惯导学习笔记

徐良春

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1 惯导机械编排

1.1 基本公式

地速：

$$\left.\frac{\mathrm{dr}}{\mathrm{dt}}\right|_e = v_e$$

哥氏方程：

$$\left.\frac{\mathrm{dr}}{\mathrm{dt}}\right|_a = \left.\frac{\mathrm{dr}}{\mathrm{dt}}\right|_b + \omega_{\mathrm{ab}} \times r$$

比力方程：

$$\left.\frac{d^2 r}{\mathrm{dt}^2}\right|_i = f + g$$

地速的哥氏方程：

$$\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i = \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e + \omega_{\mathrm{ie}} \times v_e$$

1.2 i系下速度方程

由哥氏方程和地速定义：

$$\left.\frac{dr}{\mathrm{dt}}\right|_i = \left.\frac{\mathrm{dr}}{\mathrm{dt}}\right|_e + \omega_{\mathrm{ie}} \times r = v_e + \omega_{\mathrm{ie}} \times r$$

i系速度求导：

$$\left.\frac{d^2 r}{\mathrm{dt}^2}\right|_i = \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i + \left.\frac{d(\omega_{\mathrm{ie}} \times r)}{\mathrm{dt}}\right|_i$$

又因为

$$\begin{aligned}
\left.\frac{d(\omega_{\mathrm{ie}} \times r)}{\mathrm{dt}}\right|_i &= \left.\frac{d\omega_{\mathrm{ie}}}{\mathrm{dt}}\right|_i \times r + \omega_{\mathrm{ie}} \times \left.\frac{\mathrm{dr}}{\mathrm{dt}}\right|_i \\
&= 0 \times r + \omega_{\mathrm{ie}} \times (v_e + \omega_{\mathrm{ie}} \times r) \\
&= \omega_{\mathrm{ie}} \times v_e + \omega_{\mathrm{ie}} \times (\omega_{\mathrm{ie}} \times r)
\end{aligned}$$

所以：

$$\begin{aligned}
\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i &= \left.\frac{d^2 r}{\mathrm{dt}^2}\right|_i - \left.\frac{d(\omega_{\mathrm{ie}} \times r)}{\mathrm{dt}}\right|_i \\
&= f + g - (\omega_{\mathrm{ie}} \times v_e + \omega_{\mathrm{ie}} \times (\omega_{\mathrm{ie}} \times r)) \\
&= f - \omega_{\mathrm{ie}} \times v_e + (g - \omega_{\mathrm{ie}} \times (\omega_{\mathrm{ie}} \times r)) \\
&= f - \omega_{\mathrm{ie}} \times v_e + g_l
\end{aligned}$$

投影到i系：

$$\begin{aligned}\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i^i &= f^i - \omega_{\mathrm{ie}}^i \times v_e^i + g_l^i \\ &= C_b^i f^b - \omega_{\mathrm{ie}}^i \times v_e^i + g_l^i\end{aligned}$$

1.3 e系下速度方程

地速的哥氏方程：

$$\begin{aligned}\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i &= \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e + \omega_{\mathrm{ie}} \times v_e \\ f - \omega_{\mathrm{ie}} \times v_e + g_l &= \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e + \omega_{\mathrm{ie}} \times v_e \\ \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e &= f - 2\omega_{\mathrm{ie}} \times v_e + g_l\end{aligned}$$

投影到e系：

$$\begin{aligned}\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e^e &= f^e - 2\omega_{\mathrm{ie}}^e \times v_e^e + g_l^e \\ &= C_b^e f^b - 2\omega_{\mathrm{ie}}^e \times v_e^e + g_l^e\end{aligned}$$

1.4 n系下速度方程

地速的哥氏方程：

$$\begin{aligned}\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i &= \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_n + \omega_{\mathrm{in}} \times v_e \\ f - \omega_{\mathrm{ie}} \times v_e + g_l &= \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_n + \omega_{\mathrm{in}} \times v_e \\ \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e &= f - \omega_{\mathrm{ie}} \times v_e + g_l - (\omega_{\mathrm{ie}} + \omega_{\mathrm{en}}) \times v_e \\ \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e &= f - (2\omega_{\mathrm{ie}} + \omega_{\mathrm{en}}) \times v_e + g_l\end{aligned}$$

投影到n系：

$$\begin{aligned}\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_n^n &= f^n - (2\omega_{\mathrm{ie}}^n + \omega_{\mathrm{en}}^n) \times v_e^n + g_l^n \\ &= C_b^n f^b - (2\omega_{\mathrm{ie}}^n + \omega_{\mathrm{en}}^n) \times v_e^n + g_l^n \qquad (1)\end{aligned}$$

速度更新算法：

$$\begin{aligned}v_e^n(t_k) &= v_e^n(t_{k-1}) + \int_{t_{k-1}}^{t_k} C_b^n(t) f^b(t)\mathrm{dt} + \int_{t_{k-1}}^{t_k} (g_l^n - (2\omega_{\mathrm{ie}}^n + \omega_{\mathrm{en}}^n) \times v_e^n)\mathrm{dt} \\ &\approx v_e^n(t_{k-1}) + \Delta v_f^n(t_k) + \Delta v_{g/\mathrm{cor}}^n(t_k) \qquad (2)\end{aligned}$$

式中：

$$\begin{aligned}\Delta v_f^n(t_k) &= \int_{t_{k-1}}^{t_k} C_b^n(t) f^b(t)\mathrm{dt} \\ &= \int_{t_{k-1}}^{t_k} C_{n(t_{k-1})}^{n(t)} C_{b(t_{k-1})}^{n(t_{k-1})} C_{b(t)}^{b(t_{k-1})} f^b(t)\mathrm{dt} \\ &= C_{n(t_{k-1})}^{n(t)} C_{b(t_{k-1})}^{n(t_{k-1})} \int_{t_{k-1}}^{t_k} C_{b(t)}^{b(t_{k-1})} f^b(t)\mathrm{dt} \\ &\approx (I - (0.5\boldsymbol{\zeta}_k \times)) C_{b(t_{k-1})}^{n(t_{k-1})} \Delta v_f^b(t_k)\end{aligned}$$

根据定义：

$$
\begin{aligned}
I-(0.5\boldsymbol{\zeta}_k\times) &= I-0.5\begin{pmatrix} 0 & -\boldsymbol{\zeta}_k[2] & \boldsymbol{\zeta}_k[1]\\ \boldsymbol{\zeta}_k[2] & 0 & -\boldsymbol{\zeta}_k[0]\\ -\boldsymbol{\zeta}_k[1] & \boldsymbol{\zeta}_k[0] & 0 \end{pmatrix}\\
&= \begin{pmatrix} 1 & 0.5\boldsymbol{\zeta}_k[2] & -0.5\boldsymbol{\zeta}_k[1]\\ -0.5\boldsymbol{\zeta}_k[2] & 1 & 0.5\boldsymbol{\zeta}_k[0]\\ 0.5\boldsymbol{\zeta}_k[1] & -0.5\boldsymbol{\zeta}_k[0] & 1 \end{pmatrix}
\end{aligned}
$$

又有：

$$
\begin{aligned}
\boldsymbol{\zeta}_k &= [\omega_{\mathrm{ie}}^n+\omega_{\mathrm{en}}^n]_{k-1/2}\Delta t_k\\
C_e^n &= R_y(-\varphi-\pi/2)R_z(\lambda)\\
&= \begin{pmatrix} -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi\\ -\sin\lambda & \cos\lambda & 0\\ -\cos\varphi\cos\lambda & -\cos\varphi\sin\lambda & -\sin\varphi \end{pmatrix}\\
\omega_e &= 7.2921151467\times10^{-5}\,\mathrm{rad}/s\\
\omega_{\mathrm{ie}}^e &= (\ 0\ \ 0\ \ \omega_e\ )^T\\
\omega_{\mathrm{ie}}^n &= C_e^n\omega_{\mathrm{ie}}^e\\
&= (\ \omega_e\cos\varphi\ \ 0\ \ -\omega_e\sin\varphi\ )^T\\
\omega_{\mathrm{en}}^n &= \begin{pmatrix} \dot{\lambda}\cos\varphi\\ -\dot{\varphi}\\ -\dot{\lambda}\sin\varphi \end{pmatrix}\\
&= \begin{pmatrix} v_E/(R_N+h)\\ -v_N/(R_M+h)\\ -v_E\tan\varphi/(R_N+h) \end{pmatrix}\\
R_N &= \frac{a}{(1-e^2\sin^2\varphi)^{1/2}}\\
R_M &= \frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{3/2}}\\
a &= 6378137.0\\
f &= \frac{a-b}{a}\\
&= 1.0/298.257223563
\end{aligned}
$$

位置外推：

$$
\begin{aligned}
h_{k-1/2} &= h_{k-1}-\frac{v_D(t_{k-1})\Delta t_k}{2}\\
q_{n(k-1/2)}^{e(k-1)} &= q_{n(k-1)}^{e(k-1)}\star q_{n(k-1/2)}^{n(k-1)}\\
q_{n(k-1/2)}^{e(k-1/2)} &= q_{e(k-1)}^{e(k-1/2)}\star q_{n(k-1/2)}^{e(k-1)}\\
&= q_{e(k-1)}^{e(k-1/2)}\star\left(q_{n(k-1)}^{e(k-1)}\star q_{n(k-1/2)}^{n(k-1)}\right)
\end{aligned}
$$

q_n^e用经纬高表示：

$$
q_n^e = \begin{pmatrix} \cos\left(-\frac{\pi}{4}-\frac{\varphi}{2}\right)\cos\left(\frac{\lambda}{2}\right)\\ -\sin\left(-\frac{\pi}{4}-\frac{\varphi}{2}\right)\sin\left(\frac{\lambda}{2}\right)\\ \sin\left(-\frac{\pi}{4}-\frac{\varphi}{2}\right)\cos\left(\frac{\lambda}{2}\right)\\ \cos\left(-\frac{\pi}{4}-\frac{\varphi}{2}\right)\sin\left(\frac{\lambda}{2}\right) \end{pmatrix}
$$

经度$\boldsymbol{\lambda}$的范围为$(-\boldsymbol{\pi},\boldsymbol{\pi}]$,纬度$\boldsymbol{\varphi}$的范围为$\left[-\frac{\boldsymbol{\pi}}{\mathbf{2}},\frac{\boldsymbol{\pi}}{\mathbf{2}}\right]$

当$\boldsymbol{\lambda}=\boldsymbol{\pi}$时:

$$\begin{aligned} q_n^e &= \begin{pmatrix} 0 \\ -\sin\left(-\frac{\pi}{4}-\frac{\varphi}{2}\right) \\ 0 \\ \cos\left(-\frac{\pi}{4}-\frac{\varphi}{2}\right) \end{pmatrix} \\ \varphi &= 2*\left(-\frac{\pi}{4}-\arctan\left(-\frac{q_2}{q_4}\right)\right) \end{aligned}$$

当$\boldsymbol{\varphi}=\frac{\boldsymbol{\pi}}{\mathbf{2}}$时:

$$\begin{aligned} q_n^e &= \begin{pmatrix} 0 \\ \sin\left(\frac{\lambda}{2}\right) \\ -\cos\left(\frac{\lambda}{2}\right) \\ 0 \end{pmatrix} \\ \lambda &= 2*\arctan\left(-\frac{q_2}{q_3}\right) \end{aligned}$$

否则:

$$\begin{aligned} \lambda &= 2*\arctan\left(\frac{q_4}{q_1}\right) \\ \varphi &= 2*\left(-\frac{\pi}{4}-\arctan\left(\frac{q_3}{q_1}\right)\right) \end{aligned}$$

其中:

$$\begin{aligned} q_{n(k-1/2)}^{n(k-1)} &= \begin{pmatrix} \cos\|0.5\boldsymbol{\zeta}_{k-1/2}\| \\ \frac{0.5\boldsymbol{\zeta}_{k-1/2}}{\|0.5\boldsymbol{\zeta}_{k-1/2}\|}\sin\|0.5\boldsymbol{\zeta}_{k-1/2}\| \end{pmatrix} \\ q_{e(k-1)}^{e(k-1/2)} &= \begin{pmatrix} \cos\|0.5\xi_{k-1/2}\| \\ -\frac{0.5\xi_{k-1/2}}{\|0.5\xi_{k-1/2}\|}\sin\|0.5\xi_{k-1/2}\| \end{pmatrix} \\ \boldsymbol{\zeta}_{k-1/2} &= \omega_{\text{in}}^n(t_{k-1})\Delta t_k/2 \\ \xi_{k-1/2} &= \omega_{\text{ie}}^n\Delta t_k/2 \end{aligned}$$

速度外推:

$$\begin{aligned} \Delta v_e^n(t_{k-1}) &= \Delta v_f^n(t_{k-1})+\Delta v_{g/\text{cor}}^n(t_{k-1}) \\ v_e^n(t_{k-1/2}) &= v_e^n(t_{k-1})+\frac{1}{2}\Delta v_e^n(t_{k-1}) \\ &= v_e^n(t_{k-1})+\frac{1}{2}(\Delta v_f^n(t_{k-1})+\Delta v_{g/\text{cor}}^n(t_{k-1})) \end{aligned}$$

重力和哥氏改正:

$$\begin{aligned} \Delta v_{g/\text{cor}}^n(t_k) &= [g_l^n-(2\omega_{\text{ie}}^n+\omega_{\text{en}}^n)\times v_e^n]_{k-1/2}\Delta t_k \\ g_l^n &= \begin{pmatrix} 0 & 0 & g \end{pmatrix}^T \end{aligned}$$

$$g = g_0(1+5.27094*10^{-3}\sin^2\varphi+2.32718*10^{-5}\sin^4\varphi)-3.086*10^{-6}h$$

由于

$$
\begin{aligned}
C_{b(t)}^{b(t_{k-1})} &\approx I+[\Delta\theta(t)\times] \\
\Delta\theta(t) &= \int_{t_{k-1}}^{t} \omega_{\mathrm{ib}}^{b}(t)\mathrm{dt} && (3)\\
\Delta v(t) &= \int_{t_{k-1}}^{t} f^{b}(t)\mathrm{dt} && (4)\\
\Delta\theta(t_{k-1}) &= \Delta v(t_{k-1})=0 && (5)
\end{aligned}
$$

式中：

$$
\begin{aligned}
\Delta v_f^b(t_k) &= \int_{t_{k-1}}^{t_k} C_{b(t)}^{b(t_{k-1})} f^b(t)\mathrm{dt} \\
&\approx \int_{t_{k-1}}^{t_k} (I+[\Delta\theta(t)\times]) f^b(t)\mathrm{dt} \\
&= \int_{t_{k-1}}^{t_k} f^b(t)\mathrm{dt} + \int_{t_{k-1}}^{t_k} (\Delta\theta(t)\times f^b(t))\mathrm{dt} \\
&= \Delta v(t_k) + \int_{t_{k-1}}^{t_k} (\Delta\theta(t)\times f^b(t))\mathrm{dt}
\end{aligned}
$$

又有

$$
\begin{aligned}
\Delta\theta(t)\times f^b(t) &= \Delta\theta(t)\times\Delta\dot{v}(t) \\
&= \frac{d}{\mathrm{dt}}(\Delta\theta(t)\times\Delta v(t)) - \Delta\dot{\theta}(t)\times\Delta v(t) && (6)\\
&= \frac{1}{2}\frac{d}{\mathrm{dt}}(\Delta\theta(t)\times\Delta v(t)) + \frac{1}{2}\big(\Delta\dot{\theta}(t)\times\Delta v(t)+\Delta\theta(t)\times\Delta\dot{v}(t)\big) - \Delta\dot{\theta}(t)\times\Delta v(t) \\
&= \frac{1}{2}\frac{d}{\mathrm{dt}}(\Delta\theta(t)\times\Delta v(t)) + \frac{1}{2}\big(-\Delta\dot{\theta}(t)\times\Delta v(t)+\Delta\theta(t)\times\Delta\dot{v}(t)\big) \\
&= \frac{1}{2}\frac{d}{\mathrm{dt}}(\Delta\theta(t)\times\Delta v(t)) + \frac{1}{2}\big(\Delta v(t)\times\Delta\dot{\theta}(t)+\Delta\theta(t)\times\Delta\dot{v}(t)\big) \\
&= \frac{1}{2}\frac{d}{\mathrm{dt}}(\Delta\theta(t)\times\Delta v(t)) + \frac{1}{2}(\Delta v(t)\times\omega_{\mathrm{ib}}^b(t)+\Delta\theta(t)\times f^b(t))
\end{aligned}
$$

于是

$$
\begin{aligned}
\int_{t_{k-1}}^{t_k} (\Delta\theta(t)\times f^b(t))\mathrm{dt} &= \frac{1}{2}(\Delta\theta(t_k)\times\Delta v(t_k) - \Delta\theta(t_{k-1})\times\Delta v(t_{k-1})) + \\
&\quad \frac{1}{2}\int_{t_{k-1}}^{t_k} (\Delta v(t)\times\omega_{\mathrm{ib}}^b(t)+\Delta\theta(t)\times f^b(t))\mathrm{dt} \\
&= \frac{1}{2}(\Delta\theta(t_k)\times\Delta v(t_k)) + \\
&\quad \frac{1}{2}\int_{t_{k-1}}^{t_k} (\Delta v(t)\times\omega_{\mathrm{ib}}^b(t)+\Delta\theta(t)\times f^b(t))\mathrm{dt}
\end{aligned}
$$

双子样法

假设在$t_{k-1}\sim t_k$和$t_{k-2}\sim t_{k-1}$内角速度和比力都是线性变化：

$$
\begin{aligned}
\omega_{\mathrm{ib}}^b(t) &= a+2b(t-t_{k-1}) \\
f^b(t) &= A+2B(t-t_{k-1})
\end{aligned}
$$

利用$t_{k-1}\sim t_k$和$t_{k-2}\sim t_{k-1}$的两次角增量输出求系数a和b，同理求A和B：

$$\begin{aligned}\Delta\theta(t_k) &= \int_{t_{k-1}}^{t_k}\omega_{\mathrm{ib}}^b(t)\mathrm{dt}\\ &= \int_{t_{k-1}}^{t_k}(a+2b(t-t_{k-1}))\mathrm{dt}\\ \Delta v(t_k) &= \int_{t_{k-1}}^{t_k} f^b(t)\mathrm{dt}\\ &= \int_{t_{k-1}}^{t_k}(A+2B(t-t_{k-1}))\mathrm{dt}\end{aligned}$$

代入划桨效应积分式，整理后得：

$$\frac{1}{2}\int_{t_{k-1}}^{t_k}(\Delta v(t)\times\omega_{\mathrm{ib}}^b(t)+\Delta\theta(t)\times f^b(t))\mathrm{dt} \;=\; \frac{1}{12}(\Delta v(t_{k-1})\times\Delta\theta(t_k)+\Delta\theta(t_{k-1})\times\Delta v(t_k))$$

速度更新总结：

$$\begin{aligned}v_e^n(t_k) &= v_e^n(t_{k-1})+\int_{t_{k-1}}^{t_k}C_b^n(t)f^b(t)\mathrm{dt}+\int_{t_{k-1}}^{t_k}(g_l^n-(2\omega_{\mathrm{ie}}^n+\omega_{\mathrm{en}}^n)\times v_e^n)\mathrm{dt}\\ &= v_e^n(t_{k-1})+\Delta v_f^n(t_k)+\Delta v_{g/\mathrm{cor}}^n(t_k)\\ &\approx v_e^n(t_{k-1})+(I-(0.5\boldsymbol{\zeta}_k\times))C_{b(t_{k-1})}^{n(t_{k-1})}\Delta v_f^b(t_k)+\Delta v_{g/\mathrm{cor}}^n(t_k)\\ &\approx v_e^n(t_{k-1})+(I-(0.5\boldsymbol{\zeta}_k\times))C_{b(t_{k-1})}^{n(t_{k-1})}\left(\Delta v(t_k)+\int_{t_{k-1}}^{t_k}(\Delta\theta(t)\times f^b(t))\mathrm{dt}\right)+\Delta v_{g/\mathrm{cor}}^n(t_k)\\ &= v_e^n(t_{k-1})+(I-(0.5\boldsymbol{\zeta}_k\times))C_{b(t_{k-1})}^{n(t_{k-1})}\Bigg(\Delta v(t_k)+\bigg(\frac{1}{2}(\Delta\theta(t_k)\times\Delta v(t_k))+\frac{1}{2}\int_{t_{k-1}}^{t_k}(\Delta v(t)\times\\ &\quad\omega_{\mathrm{ib}}^b(t)+\Delta\theta(t)\times f^b(t))\mathrm{dt}\bigg)\Bigg)+\Delta v_{g/\mathrm{cor}}^n(t_k)\\ &\approx v_e^n(t_{k-1})+(I-(0.5\boldsymbol{\zeta}_k\times))C_{b(t_{k-1})}^{n(t_{k-1})}\Bigg(\Delta v(t_k)+\bigg(\frac{1}{2}(\Delta\theta(t_k)\times\Delta v(t_k))+\frac{1}{12}(\Delta v(t_{k-1})\times\\ &\quad\Delta\theta(t_k)+\Delta\theta(t_{k-1})\times\Delta v(t_k))\bigg)\Bigg)+[g_l^n-(2\omega_{\mathrm{ie}}^n+\omega_{\mathrm{en}}^n)\times v_e^n]_{k-1/2}\Delta t_k\end{aligned}$$

1.5 位置微分方程

经纬高位置：

$$\dot{r}^n=\begin{pmatrix}\frac{1}{R_M+h} & 0 & 0\\ 0 & \frac{1}{(R_N+h)\cos\varphi} & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}v_n\\ v_e\\ v_d\end{pmatrix}=D^{-1}v^n$$

$$r^n(t_{k+1}) \;=\; r^n(t_k)+\frac{1}{2}\begin{pmatrix}\frac{1}{R_M+h} & 0 & 0\\ 0 & \frac{1}{(R_N+h)\cos\varphi} & 0\\ 0 & 0 & -1\end{pmatrix}_{k-\frac{1}{2}}(v_e^n(t_k)+v_e^n(t_{k+1}))\Delta t$$

1.6 姿态更新算法

欧拉角转四元数：

第一次绕Z轴转动，$\phi=\theta=0$，四元数表示为 ：$q_\psi=\cos\frac{\psi}{2}-\mathrm{k}\sin\frac{\psi}{2}$

第二次绕y轴转动, $\phi=\psi=0$, 四元数表示为 :$q_\theta=\cos\frac{\theta}{2}-\mathrm{j}\sin\frac{\theta}{2}$

第三次绕x轴转动, $\psi=\theta=0$, 四元数表示为 :$q_\phi=\cos\frac{\phi}{2}-\mathrm{i}\sin\frac{\phi}{2}$

则绕三轴转动的合成为 :$q=q_\phi q_\theta q_\psi$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}=\begin{pmatrix} \cos\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}+\sin\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}-\cos\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}+\sin\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}-\sin\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} \end{pmatrix}$$

欧拉角转DCM

$$C_b^n=\begin{pmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi+\sin\phi\sin\theta\cos\psi & \sin\phi\sin\psi+\cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi+\sin\phi\sin\theta\sin\psi & -\sin\phi\cos\psi+\cos\phi\sin\theta\sin\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{pmatrix}$$

四元数转DCM

$$C_B^A=\begin{pmatrix} q_1^2+q_2^2-q_3^2-q_4^2 & 2(q_2q_3-q_1q_4) & 2(q_2q_4+q_1q_3) \\ 2(q_2q_3+q_1q_4) & q_1^2-q_2^2+q_3^2-q_4^2 & 2(q_3q_4-q_1q_2) \\ 2(q_2q_4-q_1q_3) & 2(q_3q_4+q_1q_2) & q_1^2-q_2^2-q_3^2+q_4^2 \end{pmatrix}$$

DCM、四元数转欧拉角

$$\begin{aligned} \theta &= \tan^{-1}\frac{\sin\theta}{\cos\theta} \\ &= \tan^{-1}\frac{-c_{31}}{\sqrt{c_{32}^2+c_{33}^2}} \\ \phi &= \tan^{-1}\frac{\sin\phi}{\cos\phi} \\ &= \tan^{-1}\frac{c_{32}}{c_{33}} \\ \psi &= \tan^{-1}\frac{\sin\psi}{\cos\psi} \\ &= \tan^{-1}\frac{c_{21}}{c_{11}} \end{aligned}$$

正交的小角变换

$$C_\beta^\alpha=\begin{pmatrix} 1 & \psi_{\beta\alpha} & -\theta_{\beta\alpha} \\ -\psi_{\beta\alpha} & 1 & \phi_{\beta\alpha} \\ \theta_{\beta\alpha} & -\phi_{\beta\alpha} & 1 \end{pmatrix}=I_3-(\triangle\mathbf{\Theta}\times)$$

陀螺的输出：

$$\triangle\theta_{b(t_{k-1})b(t_k)}=\int_{t_{k-1}}^{t_k}\omega_{\mathrm{ib}}^b(t)\mathrm{dt}$$

$$C_{b(t_{k-1})}^{b(t_k)}=I-(\triangle\theta_{b(t_{k-1})b(t_k)}\times)$$

$$C_{b(t_k)}^{b(t_{k-1})}=I+(\triangle\theta_{b(t_{k-1})b(t_k)}\times)$$

四元数q_b^n的更新如下：

$$
\begin{aligned}
q_{b(k)}^{n(k-1)} &= q_{b(k-1)}^{n(k-1)} * q_{b(k)}^{b(k-1)} \\
q_{b(k)}^{n(k)} &= q_{n(k-1)}^{n(k)} * q_{b(k)}^{n(k-1)} \\
&= q_{n(k-1)}^{n(k)} * \left(q_{b(k-1)}^{n(k-1)} * q_{b(k)}^{b(k-1)} \right) \\
q_{b(k)}^{b(k-1)} &= \begin{pmatrix} \cos\|0.5\phi_k\| \\ \frac{0.5\phi_k}{\|0.5\phi_k\|}\sin\|0.5\phi_k\| \end{pmatrix} \\
\dot{\phi} &\approx w_{\mathrm{ib}}^b + \frac{1}{2}\phi \times w_{\mathrm{ib}}^b + \frac{1}{12}\phi \times (\phi \times w_{\mathrm{ib}}^b) \\
&\approx w_{\mathrm{ib}}^b + \frac{1}{2}\triangle\theta(t) \times w_{\mathrm{ib}}^b
\end{aligned}
$$

其中：

$$
\triangle\theta(t) = \int_{t_{k-1}}^{t} \omega_{\mathrm{ib}}^b(t)\mathrm{dt}
$$

所以：

$$
\begin{aligned}
\phi_k &= \int_{t_{k-1}}^{t_k} \left[\omega_{\mathrm{ib}}^b(t) + \frac{1}{2}\triangle\theta(t) \times w_{\mathrm{ib}}^b \right]\mathrm{dt} \\
&\approx \triangle\theta_k + \frac{1}{12}\triangle\theta_{k-1} \times \triangle\theta_k
\end{aligned}
$$

又有：

$$
q_{n(k-1)}^{n(k)} = \begin{pmatrix} \cos\|0.5\zeta_k\| \\ -\frac{0.5\zeta_k}{\|0.5\zeta_k\|}\sin\|0.5\zeta_k\| \end{pmatrix}
$$

为**了求ζ_k，转动角的四元数，重算内插位置：**

$$
\begin{aligned}
q_{\delta\theta} &= \left(q_{n(k-1)}^{e(k-1)} \right)^{-1} * q_{n(k)}^{e(k)} \\
q_{n(k-1/2)}^{e(k-1/2)} &= q_{n(k-1)}^{e(k-1)} * q_{0.5\delta\theta}
\end{aligned}
$$

四元数转等效旋转矢量：

$$
\begin{aligned}
q_b^a &= \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix}^T \\
\|0.5\phi\| &= \tan^{-1}\frac{\sin\|0.5\phi\|}{\cos\|0.5\phi\|} = \tan^{-1}\frac{\sqrt{q_2^2 + q_3^2 + q_4^2}}{q_1} \\
f &\equiv \frac{\sin\|0.5\phi\|}{\|\phi\|} \\
&= 0.5 * \frac{\sin\|0.5\phi\|}{\|0.5\phi\|} \\
&= \frac{1}{2}\left(1 - \frac{\|0.5\phi\|^2}{3!} + \frac{\|0.5\phi\|^4}{5!} - \frac{\|0.5\phi\|^6}{7!} + \dots \right) \\
\phi &= \frac{1}{f}\begin{pmatrix} q_2 & q_3 & q_4 \end{pmatrix}^T
\end{aligned}
$$

如果$q_1 = 0$：

$$
\phi = \pi\begin{pmatrix} q_2 & q_3 & q_4 \end{pmatrix}^T
$$