Mean response time and accuracy in the diffusion superposition model with deadline

Supplement for "Detection behavior of monkeys in complex multisensory contexts"

This online supplement provides implementation details on the derivation of the predictions for mean response time and accuracy for the diffusion superposition model (Diederich, 1995; Schwarz, 1994) with a deadline. For simplicity, we reiterate the relevant parts of the methods section here and then add code in R statistical language for the different equations. The R code (R core team, 2015) includes the necessary defaults that allow testing and deployment in other analyses.

A sequential sampling model with an additive summation mechanism

To explain our behavioral data, we propose an extension of the superposition model (Diederich, 1995; Schwarz, 1989, 1994). We first describe the basic superposition model and then its extension. These models are a class of drift diffusion models adapted to multisensory contexts. They possess considerable explanatory power. The integration mechanism corresponds to a linear summation of the activity of the two sensory channels (Diederich, 1995; Schwarz, 1989, 1994). In diffusion models, upon presentation of a stimulus, buildup of evidence is described by a noisy diffusion process $\mathbf{X}(t)$ with drift μ and variance $\sigma^2 > 0$ (e.g., Ratcliff, 1988; Smith & Ratcliff, 2004). The stimulus is 'detected' as soon as an evidence criterion $c > 0$ is met for the first time. The density $g(t)$ and distribution $G(t)$ of the first-passage times are well known ("inverse Gaussian", Cox & Miller, 1965),

$$g(t \mid c, \mu, \sigma^2) = \frac{c}{\sigma\sqrt{2\pi t^3}} \cdot \exp\left[-\frac{(c-\mu t)^2}{2\sigma^2 t}\right], \tag{1}$$

$$G(t \mid c, \mu, \sigma^2) = \Phi(\mu t \mid c, \sigma^2 t) + \exp\left(\frac{2c\mu}{\sigma^2}\right)\Phi(-\mu t \mid c, \sigma^2 t), \tag{2}$$

with $\Phi(x \mid m, s^2)$ denoting the Normal distribution with mean m and variance s^2. The expected detection time is obtained by integrating $\int_0^\infty t \times g(t \mid c, \mu, \sigma^2)\, dt$, which simplifies to

$$E[\mathbf{D}] = c/\mu. \tag{3}$$

Predictions for the detection times for unimodal stimuli A and V are, therefore, easily obtained, c / μ_A, and c / μ_V, respectively.

When two stimuli are presented simultaneously, coactivation occurs. The model assumes that the two modality-specific processes superimpose linearly, $\mathbf{X}_{AV}(t) = \mathbf{X}_A(t) + \mathbf{X}_V(t)$. The new process $\mathbf{X}_{AV}$ is, therefore, again a diffusion process with drift $\mu_{AV} = \mu_A + \mu_V$ and variance $\sigma^2_{AV} = \sigma^2_A + \sigma^2_V + 2\,\rho_{AV}\,\sigma_A\,\sigma_V$ (under the assumption that $\mathbf{X}_A$ and $\mathbf{X}_V$ are uncorrelated, the covariance term is zero). Since the drift parameters add up, $\mathbf{X}_{AV}$ reaches the response criterion earlier than any of its single constituents, resulting in faster responses to redundant stimuli, $E[\mathbf{D}_{AV}] = c / \mu_{AV}$.

What happens in asynchronous stimuli (e.g., V160A, i.e., the auditory stimulus component follows the visual stimulus component with a 160 ms delay)? During the first τ = 160 ms, sensory evidence is accumulated by the visual channel alone. If the criterion c is reached within this interval, the stimulus is detected, and a response is initiated. This case occurs with probability $P(\mathbf{D}_{V(\tau)A} \leq \tau) = G(\tau \mid c, \mu_V, \sigma^2_V)$ given by (2). If detection occurs before τ, it is expected to occur within

$$E\left[\mathbf{D}_{V(\tau)A} \,\middle|\, \mathbf{D}_{V(\tau)A} \leq \tau\right] = \frac{\int_0^\tau t \cdot g\left(t \mid c, \mu_V, \sigma_V^2\right) dt}{P\left(\mathbf{D}_{V(\tau)A} \leq \tau\right)} \quad (4)$$

The solution for the integral is given by Schwarz (1994, Eq. 6). In any other case, at $t = \tau$, the process has attained a subthreshold activation level $\mathbf{X}_V(\tau) = x < c$, with probability density described by (e.g., Schwarz, 1994, Eq. 7)

$$w(x, \tau \mid c, \mu_V, \sigma_V^2) = n(x \mid \mu_V \tau, \sigma_V^2 \tau) - \exp\left(\frac{2c\mu_V}{\sigma_V^2}\right) \cdot n(x \mid 2c + \mu_V \tau, \sigma_V^2 \tau). \quad (5)$$

with $n(x \mid m, s^2)$ denoting the normal density with mean m and variance s^2. When activation level x has already been attained, less “work” has to be done to reach the criterion c. Starting at time τ, both stimuli contribute to the buildup of activity, resulting in an aggregate process drifting with μ_{AV} towards a residual barrier $c - x$:

$$E\left[\mathbf{D}_{V(\tau)A} \,\middle|\, \mathbf{X}(\tau) = x\right] = \tau + \frac{c - x}{\mu_{AV}}. \quad (6)$$

This expectation must be integrated for all possible levels of activation x, weighted by the density (5) that activation level x has been reached by time $t = \tau$:

$$E\left[\mathbf{D}_{V(\tau)A} \,\middle|\, \mathbf{D}_{V(\tau)A} > \tau\right] = \int_{-\infty}^{c} w(x, \tau \mid c, \mu_V, \sigma_V^2) \cdot E\left[\mathbf{D}_{V(\tau)A} \,\middle|\, \mathbf{X}(\tau) = x\right] dx. \quad (7)$$

An analytic solution for the overall, unconditional expected first-passage time $E[\mathbf{D}_{V(\tau)A}]$ has been derived by Schwarz (1994, Eq. 10). The diffusion process only describes the ‘detection’ latency **D**, or processing time. To derive a prediction for the observed response time **T**, an additional variable **M** must be introduced which summarizes everything not described by the diffusion process (e.g., motor execution) and is simply added to the detection time, such that **T** = **D** + **M**. Therefore, the model prediction for the mean response time is $E[\mathbf{T}] = E[\mathbf{M}] + \mu_M$, where the additional parameter denotes the expectation of **M**. Schwarz (1994) demonstrated that the additive model accurately describes the mean response times and variances reported for Participant B.D. from Miller (1986) in a speeded response task with 13 different SOA levels.

Incorporating a deadline for accumulation

An unrealistic assumption of the model described above is that accumulation will always complete and that participants would have nearly 100 percent detection accuracy. This is because, given enough time, a diffusion process with positive drift μ will certainly reach the criterion $c > 0$, and the expected detection time $E[\mathbf{D}]$ results from integration of t times the processing time density within zero and infinity (3). From an experimental perspective, this has two implications: 1) the intensity of the stimulus components is sufficiently high to ensure detection rates of 100% and 2) the temporal window for responding is infinitely long to guarantee that all responses are collected.

If the temporal window for stimulus detection is limited by a deadline d (we assume that $d > \tau$) the proportion of correct responses is given by the distribution of the detection times at time $t = d$. For unimodal stimuli and synchronous audiovisual stimuli, this probability corresponds again to the inverse Gaussian distribution at time d, $P(\mathbf{D} \leq d) = G(t \mid c, \mu, \sigma^2)$, with $\mu = \mu_A, \mu_V, \mu_{AV}$, and $\sigma^2 = \sigma^2_A, \sigma^2_V, \sigma^2_{AV}$, depending on the modality. The expected detection time, conditional on stimulus detection before d, is again obtained by integration of $t \cdot g(t)$ within time zero and d (see Eq. 4, with τ replaced by d), the solution for the integral is given by Schwarz (1994, Eq. 6).

In bimodal stimuli with onset asynchrony $0 < \tau < d$ (e.g., V160A), it is again necessary to separately consider the intervals $[0\ldots\tau]$ and $[\tau\ldots d]$ in which drift and variance of the diffusion process amount to μ_V, σ^2_V and μ_{AV}, σ^2_{AV}, respectively. The proportion of correct detections amounts to the sum of the detections within $[0\ldots\tau]$ in which only the first stimulus contributes to the buildup of evidence, and the detections within $[\tau\ldots d]$ in which both stimuli are active. Like before, activation $\mathbf{X}(\tau)$ can take any value between negative infinity and c,

$$P\big(\mathbf{D}_{\mathrm{V}(\tau)\mathrm{A}} \leq d\big) = G(\tau \mid c, \mu_{\mathrm{V}}, \sigma^2_{\mathrm{V}}) + \int_{-\infty}^{c} w(x, \tau \mid c, \mu_{\mathrm{V}}, \sigma^2_{\mathrm{V}}) \cdot G(d - \tau \mid c - x, \mu_{\mathrm{AV}}, \sigma^2_{\mathrm{AV}})\, dx \quad (8)$$

with $w(x, \tau \mid \ldots)$ denoting again the density of the processes not yet absorbed at $t = \tau$. The integral can be simplified to the sum of four terms of the form $n(x \mid 0, 1) \times \Phi(x \mid m, s^2)$. These terms are determined using the bivariate Normal distribution with correlation ρ (BVN, Owen, 1980, Eq. 10,010.1)

$$\int_{-\infty}^{y} n(x \mid 0, 1) \cdot \Phi(ax + b)\, dx = \mathrm{BVN}\left(\frac{a}{\sqrt{1+b^2}}, y, \rho = -\frac{b}{\sqrt{1+b^2}}\right). \quad (9)$$

In a similar way, the mean detection time, conditional on stimulus detection before the deadline, can be shown to amount to

$$\begin{aligned} E\big[\mathbf{D}_{\mathrm{V}(\tau)\mathrm{A}} \mid \mathbf{D}_{\mathrm{V}(\tau)\mathrm{A}} \leq d\big] &= \frac{1}{P\big(\mathbf{D}_{\mathrm{V}(\tau)\mathrm{A}} \leq d\big)} \times \Big\{H(\tau \mid c, \mu_{\mathrm{V}}, \sigma^2_{\mathrm{V}}) + \tau \cdot \int_{-\infty}^{c} w(x, \tau \mid c, \mu_{\mathrm{V}}, \sigma^2_{\mathrm{V}}) \cdot \\ & G(d - \tau \mid c - x, \mu_{\mathrm{AV}}, \sigma^2_{\mathrm{AV}})\, dx + \int_{-\infty}^{c} w(x, \tau \mid c, \mu_{\mathrm{V}}, \sigma^2_{\mathrm{V}}) \cdot H(d - \tau \mid c - x, \mu_{\mathrm{AV}}, \sigma^2_{\mathrm{AV}})\, dx\Big\}, \quad (10) \end{aligned}$$

with $H(\tau \mid c, \mu, \sigma^2) = \int_0^{\tau} t \cdot g(t \mid c, \mu, \sigma^2)\, dt = \frac{c}{\mu}\left[\Phi(\mu t \mid c, \sigma^2 t) - \exp\left(\frac{2c\mu}{\sigma^2}\right)\Phi(-\mu t \mid c, \sigma^2 t)\right]$ (Schwarz, 1994, Eq. 6). The form of the $w \times G$ integral corresponds to the one in (8). The $w \times H$ integral consists of four $n(x \mid 0, 1)\, \Phi(x \mid m, s^2)$ terms (as before in Eq. 8) as well as four terms of the form $x \times n(x \mid 0, 1) \times \Phi(x \mid m, s^2)$ that match with Owen (1980, Eq. 10,011.1).

For the expected response time, we assumed again an SOA invariant mean residual μ_M,

$$E\big[\mathbf{T}_{\mathrm{V}(\tau)\mathrm{A}} \mid \mathbf{D}_{\mathrm{V}(\tau)\mathrm{A}} \leq d\big] = E\big[\mathbf{D}_{\mathrm{V}(\tau)\mathrm{A}} \mid \mathbf{D}_{\mathrm{V}(\tau)\mathrm{A}} \leq d\big] + \mu_{\mathrm{M}}, \quad (11)$$

which adds one additional parameter.

Libraries

The code requires an implementation of the inverse Gaussian distribution which is given by package SuppDists (Wheeler, 2013, available from CRAN). In addition, package mvtnorm (Genz et al., 2014) is used to retrieve probabilities of the bivariate Normal distribution.

```
#
# Implementation of the inverse Gaussian distribution
#
library(SuppDists) # install from CRAN first

#
# Multivariate Normal distribution
#
library(mvtnorm) # install from CRAN first
```

Helper functions

$\int n(x)\, \Phi(ax + b)\, dx$

Owen (1980, Eq. 10,010.1) provides an elegant solution for integrals of the form $\int n(x)\, \Phi(ax + b)\, dx$ which appear in the prediction of accuracy (see Eq. 9 and below). We added another function that translates $\int \exp(ux)\, n(x \mid m_1, s_1^2)\, \Phi(x \mid m_2, s_2^2)\, dx$ to this basic form.

Note that $\exp(ux)\, n(x \mid m, s^2) = \exp\left(um + \frac{u^2 s^2}{2}\right) n(x + us^2 \mid m, s^2)$ and

$$\int_{-\infty}^{y} n(x \mid m_1, s_1^2) \cdot \Phi(x \mid m_2, s_2^2)\, dx = \int_{-\infty}^{\frac{y-m_1}{s_1}} n(x \mid 0, 1) \cdot \Phi\left[\frac{(m_1 - m_2)}{s_2} \cdot x + \frac{s_1}{s_2} \middle| 0, 1\right] dx$$

```
#
# Integrate dnorm(x) pnorm(a + bx) from -Inf to y (Owen, 1980, Eq. 10,010.1)
#
owen10_010.1 = function(a, b, y)
{
	rho = -b/sqrt(1 + b*b)
	pmvnorm(lower=c(-Inf, -Inf), upper=c(a/sqrt(1+b*b), y), corr=matrix(c(1, rho, rho, 1), nrow=2))
}

#
# Integrate dnorm(x | ma, sa) * pnorm(x | mb, sb) from -Inf to y
#
owen10_010.1x = function(u, ma, sa, mb, sb, y)
{
	ma1 = ma + u*sa*sa
	exp(u*ma + u*u*sa*sa/2) * owen10_010.1(a=(ma1 - mb)/sb, b=sa/sb, y=(y - ma1)/sa)
}
```

$\int x\, n(x)\, \Phi(ax+b)\, dx$

A similar closed form solution exists for integrals of the form ∫ $x\, n(x)\, \Phi(ax + b)\, dx$ (Owen, 1980, Eq. 10,011.1),

$$\int x \cdot n(x) \cdot \Phi(ax+b)\, dx = \frac{b}{\sqrt{1+b^2}} \cdot n\left(\frac{a}{\sqrt{1+b^2}}\right) \cdot \Phi\left(x\sqrt{1+b^2} + \frac{ab}{\sqrt{1+b^2}}\right) - n(x) \cdot \Phi(ax+b)$$

We defined again two helper functions that implement this solution for integrals from $-\infty$ to y, one for the basic equation and an extended function mapping to ∫ $x \exp(ux)\, n(x \mid m_1, s_1^2)\, \Phi(x \mid m_2, s_2^2)\, dx$.

```
#
# Integrate x dnorm(x) pnorm(a + bx) from -Inf to y (Owen, 1980, Eq. 10,011.1)
#
owen10_011.1 = function(a, b, y)
{
	bb = sqrt(1 + b*b)
	b/bb * dnorm(a/bb) * pnorm(y*bb + a*b/bb) - dnorm(y)*pnorm(a + b*y)
}

#
# Integrate x exp(ux) dnorm(x | ma, sa) * pnorm(x | mb, sb) from -Inf to y
#
owen10_011.1x = function(u, ma, sa, mb, sb, y)
{
	ma1 = ma + u*sa*sa
	exp(u*ma + u*u*sa*sa/2) *
		{sa * owen10_011.1(a=(ma1 - mb)/sb, b=sa/sb, y=(y - ma1)/sa) +
			ma1 * owen10_010.1(a=(ma1 - mb)/sb, b=sa/sb, y=(y - ma1)/sa)}
}
```

$\exp(a)\, \Phi(b)$

To avoid numerical instabilities for large *a* and tiny *b*, we use an approximation of the Normal distribution function by Kiani et al. (2008). The approximation is based on an exponential function which allows to collapse it with exp(*a*).

```
#
# Numerically improved solution for exp(a) * pnorm(b) (Kiani et al., 2008)
#
exp_pnorm = function(a, b)
{
	r = exp(a) * pnorm(b)
	d = is.nan(r) & b < -5.5
	r[d] = 1/sqrt(2) * exp(a - b[d]*b[d]/2) * (0.5641882/b[d]/b[d]/b[d] - 1/b[d]/sqrt(pi))
	r
}
```

Accuracy

Unimodal and synchronous stimuli

For unimodal or synchronous stimuli, accuracy is given by the inverse Gaussian distribution at *d*.

```
#
# Accuracy in unimodal and synchronous stimuli
#
acc_sync = function(d=865, c=100, mu=0.08, sigma2=10.5)
{
	pinvGauss(d, nu=c/mu, lambda=c*c/sigma2)
}
```

For example, accuracy of Monkey S for low intensity auditory stimuli is predicted as

```
acc_sync(d=865, c=100, mu=0.08, sigma2=10.5) # about 55%.
```

Asynchronous stimuli

In asynchronous stimuli, accuracy is predicted by the sum of the inverse Gaussian distribution at onset asynchrony τ and four integrals resulting from expansion of $w(x, \tau \mid c, \mu, \sigma^2)$ and $G(d - \tau \mid c - x, \mu, \sigma^2)$:

$$
\begin{aligned}
&\int_{-\infty}^{c} w(x, \tau \,\big|\, c, \mu_{\mathrm{V}}, \sigma_{\mathrm{V}}^2) \cdot G(d - \tau \,\big|\, c - x, \mu_{\mathrm{AV}}, \sigma_{\mathrm{AV}}^2)\, dx \\
&\qquad = \int_{-\infty}^{c} n(x \,\big|\, \mu_{\mathrm{V}}\tau, \sigma_{\mathrm{V}}^2\tau)\; \Phi(\mu_{\mathrm{AV}} d' \,\big|\, c - x, \sigma_{\mathrm{AV}}^2 d')\, dx \\
&\qquad + \int_{-\infty}^{c} n(x \,\big|\, \mu_{\mathrm{V}}\tau, \sigma_{\mathrm{V}}^2\tau)\; \exp\left[\frac{2(c-x)\mu_{\mathrm{AV}}}{\sigma_{\mathrm{AV}}^2}\right] \Phi(-\mu_{\mathrm{AV}} d' \,\big|\, c - x, \sigma_{\mathrm{AV}}^2 d')\, dx \\
&\qquad - \exp\left(\frac{2c\mu_{\mathrm{V}}}{\sigma_{\mathrm{V}}^2}\right) \int_{-\infty}^{c} n(x \,\big|\, 2c + \mu_{\mathrm{V}}\tau, \sigma_{\mathrm{V}}^2\tau)\; \Phi(\mu_{\mathrm{AV}} d' \,\big|\, c - x, \sigma_{\mathrm{AV}}^2 d')\, dx \\
&\qquad - \exp\left(\frac{2c\mu_{\mathrm{V}}}{\sigma_{\mathrm{V}}^2}\right) \int_{-\infty}^{c} n(x \,\big|\, 2c + \mu_{\mathrm{V}}\tau, \sigma_{\mathrm{V}}^2\tau)\; \exp\left[\frac{2(c-x)\mu_{\mathrm{AV}}}{\sigma_{\mathrm{AV}}^2}\right] \Phi(-\mu_{\mathrm{AV}} d' \,\big|\, c - x, \sigma_{\mathrm{AV}}^2 d')\, dx
\end{aligned}
$$

with $d' = d - \tau$ (Eq. 8). The four integrands (denote them by p_2, p_3, p_4, p_5) can be rearranged to match $\exp(ux)\, n(x \mid m_1, s_1^2)\, \Phi(x \mid m_2, s_2^2)$ so that owen10_010.1x can be used for evaluation:

$$
\begin{aligned}
&\int_{-\infty}^{c} w(x, \tau \,\big|\, c, \mu_{\mathrm{V}}, \sigma_{\mathrm{V}}^2) \cdot G(d - \tau \,\big|\, c - x, \mu_{\mathrm{AV}}, \sigma_{\mathrm{AV}}^2)\, dx \\
&\qquad = \int_{-\infty}^{c} n(x \,\big|\, \mu_{\mathrm{V}}\tau, \sigma_{\mathrm{V}}^2\tau)\; \Phi(x \,\big|\, c - \mu_{\mathrm{AV}} d', \sigma_{\mathrm{AV}}^2 d')\, dx \\
&\qquad + \exp\left(\frac{2c\mu_{\mathrm{AV}}}{\sigma_{\mathrm{AV}}^2}\right) \int_{-\infty}^{c} \exp\left(-\frac{2\mu_{\mathrm{AV}}}{\sigma_{\mathrm{AV}}^2} x\right) n(x \,\big|\, \mu_{\mathrm{V}}\tau, \sigma_{\mathrm{V}}^2\tau)\; \Phi(x \,\big|\, c + \mu_{\mathrm{AV}} d', \sigma_{\mathrm{AV}}^2 d')\, dx \\
&\qquad - \exp\left(\frac{2c\mu_{\mathrm{V}}}{\sigma_{\mathrm{V}}^2}\right) \int_{-\infty}^{c} n(x \big| 2c + \mu_{\mathrm{V}}\tau, \sigma_{\mathrm{V}}^2\tau)\; \Phi(x \,\big|\, c - \mu_{\mathrm{AV}} d', \sigma_{\mathrm{AV}}^2 d')\, dx \\
&\qquad - \exp\left(\frac{2c\mu_{\mathrm{V}}}{\sigma_{\mathrm{V}}^2} + \frac{2c\mu_{\mathrm{AV}}}{\sigma_{\mathrm{AV}}^2}\right) \int_{-\infty}^{c} \exp\left(-\frac{2\mu_{\mathrm{AV}}}{\sigma_{\mathrm{AV}}^2} x\right) n(x \,\big|\, 2c + \mu_{\mathrm{V}}\tau, \sigma_{\mathrm{V}}^2\tau)\; \Phi(x \,\big|\, c + \mu_{\mathrm{AV}} d', \sigma_{\mathrm{AV}}^2 d')\, dx
\end{aligned}
$$

```
#
# Accuracy in asynchronous stimuli (Eq. 8)
#
acc_async = function(d=865, c=100, mua=0.08, sigmaa2=10.5, mub=0.27, sigmab2=45.7, tau=240)
{
	# before tau
	p1 = pinvGauss(tau, nu=c/mua, lambda=c*c/sigmaa2)

	# after tau: Integral w(x, tau) * acc_sync(d-tau, c-x, mua+mub, sigmaa2+sigmab2)
	d_ = d - tau
	p2 = owen10_010.1x(u=0,
		ma=mua*tau,
		sa=sqrt(sigmaa2*tau),
		mb=c - (mua+mub)*d_,
		sb=sqrt((sigmaa2+sigmab2)*d_),
		y=c)

	p3 = exp(2*c*(mua+mub)/(sigmaa2+sigmab2)) *
		owen10_010.1x(u=-2*(mua+mub)/(sigmaa2+sigmab2),
		ma=mua*tau,
		sa=sqrt(sigmaa2*tau),
		mb=c + (mua+mub)*d_,
		sb=sqrt((sigmaa2+sigmab2)*d_),
		y=c)

	p4 = exp(2*c*mua/sigmaa2) *
		owen10_010.1x(u=0,
		ma=2*c + mua*tau, sa=sqrt(sigmaa2*tau),
		mb=c - (mua+mub)*d_,
		sb=sqrt((sigmaa2+sigmab2)*d_),
		y=c)

	p5 = exp(2*c*mua/sigmaa2 + 2*c*(mua+mub)/(sigmaa2+sigmab2)) *
		owen10_010.1x(u=-2*(mua+mub)/(sigmaa2+sigmab2),
		ma=2*c + mua*tau,
		sa=sqrt(sigmaa2*tau),
		mb=c + (mua+mub)*d_,
		sb=sqrt((sigmaa2+sigmab2)*d_),
		y=c)

	# return value (only the first one is relevant, the others are reused for mrt_async)
	c(acc=p1 + p2 + p3 - p4 - p5, p2=p2, p3=p3, p4=p4, p5=p5)
}
```

For example, accuracy of Monkey S at A240V (low intensity) is predicted as

```
acc_async(d=865, c=100, mua=0.08, sigmaa2=10.5, mub=0.27, sigmab2=45.7, tau=240)
```

which yields about 91%.

Mean response time

Unimodal and synchronous stimuli

For synchronous or unimodal stimuli, the expected detection time, conditional on detection within the deadline, corresponds to the integral of $t * g(t)$ from 0 to d (Schwarz, Eq. 6) normalized by the probability that stimulus detection occurs at all (given by acc_sync),

$$E[\mathbf{D}_{\mathrm{AV}} \mid \mathbf{D}_{\mathrm{AV}} \leq d] = \frac{\int_0^d t \cdot g\left(t \mid c, \mu_{\mathrm{AV}}, \sigma^2_{\mathrm{AV}}\right) dt}{P(\mathbf{D}_{\mathrm{AV}} \leq d)}$$

In order to derive a prediction for the mean response time $E[\mathbf{T} \mid \mathbf{D} \leq d]$, a context-invariant residual μ_M is added to $E[\mathbf{D} \mid \mathbf{D} \leq d]$ (see Eq. 11 above). The implementation in R is straightforward. It uses acc_sync given above for the probability that stimulus detection occurs at all, as well as the helper function exp_pnorm for the second term in the expression for $E[\mathbf{D} \mid \mathbf{D} \leq d]$.

```
#
# Mean RT in unimodal and synchronous stimuli
#
mrt_sync = function(d=865, c=100, mu=0.08, sigma2=10.5, mum=322)
{
        # E(D | D < d) (Schwarz, 1994, Eq. 6)
        ed = c/mu * (pnorm(mu*d, c, sqrt(sigma2*d)) -
                exp_pnorm(2*c*mu/sigma2, (-c-mu*d)/sqrt(sigma2*d)))

        # Probability that detection occurs
        pd = acc_sync(d, c, mu, sigma2)

        # return value
        ed/pd + mum
}
```

For example, the mean reaction time of Monkey S for low intensity auditory stimuli is predicted as

```
        mrt_sync(d=865, c=100, mu=0.08, sigma2=10.5, mum=322)
```

which is about 777 ms.

Asynchronous stimuli

As before for accuracy, rather tedious calculations are required for the derivation of expected values for stimulus detection within the deadline in asynchronous stimuli, say V(τ)A (as before, we assume $\tau < d$).The expected time for stimulus detection (Eq. 10) then decomposes into a time interval [0, τ] for which the conditional detection time is given by the integral of $t * g(t)$ from 0 to τ (Schwarz, 1994, Eq. 6), and into a second time interval [τ, d] for which the conditional detection time is given by the weighted sum of the probability all possible states $X(t) < c$ (Eq. 5), multiplied by the conditional detection time within $d - \tau$ (which itself is an integral of the type in Schwarz, 1994, Eq. 6). The other terms in Equation 10 are normalization factors.

Solutions for $P\big(\mathbf{D}_{\mathrm{V}(\tau)\mathrm{A}} \le d\big)$, $H(\tau \mid c, \mu_\mathrm{V}, \sigma_\mathrm{V}^2)$ and $\int_{-\infty}^{c} w(x, \tau \mid c, \mu_\mathrm{V}, \sigma_\mathrm{V}^2) \cdot G(d - \tau \mid c - x, \mu_\mathrm{AV}, \sigma_\mathrm{AV}^2)\, dx$ have already been described above. The integral of $w(x, \tau) \times H(d - \tau)$ is expanded to four integrals of exp(ux) $n(x \mid m_1, s_1^2)$ $\Phi(x \mid m_2, s_2^2)$ and to four other integrals of the form x exp(ux) $n(x \mid m_1, s_1^2)$ $\Phi(x \mid m_2, s_2^2)$:

$$
\begin{aligned}
&\int_{-\infty}^{c} w(x, \tau \mid c, \mu_\mathrm{V}, \sigma_\mathrm{V}^2) \cdot H(d - \tau \mid c - x, \mu_\mathrm{AV}, \sigma_\mathrm{AV}^2)\, dx \\
&\quad = \frac{1}{\mu} \int_{-\infty}^{c} (c - x)\, n(x \mid \mu_\mathrm{V}\tau, \sigma_\mathrm{V}^2\tau)\, \Phi(x \mid c - \mu_\mathrm{AV} d', \sigma_\mathrm{AV}^2 d')\, dx \\
&\quad - \frac{1}{\mu} \exp\left(\frac{2c\mu_\mathrm{AV}}{\sigma_\mathrm{AV}^2}\right) \int_{-\infty}^{c} (c - x)\ \exp\left(-\frac{2\mu_\mathrm{AV}}{\sigma_\mathrm{AV}^2} x\right) n(x \mid \mu_\mathrm{V}\tau, \sigma_\mathrm{V}^2\tau)\, \Phi(x \mid c + \mu_\mathrm{AV} d', \sigma_\mathrm{AV}^2 d')\, dx \\
&\quad - \frac{1}{\mu} \exp\left(\frac{2c\mu_\mathrm{V}}{\sigma_\mathrm{V}^2}\right) \int_{-\infty}^{c} (c - x)\, n(x \mid 2c + \mu_\mathrm{V}\tau, \sigma_\mathrm{V}^2\tau)\, \Phi(x \mid c - \mu_\mathrm{AV} d', \sigma_\mathrm{AV}^2 d')\, dx \\
&\quad + \frac{1}{\mu} \exp\left(\frac{2c\mu_\mathrm{V}}{\sigma_\mathrm{V}^2} + \frac{2c\mu_\mathrm{AV}}{\sigma_\mathrm{AV}^2}\right) \\
&\quad \times \int_{-\infty}^{c} (c - x)\ \exp\left(-\frac{2\mu_\mathrm{AV}}{\sigma_\mathrm{AV}^2} x\right) n(x \mid 2c + \mu_\mathrm{V}\tau, \sigma_\mathrm{V}^2\tau)\, \Phi(x \mid c + \mu_\mathrm{AV} d', \sigma_\mathrm{AV}^2 d')\, dx
\end{aligned}
$$

The four minuends (with c) have already been solved above for the prediction of accuracy (p_2, p_3, p_4, p_5). The subtrahends can be fed directly into owen10_011.1x.

$$
\begin{aligned}
&\int_{-\infty}^{c} w(x, \tau \mid c, \mu_\mathrm{V}, \sigma_\mathrm{V}^2) \cdot H(d - \tau \mid c - x, \mu_\mathrm{AV}, \sigma_\mathrm{AV}^2)\, dx \\
&\quad = -\frac{1}{\mu_\mathrm{AV}} \int_{-\infty}^{c} x\, n(x \mid \mu_\mathrm{V}\tau, \sigma_\mathrm{V}^2\tau)\, \Phi(x \mid c - \mu_\mathrm{AV} d', \sigma_\mathrm{AV}^2 d')\, dx \\
&\quad + \frac{1}{\mu_\mathrm{AV}} \exp\left(\frac{2c\mu_\mathrm{AV}}{\sigma_\mathrm{AV}^2}\right) \int_{-\infty}^{c} x\ \exp\left(-\frac{2\mu_\mathrm{AV}}{\sigma_\mathrm{AV}^2} x\right) n(x \mid \mu_\mathrm{V}\tau, \sigma_\mathrm{V}^2\tau)\, \Phi(x \mid c + \mu_\mathrm{AV} d', \sigma_\mathrm{AV}^2 d')\, dx \\
&\quad + \frac{1}{\mu_\mathrm{AV}} \exp\left(\frac{2c\mu_\mathrm{V}}{\sigma_\mathrm{V}^2}\right) \int_{-\infty}^{c} x\, n(x \mid 2c + \mu_\mathrm{V}\tau, \sigma_\mathrm{V}^2\tau)\, \Phi(x \mid c - \mu_\mathrm{AV} d', \sigma_\mathrm{AV}^2 d')\, dx \\
&\quad - \frac{1}{\mu_\mathrm{AV}} \exp\left(\frac{2c\mu_\mathrm{V}}{\sigma_\mathrm{V}^2} + \frac{2c\mu_\mathrm{AV}}{\sigma_\mathrm{AV}^2}\right) \int_{-\infty}^{c} x\ \exp\left(-\frac{2\mu_\mathrm{AV}}{\sigma_\mathrm{AV}^2} x\right) n(x \mid 2c + \mu_\mathrm{V}\tau, \sigma_\mathrm{V}^2\tau)\, \Phi(x \mid c + \mu_\mathrm{AV} d', \sigma_\mathrm{AV}^2 d')\, dx \\
&\quad + \frac{c}{\mu_\mathrm{AV}} (p_2 - p_3 - p_4 + p_5)
\end{aligned}
$$

An implementation in R is shown on the next page. Because accuracy has to be determined anyway, the code can easily be optimized such that some of the calculations are performed only once. For this reason, acc_async returns the four components p_2, p_3, p_4, p_5 along with the predicted accuracy for asynchronous stimuli.

```
#
# Mean RT in asynchronous stimuli (Eq. 10)
#
mrt_async = function(d=865, c=100, mua=0.08, sigmaa2=10.5, mub=0.27, sigmab2=45.7, mum=322,
        tau=240)
{
        # probability that target is detected
        p = acc_async(d, c, mua, sigmaa2, mub, sigmab2, tau)

        # H(tau) (Schwarz, 1994, Eq. 6)
        Htau = c/mua * (pnorm(mua*tau, c, sqrt(sigmaa2*tau)) -
                exp_pnorm(2*c*mua/sigmaa2, (-c-mua*tau)/sqrt(sigmaa2*tau)))

        # tau * Integral w(x, tau) * G(d - tau)
        tauwG = p['p2'] + p['p3'] - p['p4'] - p['p5']

        # c/muAV * (p2 - p3 - p4 + p5)
        minuends = c/(mua + mub) * (p['p2'] - p['p3'] - p['p4'] + p['p5'])

        # New terms
        d_ = d - tau
        q1 = 1/(mua + mub) *
                owen10_011.1x(u=0,
                ma=mua*tau, sa=sqrt(sigmaa2*tau),
                mb=c - (mua+mub)*d_, sb=sqrt((sigmaa2+sigmab2)*d_), y=c)

        q2 = 1/(mua + mub) * exp(2*c*(mua+mub)/(sigmaa2+sigmab2)) *
                owen10_011.1x(u=-2*(mua+mub)/(sigmaa2+sigmab2),
                ma=mua*tau, sa=sqrt(sigmaa2*tau),
                mb=c + (mua+mub)*d_, sb=sqrt((sigmaa2+sigmab2)*d_), y=c)

        q3 = 1/(mua + mub) * exp(2*c*mua/sigmaa2) *
                owen10_011.1x(u=0,
                ma=2*c + mua*tau, sa=sqrt(sigmaa2*tau),
                mb=c - (mua+mub)*d_, sb=sqrt((sigmaa2+sigmab2)*d_), y=c)

        q4 = 1/(mua + mub) * exp(2*c*mua/sigmaa2 + 2*c*(mua+mub)/(sigmaa2+sigmab2)) *
                owen10_011.1x(u=-2*(mua+mub)/(sigmaa2+sigmab2),
                ma=2*c + mua*tau, sa=sqrt(sigmaa2*tau),
                mb=c + (mua+mub)*d_, sb=sqrt((sigmaa2+sigmab2)*d_), y=c)

        # return value
        1/unname(p['acc']) * (Htau + tauwG + minuends - q1 + q2 + q3 - q4) + mum
}
```

For Monkey S and V240A in low intensity, the function returns an expected mean RT of about 699 ms:

```
mrt_async(d=865, c=100, mua=0.08, sigmaa2=10.5, mub=0.27, sigmab2=45.7, mum=322,
tau=240)
```

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