Multiple scales methods in meteorology

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Abstract With emphasis on meteorological applications, we discuss here the fluid dynamical fundamental governing equations, their nondimensionalization including the identification of key nondimensional parameters, and a general approach to meteorological modelling based on multiple scales asymptotics.

1 Overview

In Chapter 2 we will derive the fluid mechanical conservation laws. We explore the basic principles considering "pure" fluid mechanics, i.e., we neglect the influences of gravity, Earth's rotation (Coriolis force), molecular transport, and of the so called "diabatic effects". The latter subsume all processes that involve external energy supply by radiation or conversion of energy due to condensation, chemical reactions, etc.. Gravity and the Coriolis force will be included in subsequet sections. The chapter concludes with a summary of the governing equations, now extended to also include a general set of species transport equations. These will be important, e.g., in describing (atmospheric) chemistry or moist processes.

Chapter 3 introduces the technique of multiple scales asymptotics using the (almost trivial) example of a linear oscillator. After deriving analytical solutions, we will focus on a situation which, in many ways, resembles situations arising frequently in geophysical problems: a slow background motion caused by an external force is accompanied by rapid oscillations around it, with the oscillation amplitudes generally not being small. To give some meaning to the notions of "smallness" and "rapidity", we will first nondimensionalize the oscillator equations and identify small parameters that lend themselves for comparison. By means of single and multiple scales analyses we will then try to derive simplified approximate solutions that become more and more accurate as the small parameters vanish.

One important aim of theoretical meteorology is the development of simplified model equations that describe the large variety of scale-dependent phenomena observed in atmospheric flows. Chapter 4 summarizes the basic scaling arguments that justify a unified approach to the derivation of such models based on multiple scales asymptotic techniques. We note that Keller and Ting (1951) already anticipated the foundations of this approach in an internal report of the Institute for Mathematics and Mechanics of New York University. In particular, Chapter 4 non-dimensionalization to the equations of compressible flows on a rotating sphere as a first step in building the unified multiscale modelling framework. The subsequent steps are the introduction of a quite generally applicable set of distinguished limits, and multiple-scales asymptotics. For simplicity, diabatic effects, such as radiation, water phase transitions, or turbulent transport are represented as lumped terms in the governing equations to be specified later. For extensions see Klein and Majda (2006).

Chapter 5 employs the general asymptotics-based approach to rederive the classical quasi-geostrophic model, see Pedlosky (1987).

The reader may want to consult Klein (2010) for further references.

2 Fluid mechanical conservation laws

In this chapter we will derive the fluid mechanical conservation laws. In section 2.1 we explore the basic principles considering "pure" fluid mechanics, i.e., we neglect the influences of gravity, Earth's rotation (Coriolis force), molecular transport, and of the so called "diabatic effects". The latter subsume all processes that involve external energy supply by radiation or conversion of energy due to condensation, chemical reactions, etc.. Gravity and the Coriolis force will be included later in Sections 2.3 and 2.4. Section 2.6 provides a summary of the governing equations, now extended to also include a general set of species transport equations. These will be important, e.g., in describing (atmospheric) chemistry or moist processes.

Remark: *In the present context, some quantity, say U, is conserved if the total content of U within a given, fixed control volume in space can change in time only by* exchange of U across the control volume's interface.

2.1 Pure fluid dynamics

Mass conservation During the motion of a mass parcel its mass is conserved while, in general, the parcel's volume can change. The change of density (mass per unit volume) caused by this change of volume is expressed in the law of mass conservation. The mass M of a fixed control volume Ω

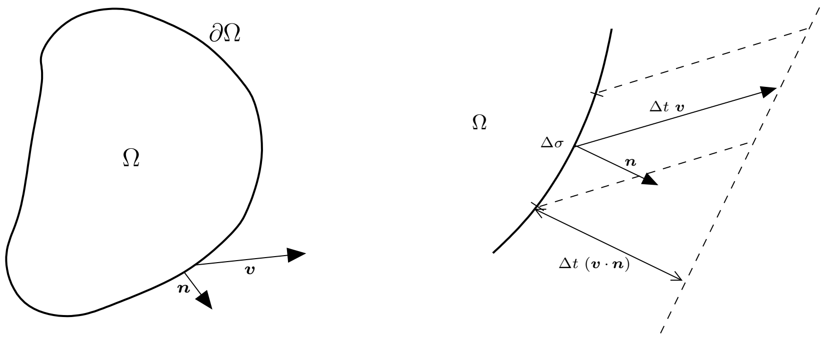

Figure 1. Change of a volume's mass with time

at time t can be expressed as an integral over the density,

$$M(t;\Omega) = \int_\Omega \varrho(t, \boldsymbol{x}) \; dV \,. \tag{1}$$

This mass will change during a time interval Δt if mass parcels cross interface $\partial\Omega$, being carried along by the flow velocity, $\boldsymbol{v}$ (see Fig. 1). The change of mass, ΔM, associated with the passage of parcels across a control surface segment $\Delta\sigma \subset \partial\Omega$, is then equal to $-\varrho\,(\Delta t\,\boldsymbol{v})\cdot\boldsymbol{n}\,\Delta\sigma$, where $\boldsymbol{n}$ is the outward pointing normal on $\Delta\sigma$. The scalar multiplication $\boldsymbol{v}\cdot\boldsymbol{n}$ selects the component of $(\Delta t\,\boldsymbol{v})$ perpendicular to $\Delta\sigma$ as the one relevant for mass transport across the surface element. By summation (integration) along the entire boundary of the control volume, and for time increments covering a finite interval $t \in [t_1, t_2]$ we find

$$M(t_2;\Omega) - M(t_1;\Omega) = -\int_{t_1}^{t_2} \int_{\partial\Omega} (\varrho\boldsymbol{v})\cdot\boldsymbol{n} \; d\sigma \, dt \,. \tag{2}$$

This is the most general formulation of the law of mass conservation which holds for arbitrary control volumes for which the integrals in (1), (2) are meaningfully defined. Notice that the above definitions merely require suitable *integrability* for the mass and momentum densities, ϱ and $\varrho\boldsymbol{v}$. These quantities need not be differentiable in either space or time for the mass balance in (2) to make sense!

If, however, we may assume differentiability of $M(t;\Omega)$ w.r.t. time, t, we may let $(t_2 - t_1) \to 0$ to find

$$\frac{dM}{dt} = -\int_{\partial\Omega} (\varrho\boldsymbol{v})\cdot\boldsymbol{n} \; d\sigma \,. \tag{3}$$

If, in addition, $\varrho\boldsymbol{v}$ satisfies the conditions of Gauß' integral theorem, (see appendix 5.6), then $\int_{\partial\Omega}(\varrho\boldsymbol{v})\cdot\boldsymbol{n}\; d\sigma = \int_{\Omega}\nabla\cdot(\varrho\boldsymbol{v})\; dV$, and

$$\int_{\Omega}\Big(\varrho_t + \nabla\cdot(\varrho\boldsymbol{v})\Big)\; dV = 0 \qquad \text{for arbitrary Gauß domains } \Omega. \tag{4}$$

This equation can hold, for continuously differentiable fields $\varrho, \boldsymbol{v}$ and for arbitrary control volumes Ω, only if pointwise the following partial differential equation is satisfied:

$$\varrho_t + \nabla\cdot(\varrho\boldsymbol{v}) = 0\,. \tag{5}$$

Adopting this differential form restricts solutions to the class of continuously differentiable fields. Yet, in practice one uses (5) as a short-hand for (2), thereby implying that whereever ϱ_t and $\nabla\cdot(\varrho\boldsymbol{v})$ are singular, their spacio-temporal integrals remain well defined. For discussions of such *weak solutions of conservation laws* see, e.g., LeVeque (1990) and Kröner (1997); for a measure theoretical approach to conservation laws see Temam and Miranville (2000).

General conservation laws The considerations of the last section lead us to the following general formal pattern of a conservation law: Let U denote an extensive conserved quantity. For extensive quantities, their "total amount", $U(t,\Omega)$, is well defined for arbitrary control volumes, Ω, that are Gauß domains, and they are additive in that

$$U\left(t;[\Omega_1\cup\Omega_2]\right) = U(t;\Omega_1) + U(t;\Omega_2) \quad \forall \quad \Omega_1,\Omega_2 \;:\; \Omega_1\cap\Omega_2 = \emptyset\,. \tag{6}$$

Let $u(t,\boldsymbol{x})$ denote the density field associated with U, so that

$$U(t;\Omega) = \int_{\Omega} u(t,\boldsymbol{x})\; dV\,, \tag{7}$$

and $\boldsymbol{f}$ its flux density. Then conservation of U in time is expressed by the
<u>*Integral Conservation Law*</u>

$$U(t_2;\Omega) - U(t_1;\Omega) = -\int_{t_1}^{t_2}\int_{\partial\Omega} \boldsymbol{f}\cdot\boldsymbol{n}\; d\sigma\, dt\,. \tag{8}$$

If, in addition, u and $\boldsymbol{f}$ are sufficiently smooth, then they satisfy the
<u>*Partial Differential Equation in Conservation Form*</u>

$$u_t + \nabla\cdot\boldsymbol{f} = 0\,. \tag{9}$$

Remark: *The integral form of the conservation law in (8) is the most general basis for the formulation of numerical methods for problems in continuum mechanics because, by construction, it allows for the correct representation of non-smooth, e.g., discontinuous, solutions. See, e.g., LeVeque (1990), Kröner (1997).*

Remark: *Partial differential equations which, in addition to divergence terms, include other expressions that have no equivalent divergence form cannot be cast in the more general integral form (8). Such equations do not describe* conservation *in the present sense.*

Energy conservation The conservation of energy follows the same pattern just described, yet here the conserved quantity can not only be exchanged by the motion of fluid parcels. Rather, the energy contained in a control volume is also changed by the work done by the pressure and other stresses when the fluid moves across or along the boundary of the considered control volume, or when heat is added through thermal conduction, radiation, and the like. Further effects, such as those due to changes in potential energy in the Earth's gravity field will be discussed in the next chapter. The energy contained in some control volume Ω is

$$E(t) = \int_{\Omega} \varrho e(t, \boldsymbol{x}) \; dV \, , \tag{10}$$

where e is the total energy per unit mass, or *specific total energy*, and ϱ is again the mass density.

Energy is transported by the motion of fluid parcels in analogy with the flux of mass considered in section 2.1. The associated contribution to the energy flux density is $\varrho e \, \boldsymbol{v}$. In addition, there are forces acting within a fluid between adjacent fluid parcels. Those forces are represented by means of a second order tensor field $(p \, \boldsymbol{id} + \boldsymbol{\tau})$, where p is the thermodynamic pressure, and $\boldsymbol{id}, \boldsymbol{\tau}(t, \boldsymbol{x}) \in I\!R^{3\times 3}$ are the unit tensor and the viscous stress tensor, respectively. The interpretation of this tensor, $(p \, \boldsymbol{id} + \boldsymbol{\tau})$, and its two contributions to the energy flux is as follows:

Consider the boundary, $\partial\Omega$, of a control volume (or some similar surface embedded in the flow domain). At any location $\boldsymbol{x} \in \partial\Omega$ the vector $(p \, \boldsymbol{id} + \boldsymbol{\tau}) \cdot \boldsymbol{n}$, with $\boldsymbol{n}$ the outward pointing normal on $\partial\Omega$, denotes the force per unit area, i.e., the stress, which the fluid within the control volume exerts onto the fluid outside. If the fluid is in motion, with flow velocity $\boldsymbol{v}(t, \boldsymbol{x})$, then $\boldsymbol{v} \cdot (p \, \boldsymbol{id} + \boldsymbol{\tau}) \cdot \boldsymbol{n}$ is the work per unit time and unit area done by the fluid inside the control volume on the fluid outside. This is the second contribution to the energy flux density to be considered here. It consists of (i) the work done by the thermodynamic pressure forces, $p\,\boldsymbol{v}\cdot\boldsymbol{n}$, and (ii) the work done by the viscous stresses, $\boldsymbol{v}\cdot\boldsymbol{\tau}\cdot\boldsymbol{n}$.

We know from experience that two bodies of finite mass and different temperature tend to exchange thermal energy so as to eventually approach states of equal temperature. Let us denote the associated energy flux per unit area by $\boldsymbol{\jmath}$.

Combining the three effects just discussed we obtain the energy conservation law, written here in its differential form as

$$(\varrho e)_t + \nabla\cdot\left([\varrho e + p]\ \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\tau} + \boldsymbol{\jmath}\right) = 0\,. \tag{11}$$

In much of the subsequent discussions we will neglect the terms, $(\boldsymbol{v}\cdot\boldsymbol{\tau} + \boldsymbol{\jmath})$, which are associated with molecular transport processes, for simplicity of exposition.

Momentum conservation The momentum of the fluid contained in a control volume is defined as

$$\boldsymbol{I}(t) = \int_\Omega \varrho\boldsymbol{v}(t,\boldsymbol{x})\ dV\,, \tag{12}$$

so that $\varrho\boldsymbol{v}$ is the momentum density. Fluxes of momentum arise again through advection, i.e., through transport by the fluid motion, and the associated flux density across a surface with unit normal $\boldsymbol{n}$ is $\varrho\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{n})$.

Newton's law of motion then states that the forces acting on some finite mass equal the rate of change of its momentum. In the present continuum mechanics we have seen that $(p\,\boldsymbol{id} + \boldsymbol{\tau})\cdot\boldsymbol{n}$ is the force per unit area which some mass of fluid within our control volume Ω exerts onto the fluid outside it (when $\boldsymbol{n}$ is the outward-pointing normal unit vector). Thus, $(p\,\boldsymbol{id}+\boldsymbol{\tau})\cdot\boldsymbol{n}$ represents a flux of momentum from the control volume to its environment, and the momentum conservation law reads

$$(\varrho\boldsymbol{v})_t + \nabla\cdot(\varrho\boldsymbol{v}\circ\boldsymbol{v} + p\,\boldsymbol{id} + \boldsymbol{\tau}) = 0\,, \tag{13}$$

where $\circ$ denotes the tensorial product.

Remark: *The divergence of $(p\,\boldsymbol{id})$ equals the pressure gradient. In cartesian co-ordinates we have*

$$\Big(\nabla\cdot(p\,\boldsymbol{id})\Big) = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot\begin{pmatrix} p & 0 & 0\\ 0 & p & 0\\ 0 & 0 & p\end{pmatrix} = \left(\frac{\partial p}{\partial x},\frac{\partial p}{\partial y},\frac{\partial p}{\partial z}\right)\,. \tag{14}$$

In the notation of co-ordinate-free tensor analysis,

$$\nabla\cdot(p\,\boldsymbol{id}) = \nabla p\,. \tag{15}$$

2.2 Equations of state

The system of mass, momentum, and energy conservation laws, (5), (13), and (11), is not closed, as the pressure, p, the stress tensor, $\boldsymbol{\tau}$, and the heat flux density, $\boldsymbol{\jmath}$, have not yet been related to the primary variables, $(\varrho, \varrho\boldsymbol{v}, \varrho e)$. For the present purposes it suffices to take the model of an ideal gas with constant specific heat capacities, endowed with Newtonian friction and Fourier-type heat conduction as an example. For this case, we introduce the temperature,

$$T = \frac{p}{\varrho R} \tag{16}$$

where $R = R^*/M$, $R^* = 8.3141\ \mathrm{J\ mol^{-1}\ K^{-1}}$ is the idal gas constant, and M the gas' molecular weight. Then we express the total energy density, ϱe, as the sum of the internal (or thermal) and the kinetic energy via

$$\varrho e = \varrho(e_{\mathrm{th}} + e_{\mathrm{kin}}) = \varrho c_v T + \frac{\varrho \boldsymbol{v}^2}{2}\,. \tag{17}$$

Here the coefficient c_v is known as the *specific heat capacity at constant volume* and it is assumed constant below.

From the ideal gas law (16) we obtain

$$\varrho e = \frac{c_v}{R}\, p + \frac{1}{2}\varrho \boldsymbol{v}^2 = \frac{p}{\gamma - 1} + \frac{1}{2}\varrho \boldsymbol{v}^2\,, \tag{18}$$

where $\gamma \equiv 1 + \frac{R}{c_v} = \frac{c_p}{c_v}$ is the isentropic exponent of the gas (and c_p is its heat capacity at constant pressure). For atmospheric air a good estimate is $\gamma = 1.4 = const.$ with variations due to admixtures of water vapor and other trace gases being of the order of a few percent at most (see the subsequent remarks).

Remark: *Generalization of these constitutive laws for mixtures of ideal gases with molecular weights $(M_1, M_2, \dots M_n)$ and mass fractions $(Y_1, Y_2, \dots Y_n)$ maintains (16) and (17) but replaces the gas constant and specific heat capacity at constant volume with*

$$(R, c_v) = \sum_i^n Y_i (R_i, c_{v,i})\,.$$

See also section 2.6.

Remark: *For an ideal gas with non-constant specific heat capacities,*

$$e_{th} = \int_0^T c_v(T')\, dT'$$

with the specific heat capacity at constant volume, $c_v(T)$, being a known function of temperature. The ideal gas law from (16) is maintained in this case with R remaining independent of temperature.

For the stress tensor $\boldsymbol{\tau}$, we assume Newton's law

$$\boldsymbol{\tau} = -\mu \left(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T - \hat{\mu} \, (\nabla \cdot \boldsymbol{v}) \, \boldsymbol{id} \right) , \tag{19}$$

where the *dynamic viscosity*, μ, and the dimensionless coefficient of bulk viscosity, $\hat{\mu}$, depend on the fluid considered, and on the thermodynamic state through the temperature (quite generally) and the pressure (for some fluids). For air, $\mu \approx 1.7 \times 10^{-5}$ kg m^{-1}s^{-1} at typical atmospheric conditions, Gill (1982). The coefficient $\hat{\mu}$ is very difficult to measure experimentally, and for lack of more precise information one often assumes $\hat{\mu} = \frac{2}{3}$, which simplifies the equations to some extent by eliminiating trace($\boldsymbol{\tau}$).

For the thermal energy flux density we adopt Fourier's law of heat conduction, so that

$$\boldsymbol{\jmath} = -\lambda \cdot \nabla T \quad . \tag{20}$$

Here λ is the thermal conductivity which again depends on the medium as well as possibly on temperature and pressure. For instance, in water we have $\lambda_{H_2O} = 0.6$ Wm^{-1}K^{-1}, while in air $\lambda_{air} = 0.023$ Wm^{-1}K^{-1}, Gill (1982)).

The constitutive laws (16)–(20), when added to the conservation laws for mass, momentum, and energy in (5), (13), and (11), yield the desired closed set of partial differential equations for the flow of an ideal gas.

2.3 The influence of gravity

In the previous section we neglected Earth's gravity. Gravity exerts a bulk force, which cannot be directly expressed as a flux divergence (although in a broad range of applications in meteorology it can!). The change of momentum caused by this bulk force is proportional to the fluid density, ϱ, and directed oppositely to some unit vector $\hat{\boldsymbol{g}}$ that points away from the Earth's center of mass. In the present notes, we will restrict to flows covering sufficiently small domains, so that we may safely assume $\hat{\boldsymbol{g}} \equiv \boldsymbol{k}$, where $\boldsymbol{k}$ is the "vertical" unit vector, perpendicular to a suitably chosen tangent plane to the Earth's surface. The factor of proportionality is the acceleration of gravity,

$$g \approx 9.81 \, \frac{\mathrm{m}}{\mathrm{s}^2} \, , \tag{21}$$

which may be considered constant here.

Remark: *Because of the shape of the earth, g actually varies at sea level around ± 0.3 percent in north-south direction and around 0.3 percent with a change of height of 10 km Gill (1982).*

To account for the influence of gravity, we must endow the momentum balance in (13) with a source term,

$$(\varrho \boldsymbol{v})_t + \nabla \cdot (\varrho \boldsymbol{v} \circ \boldsymbol{v} + p\, \boldsymbol{id} + \boldsymbol{\tau}) = -\varrho\, g\, \boldsymbol{k}\,. \tag{22}$$

In the energy balance, (11), we must account for the potential energy associated with the position of the fluid mass in the Earths gravity field. This is done by extending the constitutive law from (17), to include a potential energy term, viz.,

$$\varrho e = \varrho (e_{\text{th}} + e_{\text{kin}} + e_{\text{pot}}) = \varrho c_v T + \frac{\varrho \boldsymbol{v}^2}{2} + \varrho \Phi(\boldsymbol{x})\,. \tag{23}$$

Here $\Phi(\boldsymbol{x})$ is the Earth's geopotential, which in the present setting (tangent plane approximation) we approximate by

$$\Phi(\boldsymbol{x}) = gz \tag{24}$$

with z denoting height above sea level.

Remark: *For the present setting of flows in a tangential plane with g, $\boldsymbol{k} \equiv$ const., even the momentum equation is effectively in conservative form, too. By introducing the hydrostatic pressure P_{hy}, defined by*

$$P_{hy}(z) = g \int_z^\infty \varrho(z')\; dz' \qquad \text{and} \qquad \frac{\partial P_{hy}}{\partial z} = -g\,\varrho \tag{25}$$

we rewrite the right hand side of the momentum equation to become

$$-(\varrho\, g\, \boldsymbol{k}) = (0, 0, -\varrho\, g) = \left(0, 0, \frac{\partial P_{hy}}{\partial z}\right) = \nabla \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & P_{hy} \end{pmatrix}, \tag{26}$$

or

$$-\varrho\, g\, \boldsymbol{k} = -\nabla \cdot \boldsymbol{\Pi}_{hy}\,. \tag{27}$$

We thus obtain the momentum equation in conservative form

$$(\varrho \boldsymbol{v})_t + \nabla \cdot (\varrho \boldsymbol{v} \circ \boldsymbol{v} + p\, \boldsymbol{id} + \boldsymbol{\tau} + \boldsymbol{\Pi}_{hy}) = 0\,. \tag{28}$$

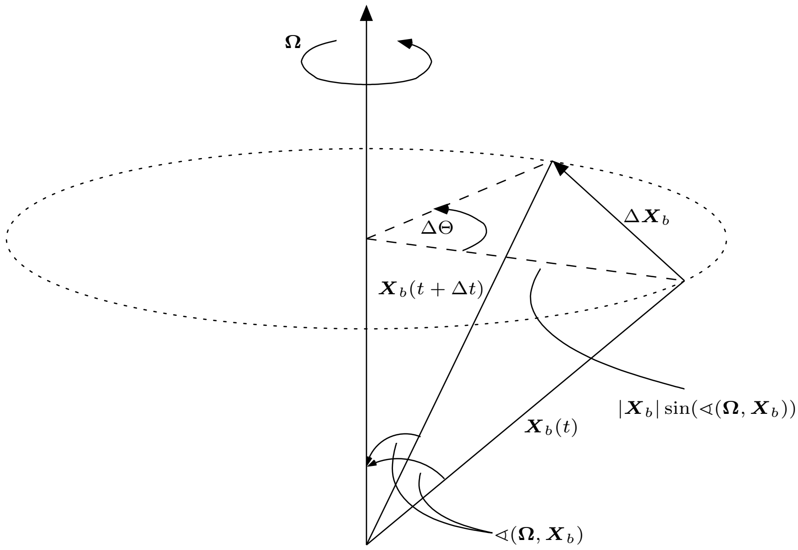

Figure 2. $\boldsymbol{X}_b$ at times t and $t + \Delta t$

2.4 The effects of Earth's rotation

Rotating frame of reference Up to now we have considered a non-accelerating coordinate system and neglected the effects of Earth's rotation. Physical phenomena are independent of the choice of a coordinate system, but their description depends on the observer and, in particular, on his choice of a coordinate system. For obvious reasons, we are interested in observers that follow Earth's rotation.

Consider some point on the Earth's surface with position vector $\boldsymbol{X}_b = \boldsymbol{X}_b(t)$ in an absolute, intertial frame of reference. Because of Earth's rotation, $\boldsymbol{X}_b$ is rotating with angular velocity $\mathbf{\Omega}$. In a small time interval Δt the vector $\boldsymbol{X}_b$ turns by an angle $\Delta\theta = |\mathbf{\Omega}|\Delta t$, where $|\mathbf{\Omega}|$ is the absolute value of $\mathbf{\Omega}$ (figure 2, see also Pedlosky (1987)). This small change of $\boldsymbol{X}_b$ is described by

$$\boldsymbol{X}_b(t+\Delta t) - \boldsymbol{X}_b(t) \equiv \Delta \boldsymbol{X}_b = \boldsymbol{n}\ |\boldsymbol{X}_b|\ \sin(\sphericalangle(\mathbf{\Omega}, \boldsymbol{X}_b))\ \Delta\theta + O\big((\Delta\theta)^2\big) \quad (29)$$

with the unit vector

$$\boldsymbol{n} = \frac{\mathbf{\Omega} \times \boldsymbol{X}_b}{|\mathbf{\Omega} \times \boldsymbol{X}_b|}\,, \quad (30)$$

pointing in direction of the change of $\boldsymbol{X}_b$ (perpendicular to $\boldsymbol{X}_b$ and $\mathbf{\Omega}$). As

$\Delta t \to 0$ we find

$$\lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{X}_b}{\Delta t} = \frac{d\boldsymbol{X}_b}{dt} = |\boldsymbol{X}_b| \ \sin(\sphericalangle(\boldsymbol{\Omega}, \boldsymbol{X}_b)) \ \frac{d\theta}{dt} \ \frac{\boldsymbol{\Omega} \times \boldsymbol{X}_b}{|\boldsymbol{\Omega} \times \boldsymbol{X}_b|}\,, \tag{31}$$

and, using $|\boldsymbol{\Omega} \times \boldsymbol{X}_b| = |\boldsymbol{\Omega}| \ |\boldsymbol{X}_b| \ \sin(\sphericalangle(\boldsymbol{\Omega}, \boldsymbol{X}_b))$,

$$\dot{\boldsymbol{X}}_b = \boldsymbol{\Omega} \times \boldsymbol{X}_b\,. \tag{32}$$

Both observers see the same vector $\boldsymbol{X}_b$ but their perception of how it changes is completely different.

Remark: *The length of $\boldsymbol{X}_b$ is constant, independent of the used coordinate system. Because of $\boldsymbol{X}_b \perp (\boldsymbol{\Omega} \times \boldsymbol{X}_b)$ it follows that*

$$\frac{d|\boldsymbol{X}_b|^2}{dt} = 2\ \boldsymbol{X}_b \cdot \frac{d\boldsymbol{X}_b}{dt} = 2\ \boldsymbol{X}_b \cdot (\boldsymbol{\Omega} \times \boldsymbol{X}_b) = 0\,. \tag{33}$$

To describe the time-dependent vector $\boldsymbol{X}_b(t)$ in a non-rotating coordinate system, the vector $\boldsymbol{X}_b(t)$ is split into a vector that describes the distance of the circle of latitude on which $\boldsymbol{X}_b(t)$ moves to the equator and two other vectors that define the position on this circle of latitude. Let $\boldsymbol{e}_\Omega$ be the unit vector in direction of the earth rotation vector $\boldsymbol{\Omega}$, then the vector describing its circle of latitude is

$$\cos[\sphericalangle(\boldsymbol{\Omega}, \boldsymbol{X}_b(0))]\ |\boldsymbol{X}_b(0)|\boldsymbol{e}_\Omega = (\boldsymbol{e}_\Omega \cdot \boldsymbol{X}_b(0))\boldsymbol{e}_\Omega\,. \tag{34}$$

The position of $\boldsymbol{X}_b(t)$ on the circle of latitude can be determined by the linear combination of a distance vector $\boldsymbol{X}_b(0)$, expressing the distance to the axis of rotation and the vector perpendicular to it. The distance vector can be computed by

$$\boldsymbol{X}_b(0) - (\boldsymbol{e}_\Omega \cdot \boldsymbol{X}_b(0))\boldsymbol{e}_\Omega = (\boldsymbol{id} - \boldsymbol{e}_\Omega \circ \boldsymbol{e}_\Omega)\boldsymbol{X}_b(0)\,. \tag{35}$$

The vector perpendicular to it with same length is (because of $\boldsymbol{e}_\Omega \ \times \boldsymbol{e}_\Omega = 0$)

$$[\boldsymbol{X}_b(0) - (\boldsymbol{e}_\Omega \cdot \boldsymbol{X}_b(0))\boldsymbol{e}_\Omega] \times \boldsymbol{e}_\Omega = \boldsymbol{X}_b(0) \times \boldsymbol{e}_\Omega\,. \tag{36}$$

Thus we get

$$\boldsymbol{X}_b(t) = (\boldsymbol{e}_\Omega \cdot \boldsymbol{X}_b(0))\boldsymbol{e}_\Omega + \cos(|\boldsymbol{\Omega}|t)(\boldsymbol{id} - \boldsymbol{e}_\Omega \circ \boldsymbol{e}_\Omega)\boldsymbol{X}_b(0) + \sin(|\boldsymbol{\Omega}|t)(\boldsymbol{X}_b(0) \times \boldsymbol{e}_\Omega)\,. \tag{37}$$

Governing equations in a rotating frame of reference With help of some tedious but straightforward computations we transform our conservation laws. In this section we use cartesian coordinates throughout, representing vectors as 3-columns, tensors as 3×3-matrices, etc.. In particular, rotation by the Earth rotation vector $\boldsymbol{\Omega}$ is represented by matrix-multiplication with the skewsymmetric matrix $\underline{\underline{\boldsymbol{\Omega}}}$ built from the components $(\Omega_x, \Omega_y, \Omega_z)^t$ of the rotation vector:

$$\underline{\underline{\boldsymbol{\Omega}}} \times \underline{\boldsymbol{u}} = \begin{pmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} \Omega_y u_z - \Omega_z u_y \\ \Omega_z u_x - \Omega_x u_z \\ \Omega_x u_y - \Omega_y u_x \end{pmatrix} . \quad (38)$$

Here vectors and tensors are represented by their coordinate tupels and matrices as indicated, and $\underline{\boldsymbol{u}}$ is the column of cartesian coordinates of $\boldsymbol{u}$.

With this notation, the mass balance in the intertial frame reads

$$\varrho_t + \underline{\nabla} \cdot (\varrho \underline{\boldsymbol{v}}) = 0 \,. \quad (39)$$

We transform the time derivative $(\partial\varrho/\partial t)$ according to (226) as

$$\frac{\partial \varrho(x,t)}{\partial t} = \frac{\partial \tilde{\varrho}}{\partial \tilde{t}} + \underline{\tilde{\boldsymbol{x}}}^T \underline{\underline{\boldsymbol{\Omega}}} (\underline{\tilde{\nabla}} \tilde{\varrho})^T \,, \quad (40)$$

and the divergence term, following (228), as

$$\begin{aligned} \underline{\nabla} \cdot (\varrho \underline{\boldsymbol{v}}) &= \underline{\nabla} \cdot \left(\varrho [\underline{\boldsymbol{v}}^{\text{rel}} + \underline{\underline{\boldsymbol{\Omega}}}\, \underline{\boldsymbol{x}}] \right) \\ &= (\underline{\nabla} \varrho) \underline{\boldsymbol{v}}^{\text{rel}} + (\underline{\nabla} \varrho)(\underline{\underline{\boldsymbol{\Omega}}}\, \underline{\boldsymbol{x}}) + \varrho \underline{\nabla} \cdot \underline{\boldsymbol{v}}^{\text{rel}} + \varrho \underline{\nabla} \cdot (\underline{\underline{\boldsymbol{\Omega}}}\, \underline{\boldsymbol{x}}) \quad (41) \\ &= (\underline{\tilde{\nabla}} \tilde{\varrho}) \underline{\tilde{\boldsymbol{v}}}^{\text{rel}} + (\underline{\tilde{\nabla}} \tilde{\varrho})(\underline{\underline{\boldsymbol{\Omega}}}\, \underline{\tilde{\boldsymbol{x}}}) + \tilde{\varrho} \underline{\tilde{\nabla}} \cdot \underline{\tilde{\boldsymbol{v}}}^{\text{rel}} \,. \end{aligned}$$

For the *scalar* quantity $\underline{\tilde{\boldsymbol{x}}}^T \underline{\underline{\boldsymbol{\Omega}}} (\underline{\tilde{\nabla}} \tilde{\varrho})^T$ we have

$$\underline{\tilde{\boldsymbol{x}}}^T \underline{\underline{\boldsymbol{\Omega}}} (\underline{\tilde{\nabla}} \tilde{\varrho})^T = \left(\underline{\tilde{\boldsymbol{x}}}^T \underline{\underline{\boldsymbol{\Omega}}} (\underline{\tilde{\nabla}} \tilde{\varrho})^T \right)^T = (\underline{\tilde{\nabla}} \tilde{\varrho}) \underline{\underline{\boldsymbol{\Omega}}}^T \underline{\tilde{\boldsymbol{x}}} \quad (42)$$

and, as $\underline{\underline{\boldsymbol{\Omega}}}$ is skew symmetric, with $\underline{\underline{\boldsymbol{\Omega}}}^T = -\underline{\underline{\boldsymbol{\Omega}}} = -\underline{\underline{\tilde{\boldsymbol{\Omega}}}}$, the mass conservation law in the rotating frame becomes

$$\begin{aligned} \varrho_t + \underline{\nabla} \cdot (\varrho \underline{\boldsymbol{v}}) &= \frac{\partial \tilde{\varrho}}{\partial \tilde{t}} + (\underline{\tilde{\nabla}} \tilde{\varrho}) \underline{\tilde{\boldsymbol{v}}}^{\text{rel}} + \tilde{\varrho} \underline{\tilde{\nabla}} \cdot \underline{\tilde{\boldsymbol{v}}}^{\text{rel}} + \left((\underline{\tilde{\nabla}} \tilde{\varrho}) \underline{\underline{\boldsymbol{\Omega}}}\, \underline{\tilde{\boldsymbol{x}}} + (\underline{\tilde{\nabla}} \tilde{\varrho})(-\underline{\underline{\boldsymbol{\Omega}}}) \underline{\tilde{\boldsymbol{x}}} \right) \\ &= \frac{\partial \tilde{\varrho}}{\partial \tilde{t}} + \underline{\tilde{\nabla}} \cdot (\tilde{\varrho} \underline{\tilde{\boldsymbol{v}}}^{\text{rel}}) \,. \end{aligned} \quad (43)$$

Thus the equation for mass conservation is invariant under the present coordinate transformation into a rotating frame.

The momentum balance, in the present notation, reads

$$(\varrho\underline{\boldsymbol{v}})_t + \left[\underline{\nabla} \cdot (\varrho\, \underline{\boldsymbol{v}}\, \underline{\boldsymbol{v}}^T)\right]^T + \underline{\nabla} p = -\varrho \underline{\nabla}\Phi - \underline{\nabla} \cdot \underline{\underline{\boldsymbol{\tau}}}\,. \tag{44}$$

As only the first two terms do change under coordinate transformations (the reader may want to verify this), we can neglect the others for the time being. For the first two terms we use the product rule to obtain

$$(\varrho\underline{\boldsymbol{v}})_t = \varrho_t\, \underline{\boldsymbol{v}} + \varrho\, \underline{\boldsymbol{v}}_t \quad \text{and} \tag{45}$$

$$\left[\underline{\nabla} \cdot (\varrho\underline{\boldsymbol{v}}\, \underline{\boldsymbol{v}}^T)\right]^T = (\underline{\nabla} \cdot (\varrho\underline{\boldsymbol{v}}))\underline{\boldsymbol{v}} + \varrho(\underline{\boldsymbol{v}}^T \underline{\nabla}^T)\underline{\boldsymbol{v}}\,. \tag{46}$$

Using mass conservation, we further have

$$\begin{aligned}(\varrho\underline{\boldsymbol{v}})_t + \left[\underline{\nabla} \cdot (\varrho\underline{\boldsymbol{v}}\, \underline{\boldsymbol{v}}^T)\right]^T &= (\varrho_t + \underline{\nabla} \cdot (\varrho\underline{\boldsymbol{v}}))\, \underline{\boldsymbol{v}} \quad + \quad \varrho \left(\underline{\boldsymbol{v}}_t + (\underline{\boldsymbol{v}}^T \underline{\nabla}^T)\underline{\boldsymbol{v}}\right) \\ &= \qquad 0 \qquad + \quad \varrho \left(\underline{\boldsymbol{v}}_t + (\underline{\boldsymbol{v}}^T \underline{\nabla}^T)\underline{\boldsymbol{v}}\right)\,,\end{aligned} \tag{47}$$

and the transformation of these terms into the rotating coordinate system yields

$$\varrho \left(\frac{\partial \tilde{\underline{\boldsymbol{v}}}^{\mathrm{rel}}}{\partial t} + ((\tilde{\underline{\boldsymbol{v}}}^{\mathrm{rel}})^T \underline{\nabla}^T)\tilde{\underline{\boldsymbol{v}}}^{\mathrm{rel}} + 2\,(\underline{\underline{\boldsymbol{\Omega}}}\, \tilde{\underline{\boldsymbol{v}}}^{\mathrm{rel}}) + \underline{\underline{\boldsymbol{\Omega}}}(\underline{\underline{\boldsymbol{\Omega}}}\, \underline{\boldsymbol{x}}) \right) + \tilde{\underline{\nabla}} p = -\varrho \tilde{\underline{\nabla}}\Phi - \tilde{\underline{\nabla}} \cdot \tilde{\underline{\underline{\boldsymbol{\tau}}}} \quad . \tag{48}$$

Physically, the term $2\,(\underline{\underline{\boldsymbol{\Omega}}}\, \tilde{\underline{\boldsymbol{v}}}^{\mathrm{rel}})$ represents the Coriolis acceleration.

The term $\underline{\underline{\boldsymbol{\Omega}}}(\underline{\underline{\boldsymbol{\Omega}}}\, \underline{\boldsymbol{x}})$ expresses the centripetal acceleration due to the rotation of the reference frame. As this term can be written as the density times the gradient of a potential (namely which one?), it is often combined with gravity term, thereby inducing a modified effective gravitational potential. The order of magnitude of the centripetal inertia may be estimated by

$$(10^{-4}\mathrm{s}^{-1})^2 \cdot 6 \cdot 10^6 \ \mathrm{m} \approx 10^{-2}\ \frac{\mathrm{m}}{\mathrm{s}^2}\,. \tag{49}$$

In contrast, the acceleration of gravity is of the order of $g \approx 10$ m s^{-2}, and the centripetal acceleration may be neglected for most practical purposes in meteorology. Notice, however, that this may be a different issue in climate models, because in long-time simulations, even small effects can accumulate and eventually induce leading-order changes.

Like the equation for the mass conservation, the equation of energy conservation does not change when introducing a rotating coordinate system. We leave the verification of this claim to the reader.

2.5 Adiabatic motions and the concept of potential temperature

Consider a flow field that is sufficiently smooth so that the differential form of the mass, momentum, and energy balances are valid, i.e.,

$$
\begin{aligned}
\varrho_t + \nabla \cdot (\varrho \boldsymbol{v}) &= 0 \\
(\varrho \boldsymbol{v})_t + \nabla \cdot (\varrho \boldsymbol{v} \circ \boldsymbol{v} + p\, \boldsymbol{id}) + 2\boldsymbol{\Omega} \times \varrho \boldsymbol{v} &= -\nabla \cdot \boldsymbol{\tau} - \varrho \left(g \nabla \Phi + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{x}\right) \\
(\varrho e)_t + \nabla \cdot ([\varrho e + p] \boldsymbol{v}) &= -\nabla \cdot (\boldsymbol{v} \cdot \boldsymbol{\tau} + \boldsymbol{\jmath})
\end{aligned}
\tag{50}
$$

and they are closed by the equation of state connecting the pressure p with the conserved quantities, $(\varrho, \varrho \boldsymbol{v}, \varrho e)$,

$$
\varrho e = \frac{p}{\gamma - 1} + \varrho \frac{\boldsymbol{v}^2}{2} + \varrho (\Phi + \Phi_\Omega) \,, \tag{51}
$$

with Φ_Ω defined such that $\nabla \Phi_\Omega = \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{x}$.

In these equations, the (molecular) stress tensor, $\boldsymbol{\tau}$, and heat flux density vector, $\boldsymbol{\jmath}$, represent transport processes of momentum and energy within the fluid that occur without any mass being exchanged. When these terms are absent, control volumes within the fluid are restricted to exchange mechanical energy only, either by advection of total energy, as represented by the flux term $\varrho e \boldsymbol{v}$, or by mechanical work, as represented by the energy flux $p\boldsymbol{v}$. Flows of this kind are called *adiabatic*.

Remark: *When considering mixtures of different fluid species, we will also require diffusion, i.e., the molecular-level transport of the individual species relative to the mean flow, to be zero for the notion of an adiabatic process to apply.*

For adiabatic, smooth flows, the conservation laws may be linearly combined to yield an evolution equation for the pressure field. Together with the equation for density, it reads

$$
\begin{aligned}
\varrho_t + \boldsymbol{v} \cdot \nabla \varrho + \varrho \nabla \cdot \boldsymbol{v} &= 0 \\
p_t + \boldsymbol{v} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{v} &= 0 \,.
\end{aligned}
\tag{52}
$$

By eliminating the velocity divergence, we obtain

$$
\frac{1}{\varrho} \frac{D\varrho}{Dt} - \frac{1}{\gamma p} \frac{Dp}{Dt} = 0 \,, \tag{53}
$$

where

$$
\frac{D}{Dt} = (\partial_t + \boldsymbol{v} \cdot \nabla) \tag{54}
$$

is the time derivative which an observer moving with the fluid (at velocity $\boldsymbol{v}$) will measure.

The above is equivalent to

$$\frac{D\Theta}{Dt} = 0 \qquad \text{where} \qquad \Theta = T_0 \left(\frac{(p/p_0)^{1/\gamma}}{\varrho/\varrho_0} \right) \tag{55}$$

is the *potential temperature*. Here ϱ_0, T_0, and $p_0 = \varrho_0 R T_0$ are some arbitrary, yet for any given flow fixed, reference values. They are introduced to render the expression of taking a non-integer power of some quantity mathematically meaningful by first non-dimensionalizing it.

If we pick ϱ_0, T_0, p_0 to denote the standard reference values of typical conditions at sea level, i.e., $p_0 = 10^5\ \mathrm{N/m^2}, T_0 = 273\ \mathrm{K}$, then Θ has a neat interpretation: Take any parcel of air at thermodynamic conditions p, ϱ, and let it undergo an adiabatic process that brings its pressure up or down to the reference pressure p_0. Then Θ is the temperature the parcel will acquire when that process is finished.

Remark: *The potential temperature is closely related to thermodynamic entropy. In the present case, one can directly be expressed as a function of the other. Thus, (55) is where our coefficient γ received its name* isentropic exponent *from.*

Remark: *Mixtures of different fluid species may undergo changes of composition when the pressure and temperature adjust during an adiabatic process. In that case, the notion of a potential temperature with exactly the same physical meaning as given above remains valid. Yet the formula in (55) becomes more involved.*

2.6 Summarizing the equations

In the sequel, we will study (i) the conservation of mass, momentum, and energy, extented by a set of species balance equations. This extension will allows us to later account for moist processes.

$$\varrho_t + \nabla \cdot (\varrho \boldsymbol{v}) = 0 \tag{56a}$$

$$(\varrho \boldsymbol{v}_{\shortparallel})_t + \nabla \cdot (\varrho \boldsymbol{v} \circ \boldsymbol{v}_{\shortparallel}) + 2(\boldsymbol{\Omega} \times \varrho \boldsymbol{v})_{\shortparallel} + \nabla_{\shortparallel} p = -(\nabla \cdot \boldsymbol{\tau})_{\shortparallel} \tag{56b}$$

$$(\varrho w)_t + \nabla \cdot (\varrho \boldsymbol{v} w) + 2(\boldsymbol{\Omega} \times \varrho \boldsymbol{v})_{\perp} + p_z = -(\nabla \cdot \boldsymbol{\tau})_{\perp} - \varrho g \tag{56c}$$

$$(\varrho e)_t + \nabla \cdot (\boldsymbol{v}\,[\varrho e + p]) = -\nabla \cdot \left(\boldsymbol{\jmath} + \boldsymbol{v} \cdot \boldsymbol{\tau} + \sum_{i=1}^{n_{\mathrm{sp}}} h_i \boldsymbol{d}_i \right). \tag{56d}$$

$$(\varrho Y_i)_t + \nabla \cdot (\varrho Y_i \boldsymbol{v}) = \varrho \omega_i - \nabla \cdot \boldsymbol{d}_i \,. \qquad (i = 1, \ldots, n_{\text{sp}} - 1) \tag{56e}$$

Here we have split the momentum equations into their horizontal "$_{\shortparallel}$" and vertical "$_{\perp}$" components anticipating that the vertical direction in meteorological applications usually plays a mathematically special role.

Also, we have added a set of transport equations for n_{sp} energy-carrying (chemical) species. The composition of the gas is described by the species' mass fractions Y_i. Potential species conversion processes, such as chemical reactions or the formation of cloud water from water vapor are represented by the source terms ω_i. The flux terms $\boldsymbol{d}_i$ cover the diffusion of species relative to the mean flow. (Why does the counter in the last equation run up to $n_{\text{sp}} - 1$ only?)

The system is closed by adding the equations of state

$$\varrho e = \varrho \int_0^T c_v(T) dT' + \varrho\, \Phi + \frac{1}{2} \varrho \boldsymbol{v}^2 + \sum_{i=1}^{n_{\text{spec}}} \varrho Y_i Q_i \,, \quad \text{and} \quad p = \varrho R T \tag{57}$$

where

$$c_v(T) = \sum_{i=1}^{n_{\text{spec}}} Y_i \, c_{v,i}(T) \,, \qquad R = \sum_{i=1}^{n_{\text{spec}}} Y_i \, R_i \,, \tag{58}$$

and

$$h_i = \int_0^T c_{p,i}(T) dT' + Q_i \,. \tag{59}$$

Here the constants R_i, Q_i are the gas constants and formation enthalpies of the species, $c_{v,i}, c_{p,i}$ are their specific heat capacities at constant volume and at constant pressure, respectively.

In addition, we have to adopt appropriate expressions for the stress tensor, heat flux density, and species diffusion fluxes, $\boldsymbol{\tau}$, $\boldsymbol{\jmath}$, and $\boldsymbol{d}$, respectively. For a Navier-Stokes fluid, the former two are given by (19) and (20). However, in practical meteorological modelling applications involving turbulence, one often replaces these fluxes with effective turbulent closure schemes so as to describe not the transport due to molecular motions but rather the transport due to turbulent fluctuations. In that case, the functional form of these "subscale" momentum and heat flux terms may take a wide variety of forms which we will not address here in detail.

3 Introduction to multiple scales asymptotics

To motivate the mathematical techniques we are going to apply to the atmospheric flow equations in later chapters, we will now analyze the simple example of a linear oscillator. After deriving analytical solutions, we will focus on a situation that in many ways resembles situations arising frequently in geophysical problems: a slow background motion caused by an external force is accompanied by rapid oscillations around it. The oscillation amplitudes are generally not small! To give some meaning to the notions of "smallness" and "rapidity", we will first nondimensionalize the oscillator equations and identify small parameters that lend themselves for comparison. By means of single and multiple scales analyses we will then try to derive simplified approximate solutions that become more and more accurate as the small parameters vanish.

3.1 Exact solutions for the linear oscillator

A typical example of a linear oscillator is a (small) piece of material with mass m attached to a spring with stiffness c. The stronger the spring, the higher the spring constant c. At time t, the mass is located at position $x = x(t)$ (see Fig. 3). If we displace it away from its equilibrium position, at which the spring's force just balances the weight of the piece, and then let the system evolve freely, it will oscillate around its equilibrium with constant amplitude. If we let the mass dive into some not-too-viscous fluid, the system will perform a damped oscillation. If the viscosity of the fluid is sufficiently high, the mass will not oscillate anymore but just move back monotonically to its equilibrium position.

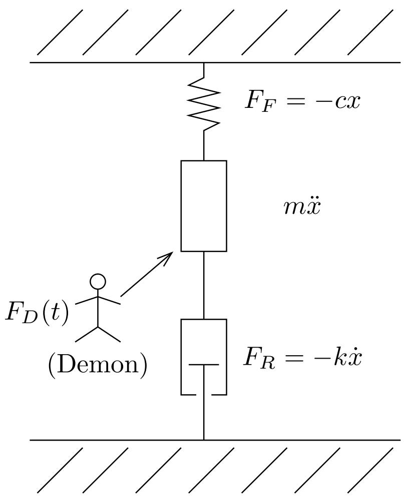

Figure 3. Damped spring–mass system

There are two forces acting *within* the system: the restoring force of the spring in the direction opposite to the *displacement* of the mass, and the frictional force $F_R = -k\dot{x}$ of the viscous fluid which acts in the direction opposite to the *motion* of the mass. Newton's law, which says that the temporal change of momentum equals the sum of all acting forces, yields an equation of motion for the system,

$$m\ddot{x} = -cx - k\dot{x} + F_D(t)\,. \tag{60}$$

Here we have included a general external force $F_D = F_D(t)$ which some demon may exert on the mass.

The solution to this second-order ordinary differential equation (ODE) is uniquely determined once initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ are prescribed. Following the theory of linear ODEs [Walter (1996)] we can construct all solutions $x(t)$ as a superposition of the general solution of the homogenous problem and one so-called *particular solution* of the inhomogenous equation, i.e.,

$$x(t) = x_h(t) + x_p(t)\,. \tag{61}$$

Free oscillations We will now derive the general solution of the spring-mass system's homogenous ODE ((60) with $F_D \equiv 0$),

$$m\ddot{x} + k\dot{x} + cx = 0\,. \tag{62}$$

To solve this equation, we choose an exponential ansatz,

$$x(t) = \exp(\omega t)\,. \tag{63}$$

Inserting, we have

$$m\left(\omega^2 \exp(\omega t)\right) + k\left(\omega \exp(\omega t)\right) + c\left(\exp(\omega t)\right) = 0\,, \tag{64}$$

and after division by $\exp(\omega t) \neq 0$ we find

$$m\omega^2 + k\omega + c = 0\,, \tag{65}$$

which is the system's *characteristic equation.* The solutions are

$$\omega_{1/2} = -\frac{1}{2}\frac{k}{m} \pm \sqrt{D} \qquad \text{with} \qquad D := \frac{k^2}{4m^2} - \frac{c}{m}\,. \tag{66}$$

If $\omega_1 = \omega_2$, i.e., if the discriminant $D = 0$, the solutions differ qualitatively from those obtained when $\omega_1 \neq \omega_2$. Generally, if ω is a k-fold multiple solution of the characteristic equation, it corresponds to k linearly independent solutions of the form

$$e^{\omega t},\ te^{\omega t}, \ldots,\ t^{k-1}e^{\omega t} \tag{67}$$

of the associated differential equation (Walter, 1996, S.173ff). One important property of linear homogenous differential equations is that the sum of two solutions is again a solution. Using this for the case $D \neq 0$ we obtain the general solution to (62),

$$x(t) = A\exp(\omega_1 t) + B\exp(\omega_2 t)\,, \tag{68}$$

where A and B are constants that remain to be determined. For the case $D = 0$ we have

$$x(t) = A\exp(\omega t) + Bt\exp(\omega t) = (A + Bt)\exp(\omega t) \tag{69}$$

again with yet unknown constants A und B.

Next we will distinguish several cases that differ w.r.t. the relative magnitudes of the spring constant, c, the damping coefficient, k and the mass, m:

1st case: $k = 0$; $c > 0$.

Thus we consider an inviscid oscillator for which

$$\omega_{1,2} = \pm i\omega_0 \qquad \text{with} \qquad \omega_0 = \sqrt{\frac{c}{m}}\,, \tag{70}$$

and

$$x(t) = A\exp(i\omega_0 t) + B\exp(-i\omega_0 t)\,. \tag{71}$$

The solution $x(t)$ is now complex-valued, but since we seek real-valued solutions, we allow $A, B \in \mathbb{C}$ to be complex as well,

$$A = A_r + iA_i, \quad B = B_r + iB_i\,. \tag{72}$$

Later we will choose A and B in such a way that the final result is again real valued and phyiscally meaningful.

Using Euler's formula, $e^{ix} = \cos x + i\sin x$, respectively, $e^{-ix} = \cos x - i\sin x$, and splitting A and B into their real and an imaginary parts, we transform (71) into

$$\begin{aligned} x(t) \quad = \quad & \Big((A_r + B_r)\cos(\omega_0 t) + (-A_i + B_i)\sin(\omega_0 t)\Big) \\ & i\,\Big((A_r - B_r)\sin(\omega_0 t) + (A_i + B_i)\cos(\omega_0 t)\Big)\,. \end{aligned} \tag{73}$$

Because sine and cosine are linearly independent, the solution is real valued if the coefficients $(A_r - B_r)$ and $(A_i + B_i)$ satisfy

$$\begin{aligned} (A_r - B_r) &= 0 \qquad \Leftrightarrow \qquad & A_r &= B_r \\ (A_i + B_i) &= 0 \qquad \Leftrightarrow \qquad & A_i &= -B_i\,. \end{aligned} \tag{74}$$

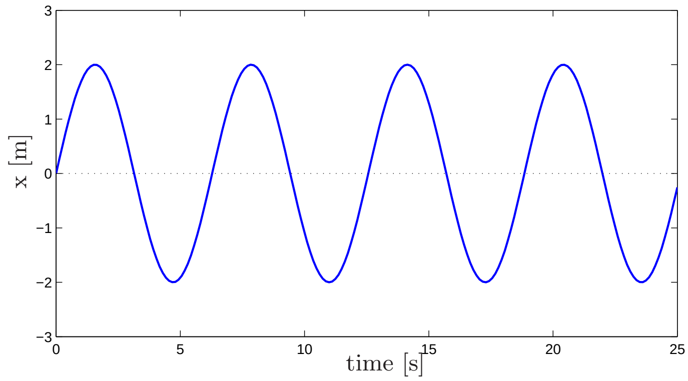

Figure 4. Exact solution for the oscillator without friction ($k = 0$) with $C = 2$ and $\omega_0 = 1$.

Now (73) can be reduced to

$$\begin{aligned} x(t) &= 2B_r \cos(\omega_0 t) + 2B_i \sin(\omega_0 t) \\ &= a \cos(\omega_0 t) + b \sin(\omega_0 t) \end{aligned} \tag{75}$$

The constants a and b in (75) can be determined from the given initial conditions and we obtain a particular solution for the considered system.

Example: *Let us chose $x_0 = x(0) = 0$, so that the mass is at its equilibrium at time $t = 0$, and $\dot{x}_0 = \frac{dx}{dt}(0) = C\omega_0$, so that it has initial velocity $C\omega_0$. Inserting, we find from (75)*

$$a = 0 \quad \textit{und} \quad b = \frac{\dot{x}_0}{\omega_0} = C\,, \tag{76}$$

and the solution reads

$$x(t) = C \sin(\omega_0 t)\,. \tag{77}$$

See Fig. 4.

2nd case: $k > 0; c > 0$.

In this case with non-zero friction, the discriminant, D, in the general solution for ω in (66) has to be examined in more detail. It may be greater than, equal, or less than zero.

a) $D < 0$ $\left(\frac{k^2}{4m^2} < \frac{c}{m} \;\Rightarrow\; \omega_{1,2} = -\frac{k}{2m} \pm i\sqrt{\frac{c}{m} - \frac{k^2}{4m^2}}\right)$

Inserting $\omega_{1,2}$ into (68) and using Euler's formula provides

$$x(t) = \exp\left(-\frac{k}{2m}t\right)\left(\tilde{a}\cos(\tilde{\omega}_a t) + \tilde{b}\sin(\tilde{\omega}_a t)\right), \tag{78}$$

where

$$\tilde{\omega}_a = \sqrt{\frac{c}{m} - \frac{k^2}{4m^2}}\,. \tag{79}$$

The coefficients $\tilde{a}$ and $\tilde{b}$ again have to be determined so as to satisfy the required initial conditions. The mass now performs a damped oscillation around its equilibrium position with a modified frequency compared to first case. See Fig. 5.

b) $D > 0$ $\left(\frac{k^2}{4m^2} > \frac{c}{m} \;\Rightarrow\; \omega_{1,2} = -\frac{k}{2m} \pm \sqrt{\frac{k^2}{4m^2} - \frac{c}{m}}\right)$

Again, we insert into (68), and A and B are constants that have to be computed using given initial conditions. We find

$$x(t) = \exp\left(-\frac{k}{2m}t\right)\left(A\exp(\tilde{\omega}_b t) + B\exp(-\tilde{\omega}_b t)\right) \tag{80}$$

where

$$\tilde{\omega}_b = \sqrt{\frac{k^2}{4m^2} - \frac{c}{m}}\,. \tag{81}$$

Comparison of the solutions in (78) and (80) shows that both cases describe a damped motion, the term $\exp(\tilde{\omega}_b t)$ "losing" against $\exp(-\frac{k}{2m}t)$ for $t \to \infty$, but there is a fundamental difference: in case a), the system with the exponential function $\exp(-\frac{k}{2m}t)$ performs a damped harmonic oscillation and passes the origin several times. In case b), however, the damping is so strong that the mass is not oscillating at all and just moves back monotonically to its equilibirium (creeping case).

c) $D = 0$ $\left(\frac{k^2}{4m^2} = \frac{c}{m} \;\Rightarrow\; \omega_{1,2} = -\frac{k}{2m}\right)$

The solution in this case is in line with the one in the case already mentioned above with general solution (69). We find

$$x(t) = \exp\left(-\frac{k}{2m}t\right)(A + Bt)\,, \tag{82}$$

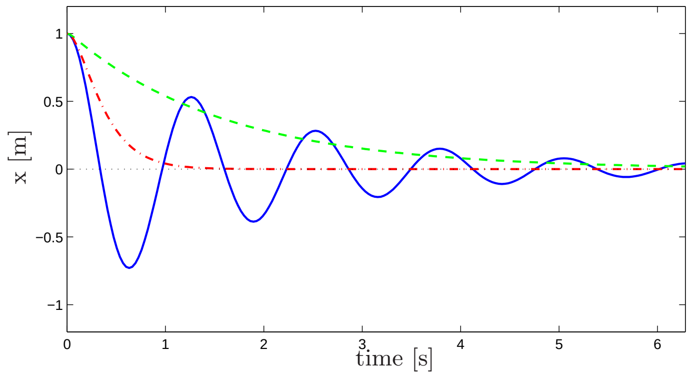

Figure 5. Exact solution for the oscillator with friction ($k > 0$): a) $D < 0$ (blue, line), b) $D > 0$ (green,dashed), c) $D = 0$ (red, point-dashed).

where again the constants A and B have to be determind from the initial conditions.

Thus, case c) is exactly "between" cases a) and b). It is called *aperiodic limiting case*. In case b) the damping is still stronger and faster than in case c) but in both cases there is no oscillation in the system.

Forced oscillations Now we account for an external force $F_D = F_D(t)$ acting on the linear oscillator by computing the particular part $x_p(t)$ of the solution of the inhomogeneous differential equation (60). We will restrict to the case of a periodic force

$$F_D(t) = F_0 \cos(\Omega t)\,. \tag{83}$$

(Why is this not a severe restriction?). Experiments show that an oscillator exposed to such a force performs a forced harmonic oscillation after some adjustment time has elapsed. The frequency of this oscillation equals that of the driving force, $F_D(t)$. Let us verify that this will in fact be a valid particular solution. We let

$$x_p(t) = A_p \sin(\Omega t) + B_p \cos(\Omega t) \tag{84}$$

with coefficients A_p and B_p remaining to be determined. Inserting the corresponding time derivatives

$$\begin{aligned} \dot{x}_p(t) &= A_p\Omega \cos(\Omega t) - B_p\Omega \sin(\Omega t) \\ \ddot{x}_p(t) &= -A_p\Omega^2 \sin(\Omega t) - B_p\Omega^2 \cos(\Omega t) \end{aligned} \tag{85}$$

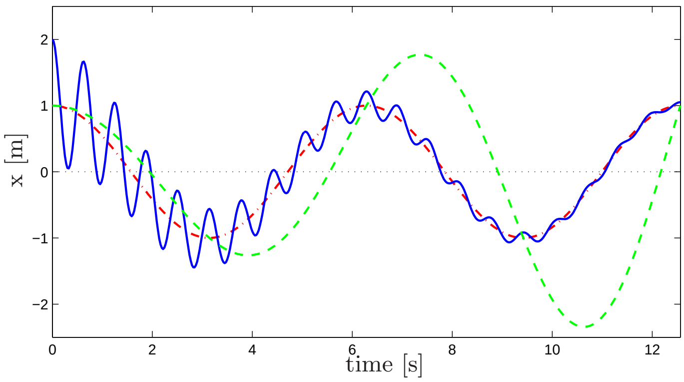

Figure 6. Exact solution for the oscillator driven by an external force. a) Superposition of fast oscillations and slow background movement (blue line) b) resonance (green dashed line) c) driving external force (red dashed-pointed line)

into equation (60) we obtain

$$\begin{aligned} F_0 \cos(\Omega t) &= -m(A_p \Omega^2 \sin(\Omega t) + B_p \Omega^2 \cos(\Omega t)) + \\ &\quad k(-B_p \Omega \sin(\Omega t) + A_p \Omega \cos(\Omega t)) + \\ &\quad c(A_p \sin(\Omega t) + B_p \cos(\Omega t)) \,. \end{aligned} \tag{86}$$

Linear independence of the sine and cosine functions allows us to split this equation, such that

$$\begin{aligned} (c - m\Omega^2)A_p - k\Omega B_p &= 0 \\ (c - m\Omega^2)B_p + k\Omega A_p &= F_0 \,. \end{aligned} \tag{87}$$

This is a system of two equations with two unknowns A_p and B_p that is easily solved. If $k \neq 0$ (2nd case), then

$$\begin{aligned} B_p &= \frac{c - m\Omega^2}{k\Omega} A_p \\ A_p &= \frac{F_0}{k\Omega} \left(1 + \left(\frac{c - m\Omega^2}{k\Omega} \right)^2 \right)^{-1} \,. \end{aligned} \tag{88}$$

If $k = 0$ and $(c - m\Omega^2) \neq 0$, then

$$A_p = 0 \qquad \text{and} \qquad B_p = \frac{F_0}{c - m\Omega^2} \tag{89}$$

and the case $k = 0$ and $(c - m\Omega^2) = 0$ yields the *resonant* solution (see for example Walter (1996)) with time-dependent coefficients

$$A_p(t) = \frac{F_0}{2m\Omega} t \qquad \text{and} \qquad B_p = 0\,. \tag{90}$$

We have found particular solutions that satisfy the inhomogenous equation for each of the parameter regimes. The sum of the homogenous and the particular solution is the general solution of the oscillator equation with external periodic forcing. For example, in case 2a) this solution reads

$$x(t) = \exp\left(-\frac{k}{2m}t\right)\left(\tilde{a}\cos(\tilde{\omega}_a t) + \tilde{b}\sin(\tilde{\omega}_a t)\right) + A_p \sin(\Omega t) + B_p \cos(\Omega t) \tag{91}$$

The constants A_p and B_p have to be defined according to equation (88) and only $\tilde{a}$ and $\tilde{b}$ are to be computed from the intial conditions. The solutions for the other cases can be derived analogously.

3.2 Dimensionless representation and small parameters

In the preceding section we were able to derive the general solution for the linear oscillator with the equation of motion

$$m\ddot{x} + k\dot{x} + cx = F_0 \cos(\Omega t) \tag{92}$$

and initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$. This solution consists of a homogeneous and an inhomogenous part. As we have seen in the second case, the homogenous solution is decaying exponentially when there is non-zero damping, so that in the longtime motion only the inhomogenous (particular) part of the solution prevails.

In the present section, we will continue our analysis by studying situations in which the free oscillation, damping, and forcing act on very different characteristic time scales. (The notion of a *characteristic scale* will hopefully be reasonably clear by the end of the section. The reader may trust her or his intuition for the time being.) An important tool for deriving concise mathematical descriptions of such *scale-separated* processes is Multiple-Scales Asymptotics, the main ideas and techniques of which will be explained here using the linear oscillator as an example.

To determine the different time scales of the system we will nondimensionalize the general equation of motion (92) in the first step.

Remarks on dimensional analysis Within the governing equation for the linear oscillator, we identify three fundamental physical dimensions

$\{\mathcal{X}_i\}_{i=1}^3$: Length $\mathcal{L}$, Time $\mathcal{T}$, and Mass $\mathcal{M}$. Each physical quantity ϕ_j that appears in this governing equation has a physical dimension that is a product of these fundamental ones, so that

$$\mathrm{Dim}(\phi_j) = \prod_{i=1}^{3} (\mathcal{X}_i)^{b_j^i} \ . \tag{93}$$

For the linear oscillator, we have

$$\begin{array}{ll}
\textbf{quantity } \phi & \textbf{physical dimension Dim}(\phi) \\
& \\
\text{dependent and independent variables} & \\
\qquad x & \qquad \mathcal{L} \\
\qquad t & \qquad \mathcal{T} \\
& \\
\text{parameters of governing equation} & \\
\qquad m & \qquad \mathcal{M} \\
\qquad k & \qquad \mathcal{M}/\mathcal{T} \\
\qquad c & \qquad \mathcal{M}/\mathcal{T}^2 \\
\qquad F_0 & \qquad \mathcal{M}\mathcal{L}/\mathcal{T}^2 \\
\qquad \Omega & \qquad 1/\mathcal{T} \\
& \\
\text{initial data} & \\
\qquad x_0 & \qquad \mathcal{L} \\
\qquad \dot{x}_0 & \qquad \mathcal{L}/\mathcal{T}
\end{array} \tag{94}$$

Once a system of concrete units is chosen based on which these fundamental dimensions shall be measured, each of the physical quantities and coefficients in the governing equations can be quantified by a sole number. The familiar SI-system is one example, where $(\mathcal{T}, \mathcal{L}, \mathcal{M})$ are measured in terms of (Second s, Meter m, Kilogram kg).

Knowing a quantity's physical dimension and the underlying system of units one can always transform these non-dimensional numbers back into measurable physical values. Obviously, there is a one-to-one map between any two different systems of units, so that the exact solutions of the governing equations will not depend on which system is chosen.

As it stands, the oscillator equation in (60) does not reveal anything besides what was built into it to begin with: Newton's law of motion for the particular case of the mechanical system in Fig. 3. To obtain a somewhat improved intuition about possible solutions one may study classes of solutions distinguished by some particular global mathematical characterization.

For any given solution of the equations one can identify "characteristic values" $[\phi_{j,\mathrm{ref}}]_{j=1}^{N}$ of the total of N physical quantities in the system which roughly describe their orders of magnitude throughout the solution or at least during a certain time interval (and within a selected region in space for pdes). These dimensional characteristic quantities may be combined into non-dimensional characteristic numbers

$$\Pi_k = \prod_{j=1}^{N} (\phi_{j,\mathrm{ref}})^{a_k^j} \,, \tag{95}$$

with the exponents a_k^j chosen so as to guarantee that the Π_k do not have a physical dimension as will be explained shortly.

These numbers are extremely useful as they provide a comparison between various quantities that may have the same physical dimension but very different physical origin. An example is the ratio of the oscillator's frequency of free, undamped oscillations, $\sqrt{c/m}$, and that of its harmonic excitation,

$$\Pi_* = \frac{\sqrt{c/m}}{\Omega} \,. \tag{96}$$

For the non-dimensional Π's to be actually non-dimensional, all the physical dimensions have to cancel exactly in the product. Using (93), we may rephrase this statement as

$$\mathrm{Dim}(\Pi_k) = \prod_{j=1}^{N} \left[\prod_{i=1}^{3} (\mathcal{X}_i)^{b_j^i} \right]^{a_k^j} = \prod_{i=1}^{3} \left[\prod_{j=1}^{N} (\mathcal{X}_i)^{b_j^i \, a_k^j} \right] = \prod_{i=1}^{3} (\mathcal{X}_i)^{\left[\sum\limits_{j=1}^{N} b_j^i \, a_k^j \right]} \equiv 1 \,. \tag{97}$$

For this equation to hold, the respective powers of each of the fundamental dimensions $\mathcal{X}_i$ must vanish independently, so that

$$\sum_{j=1}^{N} b_j^i \, a_k^j \equiv 0 \qquad (i = 1 \dots 3, \quad k \text{ arbitrary}) \,. \tag{98}$$

These are 3 linear constraints on the N-tuples $\boldsymbol{a}_k = (a_k^1, \dots, a_k^N)$, which therefore span a total space of dimension $N - 3$. This, in turn, is equivalent to the existence of a set of $N - 3$ independent characteristic numbers $\{\Pi_k\}_{k=1}^{N-3}$, which is the key statement of the famous *Buckingham's π-theorem.*

Remark: *Often this theorem is quoted from E. Buckingham (1914) as* Buckingham's Theorem. *Yet, Barenblatt (1996) acknowledges a A. Federmann*

(St. Petersburg 1911) for the first proof, and Görtler (1975) for a concise formulation.

Remark: *For further aspects of dimensional analysis, the reader may want to consult Barenblatt (1996).*

Dimensionless representation of the oscillator problem With the nine quantities from (94) we have 9-3=6 linear independent dimensionless combinations. Two of these are our new dimensionless dependent and independent variables, y and τ. Four of them are dimensionless real numbers which relate and characterize all those quantities that influence the solution in one way or the other. A possible specific choice of these quantities is

$$
\begin{array}{lllllll}
\multicolumn{7}{l}{\text{Dependent and independent variables}} \\[2ex]
y & = & \dfrac{x}{F_0/c} & ; & \mathrm{Dim}\,(y) & = & \mathcal{L}\cdot\dfrac{\mathcal{T}^2}{\mathcal{ML}}\cdot\dfrac{\mathcal{M}}{\mathcal{T}^2} \quad = \quad 1 \\[2ex]
\tau & = & \Omega t & ; & \mathrm{Dim}\,(\tau) & = & \dfrac{1}{\mathcal{T}}\cdot\mathcal{T} \quad = \quad 1 \\[3ex]
\multicolumn{7}{l}{\text{Characteristic numbers}} \\[2ex]
\mu & = & m\dfrac{\Omega^2}{c} & ; & \mathrm{Dim}\,(\mu) & = & \mathcal{M}\cdot\dfrac{1}{\mathcal{T}^2}\cdot\dfrac{\mathcal{T}^2}{\mathcal{M}} \quad = \quad 1 \\[2ex]
\kappa & = & k\dfrac{\Omega}{c} & ; & \mathrm{Dim}\,(\kappa) & = & \dfrac{\mathcal{M}}{\mathcal{T}}\cdot\dfrac{1}{\mathcal{T}}\cdot\dfrac{\mathcal{T}^2}{\mathcal{M}} \quad = \quad 1 \\[2ex]
y_0 & = & \dfrac{x_0}{F_0/c} & ; & \mathrm{Dim}\,(y_0) & = & \mathcal{L}\cdot\dfrac{\mathcal{T}^2}{\mathcal{ML}}\cdot\dfrac{\mathcal{M}}{\mathcal{T}^2} \quad = \quad 1 \\[2ex]
y_0' & = & \dfrac{\dot{x}_0}{\Omega F_0/c} & ; & \mathrm{Dim}\,(y_0') & = & \dfrac{\mathcal{L}}{\mathcal{T}}\cdot\mathcal{T}\cdot\dfrac{\mathcal{T}^2}{\mathcal{ML}}\cdot\dfrac{\mathcal{M}}{\mathcal{T}^2} \quad = \quad 1
\end{array}
\tag{99}
$$

Here μ characterizes the system's inertia, κ its damping, and y_0, y_0' the initial data that allow us to select a specific solution. In interpreting these quantities, notice that F_0/c is the static displacement of a spring with stiffness c under the effect of a (constant) force F_0.

Notice that $y : \mathbb{R}^+ \to \mathbb{R}$, is a *function*, not just a number, and τ varies all over $\mathbb{R}^+$. The original unknowns $x(t), t$ and the new ones, $y(\tau), \tau$, must

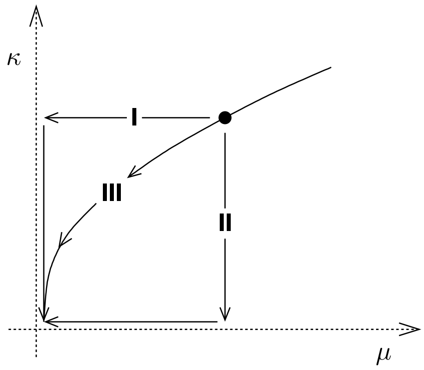

Figure 7. Different possibilities of performing a limit in an asymptotic system with small parameters μ and κ

satisfy

$$\frac{x(t)}{F_0/c} = y(\Omega t) \qquad \text{and} \qquad \Omega t = \tau\,. \tag{100}$$

Using this identity, we have

$$\dot{x}(t) \quad = \quad \frac{F_0}{c}\,\frac{dy}{d\tau}\frac{d\tau}{dt} = \frac{F_0\Omega}{c}\,y'(\tau) \tag{101}$$

$$\ddot{x}(t) \quad = \quad \frac{F_0}{c}\,\frac{d^2y}{d\tau^2}\left(\frac{d\tau}{dt}\right)^2 = \frac{F_0\Omega^2}{c}\,y''(\tau)\,, \tag{102}$$

and (92) then allows us to specify a differential equation for $y(\tau)$,

$$\frac{m\Omega^2}{c}\,y'' + \frac{k\Omega}{c}\,y' + y = \cos\tau \tag{103}$$

or

$$\mu y'' + \kappa y' + y = \cos\tau\,. \tag{104}$$

The appropriate initial conditions read

$$y(0) = y_0 = \frac{x_0 c}{F_0} \qquad \text{and} \qquad y'(0) = y_0' = \frac{\dot{x}_0 c}{F_0\Omega}\,. \tag{105}$$

Oscillators with small mass and small damping – exact solutions
Here we examine the behavior of the (dimensionless) oscillator system

$$y = y(\tau; \mu, \kappa, y_0, y_0') \tag{106}$$

for fixed initial data, but for smaller and smaller values of the inertia and damping parameters μ, κ (i.e., $\mu, \kappa \in \mathbb{R}^+;\ \mu, \kappa \ll 1$).

Distinguished limits Letting $\mu,\ \kappa \to 0$ it turns out to be important just *how* these values tend to zero: For example, if we let μ vanish first, following path I in Fig. 7, we formally get rid of the second derivative in (104) and obtain a first-order ordinary differential equation. But if we let $\kappa \to 0$ first (path II), then the system formally reduces to the undamped oscillator equation. Along the first path, we expect non-oscillatory, purely damped motions, whereas along the second, oscillations of constant amplitude about some slow mean harmonic motion should arise. To verify or reject the stated hypotheses regarding paths I and II, or to decide what behavior will emerge for some path in between, such as III, requires some deeper analyses.

The first ansatz that may come to mind when looking for solutions in the vicinity of the location $\mu = 0; \kappa = 0$ in parameter space might be a Taylor expansion with respect to the μ-κ-dependence of the solution. This would promise to provide the best-possible linear, quadratic, or higher-order polynomial approximations as long as the dependence of the solution on these parameters is sufficiently smooth.

In fact, when such an expansion exists at all, it reads

$$y = y|_{\mu,\kappa=0} + \left(\mu \frac{\partial y}{\partial \mu}\bigg|_{\mu,\kappa=0} + \kappa \frac{\partial y}{\partial \kappa}\bigg|_{\mu,\kappa=0} \right) + o(\mu,\kappa) \qquad (\mu,\kappa \to 0)\,, \tag{107}$$

where

$$\left(\frac{\partial y}{\partial \mu}, \frac{\partial y}{\partial \kappa} \right) = \mathrm{grad}_{(\mu,\kappa)} y \tag{108}$$

is the gradient (or (*Fréchet-*) derivative) of the solution with respect to our two small parameters. (For the definition of a Fréchet-derivative see, e.g., (Werner, 2000, S. 113)).

However, from our previous considerations regarding the path-dependence of the solution behavior as $\mu \to 0$ and $\kappa \to 0$ we conclude that even though there is a limit solution, $y|_{\mu,\kappa=0}(\tau) = \cos(\tau)$, it is *not* the limit of either solution found along paths I or II in the parameter space of Fig. 7. Thus we cannot decide whether the “proper behaviour” for very small μ, κ should be purely oscillatory, purely damped, or something in between. We are led to conclude that the gradient in (108) *does not exist*!

Fortunately, not all is lost. Analysis has it that even if a Fréchet-derivative does not exist, linear approximations to the solution *along straight lines*, i.e., directional derivatives, may still be well-defined. This would lead us to consider coupling μ, κ in such a way that $\kappa = \alpha\mu$ as $\mu \to 0$ for some fixed α.

We allow here for more general μ-κ–relationships which include this former one by letting (μ, κ) approach the origin of their parameter space along some parameterized trajectory. To this end, we introduce a new expansion parameter $\varepsilon \ll 1$, and two functions $\hat{\mu}(\varepsilon), \hat{\kappa}(\varepsilon)$ that should satisfy

$$(\hat{\mu}(\varepsilon), \hat{\kappa}(\varepsilon)) = o(1) \qquad (\varepsilon \to 0)\,. \tag{109}$$

Then, dropping the solution's explicit dependence on the initial data in the notation for the moment, we Taylor-expand w.r.t. ε,

$$y(\tau; \mu, \kappa) = \hat{y}(\tau; \varepsilon) = \hat{y}(\tau; \hat{\mu}(\varepsilon), \hat{\kappa}(\varepsilon)) = \hat{y}(\tau; 0) + \varepsilon \frac{\partial \hat{y}}{\partial \varepsilon}(\tau; 0) + o(\varepsilon) \qquad (\varepsilon \to 0)\,. \tag{110}$$

The mappings

$$\begin{aligned} (\hat{\mu}, \hat{\kappa}) \quad : \quad I\!\!R \to & \quad I\!\!R^2 \\ \varepsilon \mapsto & \quad (\hat{\mu}(\varepsilon), \hat{\kappa}(\varepsilon)) \end{aligned} \tag{111}$$

comprise a *distinguished limit*.

Remark: *Here $\partial\hat{y}/\partial\varepsilon(\tau; 0)$ is a generalization of the* directional *or* Gâteaux *derivative*

$$\frac{\partial}{\partial \varepsilon} \left(y(\tau; \alpha\varepsilon, \beta\varepsilon) \right)_{\varepsilon=0} \quad \alpha, \beta = const. \tag{112}$$

In general we know from functional analysis that for some mapping f

$$f \text{ Fréchet-differentiable} \quad \begin{array}{c} \Rightarrow \\ \not\Leftarrow \end{array} \quad f \text{ Gâteaux-differentiable}\,. \tag{113}$$

We conclude that asymptotic expansions based on an approximation in ε with respect to appropriate distinguished limits are more general than a multi-parameter expansions, because–as pointed out above–the latter correspond to classical Taylor-expansions in μ, κ and would require the existence of the Fréchet-derivative $\text{grad}_{\mu,\kappa} y$ at $\mu = \kappa = 0$.

Time scales In analyzing distinguished limits below, we will pay special attention to the timescales on which oscillation, damping, and background forcing will act as ε vanishes. The expressions m/k and $\sqrt{m/c}$ determine the timescales of damping and free oscillation of the system: In our solution from (91), the damping is described by $\exp(-\frac{k}{2m}t)$. Thus, t has to change by $O(m/k)$ to change the argument of the exponential function by $O(1)$. For an undamped, free oscillation, t has to change by $O(1/\omega_0) = O\left(\sqrt{m/c}\right)$

to change the arguments of sine and cosine by $O(1)$. The dimensionless quantities

$$\mathrm{K_D} = \Omega \frac{m}{k} = \mu/\kappa \qquad \text{and} \qquad \mathrm{K_O} = \Omega \sqrt{\frac{m}{c}} = \sqrt{\mu} \tag{114}$$

allow us to compare the internal damping and oscillation time scales of the oscillator with that of the external forcing. Notice that both are functions of our dimensionless mass and damping parameters, (μ, κ).

Three examples shall underline the consequences of picking particular distinguished limits in the oscillator problem:

1st case: $\mu \sim \kappa$ $\qquad$ ($\mu = \varepsilon$ and $\kappa = \varepsilon\hat{\kappa}$ with $\hat{\kappa} = \text{const.}$ as $\varepsilon \to 0$)

Here we have $\mathrm{K_D} = 1/\hat{\kappa} = \text{const.}$ and $\mathrm{K_O} = \sqrt{\varepsilon}$. Thus, as $\varepsilon \to 0$, the timescale of free oscillation becomes much shorter than that of the external forcing, while the damping timescale remains comparable to it. We verify this considering the upper graph in Fig. 8. The frequency of oscillation increases as we reduce ε from 10^{-2} to $5 \cdot 10^{-4}$, whereas the time which the oscillation needs to decay remains the same.

2nd case: $\mu \sim \kappa^2$ $\qquad$ ($\mu = \varepsilon$ and $\kappa = \sqrt{\varepsilon}\hat{\kappa}$ with $\hat{\kappa} = \text{const.}$ as $\varepsilon \to 0$)

Here, since $\mathrm{K_O} = \sqrt{\varepsilon}$ and $\mathrm{K_D} = \sqrt{\varepsilon}/\hat{\kappa}$, the oscillation and damping timescales remain comparable in the limit. As a consequence, the number of oscillations which the system performs before the oscillation is essentially damped away remains nearly independent of ε. We corroborate this by inspecting the centred graph in Fig. 8.

3rd case: $\mu \sim \kappa^3$ $\qquad$ ($\mu = \varepsilon$ and $\kappa = \varepsilon^{\frac{1}{3}}\hat{\kappa}$ with $\hat{\kappa} = \text{const.}$ as $\varepsilon \to 0$)

For this case, Fig. 8 reveals that the system does no longer oscillate at all. For small ε, the damping timescale $\mathrm{K_D} = \varepsilon^{\frac{2}{3}}/\hat{\kappa}$ is always much smaller than the timescale of oscillation $\mathrm{K_O} = \sqrt{\varepsilon}$. Therefore, the inertial motion of the oscillator is already damped before the first overshoot due to the incipient oscillation can take place. After a short initial transient, the mass is essentially "slaved" by the external forcing.

It turns out that the regime $\mu \sim \kappa^2$ is precisely the threshold that separates the regions in μ-κ–space in which, as $\varepsilon \to 0$, either an oscillatory or a purely damped motion prevails. If μ vanishes slower than κ^2, the system will oscillate, otherwise it is strongly damped as summarized in figure 9.

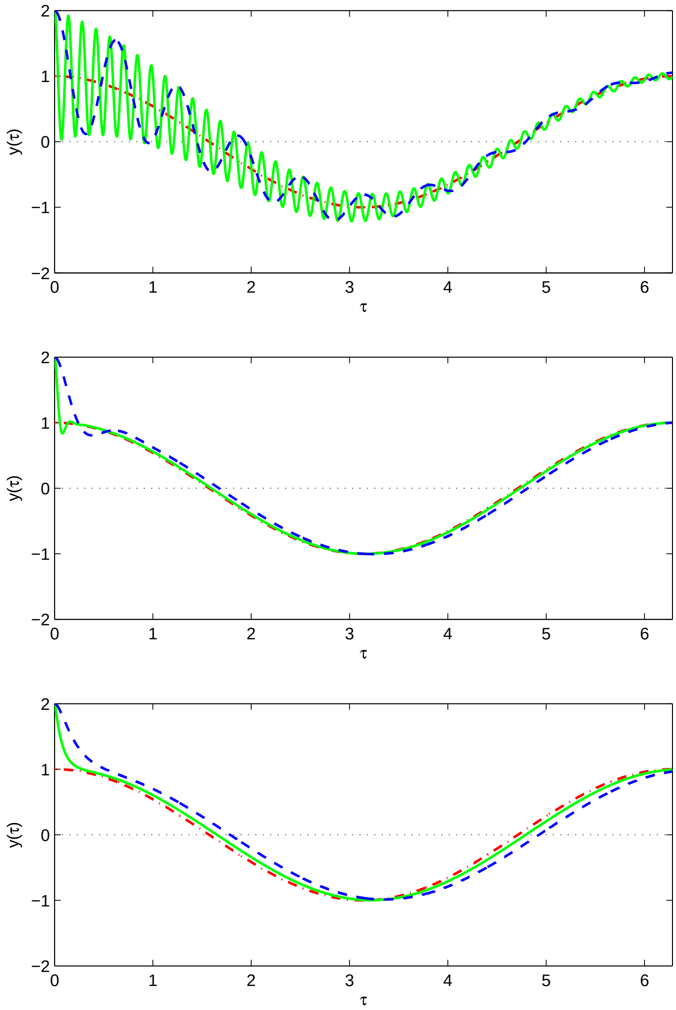

Figure 8. Impact of different choices for the distinguished limit. Exact solutions for $\varepsilon = 0.01$ (blue dashed lines), $\varepsilon = 0.0005$ (green lines) and background oscillation (red dashed-pointed lines). $\mu = \kappa = \varepsilon$ (upper), $\mu = \kappa^2 = \varepsilon$ (centre), and $\mu = \kappa^3 = \varepsilon$ (lower graph).

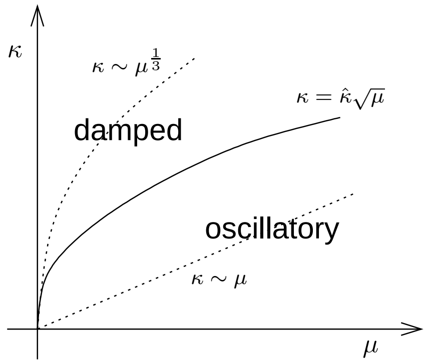

Figure 9. Ratio between mass and damping in the problem and the resulting behaviour of the solution.

If there exist solutions for which y' and y'' remain bounded while $\mu \to 0$ and $\kappa \to 0$, equation (104) reduces to

$$y = \cos\tau\,. \tag{115}$$

in the limit. The solution written in dimensional terms becomes

$$x(t) = \frac{F_0}{c}\;y(\tau) = \frac{F_0}{c}\;\cos(\Omega t)\,. \tag{116}$$

This is in line with "experimental observations" (and with the exact solution, of course). After a certain initial period of adjustment, the oscillator with external force $F_D(t) = F_0\cos(\Omega t)$ performs a periodic oscillation with angular frequency Ω. This is a limiting long-time behavior to be expected from any approximate (asymptotic) solution below, as long as $\kappa > 0$.

3.3 Regular perturbation analysis for small mass and damping

In the sequel we will derive approximate solutions to the dimensionless equation of motion (104) using techniques of asymptotic analysis. To do so, we choose case 1 from the last section, i.e., we let

$$\mu = \varepsilon\,, \quad \kappa = \varepsilon\hat{\kappa} \qquad \text{with} \qquad \hat{\kappa} = \text{const.} \quad \text{as} \quad \varepsilon \to 0\,. \tag{117}$$

Remark: *Choosing a different distinguished limit would yield different asymptotic results!*

With this distinguished limit fixed, the oscillator's governing equation reduces to

$$\varepsilon y'' + \varepsilon\hat{\kappa} y' + y = \cos\tau\,. \tag{118}$$

We consider the solution to explicitly depend only on τ and ε and denote it by $y(\tau;\varepsilon)$, dropping the "hat" notation used earlier in (110).

Slow-time asymptotic expansion For the solution $y = y(\tau;\varepsilon)$ of (118), we choose here an asymptotic ansatz of the form

$$y(\tau;\varepsilon) = y^{(0)}(\tau) + \varepsilon y^{(1)}(\tau) + \varepsilon^2 y^{(2)}(\tau) + o\big(\varepsilon^2\big) \ . \tag{119}$$

Remark: *Generally, a series of the form*

$$\sum_{n=1}^{N} \phi_n(\varepsilon)\, u_n(\underline{\boldsymbol{x}}) \tag{120}$$

with $\phi_{n+1}(\varepsilon) = o(\phi_n(\varepsilon))$ *for* $\varepsilon \to 0$ *is called an asymptotic N-term expansion of the function* u *if*

$$u(\underline{\boldsymbol{x}};\varepsilon) - \sum_{n=1}^{N} \phi_n(\varepsilon)\, u_n(\underline{\boldsymbol{x}}) = o(\phi_N(\varepsilon)) \tag{121}$$

for $\varepsilon \to 0$ *(see for example Kevorkian and Cole (1996) and Schneider (1978)). Notice that the result of the analysis* will depend on the choice of the asymptotic sequence $\{\phi_n(\varepsilon)\}_{n\in\boldsymbol{N}}$.

The chosen ansatz is a Taylor expansion in $\varepsilon = 0$ of the desired solution $y = y(\tau;\varepsilon)$, i.e., we look for the coefficients of $\varepsilon, \varepsilon^2$ etc. in

$$\begin{aligned} y(\tau;\varepsilon) &= \sum_{n=0}^{N} \frac{1}{n!}\, \varepsilon^n \left(\frac{\partial^n y}{\partial \varepsilon^n}\right)(\tau;0) + o\big(\varepsilon^N\big) \\ &= y(\tau;0) + \varepsilon(\partial_\varepsilon y)(\tau;0) + \frac{\varepsilon^2}{2}(\partial_\varepsilon^2 y)(\tau;0) + o\big(\varepsilon^2\big) \end{aligned} \tag{122}$$

Letting $y^{(0)}(\tau) := y(\tau;0)$, $y^{(1)}(\tau) := (\partial_\varepsilon y)(\tau;0)$ etc. naturally leads to (119). We proceed to determine the asymptotic behaviour of the solution y for a fixed τ and an arbitrary but small ε. Depending on the power of ε considered, one speaks of the behaviour of the solution *at a certain order*. Thus, $y^{(0)}(\tau)$ describes the behaviour at leading or zeroeth order, $y^{(1)}(\tau)$ the behaviour at first order, etc.

Inserting the time derivatives from (119) (time means the dimensionless time τ)

$$\begin{aligned} y'(\tau;\varepsilon) &= y^{(0)\prime}(\tau) + \varepsilon y^{(1)\prime}(\tau) + \varepsilon^2 y^{(2)\prime}(\tau) + o\big(\varepsilon^2\big) \\ y''(\tau;\varepsilon) &= y^{(0)\prime\prime}(\tau) + \varepsilon y^{(1)\prime\prime}(\tau) + \varepsilon^2 y^{(2)\prime\prime}(\tau) + o\big(\varepsilon^2\big) \end{aligned} \tag{123}$$

into (118) we find

$$
\begin{aligned}
0 \;=\; & \varepsilon \; \Big(y^{(0)''} + \varepsilon y^{(1)''} + \varepsilon^2 y^{(2)''} + o\big(\varepsilon^2\big)\Big) + \\
& \varepsilon\hat{\kappa}\,\Big(y^{(0)'} + \varepsilon y^{(1)'} + \varepsilon^2 y^{(2)'} + o\big(\varepsilon^2\big)\Big) + \\
& \Big(y^{(0)}(\tau) + \varepsilon y^{(1)} + \varepsilon^2 y^{(2)} + o\big(\varepsilon^2\big)\Big) - \cos\tau
\end{aligned} \tag{124}
$$

or

$$
\Big(y^{(0)} - \cos\tau\Big) + \varepsilon\,\Big(y^{(0)''} + \hat{\kappa} y^{(0)'} + y^{(1)}\Big) + \varepsilon^2\,\Big(y^{(1)''} + \hat{\kappa} y^{(1)'} + y^{(2)}\Big) = o\big(\varepsilon^2\big)\ . \tag{125}
$$

If this equation is to hold for arbitrary (but small) ε, each of the coefficients of ε^i for $(i = 1, 2, \dots)$ (the expressions in brackets) has to be zero individually. Therefore,

$$
\begin{aligned}
y^{(0)} &= \cos\tau \\
y^{(1)} &= -y^{(0)''} - \hat{\kappa} y^{(0)'} &&= \cos\tau + \hat{\kappa}\sin\tau \\
y^{(2)} &= -y^{(1)''} - \hat{\kappa} y^{(1)'} &&= (1 - \hat{\kappa}^2)\cos\tau + 2\hat{\kappa}\sin\tau\,.
\end{aligned} \tag{126}
$$

Our second-order accurate asymptotic solutions thus reads

$$
y(\tau;\varepsilon) = \cos\tau + \varepsilon\,(\cos\tau + \hat{\kappa}\sin\tau) + \varepsilon^2\,\big((1-\hat{\kappa}^2)\cos\tau + 2\hat{\kappa}\sin\tau\big) + o\big(\varepsilon^2\big)\ , \tag{127}
$$

and this is compared with the exact solution in Fig. 10.

Discussion For $\varepsilon \to 0$, the approximate asymptotic solution in (127) reduces to

$$
y(\tau;0) = y^{(0)}(\tau) = \cos\tau\,, \tag{128}
$$

and the asymptotic solution in the limit coincides with (115). Yet, there is a severe catch:

We have no degrees of freedom to meet the initial data!

Instead, the initial displacement and velocity from the leading-order asymptotics are $y^{(0)}(\tau) = 1$ and ${y^{(0)}}'(0) = 0$. Any different values for y_0 and y_0' can nowhere be accounted for. The reason is that with the present ansatz we can only see the solution after any initial transient, which *would be* determined by the initial data, has already decayed.

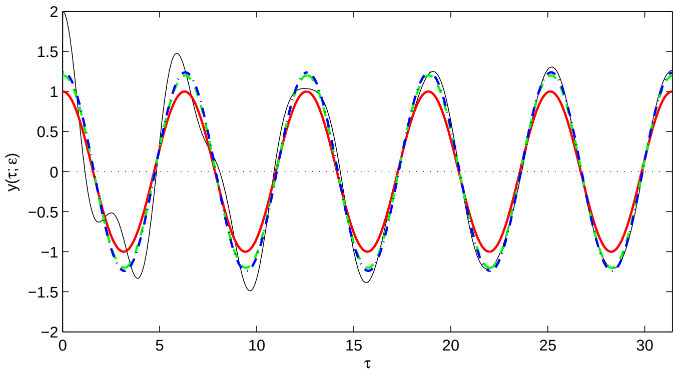

Figure 10. Asymptotic expansion of the solution in τ with $y_0 = 2$, $y_0' = 0$, $\varepsilon = 0.2$ and $\hat{\kappa} = 0.2$. Exact solution: black line, $y(\tau;\varepsilon) = y^{(0)}$: red line, $y(\tau;\varepsilon) = y^{(0)} + \varepsilon y^{(1)}$: green dashed, $y(\tau;\varepsilon) = y^{(0)} + \varepsilon y^{(1)} + \varepsilon^2 y^{(2)}$: blue dash-pointed

This can be verified in Fig. 10, even when the next two higher-order terms are included. Adding the first- and second-order contributions, makes merely a slight difference for the asymptotic approximation. It can do no more than reproduce with better accuracy the background oscillation as more and more expansion terms are included.

Fast-time asymptotic expansion Using the time coordinate $\tau = \Omega t$ we miss the fast oscillatory motions associated with the free oscillation time scale $\sqrt{m/c}$. The rescaled time variable

$$\vartheta = \frac{t}{\sqrt{m/c}} = \frac{\tau}{\sqrt{m\Omega^2/c}} = \frac{\tau}{\sqrt{\varepsilon}} \tag{129}$$

would remedy this problem. We will try out a new expansion scheme in which the unknowns will depend on ϑ instead of on τ. Since each time derivative $d/d\tau$ that appears in the governing equation in that case will produce a factor $1/\sqrt{\varepsilon}$ by the chain rule, we should expand the solution in powers of $\sqrt{\varepsilon}$ instead of in powers of ε. (What happens if we don't but use ϑ as the independent variable while expanding in powers of ε?) Thus we choose the new asymptotic expansion scheme

$$y(\tau;\varepsilon) =: y^{(0)}(\vartheta) + \sqrt{\varepsilon} y^{(1)}(\vartheta) + \varepsilon y^{(2)}(\vartheta) + o(\varepsilon) \; , \tag{130}$$

which is equivalent to

$$y(\tau;\varepsilon) = y^{(0)}\left(\frac{\tau}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}y^{(1)}\left(\frac{\tau}{\sqrt{\varepsilon}}\right) + \varepsilon y^{(2)}\left(\frac{\tau}{\sqrt{\varepsilon}}\right) + o(\varepsilon)\ . \tag{131}$$

We will need to be aware of the latter form of writing the expansion when we insert into the governing equation in (118), which is written in terms of τ. In preparation, we compute the τ-derivatives of (131) for fixed ε,

$$\begin{aligned}\left.\frac{\partial y}{\partial \tau}\right|_{\varepsilon}(\vartheta;\varepsilon) &= \frac{dy^{(0)}}{d\vartheta}\frac{d\vartheta}{d\tau} + \sqrt{\varepsilon}\frac{dy^{(1)}}{d\vartheta}\frac{d\vartheta}{d\tau} + \varepsilon\frac{dy^{(2)}}{d\vartheta}\frac{d\vartheta}{d\tau} + o\left(\varepsilon\frac{d\vartheta}{d\tau}\right) \\ &= \frac{1}{\sqrt{\varepsilon}}\frac{dy^{(0)}}{d\vartheta} + \frac{dy^{(1)}}{d\vartheta} + \sqrt{\varepsilon}\frac{dy^{(2)}}{d\vartheta} + o\big(\sqrt{\varepsilon}\big) \qquad (132) \\ \left.\frac{\partial^2 y}{\partial \tau^2}\right|_{\varepsilon}(\vartheta;\varepsilon) &= \frac{1}{\varepsilon}\frac{d^2y^{(0)}}{d\vartheta^2} + \frac{1}{\sqrt{\varepsilon}}\frac{d^2y^{(1)}}{d\vartheta^2} + \frac{d^2y^{(2)}}{d\vartheta^2} + o(1)\ . \qquad (133)\end{aligned}$$

The notation $\left.\frac{\partial y}{\partial \tau}\right|_{\varepsilon}$ shall underline that, for any solution of the oscillator problem, ε is a fixed parameter and thus held constant when differentiating. If, instead, we were to consider ε as varying in time, then the time scales, and thus the mass, spring stiffness, and the damping coefficient, would change in time, too.

We express the forcing term in (118) in terms of ϑ, and then Taylor-expand w.r.t. $\sqrt{\varepsilon}$,

$$\cos\tau = \cos(\sqrt{\varepsilon}\vartheta) = 1 - \frac{\varepsilon}{2}\vartheta^2 + O\big(\varepsilon^2\big)\ . \tag{134}$$

Inserting the appropriate derivatives and expansions into (118) we find

$$\begin{aligned}&\frac{d^2y^{(0)}}{d\vartheta^2} + \sqrt{\varepsilon}\frac{d^2y^{(1)}}{d\vartheta^2} + \varepsilon\frac{d^2y^{(2)}}{d\vartheta^2} \\ &\qquad + \hat{\kappa}\sqrt{\varepsilon}\frac{dy^{(0)}}{d\vartheta} + \hat{\kappa}\varepsilon\frac{dy^{(1)}}{d\vartheta} \\ &\qquad + y^{(0)} + \sqrt{\varepsilon}y^{(1)} + \varepsilon y^{(2)} \quad = \quad 1 - \varepsilon\frac{\vartheta^2}{2} + o(\varepsilon)\ .\end{aligned} \tag{135}$$

This yields, after collection of like powers of ε, the following hierarchy of perturbation equations,

$$
\begin{aligned}
O(1): \qquad & \frac{d^2y^{(0)}}{d\vartheta^2}+y^{(0)} &=& \ 1 \\
O\big(\sqrt{\varepsilon}\big): \qquad & \frac{d^2y^{(1)}}{d\vartheta^2}+y^{(1)} &=& -\hat{\kappa}\frac{dy^{(0)}}{d\vartheta} \\
O(\varepsilon): \qquad & \frac{d^2y^{(2)}}{d\vartheta^2}+y^{(2)} &=& -\hat{\kappa}\frac{dy^{(1)}}{d\vartheta}-\frac{\vartheta^2}{2}\,.
\end{aligned}
\tag{136}
$$

At leading order (i.e., at $O(1)$) we find the equation for an undamped oscillator with time-independent forcing. The solution is

$$
y^{(0)} = a_0\sin\vartheta + b_0\cos\vartheta + 1\,. \tag{137}
$$

Here a_0 and b_0 are constants that depend on the initial conditions.

Next we solve the first-order equation (at $O(\sqrt{\varepsilon})$) in the expansion, which becomes

$$
\frac{d^2y^{(1)}}{d\vartheta^2}+y^{(1)} = -\hat{\kappa}\left(a_0\cos\vartheta - b_0\sin\vartheta\right) \tag{138}
$$

The solution is a superposition of the homogeneous solution, $y_h^{(1)} = a_1\sin\vartheta + b_1\cos\vartheta$, and a particular solution that takes care of the right-hand side. The coefficients a_1 and b_1 will have to be computed from the initial conditions as before. To derive a particular solution, we use the technique of variation of coefficients, i.e.,

$$
y_p^{(1)} = f(\vartheta)\sin\vartheta + g(\vartheta)\cos\vartheta\,, \tag{139}
$$

where f and g remain to be determined. Inserting this ansatz into the differential equation (138) yields

$$
(\ddot{f}-2\dot{g})\sin\vartheta + (\ddot{g}+2\dot{f})\cos\vartheta = -\hat{\kappa}\left(a_0\cos\vartheta - b_0\sin\vartheta\right)\,. \tag{140}
$$

Comparing coefficients, we find the constraints

$$
\ddot{f}-2\dot{g} = \hat{\kappa}b_0 \quad \text{and} \quad \ddot{g}+2\dot{f} = -\hat{\kappa}a_0\,. \tag{141}
$$

The desired solutions are polynomials in ϑ of degree less than two, and we let $f(\vartheta) = A_f\vartheta + B_f$ and $g(\vartheta) = A_g\vartheta + B_g$. Without loss of generality, we may also assume $B_f \equiv 0$ and $B_g \equiv 0$, as these terms can be covered by the homogenous part of the solution. Solving (141), yields

$$
y_p^{(1)} = -\vartheta\frac{\hat{\kappa}}{2}\left(a_0\sin\vartheta + b_0\cos\vartheta\right) \tag{142}
$$

so that

$$y^{(1)} = y_h^{(1)} + y_p^{(1)} = a_1 \sin\vartheta + b_1 \cos\vartheta - \vartheta \frac{\hat{\kappa}}{2} (a_0 \sin\vartheta + b_0 \cos\vartheta) \ . \quad (143)$$

In the same way one may compute higher-order solutions.

It turns out that this ansatz based on the short-time variable is still not satisfactory. As seen in figure 11, although the solution improves with increasing order, this is true only for the early stages of the evolution. With increasing time the solution deteriorates dramatically, and this gets worse the higher the approximation order!

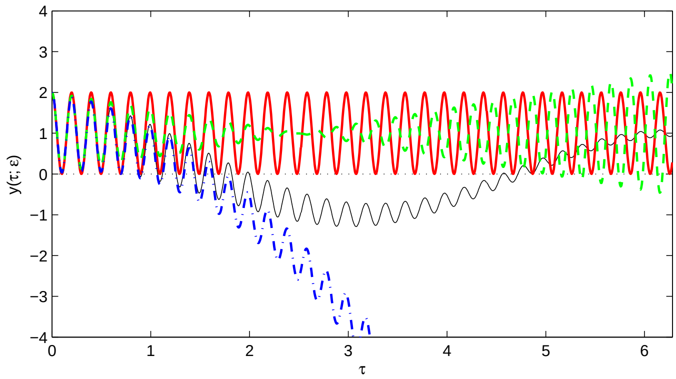

Figure 11. Asymptotic expansion of the solution in ϑ with $y_0 = 2$, $y_0' = 0$, $\varepsilon = 0.001$ and $\hat{\kappa} = 0.8$. $y(\tau;\varepsilon)$ (exact solution): black line; $y^{(0)}$: red line; $y^{(0)} + \sqrt{\varepsilon} y^{(1)}$: green dashed line; $y^{(0)} + \sqrt{\varepsilon} y^{(1)} + \varepsilon y^{(2)}$: blue dash-pointed line

Remark: *For the present short-time expansion we could have chosen an ansatz similar to (119) (i.e. with $\phi_n(\varepsilon) = \varepsilon^n$). It is easy to verifiy that this yields only the trivial solution $y(\tau;\varepsilon) \equiv 0$. This shows that the choice of the asymptotic sequence $\{\phi_n(\varepsilon)\}_{n\in N}$ is crucial when looking for an asymptotic solution.*

Remark: *As we have seen, the differential equation for $y^{(1)}$ has a resonant, amplifying solution. After some time, the term $\sqrt{\varepsilon} y^{(1)}$ becomes comparable to the previous one, $y^{(0)}$, and the whole idea of building a series with smaller and smaller corrections as the order of approximation increases fails badly. In fact, the present asymptotic approximate solution is valid only for $\vartheta = O(1)$ or, equivalently, $\tau = O(\sqrt{\varepsilon})$, i.e., for asymptotically short times on the time scale of the background forcing.*

3.4 Multiple scales analysis

The analyses based on single timescale representations, with $\tau = O(\sqrt{\varepsilon})$ and $\tau = O(1)$, respectively, were at best partially successful. They did allow us to cover the early, respectively, late development of the solution, but were definitely not valid uniformly in time. Here we consider an asymptotic expansion scheme that accounts for both of these timescales in a single sweep,

$$
\begin{aligned}
y(\tau;\varepsilon) &= y^{(0)}\left(\frac{\tau}{\sqrt{\varepsilon}},\tau\right) + \sqrt{\varepsilon}y^{(1)}\left(\frac{\tau}{\sqrt{\varepsilon}},\tau\right) + \varepsilon y^{(2)}\left(\frac{\tau}{\sqrt{\varepsilon}},\tau\right) + o(\varepsilon) \\
&= y^{(0)}(\vartheta,\tau) + \sqrt{\varepsilon}y^{(1)}(\vartheta,\tau) + \varepsilon y^{(2)}(\vartheta,\tau) + o(\varepsilon)\ .
\end{aligned}
\tag{144}
$$

Including both the time variables considered in our previous expansions, we expect this scheme to allow us to fulfill the initial conditions, $y(0;\varepsilon) = y_0$ and $y'(0;\varepsilon) = y_0'$, but to also capture the long-time behavior of the solution without resonant growth.

Remark: *The key challenge, of course, will be to find the dependencies of the expansion functions, $y^{(i)}(\vartheta,\tau)$, on* two time variables, *even though our orginal govering equation involves merely a single independent variable, τ. In fact, we will have to determine the variation of the expansion functions within the entire (ϑ,τ)–plane, because we want to obtain solutions that are valid for small, but otherwise arbitrary values of ε. Varying ε, the ratio between ϑ and τ changes, as indicated in Fig. 12. Thus, for any fixed value of τ we may be interested in $y^{(i)}(\vartheta,\tau)$ within a range of values of ϑ, and vice versa, the range being determined by the range of realistic values of ε.*

To proceed, we need to work out the time derivatives needed in (144) taking into account that the $y^{(i)}(\vartheta,\tau)$ are functions of two independent variables. Using the chain rule, we find

$$
\begin{aligned}
\left.\frac{\partial y}{\partial \tau}\right|_{\varepsilon} &= y_\vartheta^{(0)}\vartheta_\tau + y_\tau^{(0)} + \sqrt{\varepsilon}y_\vartheta^{(1)}\vartheta_\tau + \sqrt{\varepsilon}y_\tau^{(1)} + \varepsilon y_\vartheta^{(2)}\vartheta_\tau + \varepsilon y_\tau^{(2)} + o(\varepsilon\vartheta_\tau) \\
&= \frac{1}{\sqrt{\varepsilon}}y_\vartheta^{(0)} + \left(y_\tau^{(0)} + y_\vartheta^{(1)}\right) + \sqrt{\varepsilon}\left(y_\tau^{(1)} + y_\vartheta^{(2)}\right) + o\big(\sqrt{\varepsilon}\big) \\
\left.\frac{\partial^2 y}{\partial \tau^2}\right|_{\varepsilon} &= \frac{1}{\varepsilon}y_{\vartheta\vartheta}^{(0)} + \frac{1}{\sqrt{\varepsilon}}\left(2y_{\vartheta\tau}^{(0)} + y_{\vartheta\vartheta}^{(1)}\right) + \left(y_{\tau\tau}^{(0)} + 2y_{\vartheta\tau}^{(1)} + y_{\vartheta\vartheta}^{(2)}\right) + o(1)
\end{aligned}
\tag{145}
$$

Inserting into (118) one has

$$
\left(y_{\vartheta\vartheta}^{(0)} + y^{(0)} - \cos\tau\right) + \sqrt{\varepsilon}\left(2y_{\vartheta\tau}^{(0)} + y_{\vartheta\vartheta}^{(1)} + \hat{\kappa}y_\vartheta^{(0)} + y^{(1)}\right) + o\big(\sqrt{\varepsilon}\big) = 0\,. \tag{146}
$$

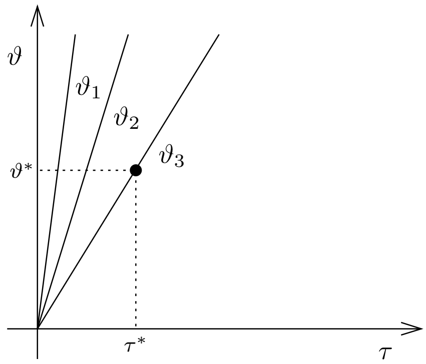

Figure 12. Relation between the time coordinates τ and ϑ for chosen ε, where $\vartheta_i = \tau/\sqrt{\varepsilon_i}$ ($\varepsilon_1 < \varepsilon_2 < \varepsilon_3$). For every τ^* there is exactly one ϑ^*.

For this equation to hold for arbitrary $\varepsilon \ll 1$ it is sufficient that the different terms in brackets vanish. Considering that, through the dependence of the $y^{(i)}$ on τ and $\vartheta = \tau/\sqrt{\varepsilon}$, these brackets implicitly do depend on ε, it is not immediately clear that the vanishing of the brackets is also necessary. But, if we *can* make all the coefficients disappear for arbitrary (ϑ, τ), then our equation is in any case satisfied. Let's give it a try!

For the different orders in $\sqrt{\varepsilon}$ this leads to

$$\begin{aligned} O(1): \quad & y^{(0)} + y^{(0)}_{\vartheta\vartheta} &=& \cos\tau \\ O\big(\sqrt{\varepsilon}\big): \quad & y^{(1)} + y^{(1)}_{\vartheta\vartheta} &=& -2y^{(0)}_{\vartheta\tau} - \hat{\kappa} y^{(0)}_{\vartheta} \,. \end{aligned} \tag{147}$$

It is of crucial importance that the first of these equations is an ODE for the ϑ–depencence of $y^{(0)}$, and that any τ–derivative of this leading-order function appears only at the next order. Thus, if the first equation is solved for the ϑ–dependence, τ remains as a parameter the influence of which will remain to be determined.

Following the same procedures as earlier in this section, we find the general solution in terms of ϑ for $(147)_1$ to be

$$y^{(0)}(\vartheta, \tau) = A(\tau)\cos\vartheta + B(\tau)\sin\vartheta + \cos\tau \,. \tag{148}$$

This is the superposition of a slow background motion and an oscillation with increasing frequency as $\varepsilon \to 0$. In addition, when $A(\tau) \equiv B(\tau) \equiv 0$ we just retrieve the leading-order solution $y^{(0)}$ from the single-scale analysis. Thus, the present result includes the previous solutions, and has extended the description of the fast oscillations to arbitrary τ. (Remember that the fast-time analysis restricted us to considering $\tau = O(\sqrt{\varepsilon})$ only!)

Considering $(147)_2$ and inserting the partial derivatives of (148) we have,

$$y^{(1)}_{\vartheta\vartheta} + y^{(1)} = (2A' + \hat{\kappa}A)(\tau)\sin\vartheta - (2B' + \hat{\kappa}B)(\tau)\cos\vartheta\,. \tag{149}$$

This equation has a resonant solution (compare the remarks in the section 3.1):

$$y^{(1)} = y^{(1)}_h - \vartheta\left(\left(A' + \frac{\hat{\kappa}}{2}A\right)(\tau)\cos\vartheta + \left(B' + \frac{\hat{\kappa}}{2}B\right)(\tau)\sin\vartheta\right) =: y^{(1)}_h + \vartheta\tilde{y}^{(1)}_p\,. \tag{150}$$

If $\tilde{y}^{(1)}_p \neq 0$, the term $\sqrt{\varepsilon}y^{(1)}(\vartheta,\tau) = \sqrt{\varepsilon}y^{(1)}_h + \tau\tilde{y}^{(1)}_p$ in the asymptotic ansatz (144) is no longer negligible compared to the term $y^{(0)}(\vartheta,\tau)$ if $\tau = O(1)$. To exclude resonant solutions like this and thus to make sure that successive terms in our approximation yield systematically smaller and smaller corrections even if $\tau = \sqrt{\varepsilon}\vartheta = O(1)$, we demand that, for fixed τ

$$\sqrt{\varepsilon}y^{(i)}\left(\frac{\tau}{\sqrt{\varepsilon}},\tau\right) \to 0 \qquad \text{as} \qquad \varepsilon \to 0\,. \tag{151}$$

Remark: *The condition (151) is known as* sub-linear growth condition. *In general we impose the condition*

$$\frac{\phi_n(\varepsilon)}{\phi_{n-1}(\varepsilon)}\,u^{(n)}\left(\frac{x_1}{\psi(\varepsilon)}, x_1, x_2, \ldots, x_m\right) \to 0 \quad \textit{as} \quad \varepsilon \to 0,\ x_i, i = 1,\ldots,m \textit{ fixed} \tag{152}$$

on the coefficients $u^{(n)}(\eta, x_1, x_2, \ldots, x_m)$ of an asymptotic expansion of (121) (with $\eta = x_1/\psi(\varepsilon) \to \infty$ for $\varepsilon \to 0$ at fixed x_1). The name "sub-linear growth condition" is motivated by the above condition for $\eta = x_1/\varepsilon$ and $\phi_n(\varepsilon) = \varepsilon^n$. In this case,

$$\lim_{\eta\to\infty}\frac{u^{(n)}(\eta, x_1, x_2, \ldots, x_m)}{\eta} = 0 \quad , \ x_i, i = 1,\ldots,m \textit{ fixed.} \tag{153}$$

and this means the $u^{(n)}$ grow slower (sub-linear) than η for $\eta \to \infty$, or for $\varepsilon \to 0$ at fixed x_1.

Demanding that $y^{(1)}$ contain no "resonant terms" of the type $\vartheta\tilde{y}^{(1)}_p$ leads to

$$A'(\tau) + \frac{\hat{\kappa}}{2}\,A(\tau) = 0 \quad \text{and} \quad B'(\tau) + \frac{\hat{\kappa}}{2}\,B(\tau) = 0\,. \tag{154}$$

These equations are solved by

$$A(\tau) = A(0)\exp\left(-\frac{\hat{\kappa}}{2}\tau\right) \quad \text{and} \quad B(\tau) = B(0)\exp\left(-\frac{\hat{\kappa}}{2}\tau\right)\,. \tag{155}$$

At this point it is reasonable to continue with the expansion of the initial conditions $y(0) = y_0$, $y'(0) = y'_0$. This will provide us with some additional information that will be needed to determine $A(0)$ and $B(0)$. The expansion of the initial conditions is done by inserting $\tau = 0$ and $\vartheta = 0$ into (144). This results in

$$\begin{aligned} y_0 &= y^{(0)}(0,0) + \sqrt{\varepsilon} y^{(1)}(0,0) + \varepsilon y^{(2)}(0,0) + o(\varepsilon) && (156) \\ y'_0 &= \left(\frac{1}{\sqrt{\varepsilon}} y^{(0)}_\vartheta + (y^{(0)}_\tau + y^{(1)}_\vartheta) + \sqrt{\varepsilon}(y^{(1)}_\tau + y^{(2)}_\vartheta) \right)(0,0) + o\big(\sqrt{\varepsilon}\big) && (157) \end{aligned}$$

Assuming initial conditions of type $y_0 = O(1)$ and $y'_0 = O(1)$ and an allowing for small but otherwise arbitrary ε, we conclude that

$$\begin{aligned} y^{(0)}(0,0) &= y_0 \\ y^{(i)}(0,0) &= 0 \qquad (i = 1, 2, ...) \\ y^{(0)}_\vartheta(0,0) &= 0 \qquad\qquad\qquad . \\ (y^{(0)}_\tau + y^{(1)}_\vartheta)(0,0) &= y'_0 \\ (y^{(i)}_\tau + y^{(i+1)}_\vartheta)(0,0) &= 0 \qquad (i = 1, 2, ...) \end{aligned} \tag{158}$$

Inserting these results into the leading-order solution from (148), and using

$$y^{(0)}_\vartheta = -A(\tau)\sin\vartheta + B(\tau)\cos\vartheta\,, \tag{159}$$

we derive initial data for A and B,

$$\begin{aligned} y^{(0)}(0,0) &= A(0) + 1 && \Rightarrow && A(0) &= y_0 - 1 \\ y^{(0)}_\vartheta(0,0) &= B(0) && \Rightarrow && B(0) &= 0\,. \end{aligned} \tag{160}$$

Thus, with (155) and (148) we obtain the leading-order multiple-scales solution,

$$y^{(0)}(\vartheta, \tau) = (y_0 - 1)\exp\left(-\frac{\hat{\kappa}}{2}\tau\right)\cos\vartheta + \cos\tau\,. \tag{161}$$

Remark: *We have used the initial conditions exclusively at $\vartheta = \tau = 0$, but not for $\vartheta = 0$ and arbitrary τ. The latter would be incorrect, because in the ϑ, τ–plane, for fixed system parameters μ, κ, i.e., for fixed ε, the solution evolves along a straight line through the origin in the ϑ, τ–plane as shown*

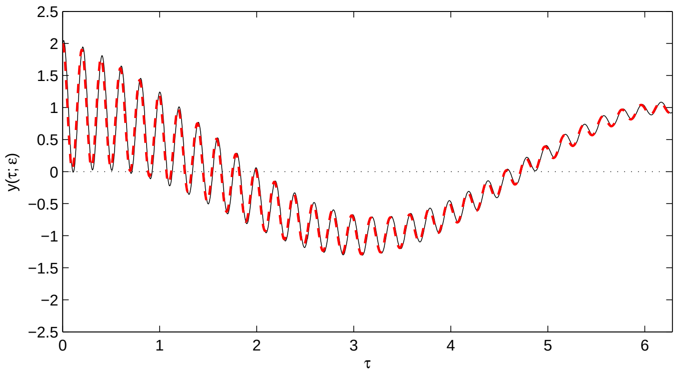

Figure 13. Asymptotic multiscale expansion of the solution in τ and ϑ with $\varepsilon = 0.001$, $\hat{\kappa} = 0.8$ and initial conditions $y_0 = 2$ and $y_0' = 10$. Exact solution: black line, $y^{(0)}(\tau; \varepsilon)$: red dashed line

in Fig. 12. Obviously, at physical time $t = 0$, we access the asymptotic solutions $y^{(i)}(\vartheta, \tau)$ at $\vartheta = \tau = 0$.

Figure 13 displays the leading-order and exact solutions for $y_0 = 2; y_0' = 0$, i.e., for a setting in which $y_0 = O_S(1)$ and $y_0' = O_S(1)$. The agreement is convincing, even after many of the fast oscillation cycles, and for times $\tau = O(1)$.

Remark: *We observe that the leading-order multiple scales solution from Fig. 13 nowhere makes explicit use of the velocity initial datum, $y(0) = y_0'$. This raises several questions:*

1. *Why is that?*
2. *How will we account for this second initial condition in the present expansion scheme?*
3. *What kind of initial data would be required to make the initial velocity show up in the leading-order solution?*

3.5 Some comments and a question

Multiple-Scales Asymptotics is

- a direct, constructive approach to approximate model reduction,
- first of all a *formal* approach; rigorous justification of any specific asymptotic expansion must generally be handled on a case-by-case basis,
- a means to systematically describe scale interactions,

Multiple-Scales Asymptotics is *not*

- a techique for deriving "the" general solution to any given pde- or related problem,
- a cure to all multiscale problems; problems involving a continuous range of interacting scales as observed, e.g., in turbulent flows, cannot be handled by multiple scales asymptotics (at least not in a straightforward fashion.

Having read this section, how would you describe a "scale", and how would you mathematically describe scale-separation in a class of problems?

4 Universal parameters, distinguished limits, and non-dimensionalization

One important aim of theoretical meteorology is the development of simplified model equations that describe the large variety of scale-dependent phenomena observed in atmospheric flows. Here we summarize the basic scaling arguments that justify a unified approach to the derivation of such models based on multiple scales asymptotic techniques. The approach was proposed by Klein (2004) and has led to or been an important part of several recent new developments, Majda and Klein (2003), Klein et al. (2004), Majda and Biello (2004), Mikusky et al. (2005), Biello and Majda (2006), Klein and Majda (2006), and references therein. Remarkably, Keller and Ting (1951) already anticipated the foundations of this approach in an internal report of the Institute for Mathematics and Mechanics of New York University.

To elucidate our main points, we restrict the discussion here to inviscid compressible flows on a rotating sphere. Diabatic effects, such as radiation, water phase transitions, or turbulent transport will be represented as lumped terms in the governing equations to be specified later. Extensions of the framework to include moist processes have been developed by Klein and Majda (2006).

4.1 Universal parameters and distinguished limits

Table 1 displays several physical variables that are characteristic of atmospheric flows, and that are valid independently of the typical length and time scales of any specific flow phenomenon: The mean sea level pressure p_{ref} is set by the requirement that it balance the weight of a vertical column of air. Thus, it is directly given by the total mass of the atmosphere which, to a very good approximation, is evenly distributed over the sphere. A reference temperature T_{ref} is set roughly by the global radiation balance which, even

Earth's radius	a	$=$	$6\cdot 10^6$	m
Earth's rotation rate	Ω	$\sim$	10^{-4}	s^{-1}
Acceleration of gravity	g	$=$	9.81	ms^{-2}
Sea level pressure	p_{ref}	$=$	10^5	Pa
Water freezing temperature	T_{ref}	$\sim$	273	K
Equator–pole temperature difference	$\Delta T\vert_{\mathrm{eq}}^{\mathrm{p}}$	$\sim$	50	K
Dry air gas constant	R	$=$	287	$\mathrm{ms}^{-2}/\mathrm{K}$
Dry air isentropic exponent	γ	$=$	1.4	

Table 1. Universal characteristics of atmospheric motions.

without greenhouse gases, would render the mean near-surface air temperature near 250K. The actual value in Table 1 is the freezing temperature of water under standard conditions, i.e., $T_{\mathrm{ref}} \sim 273\mathrm{K}$, which is about midway between the observed maximal and minimal near-surface air temperatures. The equator-to-pole air temperature difference near the surface, $\Delta T|_{\mathrm{eq}}^{\mathrm{p}}$, is a consequence of the latitudinal variation of the sun's irradiation. The dry air gas constant, R, and isentropic exponent, γ, as thermodynamic properties are also universally characteristic for atmospheric flows, because their variations due to admixtures of water vapor, trace gases, and the like are no larger than a few percent in general.

Based on these eight basic reference quantities, four independent dimensionless combinations can be composed. To define combinations with intuitive interpretations, we introduce as auxiliary quantities the pressure scale height, h_{sc}, the characteristic speed c_{ref} of barotropic[1] gravity waves, and a reference density, ϱ_{ref}, via

$$\begin{array}{rclcrl} h_{\mathrm{sc}} & = & p_{\mathrm{ref}}/(g\,\rho_{\mathrm{ref}}) & \sim & 10 & \mathrm{km} \\ c_{\mathrm{ref}} & = & \sqrt{g h_{\mathrm{sc}}} & \sim & 300 & \mathrm{ms}^{-1} \\ \rho_{\mathrm{ref}} & = & p_{\mathrm{ref}}/(R\,T_{\mathrm{ref}}) & \sim & 1 & \mathrm{kgm}^{-3}. \end{array} \tag{162}$$

[1] Atmospheric flow modes are called "barotropic" if their structure is homogeneous in the vertical direction.

Then we let

$$
\begin{aligned}
\Pi_1 &= \frac{h_{\rm sc}}{a} &\sim\ & 1.67 \cdot 10^{-3}\,, \\
\Pi_2 &= \frac{\Delta T|^{\rm p}_{\rm eq}}{T_{\rm ref}} &\sim\ & 0.18\,, \\
\Pi_3 &= \frac{c_{\rm ref}}{\Omega a} &\sim\ & 0.5\,.
\end{aligned}
\tag{163}
$$

The interpretations of Π_1 and Π_2 should be obvious, while the parameter Π_3 compares a typical barotropic gravity wave speed with the tangential speed of points on the equator as induced by Earth's rotation.

Remark: *The sound speed, $\sqrt{\gamma p_{\rm ref}/\rho_{\rm ref}}$ is comparable to the barotropic wave speed, $c_{\rm ref} = \sqrt{g h_{\rm sc}}$ according to (162).*

The parameter Π_1 is definitely quite small. Π_2 is not extremely small, yet, many successful developments in theoretical meteorology have relied on scale analysis (asymptotics) in terms of, e.g., Rossby numbers or internal wave Froude numbers with values in a similar range. Finally, for Π_3 one may argue that, even though it is less than unity, one may be hard pressed to consider it "asymptotically small". Deviating somewhat from our earlier work cited above, we will consider $\Pi_3 \ll 1$ in the present notes.

There is little hope for success with asymptotic expansions that would allow Π_1, Π_2, and Π_3 to vary *independently* in a limit process: even for the simple example of a linear oscillator such expansions in two independent parameters were found in section 3 not to exist! Thus, for the present parameters we need to adopt a distinguished limit, and we investigate the following scaling relationships below,

$$
\Pi_1 \sim \varepsilon^3\,, \qquad \Pi_2 \sim \varepsilon\,, \qquad \Pi_3 \sim \sqrt{\varepsilon}\,, \qquad \text{as} \qquad \varepsilon \to 0\,. \tag{164}
$$

These limits are compatible with the estimates in (163) for actual values of $\varepsilon \in [1/7 \ldots 1/8]$. We will adopt ε as the reference expansion parameter for asymptotic analyses below, and any additional small or large non-dimensional parameter that may be associated with singular perturbations in the governing equations is subsequently tied to ε through suitable further distinguished limits.

Remark: *Before we proceed to do so, we notice that Keller and Ting (1951) already proposed to use the acceleration ratio, $\varepsilon \sim (a\Omega^2/g)^{1/3} = (\Pi_1/\Pi_3^2)^{1/3}$, as a basic expansion parameter for meteorological modelling. When $\Pi_3 = O(1)$, this is equivalent to (164) above.*

Remark: *In contrast to (164), in my earlier work I have usually let* $\Pi_3 = O(1)$. *The present, slightly modified limit I introduce because it appears to unify current developments of planetary balanced models by my colleague Stamen Dolaptchiev with Pedlosky's derivations of the quasi-geostrophic theory in (Pedlosky (1987)).*

4.2 Nondimensionalization and general multiple scales ansatz

With p_{ref} and T_{ref}, and through the ideal gas equation of state, $\rho = p/RT$, Table 1 immediately suggests reference values for the nondimensionalization of pressure, temperature, and density. But what about velocity, length, and time?

Hydrostatic–geostrophic velocity scale Most theories for atmospheric flows rely on the assumption that typical flow speeds are small compared with the speed of barotropic gravity waves c_{ref} which, except for a factor of $\sqrt{\gamma}$, matches the speed of sound. Here and in the rest of this section we make this assumption explicit by introducing a reference speed

$$u_{\text{ref}} = \frac{gh_{\text{sc}}}{\Omega a} \frac{\Delta T|_{\text{eq}}^{\text{p}}}{T_{\text{ref}}} = c_{\text{ref}} \Pi_2 \Pi_3 \sim \varepsilon^{\frac{3}{2}} c_{\text{ref}} \tag{165}$$

for the nondimensionalization of the flow velocity. (What is a typical value for u_{ref}?) The reader may verify that the above expression has in fact the dimension of a velocity, but what is the motivation for this choice? We will resolve this question later when we (re-)derive the quasi-geostrophic theory (QG). (See also the **Remark** at the end of this section!)

The choice of a velocity scale in (165) allows us to express two classical non-dimensional parameters of theoretical meteorology (and fluid dynamics), the (barotropic) Froude and Mach numbers, in terms of our small parameter,

$$\overline{\text{Fr}} = \frac{\text{M}}{\sqrt{\gamma}} = \frac{u_{\text{ref}}}{c_{\text{ref}}} \sim \varepsilon^{\frac{3}{2}} \qquad \text{as} \qquad \varepsilon \to 0\,. \tag{166}$$

Scaling of space and time As we are interested in multiple scales problems, and will consistently take into account different characteristic lengths in our analyses, the specific choice of reference length and time scales for non-dimensionalization should not make much of a difference. We opt here to use, h_{sc}, i.e., the smallest length scale that suggests itself just from the fundamental parameters in Table 1 via equation (162), to non-dimensionalize all lengths. The associated advection time serves as a reference time. Thus,

$$\ell_{\text{ref}} = h_{\text{sc}} \qquad \text{and} \qquad t_{\text{ref}} = \frac{h_{\text{sc}}}{u_{\text{ref}}} \,. \tag{167}$$

Questions

- What would be the scaling in terms of ε of the Froude number based on a typical internal gravity wave speed*?
- What would be the scaling of the Rossby number based on our reference length, h_{sc}*?
- By what power of $1/\varepsilon$ is the internal Rossby radius larger than h_{sc}*?
- By what power of $1/\varepsilon$ is the Obhoukhov or *external* Rossby radius larger than h_{sc}?
- By what power of $1/\varepsilon$ is the Obhoukhov or *external* Rossby radius larger than the internal Rossby radius? — Compare your result with a related remark in Pedlosky (1987).
- Does the Oboukhov or *external* Rossby radius come out larger, comparable or smaller than the Earth radius, a, which is representative of the planetary scale?

* It is safe to assume that the variation of potential temperature across the troposphere is comparable to the equator-pole temperature difference, $\Delta T|_{\text{eq}}^{\text{p}}$.

4.3 Scaled governing equations and general multiple scales expansion scheme

With these scalings, the nondimensional governing equations in the rotating earth system may be written as

$$\begin{aligned}
\boldsymbol{v}_{\shortparallel,t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} \boldsymbol{v}_{\shortparallel} + w \boldsymbol{v}_{\shortparallel,z} + \varepsilon\, 2(\boldsymbol{\Omega} \times \boldsymbol{v})_{\shortparallel} + \frac{1}{\varepsilon^3} \frac{1}{\rho} \nabla_{\shortparallel} p &= \boldsymbol{Q}_{\boldsymbol{v}_{\shortparallel}} \,, \\
w_t + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} w + w w_z + \varepsilon\, 2(\boldsymbol{\Omega} \times \boldsymbol{v})_{\perp} + \frac{1}{\varepsilon^3} \frac{1}{\rho} p_z &= Q_w - \frac{1}{\varepsilon^3} \,, \\
p_t + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} p + w p_z + \gamma p \nabla \cdot \boldsymbol{v} &= Q_p \,, \\
\Theta_t + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} \Theta + w \Theta_z &= Q_\Theta \,,
\end{aligned} \tag{168}$$

where

$$\Theta = \frac{p^{1/\gamma}}{\rho} \tag{169}$$

is the dimensionless potential temperature, $\boldsymbol{k}$ is the local vertical unit vector indicating the direction of the acceleration of gravity. The terms $\boldsymbol{Q}_{\boldsymbol{v}}, Q_p$, and Q_Θ represent additional effects which in a concrete application may stem from turbulence closures or similar models for the net influence of non-resolved scales.

Klein (2004) suggested to consider the small parameter ε as introduced above as *the* general singular asymptotic expansion parameter for theoretical developments in meteorology (although suggesting a slightly different distinguished limit for Π_3 from (163)). To this end, the solution vector $\mathcal{U} = (p, \Theta, \boldsymbol{v})$ is expanded in powers of ε (or some fractional power thereof), and all expansion functions would depend on a series of space-time coordinates that are scaled again by powers of ε. The most straightforward version of such a scheme reads

$$\mathcal{U}(u, z, t, ; \varepsilon) = \sum_i \varepsilon^i \mathcal{U}^{(i)}(\ldots, \frac{t}{\varepsilon}, t, \varepsilon t, \ldots, \frac{\boldsymbol{x}}{\varepsilon}, \boldsymbol{x}, \varepsilon\boldsymbol{x}, \ldots \frac{z}{\varepsilon}, z, \ldots)\,. \tag{170}$$

In practical applications it might be necessary to work with fractional powers of ε for the scaling of the coordinates, or more general asymptotic sequences, $\phi^{(i)}(\varepsilon)$, as explained in the context of (121) in section 3.

In a number of publications, e.g., Majda and Klein (2003); Klein (2004); Klein et al. (2004); Klein (2005), we have demonstrated that a wide range of known simplified model equations of theoretical meteorology can be rederived in a unified fashion starting from the full compressible flow equations in (168) and suitable specializations of the multi-scale ansatz in (170). To derive a typical existing model, one would maintain one scaled time, one scaled horizontal coordinate, and one pair of scaled vertical coordinates, respectively, and this we consider a welcome "validation" of the approach. We will demonstrate the procedure in the next chapter.

Of course, such "validation studies" are but a first step, as (170) strongly suggests itself as the basis for systematic studies of multiple scales interactions. See Majda and Klein (2003), Majda and Biello (2004), Klein et al. (2004), Biello and Majda (2006), Klein and Majda (2006) for related developments.

Remark: *The particular choice of a reference velocity in (165) does in no way restrict our degrees of freedom in constructing simplified asymptotic models. If, for example, we were to consider flows that are inherently compressible, so that systematically $|\boldsymbol{v}| \sim c_{\text{ref}}$, then the asymptotic expansion scheme for the (dimensionless) flow velocity would simply have to read*

$$\boldsymbol{v} = \frac{1}{\varepsilon^{\frac{3}{2}}} \left(\sum_i \varepsilon^i \boldsymbol{v}^{(i)}(\ldots, \frac{u}{\varepsilon}, u, \varepsilon u, \ldots \frac{z}{\varepsilon}, z, \ldots \frac{t}{\varepsilon}, t, \varepsilon t, \ldots) \right) . \tag{171}$$

5 (Re-)derivation of the quasi-geostrophic (QG) theory

In this chapter, we employ the general asymptotics-based approach from section 4 to rederive the quasi-geostrophic model, see Pedlosky (1987).

5.1 Asymptotic expansion scheme

For the derivation of this theory, we will take the dimensionless form of the compressible flow equations from (168) as our point of departure. For simplicity of the exposition we assume adiabatic flow, dropping the source and transport terms in the governing equations. Then (168) simplifies to

$$\begin{aligned}
\varrho_t + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} \varrho + w \varrho_z + \varrho \left(\nabla_{\shortparallel} \cdot \boldsymbol{v}_{\shortparallel} + w_z \right) &= 0\,, \\
\boldsymbol{v}_{\shortparallel,t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} \boldsymbol{v}_{\shortparallel} + w \boldsymbol{v}_{\shortparallel,z} + \varepsilon\, 2 (\boldsymbol{\Omega} \times \boldsymbol{v})_{\shortparallel} + \frac{1}{\varepsilon^3} \frac{1}{\rho} \nabla_{\shortparallel} p &= 0\,, \\
w_t + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} w + w w_z + \varepsilon\, 2 (\boldsymbol{\Omega} \times \boldsymbol{v})_{\perp} + \frac{1}{\varepsilon^3} \frac{1}{\rho} p_z &= -\frac{1}{\varepsilon^3}\,, \\
\Theta_t + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} \Theta + w \Theta_z &= 0\,,
\end{aligned} \tag{172}$$

and

$$\Theta = \frac{p^{1/\gamma}}{\rho}\,. \tag{173}$$

The quasi-geostrophic theory is designed to address flows on length scales comparable to the *internal Rossby radius*, and on time scales corresponding to horizontal advection across such distances. How can we access these length and time scales within our multiple scales asymptotic scheme?

The *internal Rossby radius* is defined as the distance which an internal gravity wave would travel during a characteristic Earth rotation time. This is equivalent to requiring

$$L_{\mathrm{Ro}} = \frac{N h_{\mathrm{sc}}}{\Omega}\,, \tag{174}$$

where

$$N = \sqrt{\frac{g}{\Theta} \frac{\partial \Theta}{\partial z'}} \tag{175}$$

is the so-called Brunt-Väisälä or buoyancy frequency, and

$$N h_{\mathrm{sc}} = \sqrt{g h_{\mathrm{sc}}} \sqrt{\frac{h_{\mathrm{sc}}}{\Theta} \frac{\partial \Theta}{\partial z'}} = c_{\mathrm{ref}} \sqrt{\frac{h_{\mathrm{sc}}}{\Theta} \frac{\partial \Theta}{\partial z'}} \tag{176}$$

is a typical travelling speed of internal gravity waves. The reader may want to consult the established literature for corroboration.

Remark: *Here and below, primes mark* dimensional *variables!*

Remark: *Another interpretation of the internal Rossby radius considers it the characteristic distance which an internal gravity wave would have to travel to become affected by the Coriolis effect.*

Non-dimensionalizing L_{Ro} by our reference length, $\ell_{\text{ref}} = h_{\text{sc}}$, and using the above we find

$$\frac{L_{\text{Ro}}}{h_{\text{sc}}} = \frac{N}{\Omega} = \frac{c_{\text{ref}}}{\Omega h_{\text{sc}}} \sqrt{\frac{h_{\text{sc}}}{\Theta}\frac{\partial \Theta}{\partial z'}} \sim \frac{c_{\text{ref}}}{\Omega a}\,\frac{a}{h_{\text{sc}}} \sqrt{\frac{\Delta T|_{\text{eq}}^{\text{p}}}{T_{\text{ref}}}} = \Pi_3\,\Pi_1\,\sqrt{\Pi_2} \tag{177}$$

or, using the distinguished limits introduced earlier for the Π_i,

$$\frac{L_{\text{Ro}}}{h_{\text{sc}}} = O\left(\varepsilon^{\frac{1}{2}}\,\frac{1}{\varepsilon^3}\,\varepsilon^{\frac{1}{2}}\right) = O\left(\frac{1}{\varepsilon^2}\right)\,, \tag{178}$$

Here we have used the scalings of our fundamental parameters from Table 1 as discussed in (162)–(164), and the well established observation, [Schneider (2006); Frierson (2008)], that the equator-to-pole temperature differences are comparable to the vertical potential temperature variations across the troposphere, so that

$$\frac{h_{\text{sc}}}{\Theta}\frac{\partial \Theta}{\partial z} \sim \frac{\Delta T|_{\text{eq}}^{\text{p}}}{T_{\text{ref}}}\,. \tag{179}$$

With the estimate in (178), if we want to describe horizontal variations on scales comparable to L_{Ro}, we should use the dimensionless horizontal coordinate

$$\boldsymbol{\xi} = \frac{\boldsymbol{x}'}{L_{\text{Ro}}} = \frac{h_{\text{sc}}}{L_{\text{Ro}}}\,\frac{\boldsymbol{x}'}{h_{\text{sc}}} = \varepsilon^2\,\boldsymbol{x}\,. \tag{180}$$

We will be interested here in phenomena associated with advection over distances of L_{Ro}, so we will use the time variable

$$\tau = \frac{t'}{L_{\text{Ro}}/u_{\text{ref}}} = \frac{h_{\text{sc}}}{L_{\text{Ro}}}\,\frac{t'}{h_{\text{sc}}/u_{\text{ref}}} = \varepsilon^2\,t\,. \tag{181}$$

Finally, in order to study phenomena which occupy the full depth of the troposphere, we will use a vertical coordinate non-dimensionalized by the pressure scale height, h_{sc}, i.e., we use our original dimensionless coordinate

$$z = \frac{z'}{h_{\text{sc}}}\,. \tag{182}$$

These scalings are summarized in Fig. 14.

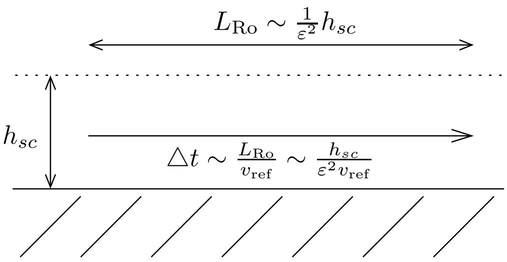

Figure 14. Length and time scales addressed by quasi-geostrophic theory.

Our asymptotic expansion scheme for the solution written in terms of non-dimensional variables will thus read

$$\mathcal{U}(t, \boldsymbol{x}, z; \varepsilon) = \sum_i \varepsilon^i \mathcal{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)\,, \qquad \mathcal{U} = (p, \Theta, \boldsymbol{v}_{\shortparallel}, w)^t\,, \tag{183}$$

which is the announced specialization of the general multiple scales expansion scheme in (170) adapted to resolve advection phenomena on the length scale of the internal Rossby radius.

5.2 Some preliminaries

When inserting this expansion into the governing equations in (172) we will have to account for the following transformation rules for the partial derivatives,

$$\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{x},z;\varepsilon} = \varepsilon^2 \left.\frac{\partial}{\partial \tau}\right|_{\boldsymbol{\xi},z;\varepsilon}\,, \qquad \nabla_{\boldsymbol{x}}\big|_{t,z;\varepsilon} = \varepsilon^2\, \nabla_{\boldsymbol{\xi}}\big|_{\tau,z;\varepsilon}\,. \tag{184}$$

Here the subscripts indicate which variables are to be held constant when carrying out the partial differentiations.

We also anticipate the following properties of the background stratification of the atmosphere in order to save us some tedious calculations:

$$\begin{aligned}
\varrho(t, \boldsymbol{x}, z; \varepsilon) &= \varrho_0(z) + \varepsilon \varrho_1(z) + \varepsilon^2 \varrho^{(2)}(\tau, \boldsymbol{\xi}, z) + o\big(\varepsilon^2\big)\,,\\
p(t, \boldsymbol{x}, z; \varepsilon) &= p_0(z) + \varepsilon p_1(z) + \varepsilon^2 p^{(2)}(\tau, \boldsymbol{\xi}, z) + o\big(\varepsilon^2\big)\,,\\
\Theta(t, \boldsymbol{x}, z; \varepsilon) &= 1 + \varepsilon \Theta_1(z) + \varepsilon^2 \Theta^{(2)}(\tau, \boldsymbol{\xi}, z) + o\big(\varepsilon^2\big)\,,\\
w(t, \boldsymbol{x}, z; \varepsilon) &= \varepsilon^3 w^{(3)}(\tau, \boldsymbol{\xi}, z) + o\big(\varepsilon^3\big)\,.
\end{aligned} \tag{185}$$

Remark: *That the leading-order thermodynamic variables are independent of time and do not vary horizontally can actually be* derived *within the* *present framework rather than having to be anticipated. The same is true for the vanishing of the leading two orders of vertical velocity,* $w^{(0)}, w^{(1)}$.

Remark: *The leading-order potential temperature must be a constant, because of the order-of-magnitude analyses of the previous subsection which restrict variations of potential temperature to* $\Delta\Theta/T_{\mathrm{ref}} = O(\varepsilon)$*. We may set this leading-order constant to* $\Theta^{(0)} \equiv 1$ *by choosing an appropriate reference temperature.*

5.3 Expansions of the governing equations

The next steps are standard procedure. We insert the expansion scheme, collect like powers of ε, and separately equate the sum of these terms to zero, so as to create the usual hierarchy of perturbation equations.

Mass conservation Expanding the mass conservation law, $(172)_1$, we find

$$
\begin{array}{lcrcl}
O\big(\varepsilon^0\big) & : & \varrho_0 \, \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{v}_{\shortparallel}^{(0)} & = & 0\,, \\
O(\varepsilon) & : & \varrho_0 \, \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{v}_{\shortparallel}^{(1)} + \dfrac{\partial}{\partial z}\Big(\varrho_0 w^{(3)}\Big) & = & 0\,.
\end{array}
\tag{186}
$$

In writing down the terms of $O(\varepsilon)$ we have already neglected $\varrho^{(1)}\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{v}_{\shortparallel}^{(0)}$ on account of (186).

Horizontal momentum balance

Splitting the Coriolis term Before expanding the momentum balances, we need to explicitly split the Coriolis term into its horizontal a vertical components.

$$
\begin{aligned}
\boldsymbol{\Omega} \times \boldsymbol{v} &= (\boldsymbol{\Omega}_{\shortparallel} + \boldsymbol{k}\Omega_{\perp}) \times (\boldsymbol{v}_{\shortparallel} + \boldsymbol{k}w) \\
&= \underbrace{(\boldsymbol{\Omega}_{\shortparallel} \times \boldsymbol{v}_{\shortparallel})}_{=(\boldsymbol{\Omega}\times\boldsymbol{v})_{\perp}} + \underbrace{(\Omega_{\perp}\boldsymbol{k} \times \boldsymbol{v}_{\shortparallel} + w\boldsymbol{\Omega}_{\shortparallel} \times \boldsymbol{k})}_{=(\boldsymbol{\Omega}\times\boldsymbol{v})_{\shortparallel}} + \underbrace{(\Omega_{\perp} w\boldsymbol{k} \times \boldsymbol{k})}_{=0,\ \text{as}\ \boldsymbol{k}\times\boldsymbol{k}=0}\,.
\end{aligned}
\tag{187}
$$

We will need the vertical component of $\boldsymbol{\Omega}$ (see Fig. 15), which we expand

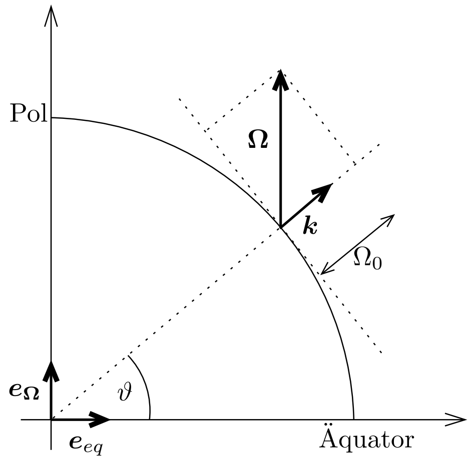

Figure 15. Splitting of the coriolis term into a horizontal and a vertical component.

as

$$
\begin{aligned}
\Omega_{\perp} &= \boldsymbol{k} \cdot \boldsymbol{\Omega} \\
&= (\boldsymbol{e}_{eq} \cos\vartheta + \boldsymbol{e}_{\boldsymbol{\Omega}} \sin\vartheta) \cdot \boldsymbol{e}_{\boldsymbol{\Omega}} |\boldsymbol{\Omega}| \\
&= |\boldsymbol{\Omega}| \sin\vartheta \\
&= |\boldsymbol{\Omega}| \sin\left(\vartheta_0 + \frac{y'}{a}\right) \\
&= |\boldsymbol{\Omega}| \sin(\vartheta_0 + \varepsilon\, \xi_2) \\
&= \underbrace{|\boldsymbol{\Omega}| \sin(\vartheta_0)}_{=:\Omega_0} + \varepsilon \underbrace{|\boldsymbol{\Omega}| \cos(\vartheta_0)}_{=:\beta} \xi_2 + o(\varepsilon) \\
&= \Omega_0 + \varepsilon\beta\xi_2 + o(\varepsilon)\ .
\end{aligned}
\tag{188}
$$

Here we have taken into account that ϑ is the arclength along a longitudinal circle divided by the radius of the reference sphere, a, introduced deviations from a reference latitude, so that $\vartheta = \vartheta_0 + y'/a$, and recalled that $h_{\mathrm{sc}}/a = \varepsilon^3$ and $\xi_2 = \varepsilon^2 y'/h_{\mathrm{sc}}$. The rest is Taylor expansion of the sine function about the reference latitude.

Since $w^{(0)} \equiv w^{(1)} \equiv 0$ we also know that $w\boldsymbol{\Omega}_{\shortparallel} \times \boldsymbol{k} = o(\varepsilon)$ and, in summary, we find

$$
\begin{aligned}
(\boldsymbol{\Omega} \times \boldsymbol{v})_{\shortparallel} &= (\Omega_0 + \varepsilon\beta\xi_2)\boldsymbol{k} \times \boldsymbol{v}_{\shortparallel} + o(\varepsilon)\ , \\
(\boldsymbol{\Omega} \times \boldsymbol{v})_{\perp} &= \boldsymbol{\Omega}_{\shortparallel} \times \boldsymbol{v}_{\shortparallel}\,.
\end{aligned}
\tag{189}
$$

Expansion of the horizontal momentum balance Consider now the horizontal momentum balance, written in terms of the new variables, $(\tau, \boldsymbol{\xi}, z)$,

$$
\boldsymbol{v}_{\shortparallel\tau} + (\boldsymbol{v}_{\shortparallel} \cdot \nabla_{\boldsymbol{\xi}})\boldsymbol{v}_{\shortparallel} + \frac{1}{\varepsilon^2}\, w\boldsymbol{v}_{\shortparallel z} + \frac{1}{\varepsilon^3}\,\frac{\nabla_{\boldsymbol{\xi}} p}{\varrho} + \frac{1}{\varepsilon}\,(\hat{\boldsymbol{\Omega}} \times \boldsymbol{v})_{\shortparallel} = 0\,. \tag{190}
$$

Using $w^{(0)} \equiv w^{(1)} \equiv 0$, and that $p^{(0)} \equiv p_0(z)$ and $p^{(1)} \equiv p_1(z)$, we immediately move to the equation at $O\big(\varepsilon^{-1}\big)$ where we find the *geostrophic balance*,

$$
\Omega_0\, \boldsymbol{k} \times \boldsymbol{v}_{\shortparallel}^{(0)} + \nabla_{\boldsymbol{\xi}} \pi^{(2)} = 0\,, \qquad \text{where} \qquad \pi^{(2)} = \frac{p^{(2)}}{\varrho_0}\,, \tag{191}
$$

i.e., the balance of the horizontal Coriolis and pressure gradient forces. Geostrophic balance implies that, at leading order, the horizontal flow direction is perpendicular to the horizontal pressure gradient.

We verify for later purposes that

$$
\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{v}_{\shortparallel}^{(0)} = 0 \qquad \text{and} \qquad \boldsymbol{v}_{\shortparallel}^{(0)} = \frac{1}{\Omega_0}\, \boldsymbol{k} \times \nabla_{\boldsymbol{\xi}} \pi^{(2)} \tag{192}
$$

by, respectively, applying $(\boldsymbol{k} \cdot (\nabla_{\boldsymbol{\xi}} \times [\,\cdot\,]))$ and $(\boldsymbol{k} \times [\,\cdot\,])$ to $(191)_1$.

Remark: *With (191) we have found a time-independent constraint on the leading order velocity and second order pressure fields. Such constraints did not exist for the original system of the compressible flow equations! The constraint implies that only if the initial data for a given flow problem satisfy the constraint, at least at the given orders, can we hope that the approximate asymptotic solution will remain close to the exact solution. This kind of "change of type" of the asymptotic limit problem relative to the original one is typical of* singular perturbation problems. *Also, we recall that we encountered a similar issue with the slow-time expansion for the linear oscillator in chapter 3.*

Vertical momentum balance From the vertical momentum balance we obtain at orders ε^{-5} to ε^{-2}

$$
\frac{\partial p^{(i)}}{\partial z} = -\varrho^{(i)} \qquad (i = 0, 1, 2, 3)\,. \tag{193}
$$

Expanding the defining equation for the potential temperature, i.e., $\varrho\Theta = p^{\frac{1}{\gamma}}$ into

$$
\begin{aligned}
O(\varepsilon^0):&\qquad & \varrho_0 &= {p_0}^{\frac{1}{\gamma}} \\
O(\varepsilon):&\qquad & \varrho_1 + \varrho_0\Theta_1 &= {p_0}^{\frac{1}{\gamma}}\frac{p_1}{\gamma p_0} \\
O(\varepsilon^2):&\qquad & \varrho^{(2)} + \varrho_1\Theta_1 + \varrho_0\Theta^{(2)} &= {p_0}^{\frac{1}{\gamma}}\left(\frac{p^{(2)}}{\gamma p_0} + \frac{(1-\gamma){p_1}^2}{2\gamma^2{p_0}^2}\right)
\end{aligned}
\tag{194}
$$

we obtain from (193)

$$
{p^{(0)}}^{-\frac{1}{\gamma}}\frac{\partial p^{(0)}}{\partial z} = -1 \tag{195}
$$

with the exact solution

$$
p_0(z) = \left(1 - \frac{\gamma-1}{\gamma}\,z\right)^{\frac{\gamma}{\gamma-1}}\,. \tag{196}
$$

In a similar way one solves the first order equation explicitly for given $\Theta_1(z)$. We leave this as an exercise.

Evolution of the potential temperature The first non-trivial asymptotic equation is extracted from the potential temperature transport equation at $O(\varepsilon)$, and it reads

$$
\left(\frac{\partial}{\partial\tau} + \boldsymbol{v}_{\shortparallel}^{(0)}\cdot\nabla_{\boldsymbol{\xi}}\right)\Theta^{(2)} + w^{(3)}\frac{d\Theta_1}{dz} = 0\,. \tag{197}
$$

5.4 Summary of the leading-order balances

Using the expansion scheme in (185), we have found first that the background structure is in hydrostatic balance, i.e.,

$$
\frac{dp_i}{dz} = -\varrho_i \qquad (i = 0, 1)\,. \tag{198}
$$

The remaining primary unknowns for description of the flow field are then

$$
\left(\pi^{(2)}, \boldsymbol{v}_{\shortparallel}^{(0)}, w^{(3)}, \Theta^{(2)}\right)(\tau, \boldsymbol{\xi}, z)\,, \tag{199}
$$

where $\pi^{(2)} = p^{(2)}/\varrho_0$, and they satisfy the following balance and transport equations:

Hydrostatic Balance

$$\frac{\partial \pi^{(2)}}{\partial z} = \Theta^{(2)} \tag{200}$$

Geostrophic Balance

$$\Omega_0 \, \boldsymbol{k} \times \boldsymbol{v}_{\shortparallel}^{(0)} + \nabla_{\boldsymbol{\xi}} \pi^{(2)} = 0 \tag{201}$$

Anelastic Constraint

$$\varrho_0 \, \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{v}_{\shortparallel}^{(1)} + \frac{\partial}{\partial z} \left(\varrho_0 w^{(3)} \right) = 0 \tag{202}$$

Potential Temperature Transport

$$\left(\frac{\partial}{\partial \tau} + \boldsymbol{v}_{\shortparallel}^{(0)} \cdot \nabla_{\boldsymbol{\xi}} \right) \Theta^{(2)} + w^{(3)} \frac{d\Theta_1}{dz} = 0 \tag{203}$$

If it were not for the appearance of the first-order divergence, $\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{v}_{\shortparallel}^{(1)}$ in (202), we would have the same number of equations as we have unknowns. As it is, the system is as yet unclosed. We will extract additional information on $\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{v}_{\shortparallel}^{(1)}$ from the next higher order horizontal momentum equation in the next section in the form of a *solvability condition* that may be interpreted as a vorticity transport equation,

First-Order Solvability Condition / Vorticity Transport Equation

$$\left(\partial_\tau + \boldsymbol{v}_{\shortparallel}^{(0)} \cdot \nabla_{\boldsymbol{\xi}} \right) (\zeta^{(0)} + \beta \xi_2) + \Omega_0 \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{v}_{\shortparallel}^{(1)} = 0 \, . \tag{204}$$

where,

$$\zeta^{(0)} = \boldsymbol{k} \cdot (\nabla_{\boldsymbol{\xi}} \times \boldsymbol{v}_{\shortparallel}^{(0)}) \, , \tag{205}$$

is the vorticity of the leading-order velocity field.

This completes the summary of the quasi-geostrophic model equations.

Remark: *We have given the QG equations here in a somewhat unusual form, sticking as closely as possible to the original equations. In this way, it remains transparent that (200) and (201) are direct consequences of the vertical and horizontal momentum balances, respectively, (202) emerges from mass conservation, and (203) from the potential temperature transport* *equation. These equations can all directly be read off the original equations at the appropriate orders in the asymptotic expansion.*

The closure for $\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{v}_{\shortparallel}^{(1)}$ in (204) emerges as a solvability condition *at $O(\varepsilon^0)$ in the horizontal momentum balance as will be shown in the next section.*

Remark: *As can see clearly in the present summary of our asymptotic limit equations, considering large spacial and long time scales only implies strong constraints on the solutions. Instead of evolution equations for the primary unknowns in the compressible flow equations (the densities of mass, momentum, and energy) we find three time independent constraints or balances! Only the potential temperature evolution equation in (203) and the vorticity transport equation in (204) have maintained the original "prognostic" (time evolution) character.*

5.5 First-order solvability condition / existence of $\nabla_{\boldsymbol{\xi}} p^{(3)}$

Consider the scaled horizontal momentum balance from (190), which we had already written in terms of our new coordinates, $(\tau, \boldsymbol{\xi}, z)$, at $O(\varepsilon^0)$,

$$(\boldsymbol{v}_{\shortparallel}^{(0)})_\tau + (\boldsymbol{v}_{\shortparallel}^{(0)} \cdot \nabla_{\boldsymbol{\xi}})\boldsymbol{v}_{\shortparallel}^{(0)} + \left[\frac{\nabla_{\boldsymbol{\xi}} p}{\varrho}\right]^{(3)} + \Omega_0\, \boldsymbol{k} \times \boldsymbol{v}_{\shortparallel}^{(1)} + \beta\xi_2 \boldsymbol{k} \times \boldsymbol{v}_{\shortparallel}^{(0)} = 0\,. \tag{206}$$

Using the fact that ϱ_0, ϱ_1 depend on z only, so that $\nabla_{\boldsymbol{\xi}}\varrho_0 = \nabla_{\boldsymbol{\xi}}\varrho_1 = 0$, and $\pi^{(i)} = p^{(i)}/\varrho_0$ we rewrite the pressure gradient term as

$$\left[\frac{\nabla_{\boldsymbol{\xi}} p}{\varrho}\right]^{(3)} = \frac{1}{\varrho_0}\nabla_{\boldsymbol{\xi}} p^{(3)} - \frac{\varrho_1}{\varrho_0^2}\nabla_{\boldsymbol{\xi}} p^{(2)} = \nabla_{\boldsymbol{\xi}}\pi^{(3)} - \nabla_{\boldsymbol{\xi}}\left(\frac{\varrho_1}{\varrho_0}\,\pi^{(2)}\right). \tag{207}$$

Next we regroup (206) into first-order "geostrophic terms" on left-hand side and terms that distort the geostrophic balance, i.e., "ageostrophic terms", on the right,

$$\begin{aligned} &\nabla_{\boldsymbol{\xi}}\pi^{(3)} + \Omega_0\, \boldsymbol{k} \times \boldsymbol{v}_{\shortparallel}^{(1)} = \\ &-\left((\boldsymbol{v}_{\shortparallel}^{(0)})_\tau + (\boldsymbol{v}_{\shortparallel}^{(0)} \cdot \nabla_{\boldsymbol{\xi}})\boldsymbol{v}_{\shortparallel}^{(0)} + \beta\xi_2 \boldsymbol{k} \times \boldsymbol{v}_{\shortparallel}^{(0)} - \nabla_{\boldsymbol{\xi}}\left(\frac{\varrho_1}{\varrho_0}\pi^{(2)}\right)\right)\,. \end{aligned} \tag{208}$$

We know that any gradient of a scalar is curl-free. In particular,

$$\boldsymbol{k} \cdot (\nabla_{\boldsymbol{\xi}} \times \nabla_{\boldsymbol{\xi}}\phi) \equiv 0 \tag{209}$$

for any scalar function $\phi(\boldsymbol{\xi})$ that is sufficiently smooth. By applying the operator $\boldsymbol{k} \cdot (\nabla_{\boldsymbol{\xi}} \times [\,\cdot\,])$ to (208) we thus eliminate the terms involving $\pi^{(2)}$

and $\pi^{(3)}$. The remaining terms become

$$
\begin{aligned}
\boldsymbol{k}\cdot\left(\nabla_{\boldsymbol{\xi}}\times\left(\Omega_0\,\boldsymbol{k}\times\boldsymbol{v}_{\shortparallel}^{(1)}\right)\right) &= \Omega_0\boldsymbol{k}\cdot\left(\boldsymbol{k}\left(\nabla_{\boldsymbol{\xi}}\cdot\boldsymbol{v}_{\shortparallel}^{(1)}\right)\right)\\
&= \Omega_0\,\nabla_{\boldsymbol{\xi}}\cdot\boldsymbol{v}_{\shortparallel}^{(1)}\,,\\
\boldsymbol{k}\cdot\left(\nabla_{\boldsymbol{\xi}}\times(\boldsymbol{v}_{\shortparallel}^{(0)})_\tau\right) &= \zeta_\tau^{(0)}\,,\\
\boldsymbol{k}\cdot\left(\nabla_{\boldsymbol{\xi}}\times\left((\boldsymbol{v}_{\shortparallel}^{(0)}\cdot\nabla_{\boldsymbol{\xi}})\boldsymbol{v}_{\shortparallel}^{(0)}\right)\right) &= \boldsymbol{v}_{\shortparallel}^{(0)}\cdot\nabla_{\boldsymbol{\xi}}\zeta^{(0)}\,,\\
\boldsymbol{k}\cdot\left(\nabla_{\boldsymbol{\xi}}\times\left(\beta\xi_2\,\boldsymbol{k}\times\boldsymbol{v}_{\shortparallel}^{(0)}\right)\right) &= \beta\,\boldsymbol{k}\cdot\boldsymbol{k}\left(\nabla_{\boldsymbol{\xi}}\xi_2\cdot\boldsymbol{v}_{\shortparallel}^{(0)}\right)\\
&= \beta v^{(0)}\\
&= \beta\left(\frac{\partial}{\partial\tau}+\boldsymbol{v}_{\shortparallel}^{(0)}\cdot\nabla_{\boldsymbol{\xi}}\right)\xi_2\,.
\end{aligned}
\tag{210}
$$

Collecting, we find the vorticity transport equation,

$$
\left(\partial_\tau+\boldsymbol{v}_{\shortparallel}^{(0)}\cdot\nabla_{\boldsymbol{\xi}}\right)(\zeta^{(0)}+\beta\xi_2)+\Omega_0\nabla_{\boldsymbol{\xi}}\cdot\boldsymbol{v}_{\shortparallel}^{(1)}=0\,,
\tag{211}
$$

as announced in (204).

5.6 Classical formulation of the QG-theory and PV transport

In (201)–(204) we have taken care to display the quasi-geostrophic balance equations in a form that reveals their close connection to the mass, momentum, and potential temperature evolution equations. This may not be the most practicable description in many applications, and it hides the central role of *potential vorticity (PV)* in the quasi-geostrophic regime.

In fact, one can rewrite (211) as a transport equation for the QG-potential vorticity,

$$
q=\zeta^{(0)}+\beta\,\xi_2+\frac{\Omega_0}{\varrho_0}\frac{\partial}{\partial z}\left(\frac{\varrho_0\Theta^{(2)}}{\Theta_1'}\right)\qquad\text{with}\qquad\Theta_1'=d\Theta_1/dz\,,
\tag{212}
$$

which then reads

$$
\left(\partial_\tau+\boldsymbol{v}_{\shortparallel}^{(0)}\cdot\nabla_{\boldsymbol{\xi}}\right)q=0\,.
\tag{213}
$$

Equipped with the additional constitutive relations

$$\begin{aligned} \boldsymbol{v}_{\shortparallel}^{(0)} &= \frac{1}{\Omega_0}\boldsymbol{k} \times \nabla_{\boldsymbol{\xi}} p^{(2)} \,, \\ \Theta^{(2)} &= \frac{\partial \pi^{(2)}}{\partial z} \,, \\ \zeta^{(0)} &= \boldsymbol{k} \cdot \left(\nabla_{\boldsymbol{\xi}} \times \boldsymbol{v}_{\shortparallel}^{(0)} \right) = \frac{1}{\Omega_0} \nabla_{\boldsymbol{\xi}}^2 \pi^{(2)} \end{aligned} \tag{214}$$

we have the QG theory in its classical form (Pedlosky (1987)): Equation (213) describes advection of potential vorticity by the leading order velocity field $\boldsymbol{v}_{\shortparallel}^{(0)}$, which can be expressed in terms of the pressure gradient $\nabla_{\boldsymbol{\xi}} \pi^{(2)}$ according to $(214)_1$. Given the (advected) PV-field, one can retrieve the pressure field solving the ellliptic equation that results from inserting $(214)_{2,3}$ into (212), viz.

$$\nabla_{\boldsymbol{\xi}}^2 \pi^{(2)} + \frac{\Omega_0^2}{\varrho_0} \frac{\partial}{\partial z} \left(\frac{\varrho_0}{\Theta_1'} \frac{\partial \pi^{(2)}}{\partial z} \right) = q - \beta \, \xi_2 \,. \tag{215}$$

Appendix

Gauß' Integral Theorem

Let $\Omega \subset I\!\!R$ be a compact subset with a smooth boundary, $\boldsymbol{n} : \partial\Omega \to I\!\!R^n$ the field of outer unit normal vectors and $U \supset \Omega$ an open subset of $I\!\!R^n$. Then for every continuous differentiable vector field $F : U \to I\!\!R^n$ the following is true

$$\int_\Omega \nabla \cdot F(x) \; dV = \int_{\partial\Omega} F(x) \cdot \boldsymbol{n} \; d\sigma$$

Proof see (Forster, 1984, S. 155)

The symbols $O()$ and $o()$ (Landau's symbols)

The symbol $O()$ is used in this text in two ways. One formulation is based on the so called Landau symbol. Here, for functions f and g the equity holds

$$f(x) = O(g(x)) \qquad \text{as} \qquad x \to a$$

if and only if $f(x)/g(x) \to const.$ for $x \to a$ (in the asymptotic sense). On the other hand the symbol is often used in the way that the statement „the quantity X is $O(\epsilon)$" means that X is of the same order of magnitude as ϵ. The particular meaning of $O(\,\cdot\,)$ will be clear from the context.

The second Landau symbol $o(\,\cdot\,)$ ("little oh"). For functions f and g we have

$$f(x) = o(g(x)) \qquad \text{as} \qquad x \to a \quad ,$$

if and only if $f(x)/g(x) \to 0$ for $x \to a$.

If for some function f we not only have $f(x) = O(g(x))$ but also $g(x) = O(f(x))$ as $x \to a$, then $f(x) = O_S(g(x))$ as $x \to a$.

Vector identities

For vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^n$ and scalars $\varphi,\ \psi \in \mathbb{R}$ the following general identities are true :

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{C} \times \boldsymbol{B}) \times \boldsymbol{A} = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}) \tag{216}$$

$$\nabla \cdot (\varphi\ \boldsymbol{A}) = \varphi \nabla \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \nabla \varphi \tag{217}$$

$$\nabla \times (\varphi\ \boldsymbol{A}) = \varphi \nabla \times \boldsymbol{A} + \nabla \varphi \times \boldsymbol{A} \tag{218}$$

$$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \nabla \times \boldsymbol{A} - \boldsymbol{A} \cdot \nabla \times \boldsymbol{B} \tag{219}$$

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} \tag{220}$$

$$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} \tag{221}$$

$$\nabla \times \nabla \varphi = 0 \tag{222}$$

$$\nabla \cdot \nabla \times \boldsymbol{A} = \mathrm{div}(\mathrm{rot}\boldsymbol{A}) = 0 \tag{223}$$

$$\nabla \cdot (\boldsymbol{A} \circ \boldsymbol{B}) = (\nabla \cdot \boldsymbol{A})\boldsymbol{B} + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} \tag{224}$$

Pressure scale height

The atmosphere is in hydrostatic balance if the vertical pressure gradient is equal to the gravity acceleration, i.e.,

$$\frac{\partial p}{\partial z} = -\varrho g$$

Starting from a reference pressure p_{ref} (e.g. a mean pressure on sea level) we can define the *pressure scale height* by the height difference at which the pressure in an atmosphere in hydrostatic balance and with constant density changes of an order of magnitude of p_{ref}

$$h_{\mathrm{sc}} := \frac{p_{\mathrm{ref}}}{\varrho_{\mathrm{ref}}\, g} \quad .$$

The thermal wind

The equation of the *thermal wind*

$$-\Omega_0\, \boldsymbol{k} \times \frac{\partial \boldsymbol{v}_{\shortparallel}^{(0)}}{\partial z} = \frac{1}{\Theta_\infty} \nabla_{\shortparallel} \Theta^{(3)}$$

does not give any information about the geostrophic wind itself, but only about it's vertical variation. The thermal wind denotes the velocity differences that result from geostrophic balance across some vertical distance, say, $\triangle z : \boldsymbol{v}_T = \triangle \boldsymbol{v} = \boldsymbol{v}(z_1) - \boldsymbol{v}(z_2)$.

Some details of the transformations into a rotating reference frame

We switch from $\mathbb{R}^3$ to a Cartesian system of coordinates $Z = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ in which $\boldsymbol{\Omega}$ and $\boldsymbol{X}_b(t)$ are defined by

$$\underline{\boldsymbol{\Omega}} := \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} \quad , \quad \underline{\boldsymbol{X}}_b(t) := \begin{pmatrix} X_{b1} \\ X_{b2} \\ X_{b3} \end{pmatrix} \quad .$$

With the additional definition of

$$\underline{\underline{\boldsymbol{\Omega}}} := \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}$$

the change with time of $\underline{\boldsymbol{X}}_b(t)$ can also be written as a matrix vector product, namely

$$\underline{\dot{\boldsymbol{X}}}_b(t) = \underline{\boldsymbol{\Omega}} \times \underline{\boldsymbol{X}}_b(t) = \underline{\underline{\boldsymbol{\Omega}}}\; \underline{\boldsymbol{X}}_b(t) = \begin{pmatrix} -\Omega_3 X_{b2}(t) + \Omega_2 X_{b3}(t) \\ \Omega_3 X_{b1}(t) - \Omega_1 X_{b3}(t) \\ -\Omega_2 X_{b1}(t) + \Omega_1 X_{b2}(t) \end{pmatrix} . \tag{225}$$

With the knowledge from the theory of ordinary differential equations we can derive equation (37) in a different way. We know that the solution of the initial value problem

$$\underline{\dot{\boldsymbol{X}}}_b(t) = \underline{\underline{\boldsymbol{\Omega}}}\; \underline{\boldsymbol{X}}_b(t) \quad , \quad \underline{\boldsymbol{X}}_b(0) = \underline{\boldsymbol{X}}_{b0}$$

is

$$\underline{\boldsymbol{X}}_b(t) = \exp(\underline{\underline{\boldsymbol{\Omega}}}t)\ \underline{\boldsymbol{X}}_b(0) \qquad \text{, wobei } \exp(\underline{\underline{\boldsymbol{\Omega}}}t) := \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!}(\underline{\underline{\boldsymbol{\Omega}}})^\nu \quad .$$

Because of

$$\underline{\underline{\boldsymbol{\Omega}}}^2 = \begin{pmatrix} -\Omega_2^2\Omega_3^2 & \Omega_1^2\Omega_2^2 & \Omega_1^2\Omega_3^2 \\ \Omega_1^2\Omega_2^2 & -\Omega_1^2\Omega_3^2 & \Omega_2^2\Omega_3^2 \\ -\Omega_1^2\Omega_3^2 & \Omega_2^2\Omega_3^2 & -\Omega_1^2\Omega_2^2 \end{pmatrix} = |\underline{\boldsymbol{\Omega}}|^2(\underline{\boldsymbol{e}}_\Omega \underline{\boldsymbol{e}}_\Omega^T - \underline{\underline{\mathbf{1}}}) \quad ,$$

$$\underline{\underline{\boldsymbol{\Omega}}}^3 = -|\underline{\boldsymbol{\Omega}}|^2\underline{\underline{\boldsymbol{\Omega}}} \quad , \quad \underline{\underline{\boldsymbol{\Omega}}}^4 = -|\underline{\boldsymbol{\Omega}}|^4(\underline{\boldsymbol{e}}_\Omega \underline{\boldsymbol{e}}_\Omega^T - \underline{\underline{\mathbf{1}}}) \quad , \quad \underline{\underline{\boldsymbol{\Omega}}}^5 = -|\underline{\boldsymbol{\Omega}}|^2\underline{\underline{\boldsymbol{\Omega}}}^3 = |\underline{\boldsymbol{\Omega}}|^4\underline{\underline{\boldsymbol{\Omega}}} \qquad \text{etc.}$$

With $\underline{\boldsymbol{e}}_\Omega = \frac{\underline{\boldsymbol{\Omega}}}{|\underline{\boldsymbol{\Omega}}|}$ and $|\underline{\boldsymbol{\Omega}}| = \sqrt{\Omega_1^2 + \Omega_2^2 + \Omega_3^2}$ we then have

$$\begin{aligned}
\underline{\boldsymbol{X}}_b(t) &= \Big[\underline{\underline{\mathbf{1}}} + t\ \underline{\underline{\boldsymbol{\Omega}}} + \frac{t^2}{2!}|\underline{\boldsymbol{\Omega}}|^2(\underline{\boldsymbol{e}}_\Omega \underline{\boldsymbol{e}}_\Omega^T - \underline{\underline{\mathbf{1}}}) - \frac{t^3}{3!}|\underline{\boldsymbol{\Omega}}|^2\underline{\underline{\boldsymbol{\Omega}}} \\
&\quad - \frac{t^4}{4!}|\underline{\boldsymbol{\Omega}}|^4(\underline{\boldsymbol{e}}_\Omega \underline{\boldsymbol{e}}_\Omega^T - \underline{\underline{\mathbf{1}}}) + \frac{t^5}{5!}|\underline{\boldsymbol{\Omega}}|^4\underline{\underline{\boldsymbol{\Omega}}} + \ldots\Big]\ \underline{\boldsymbol{X}}_b(0) \\
&= \Big[(\underline{\boldsymbol{e}}_\Omega \underline{\boldsymbol{e}}_\Omega^T) + \left(1 - \frac{|\underline{\boldsymbol{\Omega}}|^2 t^2}{2!} + \frac{|\underline{\boldsymbol{\Omega}}|^4 t^4}{4!} - \ldots\right)(\underline{\underline{\mathbf{1}}} - \underline{\boldsymbol{e}}_\Omega \underline{\boldsymbol{e}}_\Omega^T) \\
&\quad + \left(|\underline{\boldsymbol{\Omega}}|t - \frac{|\underline{\boldsymbol{\Omega}}|^3 t^3}{3!} + \frac{|\underline{\boldsymbol{\Omega}}|^5 t^5}{5!} - \ldots\right)\frac{\underline{\underline{\boldsymbol{\Omega}}}}{|\underline{\boldsymbol{\Omega}}|}\Big]\ \underline{\boldsymbol{X}}_b(0) \\
&= \left[(\underline{\boldsymbol{e}}_\Omega \underline{\boldsymbol{e}}_\Omega^T) + \cos(|\underline{\boldsymbol{\Omega}}|t)(\underline{\underline{\mathbf{1}}} - \underline{\boldsymbol{e}}_\Omega \underline{\boldsymbol{e}}_\Omega^T) + \sin(|\underline{\boldsymbol{\Omega}}|t)\frac{\underline{\underline{\boldsymbol{\Omega}}}}{|\underline{\boldsymbol{\Omega}}|}\right]\ \underline{\boldsymbol{X}}_b(0)
\end{aligned}$$

and this, because of the representation in coordinates, is just the same as the previously derived result from (37). As (225), $\underline{\underline{\boldsymbol{\Omega}}}\ \underline{\boldsymbol{X}}_b$ can be represented as $\underline{\boldsymbol{\Omega}} \times \underline{\boldsymbol{X}}_b$ and we obtain

$$\underline{\boldsymbol{X}}_b(t) = (\underline{\boldsymbol{e}}_\Omega^T \underline{\boldsymbol{X}}_b(0))\underline{\boldsymbol{e}}_\Omega + \cos(|\underline{\boldsymbol{\Omega}}|t)(\underline{\underline{\mathbf{1}}} - \underline{\boldsymbol{e}}_\Omega \underline{\boldsymbol{e}}_\Omega^T)\underline{\boldsymbol{X}}_b(0) + \sin(|\underline{\boldsymbol{\Omega}}|t)(\underline{\boldsymbol{X}}_b(0) \times \underline{\boldsymbol{e}}_\Omega) \quad .$$

Defining $\underline{\underline{\boldsymbol{R}}}(t) := \exp(\underline{\underline{\boldsymbol{\Omega}}}t)$, $\underline{\underline{\boldsymbol{R}}}(t)$ has the following properties:

- $\underline{\underline{\boldsymbol{R}}}(-t) = \underline{\underline{\boldsymbol{R}}}^T(t)$
  As $\underline{\underline{\boldsymbol{\Omega}}}$ is skew symmetric, i.e., it is $\underline{\underline{\boldsymbol{\Omega}}}^T = -\underline{\underline{\boldsymbol{\Omega}}}$, and as the $\cos(\,\cdot\,)$ is an even while the $\sin(\,\cdot\,)$ is an uneven function, we have

$$\begin{aligned}
\underline{\underline{\boldsymbol{R}}}^T(t) &= (\underline{\boldsymbol{e}}_\Omega \underline{\boldsymbol{e}}_\Omega^T)^T + \cos(|\underline{\boldsymbol{\Omega}}|t)(\underline{\underline{\mathbf{1}}} - \underline{\boldsymbol{e}}_\Omega \underline{\boldsymbol{e}}_\Omega^T)^T + \sin(|\underline{\boldsymbol{\Omega}}|t)\frac{\underline{\underline{\boldsymbol{\Omega}}}^T}{|\underline{\boldsymbol{\Omega}}|} \\
&= (\underline{\boldsymbol{e}}_\Omega \underline{\boldsymbol{e}}_\Omega^T) + \cos(-|\underline{\boldsymbol{\Omega}}|t)(\underline{\underline{\mathbf{1}}} - \underline{\boldsymbol{e}}_\Omega \underline{\boldsymbol{e}}_\Omega^T) - \sin(-|\underline{\boldsymbol{\Omega}}|t)\frac{-\underline{\underline{\boldsymbol{\Omega}}}}{|\underline{\boldsymbol{\Omega}}|} = \underline{\underline{\boldsymbol{R}}}(-t)
\end{aligned}$$

- $\underline{\underline{\boldsymbol{R}}}(-t) = \underline{\underline{\boldsymbol{R}}}^{-1}(t)$

We will now analyze how the different terms of our conservation laws change when we switch to a rotating coordinate system. To this end, we represent an arbitrary fixed vector with respect to the inertial basis $\{\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \underline{\boldsymbol{e}}_3\}$ as

$$\underline{\boldsymbol{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \underline{\boldsymbol{e}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \underline{\boldsymbol{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

in the basis $\{\underline{\tilde{\boldsymbol{e}}}_1, \underline{\tilde{\boldsymbol{e}}}_2, \underline{\tilde{\boldsymbol{e}}}_3\}$ of the rotating coordinate system. The interrelation between the coordinate systems is given in the following form :

$$\underline{\tilde{\boldsymbol{e}}}_i = \underline{\underline{\boldsymbol{R}}}(t)\underline{\boldsymbol{e}}_i \ , \qquad i = 1, 2, 3 \quad .$$

Then $\underline{\boldsymbol{x}}$ can be represented by the two bases as

$$\underline{\boldsymbol{x}} = \sum_{i=1}^{3} x_i \underline{\boldsymbol{e}}_i = \sum_{i=1}^{3} \tilde{x}_i \underline{\tilde{\boldsymbol{e}}}_i = \sum_{i=1}^{3} \tilde{x}_i \underline{\underline{\boldsymbol{R}}}(t)\underline{\boldsymbol{e}}_i \quad .$$

Multiplication from left with $\underline{\boldsymbol{e}}_k^T$ yields

$$x_k = \sum_{i=1}^{3} \tilde{x}_i (\underline{\boldsymbol{e}}_k^T \underline{\underline{\boldsymbol{R}}}(t))\underline{\boldsymbol{e}}_i = \sum_{i=1}^{3} \tilde{x}_i (\underline{\underline{\boldsymbol{R}}}(t))_{ki} \quad .$$

Hence,

$$\underline{\boldsymbol{x}} = \underline{\underline{\boldsymbol{R}}}(t)\underline{\tilde{\boldsymbol{x}}} \quad \text{und} \quad \underline{\tilde{\boldsymbol{x}}} = \underline{\underline{\boldsymbol{R}}}^{-1}(t)\underline{\boldsymbol{x}} \quad .$$

Furthermore we analyze the differential operators under the present a transformation of coordinates. For the transformation $(t, \underline{\boldsymbol{x}}) \to (\tilde{t}, \underline{\tilde{\boldsymbol{x}}})$ with $\tilde{t} = t$ it holds for a function f that

$$f(t, x) = \tilde{f}(\tilde{t}, \underline{\tilde{\boldsymbol{x}}}) = \tilde{f}(\tilde{t}(t, \underline{\boldsymbol{x}}), \underline{\tilde{\boldsymbol{x}}}(t, \underline{\boldsymbol{x}})) \quad .$$

This yields

$$\begin{aligned}
\left(\frac{\partial f}{\partial t}\right)_{\underline{\boldsymbol{x}}} &= \left(\frac{\partial \tilde{t}}{\partial t}\right)_{\underline{\boldsymbol{x}}}\left(\frac{\partial \tilde{f}}{\partial \tilde{t}}\right)_{\underline{\tilde{\boldsymbol{x}}}} &+& \sum_{i=1}^{3}\left(\frac{\partial \tilde{x}_i}{\partial t}\right)_{\underline{\boldsymbol{x}}}\left(\frac{\partial \tilde{f}}{\partial \tilde{x}_i}\right)_{\tilde{t},x_j(j\neq i)} \\
&= \frac{\partial \tilde{f}}{\partial \tilde{t}} &+& \sum_{i=1}^{3}(-\underline{\underline{\boldsymbol{\Omega}}}\,\underline{\tilde{\boldsymbol{x}}})_i\frac{\partial \tilde{f}}{\partial \tilde{x}_i} \\
&= \frac{\partial \tilde{f}}{\partial \tilde{t}} &+& \sum_{i,j=1}^{3}(-\Omega_{ij}\tilde{x}_j)\frac{\partial \tilde{f}}{\partial \tilde{x}_i} \\
&= \frac{\partial \tilde{f}}{\partial \tilde{t}} &+& \sum_{i,j=1}^{3}(\tilde{x}_j\Omega_{ji})\frac{\partial \tilde{f}}{\partial \tilde{x}_i} \\
&= \frac{\partial \tilde{f}}{\partial \tilde{t}} &+& \underline{\tilde{\boldsymbol{x}}}^T\underline{\underline{\boldsymbol{\Omega}}}(\tilde{\nabla}\tilde{f})^T
\end{aligned} \tag{226}$$

and

$$\begin{aligned}
\left(\frac{\partial f}{\partial x_i}\right)_{\tilde{t},x_k(k\neq i)} &= \left(\frac{\partial \tilde{t}}{\partial x_i}\right)_{t,x_k(k\neq i)}\left(\frac{\partial \tilde{f}}{\partial \tilde{t}}\right)_{\underline{\tilde{\boldsymbol{x}}}} + \\
&\quad + \sum_{j=1}^{3}\left(\frac{\partial \tilde{x}_j}{\partial x_i}\right)_{t,x_k(k\neq i)}\left(\frac{\partial \tilde{f}}{\partial \tilde{x}_j}\right)_{\tilde{t},\tilde{x}_k(k\neq j)} \\
&= \sum_{j=1}^{3}(\underline{\underline{\boldsymbol{R}}}^{-1}(t))_{ji}\frac{\partial \tilde{f}}{\partial \tilde{x}_j} \\
&= \sum_{j=1}^{3}(\underline{\underline{\boldsymbol{R}}}(t))_{ij}\frac{\partial \tilde{f}}{\partial \tilde{x}_j}
\end{aligned}$$

or

$$(\nabla f)^T = \underline{\underline{\boldsymbol{R}}}(t)(\tilde{\nabla}\tilde{f})^T \quad .$$

Consider now the transformation of the velocity of a particle: The position of a particle in both coordinate systems can be expressed by

$$\underline{\boldsymbol{x}}_p(t) = \sum_{i=1}^{3} x_{pi}(t)\underline{\boldsymbol{e}}_i = \sum_{j=1}^{3}\tilde{x}_{pj}(t)\underline{\tilde{\boldsymbol{e}}}_j(t)$$

and its velocity is

$$\frac{d\underline{\boldsymbol{x}}_p}{dt} = \sum_{i=1}^{3} \dot{x}_{pi}(t)\underline{\boldsymbol{e}}_i = \sum_{j=1}^{3} (\dot{\tilde{x}}_{pj}(t)\underline{\tilde{\boldsymbol{e}}}_j(t) + \tilde{x}_{pj}(t)\dot{\underline{\tilde{\boldsymbol{e}}}}_j(t))$$
$$= \underline{\dot{\boldsymbol{x}}}_p^{\text{rel}}(t) + \sum_{j=1}^{3} \tilde{x}_{pj}(t)\underline{\underline{\boldsymbol{\Omega}}}\,\underline{\tilde{\boldsymbol{e}}}_j(t)$$
$$= \underline{\dot{\boldsymbol{x}}}_p^{\text{rel}}(t) + \underline{\underline{\boldsymbol{\Omega}}}\,\underline{\boldsymbol{x}}_p(t) \quad .$$

Thus, if p denotes a particle of the fluid we look at, then $\underline{\dot{\boldsymbol{x}}}_p^{\text{rel}} = \underline{\boldsymbol{v}}_p^{\text{rel}}$ is the relevant *wind speed* and the local flow velocity $\underline{\boldsymbol{v}}(\underline{\boldsymbol{x}}, t)$ can be splitted

$$\begin{aligned} \underline{\boldsymbol{v}} = \underline{\dot{\boldsymbol{x}}} &= \underline{\boldsymbol{v}}^{\text{rel}} + \underline{\underline{\boldsymbol{\Omega}}}\,\underline{\boldsymbol{x}} \\ &= \sum_{i=1}^{3} \tilde{v}_i^{\text{rel}}\underline{\tilde{\boldsymbol{e}}}_i + \underline{\underline{\boldsymbol{\Omega}}}\,\underline{\boldsymbol{x}} \end{aligned} \tag{227}$$

into a part arising from earth's rotation and the relative wind speed. When transforming the velocity divergence we find

$$\nabla \cdot \underline{\boldsymbol{v}} = \nabla \cdot (\underline{\boldsymbol{v}}^{\text{rel}} + \underline{\underline{\boldsymbol{\Omega}}}\,\underline{\boldsymbol{x}}) = \nabla \cdot \underline{\boldsymbol{v}}^{\text{rel}} + \nabla \cdot (\underline{\underline{\boldsymbol{\Omega}}}\,\underline{\boldsymbol{x}}) \quad ,$$

whereas because of the zeros on the diagonals of the rotation matrix $\underline{\underline{\boldsymbol{\Omega}}}$

$$\nabla \cdot (\underline{\underline{\boldsymbol{\Omega}}}\,\underline{\boldsymbol{x}}) = \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}(\Omega_{ij}x_j) = \sum_{i,j=1}^{3} \Omega_{ij}\delta_{ij} = 0\,,$$

(the symbol $\delta_{ij} = 1$ if $i = j$, 0 otherwise, denotes the Kronecker-Symbol). As also $\underline{\boldsymbol{e}}_i^T\,\underline{\tilde{\boldsymbol{e}}}_j = \underline{\boldsymbol{e}}_i^T\,\underline{\underline{\boldsymbol{R}}}\,\underline{\boldsymbol{e}}_j = R_{ij}$ and $\frac{\partial}{\partial x_i} = \sum_{j=1}^{3} R_{ij}\frac{\partial}{\partial \tilde{x}_i}$, one has

$$\begin{aligned} \nabla \cdot \underline{\boldsymbol{v}}^{\text{rel}} = \sum_{i=1}^{3} \frac{\partial v_i^{\text{rel}}}{\partial x_i} &= \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\underline{\boldsymbol{e}}_i^T \sum_{j=1}^{3} \tilde{v}_j^{\text{rel}}\underline{\tilde{\boldsymbol{e}}}_j) \\ &= \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(\tilde{v}_j^{\text{rel}}(\tilde{t}(t,\underline{\boldsymbol{x}}), \underline{\tilde{\boldsymbol{x}}}(t,\underline{\boldsymbol{x}}))\right)\underline{\boldsymbol{e}}_i^T\,\underline{\tilde{\boldsymbol{e}}}_j \\ &= \sum_{i,j,k=1}^{3} R_{ik}\frac{\partial}{\partial \tilde{x}_k}\tilde{v}_j^{\text{rel}}R_{ij} \\ &= \sum_{j,k=1}^{3} \delta_{ik}\frac{\partial}{\partial \tilde{x}_k}\tilde{v}_j^{\text{rel}} = \sum_{k=1}^{3} \frac{\partial \tilde{v}_k^{\text{rel}}}{\partial \tilde{x}_k} = \tilde{\nabla} \cdot \underline{\tilde{\boldsymbol{v}}}^{\text{rel}} \quad . \end{aligned} \tag{228}$$

Furthermore, because of $\underline{\underline{\boldsymbol{R}}}^{-1}(t) = \underline{\underline{\boldsymbol{R}}}^T(t) = \underline{\underline{\boldsymbol{R}}}(-t)$ we have $(\underline{\underline{\boldsymbol{R}}}^T\underline{\underline{\boldsymbol{R}}})_{kj} = \sum_{i=1}^3 R_{ik}R_{ij} = \delta_{kj}$.

For momentum conservation the following auxiliary calculations might be useful: Transforming $(\underline{\boldsymbol{v}}_t + (\underline{\boldsymbol{v}}^T\nabla^T)\underline{\boldsymbol{v}})$ into the rotating system of coordinates and using (226) and (227) yields for the derivative with respect to time

$$
\begin{aligned}
\underline{\boldsymbol{v}}_t &= \left(\frac{\partial}{\partial t}\right)_{\underline{\boldsymbol{x}}} (\underline{\boldsymbol{v}}^{\text{rel}} + \underline{\underline{\boldsymbol{\Omega}}}\,\underline{\boldsymbol{x}}) \\
&= \left(\frac{\partial}{\partial t}\right)_{\underline{\boldsymbol{x}}} \left(\sum_{i=1}^3 \tilde{v}_i^{\text{rel}}\underline{\tilde{\boldsymbol{e}}}_i\right) \qquad + \qquad \left(\frac{\partial}{\partial t}\right)_{\underline{\boldsymbol{x}}} (\underline{\underline{\boldsymbol{\Omega}}}\,\underline{\boldsymbol{x}}) \\
&= \sum_{i=1}^3 \left(\frac{\partial \tilde{v}_i^{\text{rel}}}{\partial \tilde{t}} + \underline{\tilde{\boldsymbol{x}}}^T\underline{\underline{\boldsymbol{\Omega}}}(\tilde{\nabla}\tilde{v}_i^{\text{rel}})^T\right)\underline{\tilde{\boldsymbol{e}}}_i + \sum_{i=1}^3 \tilde{v}_i^{\text{rel}}\underline{\underline{\boldsymbol{\Omega}}}\,\underline{\tilde{\boldsymbol{e}}}_i \quad + \qquad 0 \\
&= \sum_{i=1}^3 \frac{\partial \tilde{v}_i^{\text{rel}}}{\partial t}\underline{\tilde{\boldsymbol{e}}}_i + \left(\underline{\tilde{\boldsymbol{x}}}^T\underline{\underline{\boldsymbol{\Omega}}}\tilde{\nabla}^T\right)\underline{\tilde{\boldsymbol{v}}}^{\text{rel}} + \underline{\underline{\boldsymbol{\Omega}}}\,\underline{\tilde{\boldsymbol{v}}}^{\text{rel}}
\end{aligned}
$$

Because of

$$
(\underline{\boldsymbol{v}}^{\text{rel}^T}\nabla^T)\underline{\boldsymbol{v}}^{\text{rel}} = \sum_{i=1}^3((\underline{\tilde{\boldsymbol{v}}}^{\text{rel}})^T\tilde{\nabla}^T)\tilde{v}_i^{\text{rel}}\underline{\tilde{\boldsymbol{e}}}_i \quad ,
$$

$$
\left((\underline{\underline{\boldsymbol{\Omega}}}\,\underline{\boldsymbol{x}})^T\nabla^T\right)\underline{\boldsymbol{v}}^{\text{rel}} = (\underline{\tilde{\boldsymbol{x}}}^T\underline{\underline{\boldsymbol{\Omega}}}^T\tilde{\nabla}^T)\underline{\tilde{\boldsymbol{v}}}^{\text{rel}} = -(\underline{\tilde{\boldsymbol{x}}}^T\underline{\underline{\boldsymbol{\Omega}}}\tilde{\nabla}^T)\underline{\tilde{\boldsymbol{v}}}^{\text{rel}} \quad ,
$$

$$
\begin{aligned}
(\underline{\boldsymbol{v}}^{\text{rel}^T}\nabla^T)\underline{\underline{\boldsymbol{\Omega}}}\,\underline{\boldsymbol{x}} &= \left(\sum_{i=1}^3 v_i^{\text{rel}}\frac{\partial}{\partial x_i}\right)\underline{\underline{\boldsymbol{\Omega}}}\left(\sum_{i=1}^3 x_i\underline{\boldsymbol{e}}_i\right) = \sum_{i,j=1}^3 \tilde{v}_j^{\text{rel}}\frac{\partial}{\partial \tilde{x}_j}(\underline{\underline{\boldsymbol{\Omega}}}\,\underline{\boldsymbol{e}}_i\tilde{x}_i) \\
&= \sum_{i,j=1}^3 \underline{\underline{\boldsymbol{\Omega}}}\tilde{v}_j^{\text{rel}}\delta_{ij}\underline{\boldsymbol{e}}_i = \underline{\underline{\boldsymbol{\Omega}}}\,\underline{\tilde{\boldsymbol{v}}}^{\text{rel}}
\end{aligned}
$$

and

$$
\begin{aligned}
\left((\underline{\underline{\boldsymbol{\Omega}}}\,\underline{\boldsymbol{x}})^T\nabla^T\right)(\underline{\underline{\boldsymbol{\Omega}}}\,\underline{\boldsymbol{x}}) &= -(\underline{\boldsymbol{x}}^T\underline{\underline{\boldsymbol{\Omega}}}\nabla^T)\underline{\underline{\boldsymbol{\Omega}}}\,\underline{\boldsymbol{x}} \\
&= -\sum_{i,j,k,l=1}^3 x_i\Omega_{ik}\frac{\partial}{\partial x_k}\Omega_{jl}x_l\underline{\boldsymbol{e}}_j = \sum_{i,j,k=1}^3 x_i\Omega_{ki}\Omega_{jk}\underline{\boldsymbol{e}}_j \\
&= \left(\underline{\boldsymbol{x}}^T\underline{\underline{\boldsymbol{\Omega}}}^T\underline{\underline{\boldsymbol{\Omega}}}^T\right)^T = \underline{\underline{\boldsymbol{\Omega}}}(\underline{\underline{\boldsymbol{\Omega}}}\,\underline{\boldsymbol{x}}) = \underline{\boldsymbol{\Omega}} \times (\underline{\boldsymbol{\Omega}} \times \underline{\boldsymbol{x}})
\end{aligned}
$$

we obtain

$$\begin{aligned}
\underline{\boldsymbol{v}}_t + (\underline{\boldsymbol{v}}^T \nabla^T)\underline{\boldsymbol{v}} &= \underline{\boldsymbol{v}}_t + \Big((\underline{\boldsymbol{v}}^{\text{rel}} + \underline{\underline{\boldsymbol{\Omega}}}\, \underline{\boldsymbol{x}})^T \nabla^T \Big) \Big(\underline{\boldsymbol{v}}^{\text{rel}} + \underline{\underline{\boldsymbol{\Omega}}}\, \underline{\boldsymbol{x}} \Big) \\
&= \underline{\boldsymbol{v}}_t + (\underline{\boldsymbol{v}}^{\text{rel}^T} \nabla^T)\underline{\boldsymbol{v}}^{\text{rel}} + \big((\underline{\underline{\boldsymbol{\Omega}}}\, \underline{\boldsymbol{x}})^T \nabla^T \big)\, \underline{\boldsymbol{v}}^{\text{rel}} \\
&\quad + (\underline{\boldsymbol{v}}^{\text{rel}^T} \nabla^T)(\underline{\underline{\boldsymbol{\Omega}}}\, \underline{\boldsymbol{x}}) + \big((\underline{\underline{\boldsymbol{\Omega}}}\, \underline{\boldsymbol{x}})^T \nabla^T \big)\, (\underline{\underline{\boldsymbol{\Omega}}}\, \underline{\boldsymbol{x}}) \\
&= \sum_{i=1}^{3} \frac{\partial \tilde{v}_i^{\text{rel}}}{\partial t} \underline{\tilde{\boldsymbol{e}}}_i + \sum_{i=1}^{3} ((\underline{\tilde{\boldsymbol{v}}}^{\text{rel}})^T \tilde{\nabla}^T) \tilde{v}_i^{\text{rel}} \underline{\tilde{\boldsymbol{e}}}_i \\
&\quad + 2\, \underline{\underline{\boldsymbol{\Omega}}}\, \underline{\tilde{\boldsymbol{v}}}^{\text{rel}} + \underline{\underline{\boldsymbol{\Omega}}}(\underline{\underline{\boldsymbol{\Omega}}}\, \underline{\boldsymbol{x}})
\end{aligned}$$

Multiplication with $\underline{\tilde{\boldsymbol{e}}}_k^T$ yields

$$\big(\underline{\boldsymbol{v}}_t + (\underline{\boldsymbol{v}}^T \nabla)\underline{\boldsymbol{v}} \big)_k = \frac{\partial \tilde{v}_k^{\text{rel}}}{\partial t} + ((\underline{\tilde{\boldsymbol{v}}}^{\text{rel}})^T \nabla^T) \tilde{v}_k^{\text{rel}} + 2\, (\underline{\underline{\boldsymbol{\Omega}}}\, \underline{\tilde{\boldsymbol{v}}}^{\text{rel}})_k + \big(\underline{\underline{\boldsymbol{\Omega}}}(\underline{\underline{\boldsymbol{\Omega}}}\, \underline{\boldsymbol{x}}) \big)_k$$

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