Math 348, Fall 2017 **Name:** ____________________

Midterm Exam 2

Nov 15 2017

Time Limit: 50 Minutes **EID:** ____________________

This exam contains 3 pages and 3 questions. Total of points is 20.

You are allowed to bring a non-programable calculator and one piece of A4 paper for your reference.

Please show your ID upon submission of the test paper.

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1. (6 points) Use the data given below to answer the following questions.

x	5.2	5.4	5.6
$f(x)$	10.8	11.5	12.2

(a) Assume f is a smooth function, approximate $f'(5.4)$ using the most accurate formula available. Be sure to indicate what formula you use.

Midpoint rule:

$$f'(5.4) \approx \frac{1}{0.4}\left(f(5.6) - f(5.2)\right) = \frac{1}{0.4}(12.2 - 0.8) = 3.5$$

(b) Given the additional data $f(5.8) = 13.0$, approximate $\int_{5.2}^{5.8} f(x)dx$ using the composite trapezoidal rule.

$$\int_{5.2}^{5.8} f(x)\,dx \approx \frac{0.2}{2}\left[f(5.2) + 2\left(f(5.4) + f(5.6)\right) + f(5.8)\right]$$

$$\simeq 0.1 \times (10.8 + 2 \times (11.5 + 12.2) + 13) = 7.12$$

(c) Can you use the composite Simpson's rule for the computation in part (b)? Why or why not?

5.2 5.4 5.6 5.8

No.

The number of intervals is odd, cannot use composite Simpson's rule

2. (8 points) Use the initial value problem (IVP) given below to answer the following questions.

$$y'(t) = 2y - t + 1, \quad 0 \le t \le 2$$
$$y(0) = 1$$

(a) Explain why the IVP is well-posed. [Hint: Lipschitz continuity]

$f(t,y) = 2y - t + 1$ is Lipschitz Continuous.

Since $|f(t,y) - f(t,z)| = |2(y-z)| \le 2|y-z|$, $L=2$

(b) Assume $Y_0 = 1$. Let $h = 0.1$ and use Euler's method to find Y_1 and Y_2.

$Y_1 = Y_0 + 0.1 \times f(0, 0.1) = 1 + 0.1 \times (2 - 0 + 1) = 1.3$

$Y_2 = Y_1 + 0.1 \times f(0.1, 1.3) = 1.3 + 0.1 \times (2.6 - 0.1 + 1) = 1.65$

(c) Assume $Y_0 = 1$ and $h = 0.1$ again. Use the Midpoint method to find Y_1. [Hint: $\phi(t,y) = f(t + \frac{h}{2}, y + \frac{h}{2} f(t,y))$]

$Y_1 = Y_0 + 0.1 \times f(0 + \frac{0.1}{2}, 1 + \frac{0.5}{2} f(0,0))$

$= 1 + 0.1 \times f(0.05, 1.15)$

$= 1 + 0.1 \times (2 \times 1.15 - 0.05 + 1) = 1.325$

3. (6 points) The Simpson's rule we learned in class is derived from approximating a function by its quadratic interpolating polynomial. It can also be derived through another way. Please go through (a), (b) and (c) to complete the derivation.

(a) Find c_1, c_2 and c_3 such that the following quadrature formula has the highest possible degree of precision.

$$\int_0^1 f(x)dx = c_1 f(0) + c_2 f(\frac{1}{2}) + c_3 f(1).$$

$f(x) = 1 \quad \int_0^1 1\,dx = 1 = C_1 + C_2 + C_3$

$f(x) = x \quad \int_0^1 x\,dx = \frac{1}{2} = \frac{C_2}{2} + C_3 \quad \Rightarrow C_1 = \frac{1}{6}, C_2 = \frac{2}{3}, C_3 = \frac{1}{6}$

$f(x) = x^2 \quad \int_0^1 x^2\,dx = \frac{1}{3} = \frac{C_2}{4} + C_3$

(b) Use the result in part (a) to derive a rule to approximate $\int_a^b f(x)dx$. The resulting quadrature rule is the Simpson's rule applied on the interval $[a, b]$. [Note: there is no credit for simply writing down Simpson's rule.]

$$\int_0^1 f(x)dx = \frac{1}{6}\left[f(0)+4f(\tfrac{1}{2})+f(1)\right]$$

$$[a,b] \xrightarrow{T} [0,1] \qquad x \longrightarrow y \quad \Rightarrow \quad T(x) = y = \frac{x-a}{b-a} \Rightarrow \quad x = (b-a)y+a, \quad dx = (b-a)dy$$

$$\int_a^b f(x)dx = \int_0^1 f((b-a)y+a)(b-a)dy \quad \boxed{\leftarrow \text{change of variable}}$$

$$= \frac{1}{6}\left[f(a)+4f((b-a)\tfrac{1}{2}+a)+f(b)\right](b-a)$$

$$= \frac{b-a}{6}\left[f(a)+4f(\tfrac{a+b}{2})+f(b)\right]$$

(c) Suppose the Simpson's rule on $[a, b]$ has been found from part (b), derive the error term by assuming

$$\int_a^b f(x)dx = \text{Simpson's rule} + kf^{(4)}(\xi). \quad k\cdot 24 \tag{1}$$

Precisely speaking, you can find k by applying the formula (1) with $f(x) = x^4$.

Two ways to get full score: (The first approach is easier in Calculations)

1st Approach: Step 1 (1pt) Consider $a=0$, $b=1$

Then $\int_0^1 f(x)dx = \frac{1}{6}\left[f(0)+4f(\tfrac{1}{2})+f(1)\right] + Cf^{(4)}(\xi)$ for some C

assume $f(x)=x^4 \Rightarrow \int_0^1 x^4 dx = \frac{1}{5} = \frac{1}{6}\times\left[4\times(\tfrac{1}{2})^4+1\right] + C\cdot 24$

$$= \frac{1}{6}\times\frac{5}{4} + 24C$$

$$\Rightarrow C = \frac{1}{24}\left(\frac{1}{5}-\frac{5}{24}\right) = \frac{1}{24}\left(\frac{24}{120}-\frac{25}{120}\right) = -\frac{1}{24}\cdot\frac{1}{120}$$

$$= -\frac{1}{8\times 3}\cdot\frac{1}{4\times 30} = \boxed{-\frac{1}{90}\cdot\frac{1}{2^5}}$$

Step 2 (1 pt) For the general case, use change of variable

$$\int_a^b f(x)dx = \int_0^1 f((b-a)y+a)(b-a)dy = \int_0^1 f((b-a)x+a)(b-a)dx$$

$$F(x) := f((b-a)x+a)(b-a)$$

Then use result from step 1: (1) $\int_0^1 F(x)dx = \frac{1}{6}\left[F(0)+4F\left(\frac{1}{2}\right)+F(1)\right] - \frac{1}{90}\cdot\frac{1}{2^5}\cdot F^{(4)}(\xi)$

Note $F^{(4)}(\xi) = f^{(4)}((b-a)x+a)(b-a)^5$ ⇢ Chain rule

So equation (1) $\Leftrightarrow$ $\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{1}{90}\cdot\left(\frac{b-a}{2}\right)^5\cdot f^{(4)}(\xi)$

$$K = -\frac{1}{90}\left(\frac{b-a}{2}\right)^5$$

2nd Approach: evaluate K directly.

$$\int_a^b x^4 dx = \frac{b^5-a^5}{5} = \frac{b-a}{6}\left[a^4+4\left(\frac{a+b}{2}\right)^4+b^4\right] + K\cdot\boxed{24}$$

$$\Rightarrow K = \frac{1}{24}\left(\frac{b^5-a^5}{5} - \frac{b-a}{6}\left[a^4+4\left(\frac{a+b}{2}\right)^4+b^4\right]\right) \qquad (1\text{ pt})$$

$$= \frac{1}{24}\times\left(\frac{24(b^5-a^5)}{120} - \frac{20(b-a)a^4}{120} - \frac{5(b-a)(a+b)^4}{120} - \frac{20(b-a)b^4}{120}\right)$$

$$= \frac{1}{24}\times\frac{1}{120}\times\left[4b^5-4a^5-20a^4b-20ab^4-5(b-a)(a^4+4a^3b+6a^2b^2+4ab^3+b^4)\right]$$

$$= \frac{1}{90}\times\frac{1}{2^5}\times\left[(4-5)b^5+(-4+5)a^5+(-20-5+20)a^4b+(-20+5-20)ab^4+(-30+20)a^2b^3+(30-20)a^3b^2\right.$$

$$= -\frac{1}{90}\times\frac{1}{2^5}\times\left[b^5-5ab^4+10a^2b^3-10a^3b^2+5a^4b-a^5\right]$$

$$= -\frac{1}{90}\times\frac{1}{2^5}\times(b-a)^5 \qquad (1\text{ pt})$$

Binomial formula:

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$