Lecture-7 and 8

Methods for Solving 2st Order Linear Ordinary Diff. Equations

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Second Order Linear Differential Equation

The general form of the second order linear differential equation is

$$\boxed{\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x).}$$

$$\left[\text{or } y'' + P(x)y' + Q(x)y = R(x)\right]$$

Definitions: If $R(x) = 0$, then the equation is called homogeneous otherwise nonhomogenous.

Note: Homogeneous has atleast one solution (trivial).

Aim: To find general solution for both type equations.

Existence and Uniqueness of the solution

Theorem A: Let $P(x)$, $Q(x)$ and $R(x)$ are continuous functions on some interval $[a,\ b]$ and x_0 is any point in $[a,\ b]$, then the differential equation

$$y'' + P(x)y' + Q(x)y = R(x)$$

has one and only one solution on the entire interval such that $y(x_0) = y_0$ and $y'(x_0) = y_0'$.

Note that: $y_0' \neq \dfrac{dy_0}{dx}$. Here $y_0' = \left.\dfrac{dy}{dx}\right|_{x=x_0}$

Remarks and Understanding of Th. A

In other words Theorem A states that the solution can be uniquely determined if we know the value of y and its derivative at a single point in $[a, b]$.

Ex: Show tha $y = x^2 \sin x$ and $y = 0$ are both solution of $x^2 y'' - 4xy' + (x^2 + 6)y = 0$ and satisfies the condition $y(0) = 0$ and $y'(0) = 0$. Does this contradicts Theorem A?

Ans: No. Since $P(x)$ and $Q(x)$ are not continuous, so we can not apply Theorem A.

Principle of Superposition

Theorem B: If $y_1(x)$ and $y_2(x)$ are any two solution of $y'' + P(x)y' + Q(x)y = 0,$ then $c_1y_1(x) + c_2y_2(x)$ is also a solution for any constants c_1 and c_2.

Proof:

As $\left(c_1y_1 + c_2y_2\right)'' + P(x)\left(c_1y_1 + c_2y_2\right)' + Q(x)\left(c_1y_1 + c_2y_2\right)$

$$= c_1\left[y_1'' + Py_1' + Qy_1\right] + c_2\left[y_2'' + Py_2' + Qy_2\right]$$

$$= c_1.0 + c_2.0 = 0$$

$\therefore\ (c_1y_1 + c_2y_2)$ is a solution.

Principle of Superposition

Exercise: Let $y_1(x)$ and $y_2(x)$ be two solution of $y'' + P(x)y' + Q(x)y = R(x)$. What about the following two functions ?

(a) $c_1 y_1(x) + c_2 y_2(x)$

(b) $y_1(x) - y_2(x)$

Solution:

(a) $c_1 y_1 + c_2 y_2$ will be a solution provided $c_1 + c_2 = 1.$

(b) $y_1 - y_2$ is a solution of $y'' + P(x)y' + Q(x)y = 0.$

Exercise Problems

Ex-1: If $y_1(x)$ and $y_2(x)$ are two solution of $y'' + P(x)y' + Q(x)y = 0,$ on $[a,b]$ and have a common zero in $[a,b]$, then show that one solution is constant multiple of the other.

Solution :

Let x_0 be the common zero of y_1 and $y_2 \Rightarrow y_1(x_0) = y_2(x_0) = 0$

If $y_1'(x_0) = 0$ or $y_2'(x_0) = 0,$ then by Theorem A, $y_1(x) = 0$ or $y_2(x) = 0$. Hence proved the result.

Othewise there exist a constant c such that $y_1'(x_0) = cy_2'(x_0)$ and then apply Theorem A on the solutions $Y_1(x) = y_1(x) - cy_2(x)$ and $Y_2(x) = 0.$

Wronskian of Two Solutions

Definition: If $y_1(x)$ and $y_2(x)$ are two solution of $y'' + P(x)y' + Q(x)y = 0,$ then the Wronskian of $y_1(x)$ and $y_2(x)$ is denoted as $W(y_1, y_2)$ and defined by

$$\boxed{W(y_1, y_2) = y_1 y_2' - y_2 y_1'.}$$

Important Results on Wronskian

Lemma A: If $y_1(x)$ and $y_2(x)$ are two solution of $y'' + P(x)y' + Q(x)y = 0$ on $[a, b]$, then their Wronskian $W(y_1, y_2)$ is either identically zero or never vanishes on on $[a, b]$.

Note: The lemma st a te that if the Wronskian is non zero at a single point then it is non zero thoughout the interval.

In otherwords, If the Wronskian is zero at a single point then it vanishes thoughout the interval.

Proof of Lemma A

Proof: We begin by observing that

$$W' = y_1 y_2'' - y_2 y_1''$$

Next since y_1 and y_2 are both solutions, we have

$$y_1'' + P y_1' + Q\, y_1 = 0.$$

and

$$y_2'' + P y_2' + Q\, y_2 = 0.$$

Proof of Lemma A

On multiplying the first equation by y_2 and the second by y_1 and subtracting the first from the second, we obtain

$$(y_1 y_2'' - y_2 y_1'') + P\left(y_1 y_2' - y_2 y_1'\right) = 0$$

$$\Rightarrow \frac{dW}{dx} + PW = 0$$

The general solution of this equation is $W = Ce^{-\int P dx}$

Since the exponential function is never zero, we see that W is identically zero if C = 0 or never zero if C≠0.

Important Results on Wronskian

Lemma B: If $y_1(x)$ and $y_2(x)$ are two solution of $y'' + P(x)y' + Q(x)y = 0$ on $[a,b]$, then $y_1(x)$ and $y_2(x)$ are linearly dependent if and only if their Wronskian $W(y_1, y_2)$ is either identically zero.

Note-1: The concept of Wronskian can not be used for proving two functions are Linear dependent/LI.

Note-2: The Lemma B is applicable only for proving two solutions are Linearly dependent or linearly independent.

Exercise Problems

Ex-1: Consider two function $f(x) = x^3$ and $g(x) = x^2 |x|$ on the interval $[-1, 1]$.

(a) Show that $W(f, g) = 0$.

(b) Show that f and g are linearly independent.

(c) Does (a) and (b) contradicts Lemma B ? Justify.

Solution:

(c) There are not contradicting Lemma B since f and g can not be solution of same differential equation on $[-1,1]$.

Application of Wronskian for General functions

In general we have the following results:

Given two functions $f(x)$ and $g(x)$ that are differentiable on some interval I.

(a) If for some x_0 in I, $W(f, g)(x_0) \neq 0$ then $f(x)$ and $g(x)$ are linearly independent on the interval I.

(b) If $f(x)$ and $g(x)$ are linearly dependent on I then $W(f, g)(x) = 0$ for all x in the interval I.

It DOES NOT say that if $W(f, g)(x) = 0$ then $f(x)$ and $g(x)$ are linearly dependent. In fact it is possible for two linearly independent functions to have a zero Wronskian.

Exercise Problems

Ex-1: Use the wronskian to prove that two solution of the equation $y'' + P(x)y' + Q(x)y = 0,$ on $[a,b]$ are linearly dependent if

(a) they have a common zero in $[a,b]$.

(b) they have maxima or minima at the same point in $[a,b]$.

Solution :

(a) Since $W(y_1, y_2)\big|_{x=x_0} = 0$

(b) Since $W(y_1, y_2)\big|_{x=x_0} = 0$

Fundamental Theorems

Theorem C: Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogeneous differential equation

$$y'' + P(x)y' + Q(x)y = 0 \qquad ---(1)$$

on the interval [a, b]. Then $c_1 y_1(x) + c_2 y_2(x)$ is the general solution of (1) on [a, b].

Fundamental Theorems

Theorem D: If y_g is the general solution of the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0$$

and y_p is any particular solution of the nonhomogeneous equation

$$y'' + P(x)y' + Q(x)y = R(x) \quad \text{------}(2)$$

then $y_g + y_p$ is the general solution of (2).