TDDE07 Bayesian Learning - Lab 3

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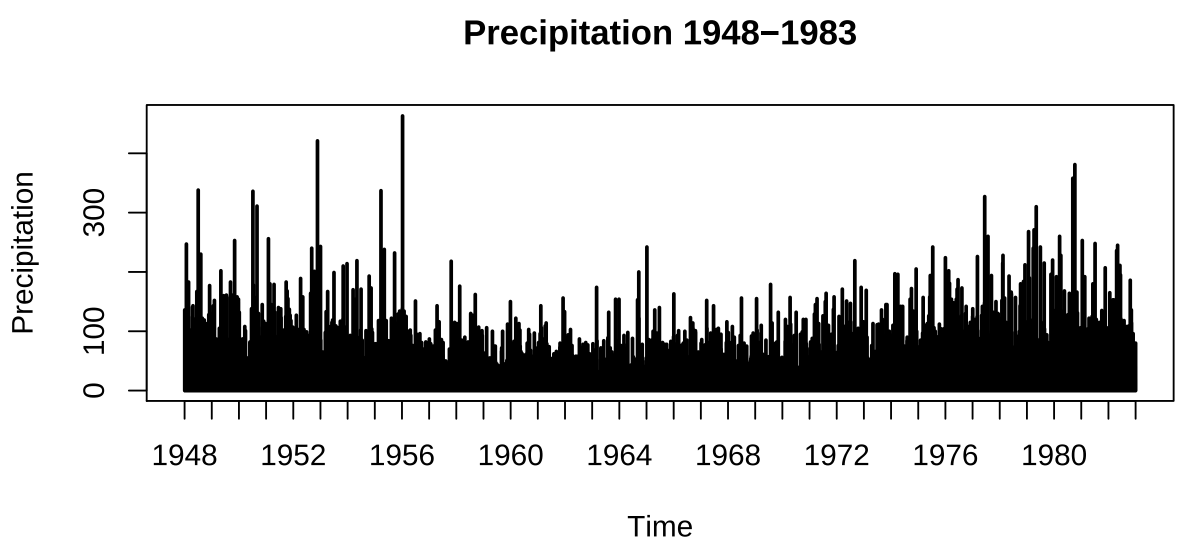
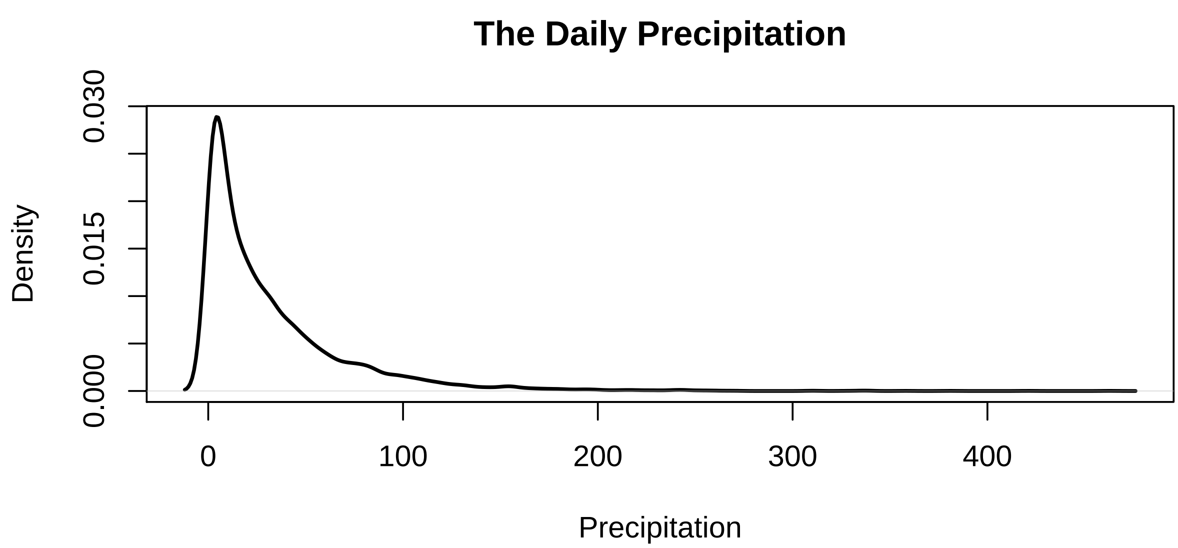
1. Normal model, mixture of normal model with semi-conjugate prior.

(a) Normal model

The historical precipitation and the kernel density of the data can be seen in Figure 1.





The convergence of the sampled parameters μ and σ^2 can be seen in Figure 2, where I have calculated the mean values of sequential draws during Gibbs sampling and plotted those values to show the autocorrelation between the draws. In this figure we see how the sampled parameters approaches constant values. The code can be found in Appendix A.

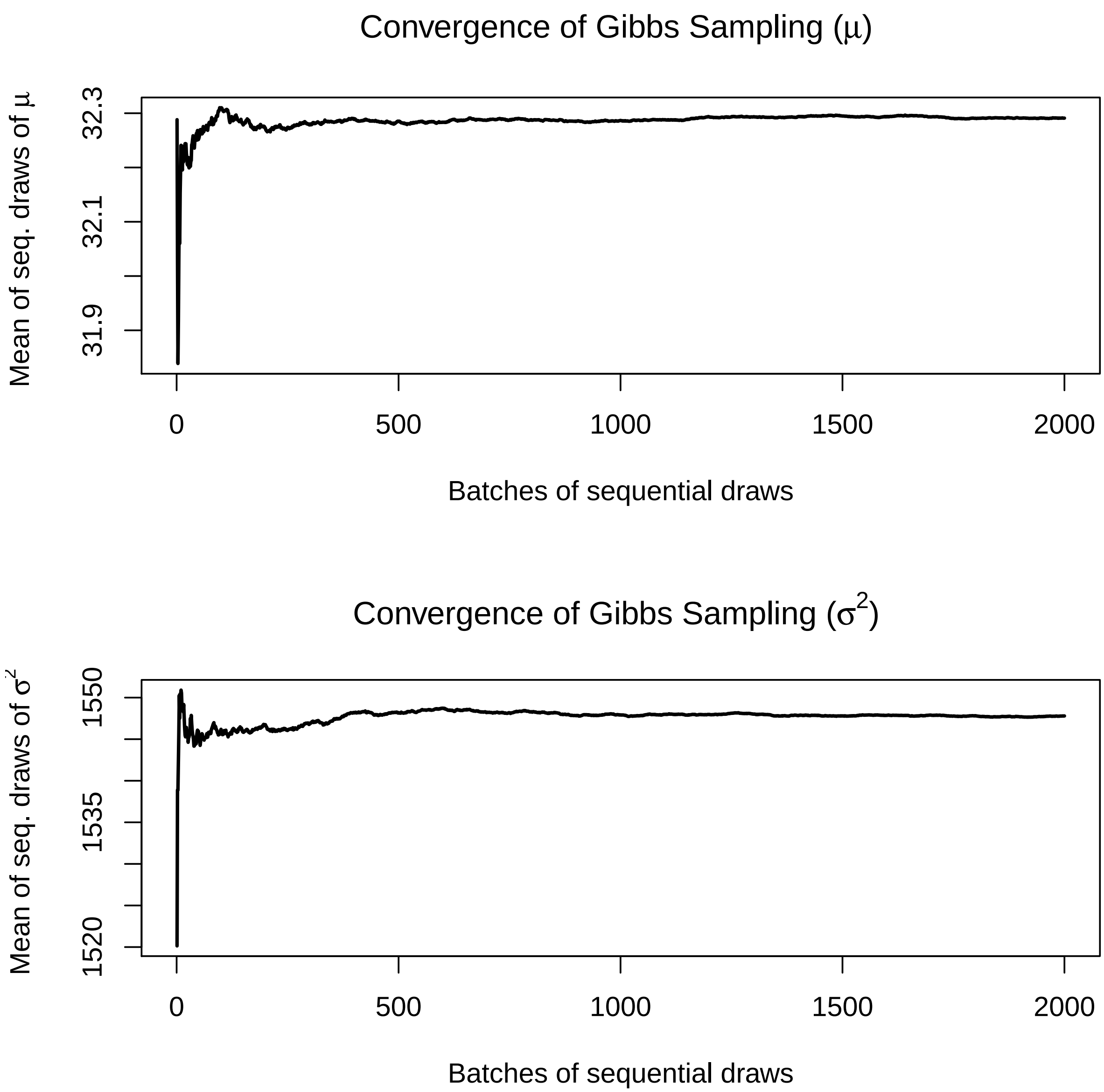


Figure 1: Predictions beta priors

(b) Mixture normal model.

I used Mattias implementation of the mixture of normals Gibbs sampler to approximate the density of the data. The results can be seen in Figure 3. In this figure we see that a mixture of normals does a pretty good job approximating the kernel density of the data. In the figure I have also plotted the convergence of the hyper parameters of the normals during the Gibbs sampling. The additions to Mattias code made to plot the convergence of the parameters during sampling can be seen in Appendix B.

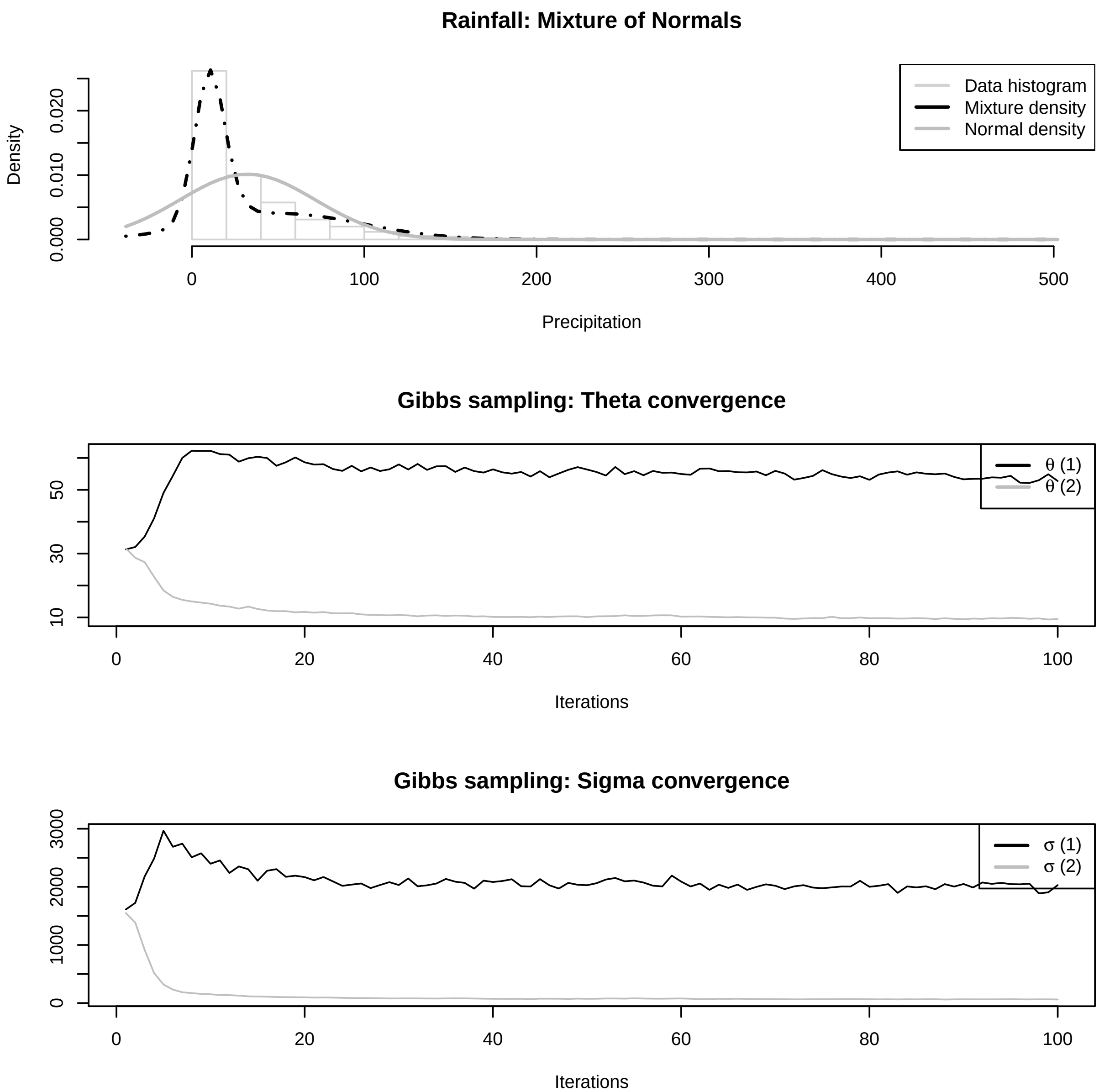


Figure 2: Mixture normal model

(c) Graphical comparison.

A comparison between the kernel density estimate of the data, a normal approximation by Gibbs sampling with parameters derived in (a) and a mixture of normals from (b) can be seen in Figure 4. Comparing a normal approximation with a mixture of normals approximation of the data - it is clear that a mixture of normals is a much better approximation of the density of the data.

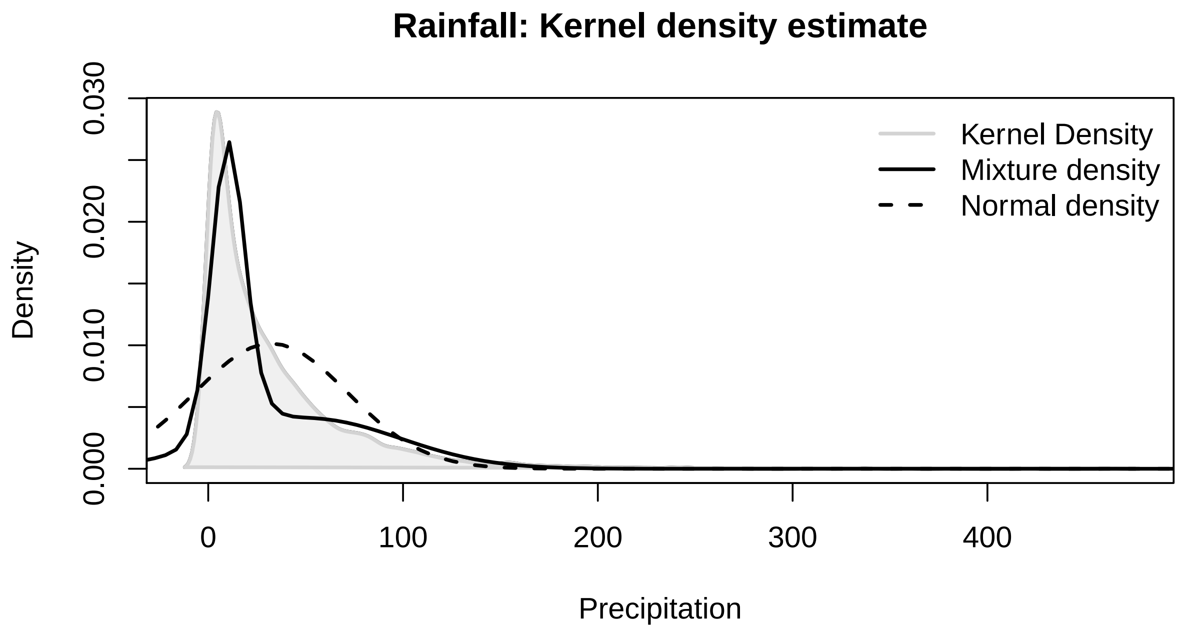


Figure 3: Mixture normal model

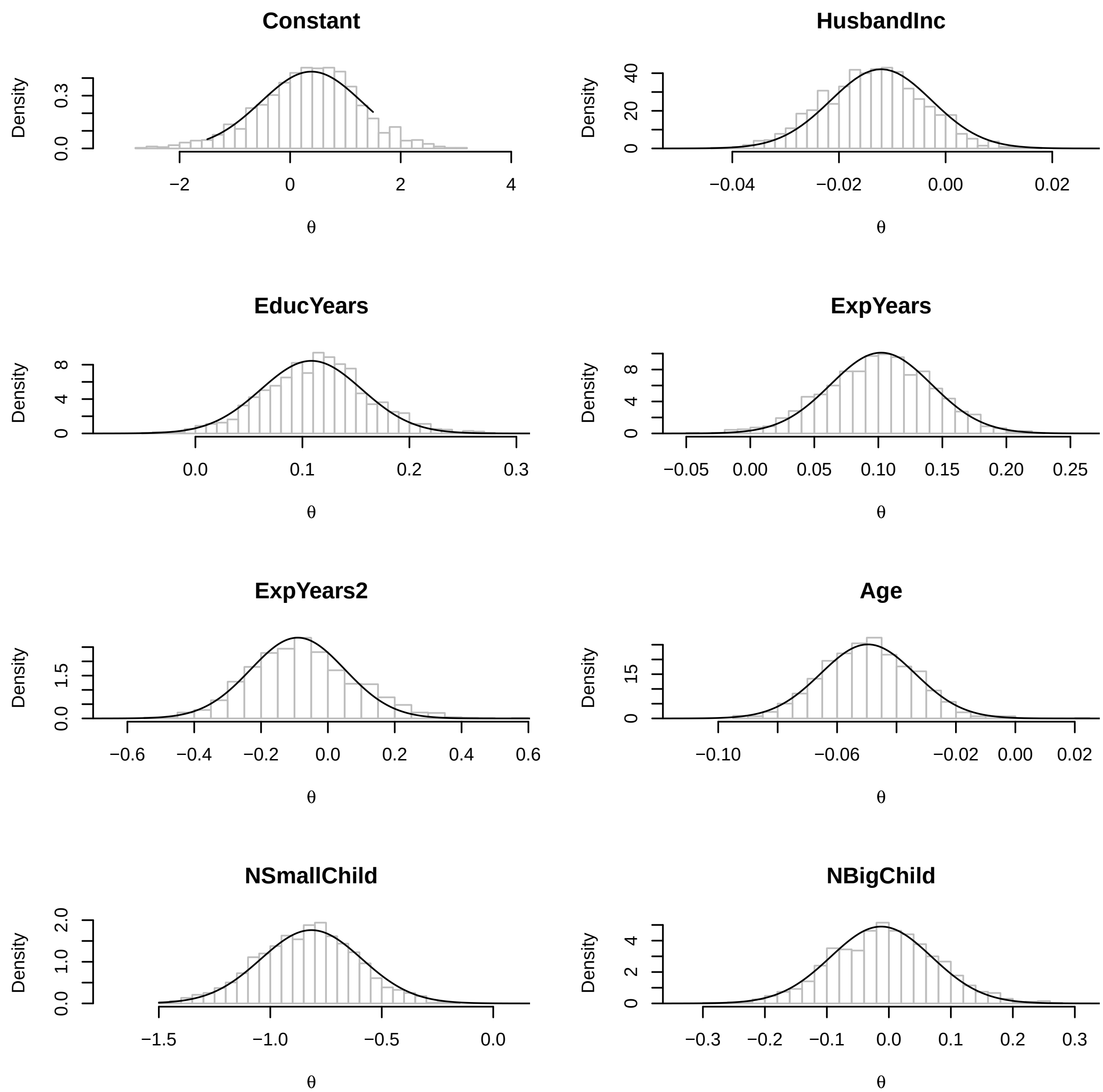
2. Probit regression

(a) and (b)

The implementation can be found in Appendix C.

(c)

A comparison between the beta parameters draws drawn during Gibbs sampling (represented as histograms) and the normal approximation of the distributions of those parameters are plotted in Figure 5. In these plots we see that beta parameters drawn during Gibbs sampling closely resembles the optim results.



Appendix A

Assignment 1

```
require(MASS)
require(geoR)

grid_w = 5
grid_h = 4

# ----------------------
#  Lab 3 - Assignment 1
# ----------------------

data = read.table("data/rainfall.txt", header=FALSE)[,1]

n = length(data)

# -----
#  (a)
# -----

pdf("plots/3_1_1_precipitation.pdf")

par(mfrow=c(2,1))

# Plot the precipitation
plot(data,
     type="h",
     lwd=2,
     ylab="Precipitation",
     xlab="Time",
     xaxt="n",
     col="black",
     main="Precipitation 1948-1983")

axis(1,
     at=seq(0, n, n/(1983-1948)),
     labels=seq(1948, 1983))

# Plot the precipitation density
prec_density = density(data)
plot(prec_density,
     type="l",
     lwd=2,
     xlab="Precipitation",
     ylab="Density",
     main="The Daily Precipitation")

dev.off()

data_mean = mean(data)
```

```r
# Prior parameters for sigma (variance)
v0 = 1
sigma_sq0 = 1 / v0

n_draws = 4000

# Initial value for sigma
sigma_sq = rinvchisq(n=1, v0, sigma_sq0)


gibbs_draws = matrix(0,n_draws,2)
for(i in 1:n_draws){
  mu = rnorm(n=1, mean=data_mean, sd=sqrt(sigma_sq/n))
  sigma_sq = rinvchisq(n=1, v0 + n, (v0*sigma_sq0 + sum((data - mu)^2))/(n + v0))
  gibbs_draws[i,] = c(mu, sigma_sq)
}


mean_draws = gibbs_draws[,1]
var_draws = gibbs_draws[,2]


# Calculate mean of batches of 2 draws to visualize the
# auto correlation between sequential draws
mean_means = c()
mean_vars = c()
for (i in 1:n_draws){
  if(i%%2 == 0){
    mean_means = c(mean_means, mean(mean_draws[i-1:i]))
    mean_vars = c(mean_vars, mean(var_draws[i-1:i]))
  }
}

# Plots displaying convergence of the Normal hyper
# parameters during sampling

pdf("plots/3_1_1_gibbs_sampl_conv.pdf")

par(mfrow=c(2,1))

# Plot the auto correlation (convergence) between draws of mu
min_mean = min(mean_means)
max_mean = max(mean_means)
plot(mean_means,
     type="l",
     ylim=c(min_mean, max_mean),
     cex=.1,
     lwd=2,
     main=expression(paste("Convergence of Gibbs Sampling ", "(", mu, ")", sep=" ")),
     xlab="Batches of sequential draws",
     ylab=expression(paste("Mean of seq. draws of ", mu, sep=" ")))
```

```r
# Plot the auto correlation (convergence) between draws of sigma
min_var = min(mean_vars)
max_var = max(mean_vars)
plot(mean_vars,
     type="l",
     ylim=c(min_var, max_var),
     cex=.1,
     lwd=2,
     main=expression(paste("Convergence of Gibbs Sampling ", "(", sigma^2, ")", sep=" ")),
     xlab="Batches of sequential draws",
     ylab=expression(paste("Mean of seq. draws of ", sigma^2, sep=" ")))

dev.off()

# -----
#  (c)
# -----

pdf("plots/3_1_3_dens_comp.pdf", height=grid_h)

kernel_density = density(data)

plot(kernel_density$x,
        kernel_density$y,
        type="l",
        cex=.1,
        lwd=2,
        ylab="Density",
        xlab="Precipitation",
        main="Rainfall: Kernel density estimate")

col1 = rgb(240, 240, 240, maxColorValue=255)
col1_b = "lightgray"
col2 = "black"
col3 = "black"

# Kernel density
polygon(kernel_density$x,
     kernel_density$y,
     col=col1,
     border=col1_b,
     lwd=2)


# Mixture of normals from (b)
# (run './template/gaussian_mixture.R' first)
lines(xGrid,
      mixDensMean,
      type="l",
      lwd=2,
      col=col2)
```

```r
# Normal density
mean = mean(mean_draws)
std_dev = sqrt(mean(var_draws))
normal_density = dnorm(xGrid, mean=mean, sd=std_dev)

lines(xGrid,
      normal_density,
      col=col3,
      lwd=2,
      lty=2)


legend("topright",
       box.lty = 0,
       legend = c("Kernel Density","Mixture density","Normal density"),
       col=c(col1_b, col2, col3),
       lty=c(1,1,2),
       lwd=2)


dev.off()
```

Appendix B

Assignment 2

```
require(mvtnorm)
require(msm)
require(MASS)
library(LaplacesDemon)

grid_w = 6
grid_h = 5

# -------
#  Lab 3
# -------


# Read data
data <- read.table("./data/WomenWork.dat", header = TRUE)

feature_labels = colnames(data)

y = as.vector(data$Work)
X = as.matrix(data[, 2:ncol(data)])

# Data spec.
n_features = ncol(X)
n_samples = nrow(X)

tau = 10


# -------------
#  (a) and (b)
# -------------


# Beta prior parameters
mu0 = rep(0, n_features)
covar0 = diag(tau^2, n_features)
Omega0 = solve(covar0)                # MV: Ni har blandat ihop Omega0 (prior precision) som är med i m
# med covar0 som är prior covariance matrix.
# Omega0 = inv(covar0)

draw_beta = function(y) {

  X_X = t(X) %*% X

  # Least squares approximate of beta
  beta_hat = ginv(X_X) %*% t(X) %*% y

  # Posterior parameters
  mu_n = ginv(X_X + Omega0)%*%(X_X%*%beta_hat+Omega0%*%mu0)
```

```r
  Omega_n = X_X + Omega0

  # Assuming sigma_sq = 1
  b_draw = rmvnorm(1, mean=mu_n, sigma=ginv(Omega_n))

  return(b_draw)
}

draw_u = function(beta) {

  # Mean of predictive distr.
  regr_mean = X %*% t(beta)

  u = rep(0, n_samples)
  for(i in 1:n_samples) {
    y_i = y[i]

    if(y_i == 0){
      # Truncate [-inf, 0)
      u[i] = rtnorm(n=1, mean=regr_mean[i], sd=1, upper=0)
    }else{
      # Truncate (0, inf]
      u[i] = rtnorm(n=1, mean=regr_mean[i], sd=1, lower=0)
    }
  }

  return(u)
}

# Initial prediction
u = rnorm(n_samples, covar0)

n_draws = 1500
beta_draws = matrix(0, n_draws, n_features)
u_draws = matrix(0, n_draws, n_samples)
for(i in 1:n_draws) {
  beta = draw_beta(u)
  u = draw_u(beta)
  beta_draws[i,] = beta
  u_draws[i,] = u
}

# Avoid first 10% of the draws
burn_in = floor(n_draws / 10)
beta_draws = beta_draws[burn_in:nrow(beta_draws),]

# -----
#  (c)
# -----


# Calculate the log posterior
LogPosteriorProbit <- function(betas, y, X, mu, Sigma){
```

```r
  # Multiply data by parameters to get predictions
  predictions <- X%*%betas;

  # Log likelihood (for probit)
  log_likelihood = sum(y*pnorm(predictions, log.p = TRUE) +
                       (1-y)*pnorm(predictions, log.p = TRUE, lower.tail = FALSE))

  # Log prior
  log_prior <- dmvnorm(betas, mu0, covar0, log=TRUE);

  # Sum of log likelihood and log prior is log posterior
  return(log_likelihood + log_prior)
}

log_posterior = LogPosteriorProbit

# Initialize as zeros
init_betas = rep(0, n_features)

opt_results = optim(init_betas,
                    log_posterior,
                    gr=NULL,
                    y,
                    X,
                    mu0,
                    covar0,
                    method=c("BFGS"),
                    control=list(fnscale=-1),
                    hessian=TRUE)

# Posterior mode (beta hat)
post_mode = opt_results$par
# Posterior covariance (J^-1(beta hat))
post_cov = -solve(opt_results$hessian)

pdf("plots/3_2_3_norm_gibbs_comp.pdf")

par(mfrow=c(4,2))

beta_grid = seq(-1.5, 1.5, 0.001)
for (i in 1:n_features) {

  # Build histogram of Gibbs draws of beta_i
  h = hist(beta_draws[,i], breaks=30, plot=FALSE)

  # Get the normal approximation for beta_i via optim.
  mean = post_mode[i]
  std_dev = sqrt(post_cov[i,i])   # MV: Här hade ni: std_dev = sqrt(post_cov[i,i]/n_features)
  norm_approx = dnorm(x=beta_grid, mean=mean, sd=std_dev)

  # Find x- and y-limits for plot
  min_x = min(c(min(beta_draws[,i]), mean-4*std_dev))
  max_x = max(c(max(beta_draws[,i]), mean+4*std_dev))
```

```
  max_y = max(c(max(norm_approx), max(h$density)))

  # Plot the histogram
  plot(h,
       freq=FALSE,
       xlim=c(min_x,max_x),
       ylim=c(0,max_y),
       border="gray",
       xlab=expression(theta),
       main=feature_labels[i+1])

  # Plot the normal approximation
  lines(beta_grid, norm_approx)
}

dev.off()
```

Appendix C

Modifications to Mattias impl. of Mixture of Normals

```
# ------------------------------
#  Lab 3 - Assignment 2 (b)
# ------------------------------

# Estimating a simple mixture of normals
# Author: Mattias Villani, IDA, Linköping University. http://mattiasvillani.com

##########    BEGIN USER INPUT #################
# Data options
data(faithful)
rawData <- faithful
x <- as.matrix(rawData['eruptions'])

# Lab 3 -
data = read.table("data/rainfall.txt", header=FALSE)[,1]
x = as.matrix(data)

# Model options
nComp <- 2    # Number of mixture components

# Prior options
alpha <- 10*rep(1,nComp) # Dirichlet(alpha)
muPrior <- rep(0,nComp) # Prior mean of theta
tau2Prior <- rep(10,nComp) # Prior std theta
sigma2_0 <- rep(var(x),nComp) # s20 (best guess of sigma2)
nu0 <- rep(4,nComp) # degrees of freedom for prior on sigma2

# MCMC options
nIter <- 100 # Number of Gibbs sampling draws

# Plotting options
plotFit <- TRUE
lineColors <- c("blue", "green", "magenta", 'yellow')
sleepTime <- 0.1 # Adding sleep time between iterations for plotting
################   END USER INPUT ###############

###### Defining a function that simulates from the
rScaledInvChi2 <- function(n, df, scale){
  return((df*scale)/rchisq(n,df=df))
}

####### Defining a function that simulates from a Dirichlet distribution
rDirichlet <- function(param){
  nCat <- length(param)
  thetaDraws <- matrix(NA,nCat,1)
  for (j in 1:nCat){
    thetaDraws[j] <- rgamma(1,param[j],1)
  }
  # Diving every column of ThetaDraws by the sum of the elements in that column.
```

```r
  thetaDraws = thetaDraws/sum(thetaDraws)
  return(thetaDraws)
}

# Simple function that converts between two different
# representations of the mixture allocation
S2alloc <- function(S){
  n <- dim(S)[1]
  alloc <- rep(0,n)
  for (i in 1:n){
    alloc[i] <- which(S[i,] == 1)
  }
  return(alloc)
}

# Initial value for the MCMC
nObs <- length(x)
# nObs-by-nComp matrix with component allocations.
S <- t(rmultinom(nObs, size = 1 , prob = rep(1/nComp,nComp)))
theta <- quantile(x, probs = seq(0,1,length = nComp))
sigma2 <- rep(var(x),nComp)
probObsInComp <- rep(NA, nComp)

# Setting up the plot
xGrid <- seq(min(x)-1*apply(x,2,sd),max(x)+1*apply(x,2,sd),length = 100)
xGridMin <- min(xGrid)
xGridMax <- max(xGrid)
mixDensMean <- rep(0,length(xGrid))
effIterCount <- 0
ylim <- c(0,2*max(hist(x)$density))

gibbs_thetas = matrix(0,nIter,2)
gibbs_sigmas = matrix(0,nIter,2)
for (k in 1:nIter){
  message(paste('Iteration number:',k))
  # Just a function that converts between different representations
  # of the group allocations
  alloc <- S2alloc(S)
  nAlloc <- colSums(S)
  print(nAlloc)
  # Update components probabilities
  w <- rDirichlet(alpha + nAlloc)

  # Update theta's
  for (j in 1:nComp){
    precPrior <- 1/tau2Prior[j]
    precData <- nAlloc[j]/sigma2[j]
    precPost <- precPrior + precData
    wPrior <- precPrior/precPost
    muPost <- wPrior*muPrior + (1-wPrior)*mean(x[alloc == j])
    tau2Post <- 1/precPost
    theta[j] <- rnorm(1, mean = muPost, sd = sqrt(tau2Post))
  }
```

```r
  gibbs_thetas[k, ] = theta

  # Update sigma2's
  for (j in 1:nComp){
    sigma2[j] <- rScaledInvChi2(1, df = nu0[j] + nAlloc[j],
       scale = (nu0[j]*sigma2_0[j] +
                  sum((x[alloc == j] - theta[j])^2))/(nu0[j] + nAlloc[j]))
  }

  gibbs_sigmas[k,] = sigma2

  # Update allocation
  for (i in 1:nObs){
    for (j in 1:nComp){
      probObsInComp[j] <- w[j]*dnorm(x[i], mean = theta[j], sd = sqrt(sigma2[j]))
    }
    S[i,] <- t(rmultinom(1, size = 1 , prob = probObsInComp/sum(probObsInComp)))
  }

  # Printing the fitted density against data histogram
  if (plotFit && (k%%1 ==0)){
    effIterCount <- effIterCount + 1
    hist(x, breaks = 20, freq = FALSE, xlim = c(xGridMin,xGridMax),
         main = paste("Iteration number",k), ylim = ylim)
    mixDens <- rep(0,length(xGrid))
    components <- c()
    for (j in 1:nComp){
      compDens <- dnorm(xGrid,theta[j],sd = sqrt(sigma2[j]))
      mixDens <- mixDens + w[j]*compDens
      lines(xGrid, compDens, type = "l", lwd = 2, col = lineColors[j])
      components[j] <- paste("Component ",j)
    }
    mixDensMean <- ((effIterCount-1)*mixDensMean + mixDens)/effIterCount

    lines(xGrid, mixDens, type = "l", lty = 2, lwd = 3, col = 'red')
    legend("topleft", box.lty = 1, legend = c("Data histogram",components, 'Mixture'),
           col = c("black",lineColors[1:nComp], 'red'), lwd = 2)
    Sys.sleep(sleepTime)
  }

}


# Calculate mean of batches of 2 draws to visualize the
# auto correlation between sequential draws
t1 = c()
t2 = c()
s1 = c()
s2 = c()
for (i in 1:nIter){
  if(i%%2 == 0){
    t1 = c(t1, mean(gibbs_thetas[,1][i-1:i]))
    t2 = c(t2, mean(gibbs_thetas[,2][i-1:i]))
```

```
    s1 = c(s1, mean(gibbs_sigmas[,1][i-1:i]))
    s2 = c(s2, mean(gibbs_sigmas[,2][i-1:i]))
  }
}


# Plots displaying convergence of the Normal hyper
# parameters during sampling

pdf("plots/3_1_2_gibbs_mixt.pdf")

par(mfrow=c(3,1))

# Plot comparison between kernel density, mixture of normals and a normal
# approximation
hist(x, breaks = 20,
     cex=.1,
     border="lightgray",
     freq = FALSE,
     xlim = c(xGridMin,xGridMax),
     xlab="Precipitation",
     ylab="Density",
     main = "Rainfall: Mixture of Normals")

lines(xGrid,
      mixDensMean,
      type = "l",
      lwd = 2,
      lty = 4,
      col = "black")

lines(xGrid,
      dnorm(xGrid, mean = mean(x), sd = apply(x,2,sd)),
      type = "l",
      lwd = 2,
      col = "gray")

legend("topright",
       box.lty = 1,
       legend = c("Data histogram","Mixture density","Normal density"),
       col=c("lightgray","black","gray"),
       lwd = 2)


# Plot the auto correlation (convergence) between draws of mu
min_t = min(c(min(t1), min(t2)))
max_t = max(c(max(t1), max(t2)))
plot(t1,
     type="l",
     ylim=c(min_t, max_t),
     cex=.1,
     lwd=2,
     main=expression(
```

```r
       paste("Convergence of Gibbs Sampling ", "(", theta, ")", sep=" ")
       ),
     xlab="Batches of sequential draws",
     ylab=expression(paste("Mean of seq. draws of ", theta, sep=" ")))

lines(t2, lwd=2, col="gray")

legend("topright",
       box.lty = 1,
       legend = c(expression(paste(theta, " (1)", sep=" ")),
                  expression(paste(theta, " (2)", sep=" "))),
       col=c("black","gray"),
       lwd = 2)

# Plot the auto correlation (convergence) between draws of sigma
min_s = min(c(min(s1), min(s2)))
max_s = max(c(max(s1), max(s2)))
plot(s1,
     type="l",
     ylim=c(min_s, max_s),
     cex=.1,
     lwd=2,
     main=expression(
       paste("Convergence of Gibbs Sampling ", "(", sigma^2, ")", sep=" ")
       ),
     xlab="Batches of sequential draws",
     ylab=expression(paste("Mean of seq. draws of ", sigma^2, sep=" ")))

lines(s2, lwd=2, col="gray")

legend("topright",
       box.lty = 1,
       legend = c(expression(paste(sigma^2, " (1)", sep=" ")),
                  expression(paste(sigma^2, " (2)", sep=" "))),
       col=c("black","gray"),
       lwd = 2)


dev.off()

#########################    Helper functions    #######################
```