Inertial Navigation

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1 INS Mechanization

1.1 Golden rule

Velocity in e frame,

$$\left.\frac{\mathrm{dr}}{\mathrm{dt}}\right|_e = v_e$$

Golden rule,

$$\left.\frac{\mathrm{dr}}{\mathrm{dt}}\right|_a = \left.\frac{\mathrm{dr}}{\mathrm{dt}}\right|_b + \omega_{\mathrm{ab}} \times r$$

Acceleration measurement,

$$\left.\frac{d^2 r}{\mathrm{dt}^2}\right|_i = f + g$$

Acceleration in i frame,

$$\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i = \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e + \omega_{\mathrm{ie}} \times v_e$$

1.2 Velocity in i Frame

$$\left.\frac{dr}{\mathrm{dt}}\right|_i = \left.\frac{\mathrm{dr}}{\mathrm{dt}}\right|_e + \omega_{\mathrm{ie}} \times r = v_e + \omega_{\mathrm{ie}} \times r$$

Take the derivative of both sides,

$$\left.\frac{d^2 r}{\mathrm{dt}^2}\right|_i = \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i + \left.\frac{d(\omega_{\mathrm{ie}} \times r)}{\mathrm{dt}}\right|_i$$

Because

$$\begin{aligned}
\left.\frac{d(\omega_{\mathrm{ie}} \times r)}{\mathrm{dt}}\right|_i &= \left.\frac{d\omega_{\mathrm{ie}}}{\mathrm{dt}}\right|_i \times r + \omega_{\mathrm{ie}} \times \left.\frac{\mathrm{dr}}{\mathrm{dt}}\right|_i \\
&= 0 \times r + \omega_{\mathrm{ie}} \times (v_e + \omega_{\mathrm{ie}} \times r) \\
&= \omega_{\mathrm{ie}} \times v_e + \omega_{\mathrm{ie}} \times (\omega_{\mathrm{ie}} \times r)
\end{aligned}$$

Then

$$\begin{aligned}
\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i &= \left.\frac{d^2 r}{\mathrm{dt}^2}\right|_i - \left.\frac{d(\omega_{\mathrm{ie}} \times r)}{\mathrm{dt}}\right|_i \\
&= f + g - (\omega_{\mathrm{ie}} \times v_e + \omega_{\mathrm{ie}} \times (\omega_{\mathrm{ie}} \times r)) \\
&= f - \omega_{\mathrm{ie}} \times v_e + (g - \omega_{\mathrm{ie}} \times (\omega_{\mathrm{ie}} \times r)) \\
&= f - \omega_{\mathrm{ie}} \times v_e + g_l
\end{aligned}$$

Projected into i frame,

$$\begin{aligned}\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i^i &= f^i - \omega_{\mathrm{ie}}^i \times v_e^i + g_l^i \\ &= C_b^i f^b - \omega_{\mathrm{ie}}^i \times v_e^i + g_l^i\end{aligned}$$

1.3 Velocity in e Frame

Using Golden rule,

$$\begin{aligned}\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i &= \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e + \omega_{\mathrm{ie}} \times v_e \\ f - \omega_{\mathrm{ie}} \times v_e + g_l &= \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e + \omega_{\mathrm{ie}} \times v_e \\ \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e &= f - 2\omega_{\mathrm{ie}} \times v_e + g_l\end{aligned}$$

Projected into e frame,

$$\begin{aligned}\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e^e &= f^e - 2\omega_{\mathrm{ie}}^e \times v_e^e + g_l^e \\ &= C_b^e f^b - 2\omega_{\mathrm{ie}}^e \times v_e^e + g_l^e\end{aligned}$$

1.4 Velocity in n Frame

Using Golden rule,

$$\begin{aligned}\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i &= \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_n + \omega_{\mathrm{in}} \times v_e \\ f - \omega_{\mathrm{ie}} \times v_e + g_l &= \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_n + \omega_{\mathrm{in}} \times v_e \\ \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e &= f - \omega_{\mathrm{ie}} \times v_e + g_l - (\omega_{\mathrm{ie}} + \omega_{\mathrm{en}}) \times v_e \\ \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e &= f - (2\omega_{\mathrm{ie}} + \omega_{\mathrm{en}}) \times v_e + g_l\end{aligned}$$

Projected into n frame,

$$\begin{aligned}\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_n^n &= f^n - (2\omega_{\mathrm{ie}}^n + \omega_{\mathrm{en}}^n) \times v_e^n + g_l^n \\ &= C_b^n f^b - (2\omega_{\mathrm{ie}}^n + \omega_{\mathrm{en}}^n) \times v_e^n + g_l^n \qquad (1)\end{aligned}$$

Updated velocity

$$\begin{aligned}v_e^n(t_k) &= v_e^n(t_{k-1}) + \int_{t_{k-1}}^{t_k} C_b^n(t) f^b(t)\mathrm{dt} + \int_{t_{k-1}}^{t_k} (g_l^n - (2\omega_{\mathrm{ie}}^n + \omega_{\mathrm{en}}^n) \times v_e^n)\mathrm{dt} \\ &\approx v_e^n(t_{k-1}) + \Delta v_f^n(t_k) + \Delta v_{g/\mathrm{cor}}^n(t_k) \qquad (2)\end{aligned}$$

where

$$
\begin{aligned}
\Delta v_f^n(t_k) &= \int_{t_{k-1}}^{t_k} C_b^n(t) f^b(t)\mathrm{dt} \\
&= \int_{t_{k-1}}^{t_k} C_{n(t_{k-1})}^{n(t)} C_{b(t_{k-1})}^{n(t_{k-1})} C_{b(t)}^{b(t_{k-1})} f^b(t)\mathrm{dt} \\
&= C_{n(t_{k-1})}^{n(t)} C_{b(t_{k-1})}^{n(t_{k-1})} \int_{t_{k-1}}^{t_k} C_{b(t)}^{b(t_{k-1})} f^b(t)\mathrm{dt} \\
&\approx (I - (0.5\boldsymbol{\zeta}_k \times)) C_{b(t_{k-1})}^{n(t_{k-1})} \Delta v_f^b(t_k)
\end{aligned}
$$

By definition,

$$
\begin{aligned}
I - (0.5\boldsymbol{\zeta}_k \times) &= I - 0.5 \begin{pmatrix} 0 & -\boldsymbol{\zeta}_k[2] & \boldsymbol{\zeta}_k[1] \\ \boldsymbol{\zeta}_k[2] & 0 & -\boldsymbol{\zeta}_k[0] \\ -\boldsymbol{\zeta}_k[1] & \boldsymbol{\zeta}_k[0] & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0.5\boldsymbol{\zeta}_k[2] & -0.5\boldsymbol{\zeta}_k[1] \\ -0.5\boldsymbol{\zeta}_k[2] & 1 & 0.5\boldsymbol{\zeta}_k[0] \\ 0.5\boldsymbol{\zeta}_k[1] & -0.5\boldsymbol{\zeta}_k[0] & 1 \end{pmatrix}
\end{aligned}
$$

Also we have

$$
\begin{aligned}
\boldsymbol{\zeta}_k &= [\omega_{\mathrm{ie}}^n + \omega_{\mathrm{en}}^n]_{k-1/2} \Delta t_k \\
C_e^n &= R_y(-\varphi - \pi/2) R_z(\lambda) \\
&= \begin{pmatrix} -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \\ -\sin\lambda & \cos\lambda & 0 \\ -\cos\varphi\cos\lambda & -\cos\varphi\sin\lambda & -\sin\varphi \end{pmatrix} \\
\omega_e &= 7.2921151467 \times 10^{-5}\,\mathrm{rad}/s \\
\omega_{\mathrm{ie}}^e &= \begin{pmatrix} 0 & 0 & \omega_e \end{pmatrix}^T \\
\omega_{\mathrm{ie}}^n &= C_e^n \omega_{\mathrm{ie}}^e \\
&= \begin{pmatrix} \omega_e\cos\varphi & 0 & -\omega_e\sin\varphi \end{pmatrix}^T \\
\omega_{\mathrm{en}}^n &= \begin{pmatrix} \dot{\lambda}\cos\varphi \\ -\dot{\varphi} \\ -\dot{\lambda}\sin\varphi \end{pmatrix} \\
&= \begin{pmatrix} v_E/(R_N + h) \\ -v_N/(R_M + h) \\ -v_E\tan\varphi/(R_N + h) \end{pmatrix} \\
R_N &= \frac{a}{(1 - e^2\sin^2\varphi)^{1/2}} \\
R_M &= \frac{a(1 - e^2)}{(1 - e^2\sin^2\varphi)^{3/2}} \\
a &= 6378137.0 \\
f &= \frac{a - b}{a} \\
&= 1.0/298.257223563
\end{aligned}
$$

Extraploating the position,

$$
\begin{aligned}
h_{k-1/2} &= h_{k-1} - \frac{v_D(t_{k-1})\Delta t_k}{2} \\
q_{n(k-1/2)}^{e(k-1)} &= q_{n(k-1)}^{e(k-1)} \star q_{n(k-1/2)}^{n(k-1)} \\
q_{n(k-1/2)}^{e(k-1/2)} &= q_{e(k-1)}^{e(k-1/2)} \star q_{n(k-1/2)}^{e(k-1)} \\
&= q_{e(k-1)}^{e(k-1/2)} \star \left(q_{n(k-1)}^{e(k-1)} \star q_{n(k-1/2)}^{n(k-1)} \right)
\end{aligned}
$$

q_n^e in terms of latitude, longitude, and altitude:

$$q_n^e = \begin{pmatrix} \cos\left(-\frac{\pi}{4}-\frac{\varphi}{2}\right)\cos\left(\frac{\lambda}{2}\right) \\ -\sin\left(-\frac{\pi}{4}-\frac{\varphi}{2}\right)\sin\left(\frac{\lambda}{2}\right) \\ \sin\left(-\frac{\pi}{4}-\frac{\varphi}{2}\right)\cos\left(\frac{\lambda}{2}\right) \\ \cos\left(-\frac{\pi}{4}-\frac{\varphi}{2}\right)\sin\left(\frac{\lambda}{2}\right) \end{pmatrix}$$

where longitude, λ, is ranged between$(-\pi,\pi]$, latitude, φ, is ranged between $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

when $\lambda=\pi$,

$$\begin{aligned} q_n^e &= \begin{pmatrix} 0 \\ -\sin\left(-\frac{\pi}{4}-\frac{\varphi}{2}\right) \\ 0 \\ \cos\left(-\frac{\pi}{4}-\frac{\varphi}{2}\right) \end{pmatrix} \\ \varphi &= 2*\left(-\frac{\pi}{4}-\arctan\left(-\frac{q_2}{q_4}\right)\right) \end{aligned}$$

when $\varphi=\frac{\pi}{2}$,

$$\begin{aligned} q_n^e &= \begin{pmatrix} 0 \\ \sin\left(\frac{\lambda}{2}\right) \\ -\cos\left(\frac{\lambda}{2}\right) \\ 0 \end{pmatrix} \\ \lambda &= 2*\arctan\left(-\frac{q_2}{q_3}\right) \end{aligned}$$

otherwise,

$$\begin{aligned} \lambda &= 2*\arctan\left(\frac{q_4}{q_1}\right) \\ \varphi &= 2*\left(-\frac{\pi}{4}-\arctan\left(\frac{q_3}{q_1}\right)\right) \end{aligned}$$

where

$$\begin{aligned} q_{n(k-1/2)}^{n(k-1)} &= \begin{pmatrix} \cos\|0.5\boldsymbol{\zeta}_{k-1/2}\| \\ \frac{0.5\boldsymbol{\zeta}_{k-1/2}}{\|0.5\boldsymbol{\zeta}_{k-1/2}\|}\sin\|0.5\boldsymbol{\zeta}_{k-1/2}\| \end{pmatrix} \\ q_{e(k-1)}^{e(k-1/2)} &= \begin{pmatrix} \cos\|0.5\xi_{k-1/2}\| \\ -\frac{0.5\xi_{k-1/2}}{\|0.5\xi_{k-1/2}\|}\sin\|0.5\xi_{k-1/2}\| \end{pmatrix} \\ \boldsymbol{\zeta}_{k-1/2} &= \omega_{\mathrm{in}}^n(t_{k-1})\Delta t_k/2 \\ \xi_{k-1/2} &= \omega_{\mathrm{ie}}^n\Delta t_k/2 \end{aligned}$$

Extraploating the velocity,

$$\begin{aligned} \Delta v_e^n(t_{k-1}) &= \Delta v_f^n(t_{k-1})+\Delta v_{g/\mathrm{cor}}^n(t_{k-1}) \\ v_e^n(t_{k-1/2}) &= v_e^n(t_{k-1})+\frac{1}{2}\Delta v_e^n(t_{k-1}) \\ &= v_e^n(t_{k-1})+\frac{1}{2}(\Delta v_f^n(t_{k-1})+\Delta v_{g/\mathrm{cor}}^n(t_{k-1})) \end{aligned}$$

Velocity correction of the gravity and coriolis terms,

$$\begin{aligned}\Delta v^n_{g/\mathrm{cor}}(t_k) &= [g^n_l - (2\omega^n_{\mathrm{ie}} + \omega^n_{\mathrm{en}}) \times v^n_e]_{k-1/2} \Delta t_k \\ g^n_l &= (\ 0\ \ 0\ \ g\ )^T\end{aligned}$$

$$g = g_0(1 + 5.27094 * 10^{-3} \sin^2\varphi + 2.32718 * 10^{-5} \sin^4\varphi) - 3.086 * 10^{-6} h$$

Since we have

$$\begin{aligned}C^{b(t_{k-1})}_{b(t)} &\approx I + [\Delta\theta(t) \times] \\ \Delta\theta(t) &= \int_{t_{k-1}}^{t} \omega^b_{\mathrm{ib}}(t)\mathrm{dt} && (3) \\ \Delta v(t) &= \int_{t_{k-1}}^{t} f^b(t)\mathrm{dt} && (4) \\ \Delta\theta(t_{k-1}) &= \Delta v(t_{k-1}) = 0 && (5)\end{aligned}$$

where

$$\begin{aligned}\Delta v^b_f(t_k) &= \int_{t_{k-1}}^{t_k} C^{b(t_{k-1})}_{b(t)} f^b(t)\mathrm{dt} \\ &\approx \int_{t_{k-1}}^{t_k} (I + [\Delta\theta(t) \times]) f^b(t)\mathrm{dt} \\ &= \int_{t_{k-1}}^{t_k} f^b(t)\mathrm{dt} + \int_{t_{k-1}}^{t_k} (\Delta\theta(t) \times f^b(t))\mathrm{dt} \\ &= \Delta v(t_k) + \int_{t_{k-1}}^{t_k} (\Delta\theta(t) \times f^b(t))\mathrm{dt}\end{aligned}$$

Furthermore,

$$\begin{aligned}\Delta\theta(t) \times f^b(t) &= \Delta\theta(t) \times \Delta\dot{v}(t) \\ &= \frac{d}{\mathrm{dt}}(\Delta\theta(t) \times \Delta v(t)) - \Delta\dot{\theta}(t) \times \Delta v(t) && (6) \\ &= \frac{1}{2}\frac{d}{\mathrm{dt}}(\Delta\theta(t) \times \Delta v(t)) + \frac{1}{2}(\Delta\dot{\theta}(t) \times \Delta v(t) + \Delta\theta(t) \times \Delta\dot{v}(t)) - \Delta\dot{\theta}(t) \times \Delta v(t) \\ &= \frac{1}{2}\frac{d}{\mathrm{dt}}(\Delta\theta(t) \times \Delta v(t)) + \frac{1}{2}(-\Delta\dot{\theta}(t) \times \Delta v(t) + \Delta\theta(t) \times \Delta\dot{v}(t)) \\ &= \frac{1}{2}\frac{d}{\mathrm{dt}}(\Delta\theta(t) \times \Delta v(t)) + \frac{1}{2}(\Delta v(t) \times \Delta\dot{\theta}(t) + \Delta\theta(t) \times \Delta\dot{v}(t)) \\ &= \frac{1}{2}\frac{d}{\mathrm{dt}}(\Delta\theta(t) \times \Delta v(t)) + \frac{1}{2}(\Delta v(t) \times \omega^b_{\mathrm{ib}}(t) + \Delta\theta(t) \times f^b(t))\end{aligned}$$

Then

$$\begin{aligned}\int_{t_{k-1}}^{t_k} (\Delta\theta(t) \times f^b(t))\mathrm{dt} &= \frac{1}{2}(\Delta\theta(t_k) \times \Delta v(t_k) - \Delta\theta(t_{k-1}) \times \Delta v(t_{k-1})) + \\ &\quad \frac{1}{2}\int_{t_{k-1}}^{t_k} (\Delta v(t) \times \omega^b_{\mathrm{ib}}(t) + \Delta\theta(t) \times f^b(t))\mathrm{dt} \\ &= \frac{1}{2}(\Delta\theta(t_k) \times \Delta v(t_k)) + \\ &\quad \frac{1}{2}\int_{t_{k-1}}^{t_k} (\Delta v(t) \times \omega^b_{\mathrm{ib}}(t) + \Delta\theta(t) \times f^b(t))\mathrm{dt}\end{aligned}$$

Assuming the angular velocity and acceleration are linear during time span $t_{k-1} \sim t_k$ and $t_{k-2} \sim t_{k-1}$

$$\begin{aligned}\omega^b_{\mathrm{ib}}(t) &= a + 2b(t - t_{k-1}) \\ f^b(t) &= A + 2B(t - t_{k-1})\end{aligned}$$

Using angular velocities and accelerations during corresponding time span to resolve the coefficients,

$$\begin{aligned}
\Delta\theta(t_k) &= \int_{t_{k-1}}^{t_k} \omega_{\mathrm{ib}}^b(t)\mathrm{dt} \\
&= \int_{t_{k-1}}^{t_k} (a + 2b(t - t_{k-1}))\mathrm{dt} \\
\Delta v(t_k) &= \int_{t_{k-1}}^{t_k} f^b(t)\mathrm{dt} \\
&= \int_{t_{k-1}}^{t_k} (A + 2B(t - t_{k-1}))\mathrm{dt}
\end{aligned}$$

Plugging into the integral above,

$$\frac{1}{2}\int_{t_{k-1}}^{t_k} (\Delta v(t) \times \omega_{\mathrm{ib}}^b(t) + \Delta\theta(t) \times f^b(t))\mathrm{dt} \;=\; \frac{1}{12}(\Delta v(t_{k-1}) \times \Delta\theta(t_k) + \Delta\theta(t_{k-1}) \times \Delta v(t_k))$$

Summary of veloctiy update,

$$\begin{aligned}
v_e^n(t_k) &= v_e^n(t_{k-1}) + \int_{t_{k-1}}^{t_k} C_b^n(t) f^b(t)\mathrm{dt} + \int_{t_{k-1}}^{t_k} (g_l^n - (2\omega_{\mathrm{ie}}^n + \omega_{\mathrm{en}}^n) \times v_e^n)\mathrm{dt} \\
&= v_e^n(t_{k-1}) + \Delta v_f^n(t_k) + \Delta v_{g/\mathrm{cor}}^n(t_k) \\
&\approx v_e^n(t_{k-1}) + (I - (0.5\boldsymbol{\zeta}_k \times))C_{b(t_{k-1})}^{n(t_{k-1})} \Delta v_f^b(t_k) + \Delta v_{g/\mathrm{cor}}^n(t_k) \\
&\approx v_e^n(t_{k-1}) + (I - (0.5\boldsymbol{\zeta}_k \times))C_{b(t_{k-1})}^{n(t_{k-1})} \left(\Delta v(t_k) + \int_{t_{k-1}}^{t_k} (\Delta\theta(t) \times f^b(t))\mathrm{dt} \right) + \Delta v_{g/\mathrm{cor}}^n(t_k) \\
&= v_e^n(t_{k-1}) + (I - (0.5\boldsymbol{\zeta}_k \times))C_{b(t_{k-1})}^{n(t_{k-1})} \Bigg(\Delta v(t_k) + \Bigg(\frac{1}{2}(\Delta\theta(t_k) \times \Delta v(t_k)) + \frac{1}{2}\int_{t_{k-1}}^{t_k} (\Delta v(t) \times \\
&\quad \omega_{\mathrm{ib}}^b(t) + \Delta\theta(t) \times f^b(t))\mathrm{dt} \Bigg) \Bigg) + \Delta v_{g/\mathrm{cor}}^n(t_k) \\
&\approx v_e^n(t_{k-1}) + (I - (0.5\boldsymbol{\zeta}_k \times))C_{b(t_{k-1})}^{n(t_{k-1})} \Bigg(\Delta v(t_k) + \Bigg(\frac{1}{2}(\Delta\theta(t_k) \times \Delta v(t_k)) + \frac{1}{12}(\Delta v(t_{k-1}) \times \\
&\quad \Delta\theta(t_k) + \Delta\theta(t_{k-1}) \times \Delta v(t_k)) \Bigg) \Bigg) + [g_l^n - (2\omega_{\mathrm{ie}}^n + \omega_{\mathrm{en}}^n) \times v_e^n]_{k-1/2} \Delta t_k
\end{aligned}$$

1.5 Position update

Position in n frame,

$$\dot{r}^n = \begin{pmatrix} \frac{1}{R_M + h} & 0 & 0 \\ 0 & \frac{1}{(R_N + h)\cos\varphi} & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_n \\ v_e \\ v_d \end{pmatrix} = D^{-1} v^n$$

$$r^n(t_{k+1}) \;=\; r^n(t_k) + \frac{1}{2} \begin{pmatrix} \frac{1}{R_M + h} & 0 & 0 \\ 0 & \frac{1}{(R_N + h)\cos\varphi} & 0 \\ 0 & 0 & -1 \end{pmatrix}_{k-\frac{1}{2}} (v_e^n(t_k) + v_e^n(t_{k+1}))\Delta t$$

1.6 Attitude Update

Quaternion in terms of Euler angle (ZYX):

Rotating with Z axis, $\phi = \theta = 0$, quaternion representation: $q_\psi = \cos\frac{\psi}{2} - \mathrm{k}\sin\frac{\psi}{2}$

Rotating with Y axis, $\phi=\psi=0$, **quaternion representation:** $\boldsymbol{q_\theta=\cos\frac{\theta}{2}-\mathrm{j}\sin\frac{\theta}{2}}$

Rotating with X axis, $\psi=\theta=0$, **quaternion representation:** $\boldsymbol{q_\phi=\cos\frac{\phi}{2}-\mathrm{i}\sin\frac{\phi}{2}}$

The whole ratation is then, $\boldsymbol{q=q_\phi q_\theta q_\psi}$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}=\begin{pmatrix} \cos\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}+\sin\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}-\cos\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}+\sin\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}-\sin\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} \end{pmatrix}$$

DCM in terms of Euler angle

$$C_b^n=\begin{pmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi+\sin\phi\sin\theta\cos\psi & \sin\phi\sin\psi+\cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi+\sin\phi\sin\theta\sin\psi & -\sin\phi\cos\psi+\cos\phi\sin\theta\sin\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{pmatrix}$$

DCM in terms of quaternion

$$C_B^A=\begin{pmatrix} q_1^2+q_2^2-q_3^2-q_4^2 & 2(q_2q_3-q_1q_4) & 2(q_2q_4+q_1q_3) \\ 2(q_2q_3+q_1q_4) & q_1^2-q_2^2+q_3^2-q_4^2 & 2(q_3q_4-q_1q_2) \\ 2(q_2q_4-q_1q_3) & 2(q_3q_4+q_1q_2) & q_1^2-q_2^2-q_3^2+q_4^2 \end{pmatrix}$$

Euler angle in terms of DCM

$$\begin{aligned} \theta &= \tan^{-1}\frac{\sin\theta}{\cos\theta} \\ &= \tan^{-1}\frac{-c_{31}}{\sqrt{c_{32}^2+c_{33}^2}} \\ \phi &= \tan^{-1}\frac{\sin\phi}{\cos\phi} \\ &= \tan^{-1}\frac{c_{32}}{c_{33}} \\ \psi &= \tan^{-1}\frac{\sin\psi}{\cos\psi} \\ &= \tan^{-1}\frac{c_{21}}{c_{11}} \end{aligned}$$

The approximation of DCM with small angle

$$C_\beta^\alpha=\begin{pmatrix} 1 & \psi_{\beta\alpha} & -\theta_{\beta\alpha} \\ -\psi_{\beta\alpha} & 1 & \phi_{\beta\alpha} \\ \theta_{\beta\alpha} & -\phi_{\beta\alpha} & 1 \end{pmatrix}=I_3-(\triangle\boldsymbol{\Theta}\times)$$

Gyroscope output,

$$\triangle\theta_{b(t_{k-1})b(t_k)}=\int_{t_{k-1}}^{t_k}\omega_{\mathrm{ib}}^b(t)\mathrm{dt}$$

$$C_{b(t_{k-1})}^{b(t_k)}=I-(\triangle\theta_{b(t_{k-1})b(t_k)}\times)$$

$$C_{b(t_k)}^{b(t_{k-1})}=I+(\triangle\theta_{b(t_{k-1})b(t_k)}\times)$$

The update of q_b^n,

$$
\begin{aligned}
q_{b(k)}^{n(k-1)} &= q_{b(k-1)}^{n(k-1)} * q_{b(k)}^{b(k-1)} \\
q_{b(k)}^{n(k)} &= q_{n(k-1)}^{n(k)} * q_{b(k)}^{n(k-1)} \\
&= q_{n(k-1)}^{n(k)} * \left(q_{b(k-1)}^{n(k-1)} * q_{b(k)}^{b(k-1)} \right) \\
q_{b(k)}^{b(k-1)} &= \begin{pmatrix} \cos\|0.5\phi_k\| \\ \frac{0.5\phi_k}{\|0.5\phi_k\|}\sin\|0.5\phi_k\| \end{pmatrix} \\
\dot{\phi} &\approx w_{\mathrm{ib}}^b + \frac{1}{2}\phi \times w_{\mathrm{ib}}^b + \frac{1}{12}\phi \times (\phi \times w_{\mathrm{ib}}^b) \\
&\approx w_{\mathrm{ib}}^b + \frac{1}{2}\triangle\theta(t) \times w_{\mathrm{ib}}^b
\end{aligned}
$$

where

$$
\triangle\theta(t) = \int_{t_{k-1}}^{t} \omega_{\mathrm{ib}}^b(t)\mathrm{dt}
$$

Accordingly,

$$
\begin{aligned}
\phi_k &= \int_{t_{k-1}}^{t_k} \left[\omega_{\mathrm{ib}}^b(t) + \frac{1}{2}\triangle\theta(t) \times w_{\mathrm{ib}}^b \right] \mathrm{dt} \\
&\approx \triangle\theta_k + \frac{1}{12}\triangle\theta_{k-1} \times \triangle\theta_k
\end{aligned}
$$

Also we have

$$
q_{n(k-1)}^{n(k)} = \begin{pmatrix} \cos\|0.5\zeta_k\| \\ -\frac{0.5\zeta_k}{\|0.5\zeta_k\|}\sin\|0.5\zeta_k\| \end{pmatrix}
$$

To calculate ζ_k, recompute q_n^e at $t_{k-1/2}$,

$$
\begin{aligned}
q_{\delta\theta} &= \left(q_{n(k-1)}^{e(k-1)} \right)^{-1} * q_{n(k)}^{e(k)} \\
q_{n(k-1/2)}^{e(k-1/2)} &= q_{n(k-1)}^{e(k-1)} * q_{0.5\delta\theta}
\end{aligned}
$$

Axis-angle representation in terms of quaternion,

$$
\begin{aligned}
q_b^a &= \left(\begin{matrix} q_1 & q_2 & q_3 & q_4 \end{matrix} \right)^T \\
\|0.5\phi\| &= \tan^{-1}\frac{\sin\|0.5\phi\|}{\cos\|0.5\phi\|} = \tan^{-1}\frac{\sqrt{q_2^2 + q_3^2 + q_4^2}}{q_1} \\
f &\equiv \frac{\sin\|0.5\phi\|}{\|\phi\|} \\
&= 0.5 * \frac{\sin\|0.5\phi\|}{\|0.5\phi\|} \\
&= \frac{1}{2}\left(1 - \frac{\|0.5\phi\|^2}{3!} + \frac{\|0.5\phi\|^4}{5!} - \frac{\|0.5\phi\|^6}{7!} + \ldots \right) \\
\phi &= \frac{1}{f}\left(\begin{matrix} q_2 & q_3 & q_4 \end{matrix} \right)^T
\end{aligned}
$$

when $q_1 = 0$,

$$
\phi = \pi\left(\begin{matrix} q_2 & q_3 & q_4 \end{matrix} \right)^T
$$