Mathematics in Physics

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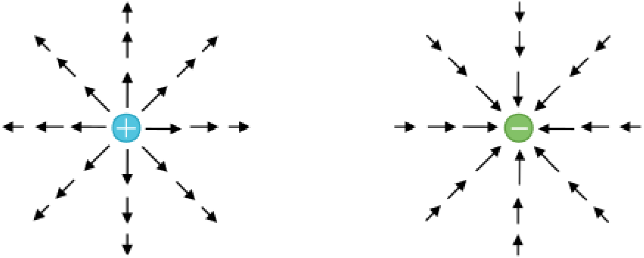
Scalars and Vectors

- Scalars - Physical quantities which can be completely described using a single number. Example- mass,speed,charge etc.
- Vectors - Physical quantities for which you need a value as well as a direction to specify it. Example- velocity, accelerati on, electric and magnetic fields etc.



Working with Vectors

Just as we can combine different scalars using addition and multiplication and so on we can do the same with vectors. N.b. there is no such thing as a vector division.

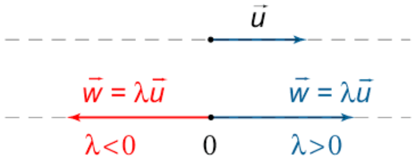
- Multiplication of a vector by a scalar or Scaling.
- Addition/Subtraction of two Vectors $\implies$ a new Vector
- Multiplication of two Vectors
    1. Scalar (Dot product)
    2. Vector (Cross product)

It is very natural to require addition, dot product and cross product of vectors.

Scaling

We can multiply a vector $\vec{u}$ by a scalar λ and get a new vector $\vec{w} = \lambda\vec{u}$.

- Such a scaling changes the magnitude of the vector $\vec{u}$ by a factor of λ
- If $\lambda < 0$ then it changes the magnitude of the vector and also reverses its direction.



Using scaling we can define unit vector in any direction $\hat{u} = \frac{\vec{u}}{|\vec{u}|}$. Some standard unit vectors are $\hat{i}$ for X direction, $\hat{j}$ for Y direction and $\hat{z}$ for Z direction in a 3 dimensional co-ordinate system.

Addition of Vectors - Triangle Law

Wind changes the angle at which rain falls. If we know how to add vectors we can find the angle. We can also find how the speed of current in a river effects the velocity of boats.

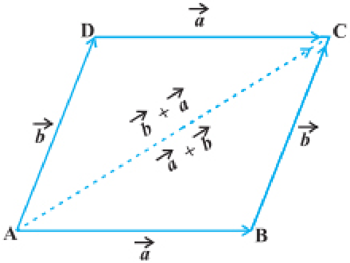


Fig. 6

$$\vec{a} = \overrightarrow{AB}, \ \vec{b} = \overrightarrow{BC}$$
$$\vec{c} = \vec{a} + \vec{b} = \overrightarrow{AC}$$
$$\vec{d} = \vec{b} + \vec{a} = \overrightarrow{AC}$$
$$|\vec{c}| = AC$$

$AC = \sqrt{(AB)^2 + (BC)^2 + 2(AB)(BC)\cos(\theta)}$ where $\theta = \angle DAB$

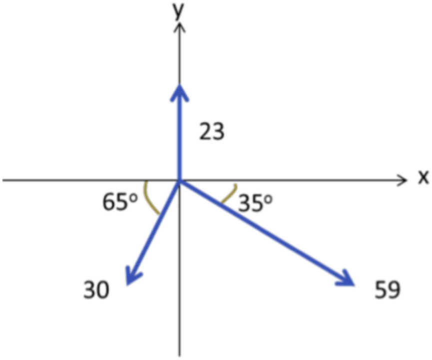
$\tan(\angle CAB) = \frac{b \sin\theta}{a + b\cos\theta}$

Properties of Vector Addition

- Commutative : $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- Associative : $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- Null Vector : $\vec{a} + \vec{0} = \vec{a}$
- Negative vector : $\vec{a} + \vec{a'} = \vec{0} \implies \vec{a'} = -\vec{a}$

Resolution of Vectors

Just as we can add two (or more) vectors to give a new vector we can also split up a vector into two or more vectors. Like the scalars this splitting or resolution of vectors in not unique



We can resolve one vector in any two of them. Say

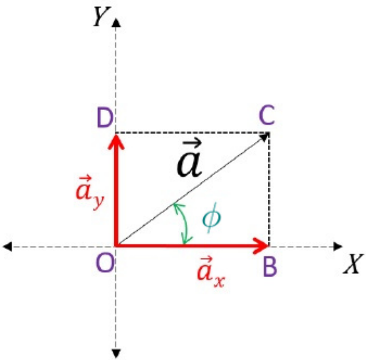
$$\vec{u} = \alpha\hat{a} + \beta\hat{b}$$

A vector equality means equality in both magnitude and direction. Equating Magnitude means

$$|\vec{u}|^2 = \alpha^2 + \beta^2 + 2\alpha\beta\cos\theta$$

Equating directions is a bit more involved specially when $\vec{a}$ and $\vec{b}$ are not perpendicular to each other. Let us for the moment focus on the case where they are orthogonal to each other.

Orthogonal Resolution of Vectors



Resolve vector $\vec{a}$ along the X and Y axis i.e. $\vec{a} = a_x\hat{i} + a_y\hat{j}$.Using results from triangle law we have

$$a^2 = a_x^2 + a_y^2$$

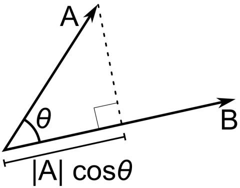
$$\tan\phi = \frac{a_y \sin(\pi/2)}{a_x + a_y \cos(\pi/2)} = \frac{a_y}{a_x}$$

Dot Product of Vectors

The dot product or the scalar product between two vectors is defined as

$$\vec{a} \cdot \vec{b} = ab\cos\theta$$

This is a definition. We need to understand the geometric meaning of dot product



The quantity $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ measures how much of the vector $\vec{a}$ is in the direction of vector $\vec{b}$. In some sense this is also a resolution and we can now use this to do resolution for non-orthogonal cases very easily

Properties of Dot products

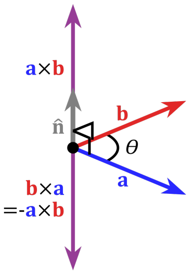
- Commutative $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- Component of $\vec{a}$ along $\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$
- Dot product of two orthogonal vectors is zero.

Cross Product of Vectors

The cross product or the vector product between two vectors is defined as

$$\vec{a} \times \vec{b} = ab \sin\theta \;\; \hat{n}$$

$\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. This is a definition. We need to understand the geometric meaning of cross product



The vector $\vec{a} \times \vec{b}$ is a vector which is always perpendicular to both $\vec{a}$ and $\vec{b}$. Geometrically $\vec{a} \times \vec{b}$ helps to find a vector which is perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. This can be very helpful.

Properties of Cross products

- Anti-Commutative $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- Geometrically $\vec{a} \times \vec{b}$ helps to find a vector which is perpendicular to the plane containing $\vec{a}$ and $\vec{b}$.
- Cross product of two parallel vectors is zero.

Product with unit vectors

If we have two vectors $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$ then what will be their dot product and cross products?

Let us first figure out what are the dot products and cross products between the unit vectors themselves.

Dot Product

$$\hat{i}\cdot\hat{i} = 1$$
$$\hat{j}\cdot\hat{j} = 1$$
$$\hat{k}\cdot\hat{k} = 1$$
$$\hat{i}\cdot\hat{j} = 0$$
$$\hat{j}\cdot\hat{k} = 0$$
$$\hat{k}\cdot\hat{i} = 0$$

Cross Product

$$\hat{i}\times\hat{i} = \vec{0}$$
$$\hat{j}\times\hat{j} = \vec{0}$$
$$\hat{k}\times\hat{k} = \vec{0}$$
$$\hat{i}\times\hat{j} = \hat{k}$$
$$\hat{j}\times\hat{k} = \hat{i}$$
$$\hat{k}\times\hat{i} = \hat{j}$$

Dot product with unit vectors

$$\vec{a} \cdot \vec{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \cdot (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$$

$$= a_x b_x(\hat{i} \cdot \hat{i}) + a_x b_y(\hat{i} \cdot \hat{j}) + a_x b_z(\hat{i} \cdot \hat{k})$$

$$+ a_y b_x(\hat{j} \cdot \hat{i}) + a_y b_y(\hat{j} \cdot \hat{j}) + a_y b_z(\hat{j} \cdot \hat{k})$$

$$+ a_z b_x(\hat{k} \cdot \hat{i}) + a_z b_y(\hat{k} \cdot \hat{j}) + a_z b_z(\hat{k} \cdot \hat{k})$$

$$= a_x b_x + a_y b_y + a_z b_z$$

Cross product with unit vectors

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= (a_y b_z - a_z b_y)\hat{i} + (a_z b_x - a_x b_z)\hat{j} + (a_x b_y - a_y b_x)\hat{k}$$