Craig interpolation

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In mathematical logic, **Craig's interpolation theorem** is a result about the relationship between different logical theories. Roughly stated, the theorem says that if a formula φ implies a formula ψ then there is a third formula ρ, called an interpolant, such that every nonlogical symbol in ρ occurs both in φ and ψ, φ implies ρ, and ρ implies ψ. The theorem was first proved for first-order logic by William Craig in 1957. Variants of the theorem hold for other logics, such as propositional logic. A stronger form of Craig's theorem for first-order logic was proved by Roger Lyndon in 1959; the overall result is sometimes called the **Craig–Lyndon theorem**.

Contents

Example

In propositional logic, let

$$\varphi = \sim(P \wedge Q) \rightarrow (\sim R \wedge Q)$$
$$\psi = (T \rightarrow P) \vee (T \rightarrow \sim R).$$

Then φ tautologically implies ψ. This can be verified by writing φ in conjunctive normal form:

$$\varphi \equiv (P \vee \sim R) \wedge Q.$$

Thus, if φ holds, then $(P \vee \sim R)$ holds. In turn, $(P \vee \sim R)$ tautologically implies ψ. Because the two propositional variables occurring in $(P \vee \sim R)$ occur in both φ and ψ, this means that $(P \vee \sim R)$ is an interpolant for the implication $\varphi \rightarrow \psi$.

Lyndon's interpolation theorem

Suppose that *S* and *T* are two first-order theories. As notation, let $S \cup T$ denote the smallest theory including both *S* and *T*; the signature of $S \cup T$ is the smallest one containing the signatures of *S* and *T*. Also let $S \cap T$ be the intersection of the two theories; the signature of $S \cap T$ is the intersection of the signatures of the two theories.

Lyndon's theorem says that if $S \cup T$ is unsatisfiable, then there is an interpolating sentence ρ in the language of $S \cap T$ that is true in all models of *S* and false in all models of *T*. Moreover, ρ has the stronger property that every relation symbol that has a positive occurrence in ρ has a positive occurrence in some formula of *S* and a negative occurrence in some formula of *T*, and every relation symbol with a negative occurrence in ρ has a negative occurrence in some formula of *S* and a positive occurrence in some formula of *T*.

Proof of Craig's interpolation theorem

We present here a constructive proof of the Craig interpolation theorem for propositional logic.[1] Formally, the theorem states:

If $\vDash\varphi \rightarrow \psi$ *then there is a* ρ *(the* interpolant*) such that* $\vDash\varphi \rightarrow \rho$ *and* $\vDash\rho \rightarrow \psi$*, where* $atoms(\rho) \subseteq atoms(\varphi) \cap atoms(\psi)$*. Here* $atoms(\varphi)$ *is the set of propositional variables occurring in* φ*, and* ⊨ *is the semantic entailment relation for propositional logic.*

Proof. Assume $\vDash\varphi \rightarrow \psi$. The proof proceeds by induction on the number of propositional variables occurring in φ that do not occur in ψ, denoted $|atoms(\varphi) - atoms(\psi)|$.

Base case $|atoms(\varphi) - atoms(\psi)| = 0$: In this case, φ is suitable. This is because since $|atoms(\varphi) - atoms(\psi)| = 0$, we know that $atoms(\varphi) \subseteq atoms(\varphi) \cap atoms(\psi)$. Moreover we have that $\vDash\varphi \rightarrow \varphi$ and $\vDash\varphi \rightarrow \psi$. This suffices to show that φ is a suitable interpolant in this case.

Next assume for the inductive step that the result has been shown for all χ where $|atoms(\chi) - atoms(\psi)| = n$. Now assume that $|atoms(\varphi) - atoms(\psi)| = n+1$. Pick a $p \in atoms(\varphi)$ but $p \notin atoms(\psi)$. Now define:

$\varphi' := \varphi[\top/p] \lor \varphi[\bot/p]$

Here $\varphi[\top/p]$ is the same as φ with every occurrence of p replaced by ⊤ and $\varphi[\bot/p]$ similarly replaces p with ⊥. We may observe three things from this definition:

$$\vDash\varphi' \rightarrow \psi \quad (1)$$

$$|atoms(\varphi') - atoms(\psi)| = n \quad (2)$$

$$\vDash\varphi \rightarrow \varphi' \quad (3)$$

From (**1**), (**2**) and the inductive step we have that there is an interpolant ρ such that:

$$\vDash\varphi' \rightarrow \rho \quad (4)$$

$$\vDash\rho \rightarrow \psi \quad (5)$$

But from (**3**) and (**4**) we know that

$$\vDash\varphi \rightarrow \rho \quad (6)$$

Hence, ρ is a suitable interpolant for φ and ψ.

QED

Since the above proof is constructive, one may extract an algorithm for computing interpolants. Using this algorithm, if $n = |atoms(\varphi') - atoms(\psi)|$, then the interpolant ρ has $O(EXP(n))$ more logical connectives than φ (see Big O Notation for details regarding this assertion). Similar constructive proofs may be provided for the basic modal logic K, intuitionistic logic and μ−calculus, with similar complexity measures.

Craig interpolation can be proved by other methods as well. However, these proofs are generally non−constructive:

- model−theoretically, via Robinson's joint consistency theorem: in presence of compactness, negation and conjunction, Robinson's joint consistency theorem and Craig interpolation are equivalent.
- proof−theoretically, via a Sequent calculus. If cut elimination is possible and as a result the subformula property holds, then Craig interpolation is provable via induction over the derivations.
- algebraically, using amalgamation theorems for the variety of algebras representing the logic.
- via translation to other logics enjoying Craig interpolation.

Applications

Craig interpolation has many applications, among them consistency proofs, model checking, proofs in modular specifications, modular ontologies.

References

1. ^ Harrison pgs. 426−427

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