Kinematics

Mainak Pal

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1 What is Kinematics?

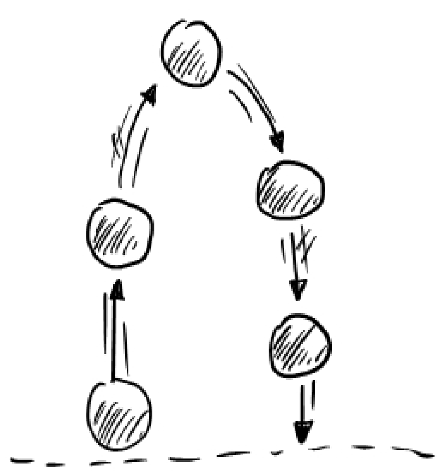
Kinematics is a subfield of classical mechanics that describes the motion of points, bodies (objects), and systems of bodies (groups of objects) without considering the forces that cause them to move.

So when we consider kinematics, we will not worry about any kinds of forces. Forces are the reason for motion. When dealing with kinematics we will simply assume the given motion and not worry about what forces cause this motion.

We will worry about forces when we consider *Dynamics* of objects. There we will compute forces and use Newton's Laws of motion to derive the motion of the objects.

1.1 Example: Projectile Motion

From experience we know how a ball moves if we throw it. Mathematically the trajectory of the ball is a parabola.



When considering Kinematics we will work with the fact that the parabolic trajectory of the ball is given by the set of equations

$$\begin{aligned} x(t) &= x_0 + v_x t \\ y(t) &= y_0 + v_y t - \frac{1}{2} g t^2 \end{aligned} \tag{1}$$

Where $x(t)$ is the horizontal co-ordinate and $y(t)$ is the vertical co-ordinate of the ball. Here $\vec{r}_0 = x_0 \hat{i} + y_0 \hat{j}$ is the initial position of the ball, $\vec{v} = v_x \hat{i} + v_y \hat{j}$ is the initial velocity of the ball and g is the constant acceleration due to gravity.

In Kinematics we don't worry about where eq.(1) comes from. We only work with them. Only when we deal with dynamics and forces we can derive the parabolic trajectory by considering the gravitational forces on the ball and the Newton's Laws of motion.

Exercise-1: Find the equation of the parabola in the X-Y plane from eq.(1) and plot it for initial velocities

(i) $\vec{v} = 5\ \hat{j}\ m/s$

(ii) $\vec{v} = \frac{1}{\sqrt{2}} \hat{i} + \frac{1}{\sqrt{2}} \hat{j}\ m/s$

2 Rest and Motion

We intuitively understand what rest and motion are: If an object does not change it's position then it is at rest and if it changes it's position then it has some motion. The only additional thing we need to talk about rest and motion in a rigorous way is the concept of a reference frame. Every statement about rest or motion has to be done with respect to some reference frame. Let us see what we mean by this using an example.

2.1 Example: Train Ride

Consider the situation that you are taking a train ride with your friend, who is sitting opposite to you. If you look out the window you see things coming towards you. Also consider a third person outside the train in the platform whom you pass by while the train is moving. Given this situation we have the following.

1. According to you the train is moving forward.
2. According to your friend the train is moving backwards, as he/she is seeing things moving away from him/her.
3. According to you, your friend is sitting still, not moving, i.e. at *rest*.
4. According to your friend, you are sitting still, not moving, i.e. at *rest*.

5. You and your friend both are moving with the train in the direction of the velocity of the train, i.e. both of you are in *motion*.

The statements 1,2 clearly don't agree with each other. So do statements 3,4 and 5. Who is right? The resolution of these contradictions is that every aspect of motion or rest is relative. Each of the statements are true w.r.t to their reference frames. Hence the conclusion: *Motion is relative.* Whenever we describe motion of an object we have to do so with respect to a reference frame (observer). Otherwise it makes no sense.

2.2 Reference Frames

Now that we have understood the importance of reference frames in describing motion, let us now define the ideas in a mathematical manner. A reference frame is a co-ordinate system with the observer sitting at the origin of the co-ordinate system. The position of the particle is a vector pointing from the origin of the reference frame to the location of the particle. So a particle can have different position vectors w.r.t. different frames at the same instant of time.

Let at time t the position vector of a particle w.r.t. frame S be $\vec{r}(t)$. The motion of the particle is encoded in the functional dependence with time. For a frame S using a 3 dimensional Cartesian co-ordinate system we can express the position of the particle in terms of unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ as

$$\vec{r}(t) = x(t)\ \hat{\imath} + y(t)\ \hat{\jmath} + z(t)\ \hat{k} \tag{2}$$

We sometimes omit the unit vectors and write $\vec{r}(t)$ as a point in the 3 dimensional space where the direction from origin to the point is implicitly assumed.

$$\vec{r}(t) = (x(t), y(t), z(t)) \tag{3}$$

Let us consider some examples of how the motion is encoded in $\vec{r}(t)$.

2.2.1 Example: Particle at Rest

If a particle is at rest w.r.t. to a frame S at the position (x_0, y_0, z_0) then we write the position vector of the particle as

$$\vec{r}(t) = x_0\ \hat{\imath} + y_0\ \hat{\jmath} + z_0\ \hat{k} = (x_0, y_0, z_0) \tag{4}$$

How do we know that this particle is at rest? Well, the position of the particle does not change w.r.t. frame S at time increases. So it must be at rest. But how do we deduce this mathematically? For this we need to evaluate the velocity of the particle. The velocity of the particle $\vec{v}(t)$ is defined as

$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t) \tag{5}$$

Differentiation of a vector function $\vec{v}(t)$ w.r.t. time t will give us a vector whose components are derivatives of the corresponding components of the position vector. So for vector functions differentiation must be done component wise.

$$\begin{aligned} \vec{v}(t) &= \frac{d}{dt}\vec{r}(t) = \frac{d}{dt}x(t)\;\hat{i} + \frac{d}{dt}y(t)\;\hat{j} + \frac{d}{dt}z(t)\;\hat{k} \\ \vec{v}(t) &= (v_x(t), v_y(t), v_z(t)) = \Big(\frac{d}{dt}x(t), \frac{d}{dt}y(t), \frac{d}{dt}z(t)\Big) \end{aligned} \tag{6}$$

For eq.(4) since the position of the particle is *constant* we find (by differentiation) that the velocity of the particle is $(0,0,0)$

2.2.2 Example: Uniform Motion

If a particle is moving only along the positive X direction with a uniform velocity v_x w.r.t. frame S then the position vector of the particle will be given by

$$\vec{r}(t) = (x_0 + v_x t)\hat{i} + y_0\hat{j} + z_0\hat{k} \tag{7}$$

Exercise: Verify the claim that for eq.(7) the particle moves with uniform velocity v_x along the X axis.

Exercise: Construct the position vector for a particle which moves along the negative Z direction with a uniform velocity v_z.

Exercise: Find the velocity of the particle which follows the parabolic trajectory in eq.(1)