Lecture-14
Methods for Solving 2st Order Linear Ordinary Diff. Equations

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The method of variation of Parameters

- Recall the nonhomogeneous equation

$$y'' + P(x)y' + Q(x)y = R(x) - - - - -(1)$$

- The Method of undetermined coefficient can not be applied here to find the particular solution unless $P(x)$ and $Q(x)$ are constants.

- The Method of Variation of Parameters is a more general method of finding the particular solution of the above equation.

- Note: If the variable y is missing then you can get directly general solution by using the concept of reduction of order.

The method of variation of Parameters

Let $$y_g(x) = c_1 y_1(x) + c_2 y_2(x)$$

be the general solution of the associated homogeneous equation:

$$y'' + P(x)y' + Q(x)y = 0 ---(2)$$

To find the general solution of (1) we need to find a particular solution $y_p(x)$ of (1), from the observation we will take y_p to be a solution which is not a part of (2), so we take

$$y_p(x) = v_1(x)y_1 + v_2(x)y_2 \quad \text{(Why ?)}$$

The method of variation of Parameters

We need to find v_1 and v_2 such that $y_p(x)$ is a solution of (1)

In order to do that we find

$$y_p' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2, \text{ where}$$

we restrict v_1 and v_2 such that

$$v_1' y_1 + v_2' y_2 = 0 ----- (A)$$

$$\Rightarrow y_p' = v_1 y_1' + v_2 y_2'$$

The method of variation of Parameters

Now we calculate,

$$y_p'' = v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2'$$

Substituting y_p, y_p', y_p'' in the equation (1) we get one more condition involving v_1 and v_2

$$v_1' y_1' + v_2' y_2' = R(x) --- \text{(B)}$$

The method of variation of Parameters

- Now we have the following linear system in

$$v_1' y_1 + v_2' y_2 = 0$$
$$v_1' y_1' + v_2' y_2' = R(x)$$

- This system has a unique solution (why?) given by

$$v_1' = \frac{-y_2 R(x)}{y_1 y_2' - y_2 y_1'} = \frac{-y_2 R(x)}{W(y_1, y_2)}$$

$$v_2' = \frac{y_1 R(x)}{y_1 y_2' - y_2 y_1'} = \frac{y_1 R(x)}{W(y_1, y_2)}$$

The method of variation of Parameters

To find v_1 and v_2 we integrate these two formulas to get

$$v_1 = \int \frac{-y_2 R(x)}{W(y_1, y_2)} dx$$

$$v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$$

Hence

$$y_p(x) = v_1(x) y_1(x) + v_2(x) y_2(x),$$

where v_1 and v_2 are given by the above formulas.

Exercise Problems

Ex-1: $y'' + y = \sec x$

Ex-2: $x^2 y'' - 2xy' + 2y = x^2$

Ex-3: $x^2 y'' - 2xy' = x^2$

Solution :

(1) $y_p = x \sin x + \cos \log(\cos x)$

(2) and (3). Hint: $R(x) = 1$

Disadvantage of this Method

- The method completely depends on the general solution of the homogeneous part is.

- The computation of v_1 and v_2 depends on the integration and sometimes it may be more complicated.