Formulas PI

• Let $\lambda = \frac{CA}{CB}$. Then, $\boxed{\vec{r}_C = \frac{1}{1+\lambda}\vec{r}_A + \frac{\lambda}{\lambda+1}\vec{r}_B}$

$C \in [AB]$

• The vector eq. of a line "ℓ" through A and B, for $A \neq B$:

$$\boxed{\forall T \in \ell,\ \exists!\, \lambda \in \mathbb{R} : \vec{r}_T = \lambda \vec{r}_A + (1-\lambda)\vec{r}_B}$$

• Euler's line: H, G, U collinear and $\vec{HG} = 2\vec{GU}$

ortho? centroid circumcentre

$[P]_{\mathcal{K}} = [\vec{OP}]_{\mathcal{B}}$

• $M_{B'B} = [id]_{BB'} = ([e_1]_{B'} | \dots | [e_n]_{B'})$ → base change from B to B'

• $[v]_{B'} = M_{B'B} \cdot [v]_B$; $[P]_{\mathcal{K}'} = M_{B'B}([P]_{\mathcal{K}} - [O']_{\mathcal{K}})$, where $\mathcal{K}(O, B)$, $\mathcal{K}'(O', B')$ and

• $\mathcal{K}, \mathcal{K}'$ have the same orientation if $\det M_{BB'} > 0$ and opposite orientations otherwise

• eq. of a plane: $S: \begin{vmatrix} x - x_P & y - y_P & z - z_P \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = 0$ (where v, w lin ind and $D(S) = \langle v, w \rangle$

Let $\begin{cases} \pi_1 : a_1x + b_1y + c_1z + d_1 = 0 \\ \pi_2 : a_2x + b_2y + c_2z + d_2 = 0 \end{cases} \Rightarrow \begin{cases} \pi_1 \parallel \pi_2 \text{ if } \operatorname{rank} M < \operatorname{rank} \tilde{M} \\ \pi_1 \cap \pi_2 = \{\ell\} \text{ (line) if } \operatorname{rank}(M) = 2 \\ \pi_1 = \pi_2 \text{ if } \operatorname{rank} M = \operatorname{rank} \tilde{M} = 1 \end{cases}$

Let $\ell_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$ and $\ell_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} + s \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$. Then:

- $\ell_1 \parallel \ell_2$ if $v = \lambda u$ and $\ell_1 \cap \ell_2 = \emptyset$
- $\ell_1 = \ell_2$ if $v = \lambda u$ and $\exists A \in \{\ell_1 \cap \ell_2\}$
- $\ell_1 \cap \ell_2 \neq \emptyset$ if ℓ_1, ℓ_2 coplanar and $\ell_1 \nparallel \ell_2$

  condition: $\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = 0$

- ℓ_1, ℓ_2 skew otherwise

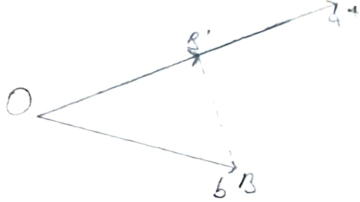
Let $S : \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \sum_{i=1}^{k} t_i \cdot \begin{bmatrix} v_{i1} \\ \vdots \\ v_{in} \end{bmatrix}$ be the eq. of S w/respect to $\mathcal{K}$. Then the eq. of S w/resp. to $\mathcal{K}'$ is: $\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = M_{B'B} \cdot \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + [O]_{\mathcal{K}'} + \sum_{i=1}^{k} t_i M_{B'B} \begin{bmatrix} v_{i1} \\ \vdots \\ v_{in} \end{bmatrix}$

in $\mathbb{R}^2$: Let $\ell : ax + by + c = 0 \Rightarrow A, B$ are on the same side of ℓ if $ax_A + by_A + c$ and $ax_B + by_B + c$ have the same sign.

Let $\pi : ax + by + cz + d = 0 \Rightarrow A, B$ —||— $ax_A + by_A + cz_A + d$ and $ax_B + by_B + cz_B + d$ —||—

$\mathbb{R}^3$

$pr_a^{\perp} : V \to \mathbb{R}$, $pr_a^{\perp}(b) = |\vec{OB'}|$



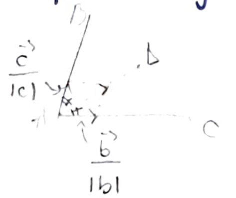
$Pr_a^{\perp} : V \to V$, $Pr_a^{\perp}(b) = \vec{OB'} = pr_a^{\perp}(b) \cdot \frac{a}{|a|}$

- $\cos \measuredangle(a,b) = \frac{pr_a^{\perp}(b)}{|b|} = \frac{pr_b^{\perp}(a)}{|a|}$

$\boxed{Pr_a^{\perp}(v) = \frac{\langle a, v \rangle}{|a|^2} \cdot a}$ (so $pr_a^{\perp}(b) = \frac{\langle a, v \rangle}{|a|}$) $\boxed{pr_a^{\perp}(b) = |b| \cdot \cos \measuredangle(a,b)}$

- $\langle a, b \rangle = |a| \cdot |b| \cdot \cos \measuredangle(a,b)$

Compute the angle bisector of an angle



$\frac{AC}{CD} = \frac{AB}{BD}$ - bisector thm.

$d(P, AB) = d(P, AC), \forall P \in AD$
this yields both the internal and external bisectors
to figure out which is which, pick the one having B and C on opposing sides

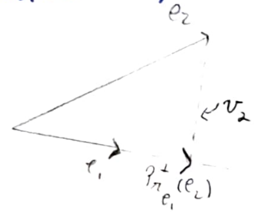
Gram-Schmidt

Let $\mathcal{K} = (O, \mathcal{B})$, $\mathcal{B} = (e_1, \dots, e_n)$. In order to construct $\mathcal{B}'$, an orthonormal frame containing e_i:

1. Construct $(v_1, \dots, v_n) \to$ orthogonal

$v_1 = e_1$

$v_2 = e_2 - \frac{\langle v_1, e_2 \rangle}{\langle v_1, v_1 \rangle} \cdot v_1$ ← $Pr_{v_1}^{\perp}(e_2)$



$v_3 = e_3 - \frac{\langle v_1, e_3 \rangle}{\langle v_1, v_1 \rangle} \cdot v_1 - \frac{\langle v_2, e_3 \rangle}{\langle v_2, v_2 \rangle} v_2$

⋮

2. Normalize $(v_1, v_2, \dots, v_n) \rightsquigarrow \mathcal{B}' = \left(\frac{v_1}{|v_1|}, \frac{v_2}{|v_2|}, \dots, \frac{v_n}{|v_n|}\right)$

so $\boxed{v_i = e_i - \sum_{k=1}^{i-1} Pr_{v_k}^{\perp}(e_i)}$

Let $\mathcal{H} : a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$. then $n(a_1, a_2, \dots, a_n) \perp \mathcal{H}$
normal vector

$\cos \measuredangle(v_1, v_2) = \frac{\langle v_1, v_2 \rangle}{|v_1| \cdot |v_2|} = \frac{\langle n_1, n_2 \rangle}{|n_1| \cdot |n_2|}$

Angles between
- 2 lines:
- 2 Hyperplanes:
- a line with $D(\ell) = \langle v \rangle$ and a hyperplane with a normal vector n → $\frac{\pi}{2} - \arccos\left(\frac{\langle v_1, n_1 \rangle}{|v_1| \cdot |n_1|}\right)$ if $\cos \measuredangle(v_1, n_1) > 0$

  $\frac{\pi}{2} - \arccos\left(\frac{\langle v_1, -n_1 \rangle}{|v_1| \cdot |n_1|}\right)$ otherwise

(in other words, $\measuredangle(v, n)$ is the complement of the angle between ℓ and $\mathcal{H}$)

Let $A(x_A, y_A)$, $B(x_B, y_B)$, $D(x_D, y_D)$ s.t. $ABCD$ parallelogram. Then

~~$\text{Area}(ABCD) = \left|[\vec{AB}, \vec{AD}]\right| = \left\|\begin{matrix} x_{\vec{AB}} & y_{\vec{AB}} \\ x_{\vec{AD}} & y_{\vec{AD}} \end{matrix}\right\| = \left|\det(M_{BB'})\right|$ where $B' = (\vec{AB}, \vec{AD})$~~

$$\text{Area}_{or}(ABCD) = [\vec{AB}, \vec{AD}] = \det\left[[\vec{AB}]_B \mid [\vec{AD}]_B\right] \Rightarrow \sin\measuredangle_{or}(v, w) = \frac{[v, w]}{|v| \cdot |w|}$$

- Cross product ($\mathbb{E}^3$ only)

$\vec{v}, \vec{w} \in V^3 \rightsquigarrow \vec{v} \times \vec{w} \in V^3$

- $|\vec{v} \times \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot \sin\measuredangle(v, w)$
- $\vec{v} \times \vec{w} \perp \vec{v}$, $\vec{v} \times \vec{w} \perp \vec{w}$
- $(\vec{v}, \vec{w}, \vec{v} \times \vec{w})$ - right oriented

- $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
- $(\vec{v_1} + \vec{v_2}) \times \vec{w} = \vec{v_1} \times \vec{w} + \vec{v_2} \times \vec{w}$
- $(\lambda \vec{v_1}) \times \vec{v_2} = \lambda(\vec{v_1} \times \vec{v_2})$
- $\vec{v} \times \vec{v} = \vec{0}$

For orthonormal frames: $\vec{v}(x_1, y_1, z_1) \times \vec{w}(x_2, y_2, z_2) = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$

$$A_{ABC} = \frac{1}{2}|\vec{AB} \times \vec{AC}|$$

$$V_{ABCDA'B'C'D'} = [\vec{AB}, \vec{AD}, \vec{AA'}]$$

$$V_{ABCD} = \frac{1}{6}[\vec{AB}, \vec{AC}, \vec{AD}] = \frac{1}{3} S_{ABC} \cdot d(D, ABC)$$

$$\boxed{[\vec{a}, \vec{b}, \vec{c}] = \vec{a} \cdot \langle \vec{b}, \vec{c} \rangle} = \vec{b} \cdot \langle \vec{c}, \vec{a} \rangle = \vec{c} \cdot \langle \vec{a}, \vec{b} \rangle$$

$$\boxed{[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{vmatrix}} \rightsquigarrow a, b, c \text{ coplanar iff } [\vec{a}, \vec{b}, \vec{c}] = 0$$

Double Cross Formula (Grassman): $\boxed{(a \times b) \times c = \langle a, c \rangle b - \langle b, c \rangle a}$

Jacobi: $(a \times b) \times c + (b \times c) \times a + (c \times a) \times b = 0$

Lagrange: $\langle a \times b, c \times d \rangle = \langle a, c \rangle \cdot \langle b, d \rangle - \langle b, c \rangle \cdot \langle a, d \rangle = \begin{vmatrix} \langle a, c \rangle & \langle a, d \rangle \\ \langle b, c \rangle & \langle b, d \rangle \end{vmatrix}$

misc: $(a \times b) \times (c \times d) = [a, c, d] b - [b, c, d] a = c \cdot [a, b, d] - d \cdot [a, b, c]$

Let ℓ be a line and $A \in \ell \Rightarrow d(P, \ell) = \dfrac{|\vec{AP} \times \vec{v}|}{|v|}$, where $D(\ell) = \langle v \rangle$.

$\langle \vec{AB}, \vec{AC} \rangle = \frac{1}{2}(b^2 + c^2 - a^2)$ → cosines theorem

$\langle \vec{AB}, \vec{AC} \rangle + bc = 2p(p - a)$

$\langle \vec{AB}, \vec{AC} \rangle - bc = (p - b)(p - c) \cdot (-2)$

||Formulas - P II||

- $Pr_{H,v}(P) = \left(I_m - \frac{v\cdot a^T}{v^T\cdot a}\right)\cdot P - \frac{a_{m+1}}{v^T\cdot a}\cdot v$
- $Pr^{\perp}_{H}(P) = \left(I_m - \frac{a\cdot a^T}{|a|^2}\right)P - \frac{a_{m+1}}{|a|^2}\cdot a$
- $Ref_{H,v}(P) = \left(I_m - 2\frac{v\cdot a^T}{v^T\cdot a}\right)P - 2\frac{a_{m+1}}{v^T\cdot a}\, v$
- $Ref^{\perp}_{H}(P) = \left(I_m - 2\frac{a\cdot a^T}{|a|^2}\right)P - \frac{a_{m+1}}{|a|^2}\, a$
- $Pr_{\ell,H}(P) = \frac{v\cdot a^T}{v^T\cdot a}P + \left(I_m - \frac{v\cdot a^T}{v^T\cdot a}\right)a$ ← arbitrary point on ℓ
- $Pr^{\perp}_{\ell}(P) = \frac{a\cdot a^T}{|a|^2}P + \left(I_m - \frac{a\cdot a^T}{|a|^2}\right)a$
- $Ref_{\ell,H}(P) = \left(2\frac{v\cdot a^T}{v^T\cdot a} - I_m\right)P + 2\left(I_m - \frac{v\cdot a^T}{v^T\cdot a}\right)a$
- Homothety $\phi_{c,\lambda}$ with center C, where $\mathcal{K} = (C, \mathcal{B})$

$$[\phi_{c,\lambda}(P)]_{\mathcal{K}} = \lambda [P]_{\mathcal{K}}$$

Isometries in $\mathbb{E}^2$

DIRECT ($\det A = 1$)	INDIRECT ($\det A = -1$)	Angle of rotation:
• identity	• reflection : $Ref^{\perp}_{\ell}$	$\cos\theta = \frac{Tr A}{2}$
• translation : $T_{\vec v}$	• glide-reflection : $T_{\vec v} \circ Ref^{\perp}_{\ell}$ ($\vec v \in D(\ell)$)	
• rotation : $Rot_{\theta, a}$		

Isometries in $\mathbb{E}^3$

DIRECT	INDIRECT	Angle of rotation:
• identity :	• reflection (in a plane)	$\cos\theta = \frac{Tr A - 1}{2}$
• translation : $Tr_{\vec v}$	• glide-reflection : $T_{\vec v} \circ Ref_{\pi,\ell}$ ($\vec v \in D(\ell)$)	
• rotation around a line : $Rot_{\theta,\ell}$	• rotation-reflection : $Rot_{\theta,\ell} \circ Ref_{\pi,\ell}$ ($\ell \perp \pi$)	
• glide rotation : $T_{\vec v} \circ Rot_{\theta,\ell}$ ($\vec v \in D(\ell)$)		

Conics

- $ax^2 + bxy + cy^2 + dx + ey + f = 0$

Eq. \ Conic	Ellipse	Hyperbola	Parabola
Canonic form	$\mathcal{E}_{a,b} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\mathcal{H} : \frac{x^2}{a} - \frac{y^2}{b} = 1$	$\mathcal{P} : y^2 = 2px$
Parametric eq.	$y(x) = \pm\frac{b}{a}\sqrt{a^2 - x^2}$	$y(x) = \pm\frac{b}{a}\sqrt{x^2 - a^2}$	$y(x) = \pm\sqrt{2px}$
Tangent with a given slope	$y = mx \pm \sqrt{m^2a^2 + b^2}$	$y = mx \pm \sqrt{m^2a^2 - b^2}$	$y = mx + \frac{p}{2m}$
Tangent at a given point	$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$	$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$	$yy_0 = p(x + x_0)$

! Ellipses have another parametric equation:

$$t \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$c_{\mathcal{E}} = \sqrt{a^2 - b^2}$

$c_{\mathcal{H}} = \sqrt{a^2 + b^2}$