Math 348, Fall 2017
Midterm Exam 1
Oct 11 2017
Time Limit: 50 Minutes

Name: ______________________

EID: ______________________

This exam contains 2 pages and 4 questions. Total of points is 20.

You are allowed to bring a non-programable calculator and one piece of A4 paper for your reference.

Please show your ID upon submission of the test paper.

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1. (6 points) Read each of the the following statements. Are they true or false statements?

   1. The relative error in approximation of $p = 0.01$ by $p^* = 0.012$ is 0.002.

      False.

   2. Suppose a continuous function f is to be evaluated at x_0 in the interval $[0, 1]$, but instead of the actual value $f(x_0)$, what we get is the approximate value $f(x_0 + \delta)$ for some positive number δ. Now assume that $x_0 + \delta$ is also in the interval $[0, 1]$ and $|f'(x)| \leq M$ for all $x \in [0, 1]$ and some $M > 0$, is it true that the error $|f(x_0) - f(x_0 + \delta)|$ is less than or equal $M\delta$?

      True. By mean value theorem, $\exists\, \xi \in (x_0, x_0+\delta)$ s.t. $|f'(\xi)| = \left|\frac{f(x_0+\delta)-f(x_0)}{\delta}\right| \leq M$

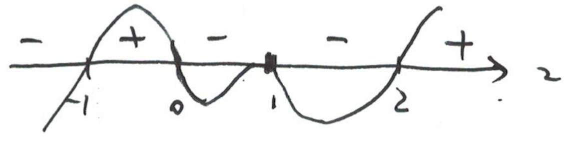
      $\Rightarrow |f(x_0+\delta) - f(x_0)| \leq M\delta$

   3. Is it true that $\frac{2h^2+h^3}{h} = O(h)$ considering h to be sufficiently small?

      True. $\lim_{h\to 0} \left| \frac{\frac{2h^2+h^3}{h}}{h} \right| = \lim_{h \to 0} |2+h| = 2 < \infty$

2. (4 points) Let $f(x) = (x + 1)x(x - 1)^2(x - 2)$. To which zero of f does the Bisection method converge when applied on the interval $[-0.5, 1.5]$? ([Hint: if you are not sure about your answer, write down the first few points that the Bisection method generates.])

   f(x):



   $\begin{cases} a = -0.5 \\ b = 1.5 \end{cases}$ $\frac{f(a) > 0}{f(b) < 0}$ $\Rightarrow$ $\frac{a+b}{2} = 0.5$ $f(0.5) < 0$

   Next $\begin{cases} a = -0.5 \\ b = 0.5 \end{cases}$ Contains only 0.

   So the Bisection method converges to 0.

3. (5 points) Considering the root-finding problem for the function $f(x) = x^2 - 2$ using Newton's method. Does Newton's method converge if it starts with $p_0 \in [1, 2]$? If it does convergence, at which order it converges? Explain your reason.

Answer. Newton's method can be seen as fixed-point iteration

for $F(x) := x - \frac{f(x)}{f'(x)} = x - \frac{x^2-2}{2x} = \frac{1}{2}x + \frac{1}{x}$

So $F'(x) = \frac{1}{2} - \frac{1}{x^2}$ and $F''(x) = \frac{1}{x^3}$

Now we can verify

① $F(\sqrt{2}) = \sqrt{2}$ and $|F'(x)| \le \frac{1}{2}$ for $x \in [1, 2]$

and ① implies ② F maps $[1,2]$ into $[1, 2]$.

③ $F'(\sqrt{2}) = \frac{1}{2} - \frac{1}{2} = 0$.

④ $|F''(x)|$ is bounded by 1 for $x \in [1, 2]$

①, ②, ③ and ④ together, we conclude that the fixed-point iteration converges qudratically.

4. (5 points) A natural cubic spline S on $[1, 3]$ is defined by

$$S(x) = \begin{cases} S_1(x) = 1 + 3x^2 - x^3, & \text{if } 1 \le x < 2 \\ S_2(x) = 5 + b(x-2) + c(x-2)^2 + d(x-2)^3, & \text{if } 2 \le x < 3. \end{cases}$$

Find b, c and d. $[1, 2]$ and $[2, 3]$

Answer b, c and d can be solved by definition of natural splines.

$$\boxed{\begin{aligned} &S_1'(x) = 6x - 3x^2, \quad S_2'(x) = b + 2c(x-2) + 3d(x-2)^2 \\ &S_1''(x) = 6 - 6x, \quad S_2''(x) = 2c + 6d(x-2) \end{aligned}}$$

We must have

① $S_1(2) = S_2(2)$

② $S_1'(2) = S_2'(2)$

③ $S_1''(2) = S_2''(2)$

} Continuity of function, derivative and second derivative

④ $S_1''(1) = 0$, $S_2''(3) = 0$ → boundary condition

① is automatically satisfied. ② implies $6 \times 2 - 3 \times 2^2 = b \Rightarrow \boxed{b = 0}$

③ implies $6 - 6 \times 2 = 2c \Rightarrow \boxed{c = -3}$

④ use $S_2''(3) = 0$ we have $2c + 6d = 0 \Rightarrow \boxed{d = -\frac{1}{3}c = 1}$