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A Critical Analysis of the Modified Equation Technique of Warming and Hyett

SIN-CHUNG CHANG

Computational Methods Branch, NASA Lewis Research Center, Cleveland, Ohio 44135

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Warming and Hyett developed a modified equation technique in which the behavior of a difference scheme is evaluated by using the coefficients of a certain modified equation. Specifically, they discovered a connection between these coefficients and the multiplication factor obtained from the von Neumann analysis. Since the dissipation and dispersion of error components are determined by the multiplication factor, the former properties can be studied using the coefficients of the modified equation. The work of Warming and Hyett represents a key step in the development of the method of modified equations. Through this work, it became clear that modified equations should be derived from the difference scheme rather than from the original differential equation. However, in order to "prove" the above connection, Warming and Hyett incorrectly interpreted their modified equations as the actual partial differential equations solved by the difference schemes. The main purpose of the current study is to investigate rigorously the above connection without using their interpretation. The result of this investigation shows that the above connection is only partially valid for multilevel schemes. In the von Neumann analysis, the multiplication factor associated with a wave number generally has $(L-1)$ roots for an L-level scheme. It is shown that the coefficients of the modified equation provide information for only the principal root. © 1990 Academic Press, Inc.

INTRODUCTION

The method of modified equations is an important tool in the design and analysis of difference schemes for linear and nonlinear time-dependent problems. Extensive lists of publications on this subject are given in two recent papers [1] by Goodman and Majda and [2] by Griffiths and Sanz-Serna. A general discussion on the theoretical foundation and applicability of this method can also be found in [2] (hereafter referred to as GS).

According to GS, the modified equation technique for the analysis of a numerical scheme consists of the construction of a modified differential equation in such a way that the numerical solutions are more accurately matched by the solutions of the modified equation than by the solutions of the original differential equation being solved by the numerical scheme. In other words, the behavior of the numerical scheme is better described by the modified equation than by the original differential equation. Since a particular solution of the modified equation cannot be specified without the necessary initial/boundary conditions, in applying the modified equa-

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tion technique, one should consider modified *problems*, i.e., the modified equation should be supplemented by the necessary initial/boundary conditions.

An exception to the above description of modified equations is the work of Warming and Hyett [3] (hereafter referred to as WH). This work represents a key step in the development of the method of modified equations. Through WH, it became clear that modified equations should be derived from the difference scheme rather than from the original differential equation. The significance of WH is also reflected by the dominant role it plays in Chapter 4 of a recent textbook by Anderson, Tannehill, and Pletcher [4]. In WH, the behavior of a given numerical scheme is evaluated by using the *coefficients* of a special modified equation constructed from the numerical scheme. *The solutions of the modified equation are not used in this evaluation.* Specifically, Warming and Hyett discovered a connection between these coefficients and the error multiplication factor which one obtains in the von Neumann analysis. Since the dissipation and dispersion of error components are determined by the multiplication factor, the former properties can be studied using the coefficients of the modified equation. Note that an exponential function appears on page 166 of WH (see also Eq. (1.27) of the current paper) as an "elementary" soution of the modified equation. However, this solution is not intended to be an approximation of a numerical solution as described in GS. It is introduced only as a part of Warming and Hyett's effort to "prove" the above connection.

The modified equations derived in WH differ from most other modified equations in two important aspects: First, the modified equations considered by Warming and Hyett contain spatial derivatives of arbitrarily high order while those considered by most authors are differential equations of finite order. Shokin is another author who considers modified equations of arbitrary order [5].

Second, the modified equations considered in WH are given a more specific interpretation than that generally given to modified equations; i.e., the former are interpreted by Warming and Hyett as the actual partial differential equations solved by difference schemes. *It should be noted that their more specific interpretation is essential for the develoment in WH, i.e., it is used to establish the connection between the multiplication factor and the coefficients of the corresponding modified equation.* This fact is stated on page 171 in WH; i.e., "But since the modified equation represents the exact partial differential equation solved by a finite-difference scheme, the amplification factor (5.1) must equal the amplification factor (4.4) of the difference scheme." The above quoted statement also indicates that Warming and Hyett's proof of the above connection, contrast to some arguments given in WH, is not heuristic in nature.

Note that the validity of the connection mentioned above is not dependent on the mesh size. As a result, Warming and Hyett's interpretation cannot be used to prove this connection unless it means to apply to difference schemes of arbitrary mesh size. With this understanding, it becomes obvious that *their interpretation really has no clear meaning since one cannot define uniquely "the actual partial differential equation" solved by a given difference scheme with a mesh of finite interval.* This can be seen by the following argument: For such a difference scheme, there are infinitely many discrete solutions corresponding to different initial/boundary conditions. Given any one of these discrete solutions, there exist infinitely many smooth functions whose values at the mesh points coincide with those of the discrete solution. These smooth functions generally do not satisfy the same partial differential equation. Q.E.D.

Since no clear definition of Warming and Hyett's interpretation is given in WH, one can only infer its meaning from how this interpretation is used. The arguments used by Warming and Hyett to establish the connection between the multiplication factor and the modified equation (pp. 166, 171 in WH) are reconstructed and presented in Section 1 of the current paper in an easier to understand manner. From the role it plays in these arguments and other statements made in WH, it appears that Warming and Hyett's interpretation means *a smooth function should satisfy a special modified equation if its values at the mesh points form a solution of the difference scheme from which the modified equation is constructed.* Using the argument which is presented earlier to demonstrate the ambiguity of Warming and Hyett's interpretation, one may conclude that the above italicized statement cannot be valid generally.

It is interesting to note that Warming and Hyett realized that their interpretation of modified equations must be qualified. A paragraph in WH (p. 165) contains the following statement "This assertion of the equivalence of the modified equation and the difference algorithm should be qualified since the modified equation contains spatial derivatives of arbitrarily high order. Thus, strictly speaking, an infinite number of boundary conditions is required to define a solution. In our analysis (Sections 3 and 5), we assume spatial periodicity to replace the required boundary conditions." However, the above qualified interpretation may be contradicted by an example given in Section 1.

In this paper we will rigorously study the connection between a multiplication factor and the corresponding modified equation without using Warming and Hyett's interpretation. The result of this study reveals that this connection is only partially valid for a class of schemes involving more than two time levels. Note that, in the von Neumann analysis, the multiplication factor associated with a wave number generally has $(L-1)$ roots for an L-level scheme. Among these roots, one is the principal root while the rest are spurious roots. The current analysis reveals that the modified equation constructed according to the procedure specified in WH provides information for only the principal root. This inadequacy is not apparent in WH since it considers only two-level model problems which have no spurious roots for the multiplication factor. In Ref. [4], however, the inadequacy became apparent when the technique developed in WH is used to study a three-level scheme, i.e., the leap-frog method. Given a wave number, there are two solutions for the relative phase error according to the von Neumann method while only one solution can be obtained by using the modified equation approach (see Eqs. (4-24) and (4-40) in [4]).

Note that another difficulty in applying the idea of modified equations to multilevel schemes is described on pages 1000–1001 of GS. For a multilevel scheme, the main difference scheme must be supplemented by other starting conditions (which are not the initial conditions—see Eq. (3.5b) in GS). Generally, the solutions of a modified equation satisfy the starting conditions at an order of correctness lower than they satisfy the main difference scheme. As a result, a modified problem cannot attain the order of correctness which one would expect if only the main difference scheme is considered. Since the roots of a multiplication factor are completely determined by the main difference scheme in the von Neumann analysis, it is obvious that the difficulty described in GS regarding multilevel schemes differs from what we describe here in both its nature and origin.

The remainder of the paper is briefly described as follows: In Section 1, we review the work of Warming and Hyett. We also point out its deficiencies and describe the correct way to understand the relation between the von Neumann analysis and the modified equation stability analysis developed in WH. Note that this section describes essentially all the key ideas of the current work with minimum mathematical details. Thus a reader who is not particularly mathematically inclined may gain enough understanding of the current work without reading beyond Section 1. In Section 2, the von Neumann analysis is applied to a class of L-level difference schemes. It is shown that the coefficients of the modified equation provide information for the principal root but not the spurious roots. Finally, in Section 3, we summarize and discuss the key results of the current investigation.

1. A Critical Review of the Work by Warming and Hyett

The linear partial differential equations considered in WH and the current paper have the form

$$\frac{\partial u}{\partial t} + \mathscr{L}_x(u) = 0, \tag{1.1}$$

where

$$\mathscr{L}_x(u) \stackrel{\text{def}}{=} \sum_{m=1}^{M} c_m \left(\frac{\partial^m}{\partial x^m}\right), \qquad M = 1, 2, 3, ... \tag{1.2}$$

The dependent variable u in Eq. (1.1) is a function of a spatial variable x and a temporal variable t. The coefficients c_m in Eq. (1.2) are assumed to be real constants. Note that $\mathscr{L}_x$ was originally defined in WH (p. 160) as a linear spatial differential operator; i.e., the coefficients c_m could be functions of x and t. However, as shown in [6], this will result in the breakdown of the modified equation derivation procedure developed in WH. No difficulty occurs in WH since it deals only with constant coefficient model problems.

In the following, the essence of WH will be discussed using a numerical example. To proceed, we consider a specific example of Eq. (1.1); i.e.,

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0, \tag{1.3}$$

where c is a real constant. Equation (1.3) may be solved numerically using the upwind difference scheme; i.e.,

$$\frac{(u_j^{n+1}-u_j^n)}{\Delta t}+\frac{c(u_j^n-u_{j-1}^n)}{\Delta x}=0, \tag{1.4}$$

where Δx and Δt are grid intervals in the x- and t-directions, respectively. u_j^n denotes the finite-difference solution at the mesh point where $x=j\,\Delta x$ and $t=n\,\Delta t$.

The work by Warming and Hyett is closely related to the von Neumann stability analysis. As a preliminary to the correct understanding of this relation, first we present several important results of the von Neumann analysis. Let

$$u_j^n=G^n e^{ikj\Delta x}, \qquad i\equiv\sqrt{-1}, \tag{1.5}$$

where G is the multiplication factor, and k the real wave number. Substituting Eq. (1.5) into Eq. (1.4), one obtains

$$G=(1-\xi)+\xi e^{-ik\Delta x}, \tag{1.6}$$

where ξ is the courant number, i.e.,

$$\xi \overset{\text{def}}{=} c\,\Delta t/\Delta x. \tag{1.7}$$

Let β be a complex variable. It is shown in Section 2 that, in a neighborhood Ψ_1 of $\beta=0$, there exists an analytic function $\alpha_1(\beta)$ such that

$$\text{(a)} \qquad e^{\alpha_1(\beta)\Delta t}=(1-\xi)+\xi e^{-\beta\Delta x}, \qquad \beta\in\Psi_1 \tag{1.8}$$

and

$$\text{(b)} \qquad \alpha_1(0)=0. \tag{1.9}$$

Note that the right side of Eq. (1.8) is reduced to that of Eq. (1.6) if $\beta=ik$. Thus

$$G=e^{\alpha_1(ik)\Delta t}, \qquad ik\in\Psi_1. \tag{1.10}$$

Using the analyticity of $\alpha_1(\beta)$ and Eq. (1.9), one concludes that

$$\alpha_1(\beta)=\sum_{p=1}^{\infty} v_1(p)\beta^p, \qquad ik\in\Psi_1, \tag{1.11}$$

where

$$v_1(p)\overset{\text{def}}{=}\frac{1}{p!}\left[\frac{d^p\alpha_1(\beta)}{d\beta^p}\right]_{\beta=0}, \qquad p=1,2,3,\ldots \tag{1.12}$$

From Eq. (1.8), one obtains

$$v_1(1)=-c, \qquad v_1(2)=\frac{c\,\Delta x(1-\xi)}{2}, \qquad v_1(3)=\frac{-c(\Delta x)^2\,(2\xi^2-3\xi+1)}{6}, \ldots. \tag{1.13}$$

Obviously, $\nu_1(1)$, $\nu_1(2)$, and $\nu_1(3)$, are all real. In fact, it is shown in Section 2 that $\nu_1(p)$ are real for all $p \geqslant 1$. This coupled with Eq. (1.11) implies that

$$\mathrm{Re}[\alpha_1(ik)] = \sum_{p=1}^{\infty} (-1)^p k^{2p} \nu_1(2p), \qquad ik \in \Psi_1 \tag{1.14}$$

and

$$\mathrm{Im}[\alpha_1(ik)] = \sum_{p=0}^{\infty} (-1)^p k^{2p+1} \nu_1(2p+1), \qquad ik \in \Psi_1, \tag{1.15}$$

where $\mathrm{Re}[\alpha_1(ik)]$ and $\mathrm{Im}[\alpha_1(ik)]$ are, respectively, the real and imaginary parts of $\alpha_1(ik)$. According to Eqs. (1.5) and (1.10), the dissipation and dispersion of an error component are determined, respectively, by $\mathrm{Re}[\alpha_1(ik)]$ and $\mathrm{Im}[\alpha_1(ik)]$. Thus it follows from Eqs. (1.14) and (1.15) that the dissipation and dispersion of an error component with $ik \in \Psi_1$ are determined, respectively, by $\nu_1(2p)$ $(p = 1, 2, ...)$ and $\nu_1(2p+1)$ $(p = 0, 1, 2, ...)$. Note that this conclusion is a direct result of the von Neumann analysis involving Eq. (1.4). No ideas from modified equations were used in this derivation.

The coefficients $\nu_1(p)$ can be evaluated using Eq. (1.12) if $\alpha_1(\beta)$ is explicitly given. As it turns out, they can also be evaluated using the modified equations derivation procedure developed in WH (hereafter, this procedure is referred to as the W–H procedure). In the Appendix, it will be shown that the coefficients $\nu_1(p)$, $p = 1, 2, 3, ...,$ are identical to the coefficients of the modified equation generated from the difference scheme (1.4). This fact coupled with Eqs. (1.14), (1.15), and (1.10) makes it possible to study the dissipative and dispersive errors of the difference scheme (1.4) by using the coefficients of the corresponding modified equation.

The argument used in WH to establish the relation between the behavior of a difference scheme and the corresponding modified equation are different from those presented above. In the following, these arguments will be discussed. Specifically, we will (i) briefly describe the W–H procedure, (ii) discuss the validity of the individual steps within the W–H procedure, (iii) provide a counterexample to Warming and Hyett's interpretation of their modified equations, (iv) explain how Warming and Hyett established the connection between the multiplication factor G of a difference scheme and the coefficients of the corresponding modified equation by using their interpretation of modified equations, and (v) reinterpret the W–H procedure in a way such that it is free of the criticisms cited in (ii) and (iii) and also provides the basis for rigorous discussions given in the later sections.

(i) *A Description of the W–H Procedure*

In the W–H procedure, Warming and Hyett consider a smooth function $u(x, t)$ such that $u_j^n \stackrel{\mathrm{def}}{=} u(j\,\Delta x, n\,\Delta t)$ satisfies Eq. (1.4) at all mesh points. Substituting the

Taylor series expansions of u_j^{n+1} and u_{j-1}^n about the mesh point $(j\,\Delta x, n\,\Delta t)$ into Eq. (1.4), one obtains

$$\left[\left(\frac{\partial u}{\partial t}\right)_j^n + c\left(\frac{\partial u}{\partial x}\right)_j^n\right] + \left[\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_j^n - \frac{c\,\Delta x}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n\right]$$
$$+\left[\frac{(\Delta t)^2}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)_j^n + \frac{c(\Delta x)^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_j^n\right] + \cdots = 0, \tag{1.16}$$

where $(\partial u/\partial t)_j^n, (\partial u/\partial x)_j^n, \ldots$, respectively, are the values of $\partial u/\partial t, \partial u/\partial x, \ldots$, at the mesh point $(j\,\Delta x, n\,\Delta t)$. In the W–H procedure, *Warming and Hyett assume that Eq. (1.16) is valid beyond the mesh points*; *i.e.*,

$$\left[\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right] + \left[\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - \frac{c\,\Delta x}{2}\frac{\partial^2 u}{\partial x^2}\right]$$
$$+\left[\frac{(\Delta t)^2}{6}\frac{\partial^3 u}{\partial t^3} + \frac{c(\Delta x)^2}{6}\frac{\partial^3 u}{\partial x^3}\right] + \cdots = 0 \tag{1.17}$$

for all $(x, t) \in \Gamma$, where Γ is a continuous domain of x and t. Note that brackets are iserted on the left sides of Eqs. (1.16) and (1.17) to indicate that a term in a partial sum of the convergent series on the left side of Eq. (1.16) or Eq. (1.17) should be the entire expression within one set of brackets [7, pp. 370–371]. This is significant since the convergence property of a series is dependent on how the terms within it are grouped [7, pp. 332–333]. Note that, except for missing brackets, Eq. (1.5) in WH (p. 164) will replace Eq. (1.17) if the Lax–Wendroff scheme; i.e., Eq. (1.3) in WH (p. 161) takes the place of the upwind scheme (1.4).

Next, in order to obtain a partial differential equation with the form (1.23), time derivatives higher than first order are "eliminated" from Eq. (1.17) by a method to be described below. The method requires repeated use of Eq. (1.17) itself. To eliminate the $\partial^2 u/\partial t$ term, Eq. (1.17) is multiplied term-by-term by the operator $-(\Delta t/2)\partial/\partial t$. The result is then added to Eq. (1.17). The resulting new equation has a mixed derivative term $-(c\,\Delta t/2)\partial^2 u/\partial t \partial x$ which, in turn, can be eliminated by applying the operator $(c\,\Delta t/2)\partial/\partial x$ to Eq. (1.17) term-by-term and adding the result to the new equation. This elimination procedure can be organized into a table as illustrated in Table I. The first two rows list the derivatives through third order and their coefficients appearing in Eq. (1.17). The subsequent rows list the coefficients of the derivative terms obtained after operation on Eq. (1.17) with the differential operators shown on the left-most column. The table is continued until the desired number of time derivatives are eliminated. For any $n = 1, 2, 3, \ldots$, the elimination procedure leads to

$$M(n) + R(n) = 0, \tag{1.18}$$

TABLE I. The Modified Equation Derivation Procedure

Partial derivatives	$\frac{\partial u}{\partial t}$	$\frac{\partial u}{\partial x}$	$\frac{\partial^2 u}{\partial t^2}$	$\frac{\partial^2 u}{\partial t\,\partial x}$	$\frac{\partial^2 u}{\partial x^2}$	$\frac{\partial^3 u}{\partial t^3}$	$\frac{\partial^3 u}{\partial t^2\,\partial x}$	$\frac{\partial^3 u}{\partial t\,\partial x^2}$	$\frac{\partial^3 u}{\partial x^3}$
Coefficients of Eq. (1.17)	1	c	$\frac{\Delta t}{2}$	0	$-\frac{c\,\Delta x}{2}$	$\frac{(\Delta t)^2}{6}$	0	0	$\frac{c(\Delta x)^2}{6}$
$-\frac{\Delta t}{2}\frac{\partial}{\partial t}$ Eq. (1.17)			$-\frac{\Delta t}{2}$	$-\frac{c\,\Delta t}{2}$	0	$-\frac{(\Delta t)^2}{4}$	0	$\frac{c\,\Delta t\,\Delta x}{4}$	0
$\frac{c}{2}\Delta t\frac{\partial}{\partial x}$ Eq. (1.17)				$\frac{c\,\Delta t}{2}$	$\frac{c^2\,\Delta t}{2}$	0	$\frac{c(\Delta t)^2}{4}$	0	$-\frac{c^2\,\Delta t\,\Delta x}{4}$
$\frac{1}{12}(\Delta t)^2\frac{\partial^2}{\partial t^2}$ Eq. (1.17)						$\frac{(\Delta t)^2}{12}$	$\frac{c(\Delta t)^2}{12}$	0	0
$-\frac{1}{3}c(\Delta t)^2\frac{\partial^2}{\partial t\,\partial x}$ Eq. (1.17)							$-\frac{c(\Delta t)^2}{3}$	$-\frac{c^2(\Delta t)^2}{3}$	0
$\left(\frac{c^2(\Delta t)^2}{3}-\frac{c\,\Delta t\,\Delta x}{4}\right)\frac{\partial^2}{\partial x^2}$ Eq. (1.17)								$\frac{c^2(\Delta t)^2}{3}-\frac{c\,\Delta t\,\Delta x}{4}$	$\frac{c^3(\Delta t)^2}{3}-\frac{c^2\,\Delta t\,\Delta x}{4}$
...									
Sum of coefficients	1	c	0	0	$\frac{c\,\Delta x(\xi-1)}{2}$	0	0	0	$\frac{c(\Delta x)^2(2\xi^2-3\xi+1)}{6}$

where

$$M(n) \stackrel{\text{def}}{=} \frac{\partial u}{\partial t} - \mu(1)\frac{\partial u}{\partial x} - \mu(2)\frac{\partial^2 u}{\partial x^2} - \cdots - \mu(n)\frac{\partial^n u}{\partial x^n} \tag{1.19}$$

and

$$R(n) \stackrel{\text{def}}{=} \text{the residual term containing derivatives higher than } n\text{th order}. \tag{1.20}$$

The coefficients $\mu(1), \mu(2), ..., \mu(n)$ are real constants. Comparing Table I and Eq. (1.19), one concludes that

$$\mu(1) = \nu_1(1), \qquad \mu(2) = \nu_1(2), \qquad \mu(3) = \nu_1(3), \tag{1.21}$$

where $\nu_1(1)$, $\nu_1(2)$, $\nu_1(3)$ are defined in Eq. (1.13). In WH, *it is implictly assumed that*

$$\lim_{n \to \infty} M(n) = 0. \tag{1.22}$$

As a result of Eqs. (1.19) and (1.22), one obtains the modified equation

$$\frac{\partial u}{\partial t} = \sum_{p=1}^{\infty} \mu(p) \frac{\partial^p u}{\partial x^p}. \tag{1.23}$$

(ii) *The validity of the W–H Procedure*

In the W–H procedure, Warming and Hyett assume the existence of a smooth function $u(x, t)$ which

(a) coincides with an exact solution of Eq. (1.4) at the mesh points, and

(b) satisfies Eq. (1.17) at least in some domain Γ on the $x-t$ plane.

In order to proceed from Eq. (1.17) to Eq. (1.18), one must also assume that the function u is such that

(c) the series formed as the results of the successive term-by-term differentiation with respect to x or t of the series in Eq. (1.17) also converge to zero in Γ.

Obviously, condition (a) does not imply conditions (b) and (c). In other words, a smooth function u which coincides with an exact solution of Eq. (1.4) at the mesh points generally does not satisfy Eq. (1.18).

The last assumption of the W–H procedure is Eq. (1.22). Since $M(1), M(2), ...,$ respectively, are the partial sums of the *different* series $(M(1) + R(1))$, $(M(2) + R(2))$, ..., in general, it is incorrect to consider Eq. (1.22) as a result of Eq. (1.18). Note that a counterexample to Eq. (1.22) will be presented immediately.

In conclusion, the above discussion shows that *a smooth function u which satisfies Eq.* (1.4) *at all mesh points generally does not satisfy the modified equation* (1.23). In other words, Warming and Hyett's interpretation of Eq. (1.23) is not supported by its derivation procedure.

(iii) *A Counterexample to the Interpretation of Warming and Hyett*

As a counterexample to Warming and Hyett's interpretation of the modified equation (1.23), we choose

$$u(x, t) = e^{(2\pi it/\Delta t)}. \tag{1.24}$$

Since $u(j\,\Delta x, n\,\Delta t) = 1$ at all mesh points, $u_j^n =^{\text{def}} u(j\,\Delta x, n\,\Delta t)$ satisfies Eq. (1.4) and any periodic condition in space. However, for this choice of u, every term on the right side of Eq. (1.23) vanishes while

$$\frac{\partial u}{\partial t} = \frac{2\pi i}{\Delta t} e^{(2\pi it/\Delta t)} \neq 0; \tag{1.25}$$

i.e., the function u does not satisfy the modified Eq. (1.23) even though $u_j^n = u(j\Delta x, n\,\Delta t)$ satisfies the difference scheme (1.4) and $u(x, t)$ satisfies any periodic condition in space. This counterexample clearly demonstrates that the modified equation generally does not represent the exact partial differential equation solved by a finite-difference equation even if spatial periodicity is assumed. Note that

$$M(n) = \frac{2\pi i}{\Delta t} e^{(2\pi it/\Delta t)}, \qquad n = 1, 2, 3, \ldots \tag{1.26}$$

if u is given by Eq. (1.24). Obviously Eq. (1.26) is inconsistent with the assumption in (1.22). Also note that counterexamples in which u is a function of both x and t will be provided in Section 2.

(iv) *The Proof Given by Warming and Hyett on the Connection between the Multiplication Factor and the Modified Equation*

Let $\alpha_1(\beta)$ be the function defined by Eqs. (1.8) and (1.9). Let

$$u(x, t) \stackrel{\text{def}}{=} e^{\alpha_1(ik)t + ikx}, \qquad ik \in \Psi_1, \tag{1.27}$$

where k is any real number. Then $u_j^n =^{\text{def}} u(j\,\Delta x, n\,\Delta t)$ satisfies Eq. (1.4) at all mesh points. If the modified Eq. (1.23) indeed represents the exact partial differential equation solved by the finite-difference scheme (1.4) then $u(x, t)$ defined in Eq. (1.27) should be a solution to Eq. (1.23). Substituting Eq. (1.27) into Eq. (1.23), one obtains

$$\alpha_1(ik) = \sum_{p=1}^{\infty} \mu(p)(ik)^p, \qquad ik \in \Psi_1. \tag{1.28}$$

Equations (1.10) and (1.28) imply that the multiplication factor G is determined by the coefficients $\mu(p)$ at least in some neighborhood of $k=0$. Since $\mu(p)$, $p = 1, 2, 3, \ldots$, are all real, Eq. (1.28) also implies that Eqs. (1.14) and (1.15) are still valid if $\nu_1(2p)$ and $\nu_1(2p+1)$ in these equations are replaced, respectively, by $\mu(2p)$ and $\mu(2p+1)$.

Note that, in the Appendix, it will be proved rigorously that

$$\mu(p) = \nu_1(p), \qquad p = 1, 2, 3, \ldots. \tag{1.29}$$

As a result, Eq. (1.28) may be obtained from Eqs. (1.11) and (1.29) without using the interpretation of the modified equation given by Warming and Hyett.

(v) *A New Interpretation for the W–H Procedure*

The previous discussion shows that:

(a) The interpretation given to the modified Eq. (1.23) by Warming and Hyett is flawed and thus Eq. (1.28) may not be proved by an argument using Eq. (1.23); and

(b) with the aid of Eq. (1.28), the application of the modified equation technique developed in WH requires as the input only the coefficients $\mu(p)$, $p = 1, 2, 3, \ldots$.

From observations (a) and (b), one concludes that the only useful information provided by the W–H procedure is the coefficients $\mu(p)$. As a result, it is unnecessary to consider the W–H procedure as a procedure to generate the modified Eq. (1.23). Instead, one may view the W–H procedure only as a procedure to yield the coefficients $\mu(p)$ from the difference Eq. (1.4).

In developing a new interpretation for the W–H procedure, we consider the following L-level difference analogue of Eq. (1.1):

$$\sum_{l=0}^{L-1} \sum_{q=-q_i}^{q_f} A_q^l u_{j+q}^{n+l} = 0$$

$$L = 2, 3, 4, \ldots; \qquad n = 0, 1, 2, \ldots; \qquad j = 0, \pm 1, \pm 2, \ldots, \tag{1.30}$$

where q_i and q_f are nonnegative integers, and A_q^l real constants. Note that the number of the spatial mesh points to the left (right) of x_j $(= j\,\Delta x)$ used in Eq. (1.30) may vary from one time level to another. Thus $-q_i$ and q_f specify the maximum range of q among all time levels such that $A_q^l \neq 0$. Also note that Eq. (1.4) and the model problems considered in WH are all special cases of Eq. (1.30) with $L = 2$.

By expanding each u_{j+q}^{n+l} into a Taylor series about the point $(j\,\Delta x, n\,\Delta t)$, and substituting the results into Eq. (1.30), one obtains

$$\sum_{p=0}^{\infty} \left[\sum_{m=0}^{p} E_{p,m} \left(\frac{\partial^p u}{\partial t^{p-m} \partial x^m} \right)_{x = j\Delta x,\, t = n\Delta t} \right] = 0, \tag{1.31}$$

where

$$E_{p,m} \stackrel{\text{def}}{=} \frac{1}{(p-m)!\,m!} \sum_{l=0}^{L-1} \sum_{q=-q_i}^{q_f} A_q^l (l\,\Delta t)^{p-m} (q\,\Delta x)^m \tag{1.32}$$

$$p = 0, 1, 2, \ldots; \qquad m = 0, 1, 2, \ldots, p.$$

Equation (1.31) implies that

$$E_{0,0} \stackrel{\text{def}}{=} \sum_{l=0}^{L-1} \sum_{q=-q_i}^{q_f} A_q^l = 0 \tag{1.33}$$

is required for consistency between Eqs. (1.1) and (1.30). In the current paper, we also assume that the coefficients A_q^l are normalized such that

$$E_{1,0} \stackrel{\text{def}}{=} \Delta t \sum_{l=1}^{L-1} \sum_{q=-q_i}^{q_f} l A_q^l = 1. \tag{1.34}$$

Note that Eq. (1.16) is a special case of Eq. (1.31). A comparison between Eqs. (1.16) and (1.31) also reveals that both Eqs. (1.33) and (1.34) are satisfied for the difference scheme (1.4).

The W–H procedure was explained using Eq. (1.4) as an example. Similarly, for any difference scheme (1.30) with $E_{0,0}=0$ and $E_{1,0}=1$, the W–H procedure can also be carried out by replacing each coefficient which is listed right below the derivative $\partial^p u/\partial t^{p-m}\, \partial x^m$ in Table I with the coefficient $E_{p,m}$. Again the elimination procedure illustrated in Table I will lead to Eq. (1.18). It is seen that the coefficients $\mu(p)$ are determined by the coefficients $E_{p,m}$. A comprehensive analysis of the W–H procedure [6] reveals that $E_{p,m}$ and $\mu(p)$ are related by a set of algebraic relations. To express these relations, one first defines the coefficients $E_{p,m}^{(n,l)}$ (n, $p=1, 2, 3, ...$; $l=0, 1, 2, ..., (n-1)$; $m=0, 1, 2, ..., p$) in terms of $E_{p,m}$ by induction; i.e.,

$$E_{p,m}^{(1,0)} \stackrel{\text{def}}{=} E_{p,m}, \qquad p=1, 2, 3, ...; \quad m=0, 1, 2, ..., p \tag{1.35}$$

$$E_{p,m}^{(n+1,0)} \stackrel{\text{def}}{=} E_{p,m}^{(n,n-1)} - E_{n+1,0}^{(n,n-1)} \cdot E_{p-n,m}$$
$$n,\ p=1, 2, 3, ...; \qquad m=0, 1, 2, ..., p \tag{1.36}$$

$$E_{p,m}^{(n,l+1)} \stackrel{\text{def}}{=} E_{p,m}^{(n,l)} - E_{n,l+1}^{(n,l)} \cdot E_{p-n+1,m-l-1}$$
$$n=2, 3, 4, ...; \qquad l=0, 1, 2, ..., (n-2);$$
$$p=1, 2, 3, ...; \qquad m=0, 1, 2, ..., p, \tag{1.37}$$

where

$$E_{p,m} \stackrel{\text{def}}{=} 0, \qquad \text{if} \quad m>p, \quad \text{or} \quad m<0, \quad \text{or} \quad p<1 \tag{1.38}$$

is assumed in Eqs. (1.36) and (1.37). It is shown in [6] that

$$\mu(p) = -E_{p,p}^{(p,p-1)}, \qquad p=1, 2, 3, ... \tag{1.39}$$

Note that the exact roles played by the coefficients $E_{p,m}^{(n,l)}$ in the W–H procedure are described in [6]. Also note that, for any given p, Eqs. (1.35) to (1.39) imply that $\mu(p)$ is completely determined by the set of coefficients $E_{p',m'}$ with $p'=1, 2, ..., p$ and $m'=0, 1, 2, ..., p'$. In other words, $\mu(p)$ is not dependent on any $E_{p',m'}$ with $p'>p$.

In the new interpretation of the W–H procedure, it is simply considered as the algebraic procedure by which the coefficients $\mu(p)$ are generated from the coefficients $E_{p,m}$ through the use of Eqs. (1.35) *to* (1.39). With this interpretation, the W–H procedure is completely free of the criticisms cited in (ii) and (iii).

With the aid of Eq. (1.29), it has been shown that the multiplication factor G for the difference scheme (1.4) may be determined in terms of the coefficients $\mu(p)$ if the wave number k is small enough. In the following section, it will be shown that a similar connection between G and $\mu(p)$ exists for any difference scheme (1.30) with $L=2$. However, this connection become more complicated for a difference scheme (1.30) with $L \geqslant 3$. In the von Neumann analysis, the multiplication factor G associated with a wave number k generally has $(L-1)$ roots for a difference scheme (1.30). As it turns out, only the principal root can be determined in terms of the coefficient $\mu(p)$ in the neighborhood of $k=0$.

2. Von Neumann Stability Analysis

In this section, we consider the difference schemes (1.30). Let

$$u_j^n \stackrel{\text{def}}{=} e^{\alpha(n\Delta t)+\beta(j\Delta x)}, \tag{2.1}$$

where α and β are complex parameters. It is easy to see that u_j^n is a solution of Eq. (1.30) if and only if

$$\sum_{l=0}^{L-1} \sum_{q=-q_i}^{q_f} A_q^l e^{(\alpha l \Delta t+\beta q \Delta x)}=0. \tag{2.2}$$

In the Von Neumann analysis, an error component may be written in the form of Eq. (2.1) with $\beta=ik$ where $i \equiv \sqrt{-1}$ and k is a real number. Given a k, the parameter α is determined by the requirement that this component be a solution of Eq. (1.30), i.e., that α be a solution of Eq. (2.2). Let

$$G \stackrel{\text{def}}{=} e^{\alpha \Delta t}. \tag{2.3}$$

Then Eq. (2.1) implies that $u_j^n=G^n e^{ijk\Delta x}$, i.e., G is the multiplication factor. In the current paper, unless specified otherwise, β is allowed to have both real and imaginary parts.

Equation (2.2) represents a relation between α and β. This relation is easier to grasp if it is rewritten as

$$\sum_{l=0}^{L-1}\left(\sum_{q=-q_i}^{q_f} A_q^l e^{\beta q \Delta x}\right) G^l=0. \tag{2.4}$$

For a given β, the above equation can be considered as an algebraic equation of G. If

$$\sum_{q=-q_i}^{q_f} A_q^{L-1} e^{\beta q \Delta x} \neq 0 \tag{2.5}$$

the algebraic equation (2.4) is of degree $(L-1)$. Thus, generally, G has $(L-1)$ roots for each value of β. Given G, the parameter α is determined by Eq. (2.3) up to an arbitrary multiple of $2\pi i/\Delta t$. In other words, for given value of β, generally there are $(L-1)$ principal solutions of α. As will be shown, an understanding of the relation between α and β in the neighborhood of $\beta=0$ is critical to the current investigation. Assuming Eqs. (1.33) and (1.34), this relation was studied by using the Implicit Function Theorem [8, p. 147]. The details are given in [6]. Let NR be the number of the simple roots (i.e., roots with multiplicity $=1$) of the algebraic Eq. (2.4) when $\beta=0$. It is shown in [6] that there exists a set of functions $\alpha_r(\beta)$ ($r=1, 2, 3, \ldots, \mathrm{NR}$) which, respectively, are bounded and analytic in some bounded neighborhoods Ψ_r of $\beta=0$. Furthermore, these functions satisfy the conditions:

(a)
$$\sum_{l=0}^{L-1} \sum_{q=-q_i}^{q_f} A_q^l e^{[\alpha_r(\beta) l \Delta t + \beta q \Delta x]} = 0, \qquad \beta \in \Psi_r, \tag{2.6}$$

(b)
$$\sum_{l=1}^{L-1} \sum_{q=-q_i}^{q_f} l A_q^l e^{[\alpha_r(\beta) l \Delta t + \beta q \Delta x]} \neq 0, \qquad \beta \in \Psi_r \tag{2.7}$$

(c) $\alpha_r(0)$, $r=1, 2, 3, \ldots, \mathrm{NR}$, are all distinct with

$$\alpha_r(0) \begin{cases} =0 & \text{if } r=1 \\ \neq 0 & \text{if } r=2, 3, 4, \ldots, \mathrm{NR} \end{cases} \tag{2.8}$$

and

$$\frac{\pi}{\Delta t} \geqslant \operatorname{Im}(\alpha_r(0)) > -\frac{\pi}{\Delta t}, \qquad r=2, 3, 4, \ldots, \mathrm{NR} \tag{2.9}$$

where $\operatorname{Im}(\alpha_r(0))$ is the imaginary part of $\alpha_r(0)$.

(d) Let

$$G_r(\beta) \stackrel{\text{def}}{=} e^{\alpha_r(\beta)\Delta t}, \qquad \beta \in \Psi_r; \quad r=1, 2, 3, \ldots, \mathrm{NR}. \tag{2.10}$$

Then $G_r(0)$, $r=1, 2, 3, \ldots, \mathrm{NR}$, represent all the simple roots of the algebraic Eq. (2.4) when $\beta=0$.

Note that (1) it is also shown in [6] that

$$(L-1) \geqslant \mathrm{NR} \geqslant 1; \tag{2.11}$$

i.e., the number of the functions $\alpha_r(\beta)$ is not more than $(L-1)$ and, at the minimum, $\alpha_1(\beta)$ exists. As a result, $\mathrm{NR}=1$ for any scheme (1.30) with $L=2$. (2) Since the number of the simple roots for a quadratic equation is either 0 or 2, Ineq. (2.11) also implies that $\mathrm{NR}=2$ for any scheme (1.30) with $L=3$. (3) Eqs. (1.8) and (1.9) may be obtained using Eqs. (2.6) and (2.8). (4) $\alpha_1(0)$ is uniquely determined by conditions (c) and (d). So is the set of $\alpha_r(0)$, $r=2, 3, \ldots, \mathrm{NR}$. Upon the specification of $\alpha_r(0)$, $r=2, 3, \ldots, \mathrm{NR}$, the functions $\alpha_r(\beta)$, $r=1, 2, 3, \ldots, \mathrm{NR}$, respectively, are unique over the neighborhoods Ψ_r.

Let k be a wave number such that $ik \in \Psi_r$ for all $r = 1, 2, 3, \ldots, \mathrm{NR}$. Since Eq. (2.6) states that Eq. (2.2) is satisfied if $\alpha = \alpha_r(\beta)$, $r = 1, 2, \ldots, \mathrm{NR}$, a comparison between Eqs. (2.10) and (2.3) reveals that $G_1(ik), G_2(ik), \ldots, G_{\mathrm{NR}}(ik)$ are all roots of the multiplication factor G. Since $\alpha_1(0) = 0$ and $\alpha_r(0) \neq 0$, $r = 2, 3, \ldots, \mathrm{NR}$, hereafter, $G_1(ik)$ will be designated as the principal root while $G_2(ik), G_3(ik), \ldots, G_{\mathrm{NR}}(ik)$ will be designated as the spurious roots.

As an example, consider the leap-frog scheme for Eq. (1.3), ie.,

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\,\Delta t} + c\,\frac{u_{j+1}^n - u_{j-1}^n}{2\,\Delta x} = 0.$$

For this scheme, $\mathrm{NR} = 2$,

$$G_1(ik) = -i\xi \sin(k\,\Delta x) + \sqrt{1 - \xi^2 \sin^2(k\,\Delta x)}$$

and

$$G_2(ik) = -i\xi \sin(k\,\Delta x) - \sqrt{1 - \xi^2 \sin^2(k\,\Delta x)},$$

where $\xi =^{\mathrm{def}} c\,\Delta t/\Delta x$.

Since every $\alpha_r(\beta)$ is analytic in Ψ_r, one has [8, p. 516]

$$\alpha_r(\beta) = \alpha_r(0) + \sum_{p=1}^{\infty} \nu_r(p)\beta^p, \qquad \beta \in \Psi_r; \quad r = 1, 2, 3, \ldots, \mathrm{NR}, \tag{2.12}$$

where

$$\nu_r(p) \stackrel{\mathrm{def}}{=} \frac{1}{p!}\left[\frac{d^p \alpha_r(\beta)}{d\beta^p}\right]_{\beta=0} \qquad p = 1, 2, 3, \ldots; \quad r = 1, 2, 3, \ldots, \mathrm{NR}. \tag{2.13}$$

Several comments can be made about Eqs. (2.12) and (2.13):

(a) Because $\alpha_1(0) = 0$, Eqs. (2.12) and (2.13), respectively, are reduced to Eqs. (1.11) and (1.12) for any scheme (1.30) with $\mathrm{NR} = 1$.

(b) Equations (2.10) and (2.12) imply that any root $G_r(ik)$ is determined by $\alpha_r(0)$ and $\nu_r(p)$, $p = 1, 2, 3, \ldots$, if $ik \in \Psi_r$. As will be shown in the Appendix, Eq. (1.29) is valid for any difference scheme (1.30). Since $\alpha_1(0) = 0$, one concludes that the principal root $G_1(ik)$ may be evaluated using the coefficients $\mu(p)$ if $ik \in \Psi_1$. However, since generally $\nu_r(p) \neq \mu(p)$, $p = 1, 2, 3, \ldots$, if $r \geqslant 2$, a spurious root may not be determined by the coefficients $\mu(p)$ even if $\alpha_r(0)$ is given.

(c) Since $\mu(p)$ are real, the relation $\nu_1(p) = \mu(p)$, $p = 1, 2, 3, \ldots$, implies that $\nu_1(p)$ are also real. It is shown in [6] that this fact can also be established without involving $\mu(p)$ by using Eqs. (2.6) to (2.8) and (2.13). Since $\nu_1(p)$ are real and $\alpha_1(0) = 0$, Eq. (2.12) can be used to show that Eqs. (1.14) and (1.15) are valid for any difference scheme (1.30). Obviously, these two equations are also valid for any difference scheme (1.30) if $\nu_1(2p)$ and $\nu_1(2p+1)$, respectively, are replaced by $\mu(2p)$ and $\mu(2p+1)$.

(d) Let

$$u_r^{(\beta)}(x, t) \stackrel{\text{def}}{=} e^{\alpha_r(\beta)t + \beta x}, \qquad \beta \in \Psi_r; \quad r = 1, 2, 3, ..., \text{NR}, \tag{2.14}$$

where $u_r^{(\beta)}$ represents a two-parameter family of functions of x and t. Equation (2.6) implies that

$$u_j^n = u_r^{(\beta)}(j\,\Delta x, n\,\Delta t) \tag{2.15}$$

is a solution of Eq. (1.30) for any $r = 1, 2, ..., \text{NR}$ and $\beta \in \Psi_r$. According to the interpretation of the modified Eq. (1.23) given by Warming and Hyett, one would expect that

$$u = u_r^{(\beta)}(x, t) \tag{2.16}$$

is a solution of Eq. (1.23) for any $r = 1, 2, ..., \text{NR}$ and $\beta \in \Psi_r$. If this were true, Eqs. (2.14), (2.16), and (1.23) would imply that, for any $r = 1, 2, ..., \text{NR}$,

$$\alpha_r(\beta) = \sum_{p=1}^{\infty} \mu(p)\beta^p, \qquad \beta \in \Psi_r. \tag{2.17}$$

Since the power series expansion of the analytic function $\alpha_r(\beta)$ is unique in Ψ_r [8, p. 413], consistency between Eqs. (2.12) and (2.17) requires that

$$\alpha_r(0) = 0 \tag{2.18}$$

and

$$\nu_r(p) = \mu(p), \qquad p = 1, 2, 3, ... \tag{2.19}$$

According to Eq. (2.8), Eq. (2.18) is false unless $r = 1$. For $r = 1$, Eq. (2.19) is simply Eq. (1.29). Thus one concludes that $u = u_r^{(\beta)}(x, t)$ satisfies the modified Eq. (1.23) if and only if $r = 1$.

3. Conclusions and Discussions

Warming and Hyett developed a modified equation technique by which the behavior of a difference scheme (1.30) with $L = 2$ may be evaluated in terms of the coefficients $\mu(p)$ of the modified equation (1.23). Specifically, they discovered that the multiplication factor G is related to $\mu(p)$ by Eqs. (1.10) and (1.28). Since the dissipation and dispersion of error components are determined by G, the former properties can be determined in terms of $\mu(p)$.

In order to "prove" the relation (1.28), Warming and Hyett interpret the modified equation (1.23) as the actual partial differential equation solved by the difference scheme (1.30). It is shown in the current paper that this interpretation is flawed even if spatial periodicity is assumed.

In the current paper, it is also shown that the W–H procedure by which the modified equation (1.23) is derived cannot be justified without imposing unrealistic assumptions. However, this difficulty may be avoided if the W–H procedure is simply considered as an algebraic procedure by which the coefficients $\mu(p)$ are generated from the coefficients $E_{p,m}$ through the use of Eqs. (1.35) to (1.39). This new interpretation of the W–H procedure also provides the basis for a rigorous proof of the important relation (1.29). Given the relation (1.29), Eq. (1.28) is simply a result of Eq. (1.11). In other words, Eq. (1.28) can be proved rigorously by an argument independent of the modified equation (1.23).

For a difference scheme (1.30) with $L > 2$, generally the multiplication factor G has several roots for a given wave number k. It is shown in the current paper that only the principal root can be evaluated in terms of the coefficient $\mu(p)$ by using Eqs. (2.10) and (1.28).

The coeffiencts $\mu(p)$ can be evaluated using the W–H procedure. With the aid of Eqs. (1.12), (1.29), and (2.6) to (2.8), they can also be evaluated using differentiation. As an example, we consider the explicit two-level scheme

$$u_j^{n+1} = B_- u_{j-1}^n + B_0 u_j^n + B_+ u_{j+1}^n, \tag{3.1}$$

where B_-, B_0, and B_+ are real constants which satisfy the consistent condition

$$B_- + B_0 + B_+ = 1. \tag{3.2}$$

For this case, it can be shown that

$$\mu(1) = \nu_1(1) = \frac{\Delta x}{\Delta t}(B_+ - B_-) \tag{3.3}$$

$$\mu(2) = \nu_1(2) = \frac{(\Delta x)^2}{2\Delta t}[(B_+ + B_-) - (B_+ - B_-)^2] \tag{3.4}$$

$$\mu(3) = \nu_1(3) = \frac{(\Delta x)^2}{6}(3B_0 - 2)\mu(1) + \frac{(\Delta t)^2}{3}[\mu(1)]^3 \tag{3.5}$$

and

$$\mu(4) = \nu_1(4) = \frac{(\Delta x)^4}{2\Delta t} B_+ B_- + \frac{(\Delta x)^2}{12}\mu(2) - \Delta t[\mu(2)]^2. \tag{3.6}$$

APPENDIX: Proof for Eq. (1.29)

In this appendix, relation (1.29) will be rigorously established for the difference schemes (1.30). As a preliminary, note that

(a) The power series $\sum_{p=0}^{\infty} z^p/p!$ converges uniformly to e^z in any bounded domain on the complex z-plane [8, pp. 409, 535].

(b) Let $g_p(z)$ and $h_p(z)$, $p=0, 1, 2, \ldots$, be functions of the complex variable z. Let the series $\sum_{p=0}^{\infty} g_p(z)$ and $\sum_{p=0}^{\infty} h_p(z)$ converge uniformly in a complex domain J. If $\zeta(z)$ and $\eta(z)$ are bounded in J, then the series

$$\sum_{p=0}^{\infty} [\zeta(z)\, g_p(z) + \eta(z) h_p(z)]$$

converges uniformly to

$$\zeta(z) \sum_{p=0}^{\infty} g_p(z) + \eta(z) \sum_{p=0}^{\infty} h_p(z)$$

in J [9, pp. 337, 428].

Since $\alpha_r(\beta)$, respectively, are bounded in the bounded domains Ψ_r, one concludes from (a) that the series

$$\sum_{p=0}^{\infty} [\alpha_r(\beta) l\, \Delta t + \beta q\, \Delta x]^p/p!,$$

respectively, converges uniformly to

$$e^{[\alpha_r(\beta) l \Delta t + \beta q \Delta x]}$$

in Ψ_r. With the aid of this fact and (b), Eq. (2.6) implies that

$$\sum_{p=1}^{\infty} \left[\sum_{m=0}^{p} E_{p,m}(\alpha_r(\beta))^{p-m} \beta^m \right] = 0, \qquad r = 1, 2, 3, \ldots, \mathrm{NR}, \tag{A.1}$$

where the coefficients $E_{p,m}$ are defined in Eq. (1.32) and the series on the left side converges uniformly in Ψ_r. Furthermore, using Eq. (A.1) and (b), one has

$$\sum_{p=1}^{\infty} \left[(\alpha_r(\beta))^{n-l} \beta^l \sum_{m=0}^{p} E_{p,m}(\alpha_r(\beta))^{p-m} \beta^m \right] = 0$$

$$n = 0, 1, , \ldots; \qquad l = 0, 1, 2, \ldots, n; \qquad r = 1, 2, 3, \ldots, \mathrm{NR}, \tag{A.2}$$

where the series on the left side converges uniformly in Ψ_r.

It is shown in [6] that, by linerly combining the equations given in (A.2), one may obtain

$$\sum_{p=1}^{\infty} \left[\sum_{m=0}^{p} E_{p,m}^{(n,l)}(\alpha_r(\beta))^{p-m} \beta^m \right] = 0$$

$$n = 1, 2, 3, \ldots; \qquad l = 0, 1, 2, \ldots, (n-1); \qquad r = 1, 2, 3, \ldots, \mathrm{NR}, \tag{A.3}$$

where the coefficients $E_{p,m}^{(n,l)}$ are functions of $E_{p,m}$ defined by Eqs. (1.35) to (1.38). Again the series on the left side of Eq. (A.3) converges uniformly in Ψ_r. Note that the linear combination procedure which leads Eq. (A.2) to Eq. (A.3) is very similar to the eliminaton procedure described in Table I; i.e., the latter is identical to the former except that the derivative $\partial^n u/\partial t^{n-l}\,\partial x^l$ in the latter is replaced by the product $[\alpha_r(\beta)]^{n-l}\beta^l$ in the former. Also note that the coefficients $E^{(n,l)}_{p,m}$ (n, $p=1, 2, 3, \ldots$; $l=0, 1, 2, \ldots, (n-1)$; $m=0, 1, 2, \ldots, p$) possess the following properties [6]:

$$\text{(a)}\qquad E^{(n,l)}_{1,0}=E_{1,0}=1 \tag{A.4}$$

$$\text{(b)}\qquad E^{(n,l)}_{p,p}=E^{(p,p-1)}_{p,p} \qquad \text{if} \quad n>p \tag{A.5}$$

$$\text{(c)}\qquad E^{(n,l)}_{p,m}=0 \qquad \text{if} \quad p\geqslant 2,\quad p>m,\quad \text{and} \quad n>p \tag{A.6}$$

$$\text{(d)}\qquad E^{(n,l)}_{n,m}=0 \qquad \text{if} \quad n\geqslant 2,\quad n>m,\quad \text{and} \quad l\geqslant m \tag{A.7}$$

For the special case $l=n-1$, with the aid of Eqs. (A.4) to (A.7) and (1.39), Eq. (A.3) can be rewritten as

$$\alpha_r(\beta)=\mu(1)\beta+\mu(2)\beta^2+\cdots+\mu(n)\beta^n-\sum_{p=n+1}^{\infty}\left[\sum_{m=0}^{p}E^{(n,n-1)}_{p,m}(\alpha_r(\beta))^{p-m}\beta^m\right]$$

$$n=1, 2, 3, \ldots; \qquad r=1, 2, 3, \ldots, \mathrm{NR}; \qquad \beta\in\Psi_r. \tag{A.8}$$

Consider the special case $r=1$. For a fixed $n>0$, let

$$f_p(\beta)\overset{\text{def}}{=}\mu(p)\beta^p, \qquad p=1, 2, 3, \ldots, n; \quad \beta\in\Psi_1, \tag{A.9}$$

and

$$f_p(\beta)\overset{\text{def}}{=}-\sum_{m=0}^{p}E^{(n,n-1)}_{p,m}(\alpha_1(\beta))^{p-m}\beta^m,$$

$$p=n+1, n+2, n+3, \ldots; \qquad \beta\in\Psi_1. \tag{A.10}$$

Then Eq. (A.8) implies that

$$\alpha_1(\beta)=\sum_{p=1}^{\infty}f_p(\beta), \qquad \beta\in\Psi_1. \tag{A.11}$$

Since $\alpha_1(\beta)$ is analytic in Ψ_1, so are the functions $f_p(\beta)$, $p=n+1$, $n+2$, $n+3, \ldots$ [7, p. 513]. Thus one has

$$f_p(\beta)=\sum_{m=0}^{\infty}a^{(p)}_m\beta^m, \qquad p=1, 2, 3, \ldots, \tag{A.12}$$

where $a^{(p)}_m$ are coefficients independent of β. Obviously,

$$a^{(p)}_m=\delta^p_m\mu(p) \qquad \text{if} \quad p=1, 2, 3, \ldots, n, \tag{A.13}$$

where δ^p_m is the Kronecker-delta symbol. Moreover, since $\alpha_1(0)=0$, the Taylor's expansion about $\beta=0$ of any $f_p(\beta)$ with $p\geqslant n+1$ has a leading term involving β^m, where $m\geqslant p$. As a result,

$$a^{(p)}_m=0 \quad \text{if} \quad p=n+1, n+2, n+3, \ldots, \quad \text{and} \quad m=0, 1, 2, \ldots, (p-1). \tag{A.14}$$

A direct result of Eqs. (A.13) and (A.14) is $a_0^{(p)}=0$, $p=1, 2, 3, \ldots$. Using the Weierstrass' theorem on double series [9, p. 430], one concludes that

(a) The series $\sum_{p=1}^{\infty} a_m^{(p)}$ converges for any $m=1, 2, 3, \ldots$.

(b)
$$\alpha_1(\beta)=\sum_{m=1}^{\infty} A_m\beta^m, \qquad \beta\in\Psi_1, \tag{A.15}$$

where

$$A_m \stackrel{\text{def}}{=} \sum_{p=1}^{\infty} a_m^{(p)}, \qquad m=1, 2, 3, \ldots. \tag{A.16}$$

According to Eqs . (A.13) and (A.14),

$$A_m=\mu(m), \qquad m=1, 2, 3, \ldots, n. \tag{A.17}$$

With the aid of Eqs. (A.17) and (2.8), a comparison between Eqs. (2.12) and (A.15) reveals that $\nu_1(p)=\mu(p)$, $p=1, 2, 3, \ldots, n$. This must be true since $\alpha_1(\beta)$ is analytic in Ψ_1. Moreover, since n is any integer $\geqslant 1$, one concludes that the relation (1.29) must be true for any difference scheme (1.30). Q.E.D.

Finally, it is noted that the argument which is used to prove Eq. (1.29) does not apply if $r>1$. This is because $\alpha_r(0)\neq 0$ if $r>1$. Thus, in general,

$$\nu_r(p)\neq\mu(p), \qquad p=1, 2, 3, \ldots; \qquad r=2, 3, 4, \ldots, \text{NR}. \tag{A.18}$$

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