Error Analysis

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1 Accuracy in Physics

Physics is the study of natural phenomenon around us and to describe it we use Mathematics, which is said to be the language of nature. We use mathematical models to describe nature but the models have limitations or assumptions which we need to keep in mind. These limitations are not due to the rules of mathematics but rather our own inability to come up with a *perfect* description of nature. These inaccuracies related to the theoretical modelling of nature are known as *theoretical assumptions*. Now, writing down a theoretical model is only the beginning in Physics. After we write down a model (say some differential equation) we try to solve the model to give some predictions about the nature.Then we perform real life experiments to test if our theoretical predictions match the experimental results we obtain in a laboratory. The central idea is that the experiments do not care and see what model we used and solved. For example we always work with frictionless surfaces in theory, but in practice no surface is exactly frictionless. It may be very smooth, *almost* frictionless, but not with *exactly* zero friction . Although in experiments we are exposed to the full reality of nature there are still inaccuracies in experiments as well. These are simply related to how precise our measurements are. There are many sources of experimental errors. Let us now use an example to see the difference between experimental errors and theoretical assumptions.

1.1 Example: Simple Pendulum

Let us start with the following: The time period of a simple pendulum of length l is given by

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{1}$$

Where g is the constant acceleration due to gravity. We can derive this equation using Newton's Laws of motion, but for now take it to be true. Having said this, let us now look at what theoretical assumptions go into *deriving* the above formula.

1.1.1 Theoretical Assumptions

1. The mechanics of the pendulum can be described by Newton's Laws of motion. The reason that we can use Newton's laws of motion and get a good answer is that the objects we are dealing with in the case of a simple pendulum is very down to Earth scenario.

   (a) The bob of the pendulum does not move fast compared to speed of light in vacuum. Hence no Special Relativity required.

   (b) The bob is not very small like atoms or molecules. So we don't need to consider any Quantum Mechanics.

   (c) The bob is not heavy like the Earth or the Sun. So we don't need to consider General Relativity.

   Instead if we had been dealing with collision of protons inside LHC in Geneva, Switzerland we would have considered special relativity and quantum mechanics. As the proton is very small (about 100,000 times smaller than an atom!) and also very fast moving (around 99.99% the speed of light!) inside LHC.

2. For simplicity we consider that the angular amplitude of oscillation of the pendulum is small enough so that the circular arc of a pendulum motion can be approximated by a straight line. If the amplitude is not small then eq.(1) is no longer valid.

3. There is no air resistance. This greatly simplifies the differential equation we get when we apply Newton's Laws of motion for the pendulum.

4. We assume that the gravity is constant at the scale of the pendulum even though the acceleration due to gravity varies with distance from the centre of the Earth.

5. We assume that the string we use to tie the bob to a suspension point is massless and not extendable. This also simplifies the differential equation.

   Once we make all these assumptions only then eq.(1) is true. All of the assumptions above are justified and are almost true in the case of our pendulum so eq.(1) is a good model for the pendulum.

1.1.2 Experimental Errors

Having derived eq.(1) now we go to a laboratory and perform experiments to verify it. One way to do it would be to take different lengths of pendulum string and measure the time period in each case. We can also change the mass of the bob at the same time to verify that the time period is independent of whatever mass we have.

1. Every measurement we make in a laboratory has some finite precision. For example we cannot measure the length of the string to be exactly 10 cm. We can only measure it with some precision however small such as (10 ± 0.01) cm or (10 ± 0.0001) cm. The same is true for time period as well. We can only take measurements for time period of the type (2 ± 0.01) sec or (2 ± 0.002) sec depending on the precision of the instrument you have in your laboratory.

   If we are measuring some physical quantity in an experiment, the measurement will always be of the type $Q \pm \delta Q$. Where Q is the central value and δQ is the precision or absolute error of the measurement. There is also something called the relative error ΔQ which is more meaningful than the absolute error δQ. We have the relation

$$\Delta Q = \frac{\delta Q}{Q} \tag{2}$$

   There is another measure for error which is frequently used, known as the percentage error. As the name suggests the percentage error in measurement of a quantity is simply $\Delta Q \times 100\%$

2. Now if we have some formula of the type $F^\alpha = P^\beta Q^\gamma R^\omega$ and we want to measure F by measuring P, Q and R then the errors made while measuring P, Q, R will carry over and result in an error in measuring F. Mathematically we proceed as follows:

$$\begin{aligned} F^\alpha &= P^\beta Q^\gamma R^\omega \\ \implies \alpha \log F &= \beta \log P + \gamma \log Q + \omega \log R \\ \implies |\alpha \frac{\delta F}{F}| &= |\beta \frac{\delta P}{P}| + |\gamma \frac{\delta Q}{Q}| + |\omega \frac{\delta R}{R}| \end{aligned} \tag{3}$$

   While computing the error we take absolute value in eq.(3). It is so because we want to compute the largest error possible considering both the $\pm$ signs that appears before δQ when we take measurement.

   Let us see how for the case of simple pendulum the errors can be computed. Say we have measured time period T for a number of lengths l and we use this to compute the acceleration due to gravity g. If we make an error of δl while measuring length and δT error while measuring time period then, how does this carry over to an error in measuring g using eq.(1)? We start with eq.(1) and proceed in the same way as eq.(3)

$$
\begin{aligned}
T &= 2\pi\sqrt{\frac{l}{g}} \\
g &= 4\pi^2\frac{l}{T^2} \\
\log g &= \log(4\pi^2) + \log l - 2\log T \\
\frac{\delta g}{g} &= \frac{\delta l}{l} + 2\frac{\delta T}{T}
\end{aligned}
\tag{4}
$$

So, now if we make 0.1% error in measuring T and 0.05% error in measuring l then the total percentage error in measuring g will be (according to eq.(4))

$$
\underbrace{\left(\frac{\delta g}{g}\times 100\%\right)}_{0.05+2\times 0.1=0.25} = \underbrace{\left(\frac{\delta l}{l}\times 100\%\right)}_{0.05} + 2\underbrace{\left(\frac{\delta T}{T}\times 100\%\right)}_{0.1} \tag{5}
$$

Exercise-1: Find the percentage error made while measuring g using eq.(1) if the percentage error in l and T are 0.4% and 0.2%.

Exercise-2: Find the percentage error made while measuring volume of a cube if the percentage error in measuring the length of the cube is 0.1%

Exercise-3: Find the percentage error made while measuring volume of a sphere if the percentage error in measuring the radius of the sphere is 0.1%

Exercise-4: Find the percentage error made while measuring kinetic energy of a particle if the mass of the particle is (1 ± 0.005) kg and its speed is (200 ± 5) m/s.

Exercise-5: In exercise-1 if the percentage error in measuring T is kept fixed at 0.2% then how precisely should you measure l to have a 0.21% accuracy in measuring g.