Spherical Co-ordinates

$P(x, y, z)$ Discuss

(r, θ, ϕ)

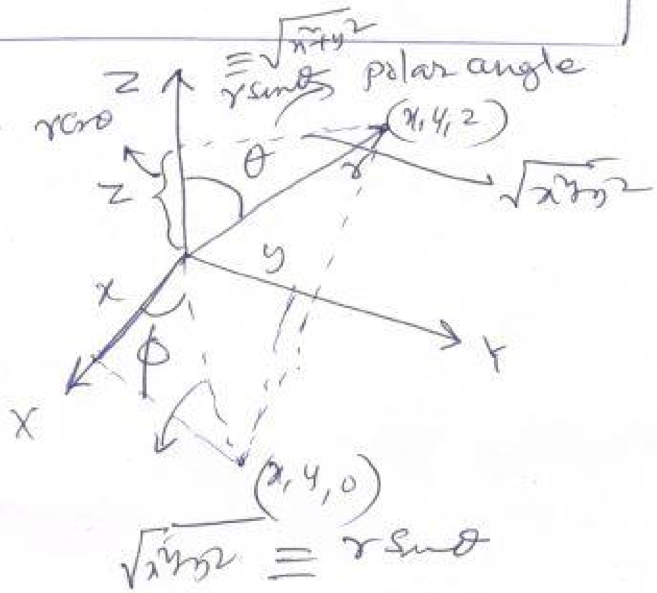
$\vec{r} = r\hat{r}$ — (1)

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ — (2) $\quad |\vec{r}| = r = \sqrt{x^2+y^2+z^2}$

$\Rightarrow \hat{r} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$

$$\Rightarrow \boxed{\hat{r} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}}$$

$\cos\theta = \frac{z}{r} = \frac{z}{\sqrt{x^2+y^2+z^2}}$



$\Rightarrow \boxed{z = r\cos\theta}$

$\boxed{x = r\sin\theta\cos\phi}$

$\boxed{y = r\sin\theta\sin\phi}$

$\boxed{\tan\phi = \frac{y}{x}}$

$\cos\theta = \frac{z}{r}$ or $\boxed{\tan\theta = \frac{\sqrt{x^2+y^2}}{z}}$

$\Rightarrow \hat{r} = \frac{\cancel{r}\sin\theta\cos\phi}{\cancel{r}}\hat{i} + \frac{\cancel{r}\sin\theta\sin\phi}{\cancel{r}}\hat{j} + \frac{\cancel{r}\cos\theta}{\cancel{r}}\hat{k}$

$$\boxed{\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}$$

$\hat{\theta} =$ rotate $\hat{r}$ along θ by $90^\circ \equiv \pi/2$

$\hat{\theta} = \sin(\theta+\pi/2)\cos\phi\,\hat{i} + \sin(\pi/2+\theta)\sin\phi\,\hat{j} + \cos(\pi/2+\theta)\,\hat{k}$

$= \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$

$$\hat\phi = \hat r \times \hat\theta$$

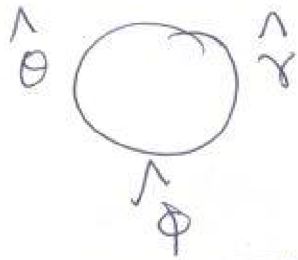
$$= \left(\sin\theta\cos\phi\,\hat x + \sin\theta\sin\phi\,\hat y + \cos\theta\,\hat z\right) \times \left(\cos\theta\cos\phi\,\hat x + \cos\theta\sin\phi\,\hat y - \sin\theta\,\hat z\right)$$

$$= -\cancel{\sin\theta\cos\theta\sin\phi\cos\phi\,\hat z} + \cos^2\theta\cos\phi\,\hat y + \cancel{\sin\theta\cos\theta\sin\phi\cos\phi\,\hat z} - \cos^2\theta\sin\phi\,\hat x + \sin^2\theta\cos\phi\,\hat y - \sin^2\theta\sin\phi\,\hat x$$

$$= \cos\phi\,\hat y - \sin\phi\,\hat x$$

$$= -\sin\phi\,\hat x + \cos\phi\,\hat y$$

$$\boxed{\hat\phi = -\sin\phi\,\hat x + \cos\phi\,\hat y}$$



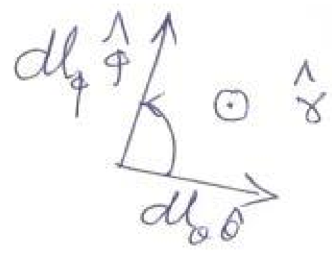
$$\boxed{\begin{aligned} \hat r \times \hat\theta &= \hat\phi \\ \hat\theta \times \hat\phi &= \hat r \\ \hat\phi \times \hat r &= \hat\theta \end{aligned}}$$

~~Line~~ el. A

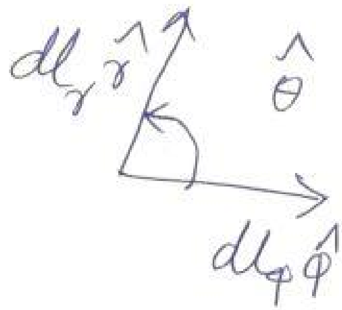
$$\vec{dl} = \underbrace{dr\,\hat r}_{dl_r} + \underbrace{r\,d\theta\,\hat\theta}_{dl_\theta} + \underbrace{r\sin\theta\,d\phi\,\hat\phi}_{dl_\phi}$$

$$d\vec a_\phi = dl_r\,\hat r \times dl_\theta\,\hat\theta$$
$$= dr\,r\,d\theta\,\hat\phi$$
$$= r\,dr\,d\theta\,\hat\phi$$

$$d\vec{a}_r = dl_\theta\,\hat{\theta} \times dl_\phi\,\hat{\phi}$$
$$= r\,d\theta\,\hat{\theta} \times r\sin\theta\,d\phi\,\hat{\phi}$$
$$= r^2\sin\theta\,d\theta\,d\phi\,\hat{r}$$

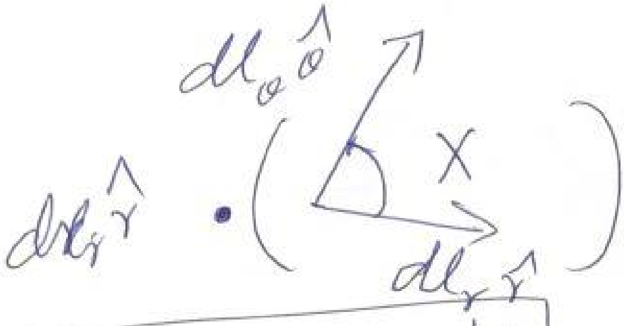


$$d\vec{a}_\theta = dl_\phi\,\hat{\phi} \times dl_r\,\hat{r}$$
$$= r\sin\theta\,d\phi\,\hat{\phi} \times dr\,\hat{r}$$
$$= r\sin\theta\,dr\,d\phi\,\hat{\theta}$$



$$\boxed{d\vec{a} = r^2\sin\theta\,d\theta\,d\phi\,\hat{r} + r\sin\theta\,dr\,d\phi\,\hat{\theta} + r\,dr\,d\theta\,\hat{\phi}}$$

volume element



$$\boxed{dV = r^2\sin\theta\,dr\,d\theta\,d\phi}$$

$$\vec{e}_1^{\,s} = \hat{r},\ \vec{e}_2^{\,s} = \hat{\theta},\ \vec{e}_3^{\,s} = \hat{\phi},\ \vec{e}_1^{\,c} = \hat{i},\ \vec{e}_2^{\,c} = \hat{j},\ \vec{e}_3^{\,c} = \hat{k}$$

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$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

$$\Rightarrow \vec{e}_\alpha^{\,s} = O \cdot \vec{e}^{\,c} \Rightarrow e^s = O e^c$$

$$\Rightarrow \boxed{\vec{e}_i^{\,s} = O_{ij}\,\vec{e}_j^{\,c}}$$

$$O^T O = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \cdot \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{O^{-1} = O^T}$$

$$e^{S} = O e^{C}$$

$$O^{T} = O^{-1}$$

$$\Rightarrow e^{C} = O^{T} e^{S}$$

$$\Rightarrow \begin{bmatrix} \vec{e}^{\,C}_1 \\ \vec{e}^{\,C}_2 \\ \vec{e}^{\,C}_3 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \vec{e}^{\,S}_1 \\ \vec{e}^{\,S}_2 \\ \vec{e}^{\,S}_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix}$$

$$\boxed{\begin{aligned} \Rightarrow \hat{x} &= \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} \\ \hat{y} &= \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi} \\ \hat{k} &= \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} \end{aligned}}$$

$$f = f\left(r(x,y,z),\ \theta(x,y,z),\ \phi(x,y,z)\right)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\cancel{2} r\frac{\partial r}{\partial x} = \cancel{2}x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{\cancel{r}\sin\theta\cos\phi}{\cancel{r}} = \sin\theta\cos\phi$$

$$\cancel{2} r\frac{\partial r}{\partial y} = \cancel{2}y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{\cancel{r}\sin\theta\sin\phi}{\cancel{r}} = \sin\theta\sin\phi$$

$$\cancel{2} r\frac{\partial r}{\partial z} = \cancel{2}z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r} = \frac{\cancel{r}\cos\theta}{\cancel{r}} = \cos\theta$$

$$\boxed{\frac{\partial r}{\partial x} = \sin\theta\cos\phi} \quad \boxed{\frac{\partial r}{\partial y} = \sin\theta\sin\phi} \quad \boxed{\frac{\partial r}{\partial z} = \cos\theta}$$

$$\tan\theta = \frac{\sqrt{x^2+y^2}}{z}$$

$$z^2\tan^2\theta = x^2 + y^2$$

$$\cancel{2}z^2\tan\theta\sec^2\theta\frac{\partial \theta}{\partial x} = \cancel{2}x \Rightarrow \frac{\partial \theta}{\partial x} = \frac{x}{z^2\tan\theta\sec^2\theta}$$

$$\Rightarrow \frac{\partial \theta}{\partial y} = \frac{y}{z^2\tan\theta\sec^2\theta}$$

$$\cancel{2}z^{\cancel{2}}\tan\theta\sec^2\theta\frac{\partial \theta}{\partial z} + \cancel{2}\cancel{z}\tan^2\theta = 0$$

$$\boxed{\frac{\partial \theta}{\partial z} = \frac{-z\tan^2\theta}{z^2\tan\theta\sec^2\theta}}$$

$$\frac{\partial\theta}{\partial x} = \frac{x}{z^2 \tan\theta \sec^2\theta}$$

$$= \frac{\cancel{r}\sin\theta\cos\phi}{r^{\cancel{2}}\cancel{\cos^2\theta}\,\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cancel{\cos^2\theta}}} = \frac{1}{r}\cos\theta\cos\phi$$

$$\boxed{\frac{\partial\theta}{\partial x} = \frac{\cos\theta\cos\phi}{r}}$$

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$$\frac{\partial\theta}{\partial y} = \frac{y}{z^2\tan\theta\sec^2\theta}$$

$$= \frac{\cancel{r}\sin\theta\sin\phi}{r^{\cancel{2}}\cancel{\cos^2\theta}\tan\theta\,\cancel{\sec^2\theta}}$$

$$= \frac{\cos\theta\sin\phi}{r}$$

$$\boxed{\frac{\partial\theta}{\partial y} = \frac{\cos\theta\sin\phi}{r}}$$

$$\frac{\partial\theta}{\partial z} = -\frac{\tan^{\cancel{2}}\theta}{r\cos\theta\cdot\cancel{\tan\theta}\sec^2\theta} = -\frac{\sin\theta}{r\cancel{\cos^2\theta\sec^2\theta}}$$

$$= -\frac{\sin\theta}{r}$$

$$\boxed{\frac{\partial\theta}{\partial z} = -\frac{\sin\theta}{r}}$$

$$\tan\phi = \frac{y}{x}$$

$$\sec^2\phi \frac{\partial\phi}{\partial x} = -\frac{y}{x^2} \Rightarrow \boxed{\frac{\partial\phi}{\partial x} = -\frac{y}{x^2\sec^2\phi}} = -\frac{r\sin\theta\sin\phi}{r^2\sin^2\theta\cos^2\phi\sec^2\phi} = -\frac{\sin\phi}{r\sin\theta}$$

$$\sec^2\phi \frac{\partial\phi}{\partial y} = \frac{1}{x} \Rightarrow \boxed{\frac{\partial\phi}{\partial y} = \frac{x}{x^2\sec^2\phi}} = \frac{1}{r\sin\theta\cos\phi\cdot\sec^2\phi} = \frac{\cos\phi}{r\sin\theta}$$

$$\boxed{\frac{\partial\phi}{\partial z} = 0}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cdot\sin\theta\cos\phi + \frac{\partial f}{\partial\theta}\frac{\cos\theta\cos\phi}{r} + \frac{\partial f}{\partial\phi}\left(-\frac{\sin\phi}{r\sin\theta}\right)$$

$$\boxed{\frac{\partial f}{\partial x} = \sin\theta\cos\phi\frac{\partial f}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial f}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial f}{\partial\phi}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\cdot(\sin\theta\sin\phi) + \frac{\partial f}{\partial\theta}\left(\frac{\cos\theta\sin\phi}{r}\right) + \frac{\partial f}{\partial\phi}\left(\frac{\cos\phi}{r\sin\theta}\right)$$

$$\boxed{\frac{\partial f}{\partial y} = \sin\theta\sin\phi\frac{\partial f}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial f}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial f}{\partial\phi}}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r}(\cos\theta) + \frac{\partial f}{\partial\theta}\left(-\frac{\sin\theta}{r}\right) + \frac{\partial f}{\partial\phi}(0)$$

$$\boxed{\frac{\partial f}{\partial z} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}}$$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \left(\sin\theta\cos\phi \frac{\partial f}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial f}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial f}{\partial \phi}\right)$$

$$\left(\sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}\right) +$$

$$\left(\sin\theta\sin\phi\frac{\partial f}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial f}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial f}{\partial \phi}\right)$$

$$\left(\sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}\right) +$$

$$\left(\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}\right)\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$

$$= \sin^2\theta\cos^2\phi\frac{\partial f}{\partial r}\hat{r} + \frac{\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial f}{\partial \theta}\hat{r} - \frac{\sin\phi\cos\phi}{r}\frac{\partial f}{\partial \phi}\hat{r}$$

$$+ \sin\theta\cos\theta\cos^2\phi\frac{\partial f}{\partial r}\hat{\theta} + \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial f}{\partial \theta}\hat{\theta} - \frac{\cos\theta\sin\phi\cos\phi}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\theta}$$

$$- \sin\theta\sin\phi\cos\phi\frac{\partial f}{\partial r}\hat{\phi} - \frac{\cos\theta\sin\phi\cos\phi}{r}\frac{\partial f}{\partial \theta}\hat{\phi} + \frac{\sin^2\phi}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi} +$$

$$\sin^2\theta\sin^2\phi\frac{\partial f}{\partial r}\hat{r} + \frac{\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial f}{\partial \theta}\hat{r} + \frac{\sin\phi\cos\phi}{r}\frac{\partial f}{\partial \phi}\hat{r} +$$

$$+ \sin\theta\cos\theta\sin^2\phi\frac{\partial f}{\partial r}\hat{\theta} + \frac{\cos^2\theta\sin^2\phi}{r}\frac{\partial f}{\partial \theta}\hat{\theta}$$

$$+ \frac{\cos\theta\sin\phi\cos\phi}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\theta} + \sin\theta\sin\phi\cos\phi\frac{\partial f}{\partial r}\hat{\phi} + \frac{\cos\theta\sin\phi\cos\phi}{r}\frac{\partial f}{\partial \theta}\hat{\phi}$$

$$+ \frac{\cos^2\phi}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi} + \cos^2\theta\frac{\partial f}{\partial r}\hat{r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial f}{\partial \theta}\hat{r}$$

$$- \sin\theta\cos\theta\frac{\partial f}{\partial r}\hat{\theta} + \frac{\sin^2\theta}{r}\frac{\partial f}{\partial \theta}\hat{\theta}$$

$$\Bigg(\sin^2\theta \cos^2\phi \frac{\partial f}{\partial r} + \frac{\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial f}{\partial \theta} - \frac{\sin\phi\cos\phi}{r}\frac{\partial f}{\partial \phi} +$$

$$\sin^2\theta \sin^2\phi \frac{\partial f}{\partial r} + \frac{\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial f}{\partial \theta} + \frac{\sin\phi\cos\phi}{r}\frac{\partial f}{\partial \phi} +$$

$$\cos^2\theta \frac{\partial f}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial f}{\partial \theta}\Bigg)\hat{r} +$$

$$\Bigg(\sin\theta\cos\theta\cos^2\phi \frac{\partial f}{\partial r} + \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial f}{\partial \theta} - \frac{\cos\theta\sin\phi\cos\phi}{r\sin\theta}\frac{\partial f}{\partial \phi} +$$

$$\sin\theta\cos\theta\sin^2\phi \frac{\partial f}{\partial r} + \frac{\cos^2\theta\sin^2\phi}{r}\frac{\partial f}{\partial \theta} + \frac{\cos\theta\sin\phi\cos\phi}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$- \sin\theta\cos\theta \frac{\partial f}{\partial r} + \frac{\sin^2\theta}{r}\frac{\partial f}{\partial \theta}\Bigg)\hat{\theta} +$$

$$\Bigg(-\sin\theta\sin\phi\cos\phi \frac{\partial f}{\partial r} - \frac{\cos\theta\sin\phi\cos\phi}{r}\frac{\partial f}{\partial \theta} + \frac{\sin^2\phi}{r\sin\theta}\frac{\partial f}{\partial \phi} +$$

$$\sin\theta\sin\phi\cos\phi \frac{\partial f}{\partial r} + \frac{\cos\theta\sin\phi\cos\phi}{r}\frac{\partial f}{\partial \theta} + \frac{\cos^2\phi}{r\sin\theta}\frac{\partial f}{\partial \phi}\Bigg)\hat{\phi}$$

$$= \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

$$\Rightarrow \boxed{\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}}$$

$$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\frac{\partial \hat{r}}{\partial r} = 0$$

$$\frac{\partial \hat{r}}{\partial \theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k} = \hat{\theta}$$

$$\frac{\partial \hat{r}}{\partial \phi} = -\sin\theta\sin\phi\,\hat{i} + \sin\theta\cos\phi\,\hat{j} = \sin\theta\,\hat{\phi}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\frac{\partial \hat{\theta}}{\partial r} = 0$$

$$\frac{\partial \hat{\theta}}{\partial \theta} = -\sin\theta\cos\phi\,\hat{i} - \sin\theta\sin\phi\,\hat{j} - \cos\theta\,\hat{k} = -\hat{r}$$

$$\frac{\partial \hat{\theta}}{\partial \phi} = -\cos\theta\sin\phi\,\hat{i} + \cos\theta\cos\phi\,\hat{j} = \cos\theta\,\hat{\phi}$$

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$* \boxed{\hat{\phi}\cdot\frac{\partial\hat{\phi}}{\partial\phi} = 0}$$

$$* \boxed{\hat{\phi}\times\frac{\partial\hat{\phi}}{\partial\phi} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}}$$

$$\frac{\partial \hat{\phi}}{\partial r} = 0$$

$$\frac{\partial \hat{\phi}}{\partial \theta} = 0$$

$$\frac{\partial \hat{\phi}}{\partial \phi} = -\cos\phi\,\hat{i} - \sin\phi\,\hat{j} =$$

$$\hat{\phi}\cdot\frac{\partial\hat{\phi}}{\partial\phi} = 0$$

$$\hat{\phi}\times\frac{\partial\hat{\phi}}{\partial\phi} = \left(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right)\times\left(-\cos\phi\,\hat{i} - \sin\phi\,\hat{j}\right)$$

$$= \cancel{\sin\phi\cos\phi\,\hat{i}\times\hat{i}} - \cos^2\phi\,\hat{j}\times\hat{i} + \sin^2\phi\,\hat{i}\times\hat{j} - \sin\phi\cos\phi\,\hat{j}\times\hat{j}$$

$$\hat{\phi}\times\frac{\partial\hat{\phi}}{\partial\phi} = \hat{k}$$

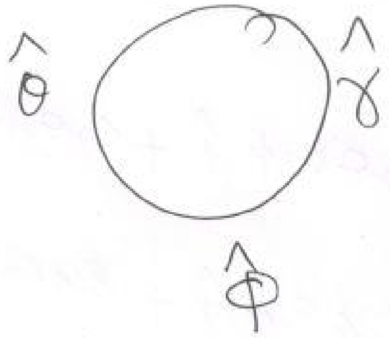
$$= \left(\sin^2\phi + \cos^2\phi\right)\hat{k} = \hat{k}$$

$$= \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

$$\hat{r} \times \hat{\theta} = \hat{\phi}$$
$$\hat{\theta} \times \hat{\phi} = \hat{r}$$
$$\hat{\phi} \times \hat{r} = \hat{\theta}$$

$$\hat{\theta} \times \hat{r} = -\hat{\phi}$$
$$\hat{\phi} \times \hat{\theta} = -\hat{r}$$
$$\hat{r} \times \hat{\phi} = -\hat{\theta}$$

$$\hat{r} \times \hat{r} = 0$$
$$\hat{\theta} \times \hat{\theta} = 0$$
$$\hat{\phi} \times \hat{\phi} = 0$$



$$\vec{V} = V_r\,\hat{r} + V_\theta\,\hat{\theta} + V_\phi\,\hat{\phi}$$

$$\vec{\nabla}\cdot\vec{V} = \left(\hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\,\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\cdot\vec{V}$$

$$= \hat{r}\cdot\frac{\partial \vec{V}}{\partial r} + \frac{1}{r}\,\hat{\theta}\cdot\frac{\partial \vec{V}}{\partial \theta} + \frac{1}{r\sin\theta}\,\hat{\phi}\cdot\frac{\partial \vec{V}}{\partial \phi}$$

$$= \hat{r}\cdot\frac{\partial}{\partial r}\left\{V_r\,\hat{r} + V_\theta\,\hat{\theta} + V_\phi\,\hat{\phi}\right\} +$$

$$\frac{1}{r}\,\hat{\theta}\cdot\frac{\partial}{\partial \theta}\left\{V_r\,\hat{r} + V_\theta\,\hat{\theta} + V_\phi\,\hat{\phi}\right\} +$$

$$\frac{1}{r\sin\theta}\,\hat{\phi}\cdot\frac{\partial}{\partial \phi}\left\{V_r\,\hat{r} + V_\theta\,\hat{\theta} + V_\phi\,\hat{\phi}\right\}$$

$$= \hat{r}\cdot\left\{\hat{r}\,\frac{\partial V_r}{\partial r} + V_r\,\frac{\partial \hat{r}}{\partial r} + V_\theta\,\frac{\partial \hat{\theta}}{\partial r} + V_\phi\,\frac{\partial \hat{\phi}}{\partial r}\right\} +$$

$$\frac{1}{r}\,\hat{\theta}\cdot\left\{V_r\,\frac{\partial \hat{r}}{\partial \theta} + \hat{\theta}\,\frac{\partial V_\theta}{\partial \theta} + V_\theta\,\frac{\partial \hat{\theta}}{\partial \theta} + V_\phi\,\frac{\partial \hat{\phi}}{\partial \theta}\right\} +$$

$$\frac{1}{r\sin\theta}\,\hat{\phi}\cdot\left\{V_r\,\frac{\partial \hat{r}}{\partial \phi} + V_\theta\,\frac{\partial \hat{\theta}}{\partial \phi} + \hat{\phi}\,\frac{\partial V_\phi}{\partial \phi} + V_\phi\,\frac{\partial \hat{\phi}}{\partial \phi}\right\}$$

$$= \hat{r}\cdot\left\{\hat{r}\,\frac{\partial V_r}{\partial r} + V_r\times 0 + \cancel{V_\theta\,\hat{\theta}} + \cancel{V_\phi\times 0}\right\} +$$

$$\frac{1}{r}\,\hat{\theta}\cdot\left\{V_r\,\hat{\theta} + \hat{\theta}\,\frac{\partial V_\theta}{\partial \theta} + \cancel{V_\theta(-\hat{r})} + V_\phi\times 0\right\} +$$

$$\frac{1}{r\sin\theta}\,\hat{\phi}\cdot\left\{V_r\cdot(\sin\theta\,\hat{\phi}) + V_\theta(\cos\theta\,\hat{\phi}) + \hat{\phi}\,\frac{\partial V_\phi}{\partial \phi} + \cancel{V_\phi(-\cos\phi\,\hat{x} - \sin\phi\,\hat{y})}\right\}$$

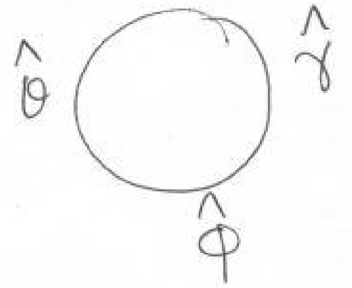
$$= \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r} + \frac{V_\theta \cos\theta}{r\sin\theta}$$

$$+ \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial \varphi}$$

$$= \frac{r^2}{r^2}\frac{\partial V_r}{\partial r} + \frac{2rV_r}{r^2} + \frac{\sin\theta\,\partial V_\theta}{r\sin\theta\,\partial\theta} + \frac{1}{r\sin\theta}V_\theta\cos\theta + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, V_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}.$$

$$\boxed{\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, V_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}}$$



$$\vec{\nabla}\times\vec{V} = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\times\left(V_r\hat{r} + V_\theta\hat{\theta} + V_\phi\hat{\phi}\right) \qquad (14)$$

$$\cancel{= \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\times(V_r\hat{r}}$$

$$= \hat{r}\times\frac{\partial}{\partial r}\left\{V_r\hat{r} + V_\theta\hat{\theta} + V_\phi\hat{\phi}\right\} +$$

$$+ \frac{1}{r}\hat{\theta}\times\frac{\partial}{\partial\theta}\left\{V_r\hat{r} + V_\theta\hat{\theta} + V_\phi\hat{\phi}\right\} +$$

$$+ \frac{1}{r\sin\theta}\hat{\phi}\times\frac{\partial}{\partial\phi}\left\{V_r\hat{r} + V_\theta\hat{\theta} + V_\phi\hat{\phi}\right\}$$

$$= \hat{r}\times\left\{\hat{r}\frac{\partial V_r}{\partial r} + V_r\frac{\partial\hat{r}}{\partial r} + \hat{\theta}\frac{\partial V_\theta}{\partial r} + V_\theta\frac{\partial\hat{\theta}}{\partial r} + \hat{\phi}\frac{\partial V_\phi}{\partial r} + V_\phi\frac{\partial\hat{\phi}}{\partial r}\right\}$$

$$+ \frac{1}{r}\hat{\theta}\times\left\{\hat{r}\frac{\partial V_r}{\partial\theta} + V_\theta\frac{\partial\hat{r}}{\partial\theta} + \hat{\theta}\frac{\partial V_\theta}{\partial\theta} + V_\theta\frac{\partial\hat{\theta}}{\partial\theta} + \hat{\phi}\frac{\partial V_\phi}{\partial\theta} + V_\phi\frac{\partial\hat{\phi}}{\partial\theta}\right\}$$

$$+ \frac{1}{r\sin\theta}\hat{\phi}\times\left\{\hat{r}\frac{\partial V_r}{\partial\phi} + V_\theta\frac{\partial\hat{r}}{\partial\phi} + \hat{\theta}\frac{\partial V_\theta}{\partial\phi} + V_\theta\frac{\partial\hat{\theta}}{\partial\phi} + \hat{\phi}\frac{\partial V_\phi}{\partial\phi} + V_\phi\frac{\partial\hat{\phi}}{\partial\phi}\right\}$$

$$= \hat{r}\times\left\{\cancel{\hat{r}\frac{\partial V_r}{\partial r}} + \cancel{V_r\times 0} + \hat{\theta}\frac{\partial V_\theta}{\partial r} + \cancel{V_\theta\times 0} + \hat{\phi}\frac{\partial V_\phi}{\partial r} + \cancel{V_\phi\times 0}\right\} +$$

$$\frac{1}{r}\hat{\theta}\times\left\{\hat{r}\frac{\partial V_r}{\partial\theta} + \cancel{\hat{\theta}V_\theta} + \cancel{\hat{\theta}\frac{\partial V_\theta}{\partial\theta}} - \hat{r}V_\theta + \hat{\phi}\frac{\partial V_\phi}{\partial\theta} + 0\right\} +$$

$$\frac{1}{r\sin\theta}\hat{\phi}\times\left\{\hat{r}\frac{\partial V_r}{\partial\phi} + \cancel{\hat{\phi}\sin\theta V_\theta} + \hat{\theta}\frac{\partial V_\phi}{\partial\phi} + \cancel{\hat{\phi}\cos\theta V_\theta} + \cancel{\hat{\phi}\frac{\partial V_\phi}{\partial\phi}} + \frac{\partial\hat{\phi}}{\partial\phi}\right.$$

(15)

$$= \frac{\partial v_\theta}{\partial r}\hat{\phi} - \frac{\partial v_\phi}{\partial r}\hat{\theta} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\hat{\phi} + \frac{v_\theta}{r}\hat{\phi} + \frac{1}{r}\frac{\partial v_\phi}{\partial \theta}\hat{r} +$$

$$+ \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\hat{r} + \frac{v_\phi}{r\sin\theta}\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$

$$= \left(\frac{\sin\theta\,\partial v_\phi}{r\sin\theta\,\partial\theta} + \frac{\cos\theta\, v_\phi}{r\sin\theta} - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right)\hat{r} +$$

$$\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{r\,\partial v_\phi}{r\,\partial r} - \frac{v_\phi \cancel{\sin\theta}}{r\cancel{\sin\theta}}\right)\hat{\theta} +$$

$$\left(\frac{r\,\partial v_\theta}{r\,\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\theta}{r}\right)\hat{\phi}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\phi\right) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{r} +$$

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}\left(r v_\phi\right)\right]\hat{\theta} +$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r v_\theta\right) - \frac{\partial v_r}{\partial \theta}\right]\hat{\phi}$$

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$$

$$= \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right) \cdot \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right)$$

$$= \hat{r}\cdot\frac{\partial}{\partial r}\left\{ \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi} \right\} +$$

$$\frac{1}{r}\hat{\theta}\cdot\frac{\partial}{\partial \theta}\left\{ \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right\} +$$

$$+ \frac{1}{r\sin\theta}\hat{\phi}\cdot\frac{\partial}{\partial \phi}\left\{ \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right\}$$

$$= \hat{r}\cdot\left\{ \frac{\partial \hat{r}}{\partial r}\frac{\partial}{\partial r} + \hat{r}\frac{\partial^2}{\partial r^2} + \frac{\partial \hat{\theta}}{\partial r}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial}{\partial \theta} + \right.$$

$$\hat{\theta}\frac{1}{r}\frac{\partial^2}{\partial r \partial \theta} + \frac{\partial \hat{\phi}}{\partial r}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} + \hat{\phi}\frac{\partial}{\partial r}\left(\frac{1}{r\sin\theta}\right)\frac{\partial}{\partial \phi} +$$

$$\left. \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial^2}{\partial r \partial \phi} \right\} +$$

$$\frac{1}{r}\hat{\theta}\cdot\left\{ \frac{\partial \hat{r}}{\partial \theta}\frac{\partial}{\partial r} + \hat{r}\frac{\partial^2}{\partial \theta \partial r} + \frac{\partial \hat{\theta}}{\partial \theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\theta}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\right)\frac{\partial}{\partial \theta} + \right.$$

$$\hat{\theta}\frac{1}{r}\frac{\partial^2}{\partial \theta^2} + \frac{\partial \hat{\phi}}{\partial \theta}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} + \hat{\phi}\frac{\partial}{\partial \theta}\left(\frac{1}{r\sin\theta}\right)\frac{\partial}{\partial \phi} +$$

$$\left. \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial^2}{\partial \theta \partial \phi} \right\} +$$

$$\frac{1}{r\sin\theta}\hat{\varphi}\left\{\frac{\partial\hat{r}}{\partial\varphi}\frac{\partial}{\partial r}+\hat{r}\frac{\partial^2}{\partial\varphi\partial r}+\frac{\partial\hat{\theta}}{\partial\varphi}\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{\theta}\frac{\partial}{\partial\varphi}\left(\frac{1}{r}\right)\frac{\partial}{\partial\theta}\right.$$

$$\hat{\theta}\frac{1}{r}\frac{\partial^2}{\partial\varphi\partial\theta}+\frac{\partial\hat{\varphi}}{\partial\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}+\hat{\varphi}\frac{\partial}{\partial\varphi}\left(\frac{1}{r\sin\theta}\right)\frac{\partial}{\partial\varphi}+$$

$$\left.\hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right\}$$

$$=\hat{r}\cdot\left\{0+\hat{r}\frac{\partial^2}{\partial r^2}+\cancel{\hat{\theta}\,0-\frac{\hat{\theta}}{r^2}\frac{\partial}{\partial\theta}}+\cancel{\frac{\hat{\theta}}{r}\frac{\partial^2}{\partial r\partial\theta}}\right.$$

$$\left.+0\,0-\cancel{\frac{\hat{\varphi}}{r^2\sin\theta}\frac{\partial}{\partial\varphi}}+\cancel{\hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\partial\varphi}}\right\}+$$

$$\frac{\hat{\theta}}{r}\cdot\left\{\hat{\theta}\frac{\partial}{\partial r}+\cancel{\hat{r}\frac{\partial^2}{\partial\theta\partial r}}+\cancel{(-\hat{r})\frac{1}{r}\frac{\partial}{\partial\theta}}\,0-\cancel{\frac{\hat{\theta}}{r^2}}\,0+\right.$$

$$\frac{\hat{\theta}}{r}\frac{\partial^2}{\partial\theta^2}+0+\cancel{\hat{\varphi}\left(-\frac{\operatorname{cosec}\theta\cos\theta}{r}\right)\frac{\partial}{\partial\varphi}}+$$

$$\left.\cancel{\hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}}\right\}+$$

$$\frac{1}{r\sin\theta}\hat{\varphi}\cdot\left\{\hat{\varphi}\sin\theta\frac{\partial}{\partial r}+\cancel{\hat{r}\frac{\partial^2}{\partial\varphi\partial r}}+\hat{\varphi}\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}+0+\right.$$

$$\left.\cancel{\hat{\theta}\frac{1}{r}\frac{\partial^2}{\partial\varphi\partial\theta}}+0+0+\hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right\}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r} +$$

$$\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1\sin\theta\,\partial^2}{r^2\sin\theta\,\partial\theta^2} + \frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} +$$

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}$$