Annexure: 1

Derivation of Expectation and Variance of Vasicek model:

Spot rate under Vasicek model:

$$dr_t = (\eta - \gamma r_t)dt + \sigma dW_t$$

We use integrating factor, e^{at}

$$d(e^{at} r_t) = e^{at} dr_t + ae^{at} r_t \ dt$$
$$d(e^{at} r_t) = e^{at} [(\eta - \gamma r_t)dt + \sigma dW_t] + ae^{at} r_t \ dt$$

$$d(e^{at} r_t) = e^{at} \eta dt + e^{at} \sigma dW_t$$

Integrating we get,

$$[e^{au} r_u]_t^T = \int_t^T \eta e^{au} du + \int_t^T e^{au} \sigma dW_u$$

$e^{aT} r_T - e^{at} r_t = [\frac{\eta e^{aT} - \eta e^{at}]}{a} + \int_t^T e^{au} \sigma dW_u$

$$e^{aT} r_T = e^{at} r_t + \frac{\eta}{a}[e^{aT} - e^{at}] + \int_t^T e^{au} \sigma dW_u$$

Spot rate under vasicek model is given by,

$$r_T = r_t e^{-a(T-t)} + \frac{\eta}{a}[1 - e^{-a(T-t)}] + \sigma \int_t^T e^{-a(T-u)} dW_u \quad \text{Equation: 1}$$

Taking expectation of equation: 1 and using Standard Brownian motion properties,

$E(r_T) = r_t e^{-a(T-t)} + \frac{\eta}{a}[1 - e^{-a(T-t)}] + \sigma \int_t^T e^{-a(T-u)} E(dW_u)$

Since $dW_u \sim N(0, dt)$ $E(dW_u) = 0$

$$\therefore E(r_T)|_t = r_t e^{-a(T-t)} + \frac{\eta}{a}[1 - e^{-a(T-t)}]$$

Now taking variance of equation: 1 and using $var(dW_u) = dt$

$var(r_T) = 0 + 0 + \sigma^2 var(\int_t^T e^{-a(T-u)} dW_u)$

$$var(r_T) = \sigma^2 \int_t^T e^{-2a(T-u)} (dW_u)^2$$

$$var(r_T) = \sigma^2 \int_t^T e^{-2a(T-u)} du$$

$$var(r_T) = \sigma^2 [\frac{e^{-2a(T-u)}}{2a}]_t^T$$

$$var(r_T) = \frac{\sigma^2}{2a} [1 - e^{-2a(T-t)}]$$

Annexure 2:

Derivation of slope of yield curve

We start with the power series expansion of $Z(r,t;T)$

$$Z(r,t;T) \approx 1 + a(r)(T-t) + b(r)(T-t)^2 + \ldots\ldots \qquad \text{Equation: 1}$$

Ignoring the higher order term with power >2

Now consider the bond price equation,

$$\frac{\partial Z}{\partial t} + \frac{1}{2}w(r,t)^2\frac{\partial^2 Z}{\partial r^2} + \big(u(r,t) - \lambda(r,t)w(r,t)\big)\frac{\partial Z}{\partial r} - rZ = 0 \qquad \text{Equation: 2}$$

Differentiating equation 1 we get,

$$\frac{\partial Z}{\partial t} = -a(r) - 2b(r)(T-t) + \ldots\ldots \qquad \text{Equation: 3}$$

$$\frac{\partial Z}{\partial r} = -\frac{\partial a(r)}{\partial r}(T-t) + \frac{\partial b(r)}{\partial r}(T-t)^2 \ \ldots\ldots \qquad \text{Equation: 4}$$

$$\frac{\partial^2 Z}{\partial r^2} = \frac{\partial^2 a(r)}{\partial r^2}(T-t) + \frac{\partial^2 b(r)}{\partial r^2}(T-t)^2 \ \ldots\ldots \qquad \text{Equation: 5}$$

Substituting equation 1, 3, 4 and 5 in 2 we get,

$$[-a(r) - 2b(r)(T-t) + \ \ldots\ldots] + \frac{1}{2}w(r,t)^2[\frac{\partial^2 a(r)}{\partial r^2}(T-t) + \frac{\partial^2 b(r)}{\partial r^2}(T-t)^2 \ \ldots\ldots] + \big(u(r,t) - \lambda(r,t)w(r,t)\big)[-\frac{\partial a(r)}{\partial r}(T-t) + \frac{\partial b(r)}{\partial r}(T-t)^2 \ \ldots\ldots] - rZ = 0$$

Collecting terms with same power of (T-t) together and ignoring the higher order terms,

$$[-a(r) - r] + \left[-2b(r) + \frac{1}{2}w(r,t)^2\frac{\partial^2 a(r)}{\partial r^2} + \big(u(r,t) - \lambda(r,t)w(r,t)\big)\frac{\partial a(r)}{\partial r} - \mathrm{r\ a(r)}\right](T-t) +$$

$$[\frac{1}{2}w(r,t)^2\frac{\partial^2 b(r)}{\partial r^2} + \big(u(r,t) - \lambda(r,t)w(r,t)\big)\frac{\partial b(r)}{\partial r} + \mathrm{r\ b(r)}](T-t)^2 = 0 \qquad \text{Equation: 6}$$

Setting each coefficients of (T-t) to zero in equation 6,so that the equation is satisfied for any value of 'r'.

$$-a(r) - r = 0$$
$$\therefore a(r) = -r$$
$$-2b(r) + \frac{1}{2}w(r,t)^2\frac{\partial^2 a(r)}{\partial r^2} + \big(u(r,t) - \lambda(r,t)w(r,t)\big)\frac{\partial a(r)}{\partial r} - \mathrm{r\ a(r)} = 0$$

Using the above result of a(r) we get $\frac{\partial a\,(r)}{\partial r} = \ -1 \quad and \quad \frac{\partial^2 a(r)}{\partial r^2} = 0$

$$-2b(r) - \big(u(r,t) - \lambda(r,t)w(r,t)\big) - \mathrm{r\ a(r)} = 0$$

$$b(r) = \frac{1}{2}\left[r^2 - \big(u(r,t) - \lambda(r,t)w(r,t)\big)\right]$$

Substituting value of a and b in equation 1:

$$Z(r,t;T) \ \approx 1 - r(T-t) + \tfrac{1}{2}\left[r^2 - \big(u(r,t) - \lambda(r,t)w(r,t)\big)\right](T-t)^2 + \ldots\ldots.$$

Equation: 7

We know,

$$\ln(1+x) \ \approx \ x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$\therefore \ \ln(Z) \ \approx (Z-1) - \tfrac{(Z-1)^2}{2} + \tfrac{(Z-1)^3}{3} - \cdots \qquad \text{Equation: 8}$$

Using equation 1:

$$Z(r,t;T) - 1 \approx a(r)(T-t) + b(r)(T-t)^2$$

$$(Z-1)^2 \ \approx a(r)^2(T-t)^2 + \ \ldots..$$

Substituting in equation 8:

$$\ln(Z) \ \approx a(r)(T-t) + b(r)(T-t)^2 - \frac{a(r)^2(T-t)^2}{2} + \cdots$$

$$-\frac{ln(Z)}{T-t} \approx -a(r) - b(r)(t-t) + \frac{a(r)^2(T-t)}{2} + \cdots$$

Substituting value of a and r in above equation we get,

$$-\frac{ln(Z)}{T-t} \approx r - \frac{1}{2}\left[r^2 - \left(u(r,t) - \lambda(r,t)w(r,t)\right)\right](T-t) + \frac{r^2(T-t)}{2} + \cdots$$

$\therefore Yield \approx r + \frac{1}{2}[(u - \lambda w)](T-t) + \cdots$ Equation: 8

From the above equation we see that slope of the yield curve = $\frac{1}{2}[(u - \lambda w)]$ which is the slope of the plot of the yield against time.