PROOFS

$\boxed{1.3}$ $\mathcal{L}$ linear $\Leftrightarrow \mathcal{L}(\alpha x+\beta y) = \alpha\mathcal{L}x+\beta\mathcal{L}y$, $\forall x,y \in C^n(I)$, $\forall \alpha,\beta\in\mathbb{R}$

$\mathcal{L}(\alpha x+\beta y) = (\alpha x+\beta y)^{(n)} + a_1(\alpha x+\beta y)^{(n-1)} + \dots + a_{n-1}(\alpha x+\beta y)' + a_n(\alpha x+\beta y)$
$(\alpha x+\beta y)' = (\alpha x)'+(\beta y)' = \alpha x'+\beta y'$. By induction: $(\alpha x+\beta y)^{(n)} = \alpha x^{(n)}+\beta y^{(n)}$ $\Big\} \Rightarrow$

$\Rightarrow \mathcal{L}(\alpha x+\beta y) = \alpha x^{(n)}+\beta y^{(n)} + a_1(\alpha x^{(n-1)}+\beta y^{(n-1)}) + \dots + a_n(\alpha x+\beta y) =$
$= \alpha(x^{(n)}+a_1x^{(n-1)}+\dots+a_nx) + \beta(y^{(n)}+a_1y^{(n-1)}+\dots+a_ny) =$
$= \alpha\mathcal{L}x+\beta\mathcal{L}y$. qed.

$\boxed{1.7}$ The set of solutions of (3) is $\operatorname{Ker}\mathcal{L}$. Thus, we only need to prove that $\underline{\dim\operatorname{Ker}\mathcal{L} = n}$. In order to do so, we need to find an isomorphism between $\operatorname{Ker}\mathcal{L}$ and $\mathbb{R}^n$ (since we know that $\dim\mathbb{R}^n = n$).

Let $\phi: \operatorname{Ker}\mathcal{L}\to\mathbb{R}^n$, $\phi(\varphi) = (\varphi(t_0), \varphi'(t_0), \dots, \varphi^{(n-1)}(t_0))$
From 1.2, we have that ϕ is bijective (φ is the sol. of an IVP). Additionally, ϕ is linear
$\Rightarrow \phi$ isomorphism between $\operatorname{Ker}\mathcal{L}$ and $\mathbb{R}^n \Rightarrow \dim\operatorname{Ker}\mathcal{L} = \dim\mathbb{R}^n = n$. qed.

$\Rightarrow \exists x_1,\dots,x_n$ linearly independent s.t. $(x_1,\dots,x_n)$ is a basis of $\operatorname{Ker}\mathcal{L}$. Then,
$\operatorname{Ker}\mathcal{L} = \{c_1x_1+\dots+c_nx_n \mid c_i\in\mathbb{R}, \forall i\in\overline{1,n}\}$. but $\operatorname{Ker}\mathcal{L}$ is the set of solutions of (3).
so $\forall x$ solution of $\mathcal{L}$, $\exists c_1,\dots,c_n\in\mathbb{R}$: $x = \sum_{i=1}^{n} c_ix_i$.

$\boxed{1.10}$ The set of solutions of $\mathcal{L}x$ is $\operatorname{Ker}\mathcal{L}+\{x_p\}$

$\boxed{1.15}$ i) $\varphi = ce^{-A(t)} \Rightarrow \overset{\exists t_0:}{\varphi(t_0)} = 0 \Leftrightarrow c=0 \Leftrightarrow \varphi = 0, \forall t$
ii) $\varphi = ce^{-A(t)}$
$a(t)\neq 0, \forall t \Rightarrow a(t)>0\ \forall t$ or $a(t)<0\ \forall t$ (since a is continuous)
$\Rightarrow (A(t))' = a(t) \Rightarrow A(t)$ strictly monotonous, so $ce^{-A(t)}$ — // —

$\boxed{2.8}$ Let m_i s.t. $M = (m_1|m_2|\dots|m_n)$. According to the existence and uniqueness thm, $\exists!\ u_i$ s.t. $u_i(t_0) = m_i \Rightarrow \exists!\ U = (u_1|u_2|\dots|u_n)$: $U(t_0) = M$
Assume $\det M\neq 0$ and U not fundamental (by contr.) $\Rightarrow \exists i,j$: u_i and u_j linearly dependent, i.e. $\exists k$ s.t. $u_j = ku_i$, $k\in\mathbb{R}$
but, since $u_i(t_0) = m_i$ and $u_j(t_0) = m_j \Rightarrow ku_i(t_0) = m_j \Rightarrow m_j = k\cdot m_i \Rightarrow$
$\Rightarrow m_i, m_j$ linearly dependent $\Rightarrow \det M = 0$, contradiction!

$\boxed{2.9}$ fundamental $\Rightarrow \det \neq 0$

~~suppose~~ $\cancel{\exists t : U(t) = 0} \Rightarrow \cancel{\exists!}$ since U is fundamental, all its columns are, by definition, linearly independent.

Assume $\exists t_0 : \det U(t_0) = 0 \Rightarrow \exists \xi \in \mathbb{R}^n : U(t_0)\xi = 0$

Let $\varphi : I \to \mathbb{R}^n, \varphi(t) = U(t)\xi \Rightarrow \varphi$ is a solution of $\begin{cases} X' = A(t)X \\ X(t_0) = 0 \end{cases}$ but the null solution is also a sol of the IVP $\Big\} \xrightarrow[\text{Th.m.}]{\text{existence \& uniqueness}}$

$\Rightarrow \varphi = 0, \forall t \Rightarrow U(t)\xi = 0, \forall t \overset{\xi \neq 0}{\Rightarrow} U(t)$ has linearly dependent columns, contradiction! $\Rightarrow \det U(t) \neq 0, \forall t \in I$

$\det \neq 0 \Rightarrow$ fundamental

Since $\det U(t) \neq 0, \forall t \Rightarrow U$ has linearly independent columns, U solution $\Big\} \therefore U$ fundamental

(or just a direct result ~~from~~ of 2.8)

$\boxed{2.10}$ i) $V'(t) = U'(t)M = A(t)U(t)M = A(t)V(t)$

$\det V = \det(U(t)M) = \det(U(t))\det M \neq 0$

ii) $V'(t) = U'(t)M(t) + U(t)M'(t)$ ~~$\Rightarrow A(t)V(t) = A(t)U(t)M(t) + U(t)M(t)$~~

$\Rightarrow \cancel{V'(t)}\ A(t)V(t) = A(t)\underbrace{U(t)M(t)}_{V(t)} + U(t)M'(t) = A(t)V(t) + U(t)M'(t) \Rightarrow$

$\Rightarrow U(t)M'(t) = 0, \forall t \Rightarrow \det(U(t)) \cdot \det(M'(t)) = 0, \forall t$ but U fundamental $\Rightarrow \det U(t) \neq 0 \forall t \Big\} \Rightarrow$

$\Rightarrow \det M'(t) = 0, \forall t \Rightarrow M(t)$ is constant

$\boxed{2.13}$ Prove that $\operatorname{Ker}\mathcal{L} \sim \mathbb{R}^n$

Fix $t_0 \in I$ and define $\phi : \ker\mathcal{L} \to \mathbb{R}^n, \phi(x) = X(t_0)$

$\boxed{2.26}$ i) $\varphi'(t) = \lambda e^{\lambda t} v = e^{\lambda t}\lambda v$

$A\varphi(t) = Ae^{\lambda t}v = e^{\lambda t}(Av) \xrightarrow[\text{corresp. to } \lambda]{v \text{ eig.}} e^{\lambda t}\lambda v \Big\} \Rightarrow \varphi'(t) = A\varphi(t)$, qed.

ii) $\lambda \in \mathbb{R} \Rightarrow \varphi(t) \in \mathbb{R} \overset{i)}{\Rightarrow} \varphi$ solution

iii) i) $\Rightarrow \varphi(t)$ satisfies $X' = AX$ $\cancel{\Rightarrow}$

~~$\varphi(t) = (e^{\alpha}\cos\beta t + e^{\alpha}$~~ $\varphi(t) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(u + iw) = \varphi_1 + i\varphi_2$

$\varphi(t)$ satisfies $X' = AX \Rightarrow (\varphi_1 + i\varphi_2)' = A(\varphi_1 + i\varphi_2) \Rightarrow \underbrace{(\varphi_1' - A\varphi_1)}_{\in \mathbb{R}} + i\underbrace{(\varphi_2' - A\varphi_2)}_{\in \mathbb{R}} = 0$

$\boxed{3.35}$ Assume, by contradiction, that $\exists\, t_0 : \varphi(t_0, \eta) = \eta^*$. Since the dependence of φ on η is not important, we write $\varphi(t,\eta)$ as $\varphi(t) \Rightarrow \varphi(t_0) = \eta^*$

Let $\psi(t) = \varphi(t+t_0)$

we have: $\begin{cases} \varphi'(t) = f(\varphi(t)) \\ \varphi(0) = \eta \end{cases}$ and want to prove that $\begin{cases} \psi'(t) = f(\psi(t)) \\ \psi(0) = \eta^* \end{cases}$

$$\left.\begin{array}{l} \psi(t) = \varphi(t+t_0) \Rightarrow \psi(0) = \varphi(t_0) = \eta^* \\ \cancel{\psi'(t) = f(\psi(t)) = f(\varphi(t+t_0)) = \varphi'(t+t_0)} \\ \psi'(t) = \varphi'(t+t_0) = f(\varphi(t+t_0)) = f(\psi(t)) \end{array}\right\} \Rightarrow \psi \text{ is the solution of } \begin{cases} x' = f(x) \\ x(0) = \eta^* \end{cases}$$

But the unique solution of the IVP is the constant function η^* $\Rightarrow$

$\Rightarrow \psi(t) = \eta^*, \forall t \in \mathbb{R} \Leftrightarrow \varphi(t+t_0) = \eta^*, \forall t \in \mathbb{R} \Rightarrow \varphi(0) = \eta^* \Rightarrow \eta^* = \eta$, contradiction!

$\boxed{3.36}$ By definition, $\dot\varphi(t) = f(\varphi(t)) \Rightarrow \lim\limits_{t\to\infty} \dot\varphi(t) = \lim\limits_{t\to\infty} f(\varphi(t)) \overset{f\text{ cont.}}{=\!=} f(\eta^*)$ (1)

$\varphi(t) = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_m \end{pmatrix}$. For each component, we apply the mean value theorem on intervals of the form $[k, k+1]$

$\exists\, \zeta_k \in [k, k+1] : \varphi(k+1) - \varphi(k) = \varphi'(\zeta_k)(k+1-k) = \varphi'(\zeta_k)$

$$\left.\begin{array}{l} \lim\limits_{k\to\infty} \varphi(k) = \lim\limits_{k\to\infty} \varphi(k+1) = \eta^* \\ \lim\limits_{k\to\infty} \dot\varphi(\zeta_k) \underset{\zeta_k \in [k,k+1]}{\overset{(1)}{=\!=}} f(\eta^*) \end{array}\right\} \Rightarrow f(\eta^*) = \eta^* - \eta^* = 0 \Rightarrow \eta^* \text{ is an equil. pt, qed}$$

$\boxed{4.2}$

i) suppose $A = P\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}$. We want to prove that $\lim\limits_{t\to\infty} e^{tA} = O_2$

$\varphi(t,\eta) = e^{tA}\eta = P\begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} P^{-1}$

i) $\lambda_{1,2} \in \mathbb{R}$. Since $\lambda_{1,2} < 0 \Rightarrow \lim\limits_{t\to\infty} e^{t\lambda_1} = \lim\limits_{t\to\infty} e^{t\lambda_2} = 0$

ii) $\lambda_{1,2} = \alpha \pm i\beta$, $\alpha < 0 \Rightarrow e^{t\lambda} = e^{t\alpha}(\cos\beta t \pm i \sin\beta t) \xrightarrow[t\to\infty]{} 0$ (sin, cos bounded)

ii) similar to i)

iii) we want to prove that $(0,0)$ is stable, i.e. $\forall\, \varepsilon > 0, \exists\, \delta > 0$: whenever $\|\eta\| < \delta$, $\|\varphi(t,\eta)\| < \varepsilon$, $\forall t \in (0,\infty)$.

We denote the euclidian norm by $\|\cdot\|_E$

We know that $\tilde H(\eta) = \|P\eta\|_E^2$ is a global f.i. We define $\|\eta\| = \|P^{-1}\eta\|$ and prove that it is a norm. Thus, since $\|\varphi(t,\eta)\| = \|P^{-1}\varphi(t,\eta)\|_E = \sqrt{\tilde H(\varphi(t,\eta))} = \sqrt{\tilde H(\eta)} = \|\eta\|, \forall t \in \mathbb{R}$, we deduce that $(0,0)$ is stable

1. triangle ineq: $\|\eta + \psi\| \le \|\eta\| + \|\psi\| \Leftrightarrow \|P^{-1}(\eta+\psi)\|_E = \underbrace{\|P^{-1}\eta + P^{-1}\psi\|_E}_{\le \|P^{-1}\eta\|_E + \|P^{-1}\psi\|_E} \le \|\eta\| + \|\psi\|$, true

2. $\|\alpha\eta\| = |\alpha|\,\|\eta\| \Leftrightarrow \|P^{-1}\alpha\eta\|_E = |\alpha|\,\|P^{-1}\eta\|_E = |\alpha|\,\|\eta\|$, True

3. $\|\eta\| = 0 \Leftrightarrow \|P^{-1}\eta\|_E = 0 \Leftrightarrow P^{-1}\eta = 0 \overset{P \text{ invertible}}{\Longleftrightarrow} \eta = 0$

$1,2,3 \Rightarrow \|\cdot\|$ norm, qed.

iv) similar to iii)

$\boxed{4.3}$ If A has the eigenvalues $\pm Bi$, then $\exists P \in \mathcal{M}_2(\mathbb{R})$: $A = P\begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix}P^{-1}$.

Then, $e^{tA} = P e^{t\begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix}} P^{-1}$. The unique sol. of the IVP $\begin{cases} \dot{x} = Ax \\ x(0) = \eta \end{cases}$ is $\varphi(t,\eta) = e^{tA}\eta$

$\Rightarrow P^{-1}\varphi(t,\eta) = e^{t\begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix}} P\eta$

Let $H(x,y) = x^2 + y^2$. Denote $\tilde{H} = H(P^{-1}\eta)$, $\forall \eta \in \mathbb{R}^2$. / We check using the definition that $\tilde{H}$ is a global f.i

we know this is a f.i of $\begin{cases} \dot{x} = -By \\ \dot{y} = By \end{cases}$

$\boxed{4.4}$ Assume η^* is an attractor (the case of a repeller is analogous)

Assume, by contradiction, that $\exists H: \mathbb{R}^2 \to \mathbb{R}$ f.i. $\Rightarrow H(\varphi(t,\eta)) = H(\eta)$, $\forall t \in [0,\infty)$, $\forall \eta \in \mathbb{R}^2 \Rightarrow \lim_{t\to\infty} H(\varphi(t,\eta)) = H(\eta)$, $\forall \eta \in \mathbb{R}^2 \Rightarrow H(\lim_{t\to\infty}\varphi(t,\eta)) = H(\eta) \Rightarrow$

$\Rightarrow H(\eta^*) = H(\eta)$, $\forall \eta \in \mathbb{R}^2 \Rightarrow H$ is constant $\Rightarrow$ contradiction!

$\boxed{4.37}$ H is a f.i. in $\Omega \Leftrightarrow H(\varphi(t,\eta)) = H(\eta)$, $\forall \eta \in \Omega$, $\forall t$ s.t. $\varphi(t,\eta) \in \Omega \Leftrightarrow$

$\Leftrightarrow H(\varphi_1(t,\eta), \varphi_2(t,\eta)) = H(\eta) \Leftrightarrow \frac{\partial H}{\partial x}\varphi(t,\eta)\cdot\dot{\varphi}_1(t,\eta) + \frac{\partial H}{\partial y}\varphi(t,\eta)\cdot\dot{\varphi}_2(t,\eta) =$

$= 0$, $\forall \eta \in \Omega$, t s.t. $\varphi(t,\eta) \in \Omega$ $(*)$

But $\begin{cases} \dot{\varphi}_1(t,\eta) = f_1(\varphi(t,\eta)) \\ \dot{\varphi}_2(t,\eta) = f_2(\varphi(t,\eta)) \end{cases} \Rightarrow (*) \Leftrightarrow \frac{\partial H}{\partial x}\cdot f_1 + \frac{\partial H}{\partial y} f_2 = 0$ in Ω

$\boxed{6.1}$ • First consider the diff. eq. $\frac{dy}{dx} = g(x_0, y_0)$. Fix $(x_0, y_0) \in \mathbb{R}^2$ and consider the solution φ of this diff eq. whose graph contains (x_0, y_0). Then

$$(1)\begin{cases} \varphi'(x) = g(x, \varphi(x)),\ \forall x \in I \\ \varphi(x_0) = y_0 \end{cases}$$

. We know that the slope of the direction field in (x_0, y_0) is $g(x_0, y_0)$. We also know that the slope of the graph of φ in (x_0, y_0) is $\varphi'(x_0)$

But $(1) \Rightarrow \varphi'(x_0) = g(x_0, \varphi(x_0)) = g(x_0, y_0)$, qed.

• Now, consider $\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$. Fix (x_0, y_0). Let $\varphi(\varphi_1, \varphi_2)$ be a solution, whose orbit contains $(x_0, y_0) \Rightarrow$

$$\Rightarrow \begin{cases} \dot{\varphi}_1(t) = f_1(\varphi_1(t), \varphi_2(t)) \\ \dot{\varphi}_2(t) = f_2(\varphi_1(t), \varphi_2(t)) \\ \varphi_1(t_0) = x_0,\ \varphi_2(t_0) = y_0 \end{cases}, \forall t \in I.$$ By definition, (x_0, y_0) is parallel to $(f_1(x_0,y_0), f_2(x_0,y_0))$.

The tangent ~~orbit~~ to the orbit $\begin{cases} x = \varphi_1(t) \\ y = \varphi_2(t) \end{cases}$ in the point (x_0, y_0) is parallel to the vector $(\dot{\varphi}_1(x_0), \dot{\varphi}_2(y_0))$.

But $\varphi_1'(t_0) = f_1(\varphi_1(t_0), \varphi_2(t_0)) = f_1(x_0, y_0)$ and $\varphi_2'(t_0) = f_2(-\,,-) = f_2(x_0, y_0)$, q.e.d.

$\boxed{8.1}$ $x_k = f^k(\eta) \Rightarrow x_{k+1} = f^{k+1}(\eta) = f(x_k)$
$\lim_{k\to\infty} x_k = \eta^* \Rightarrow \lim_{k\to\infty} x_{k+1} = \eta^*$
$\Big\} \Rightarrow \lim_{k\to\infty} f(x_k) = \eta^* \underset{f \text{ continuous}}{\Rightarrow} f(\lim_{k\to\infty} x_k) = \eta^* \Rightarrow$

$\Rightarrow f(\eta^*) = \eta^*$, q.e.d.

$\boxed{8.2}$ 1. Assume that $|f'(\eta^*)| < 1$

Then, $\exists L \in (0,1)$ s.t. $|f'(\eta^*)| < L$. Denote $\varepsilon = L - |f'(\eta^*)| < 1$

$g: \mathbb{R} \to \mathbb{R}$, $g(\eta) = |f'(\eta)|$, $\forall \eta \in \mathbb{R}$. We have that g is continuous in $\eta^* \Rightarrow$ for $\eta > 0$, $\exists \delta > 0$ s.t. if $|\eta - \eta^*| < \delta$, then $|g(\eta) - g(\eta^*)| < \varepsilon$. $\Rightarrow \exists \delta$: if $|\eta - \eta^*| < \delta$, then

$-\varepsilon < g(\eta) - g(\eta^*) < \varepsilon \Rightarrow -L + |f'(\eta)| < g(\eta) - g(\eta^*) < L - |f'(\eta^*)| \Rightarrow$

$\Rightarrow -L + |f'(\eta)| < |f'(\eta)| - |f'(\eta^*)| < L - |f'(\eta^*)| \Rightarrow |f'(\eta)| < L$, when $|\eta - \eta^*| < \delta$ (1).

$\to$ we prove that $|f(\eta) - \eta^*| \le L|\eta - \eta^*|$, when $|\eta - \eta^*| < \delta$.

For that, we use the mean value theorem $\Rightarrow \exists \xi_\eta \in (\eta, \eta^*)$ (or (η^*, η)) s.t.

$f(\eta) - f(\eta^*) = f'(\xi_\eta)(\eta - \eta^*) \xrightarrow{\eta^* \text{ fixed pt.}} f(\eta) - \eta^* = f'(\xi_\eta)(\eta - \eta^*)$

if $|\eta - \eta^*| < \delta \Leftrightarrow \eta \in (\eta^* - \delta, \eta^* + \delta) \Rightarrow \xi_\eta \in (\eta^* - \delta, \eta^* + \delta) \overset{(1)}{\Longrightarrow} f'(\xi_\eta) < L$ $\Big\} \Rightarrow$

$\Rightarrow |f(\eta) - \eta^*| \le L|\eta - \eta^*|$, when $|\eta - \eta^*| < \delta$ (2)

$\to$ we now prove by induction that $|f^k(\eta) - \eta^*| \le L^k(\eta - \eta^*)$, when $|\eta - \eta^*| < \delta$.

I $P(1)$: $|f(\eta) - \eta^*| \le L(\eta - \eta^*)$, true (from 2)

II $P(m) \Rightarrow P(m+1)$:

Let η be s.t. $|\eta - \eta^*| < \delta$. $|f^{m+1}(\eta) - \eta^*| = |f(f^m(\eta)) - \eta^*|$ $\Big\} \overset{(2)}{\Longrightarrow}$ ~~$|f^{m+1}(\eta) - \eta^*| \le L|f^m$~~

From $P(m)$: $|f^m(\eta) - \eta^*| \le L^m|\eta - \eta^*| < 1 \cdot \delta = \delta$

$\Rightarrow |f^{m+1}(\eta) - \eta^*| \le L \cdot |f^m(\eta) - \eta^*| \overset{P(m)}{\le} L \cdot L^m |\eta - \eta^*| = L^{m+1}|\eta - \eta^*|$

$\overset{\mathrm{I},\,\mathrm{II}}{\Longrightarrow}$ By induction, $\forall \eta$ s.t. $|\eta - \eta^*| < \delta$, we have $|f^n(\eta) - \eta^*| \leq L^n \cdot |\eta - \eta^*|$, where $L \in (0,1)$ $\Rightarrow$

$\Rightarrow \lim\limits_{n\to\infty} |f^n(\eta) - \eta^*| = 0 \Rightarrow \lim\limits_{n\to\infty} f^n(\eta) = \eta^* \Rightarrow \eta^*$ is an attractor, qed.

$\boxed{18.3}$ Consider $f(x) = x - \dfrac{g(x)}{g'(x)}$. Then, we need to prove that η^* is an attractor for f

1) $f(\eta^*) = \eta^* - \dfrac{\overset{0}{g(\eta^*)}}{g'(\eta^*)} = \eta^* - 0 = \eta^* \Rightarrow \eta^*$ is a fixed point

2) $f'(x) = 1 - \dfrac{(g'(x))^2 - g(x)\cdot g''(x)}{(g'(x))^2}$

$$= 1 - 1 + \frac{g(x)\cdot g''(x)}{(g'(x))^2} = \frac{g(x)\cdot g''(x)}{(g'(x))^2} \Rightarrow$$

$$\Rightarrow |f'(\eta^*)| = \frac{0\cdot g''(x)}{(g'(x))^2} = 0 < 1 \xRightarrow[\text{method}]{\text{Linearization}} \eta^* \text{ is an attractor, qed.}$$

Tip: if $|f'(\eta^*)| = 0$, η^* is called a SUPER attractor

Special List

1. $\lim_{t\to\infty} \varphi(t)$, $\lim_{t\to\infty} \varphi'(t)$ finite

Let $\lim_{t\to\infty} \varphi(t) = a$ and $\lim_{t\to\infty} \varphi'(t) = b$. suppose $b \neq 0$

I $b > 0 \Rightarrow \exists\, t_0$ s.t. $\varphi'(t) > 0, \forall t \in (t_0, \infty) \Rightarrow \varphi$ strictly increasing on $(t_0, \infty) \Rightarrow \lim_{t\to\infty} \varphi(t) = \infty$, contradiction!

II $b < 0 \Rightarrow \exists\, t_0$ s.t. $\varphi'(t) < 0, \forall t \in (t_0, \infty) \Rightarrow \varphi$ strictly decreasing on $(t_0, \infty) \Rightarrow \lim_{t\to\infty} \varphi(t) = -\infty$, contradiction!

$\Rightarrow b = 0$, qed.

2. $\dot{x} = f(x)$; $\dot{\varphi} = f(\varphi)$; $\lim_{t\to\infty} \varphi(t) = 2$

$\dot{\varphi}(t) = f(\varphi(t)) \Rightarrow \lim_{t\to\infty} \dot{\varphi}(t) = \lim_{t\to\infty} f(\varphi(t)) \overset{f \text{ cont.}}{=} f(\lim_{t\to\infty} \varphi(t)) = f(2)$

$\Rightarrow \lim_{t\to\infty} \dot{\varphi}(t) = f(2)$ (1)

$\forall k \in \mathbb{R} : \exists\, z_k \in [k, k+1)$: ~~f(k+1)-f(k)~~ $\varphi(k+1) - \varphi(k) = \varphi'(z_k)(k+1-k)$
(mean value theorem) $\Rightarrow \lim_{k\to\infty} (\varphi(k+1) - \varphi(k)) = \lim_{k\to\infty} \varphi'(z_k) \underset{(1)}{\overset{z_k \in (k, k+1)}{=\!=}} f(2)$ $\Rightarrow$

but $\lim_{k\to\infty} \varphi(k+1) = \lim_{k\to\infty} \varphi(k) \overset{\text{ip.}}{=} 2$

$\Rightarrow f(2) = 0 \Rightarrow 2$ is an equilibrium point of $\dot{x} = f(x)$

3. $\dot{X} = f(X)$, H is a first integral. ~~Suppos~~

~~Fix η~~ suppose $\exists\, \eta^*$ global attractor and fix $\eta_0 \neq \eta^*$

H f.i $\Rightarrow H(\varphi(t, \eta)) = H(\eta), \forall \eta \in \mathbb{R}^2 \Rightarrow H(\varphi(t, \eta_0)) = H(\eta_0) \Rightarrow$

$\Rightarrow \lim_{t\to\infty} H(\varphi(t, \eta_0)) = H(\eta_0)$.

η^* global attractor $\Rightarrow \forall \eta \in \mathbb{R}, \lim_{t\to\infty} \varphi(t, \eta) = \eta^* \Rightarrow \lim_{t\to\infty} \varphi(t, \eta_0) = \eta^*$ $\Big\} \Rightarrow$

$\Rightarrow \lim_{t\to\infty} H(\varphi(t, \eta_0)) = \eta^* = H(\eta_0)$. Since η_0 was chosen arbitrarily $\Rightarrow$

$\Rightarrow H(\eta) = \eta^*, \forall \eta \in \mathbb{R} \Rightarrow H$ is a constant function, contradiction!
(first integrals can't be constant by definition)

4. i) Mean Value Thm $\Rightarrow \exists\, \xi$ between x and y:

$$f(x)-f(y)=f'(\xi)(x-y) \Rightarrow |f(x)-f(y)|=|f'(\xi)|\cdot|x-y| \le \varepsilon|x-y|$$

~~ii) $-\varepsilon \le f'(\xi) \le \varepsilon$~~

~~suppose $\exists\, \tilde{\eta}^*$ f.p. $\Rightarrow f(\tilde{\eta}^*)=\tilde{\eta}^*$, $f(\eta^*)=\eta^*$, $f'(\eta^*)=$~~

~~$|\eta^*-\tilde{\eta}^*| \le \varepsilon|\eta^*-\tilde{\eta}^*$~~

~~$|f(x)-\eta^*| \le \varepsilon|x-\eta^*|$~~

ii) suppose $\exists\, \tilde{\eta}^*$ another fixed point $\xRightarrow{i)} |f(\eta^*)-f(\tilde{\eta}^*)| \le \varepsilon|\eta^*-\tilde{\eta}^*| \Rightarrow$

$\Rightarrow |\eta^*-\tilde{\eta}^*| \le \varepsilon|\eta^*-\tilde{\eta}^*| \xRightarrow{\eta^*\neq\tilde{\eta}^*} 1 \le \varepsilon$, false $(\varepsilon \in (0,1))$

$\Rightarrow \eta^*$ is the only f.p. of f

iii) prove (by induction) that $|x_k-\eta^*| \le \varepsilon^k|x_0-\eta^*|$

I $P(1)$: $|x_0-\eta^*| \le |x_0-\eta^*|$, true

II $P(k) \Rightarrow P(k+1)$: $|x_{k+1}-\eta^*| \le \varepsilon^{k+1}|x_0-\eta^*|$

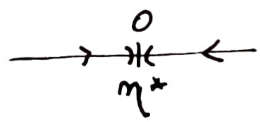
$$x_{k+1}=f(x_k) \Rightarrow |x_{k+1}-\eta^*|=|f(x_k)-\eta^*| \overset{ii}{\le} \varepsilon|x_k-\eta^*| \overset{P(k)}{\le} \varepsilon\cdot\varepsilon^k|x_0-\eta^*|$$

$\xRightarrow{I,II} |x_k-\eta^*| \le \varepsilon^k|x_0-\eta^*| \Rightarrow \lim_{k\to\infty}|x_k-\eta^*| \underset{|x_0-\eta^*|\text{ bounded}}{\overset{\varepsilon\in(0,1)}{=\!=}} 0 \Rightarrow \boxed{\lim_{k\to\infty} x_k=\eta^*},$

regardless of ~~x_0~~ the value of x_0 $\Rightarrow \eta^*$ is a global $\boxed{\text{attractor}}$

5. i) $f(0)=0 \Rightarrow 0$ is an equilibrium point

x	$-\infty$		0		∞
$f(x)$	$+$	$+\ +$	0	$-$	$-\ -$



ii) $f(2)=-5$, $f^2(2)=65$, $f^3(2)=\ldots<0 \Rightarrow (f^k(2))_{k\ge0}$ is divergent (signs alternate and abs. value increases)

$f(1)=-1$, $f^2(1)=1$, $f^3(1)=-1$, $\ldots \Rightarrow \begin{cases} f^{2k}(1)=1, \forall k\in\mathbb{Z} \\ f^{2k+1}(1)=-1, \forall k\in\mathbb{Z} \end{cases}$

iv) $f'(x)=\frac{1}{2}(-3x^2-1) \Rightarrow |f'(0)|=\frac{1}{2}<1 \xRightarrow{(i)} 0$ is an attractor fixed point $\Rightarrow$

but $\lim_{k\to\infty} f^k(1) \neq 0$

$\Rightarrow$ it's not a global attractor