Lecture-6 and 7
Methods for Solving 1st Order Ordinary Diff. Equations

Instructor: Dr. J. K. Sahoo

Linear Differential Equation

Definition: A differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x) \text{ --------------(Eqn-1)}$$

is called a linear differential equation of first order.

Note that Equation-1 can be written as

$$\left[P(x)y - Q(x)\right]dx + dy = 0$$

Linear Differential Equation

Theorem: The general solution Equation-1 is

$$y = e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx} + C\right].$$

Proof:

Since $M = P(x)y - Q(x)$ and $N = 1$

Now $\dfrac{\left[\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right]}{N} = P(x) \Rightarrow$ I.F. $= e^{\int P(x)dx}$

$\Rightarrow e^{\int P(x)dx}\left[P(x)y - Q(x)\right]dx + e^{\int P(x)dx}dy = 0$ is exact

Therefore, the solution is

$$ye^{\int P(x)dx} - \int e^{\int P(x)dx}Q(x)dx = C$$

Examples

Example-1: $x\dfrac{dy}{dx} - 3y = x^4.$

Example-2: $\dfrac{dy}{dx} + y\tan x = \sin 2x.$

Example-3: $(e^y - 2xy)\dfrac{dy}{dx} = y^2.$

Ans-1: $y = x^4 + Cx^3.$

Ans-2: $y = C\cos x - 2\cos^2 x.$

Ans-3: $xy^2 = e^y + C.$

Bernoulli's Equation

Definition: A differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is called Bernoulli's equation.

Note that: If $n = 0$ or $n = 1,$ then the differential equation becomes linear

Bernoulli's Equation

Observation: If we take the substituation $z = y^{1-n}$,

then Bernoulli's equation becomes linear and is

of the form:

$$\boxed{\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)}$$

Examples

Example-1: $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{y^2}{x} \log x$

Example-2: $x\dfrac{dy}{dx} + y = x^4 y^3$

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Ans-1: $1 = y(Cx + \log x + 1).$

Ans-2: $1 = y^2(Cx^2 - x^4).$

Problem (from Tuto-2)

Q3: If $Mx - Ny \neq 0$ and $Mdx + Ndy = 0$ is of the form

$f(xy)ydx + g(xy)xdy = 0$ then $\dfrac{1}{Mx - Ny}$ is one an

integrating factor of the differential equation.

Solution:

First multiply $\dfrac{1}{Mx - Ny}$ to the differential equation.

$$\Rightarrow \frac{f(xy)dx}{x\left[f(xy) - g(xy)\right]} + \frac{g(xy)dy}{y\left[f(xy) - g(xy)\right]} = 0.$$

Problem (from Tuto-2)

Solution (cont.):

Let $M_1 = \dfrac{f(xy)}{x\left[f(xy) - g(xy)\right]}$ and $N_1 = \dfrac{g(xy)}{y\left[f(xy) - g(xy)\right]}$.

First claim: $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$ then the proof will complete.

Now $\dfrac{\partial M_1}{\partial y} = \dfrac{fg' - gf'}{\left[f - g\right]^2}$ and $\dfrac{\partial N_1}{\partial x} = \dfrac{fg' - gf'}{\left[f - g\right]^2}$

Therefore the equation is exact.

Reduction of Order

The general second order differential equation has the form: $F(x, y, y', y'') = 0.$

Here we will consider two special types of second order equations that can be solved by first order methods.

- **Dependent variable missing**
- **Independent variable missing**

Reduction of Order

Dependent variable missing:

If y is not present, then the equation can be written as $f(x, y', y'') = 0.$

Substitute $y' = p \Rightarrow y'' = \dfrac{dp}{dx}.$

$\Rightarrow f\left(x, p, \dfrac{dp}{dx}\right) = 0$ which is first order diff. eqn.

First solve for p then for y.

Note: If both solutions are exist then the solution of the orginal second order differential equation is exist.

Examples

Example-1: $xy'' = y' + (y')^3.$

Example-3: $xy'' + y' = 4x.$

Solution-1: $x^2 + (y - c_2)^2 = c_1{}^2$

Solution-2: $y = x^2 + c_1 \log x + c_2$

Reduction of Order

Independent variable missing:

If x is not present, then the equation can be written as $f(y, y', y'') = 0.$

Substitute $y' = \dfrac{dy}{dx} = p$

$$\Rightarrow \; y'' = \frac{d^2 y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}.$$

$$\Rightarrow f\left(y, \; p, \; p\frac{dp}{dy} \right) = 0 \text{ which is first order diff. eqn.}$$

First solve for p then for y.

Examples

Example-1: $yy'' = \left(y'\right)^2$. Example-2: $y'' = k^2 y$.

Example-3: $y'' = 1 + \left(y'\right)^2$.

Solution-1: $y = c_2 e^{c_1 x}$

Solution-2: $y = c_1 e^{kx} + c_2 e^{-kx}$

THANK YOU