta t$

  I $\alpha + i\beta$ not a root => $x_p = Q_k(t)e^{\alpha t}\cos\beta t + \tilde{Q}_k(t)e^{\alpha t}\sin\beta t$

  II $\alpha + i\beta$ root of mult. m => $x_p = t^m[Q_k(t)e^{\alpha t}\cos\beta t + \tilde{Q}_k(t)e^{\alpha t}\sin\beta t]$

! note that $P_k, \tilde{P}_k$ and $Q_k, \tilde{Q}_k$ are not related

LDEs in $\mathbb{R}^n$

$$(6)\begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n + f_1(t) \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n + f_2(t) \\ \vdots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n + f_n(t) \end{cases} \rightarrow n \text{ dimensional system}$$

$(6) \Leftrightarrow$ $X' = A(t)X + f(t)$ (7), where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$

Thm 2.2: The IVP $\begin{cases} X' = A(t)X + f(t) \\ X(t_0) = \eta \end{cases}$ has a unique solution

- DEF: $U \in C^1(I, \mathcal{M}_{n\times n}(\mathbb{R}))$ is a matrix solution of $x' = A(t)x$ if $U'(t) = A(t)U(t)$

(U matrix solution $\Leftrightarrow$ its columns are solutions of the system)

- DEF: If U's columns are linearly independent, U is a fundamental matrix solution

Prop 2.8 $\forall M \in \mathcal{M}_{n\times n}(\mathbb{R})$, $\exists ! U$ matrix solution of (7) s.t. $U(t_0) = M$, for t_0 fixed

If M invertible, then U is fundamental

Thm 2.9 Let U be a matrix solution of (7). Then, U is fundamental $\Leftrightarrow$ $\det U(t) \neq 0, \forall t$

Prop 2.10 Let U be a fundamental matrix solution of (7). Then:

i) $V(t) = U(t)M$ is another fundamental matrix solution, when M invertible

ii) $M(t) = U^{-1}(t)V(t)$ is a constant matrix, when V is another fundamental mat. sol.

- DEF: Let $t_0 \in I$ be fixed. The ~~pr~~ principal matrix solution at t_0 of (7) is the unique matrix solution E : $E(t_0) = I_n$

Remark: For any fundamental matrix solution we have: $E(t) = U(t)\cdot[U(t_0)]^{-1}$

Thm 2.13: The fundamental theorem for linear homogeneous systems

The set of solutions of (7) is a linear space of dimension n. Consequently, given ~~a~~ n linearly independent solutions $u_1, \dots, u_n$, its gen. sol is:

$$\boxed{X = c_1u_1 + c_2u_2 + \dots + c_nu_n} \qquad c_1, c_2, \dots, c_n \in \mathbb{R}$$

Thm 2.14: The fundamental theorem for linear non-homogeneous systems

The general solution of (6) is

$$X = U(t)C + \int_{t_0}^{t} U(t)[U(s)]^{-1} f(s)\,ds, \; C \in \mathbb{R}^n$$

LHDEs with CC

We'll denote by $F(t)$ the principal matrix solution at $t_0 = 0$

- DEF: $\boxed{e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k}$

Thm. 2.20 The unique solution of the IVP $\begin{cases} X' = AX \\ X(0) = \eta \end{cases}$ is $\boxed{\varphi(t,\eta) = e^{At}\eta}$

Lemma 2.21: Let A, B: $AB = BA$. Then $\begin{cases} e^{At}B = Be^{At} \\ e^{(A+B)t} = e^{At} \cdot e^{Bt} \end{cases}$

Lemma 2.22: Let $A, B \in \mathcal{M}_n(\mathbb{R})$: $\exists P \in \mathcal{M}_n(\mathbb{R})$, with $A = PBP^{-1}$. Then, $e^{At} = Pe^{Bt}P^{-1}$

- DEF: The algebraic multiplicity of $\lambda \in \mathbb{C}$ eigenvalue of A is its order as root of $p(\lambda)$

  The geometric multiplicity of λ is $\dim(\operatorname{Ker}(A - \lambda I_n))$ (max. # of lin. independent eigenvectors of corresp. to λ)

Prop 2.26: Let λ be an eigenvalue of A and $v \in \mathbb{C}^n$ a corresponding eigenvector. Then

i) $\varphi(t) = e^{\lambda t} v$ satisfies $X' = AX$

ii) When $\lambda \in \mathbb{R}$, we have that $v \in \mathbb{R}^n$ and $\varphi(t) = e^{\lambda t} v$ solution of $X' = AX$

iii) When $\lambda = \alpha + i\beta \in \mathbb{C}$ and $v = u + iw \in \mathbb{C}^n$, then

- $\varphi_1(t) = \operatorname{Re}(e^{\lambda t} v) = e^{\alpha t}\cos(\beta t) u - e^{\alpha t}\sin(\beta t) w$
- $\varphi_2(t) = \operatorname{Im}(e^{\lambda t} v) = e^{\alpha t}\sin(\beta t) u + e^{\alpha t}\cos(\beta t) w$

are solutions of $X' = AX$

iv) When $\lambda \in \mathbb{R}$ eigenvalue with algebraic multiplicity 2 and geometric multiplicity 1, Let v_1 be an eigenvector and v_2 be s.t. $(A - \lambda I_n)v_2 = v_1$. Then

$$\varphi_1(t) = e^{\lambda t} v_1 \quad \text{and} \quad \varphi_2(t) = e^{\lambda t}(tv_1 + v_2)$$

are 2 solutions of $X' = AX$

- DEF: A, B are similar if $\exists P$: $\boxed{A = PBP^{-1}}$

  A is diagonalizable if it's similar to a diagonal matrix

- Property: A diagonalizable $\Leftrightarrow$ it has n linearly independent eigenvalues ($A \in \mathcal{M}_n(\mathbb{R})$)

Gen. sol of (7) when A is diagonalizable

Mth. 1. Compute $\boxed{e^{At}}$ - the principal matrix solution $\mapsto e^{At} = P\, diag(e^{\lambda_1 t}, \dots, e^{\lambda_m t}) P^{-1}$

Mth. 2. Find m linearly independent solutions : $\boxed{e^{\lambda_1 t} v_1, \dots, e^{\lambda_m t} v_m}$

Mth. 3. Do linear change of variable $\boxed{Y = P^{-1} X}$ in $X' = AX$

= LHSs with CC of dim 2 =

I uncoupled systems : $\begin{cases} x' = a_{11} x \\ y' = a_{22} y \end{cases}$

II coupled systems

Reduction to 2nd order LHDE w CC

assume $a_{12} \neq 0 \Rightarrow x' = a_{11} x + a_{12} y \overset{(*)}{\Rightarrow} \begin{cases} y = \dfrac{x' - a_{11} x}{a_{12}} & (8) \\ x'' = a_{11} x' + a_{12} y' & (9) \end{cases}$

$\cancel{y' = a_{21} x + a_{22} y \overset{*}{\Rightarrow} x' = a_{11} x + a_{12}}$ x' $\quad y' = a_{21} x + a_{22} y \quad (10)$

$8, 9, 10 \Rightarrow x'' - (a_{11} + a_{22}) x' + (a_{11} a_{22} - a_{12} a_{21}) x = 0$

$\Rightarrow x'' - \operatorname{Tr} A\, x' + \det A\, x = 0$

CHARACTERISTIC EQUATION

1. characteristic eq of $p_\lambda(A)$: $\lambda^2 - (a_{11} + a_{22}) \lambda + (a_{11} a_{22} - a_{12} a_{21}) = 0$
2. find λ_1, λ_2 roots
3. attach solutions

   I $\lambda_{1,2} \in \mathbb{R}, \lambda_1 \neq \lambda_2 \longmapsto \varphi_1 = e^{\lambda_1 t} v_1, \quad \varphi_2(t) = e^{\lambda_2 t} v_2$, where $(A - \lambda_1 I_2) v_1 = 0$, $v_1 \neq 0$

   II $\lambda_1 = \lambda_2 \in \mathbb{R} \longmapsto \varphi_1(t) = e^{\lambda_1 t} v_1, \ \varphi_2(t) = e^{\lambda_1 t}(t v_1 + v_2)$, where $(A - \lambda_1 I_2) v_2 = v_1$

   III $\lambda_{1,2} = \alpha \pm i\beta, \ \beta \neq 0 \Rightarrow \begin{cases} \varphi_1(t) = \operatorname{Re}(e^{(\alpha+i\beta)t}(v_1 + i v_2)) = e^{\alpha t}(\cos\beta t) v_1 - e^{\alpha t}(\sin\beta t) v_2 \\ \varphi_2(t) = \operatorname{Im}(e^{(\alpha+i\beta)t}(v_1 + i v_2)) = e^{\alpha t}(\sin\beta t) v_1 + e^{\alpha t}(\cos\beta t) v_2 \end{cases}$

   where $(A - (\alpha + i\beta) I_2)(v_1 + i v_2) = 0$, $v_1 + i v_2 \neq 0$
4. gen. sol : $x = c_1 \varphi_1(t) + c_2 \varphi_2(t)$

Method 3 : Compute e^{At}

Thm 2.33 Any solution of $x' = Ax$ satisfies $\boxed{\lim_{t \to \infty} x(t) = 0 \Leftrightarrow \operatorname{Re} \lambda < 0}$, $\forall \lambda$ eigenvalue

THE DYNAMICAL SYSTEM ASSOCIATED TO an AUTONOMOUS DYNAMICAL SYSTEM in $\mathbb{R}^n$

the system $\overset{(1)}{x'} = f(x)$ is autonomous, because it lacks t. Notation becomes $x' \mapsto \dot{x}$

<u>Thm 3.34</u> - Existence and uniqueness thm.

Let $f \in C^1$, $\eta \in \mathbb{R}^n$. Then, the IVP $\begin{cases} \dot{x} = f(x) \\ x(0) = \eta \end{cases}$ has a unique solution $\varphi(t,\eta)$, called the flow of the system

- DEF: $\eta^* \in \mathbb{R}$ is an equilibrium point of (1) when $\varphi(t,\eta^*) = \eta^*$, $\forall t \in \mathbb{R}$

  <u>Remark</u>: η^* equilibrium $\Leftrightarrow$ $\boxed{f(\eta^*) = 0}$

<u>Prop. 3.35</u>: Let η^* be an equilibrium of (1) and $\eta \in \mathbb{R}$, $\eta \neq \eta^*$. Then, $\varphi(t,\eta) \neq \eta^*$

<u>Thm. 3.36</u>: Assume $\boxed{\exists \lim_{t\to\infty} \varphi(t,\eta) = \eta^*} \in \mathbb{R}^n$. Then, η^* is an <u>equilibrium</u> point

ORBITS & PHASE PORTRAITS

- DEF: Let η^* be an equil. point ~~$\Rightarrow$~~

  If $A_{\eta^*} = \{\eta \in \mathbb{R}^n : \lim_{t\to\infty} \varphi(t,\eta) = \eta^*\}$ contains a <u>neighbourhood</u> of $\eta^* \Rightarrow \eta^*$ is an attractor

  If $A_{\eta^*} = \mathbb{R} \Rightarrow \eta^*$ is a global attractor

  For $t \to -\infty$, we have the notion of a repeller

- DEF: The <u>orbit</u> of the initial state η is $\boxed{\gamma_\eta = \{\varphi(t,\eta) : t \in I_\eta\}}$

  <u>Remark</u>: γ_η is the image of $\varphi(t,\eta)$

PLANAR DYNAMICAL SYSTEMS

- DEF
  - For $c \in \mathbb{R}$, the ~~curve~~ c-level curve of H is the planar curve $\boxed{\Gamma_c = \{x \in U : H(x) = c\}}$
  - H is a first integral ~~of~~ in U of $\dot{x} = f(x)$ if
    - H is not locally constant
    - $\boxed{H(\varphi(t,\eta)) = H(\eta)}$, $\forall \eta \in U$, $\forall t \in I_\eta$, $\varphi(t,\eta) \in U$
  - U is an invariant set of $\dot{x} = f(x)$ if $\gamma_\eta \subset U$, for any $\eta \in U$

!!! The orbits are contained in the level curves of a first integral !!!

- when $U = \mathbb{R}^2$, we say that H is a global first integral

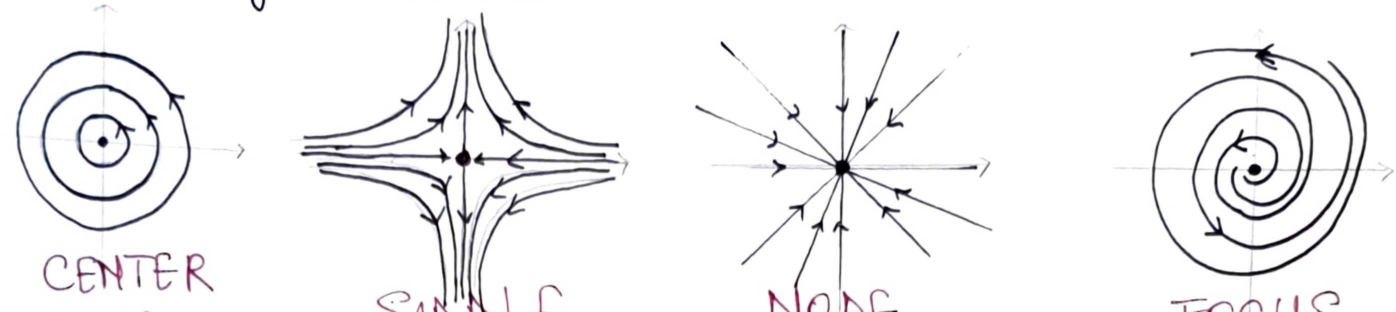
<u>Theorem 3.37</u>: H is a first integral in $\Omega \Leftrightarrow \boxed{\frac{\partial H}{\partial x}(x,y) f_1(x,y) + \frac{\partial H}{\partial y}(x,y) f_2(x,y) = 0}$, $\forall (x,y) \in \Omega$,

where $\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$

Tip: in order to find H, integrate $\frac{dy}{dx} = \frac{f_2(x,y)}{f_1(x,y)}$ and write it in a way s.t. it is well defined on U.

Remark: For $\dot{x} = Ax$ we have that $\eta^* = 0 \in \mathbb{R}^2$ is the only equilibrium $\Leftrightarrow \det A \neq 0$

= STABILITY of EQUIL. POINTS =



CENTER $\pm i\beta$ — SADDLE $\lambda_1 \cdot \lambda_2 < 0$ — NODE $\lambda_1 \cdot \lambda_2 > 0$ — FOCUS $\alpha \pm i\beta$

∘ **DEF** An equilibrium point η^* of $\dot{x} = f(x)$ is stable if $\forall \varepsilon > 0, \exists \delta > 0$: if $\|\eta - \eta^*\| < \delta$, then $\|\varphi(t,\eta) - \eta^*\| \leq \varepsilon, \forall t \in [0,\infty)$ and unstable otherwise

TYPES OF eq. pts

- node : $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \cdot \lambda_2 > 0$
- saddle : $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \cdot \lambda_2 < 0$
- center : $\lambda_{1,2} = \pm i\beta$
- focus : $\lambda_{1,2} = \alpha \pm i\beta$, $\alpha \neq 0$

→ polar coordinates

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \Rightarrow \begin{cases} \rho^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases} \Rightarrow \begin{cases} \rho\dot{\rho} = x\dot{x} + y\dot{y} \xrightarrow{\text{replace } \dot{x},\dot{y}} \dots \\ \frac{\dot{\theta}}{\cos^2\theta} = \frac{x\dot{y} - \dot{x}y}{x^2} \end{cases}$$

∘ **DEF** An orbit is periodic (closed) if $\varphi(\cdot,\eta)$ is periodic

thm 4.2

i) If $\boxed{Re(\lambda_1), Re(\lambda_2) < 0}$, the equil. point of (2): $\dot{x} = f(x)$ is a global attractor

ii) If $\boxed{Re(\lambda_1) > 0,\ Re(\lambda_2) > 0}$, —‖— is a repeller

iii) Any center is stable

iv) Any saddle is unstable

thm 4.3: Any center has a global first integral

Linearisation method for linear planar systems

Let η^* be an equilibrium point.

- $f'(\eta^*) < 0 \Rightarrow \eta^*$ is an attractor
- $f'(\eta^*) > 0 \Rightarrow \eta^*$ is a repeller

TYPE	EIGENVALUES	STABILITY
NODE	$\lambda_{1,2} < 0$	Stable
	$\lambda_{1,2} > 0$	Unstable
SADDLE	$\lambda_1 < 0 < \lambda_2$	Unstable
FOCUS	$\lambda_{1,2} = \alpha \pm i\beta, \alpha < 0$	Stable
	$\lambda_{1,2} = \alpha \pm i\beta, \alpha > 0$	Unstable
CENTER	$\lambda_{1,2} = \pm i\beta$	Stable

Prop 4.4 Let η^* be an equil point
1. If η^* is a global attractor/repeller, then the system has no global f.i.
2. If η^* is an attractor/repeller, then the system has no f.i in any neighbourhood of η^*

- NONLINEAR PLANAR SYSTEMS -

$\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases} \rightsquigarrow \boxed{Jf = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}}$ → Jacobian matrix

- DEF $\boxed{\dot{X} = Jf(\eta^*)X}$ is called the linearization of $\dot{x} = f(x)$ around η^*

- DEF Let λ_1, λ_2 be the eigenvalues of $Jf(\eta^*)$.
We say that the eq. point η^* is hyperbolic if $Re(\lambda_1), Re(\lambda_2) \neq 0$

~~5.1~~ Thm 5.1 Let η^* be a hyperbolic equil. point of $\dot{x} = f(x)$
- if $\boxed{\dot{X} = Jf(\eta^*)X}$ is an attractor/repeller, then $\boxed{\dot{X} = f(X)}$ is also an attractor/repeller
- if $\dot{x} = Jf(\eta^*)X$ is a saddle, then $\dot{x} = f(x)$ is unstable

Remark: $\dot{X} = Jf(\eta^*)X$ being a center does not provide enough information (by itself) to determine the stability of η^*.

Thm 5.2 Let η^* be an equil pt. s.t $Jf(\eta^*)$ is a center. If $\dot{x} = f(x)$ has a f.i well defined in a neighbourhood of η^*, then η^* is stable.

DIRECTION FIELD

Let $\dot{x} = f(x)$. Fix $(x_0, y_0) \in \mathbb{R}^2$ and consider the vector based in (x_0, y_0) and parallel to $v(f_1(x_0,y_0), f_2(x_0,y_0))$. The collection of all these vectors is the direction field.

- DEF Consider the diff. eq. $\frac{dy}{dx} = g(x,y)$. Let $(x_0, y_0) \in \mathbb{R}^2$. Consider the collection of vectors based in (x_0, y_0), with the slope $g(x_0, y_0)$ → this is the direction field ~~of~~ associated to the eq.

! $\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$ and $\frac{dy}{dx} = \frac{f_2}{f_1}$ have the same direction field

Property 6.1 The direction field is tangent to the orbits of the system / the solution curves of the differential equation.

NUMERICAL METHODS :

Consider the scalar ODE (ordinary differential equation) $y' = g(x,y)$ and the initial condition $y(x_0) = y_0$. We want to estimate / find the direction field and approximate the shape of the graph of the solution of the IVP.

Euler's numerical formula

Notions:

- $\varphi : [x_0, x^*] \to \mathbb{R}$ - unique solution of the IVP
- $\{x_0 < x_1 < \dots < x_{m-1} < x_m = x^*\}$ - partition of $[x_0, x^*]$
- m - # of steps
- $x_{k+1} - x_k$ - step size

Idea: We look for y_{k+1} s.t. (x_{k+1}, y_{k+1}) is on the line that contains (x_k, y_k) and has the slope $g(x_k, y_k)$ (i.e. the slope of the direction field)

$$\boxed{y_{k+1} = y_k + g(x_k, y_k)(x_{k+1} - x_k)}$$

→ for a fixed step size $h > 0$

$$\boxed{y_{k+1} = y_k + h \cdot g(x_k, y_k)}$$ → ONLY ONE ON THE EXAM

Euler's IMPROVED numerical formula

$$\boxed{y_{k+1} = y_k + h \cdot \frac{g(x_k, y_k) + g(x_{k+1}, y_k + h \cdot g(x_k, y_k))}{2}}$$

=LINEAR DIFFERENCE EQUATIONS=

$t \in \mathbb{R} \mapsto k \in \mathbb{N}$ $\quad x(t) \longmapsto x_k$ $\quad e^{rt} \longmapsto r^k$

1. ~~First order LDE~~ LDifferenceEs with CC

general solution: $x = x_h + x_p$, where

$x_h = \sum_{i=1}^{n} c_i x_i$, with x_i for $i \in \overline{1,n}$ linearly independent

- for 1st order LDiffEs w. CC., $x_h = r^k$, where r is a root of the characteristic polynomial
- for 2nd order LDiffEs w. CC
  - if $r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2 \longmapsto r_1^k, r_2^k$
  - if $r_1 = r_2 \in \mathbb{R} \longmapsto r_1^k, k r_1^k$
  - if $r_1 = \overline{r_2} \notin \mathbb{R} \longmapsto Re(r_1^k), Im(r_1^k)$

=Scalar discrete dynamical systems=

$$\boxed{x_{k+1} = f(x_k)}$$

Notation: $\begin{cases} f^0 = id;\ f^1 = f;\ f^2 = f \circ f;\ \dots,\ f^k = \underbrace{f \circ f \circ \dots \circ f}_{k \text{ times}} \\ x_k = f^k(x_0) = f^k(\eta) \end{cases}$

$\gamma_\eta = \{\eta, f(\eta), f^2(\eta), \dots\}$ - the positive orbit of the initial value η

DEF: η^* is a fixed point of f if $f(\eta^*) = \eta^*$ (so $\gamma_{\eta^*} = \{\eta^*\}$)

Thm 8.1 Let η, η^* be s.t. $\boxed{\lim_{k\to\infty} f^k(\eta) = \eta^*}$. Then, η^* is a fixed point of f

If f is invertible, we DEFine f^{-k} as $(f^{-1})^k$

DEF If $\boxed{\lim_{k\to\infty} f^k(\eta) = \eta^*}$, $\forall \eta \in V \in \mathcal{V}(\eta^*)$, then η^* is an attractor of the map f

If $V = \mathbb{R}$, η^* is a global attractor

Thm 8.2 : The linearization method

Let $f \in C^1$ and η^* be a fixed point of f

- if $\boxed{|f'(\eta^*)| < 1}$, then η^* is an attractor
- if $\boxed{|f'(\eta^*)| > 1}$, then η^* is a repeller (i.e an attractor for f^{-1})

PERIODIC POINTS

$k \in \overline{1, p-1}$

- DEF: η^* is a p-periodic point of f when η^* is a fixed point of f^p, but not of f^k, ~~$k\in 1,m$~~

<u>Remark</u>: - if η^* is a p-periodic point, then its orbit is a p-cycle: $\gamma_{\eta^*} = \{\eta^*, f(\eta^*), \dots, f^{p-1}(\eta^*)\}$

- if $\eta \in A_{\eta^*}$, then $\lim_{k\to\infty} (f^p)^k(\eta) = \eta^*$

- DEF: Let η^* be a p-periodic point of f. We say that γ_{η^*} is an attracting cycle of f when η^* is an attracting fixed point of f^p

<u>Remark</u>: Let $\eta \in A_{\eta^*}$. The unique solution of the IVP $\begin{cases} x_{k+1} = f(x_k) \\ x_0 = \eta \end{cases}$ can be split into p sequences, each converging to one of the elements ~~in~~ of the attracting p-cycle.

THE NEWTON-RAPHSON METHOD

→ used for approximating the roots of a map g

<u>Thm 8.3</u>

Let $V \subseteq \mathbb{R}$ nonempty open interval and $g: V \to \mathbb{R}$ a C^2 map s.t. $g'(x) \neq 0$, $\forall x \in V$.
Assume $\exists\, \eta^* \in V \,/\, g(\eta^*) = 0$. Then, $\exists\, \rho > 0$ s.t., whenever $|x_0 - \eta^*| < \rho$ we have $\boxed{\lim_{k\to\infty} x_k = \eta^*}$, where $\boxed{x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}}$.

CHAOTIC MAPS

- DEF: A chaotic map is a map having the following properties:
  1. $\exists$ p-cycle, $\forall p \geq 1$
  2. the butterfly effect: given $\eta \in [0,1]$ and $\delta > 0$, $\exists k \geq 1$ and $\tilde{\eta} \in [0,1]$: $|\eta - \tilde{\eta}| < \delta$ and $|f^k(\eta) - f^k(\tilde{\eta})| \geq \frac{1}{2}$
  3. a dense orbit: $\exists\, \eta \in [0,1]$: $\{f^k(\eta) : k \geq 0\}$ is dense in $[0,1]$ (i.e., $\forall x \in [0,1]$, $\exists (f^k(\eta))$ subsequence s.t $f^k(\eta)$ converges to x