Moments of Inertia

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Notes

1. *Any* senses *same for all*.
2. Densities for x-D objects:
   (a) 0-D object or particles: no definition.
   (b) 1-D object like rod: line mass density $\lambda = \frac{dM}{dx}$, dx is differential length element.
   (c) Curved 1-D object like curved rod or string: line mass density $\lambda = \frac{dM}{ds}$, ds is differential curved length element.
   (d) 2-D object like plate: surface mass density $\sigma = \frac{dM}{dA}$, dA is differential area element.
   (e) Curved 2-D object like thin shell: surface mass density $\sigma = \frac{dM}{dS}$, dS is differential curved area element.
   (f) 3-D object: volume mass density (or density) $\rho = \frac{dM}{dV}$, dV is differential volume element.
3. Moment of inertia must be defined about an axis of rotation.
4. For a discrete system of n particles with masses m_i and at distances r_i from the axis of rotation, $i = 1(1)n$, moment of inertia about the axis of rotation is $\sum\limits_{i=1}^{n} m_i r_i^2$.

   For a continuous body of mass M, the moment of inertia about an axis is $\int_M r^2 \, dM$, r is the distance of differential mass element dM from the axis of rotation. So, $I_{xx} = \int_M (y^2 + z^2) \, dM$, $I_{yy} = \int_M (z^2 + x^2) \, dM$, $I_{zz} = \int_M (x^2 + y^2) \, dM$.

Moments of inertia

1. **Homogeneous rod (1-D) of length L and mass M, axis of rotation is along any perpendicular to the rod passing through its mid-length**

   Fit a Cartesian co-ordinate system with x-axis along the length and origin at mid-length. Required moment of inertia is I_{zz}.

   For 1-D objects we can define the line mass density $\lambda(x) = \frac{dM}{dx}$ which which be constant along the length of the homogeneous rod i.e. $\lambda(x) = \lambda$. So $M = \lambda L$.

$$
\begin{aligned}
I_{\text{zz}} &= \int_M x^2 \, dM \\
&= \int\limits_{-L/2}^{L/2} \lambda x^2 \ dx \\
&= \frac{\lambda L^3}{12} \\
&= \frac{ML^2}{12}
\end{aligned}
$$

   What is the moment of inertia at any end along any perpendicular to the rod?

2. **Homogeneous circular ring (curved 1-D) of radius R and mass M, axis of rotation is perpendicular to the plane of ring passing through its center**

   Fit a cylindrical co-ordinate system with z-axis along the axis of rotation and origin at center of the ring. Required moment of inertia is I_{zz}.

   This is also an 1-D object along the arc-length(s) direction (think of a curved rod). We can define the line mass density $\lambda(s) = \frac{dM}{ds}$ which will be constant along the arc-length of the homogeneous ring i.e. $\lambda(s) = \lambda$. So $M = \lambda \int_s ds = 2\pi\lambda R$.

$$
\begin{aligned}
I_{\text{zz}} &= \int_M R^2\, dM \\
&= \int_s \lambda R^2\, ds \\
&= \int_0^{2\pi} \lambda R^2 R\ \, d\theta \\
&= 2\pi\lambda R^3 \\
&= MR^2
\end{aligned}
$$

   Here ds is a differential curved length element spanned from θ to $\theta + d\theta$. Its length comes $R\, d\theta$.

   What is the moment of inertia about any diameter of the ring?

   To evaluate I_{xx} or I_{yy} instead of doing integrations we can use *Perpendicular Axis Theorem*.

   Perpendicular Axis Theorem: If we fit a Cartesian co-ordinate system in a 2-D object with x-y plane coincident with the plane of the object, z co-ordinate of each point of the body comes zero. So, $I_{xx} = \int_M y^2\, dM$, $I_{yy} = \int_M x^2\, dM$, $I_{zz} = \int_M (x^2 + y^2)\, dM$. Clearly we see $I_{zz} = I_{xx} + I_{yy}$. This is the *Perpendicular Axis Theorem.* Note that this theorem is valid for 2-D objects and with co-ordinate system fitted in the above manner.

   In this example we arrive at $I_{xx} = I_{yy}$ from symmetry (You can rotate the ring with same easiness or hardness about any diameter − symmetry in this sense. In physical term you can say the mass distribution is symmetric about any diameter.). Using the theorem, $I_{xx} = I_{yy} = \frac{1}{2} I_{zz} = \frac{1}{2} MR^2$.

3. **Homogeneous circular disk (2-D) of radius R and mass M, axis of rotation is perpendicular to the plane of disk passing through its center**

   (Read 5 first.) Fit a cylindrical co-ordinate system with z-axis along the axis of rotation and origin at center of the disk. Required moment of inertia is I_{zz}.

   This is a 2-D object for which we can define the surface mass density $\sigma(r, \theta) = \frac{dM}{dA}$ (dA is differential surface area) which will be constant across the surface area of the homogeneous disk ($\sigma(r, \theta) = \sigma$). So $M = \sigma A$.

$$
\begin{aligned}
I_{zz} &= \int_M r^2\, dM \\
&= \int_A \sigma r^2\, dA \\
&= \sigma \int\limits_{\theta=0}^{2\pi} \int\limits_{r=0}^{R} r^2 r\, dr\, d\theta \\
&= \sigma \int\limits_{0}^{2\pi} \left(\int\limits_{0}^{R} r^3\, dr \right) d\theta \\
&= \sigma \int\limits_{0}^{2\pi} \frac{R^4}{4}\, d\theta \\
&= \frac{\pi \sigma R^4}{2} \\
&= \frac{\sigma A R^2}{2} \\
&= \frac{M R^2}{2}
\end{aligned}
$$

Here think dA is area of a differential circular strip spanning from r to $r + dr$ and θ to $\theta + d\theta$. Its area comes $r\, dr\, d\theta$ (think how).

What is the moment of inertia about any diameter of the plate?

Use *Perpendicular Axis Theorem*.

4. **Homogeneous annular disk (2-D) of internal radius R_i, external radius R_o and mass M, axis of rotation is perpendicular to the plane of disk passing through its center**

   Very similar to 3, only the limit of r will be R_i to R_o.

5. **Homogeneous rectangular plate (2-D) of length L, breadth B and mass M, axis of rotation is perpendicular to the plane of disk passing through its center**

   Fit a Cartesian co-ordinate system with x-axis along the length, y-axis along the breadth and origin at center of the plate. Required moment of inertia is I_{zz}.

   This is a 2-D object for which we can define the surface mass density $\sigma(x, y) = \frac{dM}{dA}$ (dA is differential surface area) which will be constant across the surface area of the homogeneous plate ($\sigma(x, y) = \sigma$). So $M = \sigma A$.

   Let $r(x, y)$ be the distance of point (x, y) on the plate from the axis of rotation. $r(x, y) = \sqrt{x^2 + y^2}$.

$$
\begin{aligned}
I_{\mathrm{zz}} &= \int_M r^2 \, dM \\
&= \int_A \sigma(x^2 + y^2) \, dA \\
&= \sigma \int\limits_{y=-B/2}^{B/2} \int\limits_{x=-L/2}^{L/2} (x^2 + y^2) \, dx \, dy \\
&= \sigma \int\limits_{-B/2}^{B/2} \left(\int\limits_{-L/2}^{L/2} (x^2 + y^2) \, dx \right) dy \\
&= \sigma \int\limits_{-B/2}^{B/2} \left(\frac{L^3}{12} + y^2 L \right) dy \\
&= \sigma \left(\frac{L^3 B}{12} + \frac{L B^3}{12} \right) \\
&= \frac{\sigma A (L^2 + B^2)}{12} \\
&= \frac{M (L^2 + B^2)}{12}
\end{aligned}
$$

Note how area integral is converted into a double integral above, dA can be thought of a differential rectangle of length dx and breadth dy.

Also when we are integrating about x (inside parentheses) y is treated as constant.

What are I_{xx}, I_{yy} and I_{xy}?

Perpendicular Axis Theorem will be helpful only for square plates.

6. **Homogeneous sphere (3-D) of radius R and mass M, axis of rotation is along any diameter**

   Fit a spherical co-ordinate system with z-axis along the axis of rotation and origin at center of the sphere. Required moment of inertia is I_{zz}.

   Let $\varrho(r, \theta, \varphi)$ be the distance of point (r, θ, φ) within the sphere from the axis of rotation. $\varrho(r, \theta, \varphi) = r \sin\theta$.

$$
\begin{aligned}
I_{\mathrm{zz}} &= \int_M \varrho^2 \, dM \\
&= \rho \int_V r^2 \sin^2\theta \, dV \\
&= \rho \int\limits_{\varphi=0}^{2\pi} \int\limits_{\theta=0}^{\pi} \int\limits_{r=0}^{R} r^2 \sin^2\theta r^2 \sin\theta \, dr \, d\theta \, d\varphi \\
&= \rho \int\limits_0^{2\pi} \left(\int\limits_0^{\pi} \left(\int\limits_0^{R} r^4 \sin^3\theta \, dr \right) d\theta \right) d\varphi \\
&= \rho \int\limits_0^{2\pi} \left(\int\limits_0^{\pi} \frac{R^5}{5} \sin^3\theta \, d\theta \right) d\varphi \\
&= \rho \int\limits_0^{2\pi} \frac{4R^5}{15} \, d\varphi \\
&= \frac{8\pi\rho R^5}{15} \\
&= \frac{2\rho V R^2}{5} \\
&= \frac{2MR^2}{5}
\end{aligned}
$$

Here dV is imagined as a differential shell spanned from r to $r + dr$, θ to $\theta + d\theta$ and ϱ to $\varrho + d\varrho$. Its volume comes $r^2 \sin\theta \, dr \, d\theta \, d\varphi$ (you could try to prove, but might be difficult).

Also see an alternate derivation (using Cartesian co-ordinate system) at `http://en.wikipedia.org/wiki/Moment_of_inertia`.

7. **Homogeneous thin spherical shell (curved 2-D) of radius R and mass M, axis of rotation is along any diameter**

   Fit a spherical co-ordinate system with z-axis along the axis of rotation and origin at center of the sphere. Required moment of inertia is I_{zz}.

   This is a curved 2-D object for which we can define the surface mass density $\sigma(r, \theta, \varphi) = \frac{dM}{dS}$ (dS is differential curved surface area) which will be constant across the surface area of the homogeneous shell ($\sigma(r, \theta, \varphi) = \sigma$). So $M = \sigma \int_S dS = 4\pi\sigma R^2$.

   Let $\varrho(r, \theta, \varphi)$ be the distance of point (r, θ, φ) within the sphere from the axis of rotation. $\varrho(r, \theta, \varphi) = R \sin\theta$.

$$
\begin{aligned}
I_{zz} &= \int_M \varrho^2 \, dM \\
&= \sigma \int_S R^2 \sin^2\theta \, dS \\
&= \sigma R^2 \int\limits_{\varphi=0}^{2\pi} \int\limits_{\theta=0}^{\pi} \sin^2\theta R^2 \sin\theta \, d\theta \, d\varphi \\
&= \sigma R^4 \int\limits_0^{2\pi} \left(\int\limits_0^{\pi} \sin^3\theta d\theta \right) d\varphi \\
&= \sigma R^4 \int\limits_0^{2\pi} \frac{4}{3} \, d\varphi \\
&= \frac{8\pi\sigma R^4}{3} \\
&= \frac{2MR^2}{3}
\end{aligned}
$$

Here dS is a differential thin shell element spanned from θ to $\theta + d\theta$ and ϱ to $\varrho + d\varrho$. Its area comes $R^2 \sin\theta \, d\theta \, d\varphi$.

8. **Homogeneous thick spherical shell (3-D) of internal radius R_i, external radius R_o and mass M, axis of rotation is along any diameter**

   Very similar to 6, only the limit of r will be R_i to R_o.

9. **Homogeneous solid cylinder (3-D) of height H, radius R and mass M, axis of rotation is along cylinder axis**

   Fit a cylindrical co-ordinate system with z-axis along the axis of rotation and origin at mid-height. Required moment of inertia is I_{zz}.

$$
\begin{aligned}
I_{zz} &= \int_M r^2 \, dM \\
&= \rho \int_V r^2 \, dV \\
&= \rho \int\limits_{z=-H/2}^{H/2} \int\limits_{\theta=0}^{2\pi} \int\limits_{r=0}^{R} r^2 r \, dr \, d\theta \, dz \\
&= \rho \int\limits_{-H/2}^{H/2} \left(\int\limits_0^{2\pi} \left(\int\limits_0^{R} r^3 \, dr \right) d\theta \right) dz \\
&= \rho \int\limits_{-H/2}^{H/2} \left(\int\limits_0^{2\pi} \frac{R^4}{4} \, d\theta \right) dz \\
&= \rho \int\limits_{-H/2}^{H/2} \frac{\pi R^4}{2} \, dz \\
&= \frac{\pi\rho R^4 H}{2} \\
&= \frac{\rho V R^2}{2} \\
&= \frac{MR^2}{2}
\end{aligned}
$$

Here dV is a differential shell spanned from r to $r + dr$, θ to $\theta + d\theta$ and z to $z + dz$. Its volume comes $r\,dr\,d\theta\,dz$.

For moment of inertia about a diameter at mid-height, let $\varrho(r, \theta, z)$ be the distance of point (r, θ, z) within the cylinder from the axis of rotation. $\varrho(r, \theta, z) = \sqrt{r^2 \sin^2 \theta + z^2}$. Just replace r with this ϱ in the above integral to derive this moment of inertia.

10. **Homogeneous hollow cylinder (curved 2-D) of height H, radius R and mass M, axis of rotation is along cylinder axis**

    Fit a cylindrical co-ordinate system with z-axis along the axis of rotation and origin at mid-height. Required moment of inertia is I_{zz}.

    This is a curved 2-D object for which we can define the surface mass density $\sigma(r, \theta, x) = \frac{dM}{dS}$ (dS is differential curved surface area) which will be constant across the surface area of the homogeneous cylinder ($\sigma(r, \theta, z) = \sigma$). So $M = \sigma \int_S dS = 2\pi\sigma RH$.

$$
\begin{aligned}
I_{\text{zz}} &= \int_M r^2\,dM \\
&= \sigma \int_V R^2\,dS \\
&= \sigma R^2 \int_{z=-H/2}^{H/2} \int_{\theta=0}^{2\pi} R\,d\theta\,dz \\
&= \sigma R^3 \int_{-H/2}^{H/2} \left(\int_0^{2\pi} d\theta \right) dz \\
&= \sigma R^3 \int_{-H/2}^{H/2} 2\pi\,dz \\
&= 2\pi\sigma R^3 H \\
&= MR^2
\end{aligned}
$$

    Here dS is a differential thin shell element spanned from θ to $\theta + d\theta$ and z to $z + dz$. Its area comes $R\,d\theta\,dz$.

    For moment of inertia about a diameter at mid-height, let $\varrho(r, \theta, z)$ be the distance of point (r, θ, z) within the cylinder from the axis of rotation. $\varrho(r, \theta, z) = \sqrt{R^2 \sin^2 \theta + z^2}$. Just replace r with this ϱ in the above integral to derive this moment of inertia.

11. **Homogeneous solid cone (3-D) of height H, radius R and mass M, axis of rotation is along cone axis**

    Fit a cylindrical co-ordinate system, derivation could follow 6, you can leave this one, its derivation would be little involved.

    See a derivation here: `http://openstudy.com/updates/4ecbc912e4b04e045aea6acd`