Lecture-2 and 3
Methods for Solving 1st Order Ordinary Diff. Equations

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Variables Separable

If $f(x, y) = g(x)h(y)$

$$\Rightarrow \frac{dy}{dx} = g(x)h(y)$$

$$\Rightarrow \frac{dy}{h(y)} = g(x)dx$$

Integrating both sides, we get

$$\boxed{\int \frac{dy}{h(y)} = \int g(x)dx + C,}$$ where

C is any arbitrary constant.

Examples

Ex-1: $y(x^2+1)\dfrac{dy}{dx} = x$

Ex-2: $x\dfrac{dy}{dx} + y = y^2,\ y(1) = 2$

.

Ans: (1) $\dfrac{y^2}{2} = \log\sqrt{(x^2+1)} + C$

(2) $2(y-1) = xy$

Homogeneous Equations

Definition:

A function $f(x, y)$ is called homogeneous of degree n if $\boxed{f(tx, ty) = t^n f(x, y),\ \forall x, y, t \in \mathbb{R}.}$

Verify the following Examples:

Ex-1: $x^3 + xy^2$ Ex-2: $\sin x + x$

Ex-3: $\dfrac{y}{x} + \sin\left(\dfrac{y}{x}\right)$ Ex-4: $\dfrac{x^2 - y^2}{x^2 + xy}$

Ans: Homogeneous: 1,3,4 but not 2

Homogeneous Equations

Definition:

A differential equation $M(x, y)dx + N(x, y)dy$ is called homogeneous if $M(x, y)$ and $N(x, y)$ are homogeneous of same degree.

In other words:

A differential equation $\dfrac{dy}{dx} = f(x, y)$ is called homogeneous if the function $f(x, y)$ is homogeneous of degree 0.

Method to solve Homogeneous ODE

Result:

If the differential equation $\frac{dy}{dx} = f(x, y)$ is homogeneous then it will reduce to separable form through the substituation $y = zx$.

Sketch of the Proof:

$$\text{Let } y = zx \Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx}$$

$$\therefore z + x\frac{dz}{dx} = f(x, zx) = f(1, z) \text{ (Think } x \text{ as } t\text{)}$$

$$\Rightarrow \frac{dz}{dx} = [f(1, z) - z]\frac{1}{x} = g(z)h(x) \text{(Solve for } z\text{)}$$

Examples

Ex-1: $\dfrac{dy}{dx} = \dfrac{x - y}{x + y}$

Ans: $y^2 + 2xy - x^2 = C$

Ex-2: $x^2 \cdot \dfrac{dy}{dx} = 2xy + y^2$

Ans: $Cx^2 / (1 - Cx)$

Non Homogeneous type Equations

Ex-1: $\dfrac{dy}{dx} = \dfrac{1 - xy^2}{2x^2 y}$

Motivation from previous: (you can think of $y = zx^a$)

Trial Method: Let $y = zx^a, \ a \in \mathbb{R}$.

$$\Rightarrow \frac{dz}{dx} = \frac{1 - (2a+1)z^2 x^{2a+1}}{2zx^{2a+2}}, \quad \text{(choose } a = -1/2\text{)}$$

$$\Rightarrow \frac{dz}{dx} = \frac{1}{2zx}$$

Questions

Q1: How to choose a ?

Ans: Depends on the problem

Q2: Is this method always works ?

Ans: In general No, for example

$$\frac{dy}{dx} = \frac{x + y - 1}{x - y + 1}$$

Non Homogeneous type Equations

Another Form: $\dfrac{dy}{dx} = \dfrac{ax + by + c}{dx + ey + f}$, with $ae = bd$

Procedure: Since $ae = bd \Rightarrow \dfrac{a}{d} = \dfrac{b}{e} = k$ (say)

$$\Rightarrow \frac{dy}{dx} = \frac{k(dx + ey) + c}{dx + ey + f}$$

(you can think of the substituati n $z = dx + ey$)

Ofcourse this substituation reduces to separable form

Non Homogeneous type Equations

Reduction to separable form:

Let $z = dx + ey$

$$\Rightarrow \frac{dz}{dx} = d + e\frac{dy}{dx}$$

$$\Rightarrow \frac{dz}{dx} - d = e\frac{kz + c}{z + f}$$

$$\Rightarrow \frac{dz}{dx} = e\frac{kz + c}{z + f} + d = e\frac{(k + d)z + c + fd}{z + f}$$

Examples

Ex-1: $\dfrac{dy}{dx} = \dfrac{x + y + 4}{x + y - 6}$

Solution: Let $z = x + y \Rightarrow \dfrac{dy}{dx} = \dfrac{dz}{dx} - 1$

$$\Rightarrow \frac{dz}{dx} = \frac{z + 4}{z - 6} + 1 = \frac{2z - 2}{z - 6}$$

$$\Rightarrow dz - 5\frac{dz}{z - 1} = 2dx$$

Ans: $y - x = 5\log(x + y - 1) + C$

Non Homogeneous type Equations

Another Form: $\dfrac{dy}{dx} = \dfrac{ax+by+c}{dx+ey+f}$, with $ae \neq bd$

Procedure: Substitute $x = z-h$ and $y = w-k$

$$\Rightarrow \frac{dw}{dz} = \frac{az+bw+c-ah-bk}{dz+ew+f-dh-ek}$$

Choose h and k in such way that

$$\boxed{ah+bk = c \text{ and } dh+ek = f}$$

Q: Is the above system has uniqies solution ?

Ans: Yes since the determinant $ae-bd \neq 0$.

Examples

Ex-1: $\dfrac{dy}{dx} = \dfrac{x + y + 3}{x - y + 1}$

Solution: Let $x = z - h,\ y = w - k$

$$\Rightarrow \frac{dw}{dz} = \frac{z + w + 3 - h - k}{z - w + 1 - h + k} = \frac{z + w}{z - w},\ (h = 1, k = 2)$$

Now let $w = vz \Rightarrow \dfrac{dw}{dz} = v + z\dfrac{dv}{dz}$

$$\therefore\ z\frac{dv}{dz} = \frac{1 + v}{1 - v} - v = \frac{1 + v^2}{1 - v}$$

$$\Rightarrow \tan^{-1} v - \frac{1}{2}\log(1 + v^2) = \log z + C$$

Examples

Ex-1: $\dfrac{dy}{dx} = \dfrac{x+y+1}{x+2y+3}$

Solution: Let $x = z - h,\ y = w - k$

$$\Rightarrow \frac{dw}{dz} = \frac{z+w}{z+2w}, \text{ (if we choose } h = -1, k = 2\text{)}$$

$$\Rightarrow z\frac{dv}{dz} = \frac{1+v}{1+2v} - v = \frac{1-2v^2}{1+2v}$$

$$\Rightarrow \frac{dv}{1-2v^2} + \frac{2v\,dv}{1-2v^2} = \frac{dz}{z} \text{ (Now integrate)}$$