Note on Induced Norms

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May 3, 2019

1 Vector Norms

Introduce p norm with special cases $p = 0, 1, 2, \infty$.

2 Induced Vector Norms

3 Induced Matrix Norms

An induced matrix norm is a function of an arbitrary matrix $A \in \mathbb{R}^{n \times d}$ of the form

$$||A||_{p \to q} = \sup_{\{x \in \mathbb{R}^d : ||x||_p \leq 1\}} ||Ax||_q.$$

Intuitively, the induced norm of a matrix A measures, or more precisely limits, how far it can distort a vector x. In order for this statement to make sense, we need to limit the size of applicable vectors x by limiting the q norm of x to be 1 (note that the limitation $||x||_p \leq 1$ simplifies is in practice $||x||_p = 1$). The size of the "distorted x" Ax is measured using the q norm. A common notational shorthand when $p = q$ is to write

$$||A||_p = ||A||_{p \to p}$$

For some pairs of p and q, the induced norm $||A||_{p \to q}$ is analytically directly accessible, see Table 1.

3.1 Example: Induced Norm of a 2×2 matrix

In order to get a better handle on this theoretical concept, consider the induced norm of the following matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}.$$

The applicable input vectors $\{x \in \mathbb{R}^d : ||x||_p = 1\}$ for the $|| \cdot ||_p = 1, 2$ and ∞ norm, are depicted in Figure 1.

Table 1: Analytically accessible induced norms $||A||_{p\to q}$ for domain p and co-domain q. NP hard stands for "non-deterministic polynomial-time hardness" which means not computable for our purposes.

			q	
		1	2	∞
p	1	max l_1 norm of a column	max l_2 norm of a column	max l_∞ norm of a column
	2	NP-hard	max singular value	max l_2 norm of a row
	∞	NP-hard	NP-hard	max l_1 norm of a row

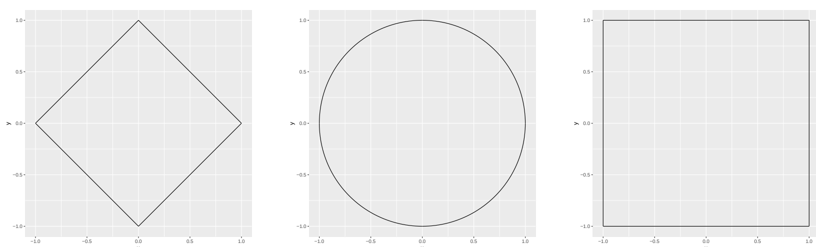

Figure 1: Applicable x on the unit circle of the l_1 (left), l_2 (center), an l_∞ norm (right). All x on the unit circle have $||x||_p = 1$, for $p = 1, 2, \infty$ respectively. Note that the applicable input vectors x are **not** specific to the choice of A but only to the choice of the p.

In order to find the supremum over all applicable input vectors, all applicable x are multiplied by A. The points Ax are depicted in Figure 2. The induced norm is defined as the maximum q norm of all points $\{Ax\}$.

We can use Table 1 to compute $||||A||_{p\to q}$ for some combinations of p and q, see values reported in Table 2. Figure 3 shows Ax with q-norm circles of radius $||A||_{p\to q}$ around the origin. The intersection of the q norm circles with $Ax : ||x||_p = 1$ shows which Ax achieves the induced norm, i.e. the $\sup_x ||Ax||_q$.

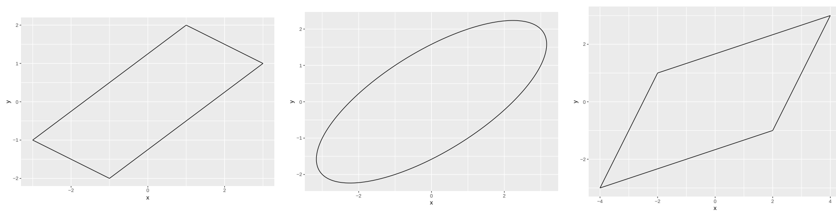

Figure 2: Projection of the unit circles by A, i.e. Ax for every point $\{x \in \mathbb{R}^d : ||x||_p = 1\}$ for $p = 1$ (left), $p = 2$ (center), and $p = \infty$ (right).

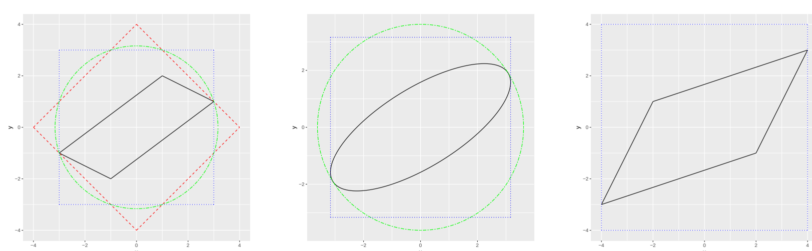

Figure 3: The colored circles are unit circles of size $||A||_{p \to q}$ for every point $x, ||x||_p = 1$ with p = 1 (left), $p = 2$ (center), an $p = \infty$ (right). The 1 norm is depicted in red, the 2 norm in green, and the ∞ norm in blue. Induced norms that are hard to compute were omitted.

Table 2: Analytically accessible results $||A||_{p \to q}$ for our example. These values match up with the size of the colored circles in Figure 3.

		q		
		1	2	∞
p	1	$\max(\lvert 1\rvert + \lvert 3\rvert, \lvert 1\rvert + \lvert 2\rvert) = 4$	$\max(\sqrt{10}, \sqrt{5}) \approx 3.16$	$\max(\lvert 3\rvert, \lvert 2\rvert) = 3$
	2	NP-hard	$\sigma_{\max} \approx 3.62$	$\max(\sqrt{10}, \sqrt{5}) \approx 3.16$
	∞	NP-hard	NP-hard	$\max(\lvert 1\rvert + \lvert 3\rvert, \lvert 1\rvert + \lvert 2\rvert) = 4$