The Effect of Goals and Environments on Human Performance in Optimal Stopping Problems (Supplementary Information)

Maime Guan and Michael D. Lee
Department of Cognitive Sciences
University of California, Irvine

Abstract

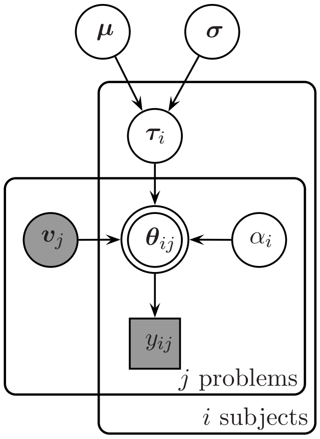
This supplementary information gives details of the formal implementation of the graphical model on which the main results in the paper are based. It also gives details of three additional modeling analyses. The first is a process-oriented variant of the basic model. The second is an examination of the lack of learning, comparing thresholds inferred from the first half versus the last half of problems. The third is a model-based analysis of the assumption that immediately preceding values do not affect thresholds.

Graphical Models

We developed all of our models as graphical models, which provides a flexible, powerful, and progressively more widely-used formalism for defining probabilistic models of cognition (Jordan, 2004; Lee & Wagenmakers, 2013). The graphical models were implemented using JAGS, software that facilitates MCMC-based computational Bayesian inference (Plummer, 2003). All of our modeling results are based on 4 chains of 2000 samples each, collected after 2000 discarded burn-in samples. The chains were verified for convergence using the standard $\hat{R}$ statistic (Brooks & Gelman, 1997).

Model for Inferring Thresholds

Figure 1 shows the graphical model for inferring thresholds. Latent parameters and observed data are represented as nodes in a graph, and the structure of the

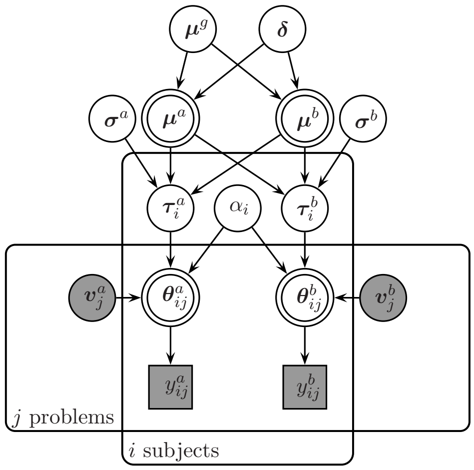


$$
\begin{aligned}
\mu_k &\sim \text{Uniform}\big(0, 100\big) \\
\sigma_k &\sim \text{Uniform}\big(0, 30\big) \\
\alpha_i &\sim \text{Uniform}\big(0, 1\big) \\
\tau_{ik} &\sim \text{TruncatedGaussian}_{(0,100)}\big(\mu_k, 1/\sigma_k^2\big), \\
\theta_{ijk} &= \begin{cases} \alpha_i & \text{if } k = \min\big(\{k' : v_{jk'} > \tau_{ik'}, v_{jk'} = \max\{v_{j1}, ..., v_{jk'}\}\}, 5\big) \\ \frac{1-\alpha_i}{4} & \text{otherwise,} \end{cases} \\
y_{ij} &\sim \text{Categorical}\big(\boldsymbol{\theta}_{ij}\big),
\end{aligned}
$$

Figure 1. Graphical model representation of the model for inferring thresholds.

graph indicates how parameters are assumed to control probabilistic processes that produce the data. Unobserved parameters are shown as unshaded nodes, while the observed participant decisions y_{ij} and values of the alternativess v_j are shown as shaded nodes. Continuous values are represented by circular nodes, while discrete values are represented by square nodes. Double-bordered nodes are deterministic, in the sense that they are defined as functions of other nodes, and are only included for semantic clarity. Bounding plates encompass parts of the graph that are replicated within the overall model.

In Figure 1, the thresholds for the ith participant correspond to the node $\boldsymbol{\tau}_i = (\tau_{i1}, \ldots, \tau_{i4})$. The threshold for this particpant in the kth position in the problem sequence, τ_{ik}, is drawn from an overarching truncated Gaussian distribution that is common to all participants for that position. These truncated Gaussians for each of the positions have means $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_4)$ and standard deviations $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_4)$. The probability of each of the five choices the ith participant can make on the jth problem are represented by $\boldsymbol{\theta}_{ij} = (\theta_{ij1}, \ldots, \theta_{ij5})$. The specific probability that the ith participant chooses the presented value on in the kth position on the jth problem corresponds to θ_{ijk}, and depends on the threshold τ_{ik}, the known presented value v_{jk}, and the accuracy of execution parameter α_i for that participant. For the maximum goal case shown in Figure 1, the value is above the threshold—which is always true for the last position, corresponding to assuming $\theta_{ij5} = 0$—and currently maximal, then $\theta_{ij} = \alpha_i$ so that option is chosen with high probability, otherwise $\theta_{ij} = (1 - \alpha_i)/4$,



$$
\begin{aligned}
\mu^g_k &\sim \text{Uniform}(0, 100)\\
\delta_k &\sim \text{Uniform}(-100, 100)\\
u^a_k &= \mu^g_k + \delta_k/2\\
u^b_k &= \begin{cases} \mu^g_k - \delta_k/2 & \text{if } a, b \text{ have same goal}\\ 100 - \left(\mu^g_k - \delta_k/2\right) & \text{if } a, b \text{ have different goal} \end{cases}\\
\alpha_i &\sim \text{Uniform}(0, 1)\\
\tau^c_{ik} &\sim \text{TruncatedGaussian}_{(0,100)}\left(\mu^c_k, 1/(\sigma^c_k)^2\right),\ c = a, b\\
\theta^c_{ijk} &= \begin{cases} \alpha_i & \text{if } k = \min\left(\left\{k' : v^c_{jk'} > \tau^c_{ik'}, v^c_{jk'} = \max\{v^c_{j1}, \ldots, v^c_{jk'}\}\right\}, 5\right)\\ \frac{1-\alpha_i}{4} & \text{otherwise},\ c = a, b \end{cases}\\
y^c_{ij} &\sim \text{Categorical}\left(\boldsymbol{\theta}^c_{ij}\right),\ c = a, b
\end{aligned}
$$

Figure 2. Graphical model representation of the model for comparing differences in thresholds across conditions, enabling the estimation of Bayes factors between experimental conditions.

and the option is unlikely to be chosen. For the minimum goal case, the value must be currently minimal and below the threshold, and the threshold for the final position is assumed to be 100. In either case, the observed choice $y_{ij} = 1$ simply follows the $\boldsymbol{\theta}_{ij}$ probabilities, as a draw from the corresponding categorical distribution.

Model for Estimating Bayes Factors

Figure 2 shows the graphical model for estimating Bayes factors, using the Savage-Dickey method. It considers two experimental conditions, involving the observed choices y^{a}_{ij} in the first condition, and y^{b}_{ij} in the second condition. The basic threshold model in Figure 1 is applied to both of the conditions, with the key addition of modeling the relationship between the mean thresholds for each position. Specifically, the mean thresholds $\boldsymbol{\mu}^{\text{a}}$ and $\boldsymbol{\mu}^{\text{b}}$ are assumed to differ by $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_4)$ around grand means $\boldsymbol{\mu}^{\text{g}}$.

In terms of comparing the mean thresholds across the two conditions, the model that asserts they are equal corresponds to $\boldsymbol{\delta} = \mathbf{0}$, while the model that asserts they are different follows the specified priors for $\boldsymbol{\delta}$. This allows the Savege-Dickey method to be applied to the prior and posterior for $\boldsymbol{\delta}$ to estimate the Bayes factors between the "same" and "different" models. These results are reported in the paper, in Figures 6 and 7, based on the simplifying approximation that the overall Bayes factor is the product of the Bayes factor for each position taken independently.

Additional Tests of Modeling Assumptions

Process-Oriented Implementation

The "accuracy of execution" approach used by the graphical model in Figure **??** is probably the simplest way to make the deterministic threshold model probabilistic, as required for Bayesian analysis. It simply assumes people choose the option dictated by their thresholds with probability α, or choose a different option with probability $1-\alpha$, with all of these mis-executions being equally likely. We also, however, considered a more process-oriented probabilistic variant of the threshold model, in which mis-executions are applied sequentially over the 5 options in each problem.

This process-oriented implementation of the threshold model starts with the first option, and follows the deterministic choice dictated by the thresholds with a high probability α'. If the first option is not chosen, the second option is then considered, and again followed with probability α', and so on for all of the options in the problem. This means that considering an option becomes conditional on not having chosen previous options. As a concrete example, if the thresholds for a problem correspond to choosing the third option, the probability of the first option being mistakenly chosen is proportional to $1 - \alpha'$, and the probability of the second option being mistakenly chosen is is proportional to $(1 - \alpha') \times (1 - \alpha')$, and the probability of choosing the third option is proportional to $(1 - \alpha') \times (1 - \alpha') \times \alpha'$. The net effect of this alternative conceptualization is the prediction that people are more likely to choose earlier options, especially when the threshold model is not followed.

Despite these theoretical differences, we found that the inferred thresholds for the process-oriented are extremely similar to those inferred for the accuracy of execution implementation. Figure 3 shows the inferred thresholds for the process-oriented implementation, and can be compared directly to those shown in Figure 5 of the paper. We also found that the Bayes factors based on the process-oriented implementation were essentially the same as those shown in Figures 6 and 7 of the main paper. We assume the similarity of the results between the implementations is because of the informativeness of the behavioral data available for each participant.

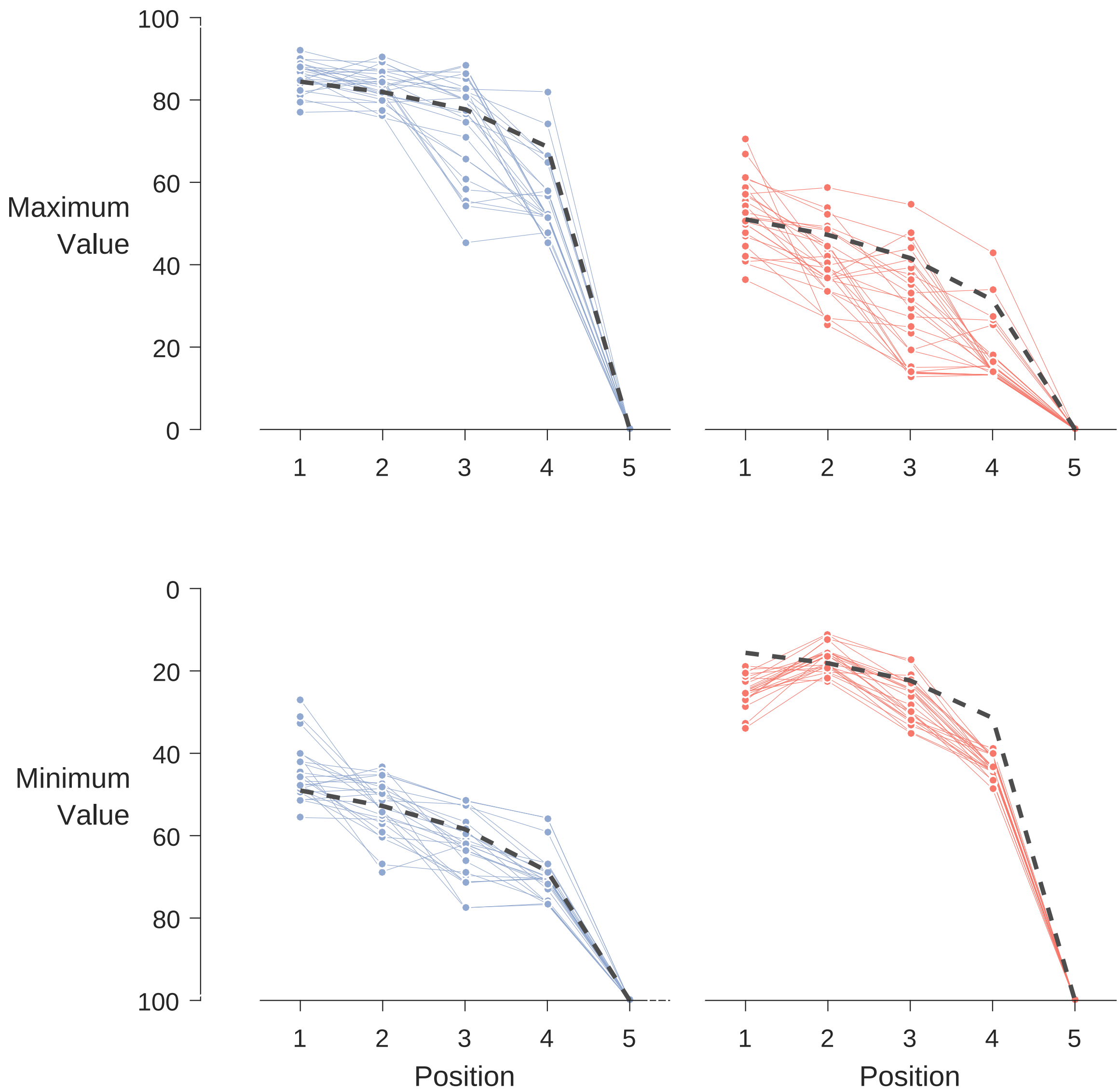


Figure 3. Marginal posterior expectations of the inferred thresholds across all participants for the four conditions, for the "process-oriented" implementation of the threshold model. The dashed line in bold is the optimal decision threshold. The top two panels show the inferred thresholds for the maximum conditions in high and low environments, from left to right. The bottom two panels show the inferred thresholds *inverted* along the horizontal axis for the minimum conditions in high and low, from left to right. We invert the direction of the minimum thresholds for ease of comparison between the conditions..

Change in Thresholds

The model used in the paper assumes the thresholds an individual participant uses do not change over the sequence of problems they complete (i.e., after completing the practice problems, there is no learning, adaptation, or self-regulation of the thresholds). Evidence for this lack of learning is presented in terms of behavioral results that show no change in accuracy, consistent with previous findings. As an additional model-based check, we applied the basic model for inferring thresholds in Figure 1 independently to the first half and second half of the problems done by each participant.

Figure **??** summarizes the results of this analysis, organized in four panels corresponding to the goal and environment manipulations. In each panel, the thresholds inferred from the first half of the problems are shown on the x-axis and the thresholds inferred from the second half of the problems are shown on the y-axis. The four different positions within the problem are shown by different markers. It is clear there is close agreement between the thresholds inferred for the first half and second half of problems, for almost all participants in all positions in all conditions. We think the largest exceptions, for the fourth threshold in a few of the conditions, arise because the behavioral data are sparse, Participants rarely get to the fourth position looking for the minimum in a low environment, for example, and exactly how many of the few cases fall in the first and second half of the problem sequence determines how much the prior dominates inference. We do not think these relatively minor exceptions provide evidence against assuming thresholds remain constant. We do think that the close agreement between the vast majority of the inferred thresholds provide evidence in favor of the constant-threshold modeling assumption.

Dependence on Previous Values

The model used in the paper assumes that the values of the previous alternative do not affect people's decision making. Evidence for this independence is provided in terms of an analysis of the behavioral data that shows the distribution of values preceding alternatives that were chosen versus not chosen. As an additional model-based check, we used the approach originally presented in Guan, Lee, and Silva (2014). This approach is based on comparing the basic model in Figure 1 with an extended model in which the thresholds can be affected by the preceding value in a problem sequence. Formally, in the extended model the threshold the ith participant uses on

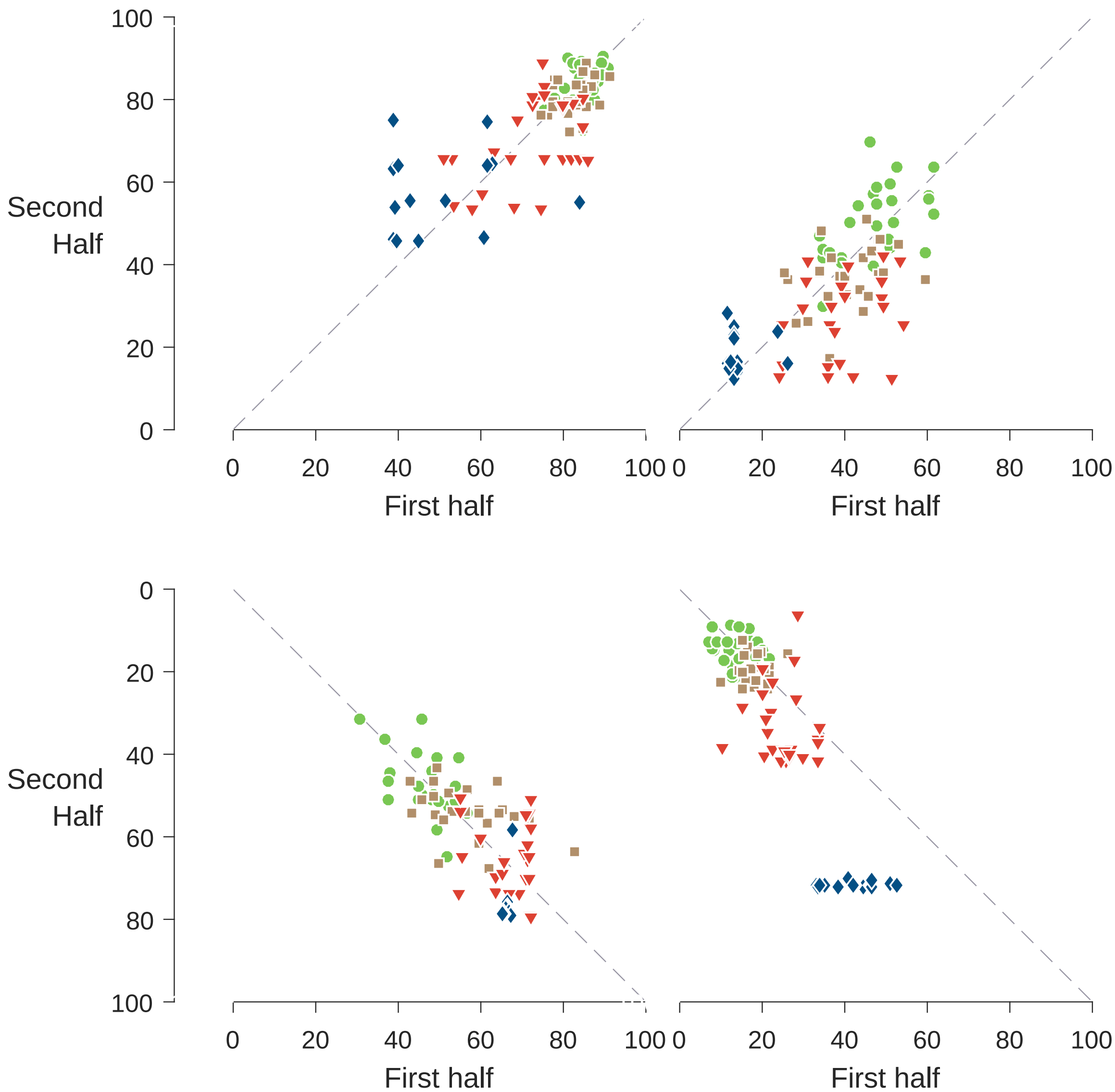


Figure 4. The posterior expectation of the inferred threshold for each participant in each position and experimental conditional, based on the first half and second half of the problems they completed. The panels correspond to the four experimental conditions, as organized in Figure 3, and the different markers correspond to the different positions (green circles are the first position, red triangles are the second position, brown squares are the third position, and blue diamonds are the fourth position).

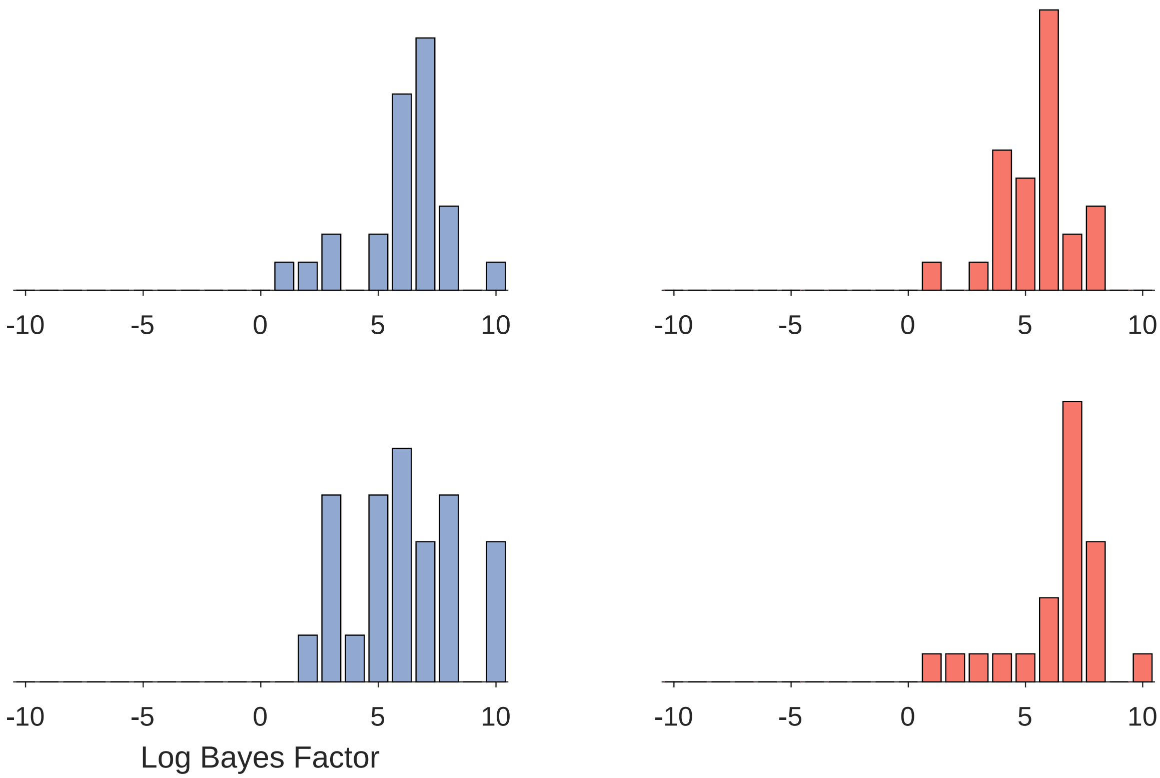


Figure 5. The distribution of log Bayes factors over all participants in each experimental condition comparing a model that assumes thresholds are independent of the previously presented value against a model that assumes a dependency. The panels correspond to the four experimental conditions, as organized in Figure 3. Positive log Bayes factors provide evidence in favor of the independence model.

the jth problem for the kth position is given by $\tau'_{ik} = \tau_{ik} + w_i \left(v_{jk} - v_{j(k-1)}\right)$, where $k = 2, 3, 4$. The parameter $w_i \sim \text{Gaussian}\big(0, 0.1\big)$ controls how the preceding value affects thresholds for the ith participant.

Intuitively, the w_i acts to increase or decrease a threshold in proportion to the difference between the current and immediately preceding value, and if $w_i = 0$ then the preceding values do not impact thresholds for the ith participant. Thus, the Savage-Dickey method can be applied to estimate Bayes factors, for each individual participant, between the model that asserts independence between thresholds and previous values and the model that asserts a dependence following the prior on w_i. Figure 5 shows the distribution of these log Bayes factors over participants in each of the four experimental conditions. Log Bayes factors greater than zero provide evidence in favor of the independence model. This is the direction of the evidence for all participants in all conditions, and often the evidence strongly favors independence, consistent with the theoretical assumption of our basic threshold model.

References

Brooks, S. P., & Gelman, A. (1997). General methods for monitoring convergence of iterative simulations. *Journal of Computational and Graphical Statistics*, *7*, 434–455.

Guan, M., Lee, M. D., & Silva, A. (2014). Threshold models of human decision making on optimal stopping problems in different environments. In *Proceedings of the 36th annual meeting of the cognitive science society. austin, tx: Cognitive science society.*

Jordan, M. I. (2004). Graphical models. *Statistical Science*, *19*, 140–155.

Lee, M. D., & Wagenmakers, E.-J. (2013). *Bayesian cognitive modeling: A practical course.* Cambridge University Press.

Plummer, M. (2003). JAGS: A program for analysis of Bayesian graphical models using Gibbs sampling. In K. Hornik, F. Leisch, & A. Zeileis (Eds.), *Proceedings of the 3rd international workshop on distributed statistical computing.* Vienna, Austria.