Kinematics

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1 What is Kinematics?

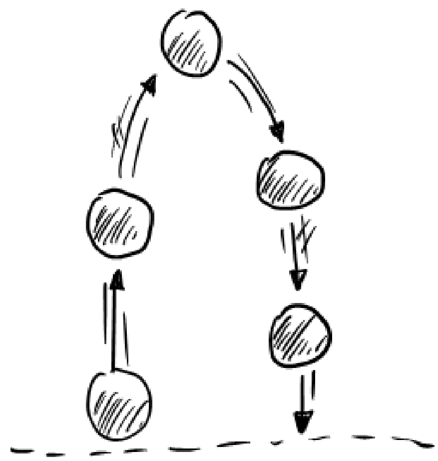
To quote Wikipedia: *Kinematics* is a subfield of classical mechanics that describes the motion of points, bodies (objects), and systems of bodies (groups of objects) without considering the forces that cause them to move.

Let us try to understand what the above statement means. It says that when we consider kinematics, we will not worry about any kinds of forces. Forces are the reason for motion. When dealing with kinematics we will simply assume the given motion and not worry about what forces cause this motion.

We will worry about forces when we consider *Dynamics* of objects. There we will compute forces and use Newton's Laws of motion to derive the motion of the objects.

1.1 Example: Projectile Motion

From experience we have an idea about how a ball moves if we throw it. If we talk in a mathematically precise manner, the trajectory of the ball is a parabola.



When considering Kinematics we will work with the given fact that the parabolic trajectory of the ball is given by the set of equations.

$$\begin{aligned} x(t) &= x_0 + v_x t \\ y(t) &= y_0 + v_y t - \frac{1}{2} g t^2 \end{aligned} \tag{1}$$

Where $x(t)$ is the horizontal co-ordinate and $y(t)$ is the vertical co-ordinate of the ball. Here $\vec{r}_0 = x_0\hat{i} + y_0\hat{j}$ is the initial position of the ball, $\vec{v} = v_x\hat{i} + v_y\hat{j}$ is the initial velocity of the ball and g is the constant acceleration due to gravity.

In Kinematics we don't worry about where eq.(1) comes from. We only work with them. Only when we deal with dynamics and forces we can derive the parabolic trajectory by considering the gravitational forces on the ball and the Newton's Laws of motion.

Exercise-1: Find the equation of the parabola in the XY plane from eq.(1) and plot it for initial velocities

(i) $\vec{v} = 5\ \hat{j}\ m/s$

(ii) $\vec{v} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\ m/s$

2 Rest and Motion

We intuitively understand what rest and motion are: If an object does not change it's position then it is at rest and if it changes it's position then it has some motion. The only additional thing we need to talk about rest and motion in a rigorous way is the concept of a reference frame. This is the case because every statement about rest or motion has to be done with respect to some reference frame. Let us see what we mean by this using an example.

2.1 Example: Train Ride

Consider the situation that you are taking a train ride with your friend, who is sitting opposite to you. If you look out the window you see things coming towards you. Also consider a third person outside the train in the platform whom you pass by while the train is moving. Given this situation we have the following.

1. According to you the train is moving forward.
2. According to your friend the train is moving backwards, as he/she is seeing things moving away from him/her.
3. According to you, your friend is sitting still, not moving, i.e. at *rest*.
4. According to your friend, you are sitting still, not moving, i.e. at *rest*.
5. According the the third person standing on the platform, you and your friend both are moving with the train in the direction of the velocity of the train, i.e. both of you are in *motion*.

The statements 1,2 clearly don't agree with each other. So do statements 3,4 and 5. Who is right? The resolution of these contradictions is that every aspect of motion or rest is relative. Each of the statements are true w.r.t to their reference frames. Hence the conclusion: *Motion is relative*. Whenever we describe motion of an object we have to do so with respect to a reference frame (observer). Otherwise it makes no sense. Think about this the next time you are using some form of transportation.

2.2 Reference Frames

Now that we have understood the importance of reference frames in describing motion, let us now define the ideas in a mathematical manner. A reference frame is a co-ordinate system with the observer sitting at the origin of the co-ordinate system. The position of the particle is a vector pointing from the origin of the reference frame to the location of the particle. So a particle can have different position vectors w.r.t. different frames at the same instant of time.

Let at time t the position vector of a particle w.r.t. frame S be $\vec{r}(t)$. The motion of the particle is encoded in the functional dependence with time. For a frame S using a 3 dimensional Cartesian co-ordinate system we can express the position of the particle in terms of unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ as

$$\vec{r}(t) = x(t)\ \hat{\imath} + y(t)\ \hat{\jmath} + z(t)\ \hat{k} \tag{2}$$

We sometimes omit the unit vectors and write $\vec{r}(t)$ as a point in the 3 dimensional space where the direction from origin to the point is implicitly assumed.

$$\vec{r}(t) = (x(t), y(t), z(t)) \tag{3}$$

Let us consider some examples of how the motion is encoded in $\vec{r}(t)$.

2.2.1 Example: Particle at Rest

If a particle is at rest w.r.t. to a frame S at the position (x_0, y_0, z_0) then we write the position vector of the particle as

$$\vec{r}(t) = x_0\ \hat{i} + y_0\ \hat{j} + z_0\ \hat{k} = (x_0, y_0, z_0) \tag{4}$$

How do we know that this particle is at rest? Well, the position of the particle does not change w.r.t. frame S at time increases. So it must be at rest. But how do we deduce this mathematically? For this we need to evaluate the velocity of the particle. The velocity of the particle $\vec{v}(t)$ is defined using the framework of differential calculus as

$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t) \tag{5}$$

Differentiation of a vector function $\vec{v}(t)$ w.r.t. time t will give us a vector whose components are derivatives of the corresponding components of the position vector. So for vector functions differentiation must be done component wise.

$$\begin{aligned} \vec{v}(t) &= \frac{d}{dt}\vec{r}(t) = \frac{d}{dt}x(t)\ \hat{i} + \frac{d}{dt}y(t)\ \hat{j} + \frac{d}{dt}z(t)\ \hat{k} \\ \vec{v}(t) &= (v_x(t), v_y(t), v_z(t)) = \left(\frac{d}{dt}x(t), \frac{d}{dt}y(t), \frac{d}{dt}z(t)\right) \end{aligned} \tag{6}$$

For eq.(4) since the position of the particle is *constant* we find (by differentiation) that the velocity of the particle is $(0, 0, 0)$

2.2.2 Example: Uniform Motion

If a particle is moving only along the positive X direction with a uniform velocity v_x w.r.t. frame S then the position vector of the particle will be given by

$$\vec{r}(t) = (x_0 + v_x t)\hat{i} + y_0\hat{j} + z_0\hat{k} \tag{7}$$

Exercise: Verify the claim that for eq.(7) the particle moves with uniform velocity v_x along the X axis.

Exercise: Construct the position vector for a particle which moves along the negative Z direction with a uniform velocity v_z starting from an initial position (x_0, y_0, z_0).

Exercise: Find the velocity of the particle which follows the parabolic trajectory in eq.(1)

2.2.3 Example: Accelerated Motion

Let us consider the following time dependence of the position vector of a particle to be

$$\vec{r}(t) = \left(x_0 + v_x t + \frac{1}{2}at^2\right)\,\hat{\imath} + y_0\,\hat{\jmath} + z_0\,\hat{k} \tag{8}$$

Exercise: Find the velocity vector $\vec{v}(t) = \dfrac{d\vec{r}}{dt}$ for eq.(8). Plot the X component of $\vec{v}(t)$ with time t.

One should find going through the above exercise that

$$\begin{aligned} v_x(t) &= v_x + at \\ v_y(t) &= 0 \\ v_y(t) &= 0 \end{aligned} \tag{9}$$

From eq.(9) we can conclude that the in the motion described by eq.(8) the particle moves along the positive X axis with a linearly increasing velocity and does not move along the Y or Z axis. This kind of a motion where the velocity itself changes with time is called an *accelerated* motion. In particular when velocity increases or decreases linearly with time we call it *uniformly accelerated* motion. To understand why we call it uniformly accelerated motion let us look at the formal mathematical definition of acceleration.

Acceleration is the time derivative of velocity vector or the second order time derivative of position vector.

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = \frac{d^2}{dt^2}\vec{r}(t) \tag{10}$$

Exercise: What is the acceleration of a particle moving with a uniform velocity?

Exercise: Verify that the two dimensional motion described by eq.(1) is an accelerated motion and find it's acceleration. Does it look familiar?

2.2.4 Example: Jerked/Jolted Motion

Consider the following motion

$$\vec{r}(t) = \left(x_0 + v_x t + \frac{1}{2}at^2 + \frac{1}{3}bt^3 + \frac{1}{4}ct^4\right)\,\hat{\imath} + y_0\,\hat{\jmath} + z_0\,\hat{k} \tag{11}$$

Exercise: Find the velocity and acceleration of this motion along the X axis and plot them with time.

If the acceleration vector $\vec{a}(t)$ itself changes with time then the motion is called a *jerked or jolted motion* and jerk is defined as the time derivative of acceleration.

$$\vec{j}(t) = \frac{d}{dt}\vec{a}(t) = \frac{d^2}{dt^2}\vec{v}(t) = \frac{d^3}{dt^3}\vec{r}(t) \tag{12}$$

Exercise: A particle which is at the origin $(0,0,0)$ at time $t = 0$ has the following instantaneous velocity

$$\vec{v}(t) = (u_x t + \frac{1}{2}a_x t^2)\ \hat{i} + \frac{1}{2}a_y t^2\ \hat{j} + u_z\ \hat{k} \tag{13}$$

Find the position vector of the particle as a function of time. How much distance does the particle cover in the n-th second of it's motion.

Exercise: A particle is at (x_0, y_0) at time $t = 0$ and has a initial velocity $\vec{v}(t = 0) = u_x\ \hat{i} + u_y\ \hat{j}$. Further it is also known that the particle has acceleration $\vec{a}(t) = -a\ \hat{j}$. Find the position vector of the particle $\vec{r}(t)$ as a function of time.

2.3 Projectile Motion

The last exercise precisely describes the case of a projectile motion and from the answer for $\vec{r}(t)$ one can find out the X and Y components of the particle. It turns out to be same as eq.(1) which for convenience is again given here

$$\begin{aligned} x(t) &= x_0 + u_x t \\ y(t) &= y_0 + u_y t - \frac{1}{2}at^2 \end{aligned} \tag{14}$$

Now let us study the projectile motion a bit more closely as it is a quite common natural motion we observe around us.

Exercise: Given the trajectory of a projectile motion as in eq.(14) with $x_0 = 0, y_0 = 0$ and considering that $y = 0$ is defined by Earth's surface, find the following

1. Angle w.r.t. X axis (call it θ) and speed of throwing the projectile (call it u). For convenience use θ, u for finding the following answers.
2. Total time of flight of the projectile
3. Total horizontal distance covered by the projectile.
4. Maximum height achieved by the projectile during flight.
5. Velocity of the projectile when it is at it's highest point.
6. Maximum speed achieved by the projectile during flight.

3 Relative Velocity