Lecture-4 and 5
Methods for Solving 1st Order Ordinary Diff. Equations

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Exact Differential Equations

Definition: A differential equation $M(x, y)dx + N(x, y)dy = 0$ is called exact differential equation if there exist a function $f(x, y)$ such that $df = M(x, y)dx + N(x, y)dy$.

Q: If you know such f then what about the solution ?

Ans: Solution will be $f(x, y) = C$.

Exact Differential Equations

Question: How to find/guess such f ?

Let see few examples:

Example-1: $(x+y+1)dx + xdy = 0$

Q: What is the function f here ?

Ans: $f(x,y) = xy + x + \dfrac{x^2}{2}$

Example-2: $\dfrac{y}{y^2}dx - \dfrac{x}{y^2}dy = 0$, $f(x,y) = \dfrac{x}{y}$

Ans: $x = Cy$.

Exact Differential Equations

Question: When such f exists ? To answer this we have the following theorem.

Theorem: The differential equation

$M(x, y)dx + N(x, y)dy = 0$ is exact

if and only if $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

Proof: Refer next slide.

Question: How to construct such f ?

Exact Differential Equations

Proof of the theorem: (Sufficient part)

Assume that the diff. equation is exact.

Claim: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial y}$

since exact $\Rightarrow \exists f$, such that $\dfrac{\partial f}{\partial x} = M$, and $\dfrac{\partial f}{\partial y} = N$

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} \text{ and } \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

So by Clairaut's Theorem, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial y}$

Exact Differential Equations

Proof of the theorem: (Necessary part)

Assume that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial y}$

We need to show that the diff. eqn. is exact

i.e., $\exists \; f$, such that $df = Mdx + Ndy$

Let $v(x, y) = \int M(x, y)dx$ (treating y as constant)

$$\Rightarrow \frac{\partial v}{\partial x} = M \text{ which implies } \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) ----------\text{(Eq-A)}$$

Exact Differential Equations

Proof of the theorem: (Necessary part continued..)

Integrating Eq-A w.r.t. x, we obtain

$$N = \left(\frac{\partial v}{\partial y}\right) + g'(y), \text{ where } g'(y) \text{ is some function of } y.$$

$$\therefore Mdx + Ndy = \frac{\partial v}{\partial x}dx + \left(\frac{\partial v}{\partial y} + g'(y)\right)dy$$

$$= d[v + g(y)]$$

$$= df, \text{ where } f = v + g(y).$$

Exact Differential Equations

Construct of $f(x, y)$:

Assume $M(x, y)dx + N(x, y)dy = 0$ is exact

$\Rightarrow \exists f$ such that $df = M(x, y)dx + N(x, y)dy$

$$\Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x, y)dx + N(x, y)dy$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial x} = M(x, y)} \quad ----------(\text{Eq-1})$$

$$\text{and} \quad \boxed{\frac{\partial f}{\partial y} = N(x, y)} \quad ----------(\text{Eq-2})$$

Exact Differential Equations

Construct of $f(x, y)$: (continued...)

From Eq-1, $f(x, y) = \int M(x, y)dx + g(y) \quad ---(\text{Eq-3})$

Diff. w.r.t. y, we obatin

$$\frac{\partial f(x, y)}{\partial y} = \frac{\partial}{\partial y} \int M(x, y)dx + g'(y)$$

$$\Rightarrow g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)dx \text{ (Using Eq-2)}$$

$$\Rightarrow g(y) = \int \left(N(x, y) - \frac{\partial}{\partial y} \int M(x, y)dx \right) dy \quad --(\text{Eq-4})$$

Exact Differential Equations

Construct of $f(x, y)$: (continued...)

Putting Eq-4 in Eq-3, we get

$$\boxed{f(x,y) = \int M(x,y)\,dx + \int\left(N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx \right) dy}$$

In this integral treat y as constant

while integrating the expression.

Examples

Ex-1: $(x^3+2xy)dx+(x^2-y)dy=0$

Solution: Here $M(x,y)=x^3+2xy, N(x,y)=x^2-y$

since $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}=2x \Rightarrow$ Exact

$$\therefore\ f(x,y)=\int Mdx+\int\left(N-\frac{\partial}{\partial y}\int Mdx\right)dy$$

As $\int Mdx=\int(x^3+2xy)dx=\dfrac{x^4}{4}+yx^2$

$$\therefore\ f(x,y)=\frac{x^4}{4}+yx^2-\frac{y^2}{2}$$

Sol: $\dfrac{x^4}{4}+yx^2-\dfrac{y^2}{2}=C$

Examples

Example:2

$$\left(x+y^{3}\right)dy+\left(y-x^{3}\right)dx=0$$

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Ans: $-\dfrac{x^{4}}{4}+xy+\dfrac{y^{4}}{4}=C$

Integrating Factors

Example: $ydx - xdy = 0$

The differential equation is not exact

Note that if you multiply the equation by $\dfrac{1}{y^2}$ then the differential equation becomes

exact. This factor /function $\dfrac{1}{y^2}$ is called integrating factor of $ydx - xdy = 0$.

Integrating Factors

Definition: A function $\mu(x, y)$ is called an integrating factor of $M(x, y)dx + N(x, y)dy = 0$ if $\dfrac{\partial(\mu M)}{\partial y} = \dfrac{\partial(\mu N)}{\partial y}$.

Q: How do you get $\mu(x, y)$?

There are some standard methods are available

Methods for Finding an Integrating Factor

Theorem-1: If $\dfrac{\partial M/\partial y - \partial N/\partial x}{N} = g(x)$, a function of x alone, then $e^{\int g(x)dx}$ is an integrating factor of $M(x,y)dx + N(x,y)dy = 0.$

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Example: $(x^2 + y^2)dx - 2xydy = 0$

Solution: I.F. $\dfrac{1}{x^2}$, Ans: $x^2 - y^2 = Cx$

Methods for Finding an Integrating Factor

Proof of Theorem-1:

Let $\mu = \mu(x)$ be an I.F of $M dx + N dy = 0$.

$$\Rightarrow \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

This term is zero.

$$\Rightarrow \mu \frac{\partial M}{\partial y} + M \frac{\partial \mu}{\partial y} = \mu \frac{\partial N}{\partial x} + N \frac{\partial \mu}{\partial x}$$

$$\Rightarrow \left[\frac{\partial M / \partial y - \partial N / \partial x}{N} \right] \partial x = \frac{\partial \mu}{\mu} = \frac{d\mu}{\mu}$$

$$\Rightarrow \int g(x)dx = \int g(x)\partial x = \log \mu$$

$$\Rightarrow \mu = e^{\int g(x)dx}$$ (Note that constant C is not required ?)

Methods for Finding an Integrating Factor

Theorem-2: If $\dfrac{\partial M/\partial y - \partial N/\partial x}{-M} = h(y)$, a function of y alone, then $e^{\int h(y)dy}$ is an integrating factor of $M(x,y)dx + N(x,y)dy = 0.$

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Example: $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$

Solution: I.F. $\dfrac{1}{y^2}$, Ans: $x^3y^3 + x^2 = Cy$

Methods for Finding an Integrating Factor

Theorem-3: If $Mx - Ny \neq 0$ and $Mdx + Ndy = 0$ is of the form $f(xy)ydx + g(xy)xdy = 0$, then

$$\frac{1}{Mx - Ny}$$ is an integrating factor.

Example: $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)xdy = 0$

Solution: I.F. $\dfrac{1}{2x^2y^2}$, Ans: $xy - \dfrac{1}{xy} + \log\left(\dfrac{x}{y}\right) = C$

Methods for Finding an Integrating Factor

Theorem-4: If $\dfrac{\partial M/\partial y - \partial N/\partial x}{Ny - Mx} = f(z)$, a function of $z = xy$, then $e^{\int f(z)dz}$ is an integrating factor of $M(x, y)dx + N(x, y)dy = 0$.

Example: $ydx + (x - 2x^2y^3)dy = 0$.

Solution: I.F. $= \dfrac{1}{x^2y^2}$, Ans: $(xy^3 + 1) = Cxy$

Exercise: $(y^2 + xy + 1)dx + (x^2 + xy + 1)dy = 0$

Methods for Finding an Integrating Factor

Theorem-5: If $\dfrac{\partial M/\partial y - \partial N/\partial x}{N - M} = f(z)$, a function of $z = x + y$, then $e^{\int f(z)dz}$ is an integrating factor of $M(x, y)dx + N(x, y)dy = 0.$

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Example: $(y^2 + xy + 1)dx + (x^2 + xy + 1)dy = 0$

THANK YOU