Lecture-10 and 11
Methods for Solving 2st Order Linear Ordinary Diff. Equations

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Homogeneous with a known solution

If we know one solution of $y'' + P(x)y' + Q(x)y = 0 \; -- \; \text{(A)}$ then the second LI solution can be determined, hence the general solution can be obtained.

Procedure:

Let y_1 be a known non zero solution of (A). Since we are looking another LI solution y_2 which implies

$\dfrac{y_2}{y_1}$ must be a non constant function.

So let $y_2 = v(x)y_1$, where $v(x)$ can be determined from the equation (A).

Homogeneous with a known solution

On Substituting $y_2, y_2{}' = vy_1' + v'y_1$ and

$y_2'' = vy_1{}'' + 2v'y_1' + v''y_1$ in equation (A), we obtain

$$v\left(y_1{}'' + Py_1{}' + Qy_1\right) + v''y_1 + v'(2y_1{}' + Py_1) = 0$$

Since y_1 is a solution which implies

$$v''y_1 + v'(2y_1{}' + Py_1) = 0$$

$\Rightarrow \dfrac{v''}{v'} = -2\dfrac{y_1{}'}{y_1} - P$. So on integration we get,

$\boxed{v = \int \dfrac{1}{y_1{}^2} e^{-\int P dx} dx}$. This formula you can use for Problem.

Exercise Problems

Ex-1: $y'' + y = 0, \qquad y_1 = \sin x$

Ex-2: $(1 - x^2)y'' + 2xy' - 2y = 0, \qquad y_1 = x$

Solution :

(a) $v = -\cot x \Rightarrow y_2(x) = -\cos x.$

(b) $v = -x - \dfrac{1}{x} \Rightarrow y_2(x) = -(x^2 + 1).$

Some More Problems

How do you solve the following problems ?

Ex-1: $y'' + y' = 0$

Ex-2: $y'' + y' - 2y = 0$

Ex-3: $x^2 y'' + xy' - y = 0.$ (guess one !)

Homogeneous with constant coefficients

Suppose $P(x)$ and $Q(x)$ are two real constants say p and q respectively. So the general form of the homegeneous equation will be

$$\boxed{y'' + py' + qy = 0},\ \ p, q \in \mathbb{R}.$$

Since the exponential function has the property that its derivatives are all constant multiples of the function itself. So this leads us to consider $y(x) = e^{mx}$ as a possible solution of $y'' + py' + qy = 0$.

$\Rightarrow \boxed{m^2 + pm + q = 0}$, called auxiliary equation.

Homogeneous with constant coefficients

Since the auxiliary or characterstic equation

$m^2 + pm + q = 0$ is a quadratic equation so

we have the following possible situations

Case-1: The two roots m_1 and m_2 are distinct.

Case-2: The roots m_1 and m_2 are equal (say m).

Case-3: The roots m_1 and m_2 are complex congugates.

Distinct Real Roots

Let the auxiliary equation

$$m^2 + pm + q = 0$$

have distinct real roots (say m_1 and m_2).

In this case we have the following two LI solutions

$$y_1 = e^{m_1 x} \text{ and } y_2 = e^{m_2 x}.$$

Hence the general solution is

$$y_g = C_1 e^{m_1 x} + C_2 e^{m_2 x}.$$

Q: Are y_1 and y_2 linearly independent ?

Exercise Problems

Ex-1: $y'' - y = 0.$

Ex-2: $y'' - 5y' + 6y = 0.$

Solution :

(1) $y_g = c_1 e^{-x} + c_2 e^{x}$

(2) $y_g = c_1 e^{2x} + c_2 e^{3x}$

Equal Real Roots

Let the auxiliary equation

$$m^2 + pm + q = 0$$

have equal real roots (say $m_1 = m_2 = m$). Here $m = -p/2$.

In this case we have only one solution $y_1 = e^{mx}$.

So using previous concept, first we calculate $v = x$.

and hence the general solution is

$$y_g = C_1 e^{mx} + C_2 x e^{mx}.$$

Exercise Problems

Ex-1: $y'' - 4y' + 4y = 0.$

Ex-2: $4y'' - 12y' + 9y = 0.$

Ex-3: $16y'' - 8y' + y = 0.$

Solution :

$$(1)\ y_g = c_1 e^{2x} + c_2 x e^{2x}$$

$$(2)\ y_g = c_1 e^{\frac{3}{2}x} + c_2 x e^{\frac{3}{2}x}$$

$$(3)\ y_g = c_1 e^{\frac{1}{4}x} + c_2 x e^{\frac{1}{4}x}$$

Complex Roots

Let the auxiliary equation

$$m^2 + pm + q = 0$$

have equal complex roots (say $m_1 = a + ib$ and $m_2 = a - ib$).

In this case we have two complex solutions

$$y_1 = e^{m_1 x} = e^{ax}(\cos bx + i\sin bx) \text{ and } y_2 = e^{m_2 x} = e^{ax}(\cos bx - i\sin bx).$$

Hence the general solution in this case can be written as

$$y_g = e^{ax}\left(c_1 \cos bx + c_2 \sin bx\right).$$

Exercise Problems

Ex-1: $y'' - 4y' + 5y = 0.$

Ex-2: $y'' + 8y = 0.$

Solution :

(1) $y_g = e^{2x}(c_1 \cos x + c_2 \sin x)$

(2) $y_g = (c_1 \cos 2\sqrt{2}x + c_2 \sin 2\sqrt{2}x)$

Euler's Equidimensional Equation

Example: Conside the following differential equation $x^2 y'' + 2xy' - 2y = 0.$

Note: This equation has variable coefficients but we can solve easily.

Q: What kind of variable coeffcicents can be solved easily or converted to constant coefficients ?

The general form of Euler's Equation is defined as

$$\boxed{x^2 y'' + pxy' + qy = 0}, \ p, q \in \mathbb{R}.$$

Euler’s Equidimensional Equation

Consider the Euler's equation

$$\boxed{x^2 y'' + pxy' + qy = 0} \;---\;(B)$$

Methodology: Let $z = \ln x$ (or $x = e^z$)

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \qquad \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{1}{x}\frac{dy}{dz}\right] = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}$$

$$\Rightarrow x^2 y'' = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

Euler's Equidimensional Equation

So the Euler's equation (B) transfered to the following problem

$$\frac{d^2 y}{dz^2} + (p-1)\frac{dy}{dz} + qy = 0$$ and the auxiliary equation is $\boxed{m^2 + (p-1)m + q = 0}$.

Note: The general solution can be derived easily (by using the earlier concept).

Exercise Problems

Ex-1: $x^2 y'' + 3xy' + 10y = 0.$

Ex-2: $2x^2 y'' + 10xy' + 8y = 0.$

Ex-3: $x^2 y'' + 2xy' - 12y = 0.$

Solution :

$$(1)\ y_g = \frac{1}{x}\left(c_1 \cos(\ln x^3) + c_2 \sin(\ln x^3)\right)$$

$$(2)\ y_g = c_1 x^{-2} + c_2 x^{-2} \ln x$$

$$(3)\ y_g = c_1 x^3 + c_2 x^{-4}$$

Tutorial Problem

Ex: The differential equation $y'' + P(x)y' + Q(x)y = 0$ with the substituation $z = \int \sqrt{Q(x)}dx$ will be transfered to constant coefficients provided

$$\frac{Q' + 2PQ}{Q^{3/2}}$$ is constant.

Exercise Problems

Ex-1: $xy'' + (x^2 - 1)y' + x^3 y = 0.$

Ex-2: $y'' + 3xy' + x^2 y = 0.$