Gradient Flow in 2D XY model

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July, 2020

Motivation : Free Vortex Density

Importance

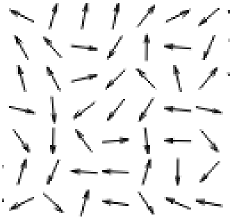
- Proliferation of free vortices →Topological phase transition . [Ref. Kosterlitz, Thouless paper]
- Continuum picture well understood ← RG for Coulomb Gas. . [Ref. Kosterlitz, Thouless paper]
- Analytical predictions of dynamic scaling depends on ρ_v^{free}[Ref. Quench dynamics in 2D XY model, arXiv:1012.0417]

For Finite Size Systems

- No well defined prescription to distinguish bound and free vortices
- Existing approaches are ad-hoc. e.g. based on Euclidean distance [Ref. Quench dynamics in 2D XY model, arXiv:1012.0417]
- Sophisticated Machine Learning algorithms such as Convolutional Neural Networks have been used.[Ref. Machine learning vortices at the Kosterlitz-Thouless transition, https://arxiv.org/abs/1710.09842]

Objective $\implies$ A simple and natural way to estimate free vortex density

2D XY model $\implies n = 2, d = 2$



$$\text{Model : } \mathcal{H} = -J \sum_{\langle xy \rangle} \vec{S}(x) \cdot \vec{S}(y) = -J \sum_{\langle xy \rangle} \cos(\theta(x) - \theta(y))$$

- ⋆ No order-disorder transition in 2D XY model (Mermin-Wagner Theorem)
- ⋆ Fluctuation of Goldstone modes restores symmetry

$$\vec{s}(x) = (\vec{\pi}_1(x), \vec{\pi}_2(x), ..., \vec{\pi}_{n-1}(x), \sigma(x)) \implies \langle |\pi(x)|^2 \rangle \propto T \ln\left(\frac{L}{a}\right)$$

2D Coulomb Gas

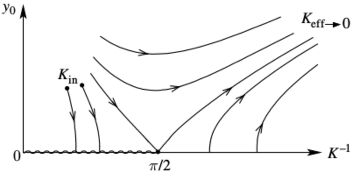
$$\mathcal{Z} \propto \mathcal{Z}_{\text{SW}}\mathcal{Z}_{\text{Q}} \text{ Where } \mathcal{Z}_{\text{SW}} = \int \mathcal{D}\phi(\vec{r}) \exp\left(-\frac{K}{2}\int d^2\vec{r}\,(\nabla\phi)^2\right)$$

$$\mathcal{Z}_{\text{Q}} = \sum_{N=0}^{\infty} y_0^N \int \prod_{i=1}^{N} d^2\vec{r}\ \exp\left(4\pi^2 K \sum_{i<j} n_i n_j\ C(\vec{r}_i - \vec{r}_j)\right)$$

RG flow equations

$$\frac{dK^{-1}}{dl} = 4\pi^3 a^4 y_0^2 + \mathcal{O}(y_0^4)$$

$$\frac{dy_0}{dl} = (2 - \pi K)y_0 + \mathcal{O}(y_0^3)$$



Still exists a phase transition

- $T < T_{\text{BKT}} \implies$ bound vortex-antivortex pairs
- $T > T_{\text{BKT}} \implies$ free vortices proliferate

Gradient Flow

Introduce a new time, say *flow time* and a new *dynamics*

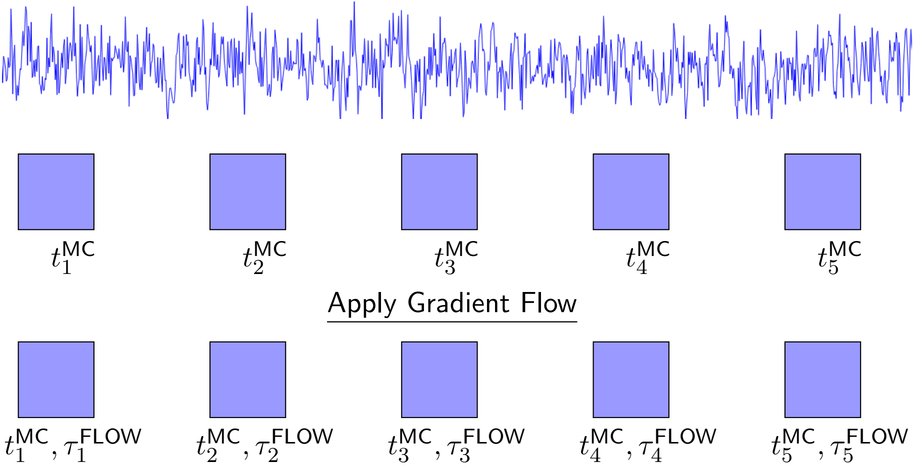
$$\frac{d}{d\tau}\theta(x,\tau) = -\frac{\delta\mathcal{H}}{\delta\theta(x,\tau)} = \underbrace{-J\sum_{y\in nn(x)}\sin(\theta(x,\tau)-\theta(y,\tau))}_{\text{2D XY}}$$

$\star$ Lattice QCD gauge field $\xrightarrow{\text{Wilson Flow}}$ Smooth and renormalized field. [Ref. Properties and uses of Wilson flow in Lattice QCD, Martin Lüscher]

$\star$ The flow freezes when we have reached a local minima of the spin configurations $\overset{\text{Expectation}}{\Longrightarrow}$ *Bound pairs killed and free vortices remain*

Algorithm

Equilibriate at temperature T



Equilibriate : Heatbath Algorithm

Adapted from heatbath updates in $U(1)$ lattice gauge theory

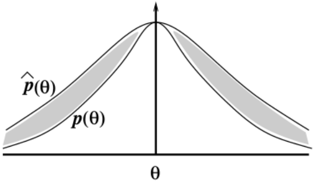
Need to Sample from

$$p(\theta(x)) = \exp\{\rho\cos(\theta(x) - \theta_{\text{eff}}(x))\}$$

Instead Sample from

$$\hat{p}(\theta(x)) = \frac{e^{\rho}}{1 + \rho(1 - \cos(\theta(x) - \theta_{\text{eff}}(x)))}$$

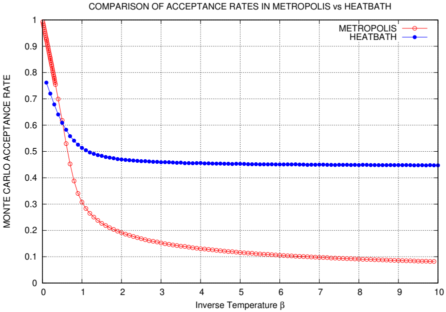
Reject some configurations to compensate



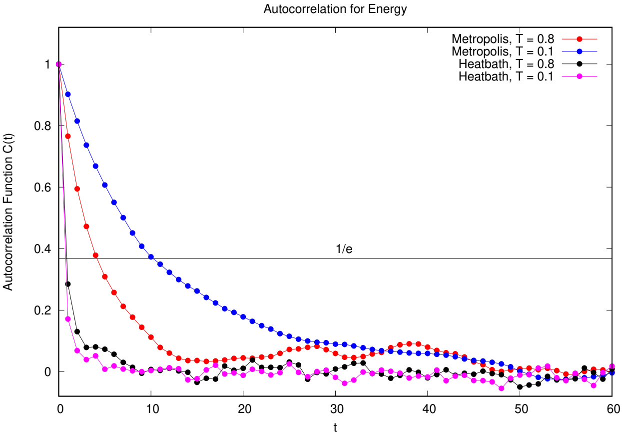
Where $\rho = J\beta|\vec{h}_{\text{eff}}(x)|$

Heatbath vs Metropolis : Acceptance Rate

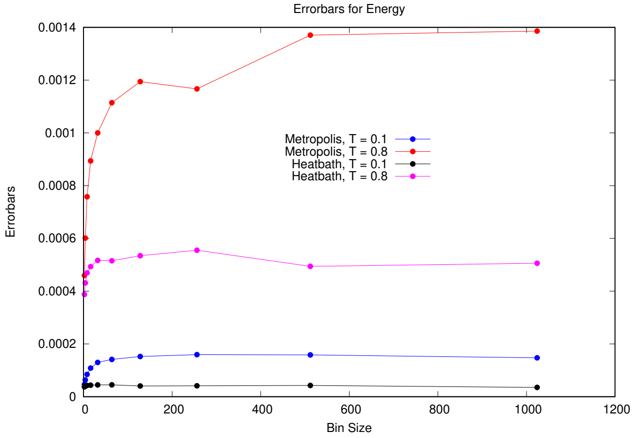
- Metropolis
  1. Proposes configurations randomly
  2. Easy to implement
  3. Low acceptance rate, specially at low temperatures
- Heatbath
  1. Makes an educated guess
  2. Implementation is more involved
  3. Moderate acceptance rate throughout temperature range



Heatbath vs Metropolis : Autocorrelation



Heatbath vs Metropolis : Binning Autocorrelation



Results - Spin Configuration $T < T_{\text{BKT}}$

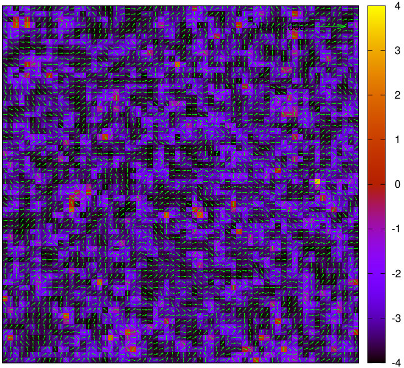


Figure: Before Gradient Flow

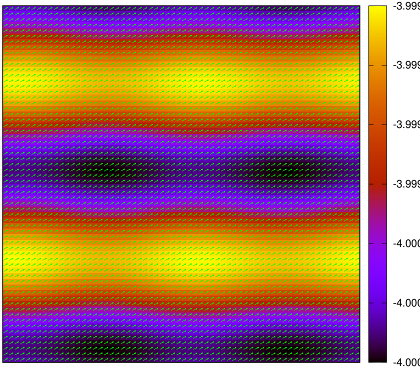


Figure: After Gradient Flow

Results - Spin Configuration $T > T_{\text{BKT}}$

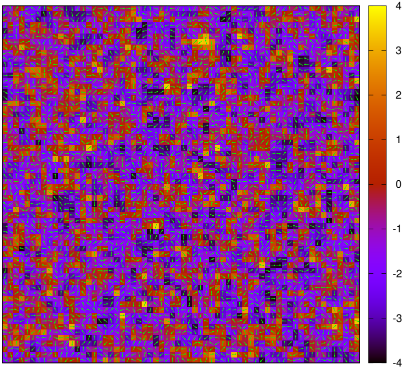


Figure: Before Gradient Flow

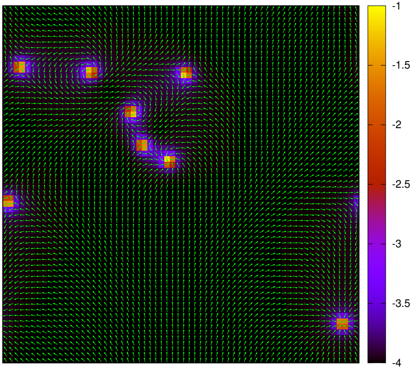
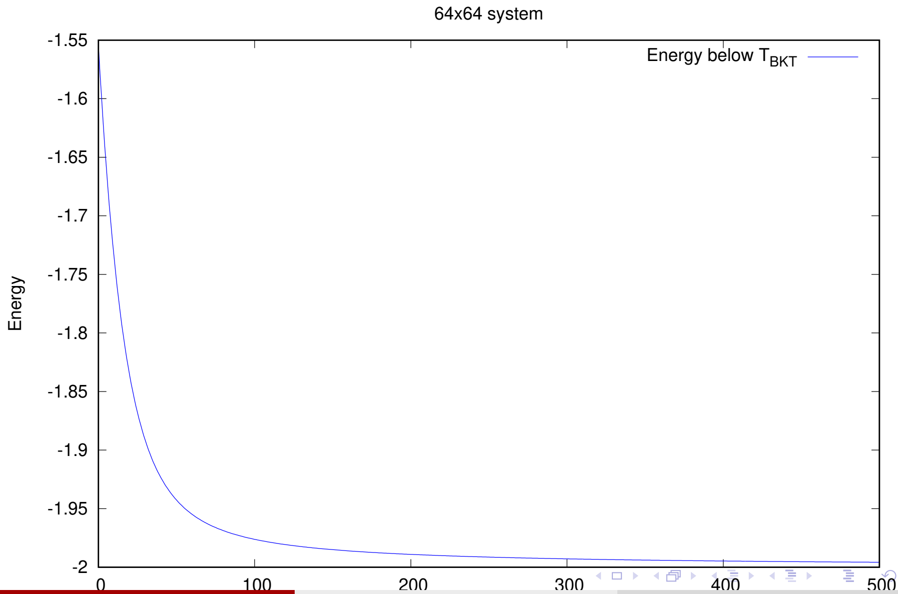
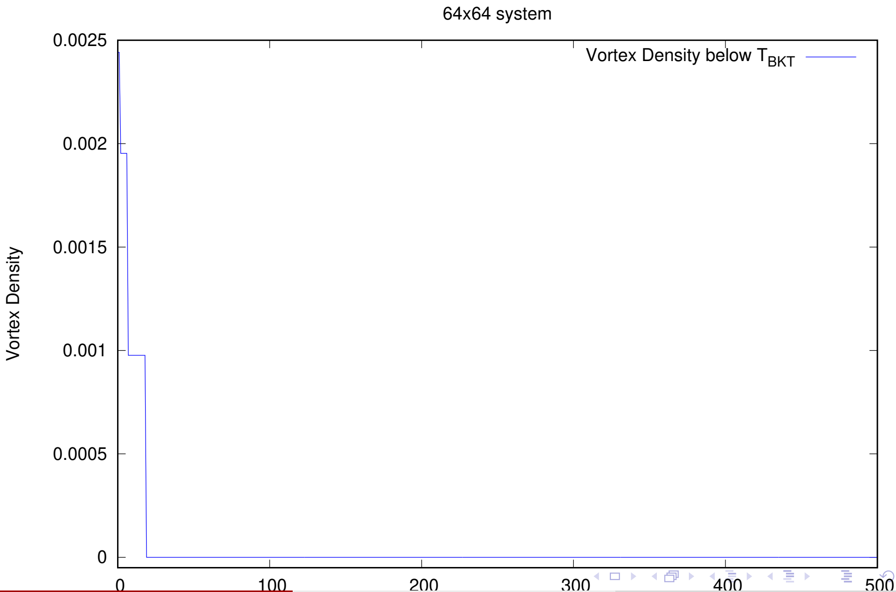


Figure: After Gradient Flow

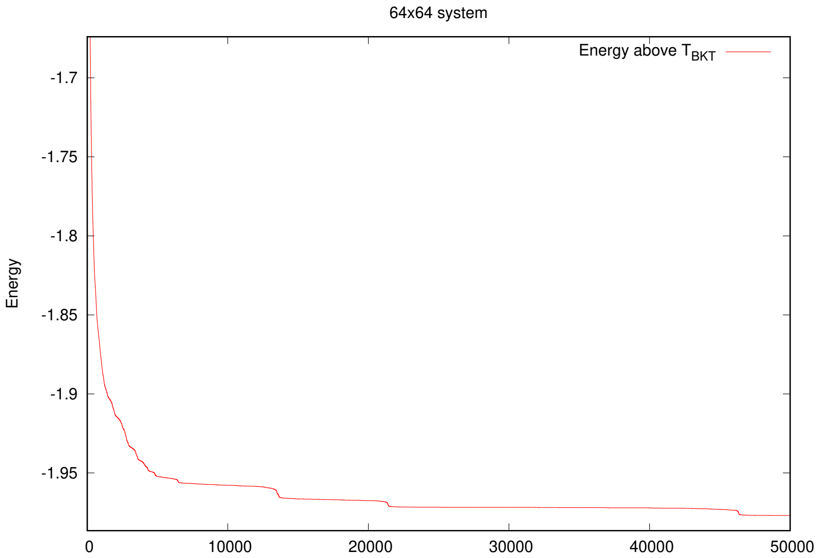
Results -Energy $T < T_{\mathrm{BKT}}$



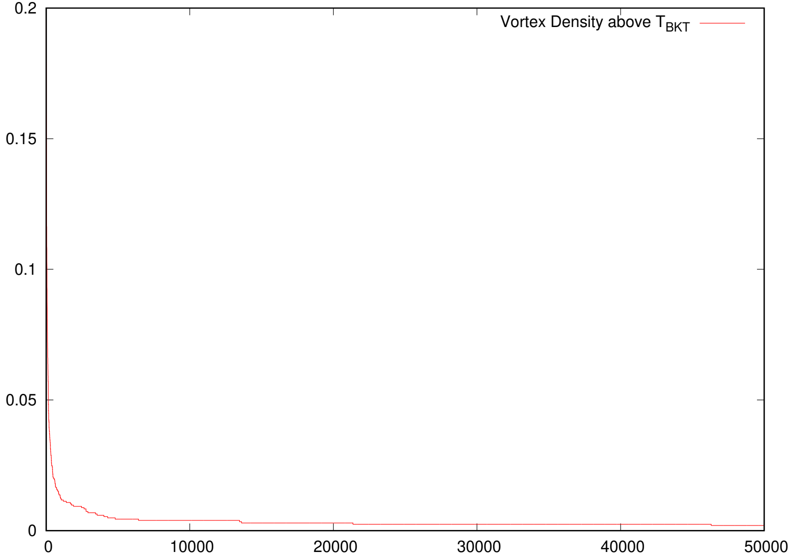
Results - Vortex Density $T < T_{\mathrm{BKT}}$



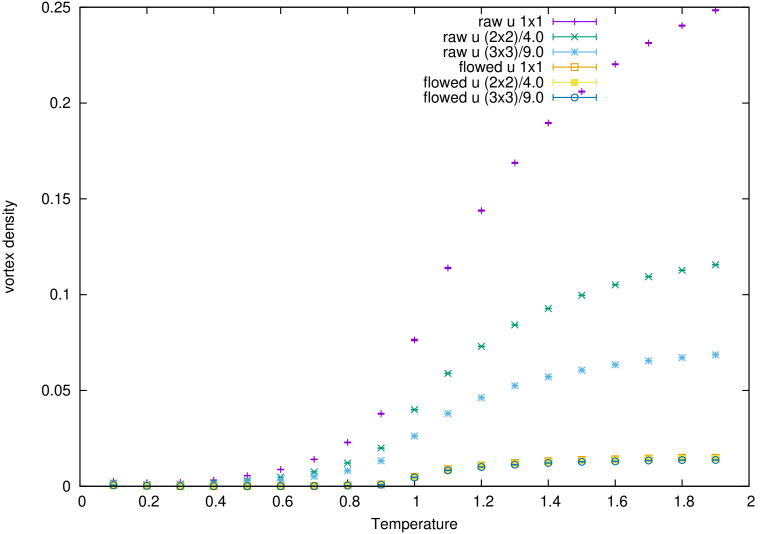
Results -Energy $T > T_{\text{BKT}}$



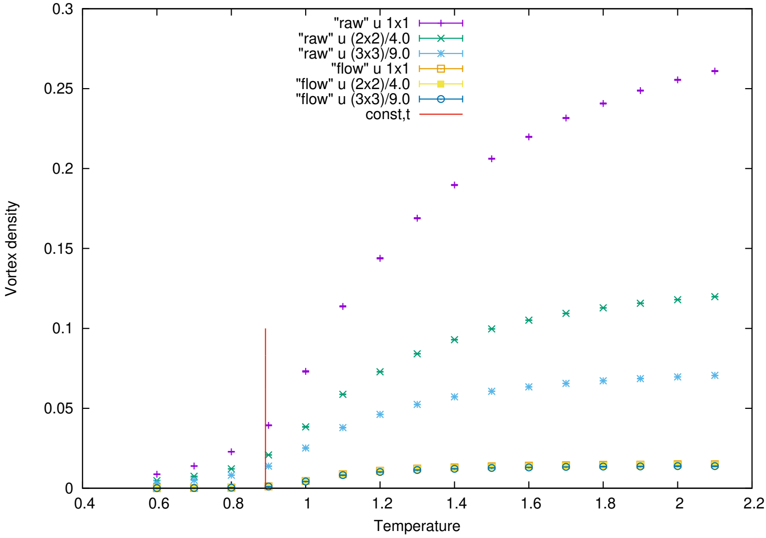
Results - Vortex Density $T > T_{\mathrm{BKT}}$



Vortex Density - Raw vs Flow - 64×64



Vortex Density - Raw vs Flow - 128×128



Future Work

- Analyze the converge of flow dynamics more systematically
- Investigate dynamic scaling in 2D XY model
- Apply the same methodology to 3D systems with topological defects.

Acknowledgements

I would like to thank **Prof. Pushan Majumdar** for everything he did for me. I feel extremely lucky to have worked with him on this project.

I would also like to thank **Prof. Arnab Sen** for the helpful discussions and suggestions related to this work. I am extremely grateful to him.

Thank You !!