DIGITAL DESIGN

CS/ECE/EEE/INSTR F215

Lecture 3
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BOOLEAN ALGEBRA

Axiomatic definitions of Boolean Algebra

- Huntington Postulates
  - Postulate 1 – Closure
    - (a) Operator +
    - (b) Operator .
  - Postulate 2 – Identity
    - (a) x+0 = 0 +x = x
    - (b) x.1 = 1.x =x
- Postulate 3 – Commutative
  - a) x+y = y+x
  - b) x.y = y.x
- Postulate 4 – Distributive
  - a) x.(y+z) = (x.y) + (x.z)
  - b) x+(y.z) = (x+y).(x+z)

- Postulate 5 - Complement
  - (a) x+x' = 1
  - (b) x.x' = 0
- Postulate 6 –
- There exists at least two elements x and y in set B, such that x≠ y

- Associative law is not part of postulates, but it can be derived from other postulates.
- No inverses for addition and multiplication – i.e. no subtraction and division.
- Complement is not available for ordinary algebra.
- Postulate 4b is not valid for ordinary algebra.

Two-valued Boolean Algebra

- It is defined over the set of two elements B={0,1}, with rules for the binary operators + and ., along with complement operator.

x	y	x.y
0	0	0
0	1	0
1	0	0
1	1	1

AND

x	y	x+y
0	0	0
0	1	1
1	0	1
1	1	1

OR

x	x'
0	1
1	0

NOT

Postulate 2 – Identity

a) (a) x+0 = 0 +x = x

b) (b) x.1 = 1.x =x

- Postulate 4 – Distributive
  - a) x.(y+z) = (x.y) + (x.z)
  - b) x+(y.z) = (x+y).(x+z)

x	y	z	y+z	x.(y+z)	x.y	x.z	(x.y) + (x.z)
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

Duality

- For the dual of the algebraic expression,
  - Interchange AND and OR
  - Replace 1's with 0's and 0's with 1's.

Postulate	Expression (a)	Dual (b)
Postulate 2	x+0 =x	x.1 =x
Postulate 3	x+y = y+x	x.y = y.x
Postulate 4	x.(y+z) = (x.y) + (x.z)	x+(y.z) = (x+y).(x+z)
Postulate 5	x+x' = 1	x.x' = 0

Theorems of Boolean Algebra

Theorem	Expression (a)	Dual (b)
Theorem 1 – Idempotency (Sameness)	x+x = x	x.x=x
Theorem 2	x+1 = 1	x.0 = 0
Theorem 3 - Involution	(x′)′ = x	
Theorem 4 - Associative	x+(y+z) =(x+y)+z	x(yz) = (xy)z
Theorem 5 – De Morgan	(x+y)′ = x′ y′	(xy)′ = x′ + y′
Theorem 6 - Absorption	x+xy = x	x(x+y) =x

Theorem 1 (a) x + x = x

Statement	Rule/postulate used	
x+x = (x+x).1	2 (b)	x.1 =x
= (x+x)(x+x′)	5 (a)	x+x' = 1
= x+xx′	4 (b)	x+(y.z) = (x+y).(x+z)
= x+0	5 (b)	x.x' = 0
= x	2 (a)	x+0 =x

Logic operations

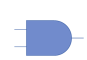
- Total 2^{2n} functions for n binary variables.
- For two variables, total 16 different functions are possible.

Boolean Functions	Operator symbol	Name
F_0 = 0		Null
F_1 = x.y	x.y	AND
F_2 = x		Transfer
F_3 = y		Transfer
F_4 = x'y	y/x	Inhibition
F_5 = x.y'	x/y	Inhibition
F_6 = x'y+xy'	x⊕ y	Ex-OR
F_7 = x+y	x+y	OR

Boolean Functions	Operator symbol	Name
F_8 = (x+y)'	x↓y	NOR
F_9 = xy + x'y'	x⊙y	Equivalence
F_{10} = x+y'	x ⊂y	Implication
F_{11} = x'+y	x ⊃ y	Implication
F_{12} = y'	y'	Complement
F_{13} = x'	x'	Complement
F_{14} = (xy)'	x↑ y	NAND
F_{15} = 1		Identity

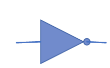
Digital Logic gates

AND



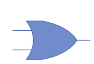
x	y	F=x.y
0	0	0
0	1	0
1	0	0
1	1	1

NOT



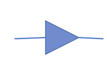
x	F =x'
0	1
1	0

OR



x	y	F=x+y
0	0	0
0	1	1
1	0	1
1	1	1

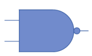
Buffer



x	F =x
0	0
1	1

Digital Logic gates

NAND

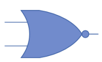


x	y	F=(x.y)'
0	0	1
0	1	1
1	0	1
1	1	0

XOR

x	y	F=x.y'+x'y F= X ⊕ Y
0	0	0
0	1	1
1	0	1
1	1	0

NOR



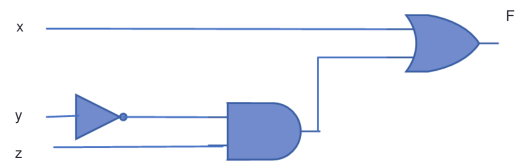
x	y	F=(x+y)'
0	0	1
0	1	0
1	0	0
1	1	0

XNOR

x	y	F=xy+x'y' F= (X⊕Y)' F= (X ⊙ Y)
0	0	1
0	1	0
1	0	0
1	1	1

Boolean Function

- An algebraic expression consisting of binary variables, constants (0 and 1), and logic operation symbols.
- F = x+y'z



Truth table for F = x+y'z

x	y	z	y'	y'z	x+y'z
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	1	0	1
1	0	1	1	1	1
1	1	0	0	0	1
1	1	1	0	0	1

Complement

- Complement of the Boolean expression can be obtained using De-Morgan's theorem.
- Or simply
  - Take dual
  - Invert the literals
- Ex. F = x'yz' + x'y'z
- Dual of F is = (x'+y+z')(x'+y'+z)
- Complement is F' = (x+y'+z)(x+y+z')

Definitions

- Literal –Ex. If x is the variable, both x and x' are literals.
- Product term –Ex. Terms such as x, xy, x'yz
- Sum term - Ex. Terms such as x, x'+y, x'+y+z
- Sum of products – SOP is the logical OR of multiple product terms.
- Ex. xy'+x'+xy'z'
- Product of sum – POS is the logical AND of the multiple OR terms.
- Ex. (x+y')(x+y+z')(y'+z')

Definitions

- Minterms – Special case product (AND) term. A minterm is a product term that contains all of the input variables that make up a Boolean expression. – Each literal no more than once.
- Maxterm – Special case sum (OR) term. A maxterm is a sum term that contains all of the input variables that make up a Boolean expression. – Each literal no more than once.

Definitions

- Canonical sum of products – It is a complete set of minterms that defines when an output variable is a logical 1.
- Each minterm corresponds to the row in the truth table where the output function is 1.

- Canonical product of sums – It is a complete set of maxterms that defines when an output variable is a logical 0.
- Each maxterm corresponds to the row in the truth table where the output function is 0.

Minterms and Maxterms

A	B	Minterms	Maxterms
0	0	m_0 = A'B'	M_0 = A+B
0	1	m_1 = A'B	M_1 = A+B'
1	0	m_2 = AB'	M_2 = A'+B
1	1	m_3 = AB	M_3 = A'+B'

Minterms and Maxterms -Properties

- Maxterm is logical complement of minterm (and vice-versa)
- For m_0 = A'B' → m_0' = (A'B')' = A+B = M_0

- Logical OR of all 2^n minterms is equal to logical 1.

$$\sum_{i=0}^{2^n-1} m_i = 1$$

- Sum = A'B' + A'B + AB' + AB
  = A'(B'+B) + A(B'+B)
  = A' + A
  = 1

Minterms and Maxterms -Properties

- Logical product of all the maxterms is equal to logical zero.

$$\prod_{i=0}^{2^n-1} M_i = 0$$

- Product = (A' + B') (A' +B)(A+B')(A+B)
  = (A')(A)
  = 0

Three variable

A	B	C	Minterms: Term	Minterms: Designation	Maxterms: Term	Maxterms: Designation
0	0	0	A'B'C'	m_0	A+B+C	M_0
0	0	1	A'B'C	m_1	A+B+C'	M_1
0	1	0	A'BC'	m_2	A+B'+C	M_2
0	1	1	A'BC	m_3	A+B'+C'	M_3
1	0	0	AB'C'	m_4	A'+B+C	M_4
1	0	1	AB'C	m_5	A'+B+C'	M_5
1	1	0	ABC'	m_6	A'+B'+C	M_6
1	1	1	ABC	m_7	A'+B'+C'	M_7

Canonical SOP and POS

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

- F(A,B,C) = A'B'C + AB'C' +AB'C +ABC' +ABC
  = $m_1+m_4+m_5+m_6+m_7$
  = $\sum (m_1 m_4 m_5 m_6 m_7)$
  = $\sum$ (1,4,5,6,7)

- F' = A'B'C' + A'BC' + A'BC
  = $\sum$ (0,2,3)

  F'= $m_0+m_2+m_3$

- F = (A+B+C) (A+B'+C)(A+B'+C')
  = $M_0M_2M_3$
  = Π ($M_0M_2M_3$)
  = Π (0,2,3)

  F= $(m_0+m_2+m_3)'$
  = $m_0' \, m_2' \, m_3'$
  = $M_0M_2M_3$

Expressing a given equation in "Sum of minterms"

- F = A+B'C

- F = A+B'C(A+A')
-   = A + AB'C+A'B'C
-   = A(B+B')+ AB'C+A'B'C
-   = AB+AB'+ AB'C+A'B'C
-   = AB(C+C') +AB'(C+C')+AB'C+A'B'C
-   = ABC + ABC'+AB'C+AB'C'+A'B'C
-   = $\sum$ (1, 4, 5, 6, 7)

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1