Lecture-12 and 13

Methods for Solving 2^{st} Order Linear Ordinary Diff. Equations

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The Method of Undetermined Coefficients

Consider a non homogeneous equation

$$y'' + P(x)y' + Q(x)y = R(x).$$

The general solution of the non homogeneous equation will be of the form

$y(x) = y_g(x) + y_p(x),$ where $y_g(x)$ is the general of the corresponding homogeneous part.

The Method of Undetermined Coefficients

The method of undetermined coefficient is a procedure of finding $y_p(x)$, when the non homogeneous equation of the form

$$y'' + py' + qy = R(x),$$

where p, q are constants and $R(x)$ is an Exponential, a Sine or Cosine or a Polynomial, or some combination of such functions.

When $R(x)$ is Exponential functions

Consider the following equation

$$y'' + py' + qy = e^{ax} \ \ (R(x) = e^{ax}) - - - - (1)$$

Case-I: (If a is not root of the auxiliary eqn)

We can assume $y_p(x) = Ae^{ax}$ (why), where the coefficient A to be determined from the equation (1).

Calculation of A:

As $y_p = Ae^{ax}$ is a solution of $y'' + py' + qy = e^{ax}$

$$\Rightarrow e^{ax} A(a^2 + pa + q) = e^{ax} \Rightarrow A = 1/(a^2 + pa + q).$$

When $R(x)$ is Exponential functions

Case-II: (If a is a simple root of the auxiliary eqn)

We can assume $y_p(x) = Axe^{ax}$ (why), where the coefficient A to be determined from the equation (1).

Calculation of A:

As $y_p = Axe^{ax}$ is a solution of $y'' + py' + qy = e^{ax}$

$$\Rightarrow e^{ax}Ax(a^2 + pa + q) + Ae^{ax}(p + 2a) = e^{ax}$$

$$\Rightarrow A = \frac{1}{p+2a} \quad \left[\text{simple root means } a^2 + pa + q = 0\right]$$

When $R(x)$ is Exponential functions

Case-III: (If a is double root of the auxiliary eqn)

$$(a^2 + pa + q = 0 \text{ and } p + 2a = 0)$$

We can assume $y_p(x) = Ax^2e^{ax}$ (why), where the coefficient A to be determined from the equation (1).

Calculation of A:

As $y_p = Ax^2e^{ax}$ is a solution of $y'' + py' + qy = e^{ax}$

$$\Rightarrow e^{ax}Ax^2(a^2 + pa + q) + Axe^{ax}(p + 2a) + 2Ae^{ax} = e^{ax}$$

$$\Rightarrow A = \frac{1}{2}$$

The Choices for $y_p(x)$

$R(x) = ke^{ax}$	$y_p(x) = Ae^{ax},\ A = k/(a^2 + pa + q)$ provided 'a' is not a root of the auxiliary equation
$R(x) = ke^{ax}$	$y_p(x) = Axe^{ax},\ A = k/(p + 2a)$ provided 'a' is a simple root of the auxiliary equation
$R(x) = ke^{ax}$	$y_p(x) = Ax^2 e^{ax},\ A = k/2$ provided 'a' is a double root of the auxiliary equation

Exercise Problems

Ex-1: $y'' + 2y' + y = 5e^{2x}$

Ex-2: $4y'' - 4y' + y = 2e^{x/2}$

Ex-3: $y'' - 5y' + 6y = e^{3x}$

Ans-1: $y_p(x) = \frac{1}{2}xe^{x}$

Ans-2: $y_p(x) = x^2e^{x}$

Ans-3: $y_p(x) = xe^{x}$

When $R(x)$ is Sine or Cosine functions

If the equation is of the form

$$y'' + p\,y' + qy = \sin bx \text{ or } \cos bx \;---\; (2)$$

Then $y_p(x)$ is given in the following table

$R(x) = \sin bx \text{ or } \cos bx$	$y_p(x) = A\sin bx + B\cos bx$, provided '*sin bx , cos bx*' are not the part of y_g
$R(x) = \sin bx \text{ or } \cos bx$	$y_p(x) = x(A\sin bx + B\cos bx)$, provided '*sin bx , cos bx*' are part of y_g

where *A*, *B* are the undetermined coefficients which should be determined from the equation (2).

When *R(x)* is a Polynomial

If the equation is of the form

$$y'' + p\,y' + qy = a_0 + a_1 x + \qquad + a_n x^n$$

Then $y_p(x)$ is given in the following table

$R(x) = a_0 + a_1 x + \quad + a_n x^n$	$y_p = A_0 + A_1 x + \quad + A_n x^n$ provided $q \neq 0$.
$R(x) = a_0 + a_1 x + \quad + a_n x^n$	$y_p = x(A_0 + A_1 x + \quad + A_n x^n)$ If $q=0$, $p \neq 0$.

where A_0 , A_1 , ….A_n are the undetermined coefficients.

Exercise Problems

Ex-1: $y'' + 2y' + y = x^2 + 1$

Ex-2: $4y'' - 4y' = x - 1$

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When *R(x)* is a combination of these functions

The choice of $y_p(x)$ is given as per the followings:

$R(x) = xe^{ax}$	$y_p(x) = (Ax + B)e^{ax}$ provided e^{ax} is not part of the solution $y_g(x)$
$R(x) = xe^{ax}$	$y_p(x) = x(Ax + B)e^{ax}$ provided e^{ax} is part of $y_g(x)$ but not xe^{ax}

When *R(x)* is a combination of these functions

The choice of $y_p(x)$ is given as per the followings:

$R(x) = e^{ax} \sin bx$	$y_p(x) = e^{ax}(A \sin bx + B \cos bx)$ provided $e^{ax} \sin bx$ is not part of the solution $y_g(x)$
$R(x) = e^{ax} \sin bx$	$y_p(x) = xe^{ax}(A \cos bx + B \sin bx)$ provided $e^{ax} \sin bx$ is part of y_g

When $R(x)$ is a combination of these functions

The choice of $y_p(x)$ is given as per the followings:

$R(x) = x \sin bx$	$y_p(x) = (Ax+B)\sin bx + (Cx+D)\cos bx$ provided $\pm i$ is not root of the auxiliary eqn
$R(x) = xe^{ax} \sin bx$	$y_p(x) = (Ax+B)e^{ax}\sin bx + (Cx+D)\, e^{ax}\cos bx$ provided $a \pm ib$ is not root of the auxiliary eqn

Note: If $R(x)$ is combination of more functions then take their respective choice and combine them together.

Exercise Problems

Find the general solution of the following problem:

(1) $y'' - 3y' - 4y = 3e^{2x}$

(2) $y'' + 4y = 3\sin x$

(3) $y'' + y' = 10x^4 + 2$

(4) $y'' - 4y' - 12y = xe^{4x}$

(5) $y'' - 4y = e^{3x}\sin x$