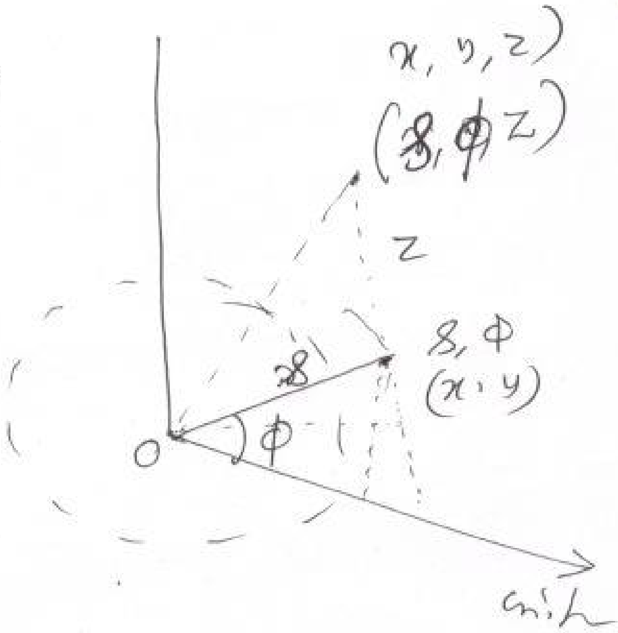
Cylindrical

$$z = z$$
$$x = s\cos\phi \qquad \boxed{s^2 = x^2 + y^2}$$
$$y = s\sin\phi \qquad \boxed{\tan\phi = \frac{y}{x}}$$



$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$\vec{r} = s\hat{s} + z\hat{k}$$
$$\hat{s} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$
$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$
$$\hat{k} = \hat{k}$$

O = orthogonal Matrix

$$\begin{bmatrix} \hat{s} \\ \hat{\phi} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

$$O^T O = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boxed{O^T = O^{-1}}$$

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{s} \\ \hat{\phi} \\ \hat{k} \end{bmatrix}$$

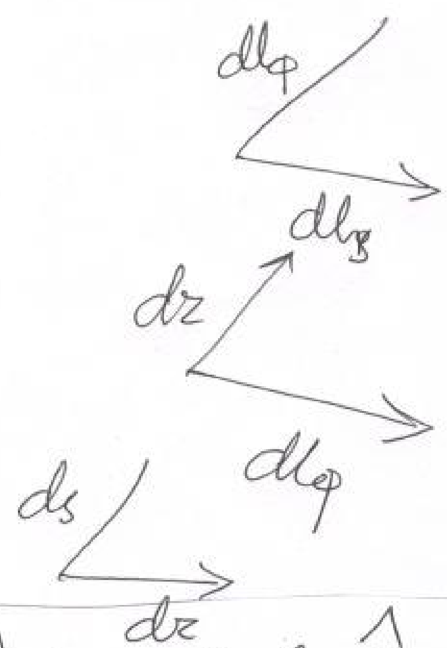
$$\boxed{\begin{aligned} \hat{i} &= \cos\phi\,\hat{s} - \sin\phi\,\hat{\phi} \\ \hat{j} &= \sin\phi\,\hat{s} + \cos\phi\,\hat{\phi} \end{aligned}}$$

Line element $\vec{dl} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{k}$

$$d\vec{a}_{k} = dl_s\,\hat{s} \times dl_\phi\,\hat{\phi}$$
$$= s\,ds\,d\phi\,\hat{k}$$

$$d\vec{a}_s = dl_\phi\,\hat{\phi} \times dz\,\hat{k}$$
$$= s\,d\phi\,dz\,\hat{s}$$

$$d\vec{a}_\phi = ds\,dz\,\hat{\phi}$$



$$\boxed{d\vec{a} = s\,d\phi\,dz\,\hat{s} + ds\,dz\,\hat{\phi} + s\,ds\,d\phi\,\hat{k}}$$

$$\boxed{dV = s\,ds\,d\phi\,dz}$$

$$\cancel{f = f(x(s,\phi,z),\ y(s,\phi,z),\ z(s,\phi,z))}$$

$$\cancel{\frac{\partial f}{\partial x} = \partial f}$$

$$f = f(s(x,y,z),\ \phi(x,y,z),\ z(\cancel{x},\cancel{y},z))$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} + \cancel{\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}}$$
$$= \frac{\partial f}{\partial s}\cos\phi + \frac{\partial f}{\partial \phi}\cancel{\sin\phi}\left(-\frac{\sin\phi}{s}\right)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y}$$
$$= \sin\phi\frac{\partial f}{\partial s} + \frac{\cos\phi}{s}\frac{\partial f}{\partial \phi}$$

$$s^2 = x^2 + y^2$$

$\boxed{\frac{\partial s}{\partial x} = \cos\phi}$ ~~21~~ 3

$$2s\frac{\partial s}{\partial x} = 2x \Rightarrow \frac{\partial s}{\partial x} = \frac{x}{s} = \frac{s\cos\phi}{s} = \cos\phi$$

$$2s\frac{\partial s}{\partial y} = 2y \Rightarrow \boxed{\frac{\partial s}{\partial y} = \sin\phi}$$

$$\tan\phi = \frac{y}{x}$$

$$\sec^2\phi\frac{\partial \phi}{\partial x} = -\frac{y}{x^2}$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = -\frac{y}{x^2\sec^2\phi}$$

$$= -\frac{s\sin\phi}{\cancel{s^2\cos^2\phi\sec^2\phi}}$$

$$= -\frac{1}{s}\sin\phi \quad \boxed{\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{s}}$$

$$\Rightarrow \sec^2\phi\frac{\partial \phi}{\partial y} = \frac{1}{x}$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{x\sec^2\phi}$$

$$= \frac{1}{s\cos\phi\sec^2\phi}$$

$$= \frac{1}{s\sec\phi}$$

$$\boxed{\frac{\partial \phi}{\partial y} = \frac{\cos\phi}{s}}$$

$$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\frac{\partial \hat{s}}{\partial s} = 0$$

$$\frac{\partial \hat{s}}{\partial \phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} = \hat{\phi}$$

$$\frac{\partial \hat{s}}{\partial z} = 0$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\frac{\partial \hat{\phi}}{\partial s} = 0$$

$$\frac{\partial \hat{\phi}}{\partial \phi} = -\cos\phi\,\hat{x} - \sin\phi\,\hat{y} = -\hat{s}$$

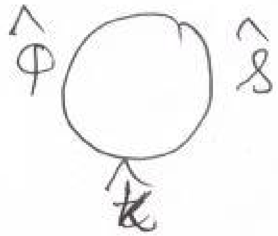
$$\frac{\partial \hat{\phi}}{\partial z} = 0$$

$$\hat{z} = \hat{z}$$

$$\frac{\partial \hat{z}}{\partial s} = 0$$

$$\frac{\partial \hat{z}}{\partial \phi} = 0$$

$$\frac{\partial \hat{z}}{\partial z} = 0$$



$$\hat{s} \times \hat{\phi} = \hat{z}$$

$$\hat{\phi} \times \hat{z} = \hat{s}$$

$$\hat{z} \times \hat{s} = \hat{\phi}$$

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$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \left(\cos\phi \frac{\partial f}{\partial s} - \frac{\sin\phi}{s}\frac{\partial f}{\partial \phi}\right)\left(\cos\phi\,\hat{s} - \sin\phi\,\hat{\phi}\right) +$$

$$\left(\sin\phi \frac{\partial f}{\partial s} + \frac{\cos\phi}{s}\frac{\partial f}{\partial \phi}\right)\left(\sin\phi\,\hat{s} + \cos\phi\,\hat{\phi}\right) + \frac{\partial f}{\partial z}\hat{k}$$

$$= \left(\cos^2\phi \frac{\partial f}{\partial s} - \frac{\sin\phi\cos\phi}{s}\frac{\partial f}{\partial \phi}\right)\hat{s} + \left(-\sin\phi\cos\phi\frac{\partial f}{\partial s} + \frac{\sin^2\phi}{s}\frac{\partial f}{\partial \phi}\right)\hat{\phi}$$

$$+ \left(\sin^2\phi \frac{\partial f}{\partial s} + \frac{\sin\phi\cos\phi}{s}\frac{\partial f}{\partial \phi}\right)\hat{s} + \left(\sin\phi\cos\phi\frac{\partial f}{\partial s} + \frac{\cos^2\phi}{s}\frac{\partial f}{\partial \phi}\right)\hat{\phi}$$

$$+ \frac{\partial f}{\partial z}\hat{k}$$

$$= \left(\cos^2\phi\frac{\partial f}{\partial s} - \cancel{\frac{\sin\phi\cos\phi}{s}\frac{\partial f}{\partial \phi}} + \sin^2\phi\frac{\partial f}{\partial s} + \cancel{\frac{\sin\phi\cos\phi}{s}\frac{\partial f}{\partial \phi}}\right)\hat{s} +$$

$$\left(-\cancel{\sin\phi\cos\phi\frac{\partial f}{\partial s}} + \frac{\sin^2\phi}{s}\frac{\partial f}{\partial \phi} + \cancel{\sin\phi\cos\phi\frac{\partial f}{\partial s}} + \frac{\cos^2\phi}{s}\frac{\partial f}{\partial \phi}\right)\hat{\phi} +$$

$$\frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{\partial f}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \left(\hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{k}\frac{\partial}{\partial z}\right) f$$

$$\Rightarrow \boxed{\vec{\nabla} = \hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{k}\frac{\partial}{\partial z}}$$

$$\boxed{\vec{\nabla} f = \hat{s}\frac{\partial f}{\partial s} + \hat{\varphi}\frac{1}{s}\frac{\partial f}{\partial \varphi} + \hat{u}\frac{\partial f}{\partial z}}$$

$\vec{\nabla}\cdot\vec{V}$ ⑥

$$= \left(\hat{s}\frac{\partial}{\partial s} + \hat{\varphi}\frac{1}{s}\frac{\partial}{\partial\varphi} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(V_s\hat{s} + V_\varphi\hat{\varphi} + V_z\hat{k}\right)$$

$$= \hat{s}\cdot\frac{\partial}{\partial s}\left\{V_s\hat{s} + V_\varphi\hat{\varphi} + V_z\hat{k}\right\} +$$

$$\frac{1}{s}\hat{\varphi}\cdot\frac{\partial}{\partial\varphi}\left\{V_s\hat{s} + V_\varphi\hat{\varphi} + V_z\hat{k}\right\} +$$

$$\hat{k}\cdot\frac{\partial}{\partial z}\left\{V_s\hat{s} + V_\varphi\hat{\varphi} + V_z\hat{k}\right\}$$

$$= \hat{s}\cdot\left\{\hat{s}\frac{\partial V_s}{\partial s} + \cancel{V_s\frac{\partial\hat{s}}{\partial s}} + \cancel{\hat{\varphi}\frac{\partial V_\varphi}{\partial s}} + \cancel{V_\varphi\frac{\partial\hat{\varphi}}{\partial s}} + \cancel{\hat{k}\frac{\partial V_z}{\partial s}} + \cancel{V_z\frac{\partial\hat{k}}{\partial s}}\right\}$$

$$\frac{1}{s}\hat{\varphi}\cdot\left\{\cancel{\hat{s}\frac{\partial V_s}{\partial\varphi}} + V_s\frac{\partial\hat{s}}{\partial\varphi} + \hat{\varphi}\frac{\partial V_\varphi}{\partial\varphi} + V_\varphi\frac{\partial\hat{\varphi}}{\partial\varphi} + \cancel{\hat{k}\frac{\partial V_z}{\partial\varphi}} + \cancel{V_z\frac{\partial\hat{k}}{\partial\varphi}}\right\}$$

$$+ \hat{k}\cdot\left\{\cancel{\hat{s}\frac{\partial V_s}{\partial z}} + \cancel{V_s\frac{\partial\hat{s}}{\partial z}} + \cancel{\hat{\varphi}\frac{\partial V_\varphi}{\partial z}} + \cancel{V_\varphi\frac{\partial\hat{\varphi}}{\partial z}} + \hat{k}\frac{\partial V_z}{\partial z} + \cancel{V_z\frac{\partial\hat{k}}{\partial z}}\right\}$$

$$= \hat{s}\cdot\left\{\hat{s}\frac{\partial V_s}{\partial s} + 0\cdots\right\} +$$

$$\frac{1}{s}\hat{\varphi}\cdot\left\{0 + V_s\hat{\varphi} + \hat{\varphi}\frac{\partial V_\varphi}{\partial\varphi} + \cancel{-V_\varphi\hat{s}}\right\} + \frac{\partial V_z}{\partial z}$$

$$= \frac{s}{s}\frac{\partial V_s}{\partial s} + \frac{1}{s}V_s + \frac{1}{s}\frac{\partial V_\varphi}{\partial\varphi} + \frac{\partial V_z}{\partial z}$$

$$= \frac{1}{s}\frac{\partial}{\partial s}(sV_s) + \frac{1}{s}\frac{\partial V_\varphi}{\partial\varphi} + \frac{\partial V_z}{\partial z}$$

$$\boxed{\vec{\nabla}\cdot\vec{V} = \frac{1}{s}\frac{\partial}{\partial s}(sV_s) + \frac{1}{s}\frac{\partial V_\varphi}{\partial\varphi} + \frac{\partial V_z}{\partial z}}$$

$$\vec{\nabla} \times \vec{V}$$

$$= \left(\hat{s}\frac{\partial}{\partial s} + \hat{\varphi}\frac{1}{s}\frac{\partial}{\partial \varphi} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{V}$$

$$= \hat{s} \times \frac{\partial}{\partial s}\vec{V} + \frac{1}{s}\hat{\varphi} \times \frac{\partial \vec{V}}{\partial \varphi} + \hat{k} \times \frac{\partial \vec{V}}{\partial z}$$

$$= \hat{s} \times \frac{\partial}{\partial s}\left\{\hat{s}V_s + \hat{\varphi}V_\varphi + \hat{k}V_z\right\} +$$

$$\frac{1}{s}\hat{\varphi} \times \frac{\partial}{\partial \varphi}\left\{\hat{s}V_s + \hat{\varphi}V_\varphi + \hat{u}V_z\right\} +$$

$$\hat{u} \times \frac{\partial}{\partial z}\left\{\hat{s}V_s + \hat{\varphi}V_\varphi + \hat{u}V_z\right\}$$

$$= \hat{s} \times \left\{V_s\frac{\partial \hat{s}}{\partial s} + \frac{\partial V_s}{\partial s}\hat{s} + \frac{\partial \hat{\varphi}}{\partial s}V_\varphi + \hat{\varphi}\frac{\partial V_\varphi}{\partial s} + \hat{u}\frac{\partial V_z}{\partial s} + V_s\frac{\partial \hat{u}}{\partial s}\right\}$$

$$\frac{1}{s}\hat{\varphi} \times \left\{V_s\frac{\partial \hat{s}}{\partial \varphi} + \hat{s}\frac{\partial V_s}{\partial \varphi} + \frac{\partial \hat{\varphi}}{\partial \varphi}V_\varphi + \hat{\varphi}\frac{\partial V_\varphi}{\partial \varphi} + \hat{u}\frac{\partial V_z}{\partial \varphi} + \frac{\partial \hat{u}}{\partial \varphi}V_z\right\}$$

$$\hat{u} \times \left\{V_s\frac{\partial \hat{s}}{\partial z} + \hat{s}\frac{dV_s}{\partial z} + \frac{\partial \hat{\varphi}}{\partial z}V_\varphi + \hat{\varphi}\frac{\partial V_\varphi}{\partial z} + \hat{u}\frac{\partial V_z}{\partial z} + \frac{\partial \hat{u}}{\partial z}V_z\right\}$$

$$= \hat{s} \times \left\{\cancel{V_s \times 0} + \cancel{\frac{\partial V_s}{\partial s}\hat{s}} + \cancel{0 \times V_\varphi} + \hat{\varphi}\frac{\partial V_\varphi}{\partial s} + \hat{u}\frac{\partial V_z}{\partial s} + 0\right\} +$$

$$\frac{\hat{\varphi}}{s} \times \left\{\cancel{V_s\hat{\varphi}} + \hat{s}\frac{\partial V_s}{\partial \varphi} + (-\hat{s})V_\varphi + \cancel{\hat{\varphi}\frac{\partial V_\varphi}{\partial \varphi}} + \hat{u}\frac{\partial V_z}{\partial \varphi} + 0\right\} +$$

$$\hat{u} \times \left\{\cancel{V_s \times 0} + \hat{s}\frac{\partial V_s}{\partial z} + \cancel{0} + \hat{\varphi}\frac{\partial V_\varphi}{\partial z} + \cancel{\hat{u}\frac{\partial V_z}{\partial z}} + 0\right\}$$

$$= \frac{\partial V_\varphi}{\partial s}\hat{z} - \frac{\partial V_z}{\partial s}\hat{\varphi} - \frac{1}{s}\frac{\partial V_s}{\partial \varphi}\hat{u} + \frac{V_\varphi}{s}\hat{u} + \frac{1}{s}\frac{\partial V_z}{\partial \varphi}\hat{s} +$$

$$+ \frac{\partial V_s}{\partial z}\hat{\varphi} - \frac{\partial V_\varphi}{\partial z}\hat{s}$$

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$$= \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\phi} +$$

$$\left(\frac{s\,\partial v_\phi}{s\,\partial s} + \frac{v_\phi}{s} - \frac{1}{s}\frac{\partial v_s}{\partial \phi}\right)\hat{u}$$

$$= \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\phi} +$$

$$\frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{u}$$

$$\boxed{\vec{\nabla}\times\vec{V} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{u}}$$

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla}$$

$$= \left(\hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{u}\frac{\partial}{\partial z}\right)\cdot\left(\hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{u}\frac{\partial}{\partial z}\right)$$

$$= \hat{s}\cdot\frac{\partial}{\partial s}\left(\hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{u}\frac{\partial}{\partial z}\right) +$$

$$\frac{1}{s}\hat{\phi}\cdot\frac{\partial}{\partial \phi}\left(\hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{u}\frac{\partial}{\partial z}\right) +$$

$$\hat{u}\cdot\frac{\partial}{\partial z}\left\{\hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{u}\frac{\partial}{\partial z}\right\}$$

$$= \hat{s}\cdot\left\{\cancel{\frac{\partial\hat{s}}{\partial s}\frac{\partial}{\partial s}} + \hat{s}\frac{\partial^2}{\partial s^2} + \cancel{\frac{\partial\hat{\phi}}{\partial s}\frac{1}{s}\frac{\partial}{\partial \phi}} + \cancel{\hat{\phi}\frac{\partial}{\partial s}\left(\frac{1}{s}\right)\frac{\partial}{\partial \phi}} + \cancel{\hat{\phi}\frac{1}{s}\frac{\partial^2}{\partial s\partial \phi}} + \cancel{\frac{\partial\hat{u}}{\partial s}\frac{\partial}{\partial z}} + \cancel{\hat{u}\frac{\partial^2}{\partial s\partial z}}\right\} +$$

$$\frac{1}{s}\hat{\phi}\cdot\left\{\frac{\partial\hat{s}}{\partial \phi}\frac{\partial}{\partial s} + \cancel{\hat{s}\frac{\partial^2}{\partial \phi\partial s}} + \cancel{\frac{\partial\hat{\phi}}{\partial \phi}\frac{1}{s}\frac{\partial}{\partial \phi}} + \cancel{\hat{\phi}\frac{\partial}{\partial \phi}\left(\frac{1}{s}\right)\frac{\partial}{\partial \phi}} + \hat{\phi}\frac{1}{s}\frac{\partial^2}{\partial \phi^2} + \cancel{\frac{\partial\hat{u}}{\partial s}\frac{\partial}{\partial z}} + \cancel{\hat{u}\frac{\partial^2}{\partial \phi\partial z}}\right\} +$$

$$\hat{u}\cdot\left\{\cancel{\frac{\partial\hat{s}}{\partial z}\frac{\partial}{\partial s}} + \cancel{\hat{s}\frac{\partial^2}{\partial z\partial s}} + \cancel{\frac{\partial\hat{\phi}}{\partial z}\frac{1}{s}\frac{\partial}{\partial \phi}} + \cancel{\hat{\phi}\frac{\partial}{\partial z}\left(\frac{1}{s}\right)\frac{\partial}{\partial \phi}} + \cancel{\hat{\phi}\frac{1}{s}\frac{\partial^2}{\partial z\partial \phi}} + \cancel{\frac{\partial\hat{u}}{\partial z}\frac{\partial}{\partial z}} + \hat{u}\frac{\partial^2}{\partial z^2}\right\}$$

$$= \frac{s\,\partial^2}{s\,\partial s^2} + \frac{1}{s^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{s}\frac{\partial}{\partial s}$$

$$= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

$$= \boxed{\nabla^2 = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}}$$