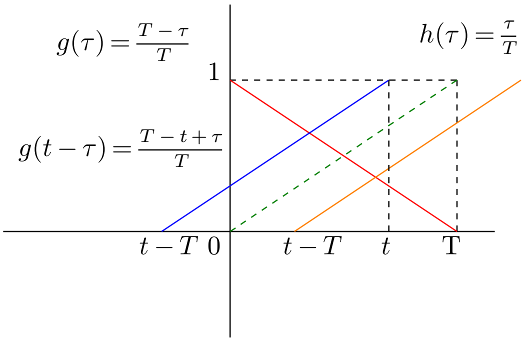
Communication System

Liangchun Xu

1 Convolution

1.1 Convolution of two triangle impulses



when $0 \leqslant t \leqslant T$,

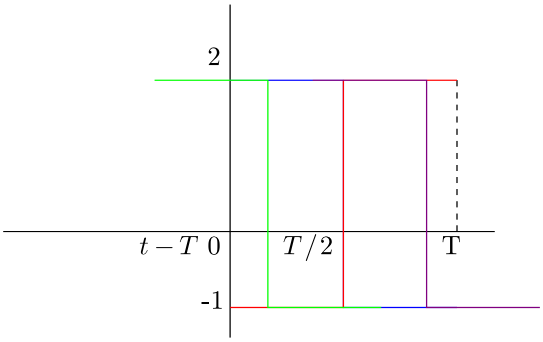
$$\begin{aligned}
\int_{-\infty}^{\infty} h(\tau)g(t-\tau)d\tau &= \int_0^t \frac{\tau}{T} * \frac{T-t+\tau}{T} d\tau \\
&= \frac{1}{T^2}\int_0^t ((T-t)\tau + \tau^2)d\tau \\
&= \frac{1}{T^2}\left((T-t)\frac{\tau^2}{2} + \frac{\tau^3}{3}\right)\Big|_0^t \\
&= \frac{t^2}{2T} - \frac{t^3}{6T^2}
\end{aligned}$$

when $T < t \leqslant 2T$,

$$\begin{aligned}
\int_{-\infty}^{\infty} h(\tau)g(t-\tau)d\tau &= \int_{t-T}^T \frac{\tau}{T} * \frac{T-t+\tau}{T} d\tau \\
&= \frac{1}{T^2}\int_{t-T}^T ((T-t)\tau + \tau^2)d\tau \\
&= \frac{1}{T^2}\left((T-t)\frac{\tau^2}{2} + \frac{\tau^3}{3}\right)\Big|_{t-T}^T \\
&= \frac{(2T-t)^2}{2T} - \frac{(2T-t)^3}{6T^2} \\
&= \frac{(t-T)^3}{6T^2} + \frac{5T}{6} - \frac{t}{2}
\end{aligned}$$

$$h(t) * g(t) = \begin{cases} \frac{t^2}{2T} - \frac{t^3}{6T^2}, 0 \leqslant t \leqslant T \\ \frac{(t-T)^3}{6T^2} + \frac{5T}{6} - \frac{t}{2},\ T < t \leqslant 2T \\ 0,\ \text{others} \end{cases}$$

1.2 Convolution of two step impulses



when $0 \leqslant t \leqslant T/2$,

$$\begin{aligned}\int_{-\infty}^{\infty} h(\tau)g(t-\tau)d\tau &= \int_0^t 2*(-1)d\tau \\ &= -2t\end{aligned}$$

when $T/2 < t \leqslant T$,

$$\begin{aligned}\int_{-\infty}^{\infty} h(\tau)g(t-\tau)d\tau &= \int_0^{t-T/2} 2*2d\tau + \int_{t-T/2}^{T/2} 2*(-1)d\tau + \int_{T/2}^{t} (-1)*(-1)d\tau \\ &= 4\left(t-\frac{T}{2}\right) - 2\left(\frac{T}{2} - \left(t-\frac{T}{2}\right)\right) + t - \frac{T}{2} \\ &= 7t - \frac{9T}{2}\end{aligned}$$

when $T < t \leqslant 3T/2$,

$$\begin{aligned}\int_{-\infty}^{\infty} h(\tau)g(t-\tau)d\tau &= \int_{t-T}^{T/2} 2*2d\tau + \int_{T/2}^{t-T/2} 2*(-1)d\tau + \int_{t-T/2}^{T} (-1)*(-1)d\tau \\ &= 4\left(\frac{T}{2} - (t-T)\right) - 2\left(\left(t-\frac{T}{2}\right) - \frac{T}{2}\right) + T - \left(t-\frac{T}{2}\right) \\ &= \frac{19T}{2} - 7t\end{aligned}$$

when $3T/2 < t \leqslant 2T$,

$$\begin{aligned}\int_{-\infty}^{\infty} h(\tau)g(t-\tau)d\tau &= \int_{t-T}^{T} 2*(-1)d\tau \\ &= -2(T-(t-T)) \\ &= 2t - 4T\end{aligned}$$

$$h(t)*g(t) = \begin{cases} -2t, 0 \leqslant t \leqslant T/2 \\ 7t - \frac{9T}{2},\ T/2 < t \leqslant T \\ \frac{19T}{2} - 7t, T < t \leqslant 3T/2 \\ 2t - 4T, 3T/2 < t \leqslant 2T \\ 0, \text{others} \end{cases}$$

1.3 Convolution of two sinusoid impulses

when $0 \leqslant t \leqslant T$,

$$\begin{aligned}
\int_{-\infty}^{\infty} h(\tau)g(t-\tau)d\tau &= \int_0^t \frac{A}{T}sin(2\pi f_0\tau) * \frac{A}{T}sin(2\pi f_0(T-(t-\tau)))d\tau \\
\text{because}\, T &= \frac{1}{2f_0}, \\
\int_{-\infty}^{\infty} h(\tau)g(t-\tau)d\tau &= \frac{A^2}{T^2}\int_0^t sin(2\pi f_0\tau)sin(2\pi f_0(t-\tau))d\tau \\
&= \frac{A^2}{2T^2}\int_0^t (cos(2\pi f_0(t-2\tau)) - cos(2\pi f_0 t))d\tau \\
&= \frac{A^2}{2T^2}\left(\frac{1}{4\pi f_0}sin(2\pi f_0(2\tau - t)) - \tau cos(2\pi f_0 t)\right)\Bigg|_0^t \\
&= \frac{A^2}{2T^2}\left(\frac{1}{4\pi f_0}2sin(2\pi f_0 t) - t cos(2\pi f_0 t)\right) \\
&= \frac{A^2}{2T^2}\left(\frac{1}{2\pi f_0}sin(2\pi f_0 t) - t cos(2\pi f_0 t)\right) \\
&= 2A^2 f_0^2\left(\frac{1}{2\pi f_0}sin(2\pi f_0 t) - t cos(2\pi f_0 t)\right) \\
&= \frac{A^2 f_0}{\pi}sin(2\pi f_0 t) - 2A^2 f_0^2 t cos(2\pi f_0 t)
\end{aligned}$$

when $T < t \leqslant 2T$,

$$\begin{aligned}
\int_{-\infty}^{\infty} h(\tau)g(t-\tau)d\tau &= \frac{A^2}{T^2}\int_{t-T}^{T} sin(2\pi f_0\tau)sin(2\pi f_0(t-\tau))d\tau \\
&= \frac{A^2}{2T^2}\left(\frac{1}{4\pi f_0}sin(2\pi f_0(2\tau - t)) - \tau cos(2\pi f_0 t)\right)\Bigg|_{t-T}^{T} \\
&= \frac{A^2}{2T^2}\left(\frac{1}{4\pi f_0}(sin(2\pi f_0(2T-t)) - sin(2\pi f_0(t-2T))) - (2T-t)\, cos(2\pi f_0 t)\right) \\
&= \frac{A^2}{2T^2}\left(\frac{1}{2\pi f_0}sin(2\pi f_0(2T-t)) - (2T-t)\, cos(2\pi f_0 t)\right) \\
\text{because}\, T &= \frac{1}{2f_0}, \\
\int_{-\infty}^{\infty} h(\tau)g(t-\tau)d\tau &= \frac{A^2}{2T^2}\left(-\frac{1}{2\pi f_0}sin(2\pi f_0 t) - (2T-t)\, cos(2\pi f_0 t)\right) \\
&= 2A^2 f_0^2\left(-\frac{1}{2\pi f_0}sin(2\pi f_0 t) - \left(\frac{1}{f_0} - t\right) cos(2\pi f_0 t)\right) \\
&= -\frac{A^2 f_0}{\pi}sin(2\pi f_0 t) - (2A^2 f_0 - 2A^2 f_0^2 t)\, cos(2\pi f_0 t)
\end{aligned}$$

$$h(t) * g(t) = \begin{cases} \frac{A^2}{2T^2}\left(\frac{1}{2\pi f_0}sin(2\pi f_0 t) - t cos(2\pi f_0 t)\right), 0 \leqslant t \leqslant T \\ \frac{A^2}{2T^2}\left(-\frac{1}{2\pi f_0}sin(2\pi f_0 t) - (2T-t)\, cos(2\pi f_0 t)\right),\ T < t \leqslant 2T \\ 0,\ \text{others} \end{cases}$$

1.4 Correlation of two sinusoid impulses

$$\begin{aligned}
\int_0^t h^2(\tau)d\tau &= \frac{A^2}{T^2}\int_0^t sin^2(2\pi f_0\tau)d\tau \\
&= \frac{A^2}{T^2}\int_0^t \left(\frac{1 - cos(4\pi f_0\tau)}{2}\right)d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A^2}{2T^2}\left(\tau - \frac{1}{4\pi f_0} sin(4\pi f_0 \tau)\right)\Bigg|_0^t \\
&= \frac{A^2}{2T^2}\left(t - \frac{1}{4\pi f_0} sin(4\pi f_0 t)\right)
\end{aligned}
$$

1.5 Correlation properties

$$
\begin{aligned}
h(n-k) * g(n-k) &= h(\tau) * g(\tau) \\
&= h(n) * g(n)
\end{aligned}
$$