|| Partial 1 prep ||

Chapter 1 - Affine space

- Characterization of equipolence relation

DEF -Two ordered ~~set~~ pairs of points are called equipollent and we write $(A,B) \sim (C,D)$ if AD and BC have the same midpoints

-The following statements are $\Leftrightarrow$

Ch. TE
1. $(A, B) \sim (C, D)$
2. $(B, A) \sim (D, C)$
3. $ABDC$ parallelogram (possibly degenerate)
4. $|AB\rangle = |CD\rangle$ and $[AB] \equiv [CD]$
5. $|AB\rangle = |CD\rangle$ and $|AB| = |CD|$

well-defined

- The set of vectors V with addition is an abelian group (assuming it is known that addition is well-defined)

$V \leftarrow \frac{E \times E}{\sim}$

1) stable part - known

2) associativity: $\forall a,b,c$ vectors, $a+(b+c) = (a+b)+c$

Fix point $A \Rightarrow \exists! B,C,D$ s.t. $a = \vec{AB}$, $b = \vec{BC}$, $c = \vec{CD} \Rightarrow$

$$\Rightarrow \begin{cases} a+(b+c) = \vec{AB} + (\vec{BC} + \vec{CD}) = \vec{AB} + \vec{BD} = \vec{AD} \\ (a+b)+c = (\vec{AB} + \vec{BC}) + \vec{CD} = \vec{AC} + \vec{CD} = \vec{AD} \end{cases} \Rightarrow a+(b+c) = (a+b)+c,\ \forall a,b,c$$

3) commutativity: Let a,b be vectors

Since $a+b$ is constructed as the diagonal of a parallelogram, with a and b as represented on the sides, the construction is symmetric and does not depend on the order of a and b in the sum

4) neutral element: ~~$a+0=0+a$~~ Let a be a vector ~~and~~ Fix point $A \Rightarrow \exists! B : a = \vec{AB}$

~~Define~~ $0 = \vec{AA}$ ~~$\Rightarrow a+0 = \vec{AB} + \vec{BB}$~~ $\Rightarrow a+0 = \vec{AB} + \vec{BB} = \vec{AB} = \vec{AA} + \vec{AB} = 0+a$

$\Rightarrow a+0 = 0+a = a,\ \forall a$

5) symmetric element: $\begin{cases} a+(-a) = \vec{AB} + (-\vec{AB}) = \vec{AB} + \vec{BA} = \vec{AA} = 0 \\ (-a)+a = (-\vec{AB}) + \vec{AB} = \vec{BA} + \vec{AB} = \vec{BB} = 0 \end{cases}$

$\Rightarrow \forall a, \exists! -a$ s.t. $a+(-a) = (-a)+a = 0$

1,2,3,4,5 $\Rightarrow$ $(V,+)$ is an abelian group

Characterization of lines and planes via vectors

- Let $S \subseteq E$ and $O \in S$ $\quad \phi_O(A) = \vec{OA}$
  1. S is a line $\Leftrightarrow$ $\phi_O(S)$ is a 1D vector space
  2) $\vec{OA}, \vec{OB}$ are linearly dependent $\Leftrightarrow$ O, A, B are collinear
  3. S is a plane $\Leftrightarrow$ $\phi_O(S)$ is a 2D vector space
  4. $\vec{OA}, \vec{OB}, \vec{OC}$ are dependent $\Leftrightarrow$ O, A, B, C are coplanar
  5. If S is a line or a plane, then $\phi_O(S)$ is independent of the choice of O in S

Translation maps and stuff?

- Operations with vectors, points and scalars:
  - $\square + \square : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$: addition of 2 vectors
  - $\square \cdot \square : \mathbb{R} \times \mathbb{V} \to \mathbb{V}$: multiplication of a scalar and a vector
  - $\square + \square : \mathbb{V} \times \mathbb{E} \to \mathbb{E}$: addition of a vector to a scalar.

    if $a = \vec{OX}$, then $a + O = X$

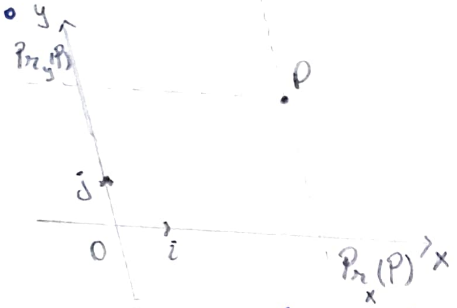
Thus, the vectors in $\mathbb{V}$ act as on the set of points $\mathbb{E}$ by translations

- A real affine space $\mathbb{A}$ is a triple $(\mathbb{P}, \mathbb{V}, t)$ which satisfies the following axioms
  (points, vectors, translation map)
  1) $\forall A, B \in \mathbb{P}, \exists! a \in \mathbb{V} : B = t(a, A)$
  2) $\forall A \in \mathbb{P}$ and $a, b \in \mathbb{V} : t(a, t(b, A)) = t(a+b, A)$
- $\dim \mathbb{A} = \dim \mathbb{V}$ $\qquad$ ~~$\mathbb{A}$~~ $\qquad$ - $D(\mathbb{A}) := \mathbb{V}$
- $(\mathbb{P}, \mathbb{V}, \phi_O)$ ~~$(\mathbb{E})$~~ - Affine space (where $\mathbb{P}$-set of points in $\mathbb{E}$, $\mathbb{V}$-set of geometric vectors, $\phi_O : \mathbb{E} \to \mathbb{V}$, $\phi_O(A) = \vec{OA}$ is a bijection)

(*) Let $\vec{CD}$ be a vector $\Rightarrow \exists! A : (C,D) \sim (O, A) \Leftrightarrow \exists! A^{(*)}$ s.t. $\vec{CD} = \vec{OA} = \phi_O(A) \Rightarrow \phi_O$ bijective

Chapter 2 - Cartesian coordinates

- A frame in $\mathbb{E}^2$ is a pair $\mathcal{K} = (O, \mathcal{B})$, where $O \in \mathbb{E}^2$ and $\mathcal{B} = (i, j)$ is a basis of $\mathbb{V}^2$.
- $[P]_{\mathcal{K}} = [\vec{OP}]_{\mathcal{B}}$



projection on Ox along Oy

$Pr_x : \mathbb{E}^2 \to Ox,\ Pr_x(P) = X(x_P, 0)$
$Pr_y : \mathbb{E}^2 \to Oy,\ Pr_y(P) = Y(0, y_P)$ } $\to$ projections for points

$pr_x : \mathbb{V}^2 \to \mathbb{R},\ pr_x(a) = a_x$
$pr_y : \mathbb{V}^2 \to \mathbb{R},\ pr_y(a) = a_y$ } $\to$ projections for vectors

projection on the 2nd component

Changing reference frames

Let $\mathcal{B} = (v_1, \dots, v_n)$, $\mathcal{B}' = (v_1', \dots, v_n')$ - bases of $\mathbb{V}$

$M_{\mathcal{B}'\mathcal{B}} = [id]_{\mathcal{B}\mathcal{B}'} = ([v_1]_{\mathcal{B}'} | \dots | [v_n]_{\mathcal{B}'})$

$\forall v \in \mathbb{V} : \boxed{[v]_{\mathcal{B}'} = M_{\mathcal{B}'\mathcal{B}} \cdot [v]_{\mathcal{B}}} \to$ change the coordinates of a vector

$\forall P \in \mathbb{E}^n : \boxed{[P]_{\mathcal{K}'} = M_{\mathcal{B}\mathcal{B}'} \cdot ([P]_{\mathcal{K}} - [O']_{\mathcal{K}})} \to$ -"- point (where $\mathcal{K} = (O, \mathcal{B})$, $\mathcal{K}' = (O', \mathcal{B}')$)

Orientation

2 bases $\mathcal{E}$ and $\mathcal{F}$ are said to have the same orientation if $\det(M_{\mathcal{E}\mathcal{F}}) > 0$ (or $\det(M_{\mathcal{F}\mathcal{E}}) > 0$).
If $\det(M_{\mathcal{E}\mathcal{F}}) < 0$, they have opposite orientation.

Thm: Let $\mathcal{K} = (O, \mathcal{B})$ and $\mathcal{K}' = (O', \mathcal{B}')$. Then, $\forall P \in \mathbb{A}^n$:

$$[P]_{\mathcal{K}'} \underset{(1)}{=} M_{\mathcal{B}'\mathcal{B}}([P]_{\mathcal{K}} - [O']_{\mathcal{K}}) \underset{(2)}{=} M^{-1}_{\mathcal{B}\mathcal{B}'}([P]_{\mathcal{K}} - [O']_{\mathcal{K}}) \underset{(3)}{=} M_{\mathcal{B}'\mathcal{B}}[P]_{\mathcal{K}} + [O]_{\mathcal{K}'}$$

Proof: $[P]_{\mathcal{K}'} = [\vec{O'P}]_{\mathcal{B}'} = M_{\mathcal{B}'\mathcal{B}} \cdot [\vec{O'P}]_{\mathcal{B}} = M_{\mathcal{B}'\mathcal{B}}([\vec{O'O}]_{\mathcal{B}} + [\vec{OP}]_{\mathcal{B}}) = M_{\mathcal{B}'\mathcal{B}}([\vec{OP}]_{\mathcal{B}} - [\vec{OO'}]_{\mathcal{B}}) =$

$= M_{\mathcal{B}'\mathcal{B}}([P]_{\mathcal{K}} - [O']_{\mathcal{K}}) \overset{def}{\Rightarrow} (1)$

$(1),\ M_{\mathcal{B}\mathcal{B}'} = M^{-1}_{\mathcal{B}'\mathcal{B}} \Rightarrow (2)$

$$M_{\mathcal{B}'\mathcal{B}} \cdot (-[O']_{\mathcal{K}}) = M_{\mathcal{B}'\mathcal{B}} \cdot (-[\vec{OO'}]_{\mathcal{B}}) = M_{\mathcal{B}'\mathcal{B}} \cdot [\vec{O'O}]_{\mathcal{B}} = \underbrace{M_{\mathcal{B}'\mathcal{B}} \cdot (M_{\mathcal{B}\mathcal{B}'}}_{I_n} \cdot \underbrace{[\vec{O'O}]_{\mathcal{B}'}}_{[O]_{\mathcal{K}'}}) = [O]_{\mathcal{K}'} \overset{(1)}{\Rightarrow} (3)$$

Chapter 3

- Lines in $\mathbb{A}^2$

$\Delta(S)$ - direction space of the line S ($\Delta(S) = \overset{\text{when } S \text{ is a line}}{\Phi_a^2(S)}$)

- Parametric equation of a line

Let $P, Q \in S$, S line and $\vec{PQ} = v$. Then

$$S: \begin{cases} x = x_p + t v_x \\ y = y_p + t v_y \end{cases} \quad \text{or} \quad S: \begin{cases} x = x_Q + t v_x \\ y = y_Q + t v_y \end{cases}$$

(initial point obviously does not matter, as long as it belongs to the line)

or

$$S: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_p \\ y_p \end{bmatrix} + t \cdot \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

- Cartesian equation: $\dfrac{x - x_p}{v_x} = \dfrac{y - y_p}{v_y}$ (also called symmetric equation)

- Linear equation: $ax + by + c = 0$ (— " — implicit equation)

then $\Delta(\ell): ax + by = 0$

- Explicit equation: $y = mx + n$

slope ONLY in $\mathbb{A}^2$

- Relative positions:

$$\begin{cases} \ell_1: a_1x + b_1y + c_1 = 0 \\ \ell_2: a_2x + b_2y + c_2 = 0 \end{cases}$$

If $\ell_1 = \lambda \ell_2$, $\ell_1 = \ell_2$

If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$, $\ell_1 \parallel \ell_2$

$\ell_1 \cap \ell_2 = \{P\}$, $P \in \mathbb{P}$ otherwise

- Planes in $\mathbb{A}^3$

Let $P(x_p, y_p, z_p) \in \overset{S \text{ plane with basis } B = (v, w)}{S}$. Then

parametric eq $S: \begin{cases} x = x_p + s v_x + t w_x \\ y = y_p + s v_y + t w_y \\ z = z_p + s v_z + t w_z \end{cases}$ or $S: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + s \cdot \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} + t \cdot \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$

eq. using determinant $S: \begin{vmatrix} x - x_p & y - y_p & z - z_p \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = 0$

linear eq: $ax + by + cz + d = 0 \Rightarrow \Delta(\pi): ax + by + cz = 0$

$\Updownarrow$

$\dfrac{x}{\alpha} + \dfrac{y}{\beta} + \dfrac{z}{\gamma} = 1$, where $\alpha = -\dfrac{d}{a}$, $\beta = -\dfrac{d}{b}$, $\gamma = -\dfrac{d}{c}$ and

$\pi \cap Ox = \{A(\alpha, 0, 0)\}$, $\pi \cap Oy = \{B(0, \beta, 0)\}$, $\pi \cap Oz = \{C(0, 0, \gamma)\}$

Rel. positions:

Let $\begin{cases} \pi_1 : a_1x+b_1y+c_1z+d_1=0 \\ \pi_2 : a_2x+b_2y+c_2z+d_2=0 \end{cases} \Rightarrow \begin{cases} \pi_1 \parallel \pi_2 \text{ if } \operatorname{rank}(M) < \operatorname{rank} \tilde{M} \\ \pi_1 \cap \pi_2 = \{\ell\}, \ \ell \text{ line, if } \operatorname{rank}(M) = 2 \\ \pi_1 = \pi_2 \text{ if } \operatorname{rank} M = \operatorname{rank} \tilde{M} = 1 \end{cases}$

- Lines in $\mathbb{A}^3$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$
parametric eq.

$$\frac{x-x_a}{v_x} = \frac{y-y_a}{v_y} = \frac{z-z_a}{v_z}$$
symmetric eq.

$$\begin{cases} a_1x+b_1y+c_1z+d_1=0 \\ a_2x+b_2y+c_2z+d_2=0 \end{cases}$$
linear eq

- Relative positions of 2 lines → tip: to find $D(\ell)$, if ℓ is given as the $\cap$ of 2 planes, we can just compute the cross prod of the normal vectors of the 2 planes

$$\ell_1 : \begin{cases} x = x_1 + t v_x \\ y = y_1 + t v_y \\ z = z_1 + t v_z \end{cases} \qquad \ell_2 : \begin{cases} x = x_2 + s u_x \\ y = y_2 + s u_y \\ z = z_2 + s u_z \end{cases}$$

$\Rightarrow$ - $\ell_1 \parallel \ell_2$ if $v = \lambda u$

- $\ell_1 = \ell_2$ if $v = \lambda u$ and $\exists A \in \{\ell_1 \cap \ell_2\}$

- ℓ_1 and ℓ_2 intersect in one point if ℓ_1, ℓ_2 coplanar and $\ell_1 \nparallel \ell_2$

$$\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix} = 0$$

- ℓ_1 and ℓ_2 are skew relative to each other otherwise

- Relative positions of a line and a plane in $\mathbb{A}^3$

Let $\pi : ax+by+cz+d=0$ and $\ell : \begin{cases} x = x_0 + t v_x \\ y = y_0 + t v_y \\ z = z_0 + t v_z \end{cases}$

I $\ell \parallel \pi$ if $\forall P(x_p, y_p, z_p) \in \ell$, $ax_p + by_p + cy_p + zy_p \neq 0$

II ℓ punctures P if $\exists! \dots$ s.t. — " — $= 0$

III $\ell \subset \pi$ if $\forall P \in \ell$, — " — $= 0$

Thm: Let S, T be $\parallel$ affine subsp. of $\mathbb{A}^n$, with $\dim(S) \leq \dim(T)$

If $S \cap T \neq \emptyset$, then $S \subseteq T$ (with eq. if $\dim S = \dim T$)

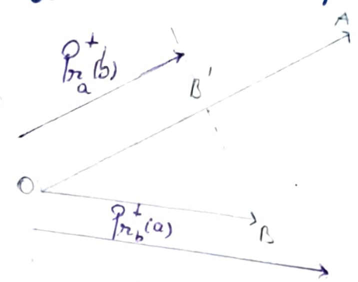
- Changing the reference frame

If $\mathcal{R}(O, \mathcal{B})$, $\mathcal{R}'(O', \mathcal{B}')$ are frames and $S : \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + t_1 \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{bmatrix} + \dots + t_k \begin{bmatrix} v_{k1} \\ v_{k2} \\ \vdots \\ v_{kn} \end{bmatrix}$ is the parametric eq of S with respect to $\mathcal{R}$, then the eq. of S with respect to $\mathcal{R}'$ is

$$S : \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = M_{\mathcal{B}'\mathcal{B}} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} + [O]_{\mathcal{R}'} + \sum_{i=1}^{m} t_i \cdot M_{\mathcal{B}'\mathcal{B}} \cdot \begin{bmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{in} \end{bmatrix}$$

Chapter 4

- for 2 vectors $\vec{OA}$ and $\vec{OB}$, the unoriented angle between a and b - $\measuredangle(a,b)$ - is the angle $\measuredangle AOB$



- orthogonal projection maps

  $pr_a^{\perp}: \mathbb{V} \to \mathbb{R}$, $pr_a^{\perp}(b) = |\vec{OB'}|$ and

  $Pr_a^{\perp}: \mathbb{V} \to \mathbb{V}$, $Pr_a^{\perp}(b) = \vec{OB'} = pr_a^{\perp}(b) \cdot \frac{a}{|a|}$

  (take the direction of a and "make" a unit vector, then multiply with the length of the new vector)

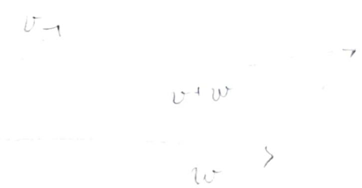
- $\cos \measuredangle(a,b) = \frac{pr_a^{\perp}(b)}{|b|} = \frac{pr_b^{\perp}(a)}{|a|}$ | remember: $Pr_a^{\perp}(v) = \frac{a \cdot v}{\|a\|^2} \cdot a$ → $Pr_a^{\perp}$ OF $\frac{\langle a, v \rangle}{\langle a, a \rangle} \cdot a$

- orthonormal frames

  A basis $\mathcal{B} = (e_1, \dots, e_n) \in \mathbb{V}^n$ is called orthogonal if its vectors are mutually orthogonal.
  If, additionally, all vectors are unit vectors, the basis is called orthonormal

  - Note that $pr_{e_i}^{\perp}(a) = |a| \cos \measuredangle(a, e_i)$, thus

$$[P]_{\mathcal{K}} = \begin{bmatrix} |a| \cos \measuredangle(a, e_1) \\ \vdots \\ |a| \cos \measuredangle(a, e_n) \end{bmatrix} = [\vec{OP}]_{\mathcal{B}}$$



Proof well-defined & linearity

well defined: $\forall\, v \in \mathbb{V}, \exists!\, w \in \mathbb{V}: Pr_a^{\perp}(v) = w$. This is true, since

Let $a = \vec{OA} \Rightarrow \exists!\, B: v = \vec{OB} \Rightarrow \exists!\, B': BB' \perp \vec{OA}$. Thus, $w = \vec{OB'}$ is unique

linear: It is enough to show that $pr_a^{\perp}$ is ~~pos~~ linear, since then

$Pr_a^{\perp}(\alpha v + \beta w) = pr_a^{\perp}(\alpha v + \beta w) \frac{a}{|a|} = \alpha\, pr_a^{\perp}(v) \frac{a}{|a|} + \beta\, pr_a^{\perp}(w) \frac{a}{|a|} = \alpha Pr_a^{\perp}(v) + \beta Pr_a^{\perp}(w)$

WRONG!

Prove for $Pr_a^{\perp}$ directly

but $pr_a^{\perp}(v) = |v| \cos \measuredangle(a, v)$! multiply by $\frac{a}{|a|}$ to make this valid for $Pr_a^{\perp}$

1. $pr_a^{\perp}(\alpha v) = \alpha\, pr_a^{\perp}(v)$

$pr_a^{\perp}(\alpha v) = |\alpha v| \cos(\measuredangle(a, \alpha v)) = |\alpha| \cdot |v| \cdot \cos \measuredangle(a, \alpha v)$

I $\alpha > 0 \Rightarrow |\alpha| = \alpha$ and v and αv have the same direction $\Rightarrow \cos\measuredangle(a, \alpha v) = \cos\measuredangle(a, v)$

II $\alpha < 0 \Rightarrow |\alpha| = -\alpha$ and $\cos \measuredangle(a, \alpha v) = -\cos\measuredangle(a, v) \Rightarrow$

III $\alpha = 0 \Rightarrow pr_a^{\perp}(\alpha v) = 0 = \alpha \cdot pr_a^{\perp}(\alpha v)$

I, II, III $\Rightarrow pr_a^{\perp}(\alpha v) = \alpha\, pr_a^{\perp}(v)$, $\forall\, \alpha \in \mathbb{R}$

this applies to $Pr_a^{\perp}(v)$

2. $pr_a^{\perp}(v + w) = pr_a^{\perp}(v) + pr_a^{\perp}(w)$

Since $pr_a^{\perp}(v) = \left(\frac{a \cdot v}{\|a\|^2}\right) \cdot a \Rightarrow pr_a^{\perp}(v + w) = \left(\frac{(v + w) \cdot a}{\|a\|^2}\right) \cdot a = \frac{a}{\|a\|^2}(v \cdot a + w \cdot a)$

Also can't use the scalar product for the proof, since this is used to prove the bilinearity of the scalar product...

Scalar product

$\langle a,b\rangle = a\cdot b = |a|\cdot|b|\cdot\cos(\sphericalangle(a,b))$

Properties (the scalar product is)

- bilinear : $\langle av+bw, u\rangle = a\langle v,u\rangle + b\langle w,u\rangle$ and $\langle u, av+bw\rangle = a\langle u,v\rangle + b\langle u,w\rangle$
- symmetric: $\langle u,v\rangle = \langle v,u\rangle$
- positive definite: $\langle v,v\rangle > 0, \forall v \neq 0$
- it recognizes right angles and unit lengths:

$u \perp v \Leftrightarrow \langle u,v\rangle = 0$ and $|v| = 1 \Leftrightarrow \langle v,v\rangle = 1$

Proof: S: $\langle u,v\rangle = |u|\cdot|v|\cdot\cos\sphericalangle(u,v) = |v|\cdot|u|\cdot\cos\sphericalangle(v,u) = \langle v,u\rangle$

pos.def: $\cos\sphericalangle(a,a) = \cos 0 = 1 \Rightarrow \langle a,a\rangle = |a|^2 > 0, \forall a \neq 0$

bilinearity: $\langle a,b\rangle = |a|\,|b|\cos\sphericalangle(a,b) = |a|\,|b|\,\frac{pr_{\frac{1}{b}}(a)}{|a|} = |b|\cdot pr_{\frac{1}{b}}(a)$

linear map

Gram-Schmidt algorithm → used for translating stuff from a reference frame to another

non-orthonormal   orthonormal

Let $\mathcal{K} = (O, \mathcal{B})$, $\mathcal{B} = (e_1, \dots, e_n)$. We want to construct $\mathcal{B}'$, an orthonormal basis, starting from $\mathcal{B}$.

1. Construct $(e_1', e_2', \dots, e_n')$

subtract the magnitude necessary for obtaining a right angle between e_1' and e_2

$$e_1' = e_1$$
$$e_2' = e_2 - \frac{\langle e_1', e_2\rangle}{\langle e_1', e_1'\rangle} e_1'$$
$$e_3' = e_3 - \frac{\langle e_1' e_3\rangle}{\langle e_1', e_1'\rangle} e_1' - \frac{\langle e_2' e_3\rangle}{\langle e_2', e_2'\rangle} e_2'$$
$$e_4' = e_4 - \frac{\langle e_1', e_4\rangle}{\langle e_1', e_1'\rangle} e_1' - \frac{\langle e_2', e_4\rangle}{\langle e_2', e_2'\rangle} e_2' - \frac{\langle e_3', e_4\rangle}{\langle e_3', e_3'\rangle} e_3'$$
$$\vdots$$

2. Normalize the vectors

$$\mathcal{B}' = \left(\frac{e_1'}{|e_1'|}, \frac{e_2'}{|e_2'|}, \dots, \frac{e_n'}{|e_n'|}\right)$$

- **Normal vectors** ? J operator?

Let $\mathcal{H}: a_1x_1 + a_2x_2 + \ldots + a_mx_m = b$ be a hyperplane with respect to an orthonormal frame $\mathcal{K}$.
Then, $n(a_1, a_2, \ldots, a_n)$ is ~~called~~ a normal vector of $\mathcal{H}$ (i.e. $v \perp w$, $\forall w \subset \mathcal{H}$)

- **Angles between lines and hyperplanes**

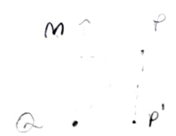
$$\cos \measuredangle(v_1, v_2) = \frac{\langle v_1, v_2 \rangle}{|v_1| \cdot |v_2|} = \frac{\langle n_1, n_2 \rangle}{|n_1| \cdot |n_2|}.$$

So:
Let ℓ_1, ℓ_2 be lines with direction vectors v_1, v_2 respectively and $\mathcal{H}_1, \mathcal{H}_2$ be hyperplanes with normal vectors n_1, n_2 respectively. Then:

- ℓ_1, ℓ_2 define 2 supplementary angles $\measuredangle(v_1, v_2)$ and $\measuredangle(-v_1, v_2)$:
$$\cos \measuredangle(v_1, v_2) = \frac{\langle v_1, v_2 \rangle}{|v_1| \cdot |v_2|}$$
- $\mathcal{H}_1, \mathcal{H}_2$ —II— :
$$\cos \measuredangle(n_1, n_2) = \frac{\langle n_1, n_2 \rangle}{|n_1| \cdot |n_2|}$$

3. ℓ_1 and $\mathcal{H}_1$ —II— : if $\cos \measuredangle(v_1, n_1) > 0$, then $\measuredangle(v_1, n_1)$ is acute ~~and~~ ~~their~~ the angle between ℓ_1 and $\mathcal{H}$ is
$$\frac{\pi}{2} - \arccos\left(\frac{\langle v_1, n_1 \rangle}{|v_1| \cdot |n_1|}\right)$$
(if $\cos \measuredangle(v_1, n_1) \leq 0$, replace n_1 with $-n_1$

- **Distances**
  - distance from a point to a hyperplane



Let $P(p_1, \ldots, p_m)$ and $\mathcal{H}: a_1x_1 + a_2x_2 + \ldots + a_mx_m = b$. Then
$$d(P, \mathcal{H}) = \frac{|a_1p_1 + a_2p_2 + \ldots + a_mp_m - b|}{\sqrt{a_1^2 + a_2^2 + \ldots + a_m^2}}$$

Proof: drop a perpendicular line from P to $\mathcal{H}$, intersecting $\mathcal{H}$ in $P' \Rightarrow d(P, \mathcal{H}) = |PP'|$.
Now let $Q \subset \mathcal{H}$ and consider the normal vector $n(a_1, \ldots, a_n)$. Then
$$|PP'| = pr_n^{\perp}(\overrightarrow{QP}) \Rightarrow d(P, \mathcal{H}) = \left|\frac{\langle n, \overrightarrow{QP} \rangle}{\langle n, n \rangle} \cdot n\right| = \frac{|\langle n, \overrightarrow{QP} \rangle|}{|n|} = \frac{|\langle n, \overrightarrow{OP} - \overrightarrow{OQ} \rangle|}{|n|} =$$
$$= \frac{|\langle n, \overrightarrow{OP} \rangle - \langle n, \overrightarrow{OQ} \rangle|}{|n|} =$$

$\overrightarrow{OQ}(x_{1Q}, x_{2Q}, \ldots, x_{nQ}) \Rightarrow \langle n, \overrightarrow{OQ} \rangle = (x_{1Q}, x_{2Q}, \ldots, x_{nQ}) \cdot (a_1, a_2, \ldots, a_n) \overset{Q \in \mathcal{H}}{=\!=} b$

$\overrightarrow{OP}(p_1, p_2, \ldots, p_m) \Rightarrow \langle \overrightarrow{OP}, n \rangle = a_1p_1 + \ldots + a_mp_m$

$n(a_1, \ldots, a_m) \Rightarrow |n| = \sqrt{a_1^2 + \ldots + a_n^2}$

$\Rightarrow$ q.e.d.

Chapter 5

- Area of parallelogram spanned by $\overrightarrow{AB}^{\,v}, \overrightarrow{AC}^{\,w}$: $Area(ABDC) = [v, w]$

$B' = (\overrightarrow{AB}, \overrightarrow{AC})$

$$[v, w] = \begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix} = |\det M_{B B'}|$$

- $$\sin_{or} \measuredangle(v, w) = \frac{[v, w]}{|v| \cdot |w|} = \frac{Area_{or}(ABDC)}{|AB| \cdot |AC|} = \frac{2 Area(ABC)}{|AB| \cdot |AC|}$$

proof: $Area_{or}(ABDC) = [v, w]$

◦ Cross product

Isom. $\vec{v}, \vec{w} \in \mathbb{V}^3 \to \vec{v} \times \vec{w} \in \mathbb{V}^3$

- $\vec{v} \times \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \sin\measuredangle(\vec{v}, \vec{w})$
- $\vec{v} \times \vec{w} \perp \vec{v}$, $\vec{v} \times \vec{w} \perp \vec{w}$
- $(\vec{v}, \vec{w}, \vec{v} \times \vec{w})$ is right oriented

- $(\vec{v}_1 + \vec{v}_2) \times \vec{w} = \vec{v}_1 \times \vec{w} + \vec{v}_2 \times \vec{w}$
- $(\alpha \vec{v}_1) \times \vec{w} = \alpha(\vec{v}_1 \times \vec{w})$
- $(\vec{v} \times \vec{w}) = -\vec{w} \times \vec{v}$
- $\vec{v} \times \vec{v} = \vec{0}$

(when the frame is orthonormal):

$$\vec{v}(x_1, y_1, z_1) \times \vec{w}(x_2, y_2, z_2) = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$A_{ABC} = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$V_{parallelepiped} = [\overrightarrow{AB}, \overrightarrow{AA'}, \overrightarrow{AD}] = \overrightarrow{AB} \cdot (\overrightarrow{AA'} \times \overrightarrow{AD}) = \begin{array}{ccc} a_1 a_2 a_3 & b_1 b_2 b_3 & c_1 c_2 c_3 \end{array} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$V_{tetrahedron} = \frac{1}{6} [\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AA'}]$$

Jacobi: $(a \times b) \times c + (b \times c) \times a + (c \times a) \times b = 0$

- Double cross Formula: $(a \times b) \times c = \langle a, c \rangle \cdot b - \langle b, c \rangle \cdot a$

proof: (using coordinates and every ounce of patience in my body)

Let $a(a_1, a_2, a_3)$, $b(b_1, b_2, b_3)$, $c(c_1, c_2, c_3)$

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k \quad / \times c$$

$$\Rightarrow (a \times b) \times c = \begin{vmatrix} i & j & k \\ a_2 b_3 - a_3 b_2 & a_3 b_1 - b_3 a_1 & a_1 b_2 - a_2 b_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = (a_3 b_1 c_3 - a_1 b_3 c_3 - a_1 b_2 c_2 + a_2 b_1 c_2) i +$$

$$(a_1 b_2 c_1 - a_2 b_1 c_1 - a_2 b_3 c_3 + a_3 b_2 c_3) j +$$

$$(a_2 b_3 c_2 - a_3 b_2 c_2 - a_3 b_1 c_1 + a_1 b_3 c_1) k$$

$$\langle a, c \rangle b - \langle b, c \rangle a = (a_1 b_1 c_1 + a_2 b_1 c_2 + a_3 b_1 c_3) i + (a_1 b_2 c_1 + a_2 b_2 c_2 + a_3 b_2 c_3) j + (a_1 b_3 c_1 + a_2 b_3 c_2 + a_3 b_3 c_3) j -$$

$-(a_1b_1c_1 + a_1b_2c_2 + a_1b_3c_3)i - (a_2b_1c_1 + a_2b_2c_2 + a_2b_3c_3)j - (a_3b_1c_1 - a_3b_2c_2 - a_3b_3c_3)k =$

(it's just tedious, but at least the proof is pretty straightforward)

- Distance between a point and a line: $d(\ell, P) = \dfrac{|\vec{AP} \times v|}{|v|}$, where $v \in D(\ell)$ and $A \in \ell$
- Common $\perp$ of 2 skew lines:

~~Find~~ For ℓ_1, ℓ_2 skew lines, find d s.t. $\begin{cases} d \perp \ell_1 \,;\, d \perp \ell_2 \\ d \cap \ell_1 \neq \emptyset,\ d \cap \ell_2 \neq \emptyset \end{cases}$

the direction of d is given by $\ell_1 \times \ell_2$

Let $\ell_1 \cap d = \{A\}$, $\ell_2 \cap d = \{B\}$ and π_1, π_2 planes ~~s.t. $A, \ell_1, d \subset \pi_1$ and $B, \ell_2, d \subset \pi_2$~~ given by:

$\pi_1 \leftarrow A, \vec{v_1}, \vec{v_1} \times \vec{v_2}$

$\pi_2 \leftarrow B, \vec{v_2}, \vec{v_1} \times \vec{v_2}$

where $\vec{v_1}, \vec{v_2}$ direction vectors of ℓ_1, ℓ_2 respectively

|| Partial II Prep ||

Chapter 6 — Affine maps

Def. A map $\phi: \mathbb{A}^n \to \mathbb{A}^m$ is an **affine map** $\Leftrightarrow \exists A \in \mathcal{M}_{m,n}(\mathbb{R}), b \in \mathcal{M}_{m,1}(\mathbb{R}) : [\phi(P)]_{\mathcal{K}'} = A \cdot [P]_{\mathcal{K}} + b$

Prop. Let $\phi: \mathbb{E}^n \to \mathbb{E}^m$ be an affine map. If a line ℓ is mapped onto a line ℓ' under ϕ, then ϕ preserves the oriented ratio on ℓ, i.e, if A, B, C collinear, then

$$\frac{\overrightarrow{AC}}{\overrightarrow{AB}} = \frac{\overrightarrow{\phi(A)\phi(C)}}{\overrightarrow{\phi(A)\phi(B)}}$$

Prop. ϕ is an affine transformation $\Leftrightarrow$

1. ϕ is injective
2. ϕ preserves lines
3. ϕ preserves the oriented ratio on lines

Homogeneous maps and coordinates

so the exact same coordinates, plus an extra "1"

Def. Let $\mathcal{K}$ be a reference frame in $\mathbb{A}^n$. The homogeneous coordinates of $P(p_1, \dots, p_n)$ are $(p_1, p_2, \dots, p_n, 1)$

Def. The homogeneous matrix of $\phi: \mathbb{A}^n \to \mathbb{A}^m$, defined with respect to $\mathcal{K}$ and $\mathcal{K}'$ by $\phi(x) = Ax + b$, is

$$\hat{\mathcal{M}}_{\mathcal{K}'\mathcal{K}}(\phi) = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

- Advantages of working with homogeneous matrices:

1. Composition of affine maps:

$$\hat{\mathcal{M}}_{\mathcal{K}''\mathcal{K}}(\psi \circ \phi) = \begin{pmatrix} A' & b' \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} = \hat{\mathcal{M}}_{\mathcal{K}''\mathcal{K}'}(\psi) \cdot \hat{\mathcal{M}}_{\mathcal{K}'\mathcal{K}}(\phi)$$

2. Computation of the homogeneous coordinates of $\phi(x)$:

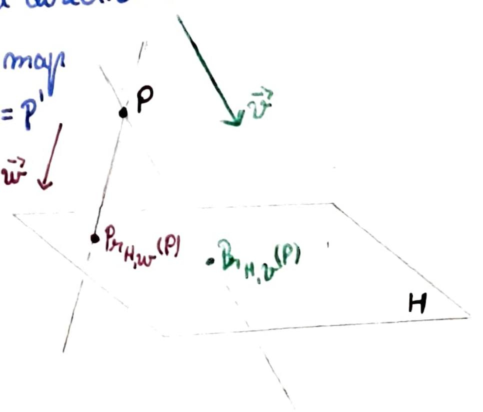
$$\begin{bmatrix} \phi(x) \\ 1 \end{bmatrix} = \hat{\mathcal{M}}_{\mathcal{K}'\mathcal{K}}(\phi) \cdot \begin{bmatrix} x \\ 1 \end{bmatrix}$$

Parallel projections in a hyperplane

Def. Let $H: a_1x_1 + \dots + a_nx_n \underset{+a_{n+1}}{} = 0$ be a hyperplane, a point P and a direction vector $\vec{v}$ NOT PARALLEL to H.

Then, the projection on the hyperplane H parallel to v is the map

$$Pr_{H,v}: \mathbb{A}^n \to \mathbb{A}^n \text{ s.t. } Pr_{H,v}(P) = P'$$

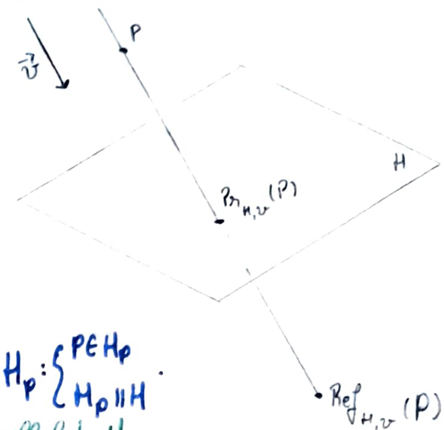


$$\boxed{Pr_{H,v}(P) = \left(I_m - \frac{v\cdot a^T}{v^T\cdot a}\right)\cdot P - \frac{a_{m+1}}{v^T\cdot a}\cdot v} \quad \text{where } a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

parallel reflections in a hyperplane

Def Same setup as for parallel projections. $\exists!\, P'$ s.t. $Pr_{H,v}(P)$ is the midpoint of $[PP']$. We denote $P' = Ref_{H,v}(P)$ and call the map the reflection in the hyperplane H parallel to v

$$\boxed{Ref_{H,v}(P) = \left(I_m - 2\frac{va^T}{v^Ta}\right)P - 2\,\frac{a_{m+1}}{v^T\cdot a}\cdot v}$$

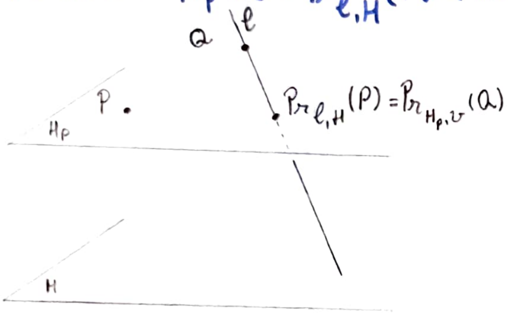


Parallel projections on a line

Def - simplified

Let H be a hyperplane ~~$P\notin H$~~ and ℓ be a line, $\ell \nparallel H$. Then, $\exists!\, H_P : \begin{cases} P\in H_P \\ H_P \parallel H \end{cases}$.

We denote $H_P \cap \ell = Pr_{\ell,H}(P) \to$ the projection on the line ℓ parallel to H

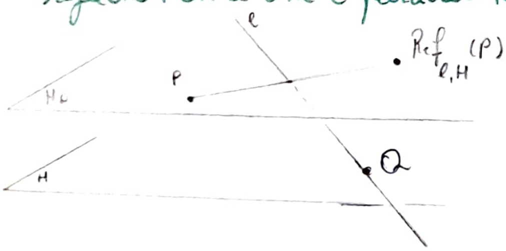


$$\boxed{Pr_{\ell,H}(P) = \frac{v\cdot a^T}{v^T\cdot a}P + \left(I_m - \frac{v\cdot a^T}{v^T\cdot a}\right)Q}$$

where $\begin{cases} \vec{v} \text{ is a direction vector of } \ell \\ Q \text{ is a point on } \ell \end{cases}$ (notice how $Pr_{\ell,H}(P) = Pr_{H_P,\ell}(Q)$)

Parallel reflexion on a line

Def reflection on a line ℓ parallel to H



$$\boxed{Ref_{\ell,H}(P) = \left(2\frac{v\cdot a^T}{v^T\cdot a} - I_m\right)P + 2\left(I_m - \frac{v\cdot a^T}{v^T\cdot a}\right)Q}$$

Chapter 7 - Isometries

Def
An isometry is a map $\phi : \mathbb{E}^n \to \mathbb{E}^n$ which preserves distances: $d(\phi(P), \phi(Q)) = d(P,Q)$

Prop.
Isometries are affine transformations

Prop.
A composition of 2 isometries is an isometry

Prop
Let $\phi(x) = Ax+b$. The following are $\Leftrightarrow$

1. ϕ is an isometry
2. ϕ preserves the length of vectors
3. ϕ preserves the scalar product
4. $A^{-1} = A^T$

Def
$A \in \mathcal{M}_n(\mathbb{R})$ is called orthogonal ($A \in O(n)$) if $\boxed{A^T \cdot A = I_n}$

Def
$SO(n) = \{A \in \mathcal{M}_n(\mathbb{R}) \mid A \in O(n) \text{ and } \det A = 1\}$.

Def
Let $\phi = Ax+b$ be an isometry.

- $A \in SO(n) \Rightarrow \phi$ is a displacement / direct isometry
- $A \notin SO(n)$ ($\det A = -1$) $\Rightarrow \phi$ is an indirect isometry

Isometries in dim 2

Theorem - Chasles
In $\mathbb{E}^2$, the isometries are either

Direct
- the identity
- a translation : $T_{\vec{v}}$
- a rotation : $Rot_{\theta,Q}$

or

Indirect
- a reflection in ℓ : $Ref^{\perp}_{\ell}$
- a glide-reflection in ℓ : $T_{\vec{v}} \circ Ref^{\perp}_{\ell}$ ($\vec{v} \in D(\ell)$)

!tip! use fixed points in order to differentiate between isometries

Lemma
If ϕ is a rotation in $\mathbb{E}^2$, then $\boxed{\cos\theta = \frac{Tr A}{2}}$

Prop
All direct isometries ~~are of the form~~ $\phi = Ax+b$ have the property that:

$$\boxed{A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$$

Isometries in dim. 3

Prop

All direct isometries in $\mathbb{E}^3$ $\phi = Ax+b$ that fix a point are either the identity or a rotation.

For a rotation, the angle θ is such that:

$$\boxed{\cos\theta = \frac{\operatorname{Tr} A - 1}{2}}$$

Classification

In $\mathbb{E}^3$, the isometries are either

Direct
- the identity :
- a translation : $T_{\vec{v}} : P \mapsto P + \vec{v}$
- a rotation (around a line) : $Rot_{\theta,\ell}$
- a glide rotation / helical displacement : $T_{\vec{v}} \circ Rot_{\theta,\ell}$ $(\vec{v} \in D(\ell))$

or

INDIRECT
- a reflection (in a plane)
- a glide reflection : $T_{\vec{v}} \circ Ref_{\pi,\ell}$ $(\vec{v} \in D(\ell))$
- a rotation reflection : $Rot_{\theta,\ell} \circ Ref_{\pi,\ell}$ (but only when $\boxed{\ell \perp \pi}$)

Prop. Euler-Rodrigues

Let v be a unit vector and $\theta \in \mathbb{R}$. Then

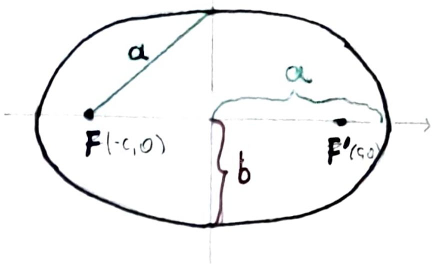
$$\boxed{Rot_{v,\theta}(x) = \cos(\theta)\cdot x + \sin(\theta)\cdot(v \times x) + (1-\cos(\theta))\langle v, x\rangle \cdot v}$$

Chapter 9 - Conics

DEF
- A quadratic curve / conic in $\mathbb{E}^2$ is a curve defined by a quadratic equation:

$$\boxed{ax^2 + bxy + cy^2 + dx + ey + f = 0}$$

1. Ellipse



DEF
- Locus of points M where $MF + MF' = 2a$

$$\boxed{\mathcal{E}_{a,b} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

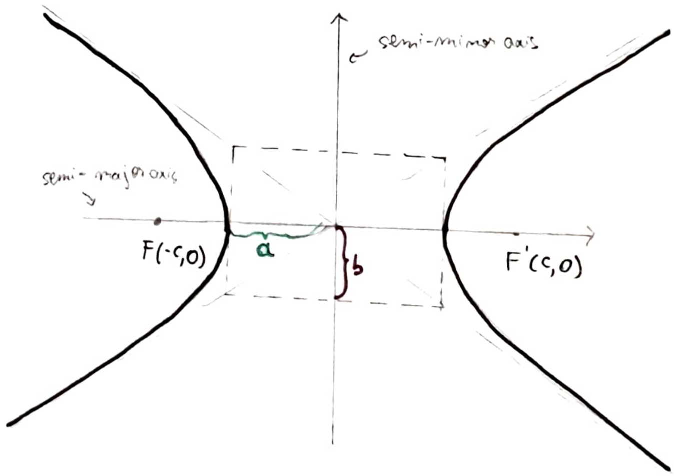
$F(-c,0)$, $F'(0,c) \to$ focal points $\Rightarrow c = \sqrt{a^2 - b^2}$

DEF
- $\varepsilon = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}} \in [0,1) \to$ the eccentricity of $\mathcal{E}_{a,b}$

- Parameterization
  - $\phi : [-a,a] \to \mathbb{E}^2$, $\phi(x) = (x, \frac{b}{a}\sqrt{a^2 - x^2}) \to$ 'northern part'
  - — " — $\phi'(x) = (x, -\frac{b}{a}\sqrt{a^2 - x^2}) \to$ 'southern part'
  - $t \longmapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to$ elliptical motion

- Tangents
  - tangents with a given slope "k": $\boxed{y = kx \pm \sqrt{a^2k^2 + b^2}}$
  - tangent at a given point: $\boxed{T_{(x_0,y_0)}\mathcal{E}_{a,b} : \frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1}$

2. Hyperbola



DEF
- Locus of points M where $|MF - MF'| = 2a$

$$\boxed{\mathcal{H} : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \Rightarrow c = \sqrt{a^2 + b^2}$$

$\varepsilon = \frac{c}{a} = \sqrt{1 + \frac{b^2}{a^2}} \in (1, \infty) \to$ eccentricity of $\mathcal{H}$

- Parametric eq's
  - $y(x) = \pm \frac{b}{a}\sqrt{x^2 - a^2}$

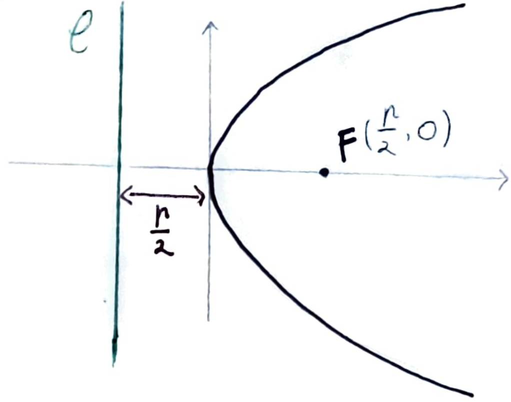
- Tangents
  - tangents with a given slope "k": $y = kx \pm \sqrt{k^2a^2 - b^2}$
  - tangent at a given point: $T_{(x_0,y_0)}\mathcal{H} : \frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$

3. Parabola

∘ Def

Locus of points M where $d(M, \ell) = MF = \frac{p}{2}$ (directrix)

$$\boxed{P : y^2 = 2px}$$



∘ Parametric equations

- $y(x) = \pm\sqrt{2px}$

∘ Tangent lines

- tangent (unique!) with a given slope: $\boxed{kx + \frac{p}{2k}}$
- tangent at a given point: $\boxed{yy_0 = p(x + x_0)}$

Chapter 10 - Classification of conics and quadrics (hyperquadrics)

Def
- A hyperquadric Q in E^n has the following equation:

$$\boxed{Q: \sum_{i,j=1}^{n} a_{i,j} x_i x_j + \sum_{i=1}^{n} b_i x_i + c = 0}$$

$$\boxed{Q: x^T \cdot M(Q) \cdot x + b^T \cdot x + c = 0}$$ → matrix form, where

$M(Q) = (q_{ij})_{i \in \overline{1,n}, j \in \overline{1,n}}$, $q_{ii} = a_{ii}$, $q_{ij} = q_{ji} = \frac{a_{ij}+a_{ji}}{2}$

$b\begin{pmatrix} b_1 \\ b_n \end{pmatrix}$, $x\begin{pmatrix} x_1 \\ x_n \end{pmatrix}$

Reducing to a canonical form:

- Step 1. - Rotation
  - Construct a basis B' of eigenvectors of $M(Q)$ and change the frame from $K(O,B)$ to $K(O,B')$ => $D = M_{BB'}^{-1} \cdot M(Q) \cdot M_{BB'}$; $x = M_{BB'} \cdot y$

$\cancel{Q: x^T M_{BB'} \cdot D \cdot M_{B'B} x + b^T \cdot M_{B'B}}$

$$\Rightarrow \boxed{Q: y^T \cdot D \cdot y + v^T \cdot y + c = 0}$$, where D -> diagonal matrix of eigenvalues; y -> coordinates of x in B'

- Step 2. - Translation
  - Force squares in the coordinates of y => isometric canonical form

Classification of conics

- Example - ellipse

$C: 73x^2 + 72xy + 52y^2 - 10x + 55y + 25 = 0$

$$\Rightarrow C: (x\ y) \cdot \underbrace{\begin{pmatrix} 73 & 36 \\ 36 & 52 \end{pmatrix}}_{M(C)} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{(-10\ 55)}_{b^T} \begin{pmatrix} x \\ y \end{pmatrix} + 25 = 0$$

. Find eigenvalues

$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where $\lambda_{1,2}$ eigenvals of $M(C)$

$\det\begin{pmatrix} 73-\lambda & 36 \\ 36 & 52-\lambda \end{pmatrix} = (73-\lambda)(52-\lambda) - 36^2 = (\lambda-100)(\lambda-25) \Rightarrow \lambda_1 = 100, \lambda_2 = 25$

. Find eigenvectors

$M(C) \cdot u_1 = 100 u_1 \Rightarrow \begin{pmatrix} -27 & 36 \\ 36 & -48 \end{pmatrix} u_1 = 0 \Rightarrow u_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \rightarrow \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ (normalized!)

$M(C) \cdot u_2 = 25 u_2 \Rightarrow u_2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \longrightarrow \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ (normalized!)

. Construct B' so that it is right oriented

For $B' = \left(e_1'\begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix}, e_2'\begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}\right)$, we have $M_{BB'} = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix} \Rightarrow \det M_{BB'} = \frac{1}{25}(-16-9) = -1 < 0 \Rightarrow$ left oriented

$\Rightarrow B' = \left(e_2'\begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}, e_1'\begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix}\right)$, so $M_{BB'} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$

. Perform rotation

$$C: (x'\ y') \cdot \begin{pmatrix} 25 & 0 \\ 0 & 100 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} + \underbrace{(-10\ 55) \cdot \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}}_{b'^T = (-50\ 25)} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} + 25 = 0$$

$\Rightarrow Q: 25x'^2 + 100y'^2 - 50x' + 25y' + 25 = 0 \Rightarrow x'^2 + 4y'^2 - 2x' + y' + 1 = 0$

◦ force squares

$\Rightarrow (x'-1)^2 + 4y'^2 + y' + \frac{1}{16} - \frac{1}{16} = 0 \Rightarrow (x'-1)^2 + (2y' + \frac{1}{4})^2 - \frac{1}{16} = 0$

◦ canonical form

$$\frac{(x'-1)^2}{\frac{1}{16}} + \frac{(2y'+\frac{1}{4})^2}{\frac{1}{16}} = 1 \xrightarrow[y''=2y'+\frac{1}{4}]{x''=x'-1} \frac{x''^2}{(\frac{1}{4})^2} + \frac{(y'')^2}{(\frac{1}{4})^2} = 1 \to \text{ellipse}$$

Non-isometric methods

Lagrange's Method

1. Step 1 - Eliminate mixed terms by completing the squares
2. Step 2 - Eliminate linear terms by completing the squares
3. Step 3 - Reach the form $ax^2+by^2+c=0$ or $ax^2+by+c=0$

r = rank Q	$(p, r-p)$ ← positive eigenvals.	equation	classification
2	(0,2) or (2,0)	$x^2+y^2+1=0$	imaginary ellipse
2	(1,1)	$x^2-y^2+1=0$	hyperbola
2	(2,0) or (0,2)	$x^2+y^2-1=0$	ellipse
2	(0,2) or (2,0)	$x^2+y^2=0$	2 complex lines
2	(1,1)	$x^2-y^2=0$	two real lines
1	(0,1) or (1,0)	$x^2+1=0$	2 complex lines
		$x^2-1=0$	2 real lines
		$x^2=0$	a real double-line
1	(0,1) or (1,0)	$x^2-y=0$	parabola

Direct evaluation

$\hat{\Delta}$	Δ	T	curve C
$\hat{\Delta}=0$	$\Delta>0$	-	A point
	$\Delta=0$	-	2 lines or $\emptyset$
	$\Delta<0$	-	2 lines
$\hat{\Delta}\neq0$	$\Delta>0$	$\hat{\Delta}T<0$	An ellipse
	$\Delta>0$	$\hat{\Delta}T>0$	$\emptyset$
	$\Delta=0$	-	A parabola
	$\Delta<0$	-	A hyperbola

where $\hat{\Delta} = \det(\hat{Q})$

$\Delta = \det(Q)$

$T = \mathrm{Tr}\, Q$

$$\hat{Q} = \begin{pmatrix} q_{11} & q_{12} & b_1 \\ q_{21} & q_{22} & b_2 \\ b_1 & b_2 & c \end{pmatrix}$$

Q ↘ ← b

b^T

Chapter 11

1. Ellipsoid

$$\boxed{\mathcal{E}_{a,b,c} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$$

↳ equation

$$\boxed{T_{(x_0,y_0,z_0)} \mathcal{E}_{a,b,c} : \frac{x x_0}{a^2} + \frac{y y_0}{b^2} + \frac{z z_0}{c^2} = 1}$$

↳ tangent

◦ Parametric eq.

$$\begin{cases} x(\theta_1, \theta_2) = a \cos \theta_1 \cos \theta_2 \\ y(\theta_1, \theta_2) = b \sin \theta_1 \cos \theta_2 \\ z(\theta_1, \theta_2) = c \sin \theta_2 \end{cases}$$

$$\theta_1 \in [0, 2\pi), \ \theta_2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

2. Elliptic Cone

$$\boxed{C_{a,b,c} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1}$$

$$\boxed{T_p C_{a,b,c} : \frac{x_p x_0}{a^2} + \frac{y_p y_0}{b^2} - \frac{z_p z_0}{c^2} = 1}$$

◦ Parametric eq.

$$\begin{cases} x(\theta, h) = h a \cos \theta \\ y(\theta, h) = h b \sin \theta \\ z(\theta, h) = h c \end{cases}$$

$$\theta \in [0, 2\pi), \ h \in \mathbb{R}$$