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Review

Numerical analysis perspective in structural shape optimization: A review post 2000

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ABSTRACT

This review presents developments in structural shape optimization post 2000 from perspective of numerical analysis techniques. Traditional shape optimization with FEM has undergone considerable transformation as developments in CAD, numerical analysis techniques and optimization algorithms have contributed significantly in improving it. Mesh dependency and inconsistent description of the geometry for design and analysis models remained major challenges in traditional FEM based shape optimization. To improve mesh based shape optimization, in-plane regularization and out-of plane filtering, vertex morphing, traction method in node based shape optimization, mesh morphing and adaptive mesh refinement techniques are discussed. Alternative numerical techniques have also been discussed briefly from shape optimization perspectives which includes modified versions of FEM like eXtended FEM (XFEM) with level sets, Fixed Grid (FG) FEM/Eulerian approach, interface enriched generalized FEM (IGFEM) and finite cell method (FCM), isogeometric analysis (IGA) and its variants like eXtended IGA (XIGA) and IGABEM and meshless methods (MMs) like element free Galerkin (EFG) and reproducing Kernal particle method (RKPM). These numerical techniques have different mathematical background and hence possess different capabilities and limitations. The present work highlights these differences and compares them in context of shape optimization. Critical observations and future research recommendations are presented before concluding remarks.

1. Introduction and background

Simulation-based design frameworks have become more popular in last few decades as they evaluate performance of the proposed design or establish relative merits of alternative designs at early phase of product development. The traditional product design process has been considerably improved by extensive use of various numerical techniques like finite element method (FEM), boundary element method (BEM), computational fluid dynamics (CFD) and some relatively new techniques including meshless methods (MMs) and isogeometric analysis (IGA). To optimize engineering systems/structures, these numerical techniques are coupled with deterministic and stochastic optimization algorithms. In this context, structural optimization has proved extremely important in satisfying ever increasing demands of designing better products while dealing with size, shape and topology of structures. With landmark article by Schmit in 1960, [1]began an era of extensive research in this field which initiated and indicated the possibility of coupling various numerical simulation techniques with non-linear mathematical programming to generate automated optimum design capabilities to solve much broader range of problems. Apart from academic research, several big corporations in the domain of aircraft, automotive, machine tools and other industrial products have put their research interest in translating structural optimization theory into tangible benefits over the years. At the outset of discussion, it is important to differentiate scope of different structural optimization problems i.e. sizing, shape and topology optimization [2].

a) Sizing optimization: The aim is to determine optimum thickness or cross-sectional area of structural member for minimizing cost or maximizing design performance.
b) Shape optimization: It deals with smooth modifications of domain boundaries to locate optimum geometric profile when design topology remains fixed.

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c) Topology optimization: It works with maximum design flexibility and determines optimum material distribution through generation and merging cavities.

Compared to sizing optimization, shape optimization is considered more significant as it improves structural conditions like stress distribution and structural weight. During the entire shape design optimization process, the design domain keeps on changing through design variable updates and subsequent internal and/or external boundary variations [3]. Formally, shape optimization is defined as an *iterative process of determining optimum geometric configuration by integrating design, analysis and optimization activities under prescribed design constraints*. A typical structural shape optimization problem with loading and boundary conditions is shown in Fig. 1.Here, the problem domain is Ω and its boundary is Γ. Design variables are the key points on the boundary of the structure.

The shape design optimization problem can be defined as constrained non-linear optimization problem as follows,

$$\begin{aligned} &\text{Min.} f(X) = min(area\,(\Omega)) \\ &\text{s.t} : g_j(X) \leq g_j^{max} \quad j = 1, 2, \ldots m \\ &\quad X = (x_1, x_2, \ldots . x_n)^T \\ &\quad x_i^l \leq x_i \leq x_i^u, \quad i = 1, 2, \ldots n \end{aligned} \tag{1.1}$$

The procedural steps for shape optimization include the description of the geometry and modification through the design variables, evaluating structural performance through numerical analysis techniques and employing an optimization algorithm to determine optimum shape [4]. The flowchart for shape optimization process is shown in Fig. 2.

The shape optimization procedures can be generally classified into - *parametric* or *non-parametric* approaches. The parametric approach can be naturally linked to CAD geometry wherein a set of geometric parameters called dimensions like distances, radii etc. are used to define the shape. The parametric technique searches the space spanned by constrained design variables to minimize or maximize the externally defined objective function. The technique evolved from the early work of Imam [5], Braibant and Fluery [6] and Bennett and Botkin [7]. Instead, the non-parametric approach represents the geometry by the nodal positions of surface nodes in an FE model. Throughout the optimization process, the nodal positions are updated by displacing the nodes through some scalar amount along the direction of a predetermined displacement vector.. Generally, the outer normal on the design node is considered as optimization displacement vector. This approach allows modifications of every node of a set of design nodes during optimization; hence the design space includes all possible combinations of FE discretization model [8]. The technique is parameter free and useful in dealing with large-scale optimization problem efficiently [9]. Free-form implicit representation with level sets can be considered as a kind of non-parametric technique.

1.1. Challenges in shape optimization

The complexity of shape optimization problem lies in integrating the aforementioned activities related to design, analysis and optimization.

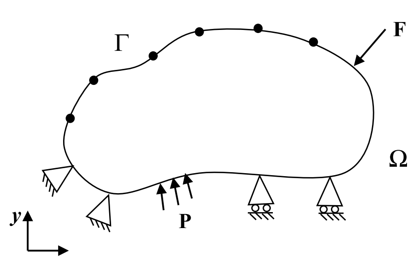

Fig. 1. 2D shape optimization.

Deficiencies of numerical analysis techniques also preclude success of structural shape optimization as solution accuracy and computational time for shape optimization largely depends on them. As FEM is extensively used for structural analysis, the resulting framework also suffers from FEM related issues like mesh distortion and subsequent remeshing associated with large shape changes, discontinuous stress values across element boundaries due to linear approximating field function and solution sensitivity for node distribution in the domain. These issues become more severe during the shape optimization process. The major challenges in the traditional shape optimization process are the following,

a) *Inconsistent geometry description for design and analysis model:* The major bottleneck in traditional FEM based shape optimization is different mathematical representations used for geometry description in design and analysis models. The design model is represented through Computer Aided Geometric Design (CAGD) elements which include Bezier, B-spline, Non Uniform Rational B-spline (NURBS) curves and surfaces. For analysis, the design model is transformed into analysis compatible model and discretized with nodes and elements. Such inconsistency in the process not only leads to issues related to accuracy but also increases computational time and efforts [10–12].
b) *Selection of design variables:* In selecting shape design variables, two often contradictory considerations are - to minimize the total number of design variables and to allow for as much geometric freedom as possible in order to capture the true optimum configuration [5].
c) *Domain remeshing*: At every step of optimization process, geometries vary and accordingly the analysis model needs to be updated. With conventional FEM, it is difficult to adapt to those domain alterations as the initial mesh may not be qualitatively good enough to produce accurate solutions due to mesh distortion. In this situation, remeshing becomes inevitable which adds substantial computational efforts in iterative shape optimization process [13,14].
d) *Mesh topology:* Design sensitivity computation is extremely important in gradient-based optimization techniques as it decides the convergence behavior of the overall optimization problem. In the shape optimization framework based on FEM and classical gradient-based optimization technique, maintaining mesh topology for initial and perturbed geometries is devisable to avoid 'numerical noise' for sensitivity computation and subsequent optimization process [15–17].
e) *Solution Accuracy:* Linear interpolation functions used in traditional FEM lead to discontinuous secondary field variables like stress across the element boundaries and thus require post-processing techniques. As sensitivity computations involve derivatives of performance measures like stress with respect to the design variables alterations, the solution accuracy of the numerical analysis becomes more important. Higher order continuity in the field function approximation improves the quality of solution, the sensitivity information and the stability of the overall optimization process [15].

Most of the abovementioned deficiencies are related to FEM. Hence, to mitigate these issues, alternative numerical analysis techniques have been employed such as Boundary Element Method (BEM) [18], Fixed Grid FEM (FGFEM) [19], eXtended FEM (XFEM) with level sets [20], meshless EFG method [21], IGA [11], IGA boundary element method (IGABEM) [22], eXtended IGA (XIGA) [23]. These numerical analysis frameworks have their own capabilities and concerns based on their mathematical background. These techniques are briefly discussed with their merits and demerits in subsequent sections.

During last four decades, several review articles have also been published in this field. It is essential to analyze these reviews in order to understand researcher's area of interest and subject focused. A brief summary of these review articles is presented in Table 1. The review span for the listed review articles is approximated on the basis of its year

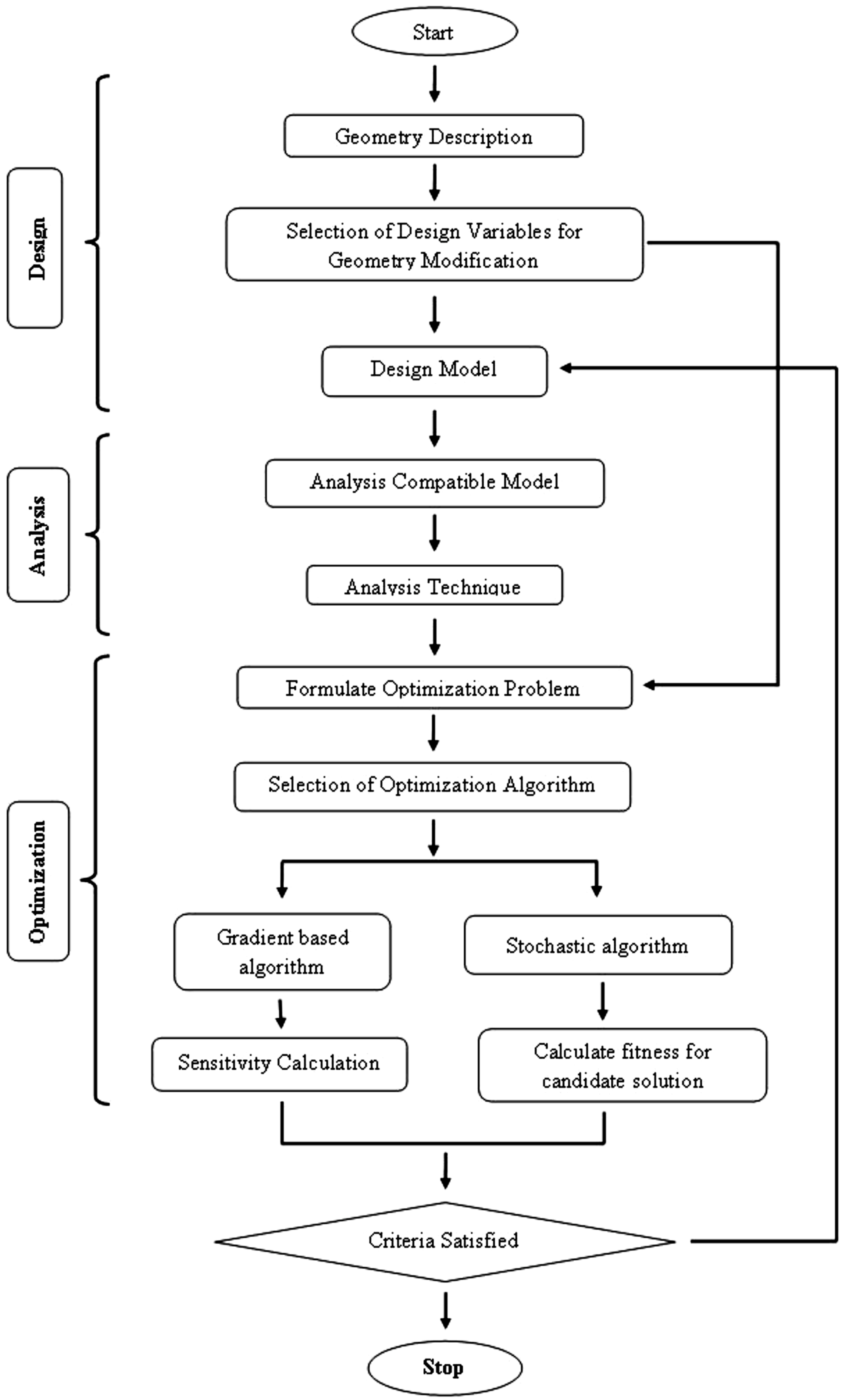

Fig. 2. Shape optimization flowchart.

of publication and cited articles.

The results of Table 1 indicate the following areas of study to be common,

a) Computational aspects of shape optimization like geometry description, modification and parametrization, sensitivity analysis and gradient-based optimization techniques are studied extensively
b) FEM related issues e.g. mesh generation, refinement and sensitivity analysis are also studied at length

Table 1
Summary of literature reviews in shape optimization.

Reference	Review span	Subject focused	Areas of interest for review
Wang et al. 2018 [24]	2005-2018	IGA based size, shape and topology optimization	a) Geometry parametrization and sensitivity analysis in shape optimization b) Solid Isotropic Material with Penalization (SIMP), level set methods, moving morphable components and phase field model in topology optimization
Daxini and Prajapati 2017 [25]	2000-2017	Shape optimization techniques based on MMs	a) MM like EFG and RKPM as an alternative numerical analysis technique and DSA with MM b) Linear elastic, thermoelastic, structure dynamics and frictional contact problems in shape optimization with MM
Saitou et al. 2005 [26]	1980-2005	Structural optimization and its relation to mechanical product development	a) Geometry parametrizations in size, shape and topology problems b) Approximation methods, optimization techniques and integration with non-structural issues like cost, manufacturing and assembly
Hsu 1994 [27]	1987-1993	Developments in shape optimization	a) Geometry representation, sensitivity analysis, structural analysis techniques and optimization algorithms b) Challenges in extending shape optimization from 2D to 3D problems c) Zero-order optimization algorithms
Haftka and Grandhi 1986 [13]	1970s to Mid 1980s	Developments in shape optimization for 2D and 3D problems	a) FEM related issues in shape optimization process b) Design variables selection and automated mesh generation c) Sensitivity analysis techniques
Ding 1986 [3]	1970s to Mid 1980s	Numerical and analytical methods of shape optimization	a) Geometry representation and parametrization techniques b) FE mesh generation, refinement and sensitivity analysis c) Representative examples in 2D and 3D shape optimization

c) Alternative numerical analysis techniques like IGA and MMs in shape optimization are also reviewed (separately)

The objective of this work is to present developments, post 2000, in the field of shape optimization from the perspective of numerical analysis techniques. The present study may not be exhaustive in content because shape optimization is a broad subject covering aeronautical, automobile, mechanical, civil and electrical domains. The scope of the present work is restricted to shape optimization applications in mechanical and automotive fields.

The layout of the remaining article is as follows. Section 2 presents a detailed analysis of shape optimization literature with particular emphasis on employed numerical analysis techniques. Section 3 present shape optimization approaches based upon FEM and its modified versions along with summary of literature. Section 4 presents IGA and its extended versions like XIGA, IGABEM along with summary of literature. Section 5 presents MMs based shape optimization along with summary of its literature. Section 6 presents observations and recommendations followed by concluding remarks in Section 7.

2. Analysis of literature

The data of the research articles, published in reputed journals across the world was extracted from the year 2000 till 19th December, 2020 from the Scopus search engine using the keyword 'shape optimization' appearing in title of articles. Subsequently, 1464 articles were found in the database. Fig. 3 below shows research trend in the field of shape optimization in last two decades. The general trend shows an increase in number of publications over the years which indicates increased research interest in this field. Another keyword, 'shape design optimization' was also used to fetch more relevant articles in the field and another 27 articles were added. For the present work, the articles related to topological shape optimization are not considered as the primary focus of this work is solely structural shape optimization.

After scrutinization, a total of 209 research articles were identified and referred to in preparation of this review. Detailed analysis of the selected articles post 2000 was carried out from the perspective of numerical analysis techniques and the same is shown in Fig. 4. For convenience, numerical analysis techniques are broadly grouped into FEM based, BEM based, IGA based and MM based techniques. The numerical analysis techniques which do not fall in any of these categories are part of other techniques.. A brief discussion of the alternative analysis techniques in context of shape optimization are presented in the subsequent sections.

Based on the analysis of collected literature, we identified journals publishing articles on structural shape optimization (Table 2).

3. FEM based shape optimization

The development in structural optimization field roughly began during the same timeframe as FEM. However, due to the fact that structural optimization needs repetitive structural analysis for candidate designs, development in structural optimization lags behind numerical techniques. Today's high-end commercial tools based on FEM e.g. ANSYS, ABAQUS, MSC Nastran provide solutions for structural optimization problems through their dedicated modules. Being the most established numerical technique, FEM has been used extensively for shape optimization since early 1970s. Refer well-known review articles [3,13] for more details. However, inherent issues related to mesh-based design frameworks like mesh distortion resulting from large shape variations and inconsistent geometry description for design and analysis models compelled researchers to modify existing framework or develop more robust numerical techniques.

To alleviate these issues, various techniques have been developed which include design parametrization techniques, filtering and regularization methods, traction method, mesh morphing and adaptive mesh refinement. These techniques are briefly discussed here.

3.1. *Improving mesh-based framework*

3.1.1. *Node based shape optimization*

The earliest shape optimization based on FEM used coordinates of boundary nodes of FE mesh as design variables [28]. Although individual node movements can cover more design options and avoids shape design parametrization, this method encountered several issues such as increased number of design variables which leads to increased computational burden and the infeasible and jagged geometric shapes which are meaningless. To alleviate these issues, geometry parameterization techniques were developed during 1980s and 90s as follows: a) popular design element concept [5,6,29,30], b) polynomial equations [31–33],

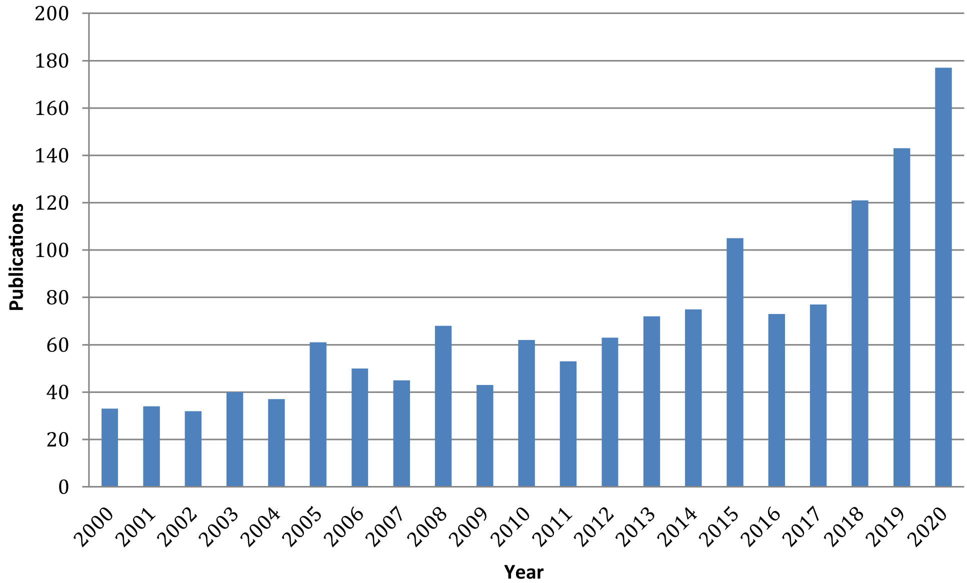

Fig. 3. Year wise publications – Scopus database.

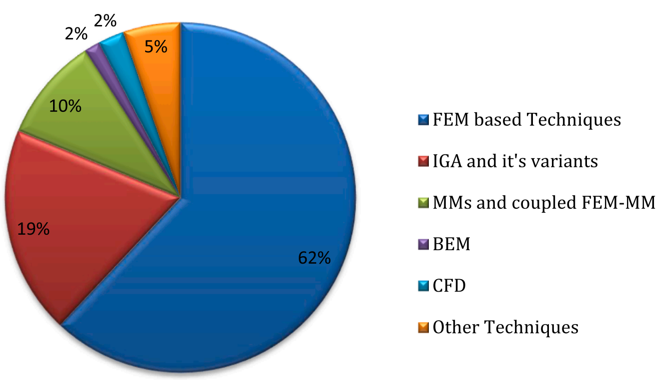

Fig. 4. Analysis techniques in shape optimization.

Table 2
List of journals involved in publishing.

Sr. No.	Journal Name	No. of articles	%
1	Structural and Multidisciplinary Optimization	67	32.06
2	Computer Methods in Applied Mechanics and Engineering	40	19.14
3	International Journal for Numerical Methods in Engineering	23	11.00
4	Computers and Structures	11	5.26
5	Computational Mechanics	7	3.35
6	Journal of Mechanical Science and Technology	4	1.91
7	International Journal of Mechanical Sciences	4	1.91
8	Engineering Optimization	3	1.44
9	Optimization and Engineering	2	0.96
10	Procedia Structural Integrity	2	0.96
11	Advances in Mechanical Engineering	2	0.96
12	Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science	2	0.96
13	Engineering with Computers	2	0.96
14	Engineering Analysis with Boundary Elements	2	0.96
15	Mechanics Based Design of Structures and Machines	2	0.96
16	Other journals	36	17.22
	TOTAL	209	100.00

c) spline representation i.e. Bezier, B-spline, Cubic spline [6,34,35] and d) natural design variables – fictitious loads [36,37]. In general, these techniques help in reducing number of design variables and achieving smoother geometric shapes. These parametrization techniques are well-known; further details can be found in other review articles [3,13].

As mentioned earlier, node based shape optimization using FE meshes generates impractical jagged geometric shapes. This issue stems from non-smooth nature of shape derivatives. To avoid numerical instabilities, filter methods and regularization techniques are also used for gradient-based optimization schemes. In-plane and out-of plane regularization are applied by separating coordinates of FE nodes into two groups based on coordinates defining geometry (normal coordinates-also work as a set of possible optimization variables) and shape of elements (tangential and internal coordinates). The normal direction coordinates are treated with out-of plane regularization while tangential and internal coordinates are treated with in-plane regularization which is also denoted as mesh regularization. In-plane regularization ensures reliable FE meshes with acceptable values of edge and face angles and aspect ratio for 2D and 3D elements. This strategy ensures smooth gradient fields and smooth geometry updates. In general, mesh regularization techniques are categorized into geometrical and mechanical approaches. Form finding based mechanical approach for mesh regularization is often used which is based on determining free-form equilibrium shape membrane subjected to certain stress field [38–40].

Vertex morphing is another technique for controlling the shape in a node based optimization framework by combining out-of-plane filtering and in-plane regularization. The concept is based on introducing a control field and linking it to the geometry by a linear map. Later, the optimization problem is formulated in the control space and after every iteration the modified control space is used to control the geometry of problem domain. Here, normal and tangential components of coordinates are treated simultaneously and hence design alterations can be performed without any additional mesh regularization. This technique is robust and can be integrated in the optimization process independently [41].

With reference to non-parametric shape optimization, *Traction method* (*H^1 gradient method*) was developed to compensate for non-smooth shape gradient by applying gradient method in Hilbert space [42–44]. In this approach, domain variation leading to minimizing the objective function is obtained as a solution to boundary value problem of a linear elastic continuum defined in the design domain and loaded with distributed external force or traction in proportion to the negative shape gradient on design boundary. This approach results in design domain reshaped with smoother boundary for one-time differentiability than that obtained with direct gradient method wherein boundary is moved in proportion to the shape gradient.

3.1.2. Mesh morphing

The term *morphing* is used for special effects in motion pictures and animations which change an image through seamless transition. Mesh morphing algorithms can be mesh-based or meshless [45]. Mesh-based algorithms use mesh topology of existing mesh being morphed while updating node locations in computational space. Meshless algorithms work on extracted grid point locations from grid cell structure and disregard mesh topology during morphing. In FEM based shape optimization framework, mesh morphing changes FE models without adding or removing nodes or elements. It alters the node locations and element shapes to generate new FE models that adapt to changes in geometry during optimization process. The mesh-based morphing attempts to maintain the constant mesh topology while altering the node locations which doesn't disturb the computation of sensitivity and the optimization process. However, it must be noted that there are limitations on geometry alterations while working with constant mesh topology. In practice, meshless based Radial Basis Functions (RBFs) prove much useful morphing technique as these functions are the mathematical tools used to interpolate some know field in multidimensional scattered data points. The grid cell structure is disregarded during the morphing step in RBFs. However, sometimes it leads to distorted grid cells with new coordinates of nodes and old cell structure. RBF Morph is a commercial tool which alters starting point mesh into new geometries through morphing and evaluates arrange of shapes to determine optimal configuration. In general, mesh morphing proves computationally more efficient than complete remeshing for small changes in geometries (locally) during optimization. Again, the morphed mesh is qualitatively equally good as old mesh and hence preserves solution accuracy [46]. This subject has remained field of intensive research in computational mechanics and computer graphics. More details related to various mesh-based and meshless morphing algorithms can be referred from [45].

3.1.3. Adaptive mesh refinement

Adaptive FEM with *h, p* and *r* refinements are not new in shape optimization and dates back to 1980s [7,47,48] and still in practice [49]. Here, *h* refinement refers to element size, *p* refinement refers to polynomial order of element and *r* refinement refers to node repositioning. In practice, *h* method is frequently used wherein the order of the shape functions remains fixed and adaptivity is implemented through smaller sized elements. In contrast, *p* method uses coarse mesh with large elements and adaptivity is implemented by increasing the order of the polynomial. From the perspective of shape optimization, p method offers some benefits: first, bigger elements can accommodate large shape changes before they distort to level of degeneracy; second, mesh topology remains fixed during adaptivity process [50]. The *r* method usually works with *h* and *p* method. In a distinctive way, variational *r*-adaption based shape optimization was demonstrated in [51] based on the principle of minimum energy and its associated optimal mesh. If geometric constraints of retaining original geometric profile are removed, the variational *r*-adaption method can be used to compute equilibrium shapes which minimize energy of system.

In adaptive mesh refinement technique, reliable and accurate error estimation (priori and posteriori) and appropriate mesh refinement strategy are the two most important components. Error estimation may be carried out globally or locally. The global error estimator works on strain energy error in an averaging sense for the whole domain while local error estimator works on quality of numerical solution locally e.g. stress or displacement in particular area. Ideally, mesh refinement occurs only for those areas where relatively large errors are indicated by error estimation. Posteriori global and local error estimators are often employed due to its better efficiency [52]. Adaptive mesh refinement techniques are not widely used in industrial applications, as mesh refinement needs access to the exact geometry that means it requires seamless and automatic communication with CAD which does not exist [53].

3.2. XFEM with level sets

To model crack problems without cumbersome remeshing, Moes et al. [54] proposed a novel numerical technique i.e. XFEM based on pure displacement-based approximation enriched near crack by discontinuous fields and near crack tip by asymptotic fields through partition of unity. The technique is capable of modeling entire crack independent of mesh and avoids remeshing. Here, FE mesh need not to conform to internal boundaries. In this case, the modified approximation for displacement considering discontinuity is as follows,

$$u^h(x) = \sum_i N_i(x)u_i + \sum_j a_j N_j(x)\, H(x) \tag{3.1}$$

where $N_i(x)$and $N_j(x)$ are the standard FE shape functions, a_j are the additional degrees of freedom related to enrichment, $H(x)$ is the Heaviside function embedded to model discontinuity due to crack. It is to be noted that the element near discontinuity only needs to be modeled with modified approximation fields and the other elements remain unchanged. On the other hand, level set method is a numerical approach for tracking moving interfaces [55]. The concept is based on implicit representation of interfaces as a level set of higher dimension function. Sukumar et al. [56] firstly demonstrated modeling of holes and inclusions through level sets within XFEM framework. Later, this combination proved quite promising in tackling shape and topology optimization problems as level sets facilitates flexible geometry description including topology modifications over fixed non-conforming mesh of XFEM. This approach is free from remeshing and competes well with other fixed grid/Euleiran grid based shape optimization techniques.

3.3. Fixed Grid based techniques

In traditional FEM (i.e. standard Lagrangian approach), body fitted mesh deforms with problem geometry and results in distorted initial mesh during shape optimization process which impairs solution accuracy of FEM. Remeshing becomes inevitable in such situations which is tedious and time consuming task. In topology optimization, 'fixed' finite element models are used to generate internal cavities by identifying less stress finite elements through homogenization approach [57].

To eradicate mesh distortion issue in shape optimization, FG-FEM/ Eulerian grid technique can be employed which maintains initial mesh geometry throughout the optimization process. FG-FEM concept can be employed by discretizing the design space with smaller regular elements

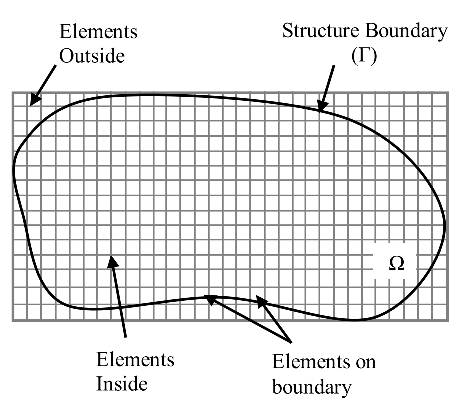

Fig. 5. Typical mesh in FG-FEM.

and superimposing it on the structure's geometry (Fig. 5). During shape optimization, the structural boundaries are allowed to move and the material properties of each finite element are then altered according to the amount of material it contains which is often termed as area ratio α [58]. Another term used for material property of individual finite element is shape density in similar sense. The participation of individual element is determined by its shape density i.e. if the element belongs to structural domain then it is assigned full shape density. In another case if it is outside the structural domain it is assigned zero shape density and if an element is on the edge, it has shape density between full magnitude and zero [59]. This technique eliminates mesh distortion completely as boundary changes are efficiently reflected in structures through changes in material properties/shape density of fixed FE grid [60]. It must be noticed that the design space discretized by fixed regular grid must cover all possible design combinations resulting from design variable alterations to determine optimal geometric configuration.

Interfaced Enriched Generalized Finite Element Method (IGFEM) is another variant which is formulated on fixed grid and avoids mesh distortion and remeshing problems. The technique was proposed by Soghrati et al. [61] to model problems with discontinuity in gradient field which is also referred as weak discontinuity by applying generalized degree of freedoms to the interface nodes of non-conforming mesh. Typical examples of weak discontinuity include problems with solution field that are C° continuous. Najafi et al. [62] demonstrated gradient-based shape optimization with IGFEM in structural and thermal fields. Later, the concept of IGFEM was extended by combining NURBS with IGFEM to take advantage of Eulerian approach and accurate geometry representation by NURBS [63]. The combined approach was applied to solve structural optimization problem with gradient-based optimization technique.

3.4. Finite cell method (FCM)

FCM is considered as an extension of classical FEM and was proposed by Parvizian et al. [64]. The technique falls in the category of embedded domain methods or fictitious domain methods wherein problem domain under consideration is extended beyond their complex boundaries into a bigger embedding domain of simple geometry which is then discretized through a simple structured grid. In this way, the technique is free from a boundary conforming mesh and thus eliminates mesh dependent results in shape optimization (Fig. 6).

Geometry representation is carried out implicitly e.g. level set method which allows topological modifications over fixed grid [65]. The method employs higher order Legendre shape functions for field variable approximation and adaptive integration schemes to take into account discontinuity in integrand within cells being cut by problem boundary. Higher order Gauss quadrature or quadtree technique can be adopted for adaptive integration by decomposing elements cut by geometry into sub-cells. In context of shape optimization, FCM offers several advantages like any structure of complex geometry can be optimized by embedding it in the simple regular fictitious domain and fixed mesh avoids mesh updating issues completely [66]. Comprehensive details on FCM can be referred from [67].

Contribution of FEM based techniques in shape optimization is presented in Fig. 7 and brief details of publications are summarized in Table 3.

4. IGA based shape optimization

The fundamental requirement for success of structural shape optimization is seamless integration of design, analysis and optimization models. To mitigate issues related to inconsistent geometry description techniques for design and analysis models, various techniques integrating CAD/Analysis had been proposed in literature which includes the use of B-splines, IGA, NURBS enhanced FEM and parametric-based implicit boundary representation. A Comprehensive review of these techniques can be referred from [184]. Usage of CAD fundamentals in FEM is not new but dates back to [185,186] where uniform B-spline Finite Element (BSFE) environment i.e. using B-spline functions as base functions in FEM was used for both design and analysis. Advancing remarkably in this direction, an alternative numerical technique was proposed by Hughes et al [11] i.e. IGA. This technique has many similarities with traditional FEM but at the same time it differs in some important aspects. The unique feature of IGA is its ability to unify design and analysis model through NURBS basis functions to accurately model geometry and the approximation of solution. The exact CAD geometry has been modeled through NURBS surfaces and subsequently NURBS elements. Later, the same functions are used for analysis purpose in regard to isoparametric concept. In this way, IGA integrates two different fields i.e. design and analysis and regarded as a fundamental step forward in numerical techniques. It provides evident benefits in structural shape optimization problems which are as follows [11]:

- The unification of design and analysis model is established
- Exact geometry representation and higher order field variable approximation improves overall accuracy
- Reduced design to analysis time
- NURBS facilitates flexible and efficient shape control which perfectly qualify for shape optimization and generates smoother optimal shapes
- Simple refinement strategies which do not require any further communication to CAD model
- Exact geometry is maintained throughout the process

In view of abovementioned benefits, in their seminal article, Hughes et al. [11] mentioned IGA based shape optimization as one of the potential research area. Since then this field has received renewed interest in last one decade.

4.1. Bezier, B-spline, NURBS and T-spline

The shape design problems falling in the category of *Ab initio* design e.g. 'skin' of car bodies, ship hull etc. cannot be formulated completely using quantitative criteria but needs combination of computational and heuristic methods. Bezier curves and surfaces provide an attractive alternate method of defining free-form curves and surfaces. Bezier curve is uniquely defined by the vertices of the open polygon and the curve points are given by,

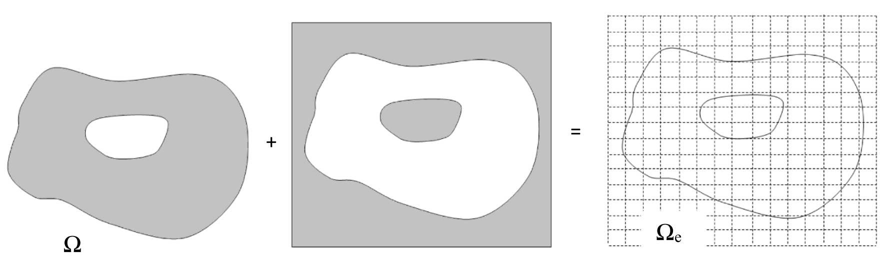

Fig. 6. Computational domain Ω in embedded domain Ω_e.

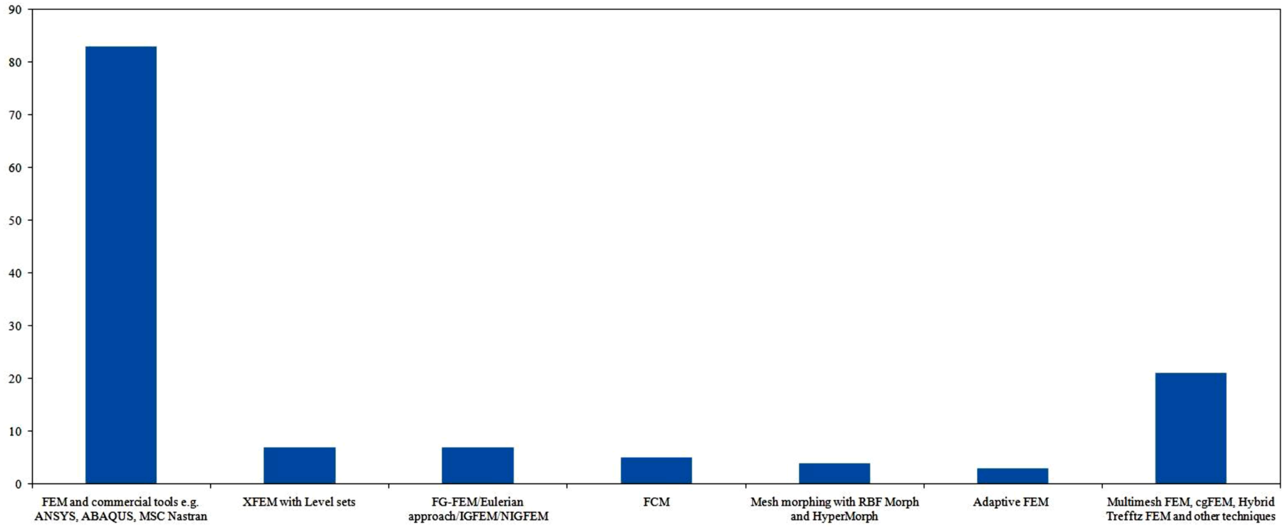

Fig. 7. FEM based techniques in shape optimization.

$$P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t) \qquad 0 < t < 1 \tag{4.1}$$

where, $J_{n,i}(t)$ is the Bernstein basis function and it is given by,

$$J_{n,i}(t) = nC_i t^i (1-t)^{n-i} \tag{4.2}$$

where, n is degree of polynomial and i is the particular vertex. The resulting curve has interesting properties like the curve lies within the convex hull of defining polygon, the curve follows defining polygon and exhibits variation diminishing property. However, some characteristics of this type of curve limits their flexibility like the polynomial order of Bezier curve is always decided by the number of polygon vertices, for instance, $n+1$ polygon vertices define n^{th} order curve. Another limitation of this type of blending function is the lack of local control which means location of each control point influence the curve geometry [187]. In contrast, B-spline curve is defined with non-global basis function which is useful in controlling the curve shape locally. They also possess flexibility of changing order of resulting curve without changing number of polygon vertices. As NURBS and other advanced representations like T-splines uses same fundamentals of B-spline, this section provides concepts of B-spline and NURBS basis functions briefly. In general, B-spline generates a single piecewise parametric polynomial curve through any number of control points with desired order of polynomial. In this way, B-spline possesses required flexibility (unlike Bezier curve) of choosing degree of curve independent of number of control points. B-spline basis functions are defined in parametric space on so called *knot vector*, $k(\xi)=\{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}^T$ where $\xi \in [0,1]$, ξ_i is i^{th} knot, n is number of basis functions and p is the polynomial order of B-spline. Based on the difference of two consecutive knot values, the knot vector is either uniform or non-uniform. Based on the knot vector $k(\xi)$, B-spline basis functions, $N_{i,p}(\xi)$, of order p is defined using Cox-de boor recursive relations,

$$\text{For } p = 0,\ N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \le \xi \le \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

$$\text{For } p \ge 1, \quad N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \tag{4.4}$$

It is worth to note that evaluating above functions, ratios of the form 0/0 are taken as zero. Multivariate B-spline basis functions i.e. bivariate and trivariate can be derived using tensor product of univariate functions. For d dimensional space, n control points and n B-spline basis function, p^{th}order B-spline curve is defined as,

$$C(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi) B_i \tag{4.5}$$

where B_i coefficients of the basis functions are the control points of the curve.

The important properties of B-spline basis functions include,

a) convex hull property,
b) partition of unity, i.e. for $\forall\xi$, $(\sum_{i=1}^{n} N_i(u) = 1)$,
c) non-negativity over entire domain,
d) local support property $(N_{i,k}(u) = 0 \quad if \quad u \notin \quad [u_i, u_{i+k+1}])$,
e) continuity and ease of refinement
f) these functions are approximants and not interpolants
g) they possess weak Kronecker delta property and hence imposing essential boundary conditions requires extra care in analysis.

NURBS are considered as an extension of B-spline by introducing weights for basis functions to deflect curve/surface towards or away from its control points. This makes NURBS most versatile and flexible technique with all desirable geometric properties. Major strengths of NURBS are as follows, they are convenient free-form surface modeling technique, they can represent all conic sections and there are many efficient and numerically stable algorithms exist to generate NURBS objects. The useful mathematical properties of NURBS include ability to be refined through knot insertion, C^{p-1}continuity for order p NURBS and the convex hull property [53]. These features make NURBS most appealing computational geometry technique within framework of IGA. The NURBS basis functions (rational functions) are defined as,

$$R_{i,p}(\xi) = \frac{N_{i,p}(\xi)\, w_i}{\sum_{i=1}^{n} N_{i,p}(\xi) w_i} \tag{4.6}$$

Where $w_i \ge 0$ and termed as weight associated with particular control point vector. Similar to B-spline basis functions, multivariate NURBS basis functions can be derived through the tensor product of univariate NURBS basis functions. It is to be note that these multivariate functions possess all properties of their univariate ingredients. Similar to B-spline curve (4.3), NURBS curve with B_i control points and basis function $R_{i,p}(\xi)$ is defined as,

$$C(\xi) = \sum_{i=1}^{n} R_{i,p}(\xi) B_i \tag{4.7}$$

In IGA, the exact geometry is captured through NURBS basis

Table 3
Summary of FEM based techniques in shape optimization.

References	Problem Type	Analysis Technique	Optimization Technique	Numerical Application
Zhu S. et al., 2020 [68]	Structural Problem	FEM	GA	aluminum alloy spherical shells with gusset joints
Lemared S. et al., 2020 [69]	Structural Problem	FEM with NASTRAN	NASTRAN optimization solver	Space telescope mirror
Gu X. et al., 2020 [70]	Structrual problem considering fatigue - shape memory alloy (SMA)	FEM with Abaqus (Simulia)	Tosca Structure (Simulia)	Plate with a hole in centre, self-expanding SMA stand
Nonogawa M. et al., 2020 [71]	Structural Problem	ABAQUS - commercial FE tool	OPTISHAPE-TS using H^1 gradient method	Running shoe
Wang W. et al., 2020 [72]	Thermal problem	FEM with COMSOL	COBYLA - derivative-free method	Cold plate channels
Etling T. et al., 2020 [73]	—	FEM with restricted mesh deformation	Gradient based technique	—
San B. et al., 2020 [74]	Structural problem - Elastic, hyperelastic and nonhomogeous elastic	Analytical method and FEM	Gradient based technique	Vertical hanging bar
Dong Y. et al, 2020 [75]	Structural Problem	FEM	Adaptive simulated annealing	Fabric rubber seal for aircraft door seal
Porziani S. et al, 2020 [76]	Structural Problem – Linear Elasticity	ANSYS workbench FEA tool with RBF morph	Biological Growth Method (BGM)	Cantilever beam, Turbine blade
Garcia-Andres X.et al, 2020 [77]	Structural Problem– Dynamics	FEM	GA	Cross section of Railway wheel
Nguyen S. et al, 2020 [78]	Structural Problem	FEM with trimmed hexahedral mesh	Gradient-based technique	Two-bar truss design problem, L shaped beam
Chen L. et al, 2020 [79]	Structural Problem	Adaptive T-spline FCM	MMA	Plate with a circular hole, torque arm and bracket geometry
Cai S..et al, 2020 [80]	Structural Problem	FCM with fixed Eulerian mesh and Smoothly deformable implicit curve	Globally Convergent Method of Moving Asymptots	Cantilever beam, Cantilever beam with a hole, displacement inverter and push gripper
Florio C. S., 2019 [81]	Structural Problem - Linear Elasticity	FEM	Biological based (evolutionary) Gradient less algorithm	Cantilever Beam; the central hole of a biaxially loaded infinite plate; both surfaces of a hollow cylinder under combined compression, torsion, and bending loads
Hafner C. et al., 2019 [82]	Structural Problem	XFEM	Gradient based technique	Wrench; Fidget Spinner; Motor Housing
Shi J.-X. et al. , 2019 [9]	Structural Problem - Linear Elasticity	FEM	H^1 gradient method (The traction method)	shear panel dampers with and without hole
Agarwal D. et al. , 2019 [83]	Structural Problem	FEM; CFD	Sequential Least Square Programming	cantilever beam; S-bend
Kuci E et al. , 2019 [84]	Electro-Mechanical application	FEM	MMA	Permanent magnets synchronous machines (PMSM)
Shimoda M. et al. , 2019 [85]	Structural Problem	FEM	H^1 gradient method (The traction method)	Solid structures (rectangular solid structure, stool-type solid structure); Shell structures (rectangular box-like shell structure, stool-type shell structure); Frame structures (tower-like frame structure, twin-tower-like frame structure, stool-type frame structure)
Doganay O.T. et al. , 2019 [86]	Structural problem - ceramic material	FEM	Gradient-based technique – Bi-objective descent algorithm	—
Frohlich B. et al. , 2019 [87]	Structural Problem	FEM	SQP	Beam; An adaptive structure
Ertl F.-J. et al. , 2019 [88]	Structural Problem	FEM with CalculiX	Gradient projection method	Fillet; Turbine blade
Zhao Z. et al. , 2019 [89]	Structural Problem-Shell structure	FEM	Improved form-finding method (IFFM)	Spherical; Cylindrical reticulated shell structures
Sharma A. & Rangarajan R. , 2019 [90]	Structural problems - Contact Problems	FEM	Gradient descent algorithm	Circular membrane in contact with flat and spherical obstacle
He G. et al. , 2019 [91]	Structural Problem	ANCF FEM	SQP	ANCF beam element
Xu W. et al. , 2019 [66]	Structural Problem - Linear Elasticity Problem	FCM with smoothly deformable implicit curve	MMA	Plate with hole
Dokken J. S. et al., 2018 [92]	Thermal Problem	Multimesh FEM	Steepest descent algorithm	Heat emitting wires; The drag minimization in Stokes flow; The orientation of 25 objects in a Stokes flow
Porziani S. et al., 2018 [93]	Structural Problem - Linear Elasticity	ANSYS Workbench FEA tool with RBF Morph	Biological Growth Method (BGM)	Bracket; Pin with a circular fillet
Groth C. et al. , 2018 [94]	Structural Problem - Linear Elasticity		Deepest descent algorithm	Cantilever beam; Structural bracket

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Table 3 (*continued*)

References	Problem Type	Analysis Technique	Optimization Technique	Numerical Application
		ANSYS Workbench FEA tool with RBF Morph		
Gunther U. et al. , 2018 [95]	Structural Problem	FEM	SQP & SLP	Chamber profile
Santhosh R. et al. , 2018 [96]	Structural Problem	Analysis with low fidelity model and validated through FEA	Gradient based technique	Shallow domes
Schmitt O. & Steinmann P. , 2018 [97]	Structural Problem	FEM	Steepest descent algorithm	Arch support; Strutting;
Zhang W. & Niu C. , 2018 [98]	Structural Problem-Contact Mechanics problems	FEM	Globally Convergent Method of Moving Asymptotes (GCMMA)	1D elastic bar-rigid wall contact problem; Hertz contact problem; an assembled aero-engine structure for leak proofness
Murai D. et al. , 2018 [99]	Structural and thermal problem	FEM - COMSOL Multiphysics	Gradient based technique	Steady-state heat conduction problem; Linear elasticity problem; Cantilever beam
Sun Q. et al. , 2018 [100]	Structural Problem	FEM	GA	Involute gear
Hu P. et al. , 2018 [101]	Structural Problem	FEM	surrogate modeling method; GA	Thin walled beam
Giacomini M. et al., 2017 [102]	Structural Problem - Linear Elasticity	Pure displacement formulation & Dual mixed formulation - FEA	Boundary Variation Algorithm	Cantilever Beam
Georgioudakis M. et al. , 2017 [103]	structural components under fatigue	XFEM with level set description	Differential evolution (DE)	Fillet geometry
Najafi A.R. et al. , 2017 [63]	Structural Problem	NURBS-based Interface enriched Generalized Finite Element Method (NIGFEM)	Gradient based technique	Notch specimen; Cantilever beam
Noel L. & Duysinx P. , 2017 [104]	Structural Problem	XFEM with level set description	MMA	Single inclusion microstructures under hydrostatic and shear loadings; Microstructures reinforced by stiff inclusions
Schmitt O. & Steinmann P. , 2017 [105]	Structural Problem - Linear Elasticity	FEM	Augmented Lagrangian method	Sine curve; Infinite plate with circular hole (2D); cube with a spherical hole (3D)
Jiang L. et al. , 2017 [106]	Structural Problem	FEM	Downhill simplex method	Flywheel
Riehl S. & Steinmann P. , 2017 [107]	Structural Problem	FEM - Background mesh and adaptive mesh refinement	Augmented Lagrangian method	L-shape under uniaxial loading; Hook profile; Cantilever beam
Marco O. et al. , 2017 [108]	Structural Problem	Cartesian grid FEM (cgFEM)	SQP	Thick-wall infinite cylinder loaded with internal pressure (defined by 1 & 4 design variables); Connecting rod
Tabatabaei S. et al. , 2016 [109]	Electric structural application	Analytical, Experimental and FEA	Multi-objective artificial immune system (AIS)	cantilever piezoelectric energy harvester
Schmitt O. et al. , 2016 [110]	Structural problem - Manufacturing considerations	FEM in pyOpt (FEM program in Python)	Sparse Nonlinear OPTimizer (SNOPT)	Shaft; Piston
Landkammer P. & Steinmann P. , 2016 [111]	metal forming	FEM	Gradient-based - Non-invasive heuristic approach	isotropic elasto-plasticity; orthotropic elasto-plasticity; solid-shell-elements; The ring compression process; extreme bending of an elasto-plastic cantilever beam
Noel L. et al. , 2016 [112]	Structural Problem	XFEM with level set description	MMA	Bar made of quadrangular elements; an infinite plate; complex geometries
Mohite P.M. & Upadhyay C.S., 2015 [49]	Structural Problem - Linear Elasticity	Adaptive FEM	complex search method algorithm	laminated composite plate
Najafi A. R. et al. , 2015[62]	Thermal Problem	Interface-enriched Generalized Finite Element Method (IGFEM)	Gradient based technique	Thermal verification problem: circular inclusion; Structural verification problem: elliptical inclusion; Thermal application: microvascular materials; Structural application: particulate composites
Liu Y. & Shimoda M. , 2015 [113]	Structural Problem-Shell structure	FEM	Gradient based technique	square plate; roof shell; L-shaped bracket; dome-shaped shell
Liu Y. et al. , 2015 [114]	Structural Problem	FEM	H^1 gradient method (The traction method)	stiffened circular plate; tiffened thin-walled box
Tanaka N. et al. , 2015 [115]	Structural Problem	FEM	Firefly algorithm (FA)	Free-form surface shell
Zhang W. et al. , 2015 [116]	Structural Problem	Weighted B-spline Finite cell method (FCM) combined with Level set function	GCMMA	Infinite plate with a circular hole; curved beam; torque arm; bracket
Shintani K. & Azegami H., 2014 [117]	Structural Problem	FEM with RADIOSS	H^1 gradient method (The traction method)	Brake model
Cai S. Y. et al., 2014 [65]	Structural – Linear elasticity	B-spline FCM	GCMMA	Torque arm, bracket geometry
		FEM	MMFD	hull structures

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Table 3 (*continued*)

References	Problem Type	Analysis Technique	Optimization Technique	Numerical Application
Choi M.-J. et al. , 2013 [118]	Thermo-elastoplasticity problems			
Hunkeler S. et al. , 2013 [119]	Structural Problem	FEM - SFE CONCEPT	Non gradient-based optimization technique	The front rail of a standard passenger car; prismatic beam example
Nasuf A. et al. , 2013 [120]	Structural Problem	FEM	Grammatical Evolution (GE); GA	A crane hook
Azegami H. et al. , 2013 [121]	Structural Problem-link mechanism	FEM	H^1 gradient method (The traction method)	A piston-crank mechanism
Ammar A. et al. , 2013 [122]	—	FEM and Proper Generalized Decomposition (PGD)	Gradient based technique	Quadrilateral with one per ametrized edge. Multicriteria shape optimization
Wang D. & Zhang W. , 2013 [123]	Structural Problem	FEM- material perturbation method (MPM) using fixed mesh	Gradient based technique	Connecting rod; dual-web disk; 3D thick plate with an elliptic hole; thin-walled cylindrical tube
Ou H. et al. , 2013 [124]	Structural Problem- Contact Mechanics	FEM	Direct approach	N-shaped component; Wedge contact; Turbine fir-tree root
Martini K. et al. , 2013 [125]	Structural Problem-Elastic Multibody system	FEM	Durability-based shape optimization (Rainflow counting method)	steering system
Wilke D.N. et al. , 2013 [126]	Structural Problem	FEM	Gradient based optimization algorithm - sequential spherical approximation algorithm	Bow-tie structure; cantilever beam; Spanner design
Firl M. et al. , 2013 [39]	Structural Problem	FE-based parametrization with filter methods and mesh regularization	Gradient based technique	Quadratic plate with corner support; L-shaped cowling
Kruzelecki J. & Proszowski R. , 2012 [127]	Structural Problem	FEM	SA algorithm	Thin-walled pressure vessel head
Kasolis F. et al. , 2012 [128]	Structural optimization	Fixed mesh FEM	Gradient based technique	acoustic horn
Wei Y. et al. , 2012 [129]	Structural Problem- Contact Mechanics problems	Feasible Direction Interior Point Method (FDIPM) with FEM, ABAQUS and Augmented Lagrange Multiplier Method (ALMM)	FDIPM	Circular plate in contact with rigid surface; Thick cylinder in contact with rigid surface;
Gerzen N. et al. , 2012 [130]	Structural Problem	FEM	SQP	Splicing plate, Plate with a hole
Firl M. & Bletzinger K.-U. , 2012 [131]	Structural Problem-Thin walled structures	FE based parametrization with non-linear kinematics	Gradient based technique	Quadratic plate with central loading, Tunnel shell
Lee G. et al. , 2012 [132]	Structural Problem	Mesh Morphing- HyperMorph	Evolutionary Optimization	Mobile Phone
Ozturk U.E. , 2011 [133]	fatigue based shape optimization	FEM - ABAQUS and FEMFAT	OPTISTRUCT	oil sump of a four wheel drive vehicle;
Lu B. et al. , 2011 [134]	Frictional Contact Problem	FEM	direct search methods (modified simplex, random direct search and enhanced Powell's)	upsetting of a cylinder, forging of aerofoil sections; a forward extrusion of components
Edke M.S. & Chang K.-H. , 2011 [135]	Fracture Problem	XFEM with level set description	SQP	Connecting rod
Le C. et al. , 2011 [136]	Structural Problem	FEM	Gradient based technique	2D- plane stress fillet design; 2D- plane stress hole design; 3D- fillet design; 3D- tube under torsion
Espath L.F.R. et al. , 2011 [137]	Structural Problem-Shell structure	FEM	SQP	circular arch; square plate - simply supported at the corners; conical shaped shell; a quarter of a circular plate with a hole
Rodenas J. J. et al. , 2011 [138]	Structural Problem	FEM	Differential Evolution (DE)	Pipe cross-section; Hook; Gravity dam
Pathak K. K. et al. , 2010 [139]	Structural Problem - Linear Elasticity	FEM	Gradient less – ANN	Cantilever beam; Circular plate with square hole; Fixed ends beam; Column
Brujic D. et al. , 2010 [140]	Structural Problem	FEM through MSC/Patran, MSC/P-Thermal and MSC/Nastran	Multi-Objective Genetic Algorithm (MOGA)	Gas turbine disk
Sonmez F. O., 2009 [141]	Structural Problem - Linear Elasticity	FEM	direct search simulated annealing	Shoulder fillets for flat and round bars
Schafer C. & Finke E., 2008 [142]	Structural Problem	FEM with ABAQUS/Standard	Design of Experiments and regression analysis	Steel wheels
Pedersen P. , 2008 [143]	Structural Problem	FEM	Shape variation through analytical shape representation	Fillet; T-head Fillet; cavities in tension
Gustafsson E. & Stromberg N. , 2008 [144]	Thermo-mechanical problem	FEM with ABAQUS	sequential linear programming	Beam
Peng D. & Jones R. , 2008 [145]	Structural Problem	FEM	Biological optimization algorithm	A square plate with a circular hole; A rib stiffened structure in an aircraft component
Yıldız A. R. et al. , 2007 [146]	Structural Problem - Linear Elasticity	FEM	Hybrid technique - GA - Robust Parameter Design through Taguchi	A welded beam design; Disc brake design; Side door bracket of the vehicle
	Structural Problem - Linear Elasticity	FEM	conjugate gradient method with symmetric Gauss–Seidel	Plate; Connecting rod

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Table 3 (*continued*)

References	Problem Type	Analysis Technique	Optimization Technique	Numerical Application
Silva C.A.C. & Bittencourt M.L. , 2007 [147]			preconditioner (CGGS)	
Sonmez F.O. , 2007 [148]	Structural Problem	FEM	Direct Simulated Annealing (DSA)	Eccentrically loaded plate; A hook; pin-joint; torque arm
Miegroet L. V. & Duysinx P., 2007 [149]	Structural Problem	XFEM with level set description	Sequential Convex Programming Optimization Algorithm	2D-fillet in tension
Kegl M. & Brank B. , 2006 [150]	Structural Problem	FEM	Gradient based technique	A pressure vessel strip; shell structure with variable thickness; truss-stiffened shell structure ; Linear and Non-linear structural model
Edke M. S. & Chang K. H. , 2006 [151]	Structural Problem	FEM	SQP	Simple Beam; Airplane torque tube
Meske R. et al. , 2006 [152]	Structural Problem	FEM	Optimality criteria - Gradient less technique	Cantilever Beam; Simply supported beam
Duysinx P. et al. , 2006 [20]	Structural Problem	XFEM with level set description	Sequential Convex Programming Optimization Algorithm	Plate with an Elliptical Hole
Meske R. et al. , 2005 [8]	Structural Problem	FEM	Optimality criteria - Gradient less technique	Infinite plate with circular hole; Connecting Rod; Hyperelastic torque support; The contact surface of an interference fit
Liu J.-S. et al. , 2005 [153]	Structural Problem	FEM	Metamorphic development (MD)	Centrally suspended object; Turbine disk
Wang X. et al. , 2005 [154]	Thermo-mechanical Problem	FEM	MMA	Stiffeners (cross-shaped, x-shaped and star-shaped)
Pedersen P. & Pedersen N.L. , 2005 [155]	Structural Problem	FEM	Optimality criteria	plate membrane problems; plate bending problems
Lee T. H. & Lee K. , 2005 [156]	Structural Problem	FEM	SQP	Funnel in cathode ray tubes
HauBler P. & Albers A. , 2005 [157]	Structural Problem-Dynamics	FEM	TOSCA.shape	Plate with hole; multibody system
Cho J. R. et al. , 2005 [158]	Structural Problem-Contact Mechanics problems	FEM	ANN	automobile tire P205/65R14
Park K. J. et al. , 2005 [159]	Structural Problem	FEM with ABAQUS	dynamic response optimization - GENESIS	Bracket; Connecting rod
Wu Z. , 2005 [160]	Structural Problem	FEM	GA	Plate with hole; Fillet
Daniel N. Wilke et al. , 2005 [161]	Structural Problem - Linear Elasticity	Triangular finite elements - unstructured remeshing	Dynamic Q and SQP	cantilever beam; full spanner; Michell-like structure
Kim N. H. & Chang Y., 2005 [59]	Structural Problem	Fixed grid Finite Element Analysis (FG-FEA)	MMFD	Boundary geometry of the structure; Torque arm
Norato J. et al. , 2004 [162]	Structural Problem - Linear Elasticity	Fictitious domain method with FEA	MMA	Design of a plate with an elliptical hole; The compliance problem; Stress concentration problem
McDonald M. & Heller M. , 2004 [163]	fatigue based shape optimization	FEM	Gradient less algorithm	constrained holes in plates
Inzarulfaisham A.R. & Azegami H. , 2004 [164]	Structural Problem - Linear Elasticity	FEM	Traction method	Beam-like continuum problem; Plate-like continuum problem
Garcia M.J. & Gonzalez C.A. , 2004 [165]	Structural Problem	FG-FEA	Evolution strategy (ES)	Cantilever beam; Michell-type structure; Spanner; cantilever beam with multiple load; Spanner-type structure with multiple loads
Camprubi N. et al. , 2004 [166]	Structural Problem-Shell structure	FEM	Gradient based technique	Paraboloidal shell; minimization of structural weight
Han J. & Yamazaki K. , 2004 [167]	Structural Problem-Contact Mechanics problems	FEM - GENESIS	GENESIS	Bolt–nut fastening structure,
Thoutireddy P. & Ortiz M. , 2004 [51]	Structural Problem	FEM	Variational r-adaption for minimum potential energy	Shape optimization of elastic inclusion
Shen J. & Yoon D. , 2004 [168]	Structural Problem	FEM	OptiStruct	Control arm;a simple beam; a curved beam; a curved connector
Lu K.-J. & Kota S., 2003 [169]	Structural Problem	FEM	GA	Antenna Reflector – Beam Shaping; Antenna Reflector – Beam Steering
Woon S.Y. et al. , 2003 [60]	Structural Problem - Linear Elasticity	FG-FEA	GA	Beam 1 - Long cantilevered beam; Bridge – simply supported; Beam 2 – circular support; Simple spanner
Barbarosie C. , 2003 [170]	Thermal & Linear Elasticity Problems	FEM	Gauss Seidel Iterative algorithm	Heat Conduction Problem, Linear Elasticity Problem
Shen J. & Yoon D. , 2003 [171]	Structural Problem	FEM	OptiStruct	Control arm; Engine mount bracket
Woon S.Y. et al. , 2003 [58]	Structural Problem	FG-FEA	GA	Cantilevered beam; Round support

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Table 3 (*continued*)

References	Problem Type	Analysis Technique	Optimization Technique	Numerical Application
Fourie P. C., Groenwold A. A. , 2002 [172]	Structural Problem - Linear Elasticity	FEM	PSO	Torque arm
de Freitas J.A.T. & Cismasiu I. , 2002 [173]	Structural Problem - Linear Elasticity	hybrid Trefftz finite element method	DQP	Cantilever plate test; Fillet; Connecting Rod
Woon S.Y. et al. , 2001 [174]	Structural Problem - Linear Elasticity	FEM with STRAND6	GA	Simple spanner head; Simple spanner head with mirroring function; Flange webbing
Hilding D. et al. , 2001 [175]	Structural Problem- Contact Mechanics problems	Adaptive FEM	Sequential Convex Programming Optimization Algorithm	Optimizing the contact pressure: Rivet; Connecting element Minimizing the effective stress: Gas turbine engine
Marburg S. & Hardtke H.-J.,2001 [176]	Structural Problem	FEM	Multigrid solution method	Vehicle hat-shelf
Schleupen A. et al. , 2000 [52]	Structural Problem	Adaptive FEM	SQP	A thick beam with varying height; A quadratic membrane with a circular hole; The clamped beam
Kress G.R. , 2000 [177]	Structural Problem	FEM	Gradient based technique	Flywheel
Herskovits J. et al. , 2000 [178]	Nonlinear elastic solids in contact.	FEM	Herskovits' feasible direction interior point algorithm	Portal frame design; Perforated plate design; Tyre design
Srikanth A. & Zabaras N. , 2000 [179]	Frictional Contact Problem	FEM	Gradient based technique	Plane strain tension example; Upset forging of a cylindrical workpiece; design in upset forging to minimize barreling; design for efficient near-net shape manufacturing in upset forging
Choi K. K.& Duan W. , 2000 [180]	Structural Problem-Non-Linear Elastic Problems	FEM- Mixed variational principle (MVP) and the total Lagrangian formulation	MMFD	Engine mount; Bushing
Rohan E. & Whiteman J.R. , 2000 [181]	Structural Problem – Elasto-plastic	FEM	Bundle trust algorithm	2D elasto-plastic body
Souli M. et al. , 2000 [182]	Fluid Problem	FEM	Goldfarb's algorithm	stationary and irrotational flow
Givoli D. & Demchenko T. , 2000 [183]	Structural Problem - Elasticity Problem	Boundary-Perturbation Finite Element (BPFE) method	—	The wave scattering problem

functions and the same functions are used for field variable approximation during analysis. The physical domain is mapped with parametric domain through univariate/bivariate NURBS basis function ϕ_i,

$$x = \sum_{i=1}^{n} \varphi_i(\xi)\, B_i \tag{4.8}$$

and the unknown field variable approximation is performed through same shape functions as used for geometry,

$$u^h(x) = \sum_{i=1}^{n} \varphi_i(\xi)\, u_i \tag{4.9}$$

where u_i denotes the value of field variables at a particular control point B_i. However, it is important to note that u_i does not usually represent physical nodal value of field variable as conventional FEA. This is due to fact that NURBS basis functions are approximants and not interpolants [188].

T-splines are the alternative computational geometry technique developed in [189] and used within IGA framework [53] to eradicate issues of NURBS. With NURBS, gaps and overlaps at surface intersections can't be avoided. T-splines i.e. a generalized form of non-uniform B-spline which permits T-junctions are the robust technique in sewing together the adjacent patches. They also facilitate efficient local refinement [190]. T-junctions are conceptually similar to 'hanging nodes' in FEM. If T-mesh is without T-junction, T-spline reduces to B-spline. In shape optimization, T-spline based IGA approach has been adopted in [191].

4.2. Bezier triangles and Bezier tetrahedra

To overcome the difficulty of representing complex geometries with tensor product NURBS, IGA with triangulations has come out as an alternative technique. Bezier triangle based isogeometric shape optimization was proposed by Wang et al. [192] for auxetic materials followed by Lopez et al. [193] for linear elastic structures. Here, the design domain and physical fields are defined with Bezier triangles. From the complex problem domain boundary defined with B-spline, coarse Bezier triangulation is generated. The procedure for triangular parametrization through B-spline and NURBS can be referred from [194,195]. The resulting coarse mesh is used to maintain mesh validity and mesh movement while analysis is performed with fine mesh after refining the coarse mesh. The technique has been extended to solve 3D shape optimization problems by using Bezier tetrahedra for interior domains [196] which bridges the gap between surface CAD models and volumetric parameterization. The required tetrahedral mesh is generated from NURBS by the mesh generating tool Gmsh. The resulting shape optimization framework provides CAD compatible geometry for 3D models.

4.3. eXtended IGA (XIGA)

XFEM with level sets has been successfully used in shape optimization to eradicate issues related to mesh distortion and mesh refinement. However, some inherent issues related to FEM still exist in this framework which includes the use of Langrangian basis functions as shape functions. A step forward to eliminate this issue is a numerical approach combining IGA and local enrichment of shape functions by adding few degrees of freedom to selected control points which possesses benefits of IGA and XFEM both [197,198]. With this combined approach the crack is represented independent from mesh and free from mesh alignment issues in crack propagation analysis. In this case, the standard generalized form of locally enriched shape functions in XIGA is as follows [198]

$$u^h(\xi) = \sum_{i=1}^{n_{en}} \varphi_i(\xi) u_i + \sum_{j=1}^{n_{cf}} \varphi_j(\xi) H(\xi) d_j + \sum_{k=1}^{n_{ct}} \varphi_k(\xi) \left(\sum_{\alpha=1}^{4} Q_\alpha(\xi) c_k^\alpha \right) \quad (4.10)$$

where $\phi_{i,j,k}$ are NURBS basis functions, d_j and c_k^α are vectors of additional degrees of freedom, $H\ (\xi)$ is Heaviside function, and $Q_\alpha$crack tip enrichment functions. Enriched IGA has evident benefits like exact geometry representation and improved solution accuracy in case of discontinuous or singular problems with only few additional degrees of freedom. Although XIGA has found promising applications in static and dynamic fracture mechanics problems [199], it has attracted researchers' attention for shape optimization field also. As computational mesh remains fixed during entire optimization process, the major limitation of traditional shape optimization technique is eliminated. Moreover, exact geometry representation generates smoother design boundaries in optimum solution. Thus, XIGA has potential to alleviate issues of traditional shape optimization process.

4.4. IGA-BEM

In traditional IGA (based on FEA), domain parameterization remains a challenging task because NURBS basis functions describes only boundaries of the structure while analysis requires volumetric meshes from parameterization of boundary surfaces. To circumvent this issue, Li and Qian [200], and Simpson et al. [201] proposed boundary integral based approach to IGA. In the proposed combination, both NURBS based geometry representation and BEM deals with quantities entirely based on problem boundary which results in much tighter integration of design and analysis models. This characteristic makes IGA-BEM much suitable numerical technique for shape optimization problems. The advantages offered by this technique in shape optimization are evident e.g. unified design and analysis model calls off interaction with CAD model during the optimization process; reduce problem dimensionality and smoother boundaries in optimum shapes. Since its introduction, it has been used in shape optimization problems successfully. However, the downside of this approach is the need of fundamental solution for the problem which is complex task in many cases.

Contribution of IGA and its variants in shape optimization is presented in Fig. 8 and brief details of publications are summarized in Table 4.

5. MM based shape optimization

Originated in the late seventies, this group of numerical methods experienced fast development in last two decades with considerable success in almost all fields of engineering including structural optimization. In general, meshless formulation may be considered from following categories i.e. Galerkin based meshless techniques and the collocation meshless techniques. The former category of meshless techniques is developed on the basis of weak formulation of PDEs. In their basic approach, element connectivity is not required for approximating the field functions. Support domains of circular and rectangular shapes are used to determine nodes that are used for interpolation of field variable at a particular point. For the purpose of domain integration, back ground cells/mesh is used with classical Gauss Quadrature technique (Fig. 9).

Galerkin based techniques can be further classified based on global or local weak formulation used for deriving discrete system of equations e.g. EFG is based on global weak form while Meshless Local Petrov-Galerkin (MLPG) method uses local weak form. On the contrary, Collocation meshless methods are based on strong form of PDEs. With the ease of construction of smooth meshless shape functions, PDEs can be solved directly at the collocation points without using any special integration and essential boundary condition enforcement technique. The important features of MMs are as follows:

a) Field functions are defined with a set of arbitrary distributed nodes and support domain;
b) Mesh used in background is for integration purpose, no mesh alignment sensitivity
c) No remeshing is required especially in case of large deformation, moving discontinuity problems and structural optimization problems;
d) Possible to construct shape functions of required order continuity for field variable approximation;
e) No post processing required for smooth derivatives of unknowns e.g. stress which is secondary field variable in structure mechanics problems;
f) Higher convergence rate than conventional grid based methods due to improved accuracy of field variable approximation;

Although meshless techniques were developed to deal with special class of structure and fracture mechanics problems where traditional numerical techniques were struggling to provide accurate solution even after successive remeshing. Some features of MMs make them much appropriate numerical analysis technique in shape optimization problems, e.g.

- MMs perform field variable approximation and numerical integration separately. Unlike FEM where shape functions are constructed at element level natural coordinates and later transformed to the global Cartesian coordinates, meshless shape functions are constructed using only nodal points and its associated support domains at the global Cartesian coordinates directly without defining elements/ mesh. Thereby, MMs are free from mesh distortion and remeshing issue. For domain integration, classical Gauss quadrature technique with back ground cell structure is used generally.

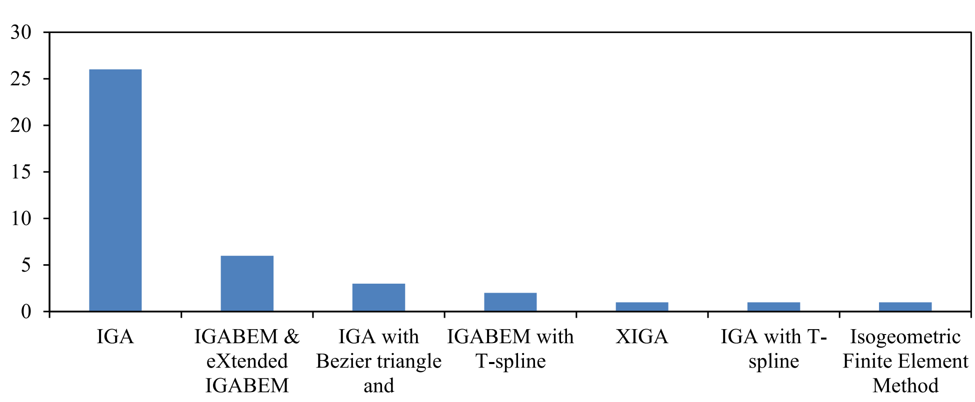

Fig. 8. IGA and its variants in shape optimization.

Table 4
Summary of IGA and its variants in shape optimization.

References	Problem Type	Analysis Technique	Optimization Technique	Numerical Application
Lopez J. et al. , 2020 [196]	Structural Problem	IGA with Bezier tetrahedral mesh	MMA	Cantilever beam; Cube with a hole; Hammer
Ummidivarapu V. et al. , 2020 [202]	Acoustic problem - Single and multi frequency	IGA	TLBO	Acoustic horn
Mostafa Shaaban A. et al. , 2020 [203]	Acoustic problems - time harmonic wave propagation	IGABEM and eXtended IGABEM (XIBEM)	PSO	Duct problem, infinite vertical noise barrier, acoustic horn
Kumar D. et al. , 2020 [204]	Structural Problem – Auxetic structures	IGA	GA	missing rib structure with four ligaments under different loading conditions
Yoon M. et al. , 2020 [205]	Thermoelastic Problem	IGA-BEM	MMFD	Rotor shaped dish, hollow cylinder problem, bellow-shaped structure
Aminzadeh M. et al. , 2020 [206]	Structural Problem	IGA	Gradient based technique	Steel slit dampers
Chen L. L. et al., 2019 [207]	Structural Problem	IGA-BEM	MMA	Sphere model; submarine model; Vase
López J. et al. , 2019 [193]	Structural Problem - Linear Elasticity	IGA based on Bezier triangles	MMA	Cantilever beam; Open spanner; Torque arm; Bracket
Hirschler T. et al. , 2019 [208]	Structural Problem-Shell structure	IGA	Sequential Least Square Programming	Pinched cylinder; Pinched hemisphere
Hirschler T. et al. , 2019 [209]	Structural Problem	IGA	SLSQP	Stiffened roof; Curved wall; Wing
Li S. et al. , 2019 [210]	Structural Problem	IGA-BEM	DE	Open spanner; Torque arm; Spigot
Weeger O. et al. , 2019 [211]	Structural Problem - Nonlinear	IGA	SLSQP, MMA and Preconditioned Truncated Newton (PTN)	Cantilever beam; curved beam
Lieu Q.X. & Lee J. , 2019 [212]	Vibration Problems	IGA	Adaptive hybrid evolutionary firefly algorithm (AHEFA)	Rectangular plate; Square plate; Circular plate; Annual plate with fully clamped outer edge
Chamoin L. & Thai H.P. , 2019 [213]	Structural Problem	IGA	Adaptive algorithm: fixed-point iteration method (alternating direction algorithm)	2D NACA airfoil profile; Plate with hole; 3D hollow cylinder
S.H. Sun et al. , 2018 [214]	Structural Problem - Linear Elasticity	IGA-BEM	PSO	Cantilever Beam; Fillet; A square plate with one circular cutout; connecting rod
Wang C. et al. , 2018 [23]	Structural Problem - Linear Elasticity	XIGA	Chaotic PSO	A square plate with one circular hole; Cantilever beam; 2D-fillet in tension; 2D suspension arm; A square plate with a complex cutout
Ding C. et al. , 2018 [215]	Structural Problem - Linear Elasticity	IGA with Indirect Factorization Updating (IFU)	MMA	Cantilever beam; Open-end spanner; continuous thickness optimization of car-roof
Ahn S.-H. et al. , 2018 [216]	nanoscale structures	IGA	Gradient based optimization algorithm	Parabolic arch design; Optimal curvature of curved graphene; Optimal shell form under distributed load
Bandara K. & Cirak F., 2017 [217]	Structural Problem	IGA	MMA	Thin shell structures:Thin strip; shell roof over a rectangular domain; Freeform architectural shell roof
Herrema A. J. et al. , 2016 [218]	Structural Problem - Elasticity Problem	IGA	Rhino based Grasshopper Algorithm	Tube Profile; Wind Turbine Blade
Lian H. et al. , 2016 [219]	Structural Problem - Linear Elasticity	IGA-BEM	MMA	Shape sensitivity analysis: Lamé problem; Kirsch problem; Shape optimization: Cantilever beam; Fillet; Connecting rod; Spanner
Lian H. et al. , 2016 [220]	Structural Problem - Linear Elasticity	IGA-BEM (using T-Splines)	MMA	Shape sensitivity analysis: spherical cavity; Shape optimization: Cantilever beam; Hammer; Chair
Wang C. et al. , 2016 [192]	Structural Problem - Linear Elasticity	IGA based on Bezier triangles	MMA	A Plate With A Hole; A Cantilever beam With One Hole; A Cantilever beam With three Holes
Radaelli G. & Herder J. L. , 2016 [221]	Structural Problem	IGA	Nelder-Mead simplex, Trust Region Reflective, Interior Point, Active Set and SQP	compliant beam
Kang P. &Youn S.-K. , 2016 [222]	Structural Problem	IGA	MMA	Trimmed shell structures:A shell architecture; A trimmed doubly curved strip
Choi M.-J. & Cho S., 2015 [223]	Thermal Problem	IGA	MMFD	Heat isolator; Cooling fin;
Ha Y.D. , 2015 [224]	Structural Problem - Linear Elasticity	IGA	MMFD	Hemispherical shell; cantilever shell;
Fubeder D. et al. , 2015 [225]	Structural Problem	IGA	Gradient descent algorithm	Example with geometric constraint (Area Maximization); Compliance minimization in linear elasticity
Kostas K. V. et al. , 2015 [226]	Structural Problem	IGABEM (using T-Splines)	simplex deterministic algorithm, evolutionary algorithm	ship-hull model
Wang Z.-P. & Turteltaub S. , 2015 [227]	Structural Problem	IGA	Iterative descent method	an orifice in a large plate; benchmark problem under quasi-static loading; cantilever plate;
He G. et al. , 2014 [228]	Structural Problem - Linear Elasticity	IGA & FCM	MMA	cantilever beam

(*continued on next page*)

Table 4 (*continued*)

References	Problem Type	Analysis Technique	Optimization Technique	Numerical Application
Kiendl J. et al. , 2014 [229]	Structural Problem-Shell structure	IGA	Gradient based optimization algorithm	parabolic arch
Koo B. et al. , 2013 [230]	Structural Problem - Linear Elasticity	IGA	MMFD	a quarter model of circular disk with a hole at the center; L-Shape Stiffener; Gripping Tool
Azegami H. et al. , 2013 [231]	Structural Problem - Linear Elasticity	Isogeometric Finite Element Method	H^1 gradient method (The traction method)	Plate with a hole in plane stress; Three-dimensional cantilever body
Manh N. D. et al. , 2011 [232]	Structural Problem- Vibration Problems	IGA	Gradient based optimization algorithm	Pear-shaped region; Harmonic drums; CEG drums
Ha S.-H. et al. , 2010 [191]	Structural Problem - Elasticity Problem	IGA with T-spline basis functions	Gradient based optimization algorithm	Bracket model
Qian X. , 2010 [233]	Structural Problem	IGA	MMA	Plate with a hole; Cantilever beam; Open spanner;
Nagy A.P. et al. , 2010 [234]	Structural Problem - Elasticity Problem	IGA	MMA	semi-circular arch (curved beam)
Cho S. & Ha S.-H., 2009 [235]	Structural Problem - Linear Elasticity	IGA	MMFD	Plane elasticity problems; A quarter model of circular disk with a hole at the center; L-shape stiffener model; connecting rod
Wall W. A. et al. , 2008 [4]	Structural Problem - Linear Elasticity	IGA	MMA	Cantilever Beam; Plate with hole; Open spanner

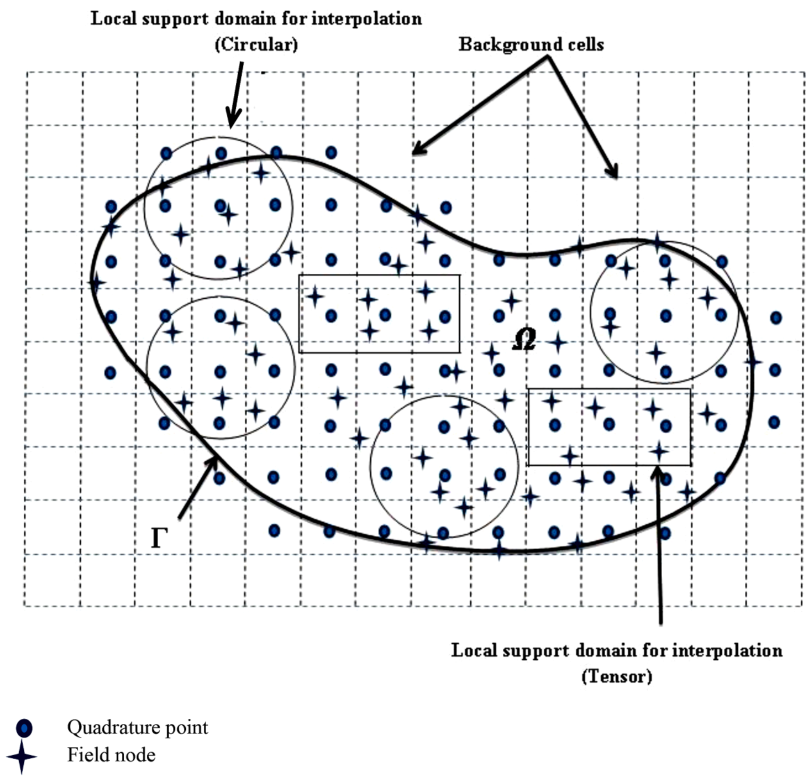

Fig. 9. Field nodes, Quadrature points, Support domains and back ground cells for integration in MMs.

- The background cell structure/mesh is solely used for numerical integration purpose and it does not impair the solution accuracy when it gets distorted. The only condition is that the cells remain convex with positive Jacobian [236].
- The meshless shape functions generate higher order field variable continuity (both primary and secondary) which improves design sensitivity analysis in gradient-based shape optimization technique and overall solution accuracy [15].
- Meshless solution is minimally affected by irregular placement of nodes which is quite common situation in shape optimization problems [237].

The literature shows successful applications of EFG and RKPM in shape optimization for linear elastic, thermoelastic, structure dynamics and frictional contact problems. The following subsections provide brief details of these techniques.

5.1. EFG

This technique falls in the category of Galerkin based meshless methods and uses global weak form of governing PDEs for deriving discrete system of equations. The technique was proposed by Belytschko et al. [237]by employing Moving Least Square (MLS) shape functions for field variable approximation and back ground cell structure/mesh with classical Gauss quadrature technique for numerical integration. The important properties of MLS shape functions are as follows:

a) Possible to construct shape functions of desired order continuity for field variable approximation by appropriately selecting weight functions.

b) Kronecker delta property is not satisfied i.e. $\varphi_i(x_j) = \begin{cases} 1 & when\ i = j \\ 0 & when\ i \neq j \end{cases}$ which means essential boundary conditions (EBCs) are not imposed in straightway and needs attention.
c) Singularity issue in case moment matrix is not invertible in evaluating shape functions
d) These functions do not form interpolants

The field function *u(x)* is approximated using MLS series function (Fig. 10) as,

$$u^h(x) = \sum_i^n \varphi_i(x) u_i \tag{5.1}$$

where MLS shape functions are given by,

$$\varphi_i(x) = \sum_j^m p_j(x)\left(A^{-1}(x)B(x)\right)_{ji} = p^T A^{-1} B_i \tag{5.2}$$

where,

$$A(x) = \sum_i^n p^T w(x) p, \tag{5.2a}$$

$$B(x) = p^T w(x) \tag{5.2b}$$

In (5.2a) and (5.2b),*p(x)*are monomial basis functions of order *m* while *w(x)* is a weight function with compact support with non-zero value. Weight functions play an important role in constructing globally continuous MLS shape functions. Commonly used weight functions are the exponential and spline functions of different order. In (5.2a), *A (x)* is the moment matrix which must remain invertible for avoiding singularity issues in EFG formulation. As mentioned above, MLS shape functions do not satisfy Kronecker delta criterion, EBCs are imposed with special techniques like penalty approach [238], Lagrange multiplier [239], coupling with FEM [240] and modified variational principle [241]. From practical viewpoint, penalty approach is more popular because it doesn't increase number of unknowns in the system and results in definite and banded system matrices which reduces computational efforts. However, selection of proper penalty parameter is a crucial task in generating solution with acceptable accuracy.

5.2. RKPM

RKPM is another popular meshfree approach to solve PDEs which uses reproducing kernel approximants to construct shape functions through a set of scattered nodes without defining element connectivity [242]. The method was proposed by introducing a correction function to Smoothed Particle Hydrodynamics (SPH) approximation to improve it near boundaries. The approximation function for RKPM is represented in integral form as follows,

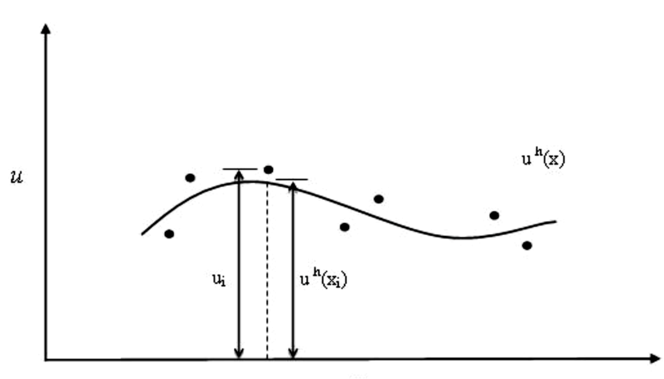

Fig. 10. MLS Shape function.

$$u^h(x) = \int_{\Omega_s} u(\xi) C(x,\xi) \widehat{W}(x-\xi,h) d\xi \tag{5.3}$$

where$\widehat{W}(x - \xi, h)$is the kernel function or weight function and h is termed as smoothing length in SPH formulation which is often termed as compact support domain, *C(x, ξ)* is correction function. Kernal functions $\widehat{W}$are non-zero only over a compact domain/smoothing domain which gives approximation a local character. Most weight functions are bell shaped and commonly used weight functions are Gaussian or spline functions of different order.

The domain Ω_S is discretized by a set of particles [$x^1, x^2,..,x^n$] and using trapezoidal rule, (5.3) is rewritten as,

$$u^h(x) = \sum_{i=1}^n C(x; x - x_i) \widehat{W}(x - x_i)\, u(x_i)\, \Delta x_i \tag{5.4}$$

The correction function *C(x, x-x$_i$)* is constructed by monomial basis as follows,

$$C(x, x - x_i) = q(x)^T H(x - x_i) \tag{5.5}$$

where, *H(x - x$_i$)* and *q(x)* are basis function and coefficient vectors respectively. To get shape functions, (5.5) is substituted in (5.4),

$$u^h(x) = \sum_{i=1}^n \varphi_i(x)\, u_i \tag{5.6}$$

where, $\phi_i(x)$ is termed as RKPM shape functions and it is given by,

$$\varphi_i(x) = \ q(x)^T H(x - x_i) \widehat{W}(x - x_i) \tag{5.7}$$

like MLS shape functions of EFG, RKPM shape functions do not satisfy Kronercker delta criterion. EBCs can't be enforced simply like FEM and needs special attention. Domain integration is usually performed with classical Gauss quadrature using back ground cells/mesh.

5.3. Coupled FEM-MM based shape optimization

Although meshless techniques produced encouraging results in shape optimization due to its inbuilt flexibility, their use is limited owing to increased computational efforts resulting from their complex shape functions involving matrix inversions (e.g. see (5.2)) and higher order integration techniques. Again, shape optimization needs structural analysis to be carried out repetitively; hence it is highly desirable to improve computational efficiency in meshless based shape optimization. To take benefits of FEM in terms of its computational simplicity and MM in terms of its improved solution accuracy, coupled FEM-MM approach is developed. In coupled FEM-MM approach, only a part of problem domain (usually highly stressed area where better approximation is sought) is modeled with MM and the rest of the domain is modeled with FEM. In shape optimization, coupled FEM-MM not only improves computational efficiency but also eliminates mesh distortion issue [243]. If FE mesh is used along EBCs and FE shape functions are combined with EFG shape functions, EBCs can be imposed similar to FEM without using any special techniques [15]. Various coupling techniques have been proposed in the literature which includes master slave coupling, coupling via ramp functions, bridging domain method and coupling with Lagrange multiplier. Comprehensive details of these techniques can be referred from [244].

Contribution of MM and coupled FEM-MM in shape optimization is presented in Fig. 11 and brief details of publications are summarized in Table 5.

6. Observations and recommendations

This section provides critical observations of present review.

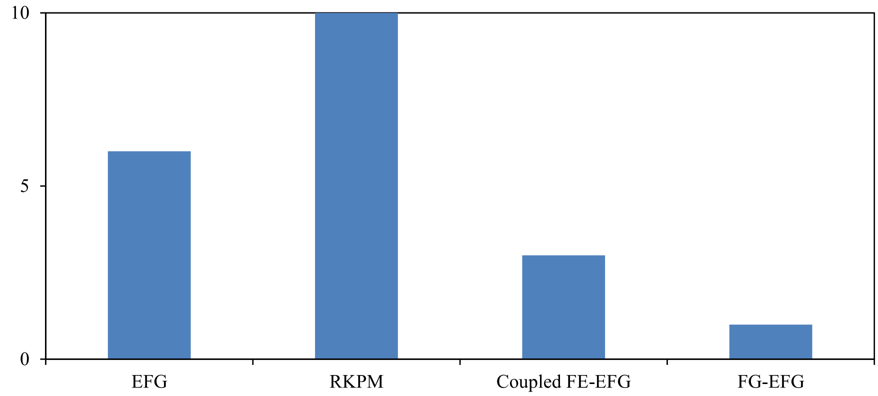

Fig. 11. MM and coupled FEM-MM in shape optimization.

Table 5
Summary of MMs in shape optimization.

References	Problem Type	Analysis Technique	Optimization Technique	Numerical Application
Rohit G. R. et al., 2020 [245]	Structural Problem	Coupled FE-EFG	PSO	Cantilever beam, Fixed-Fixed beam
Daxini S. D. & Prajapati J. M., 2019 [246]	Structural Problem - Linear Elasticity	EFG	ABC	Cantilever beam and Fillet geometry
Daxini S. D. &Prajapati J. M. , 2019 [247]	Structural Problem - Linear Elasticity	EFG	PSO	Cantilever beam; Fixed–fixed beam; Fillet
Gong S. G. et al. , 2009 [243]	Structural Problem - Linear Elasticity	Coupled FE-EFG	Gradient based optimization algorithm	Thick Cylinder; Cantilever beam; Fillet
Zhang J. et al. , 2008 [248]	Structural Problem - Linear Elasticity	RKPM	Gradient based optimization algorithm	Cantilever Beam; Shape optimization of an arc; Fillet
Zou W. et al. , 2007 [249]	Structural Problem - Linear Elasticity	Truly Meshless Method – RKPM with PUQ	SQP	Fillet; Portal frame
Bobaru F. & Rachakonda S., 2006 [250]	Thermal Problem	Coupled FG-EFG	SQP	Fins
Zhang Z. Q. et al. , 2005 [251]	Structural Problem - Linear Elasticity	RKPM	Multi-Family Genetic Algorithm (MFGA)	Bracket; Plate subjected to tension; Connection Rod
Bobaru F. & Rachakonda S., 2004 [252]	Thermal Problem	EFG	SQP	Fins
Bobaru F. & Rachakonda S., 2004 [253]	Thermal Problem	EFG	SQP	Fins
D. Lacroix & Ph. Bouillard, 2003 [15]	Structural Problem - Linear Elasticity	Coupled FE-EFG	Gradient based optimization algorithm	Membrane with an elliptic hole
Kim N. H. et al. , 2003 [254]	Structural Problem - Linear Elasticity	RKPM	SQP	Torque arm; Bracket; Road arm
Bobaru F. & Mukherjee S., 2002 [236]	Thermo-elastic Problems	EFG	SQP	Thermal fin; Uniformly loaded beam
Grindeanu I. et al. , 2002 [255]	Structural Problem - Linear Elasticity	RKPM	Gradient based optimization algorithm	Engine Mount; Elastic Beam
Bobaru F. & Mukherjee S., 2001 [21]	Structural Problem - Linear Elasticity	EFG	SQP	DSA for Pulling bar; Lame's problem; Shape optimization of Fillet
Kim N. H. et al. , 2001 [256]	Frictional Contact Problem	RKPM	SQP	Sheet Metal Stamping
Kim N. H. et al. , 2001 [257]	Structural Problem	RKPM	SQP	Vehicle bumper
Kim N. H. et al. , 2000 [258]	Structural Problem-Frictional Contact Problem	RKPM	SQP	Door Seals
Kim N. H. et al. , 2000 [259]	Structural Problem-Frictional Contact Problem	RKPM	SQP	Metal Ring Contact Problem
Kim N. H. et al. , 2000 [260]	Structural Problem-Multibody Frictional Contact Problem	RKPM	SQP	Windshield wiper

Alternative numerical analysis techniques employed in shape optimization techniques are compared in Table 6.

To highlight contribution of major numerical simulation techniques in terms of their frequency of application, quantitative analysis of collected literature is carried out and same is shown in Table 7.

a) The quantitative analysis reveals the fact that FEM based numerical simulation techniques have been extensively used for shape optimization. The reason behind that are many –
1) FEM is well understood, well established and versatile numerical technique with more than seventy years of development history and intensive research
2) Conceptually simple and easy to implement
3) Availability of plenty of commercial software with high-end capabilities e.g. ABAQUS, ANSYS, MSC/NASTRAN etc. to deal with almost all types of engineering problems including structural optimization

b) In spite of aforementioned benefits, the inherent limitations of FEM lies in mesh dependent solutions and approximated geometry description. To eliminate mesh distortion and subsequent remeshing issues, fixed grid based approaches including XFEM with level sets, FG-FEM/Eulerian approach/IGFEM and NIGFEM have been used for shape optimization frequently.

Table 6
Comparison of numerical analysis techniques.

Analysis technique	Geometry description	Shape functions	Nature of shape function	Numerical integration	Field variable continuity
FEM	Approximated form	Lagrangian polynomial (Most cases low order)	Interpolation	Gauss Quadrature	Lower
XFEM	Approximated form	Enriched Lagrangian functions in area of discontinuity	Interpolation	Gauss Quadrature	Improved at desired location
IGA	Exact geometry	NURBS / B-spline basis function	Approximation	Gauss Quadrature	Higher
XIGA	Exact geometry	Enriched NURBS basis functions in area of interest	Approximation	Gauss Quadrature	Higher
IGA-BEM	Exact boundary	NURBS / B-spline basis function	Approximation	Gauss Quadrature	Higher
EFG	Approximated form – more flexible	Moving Least Square (MLS)	Approximation	Gauss Quadrature / Stabilized Conforming Nodal Integration (SCNI)	Higher
RKPM	Approximated form – more flexible	Reproducing kernel functions	Approximation	Gauss Quadrature / Stabilized Conforming Nodal Integration (SCNI)	Higher

Table 7
Quantitative analysis of numerical simulation techniques.

FEM based techniques			IGA and its variants			MMs – EFG & RKPM		
Analysis technique	No. of articles	References	Analysis technique	No. of articles	References	Analysis technique	No. of articles	References
FEM and its commercial tools like ABAQUS, FEMFAT, ANSYS, COMSOL Multi physics, GENESIS, SFE CONCEPT, MSC/Patran, MSC/Nastran, RADIOSS, STRAND6, NASTRAN	83	[8,9,51,68-72,75,77,78,81, 83-87,89,90,95,97-101,105, 106,111,113-115,117-121, 124-127,130,133,134, 136-148,150-160,163,164, 166-172,174,176-179,181, 182]	IGA	26	[4,202,204,206, 208,209,211-213, 215-218,221 -225,227-230, 232-235]	EFG	6	[21,236, 246,247, 252,253]
XFEM and XFEM with Level sets	7	[20,82,103,104,112,135, 149]	IGA with T-spline basis	1	[191]	FG-EFG	1	[250]
FG-FEM/Eulerian approach/ IGFEM/NIGFEM	7	[58-60,62,63,128,165]	IGABEM & eXtended IGABEM	6	[203,205,207, 210,214,219]	RKPM	9	[248,251, 254-260]
FCM	5	[65, 66,79,80,116]	IGABEM with T-spline basis	2	[220,226]	Truly meshless method using PUQ with RKPM	1	[249]
Mesh morphing with RBF Morph and HyperMorph	4	[76,93, 94,132]	XIGA	1	[23]	FE-EFG	3	[15,243, 245]
Adaptive FEM	3	[49,52,175]	IGA with Bezier triangles and tetrahedral mesh	3	[192,193,196]			
Multimesh FEM, cgFEM, Hybrid Trefftz FEM and other techniques	21	[39,73,74,88,91,92,96,102, 107-110,122,123,129,131, 161,162,173,180,183]	Isogeometric Finite Element Method	1	[231]			
	130			40			20	

c) Recently, FCM in combination of some implicit function or level-set function has attracted researcher's interest for shape optimization as structure is modeled implicitly and analysis is performed with fixed grid, this combination avoids tedious mesh updating process completely.
d) Fully integrated mesh morphing algorithms can evaluate arrange of alternative configurations by automatically altering the starting point mesh to determine optimal geometric profile. RBF Morph and Hyper Morph are the popular commercial tools used in product design and development.
e) IGA has accelerated the developments in shape optimization field soon after its development in 2005. The robustness of IGA makes it an ideal choice numerical analysis technique for shape optimization. Transformation of CAD model into analysis compatible FEM model is not only time consuming but also error prone. IGA efficiently eradicates these issues and outperforms traditional FEM from several other aspects. IGA offers unified design and analysis models described through NURBS based solid elements, geometry exactness throughout the optimization process, ease of refinement, reduced design to analysis time and improved solution accuracy.
f) The concept of IGA was developed in context of FEM; hence there exist a major limitation in terms of need of volumetric parametrization to perform analysis. This task is non-trivial. To naturally combine CAD and numerical analysis, IGABEM was developed in 2011 which eliminates above issue as analysis problem is formulated with boundary integral equations while boundary representation and approximation of physical fields is carried out with NURBS. This technique provides close integration between design and analysis. More recent development in IGA based numerical simulation technique is XIGA in which isogeometric formulation based on NURBS basis functions are enriched via XFEM. In shape optimization, this approach results in accurate boundary representation and eliminates remeshing.
g) MMs like EFG & RKPM based shape optimization techniques attracted researchers' attention from early 2000. Exemplary field variable approximation and improved sensitivities, free from remeshing issues and less sensitive results for irregular node arrangements are some of the key benefits offered by MMs in context of shape optimization. However, the complex nature of shape functions involving matrix inversions (see (5.2)) and higher order integration

techniques increases computational burden of numerical analysis which limits its use.

h) From perspective of optimization techniques, two different categories of algorithms are employed i.e. *local gradient-based* and *global derivative free*. The former category includes classical deterministic algorithms like SQP, MMA, MMFD, SLSQP, Gradient Descent etc. needing sensitivity computations based on rigorous mathematical formulations. In general, these algorithms converge faster and thus need less number of function evaluations which reduces overall computational burden. This feature makes them more beneficial in structural shape optimization. However, the drawback of these techniques is that the solution can trap in local minima easily as these techniques are sensitive to initial guess. Moreover, computing design sensitivities for objective functions and constraint adds substantial complexity in the process. The sensitivities are calculated numerically, semi-analytically, analytically or automatic differentiation. Although conventional mathematical techniques like finite difference method (FDM), differentiating discrete system equations, differentiating continuum equations are often used for sensitivity computation; automatic differentiation has also attracted much research interest. It is differentiation of computer code itself, which means transforming a given computer code, written in any high-level programming language like C, C++ or Fortran, into a new program which is capable of computing not only the original function but also its derivatives. ADIFOR (Automatic Differentiation of Fortran) and ADOL-F (Automatic Differentiation by Over Loading) are the automatic differentiation tools developed for Fortran codes and ADOL-C is the computational differentiation tool available for codes written in C/C++. The automatic differentiation is fast, accurate and capable of computing first and higher order derivatives [261–264]. In general, the error associated with sensitivity computation affects overall optimization process and the convergence behavior of deterministic algorithms.

i) The second category of optimization algorithms i.e. *global - derivative free* stochastic techniques have advantage of generating near global optimal solution by searching the whole design space. These techniques are free from sensitivity computation and associated computational burden. Moreover, these techniques are not sensitive to initial guess and can escape local optima. In general, these techniques exhibit slower convergence and hence computationally expensive due to large number of function evaluations and more number of iterations. Modern meta-heuristic algorithms based on successful biological systems are typical examples of these techniques. Again these algorithms are classified into population based techniques and single-solution based techniques. GA, PSO and ABC are the population based stochastic optimization algorithms while SA, Variable Neighborhood Search (VNS) and Tabu Search (TS) are the single solution based algorithms. Diversification (global search) and intensification (local search) are the two most important attributes of these techniques. The population based algorithms are more global search oriented while the single solution based algorithms are more local search oriented. Balanced global and local search processes lead to efficient working of these techniques. In general, these techniques are preferred less in shape optimization due to its higher computational cost resulting from large number of function evaluations.

6.1. Recommendations

- Even after five decades of development history, application of shape optimization to real world industrial problems has still remained challenging due to lack of robustness in handling large boundary variations leading to degeneracy of design models. Hence, from perspective of industrial applications, it is highly desirable to develop CAD centric automated and robust shape optimization module which requires CAD based design parametrization, automatic mesh generation, CAD based DSA and optimization. In traditional FEM based framework, the optimized mesh model must be translated back to CAD model. This mesh-to-CAD task is not straightforward and needs designer's interaction. The CAD centric design optimization approach eliminates all such issue, improves design productivity and fulfills the designer's ambition of working with more integrated approach. Literature review shows some early works in this direction by Hardee et al. [265] using FEM and Pro/-Engineer framework and Grindeanu et al. [255] using RKPM and Pro/Engineer framework. Some recent literature addressing CAD based shape optimization approach include optimization of CAD models with XFEM by Hanfer et al. [82], refining CAD parametrization through addition of new CAD features by Agarwal et al. [83], computing design velocity using adjoint method for CAD based shape optimization by Agarwal et al. [266], realizing robust design and multidisciplinary optimization with CAD as principal repository for product data by Brujic et al. [140].
- In a recent development, Li and Sederberg [267] proposed S-spline curves and surfaces which are claimed to have simpler concept and implementation than T-spline local refinement by avoiding unwanted control points generated with T-splines. The S-spline curves and surfaces show promising applications in CAD and IGA. Application of S-spline in shape optimization will be a natural application, as suggested by authors, which has not been explored in research yet.
- A novel FEM based numerical approach, termed as Multimesh FEM, has been proposed in shape optimization [92]. The concept is based on discretizing the problem domain with multiple independent overlapping meshes which are freely rotated, scaled or translated with lower computational efforts without compromising mesh quality. Nitsche based FEM is used to weakly enforce continuity over non-matching mesh interfaces. With the free movement of mesh, large shape changes can also be adapted without mesh distortion and remeshing. The technique is fairly new and applied to steady state thermal shape optimization problem. Further investigation on Multimesh FEM will be interesting research field.

7. Conclusion

In present work, shape optimization techniques post 2000 is reviewed from the perspective of numerical analysis techniques. Broad range of analysis techniques have been employed in studying shape design optimization problems including analytical and numerical approaches. However, the present study is focused on numerical techniques which are most popular and promising. For convenience, these techniques were grouped into FEM based techniques, IGA and its variants, MMs and coupled FEM-MM for discussion. In context of shape optimization, these techniques were briefly discussed and compared. Quantitative analysis of collected literature was carried out to identify popularity of these techniques. FEM based numerical techniques found clearly dominating the research field owing to its exceptionally good development history and versatility. Within the framework of FEM based shape optimization, in-plane and out-of-plane regularization techniques, vertex morphing technique and traction method in context of node-based shape optimization to reduce effects of mesh distortion and improve boundary smoothness were discussed. Moreover, mesh morphing techniques to avoid cumbersome remeshing issue and adaptive mesh refinement for improving solution accuracy and computational efficiency were also discussed. To avoid mesh distortion and remeshing issue, applications of fixed-grid variants of FEM i.e. XFEM with level sets, FG-FEM/Eulerian approach, IGFEM and NIGFEM were discussed.

Since last one decade, IGA and its variants like XIGA and IGABEM have triggered renewed interest in the field of structural optimization due to the benefits like close integration of design and analysis models, improved solution accuracy, exact geometry throughout the optimization process, reduced design to analysis time and robust framework. With their unique features, IGA based techniques can be used to design and optimize more complicated geometries, time dependent and non-linear problems in structural and non-structural domains.

As an alternative numerical technique, MMs like EFG and RKPM were also used successfully in shape optimization providing advantage like eliminating mesh distortion and remeshing issues completely. Moreover, higher order continuity of field variable approximation in meshless results improves design sensitivities and overall solution accuracy. The major barrier of these techniques is their higher computational cost resulting from complex shape functions and higher order integration schemes. By coupling FEM with MM, these issues can be alleviated partially.

CRediT authorship contribution statement

Bhavik D. Upadhyay: Data curation, Writing – original draft, Formal analysis, Writing – review & editing. **Sunil S. Sonigra:** Visualization, Supervision. **Sachin D. Daxini:** Conceptualization, Methodology, Writing – original draft, Writing – review & editing.

Declaration of Competing Interest

The authors declare that there is no conflict of interests regarding the publication of this article.

References

[1] Schmit LA. Structural design by systematic synthesis. In: Proc. In: 2nd conference on electronic computation. New York: ASCE; 1960. p. 105–22.
[2] Bendsøe MP, Sigmund O. Topology optimization: theory, methods, and applications. 2nd edn. New York: Springer; 2002.
[3] Ding Y. Shape optimization of structures: a literature survey. Comput Struct 1986;24(6):985–1004.
[4] Wall WA, Frenzel MA, Cyron C. Isogeometric structural shape optimization. Comput Meth Appl Mech Eng 2008;197(33-40):2976–88.
[5] Imam MH. Three-dimensional shape optimization. Int J Numer Methods Eng 1982;18(5):661–73.
[6] Braibant V, Fleury C. Shape optimal design using B-splines. Comput Methods Appl Mech Eng 1984;44:247–67.
[7] Bennett JA, Botkin ME. Structural shape optimization with geometric description and adaptive mesh refinement. AIAA J 1985;23(3):458–64.
[8] Meske R, Sauter J, Schnack E. Nonparametric gradient-less shape optimization for real-world applications. Struct Multidiscipl Optim 2005;30(3):201–18.
[9] Shi J-X, Kozono S, Shimoda M, Takino M, Wada D, Liu Y. Non-parametric shape design optimization of elastic-plastic shear panel dampers under cyclic loading. Eng Struct 2019;189:48–61.
[10] Schramm U, Pilkey WD. Coupling of geometric descriptions and finite element using NURBs-a study in shape optimization. Finite Elem Anal Des 1993;15(1): 11–34.
[11] Hughes TJR, Cottrell JA, Bazilevs Y. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. Comput Methods Appl Mech Eng 2005;194:4135–95.
[12] Seo Y-D, Kim H-J, Youn S-K. Shape optimization and its extension to topological design based on isogeometric analysis. Int J Solids Struct 2010;47:1618–40.
[13] Haftka RT, Grandhi RV. Structural shape optimization: a survey. Comput Methods Appl Mech Eng 1986;57(1):91–106.
[14] Jang GW, Kim YY, Choi KK. Remesh-free shape optimization using the wavelet-Galerkin method. Int J Solids Struct 2004;41:6465–83.
[15] Lacroix D, Bouillard P. Improved sensitivity analysis by a coupled FE-EFG method. Comput Struct 2003;81(26-27):2431–9.
[16] Keulen FV, Haftka RT, Kim NH. Review of options for structural design sensitivity analysis. Part 1: linear systems. Comput Methods Appl Mech Eng 2005;194: 3213–43.
[17] Kim NH, Choi KK. Structural sensitivity analysis and optimization 1: linear systems. New York: Springer; 2005.
[18] Mota Soares CA, Rodrigues HC, Oliveira Faria LM, Haug EJ. Optimization of the geometry of shafts using boundary elements. ASME J Mech Trans Autom Des 1984;106(2):199–203.
[19] Garcia-Ruiz MJ, Steven GP. Fixed grid finite element in elasticity optimization. Eng Comput 1999;16(2):145–64.
[20] Duysinx P, VanMiegroet L, Jacobs T, Fleury C. Generalized shape optimization using X-FEM and level set methods. In: Bendsøe MP, Olhoff N, Sigmund O (Eds) IUTAM symposium on topological design optimization of structures, machines and materials Solid Mech Appl 2006;137:23–32.
[21] Bobaru F, Mukherjee S. Shape sensitivity analysis and shape optimization in planar using the element-free Galerkin method. Comput Meth Appl Mech Eng 2001;190:4319–37.
[22] Li K, Qian X. Isogeometric analysis and shape optimization via boundary integral. Comp-Aided Des 2011;43(11):1427–37.
[23] Wang C, Yu T, Shao G, Nguyen TT, Bui TQ. Shape optimization of structures with cutouts by an efficient approach based on XIGA and chaotic particle swarm optimization. Eur J Mec 2019;74:176–87. A/Solids.
[24] Wang Y, Wang Z, Xia Z, Poh LH. Structural design optimization using isogeometric analysis: a comprehensive review. Comput Model Eng Sci 2018;117 (3):455–507.
[25] Daxini SD, Prajapati JM. Parametric shape optimization techniques based on Meshless methods: a review. Struct Multidiscip Optim 2017;56(5):1197–214.
[26] Saitou K., Izui K., Nishiwaki S., Papalambros P. A survey of structural optimization in mechanical product development. (2005) pp: 214-226.
[27] Hsu YL. A review of structural shape optimization. Comput Ind 1994;25(1):3–13.
[28] Zienkiewicz OC, Campbell JS. Shape optimization and sequential linear programming. In: Gallagher RH, Zienkiewicz OC, editors. Optimum structural design; 1973. p. 109–26.
[29] Botkin ME. Shape optimization of plate and shell structures. AIAA J 1982;20(2): 268–73.
[30] Wang SY, Sun Y, Gallagher RH. Sensitivity analysis in shape optimization of continuum structures. Comp Struct 1985;20(5):855–67.
[31] Bhavikatti SS, Ramakrishnan CV. Optimum shape design of shoulder fillets in Tension bars and T-heads. Int J Mech Sci 1979;21:29–39.
[32] Bhavikatti SS, Ramakrishnan CV. Optimum shape design of rotating disks. Comput Srrucr 1980;11:397–401.
[33] Prasad, Emerson JF. Optimal structural remodeling of multi-objective systems. Compur Srrucr 1984;18(4):619–28.
[34] Annicchiarico W, Cerrolaza M. Finite elements, genetic algorithms and b-splines: a combined technique for shape optimization. Finite Elem Anal Des 1999;33: 125–41.
[35] Cerrolaza M, Annicchiarico W, Martinez M. Optimization of 2D boundary element models using β-splines and genetic algorithms. Eng Anal Bound Elem 2000;24: 427–40.
[36] Belegundu AD, Rajan SD. A shape optimization approach based on natural design variables and shape functions. Comput Methods Appl Mech Eng 1988;66:87–106.
[37] Tortorelli DA. A geometric representation scheme suitable for shape optimization. Mech Struct Mach 1993;21(1):95–121.
[38] Stavropoulou E, Hojjat M, Bletzinger KU. In-plane mesh regularization for node-based shape optimization problems. Comput Methods Appl Mech Eng 2014;275: 39–54.
[39] Firl M, Wüchner R, Bletzinger K-U. Regularization of shape optimization problems using FE-based Parametrization. Struct Multidisc Optim 2013;47: 507–21.
[40] Linhard J, Bletzinger K. Tracing the equilibrium – recent advances in numerical form finding. Int. J. Space Struct. 2010;25:107–16.
[41] Hojjat M, Stavropoulou E, Bletzinger K-U. The vertex morphing method for node-based shape optimization. Comput Methods Appl Mech Eng 2014;268:494–513.
[42] Azegami H. A solution to domain optimization problems. Trans Jpn Soc Mech Eng, Ser A 1994;60:1479–86. . (in Japanese).
[43] eds. Azegami H, Shimoda M, Katamine E, Wu ZC. A domain optimizationtechnique for elliptic boundary value problems. In: Hernandez S, El-Sayed M, Brebbia CA, editors. Computer aided optimization design of structures IV, structural optimization, 14. Southampton: Computational Mechanics Publications; 1995. p. 51–8. eds.
[44] Azegami H, Wu ZC. Domain optimization analysis in linear elastic problems (Approach using traction method). JSME Int J, Ser A 1996;39:272–8.
[45] Staten ML, Owen SJ, Shontz SM, Salinger AG, Coffey TS. A comparison of mesh morphing methods for 3D shape optimization. In: Proceedings of the 20th international meshing roundtable. Berlin, Heidelberg: Springer; 2011. p. 293–311.
[46] Saalfeld B, Rutten M, Saalfeld S, Kunemund J. Improved mesh morphing based on radial basis functions. Eur Congress Comput Methods Appl Sci Eng (ECCOMAS 2012), 2012:1–13.
[47] Kikuchi N, Chung KY, Torigaki T, Taylor JE. Adaptive finite element methods for shape optimization of linearly elastic structures. The optimum shape. Boston, MA: Springer; 1986. p. 139–69.
[48] Canales J, Tárrago JA, Hernández A. An adaptive mesh refinement procedure for shape optimal design. Adv Eng Software 1993;18(2):131–45.
[49] Mohite PM, Upadhyay CS. Adaptive finite element based shape optimization in laminated composite plates. Comput Struct 2015;153:19–35.
[50] Salagame RR, Belegundu AD. Shape optimization with p adaptivity. AIAA J 1995; 33(12).
[51] Thoutireddy P, Ortiz M. A variational r adaption and shape optimization method for finite deformation elasticity. Int J Numer Methods Eng 2004;61(1):1–21.
[52] Schleupen A, Maute K, Ramm E. Adaptive FE-procedures in shape optimization. Struct Multidisc Optim 2000;19:282–302.
[53] Bazilevs Y, Calo VM, Cottrell Evans JA, Hughes TJR, Lipton S, Scott MA, Sederberg TW. Isogeometric analysis using T-splines. Comput Methods Appl Mech Eng 2010;199:229–63.
[54] Moës N, Dolbow J, Belytschko T. A finite element method for crack growth without remeshing. Int J Numer Methods Eng 1999;46(1):131–50.
[55] Osher S, Sethian JA. Fronts propagating with curvature dependent speed: algorithms based on Hamilton–Jacobi formulations. J Comput Phys 1988;79: 12–49.

[56] Sukumar N, Chopp DL, Moës N, Belytschko T. Modeling holes and inclusions by level sets in the extended finite-element method. Comput Methods Appl Mech Eng 2001;190(46-47):6183–200.
[57] Suzuki K, Kikuchi N. A homogenization method for shape and topology optimization. Comput Methods Appl Mech Eng 1991;93(3):291–318.
[58] Woon SY, Tong L, Querin OM, Steven GP. Knowledge-based algorithms in fixed-grid GA shape optimization. Int J Numer Meth Eng 2003;58:643–60. a.
[59] Kim NH, Chang Y. Eulerian shape design sensitivity analysis and optimization with a fixed grid. Comput Meth Appl Mech Eng 2005;194(30-33 SPEC. ISS): 3291–314.
[60] Woon SY, Querin OM, Steven GP. On improving the GA step-wise shape optimization method through the application of the Fixed Grid FEA paradigm. Struct Multidiscipl Optim 2003;25(4):270–8. b.
[61] Soghrati S, Aragón AM, Armando Duarte C, Geubelle PH. An interface enriched generalized FEM for problems with discontinuous gradient fields. Int J Numer Methods Eng 2012;89(8):991–1008.
[62] Najafi AR, Safdari M, Tortorelli DA, Geubelle PH. A gradient-based shape optimization scheme using an interface-enriched generalized FEM. Comput Meth Appl Mech Eng 2015;296:1–17.
[63] Najafi AR, Safdari M, Tortorelli DA, Geubelle PH. Shape optimization using a NURBS-based interface-enriched generalized FEM. Int J Numer Methods Eng 2017;111(10):927–54.
[64] Parvizian J, Duster A, Rank E. Finite cell method. Comput Mech 2007;41:121–33.
[65] Cai SY, Zhang WH, Zhu JH, Gao T. Stress constrained shape and topology optimization with fixed mesh: a B spline finite cell method combined with level set function. Comput Method Appl M 2014;278:361–87.
[66] Xu W, Wang H, Cai S, Ma H. Fixed-grid hole-shape optimization for opening structures using smoothly deformable implicit curve. Adv Mech Eng 2019;11(2): 1–9.
[67] Schillinger D, Ruess M. The finite cell method: a review in the context of higher-order structural analysis of CAD and image-based geometric models. Arch Comput Meth Eng 2015;22:391–455.
[68] Zhu S, Ohsaki M, Guo X, Zeng Q. Shape optimization for non-linear buckling load of aluminum alloy reticulated shells with gusset joints. Thin–Walled Structures, 154. ELSEVIER; 2020, 106830.
[69] Lemared S, Ferrari M, Jeu CD, Dufour T, Soulier N, Hugot E. Stress mirror polishing for future large lightweight mirrors: design using shape optimization. Opt Express 2020;28(9):14055–71.
[70] Gu X, Cao Y, Zhu J, Wang J, Zhang W, Moumni Z. Shape optimization of SMA structures with respect to fatigue. Mater Des 2020;189:108456.
[71] Nonogawa M, Takeuchi K, Azegami H. Shape optimization of running shoes with desired deformation properties. Struct Multidiscipl Optim 2020;62:1535–46.
[72] Wang W, Tian X, Qian S, Wang C, Wang M, Gao G, Liu H. Secondary shape optimization of topological boundary of cold plate channels. Meccanica 2020;55 (1):19–33.
[73] Etling T, Herzog R, Loayza E, Wachsmuth G. First and second order shape optimization based on restricted mesh deformations. SIAM J Sci Comput 2020;42 (2):A1200–25.
[74] San B, Waisman H, Harari I. Analytical and numerical shape optimization of a class of structures under mass constraints and self-weight. J Eng Mech 2020;146 (1):04019109.
[75] Dong Y, Yo X, Xu X. Cross section shape optimization design of fabric rubber seal. Compos Struct 2020. https://doi.org/10.1016/j.compstruct.2020.113047.
[76] Porziani S, Groth C, Waldman W, Biancolini M. Automatic shape optimisation of structural parts driven by BGM and RBF mesh morphing. Int J Mech Sci 2020; 189:105976.
[77] Garcia-Andres X, Gutierrez-Gil J, Martinez-Casas J, Denia F. Wheel shape optimization approaches to reduce railway rolling noise. Struct Multidiscipl Optim 2020;62(5):2555–70.
[78] Nguyen S, Kim H. Stress-constrained shape and topology optimization with the level set method using trimmed hexahedral meshes. Comput Meth Appl Mech Eng 2020;366:113061.
[79] Chen L, Zhang W, Meng L, Feng S. An adaptive T-spline finite cell method for structural shape optimization. Struct Multidiscipl Optim 2020;61(5):1857–76.
[80] Cai S, Zhang W. An adaptive bubble method for structural shape and topology optimization. Comput Meth Appl Mech Eng 2020;360:112778.
[81] Florio CS. Selection of the scaling factor in finite element-based gradientless shape optimization for a consistent step size. Struct Multidiscipl Optim 2019;59: 713–30.
[82] Hafner C, Schumacher C, Knoop E, Auzinger T, Bickel B, Bächer M. X-CAD: optimizing CAD models with extended finite elements. ACM Trans Graphics 2019; 38(6). Article 157 (1-15).
[83] Agarwal D, Robinson TT, Armstrong CG, Kapellos C. Enhancing CAD-based shape optimisation by automatically updating the CAD model's parameterization, 59. Structuraland Multidisciplinary Optimization; 2019. p. 1639–54.
[84] Kuci E, Henrotte F, Duysinx P, Geuzaine C. Combination of topology optimization and Lie derivative-based shape optimization for electro-mechanical design. Struct Multidiscipl Optim 2019;59(5):1723–31.
[85] Shimoda M, Nagano T, Shi J-X. Non-parametric shape optimization method for robust design of solid, shell, and frame structures considering loading uncertainty. Struct Multidiscipl Optim 2019;59(5):1543–65.
[86] Doganay OT, Gottschalk H, Hahn C, Klamroth K, Schultes J, Stiglmayr M. Gradient based biobjective shape optimization to improve reliability and cost of ceramic components. Optim Eng 2019:1–29.
[87] Frohlich B, Gade J, Geiger F, Bischoff M, Eberhard P. Geometric element parameterization and parametric model order reduction in finite element based shape optimization. Comput Mech 2019;63(5):853–68.
[88] Ertl F-J, Dhondt G, Bletzinger K-U. Vertex assigned morphing for parameter free shape optimization of 3-dimensional solid structures. Comput Methods Appl Mech Eng 2019;353:86–106.
[89] Zhao Z, Wu J, Liu H, Liang B, Zhang N. Shape optimization of reticulated shells with constraints on member instabilities, 51. Engineering Optimization; 2019. p. 1463–79.
[90] Sharma A, Rangarajan R. A shape optimization approach for simulating contact of elastic membranes with rigid obstacles. Int J Numer Methods Eng 2019;117(4): 371–404.
[91] He G, Gao K, Jiang J, Liu R, Li Q. Shape optimization of a flexible beam with a local shape feature based on ANCF. J Adv Mech Des, Syst Manuf 2019;13(3).
[92] Dokken JS, Simon W, Funke SW, Johansson A, Schmidt S. Shape optimization using the finite element method on multiple meshes with Nitsche coupling. SIAM J Sci Comput 2018;41(3):A1923–48.
[93] Porziani S, Groth C, Biancolini ME. Automatic shape optimization of structural components with manufacturing constraints. Procedia Struct Integrity 2018;12: 416–28. Publisher: Elsevier B.V.
[94] Groth C, Chiappa A, Biancolini ME. Shape optimization using structural adjoint and RBF mesh morphing. Procedia Struct Integrity 2018;8:379–89. Publisher: Elsevier B.V.
[95] Gunther U, Hess W, Horn BM, Luthen H. A holistic topology and shape optimization approach with an application to steel profiles. Struct Multidiscipl Optim 2018;57(3):1267–81.
[96] Santhosh R, Ismail S, Jain PC, Anjaneyulu PSR. Shape optimization of shallow domes subjected to external pressure. Struct Multidiscipl Optim 2018;57(2): 903–8.
[97] Schmitt O, Steinmann P. Control of minimum member size in parameter-free structural shape optimization by a medial axis approximation. Comput Mech 2018;61(6):717–27.
[98] Zhang W, Niu C. A linear relaxation model for shape optimization of constrained contact force problem. Comput Struct 2018;200:53–67.
[99] Murai D, Kawamoto A, Kondoh T. Shape optimization using time evolution equations. Int J Numer Methods Eng 2018;115(11):1371–82.
[100] Sun Q, Sun Y, Li L. Strength analysis and tooth shape optimization for involute gear with a few teeth. Adv Mech Eng 2018;10(1):1–11.
[101] Hu P, Yang L, Li B. Skeleton-section template parameterization for shape optimization. J Mech Des, Trans ASME 2018;(12):140.
[102] Giacomini M, Pantz O, Trabelsi K. Volumetric expressions of the shape gradient of the compliance in structural shape optimization. 2017. arXiv preprint arXiv: 1701.05762.
[103] Georgioudakis M, Lagaros ND, Papadrakakis M. Probabilistic shape design optimization of structural components under fatigue. Comput Struct 2017;182: 252–66.
[104] Noel L, Duysinx P. Shape optimization of microstructural designs subject to local stress constraints within an XFEM-level set framework. Struct Multidiscipl Optim 2017;55(6):2323–38.
[105] Schmitt O, Steinmann P. On curvature approximation in 2D and 3D parameter-free shape optimization. Struct Multidiscip Optim 2017;55(5):1655–69.
[106] Jiang L, Zhang W, Ma GJ, Wu CW. Shape optimization of energy storage flywheel rotor. Struct Multidiscipl Optim 2017;55(2):739–50.
[107] Riehl S, Steinmann P. On structural shape optimization using an embedding domain discretization technique. Int J Numer Methods Eng 2017;109(9): 1315–43.
[108] Marco O, Rodenas JJ, Albelda J, Nadal E, Tur M. Structural shape optimization using Cartesian grids and automatic h -adaptive mesh projection. Struct Multidiscipl Optim 2018;58(1):61–81.
[109] Tabatabaei S, Behbahani S, Rajaeipour P. Multiobjective shape design optimization of piezoelectric energy harvester using artificial immune system. Microsyst Technol 2016;22(10):2435–46.
[110] Schmitt O, Friederich J, Riehl S, Steinmann P. On the formulation and implementation of geometric and manufacturing constraints in node-based shape optimization. Struct Multidisc Optim 2016;53:881–92.
[111] Landkammer P, Steinmann P. A non-invasive heuristic approach to shape optimization in forming. Comput Mech 2016;57(2):169–91.
[112] Noel L, Miegroet LV, Duysinx P. Analytical sensitivity analysis using the extended finite element method in shape optimization of bimaterial structures. Int J Numer Methods Eng 2016;107(8):669–95.
[113] Liu Y, Shimoda M. Non-parametric shape optimization method for natural vibration design of stiffened shells. Comput Struct 2015;146:20–31.
[114] Liu Y, Shimoda M, Shibutani Y. Parameter-free method for the shape optimization of stiffeners on thin-walled structures to minimize stress concentration. J Mech Sci Technol 2015;29(4):1383–90.
[115] Tanaka N, Honma T, Yokosuka Y. Structural shape optimization of free-form surface shell and property of solution search using firefly algorithm. J Mech Sci Technol 2015;29(4):1449–55.
[116] Zhang W, Zhao L, Cai S. Shape optimization of Dirichlet boundaries based on weighted B-spline finite cell method and level-set function. Comput Meth Appl Mech Eng 2015;294:359–83.
[117] Shintani K, Azegami H. Shape optimization for suppressing brake squeal. Struct Multidisc Optim 2014;50(6):1127–35.
[118] Choi MJ, Kim MG, Cho S. Shape-design optimization of hull structures considering thermaldeformation. Proc Inst Mech Eng Part C J Mech Eng Sci 2014; 228(13):2266–77.

[119] Hunkeler S, Duddeck F, Rayamajhi M, Zimmer H. Shape optimisation for crashworthiness followed by a robustness analysis with respect to shape variables: example of a front rail. Struct Multidisc Optim 2013;48(2):367–78.
[120] Nasuf A, Bhaskar A, Keane AJ. Grammatical evolution of shape and its application to structural shape optimization. Struct Multidisc Optim 2013;48(1):187–99.
[121] Azegami H, Zhou L, Umemura K, Kondo N. Shape optimization for a link mechanism. Struct Multidisc Optim 2013;48(1):115–25.
[122] Ammar A, Huerta A, Chinesta F, Cueto E, Leygue A. Parametric solutions involving geometry: a step towards efficient shape optimization. Comput Meth Appl Mech Eng 2013;268:178–93.
[123] Wang D, Zhang W. A general material perturbation method using fixed mesh for stress sensitivity analysis and structural shape optimization. Comput Struct 2013;129:40–53.
[124] Ou H, Lu B, Cui ZS, Lin C. A direct shape optimization approach for contact problems with boundary stress concentration. J Mech Sci Technol 2013;27(9):2751–9.
[125] Martini K, Tobias C, Eberhard P. Durability-based shape optimization with application to a steering system. Eng Optim 2013;45(3):337–55.
[126] Wilke DN, Kok S, Groenwold AA. Relaxed error control in shape optimization that utilizes remeshing. Int J Numer Meth Eng 2013;94:273–89.
[127] Kruzelecki J, Proszowski R. Shape optimization of thin-walled pressure vessel end closures. Struct Multidisc Optim 2012;46(5):739–54.
[128] Kasolis F, Wadbro E, Berggren M. Fixed-mesh curvature-parameterized shape optimization of an acoustic horn. Struct Multidisc Optim 2012;46(5):727–38.
[129] Wei Y, Rezgui A, Yao Z, Wang P. A comparative analysis of contact algorithms in contact shape optimization problems. Optim Eng 2012;13(4):595–623.
[130] Gerzen N, Materna D, Barthold F-J. The inner structure of sensitivities in nodal based shape optimization. Comput Mech 2012;49(3):379–96.
[131] Firl M, Bletzinger K-U. Shape optimization of thin walled structures governed by geometrically nonlinear mechanics. Comput Meth Appl Mech Eng 2012;237-240:107–17.
[132] Lee G, Park J, Choi D-H. Shape optimization of mobile phone folder module for structural strength. J Mech Sci Technol 2012;26(2):509–15.
[133] Ozturk UE. Efficient method for fatigue based shape optimization of the oil sump carrying a differential case in four wheel drive vehicles. Struct Multidisc Optim 2011;44(6):823–30.
[134] Lu B, Ou H, Long H. Die shape optimisation for net-shape accuracy in metal forming using direct search and localised response surface methods. Struct Multidisc Optim 2011;44(4):529–45.
[135] Edke MS, Chang K-H. Shape optimization for 2-D mixed-mode fracture using extended FEM (XFEM) and level set method (LSM). Struct Multidisc Optim 2011;44(2):165–81.
[136] Le C, Bruns T, Tortorelli D. A gradient-based, parameter-free approach to shape optimization. Comput Meth Appl Mech Eng 2011;200(9-12):985–96.
[137] Espath LFR, Linn RV, Awruch AM. Shape optimization of shell structures based on NURBS description using automatic differentiation. Int J Numer Methods Eng 2011;88(7):613–36.
[138] Rodenas JJ, Bugeda G, Albelda J, Onate E. On the need for the use of error-controlled finite element analyses in structural shape optimization processes. Int J Numer Methods Eng 2011;87(11):1105–26.
[139] Pathak KK, Sehgal DK. Gradientless shape optimization using artificial neural networks. Struct Multidisc Optim 2010;41(5):699–709.
[140] Brujic D, Ristic M, Mattone M, Maggiore P, De Poli GP. CAD based shape optimization for gas turbine component design. Struct Multidisc Optim 2010;41(4):647–59.
[141] Sonmez FO. Optimal shape design of shoulder fillets for flat and round bars under various loadings. Proc Inst Mech Eng, Part C 2009;223(8):1741–54.
[142] Schafer C, Finke E. Shape optimisation by design of experiments and finite element methods-an application of steel wheels. Struct Multidisc Optim 2008;36(5):477–91.
[143] Pedersen P. Suggested benchmarks for shape optimization for minimum stress concentration. Struct Multidisc Optim 2008;35(4):273–83.
[144] Gustafsson E, Stromberg N. Shape optimization of castings by using successive response surface methodology. Struct Multidisc Optim 2008;35(1):11–28.
[145] Peng D, Jones R. An approach based on biological algorithm for three-dimensional shape optimisation with fracture strength constrains. Comput Meth Appl Mech Eng 2008;197(49-50):4383–98.
[146] Yıldız AR, Öztürk N, Kaya N, Öztürk F. Hybrid multi-objective shape design optimization using Taguchi's method and genetic algorithm. Struct Multidisc Optim 2007;34(4):317–32.
[147] Silva CAC, Bittencourt ML. Velocity fields using NURBS with distortion control for structural shape optimization. Struct Multidisc Optim 2007;33(2):147–59.
[148] Sonmez FO. Shape optimization of 2D structures using simulated annealing. Comput Meth Appl Mech Eng 2007;196(35-36):3279–99.
[149] Miegroet LV, Duysinx P. Stress concentration minimization of 2D filets using X-FEM and level set description. Struct Multidisc Optim 2007;33(4-5):425–38.
[150] Kegl M, Brank B. Shape optimization of truss-stiffened shell structures with variable thickness. Comput Meth Appl Mech Eng 2006;195(19-22):2611–34.
[151] Edke MS, Chang KH. Shape optimization of heavy load carrying components for structural performance and manufacturing cost. Struct Multidisc Optim 2006;31(5):344–54.
[152] Meske R, Lauber B, Schnack E. A new optimality criteria method for shape optimization of natural frequency problems. Struct Multidisc Optim 2006;31(4):295–310.
[153] Liu J-S, Parks GT, Clarkson PJ. Topology/shape optimisation of axisymmetric continuum structures - a metamorphic development approach. Struct Multidisc Optim 2005;29(1):73–83.
[154] Wang X, Oguamanam DCD, Hansen JS. Shape optimization of stiffeners in stiffened composite plates with thermal residual stresses. Struct Multidisc Optim 2005;30(1):38–42.
[155] Pedersen P, Pedersen NL. An optimality criterion for shape optimization in eigen frequency problems. Struct Multidisc Optim 2005;29(6):457–69.
[156] Lee TH, Lee K. Multi-criteria shape optimization of a funnel in cathode ray tubes using a response surface model. Struct Multidisc Optim 2005;29(5):374–81.
[157] HauBler P, Albers A. Shape optimization of structural parts in dynamic mechanical systems based on fatigue calculations. Struct Multidisc Optim 2005;29(5):361–73.
[158] Cho JR, Shin SW, Yoo WS. Crown shape optimization for enhancing tire wear performance by ANN. Comput Struct 2005;83(12-13):920–33.
[159] Park KJ, Lee JN, Park GJ. Structural shape optimization using equivalent static loads transformed from dynamic loads. Int J Numer Methods Eng 2005;63(4):589–602.
[160] Wu Z. An efficient approach for shape optimization of components. Int J Mech Sci 2005;47(10):1595–610.
[161] Wilke DN, Kok S, Groenwold AA. A quadratically convergent unstructured remeshing strategy for shape optimization. Int J Numer Methods Eng 2006;65(1):1–17.
[162] Norato J, Haber R, Tortorelli D, Bendsoe MP. A geometry projection method for shape optimization. Int J Numer Methods Eng 2004;60(14):2289–312.
[163] McDonald M, Heller M. Robust shape optimization of notches for fatigue-life extension. Struct Multidisc Optim 2004;28(1):55–68.
[164] Inzarulfaisham AR, Azegami H. Solution to boundary shape optimization problem of linear elastic continua with prescribed natural vibration mode shapes. Struct Multidisc Optim 2004;27(3):210–7.
[165] Garcia MJ, Gonzalez CA. Shape optimisation of continuum structures via evolution strategies and fixed grid finite element analysis. Struct Multidisc Optim 2004;26(1-2):92–8.
[166] Camprubi N, Bischoff M, Bletzinger K-U. Shape optimization of shells and locking. Comput Struct 2004;82(29-30):2551–61.
[167] Han J, Yamazaki K. A novel technique of shape optimization for the contact parts of bolt-nut fastening structures, 36. Engineering Optimization; 2004. p. 691–703.
[168] Shen J, Yoon D. A freeform shape optimization of complex structures represented by arbitrary polygonal or polyhedral meshes. Int J Numer Methods Eng 2004;60(15):2441–66.
[169] Lu KJ, Kota S. Design of compliant mechanisms for morphing structural shapes. J Intell Mater Syst Struct 2003;14(6):379–91.
[170] Barbarosie C. Shape optimization of periodic structures. Comput Mech 2003;30(3):235–46.
[171] Shen J, Yoon D. A new scheme for efficient and direct shape optimization of complex structures represented by polygonal meshes. Int J Numer Methods Eng 2003;58(14):2201–23.
[172] Fourie PC, Groenwold AA. The particle swarm optimization algorithm in size and shape optimization. Struct Multidisc Optim 2002;23(4):259–67.
[173] De Freitas JAT, Cismasiu I. Shape optimization with hybrid-Trefftz displacement elements. Int J Numer Methods Eng 2002;53(2):473–98.
[174] Woon SY, Querin OM, Steven GP. Structural application of a shape optimization method based on a genetic algorithm. Struct Multidisc Optim 2001;22:57–64.
[175] Hilding D, Torstenfelt B, Klarbring A. A computational methodology for shape optimization of structures in frictionless contact. Comput Methods Appl Mech Eng 2001;190:4043–60.
[176] Marburg S, Hardtke H-J. Shape optimization of a vehicle hat-shelf: improving acoustic properties for different load cases by maximizing first eigenfrequency. Comput Struct 2001;79:1943–57.
[177] Kress G.R. Shape optimization of a flywheel. Struct Multidisc Optim vol: 19 pp: 74-81.
[178] Herskovits J, Leontiev A, Dias G, Santos G. Contact shape optimization: a bilevel programming approach. Struct Multidisc Optim 2000;20:214–21.
[179] Srikanth A, Zabaras N. Shape optimization and preform design in metal forming processes. Comput Methods Appl Mech Eng 2000;190:1859–901.
[180] Choi KK, Duan W. Design sensitivity analysis and shape optimization of structural components with hyperelastic material. Comput Methods Appl Mech Eng 2000;187:219–43.
[181] Rohan E, Whiteman JR. Shape optimization of elasto-plastic structures and continua. Comput Methods Appl Mech Eng 2000;187:261–88.
[182] Souli M, Zolesio JP, Ouahsine A. Shape optimization for non-smooth geometry in two dimensions. Comput Methods Appl Mech Eng 2000;188:109–19.
[183] Givoli D, Demchenko T. A boundary-perturbation finite element approach for shape optimization. Int J Numer Methods Eng 2000;47:801–19.
[184] Lian H, Bordas SPA, Sevilla R, Simpson RN. Recent developments in CAD/analysis integration. 2012.
[185] Kagan P, Fischer A, Bar-Yoseph PZ. New B-Spline finite element approach for geometrical design and mechanical analysis. Int J Numer Methods Eng 1998;41:435–58.
[186] Kagan P, Fischer A. Integrated mechanically based CAE system using B-Spline finite element. Comput-Aided Des 2000;32:539–52.
[187] Rogers DF, Adams JA. Mathematical elements for computer graphics. New York: McGraw-Hill; 1976.
[188] Nguyen VP, Anitescu C, Bordas SP, Rabczuk T. Isogeometric analysis: an overview and computer implementation aspects. Math Comput Simul 2015;117:89–116.

[189] Sederberg TW, Zheng J, Bakenov A, Nasri A. T-splines and TNURCCSs. ACM Trans Graph 2003;22(3):477–84.
[190] Cottrell JA, Hughes TJR, Bazilevs Y. Isogeometric analysis – towards integration of CAD and FEA. John Wiley & Sons, Ltd; 2009.
[191] Ha SH, Choi KK, Cho S. Numerical method for shape optimization using T-spline based isogeometric method. Struct Multidisc Optim 2010;42(3):417–28.
[192] Wang C, Xia S, Wang X, Qian X. Isogeometric shape optimization on triangulations. IDETC/CIE; 2016.
[193] López J, Anitescu C, Valizadeh N, Rabczuk T, Alajlan N. Structural shape optimization using Bézier triangles and a CAD-compatible boundary representation. Eng Comput 2019:1–16.
[194] Noah Jaxon N, Qian X. Isogeometric analysis on triangulations. Comput Aided Des 2014;46:45–57.
[195] Engvall L, Evans JA. Isogeometric triangular Bernstein–Bézier discretizations: automatic mesh generation and geometrically exact finite element analysis. Comput Method Appl Mech Eng 2016;304:378–407.
[196] Lapez J, Anitescu C, Rabczuk T. CAD-compatible structural shape optimization with a movable Bezier tetrahedral mesh. Comput Meth Appl Mech Eng 2020;367: 113066.
[197] Luycker ED, Benson DJ, Belytschko T, Bazilevs Y, Hsu MC. X-FEM in isogeometric analysis for linear fracture mechanics. Int J Numer Methods Eng 2011;87(6): 541–65.
[198] Ghorashi SS, Valizadeh N, Mohammadi S. Extended isogeometric analysis for simulation of stationary and propagating cracks. Int J Numer Methods Eng 2012; 89(9):1069–101.
[199] Bui QT. Extended isogeometric dynamic and static fracture analysis for cracks in piezoelectric materials using NURBS. Comput Meth Appl Mech Eng 2015;295: 470–509.
[200] Li K, Qian X. Isogeometric analysis and shape optimization via boundary integral. Comput Aided Des 2011;43(11):1427–37.
[201] Simpson RN, Bordas SPA, Trevelyan J, Rabczuk T. A two dimensional isogeometric boundary element method for elastostatic analysis. Comput Methods Appl Mech Engrg 2012;209-212:87–100.
[202] Ummidivarapu VK, Voruganti HK, Khajah T, Bordas SPA. Isogeometric shape optimization of an acoustic horn using the teaching-learning-based optimization (TLBO) algorithm. Comput Aided Geometric Des 2020;80:101881.
[203] Mostafa Shaaban A, Anitescu C, Atroshchenko E, Rabczuk T. Shape optimization by conventional and extended isogeometric boundary element method with PSO for two-dimensional Helmholtz acoustic problems. Eng Anal Boundary Elem 2020;113:156–69.
[204] Kumar D, Poh LH, Quek ST. Isogeometric shape optimization of missing rib auxetics with prescribed negative Poisson's ratio over large strains using genetic algorithm. Int J Mech Sci 2020. https://doi.org/10.1016/j.ijmecsci.2020.106169.
[205] Yoon M, Lee J, Koo B. Shape design optimization of thermoelasticity problems using isogeometric boundary element method. Adv Eng Softw 2020:149.
[206] Aminzadeh M, Kazemi H, Tavakkoli S. A numerical study on optimum shape of steel slit dampers. Adv Struct Eng 2020;23(14):2967–81.
[207] Chen LL, Lian H, Liu Z, Chen HB, Atroshchenko E, Bordas SPA. Structural shape optimization of three dimensional acoustic problems with isogeometric boundary element methods. Comput Meth Appl Mech Eng 2019;355:926–51.
[208] Hirschler T, Bouclier R, Duval A, Elguedj T, Morlier J. Isogeometric sizing and shape optimization of thin structures with a solid-shell approach. Struct Multidisc Optim 2019;59(3):767–85.
[209] Hirschler T, Bouclier R, Duval A, Elguedj T, Morlier J. The embedded isogeometric Kirchhoff-Love shell: From design to shape optimization of non-conforming stiffened multipatch structures. Comput Meth Appl Mech Eng 2019; 349:774–97.
[210] Li S, Trevelyan J, Wu Z, Lian H, Wang D, Zhang W. An adaptive SVD-Krylov reduced order model for surrogate based structural shape optimization through isogeometric boundary element method. Comput Meth Appl Mech Eng 2019;349: 312–38.
[211] Weeger O, Narayanan B, Dunn ML. Isogeometric shape optimization of nonlinear, curved 3D beams and beam structures. Comput Meth Appl Mech Eng 2019;345: 26–51.
[212] Lieu QX, Lee J. An isogeometric multimesh design approach for size and shape optimization of multidirectional functionally graded plates. Comput Meth Appl Mech Eng 2019;343:407–37.
[213] Chamoin L, Thai HP. Certified real-time shape optimization using isogeometric analysis, PGD model reduction, and a posteriori error estimation. Int J Numer Methods Eng 2019;119(3):151–76.
[214] Sun SH, Yu TT, Nguyen TT, Atroshchenko E, Bui TQ. Structural shape optimization by IGABEM and particle swarm optimization algorithm. Eng Anal Boundary Elem 2018;88:26–40.
[215] Ding C, Cui X, Huang G, Li G, Tamma KK, Cai Y. A gradient-based shape optimization scheme via isogeometric exact reanalysis. Eng Comput (Swansea, Wales) 2018;35(8):2696–721.
[216] Ahn SH, Choi M-J, Cho S. Isogeometric shape design optimization of nanoscale structures using continuum-based shell theory considering surface effects. Int J Mech Sci 2018;141:9–20.
[217] Bandara K, Cirak F. Isogeometric shape optimisation of shell structures using multiresolution subdivision surfaces. CAD Comput Aided Des 2018;95:62–71.
[218] Herrema AJ, Wiese NM, Darling CN, Ganapathysubramanian B, Krishnamurthy A, Hsu M-C. A framework for parametric design optimization using isogeometric analysis. Comput Meth Appl Mech Eng 2017;316:944–65.
[219] Lian H, Kerfriden P, Bordas SPA. Implementation of regularized isogeometric boundary element methods for gradient-based shape optimization in two-dimensional linear elasticity. Int J Numer Methods Eng 2016;106(12):972–1017.
[220] Lian H, Kerfridenc P, Bordas SPA. Shape optimization directly from CAD: an isogeometric boundary element approach using T-splines. Comput Meth Appl Mech Eng 2017;317:1–41.
[221] Radaelli G, Herder JL. Shape optimization and sensitivity of compliant beams for prescribed load-displacement response. Mech Sci 2016;7(2):219–32.
[222] Kang P, Youn S-K. Isogeometric shape optimization of trimmed shell structures. Struct Multidisc Optim 2016;53(4):825–45.
[223] Choi M-J, Cho S. A mesh regularization scheme to update internal control points for isogeometric shape design optimization. Comput Meth Appl Mech Eng 2015; 285:694–713.
[224] Ha YD. Generalized isogeometric shape sensitivity analysis in curvilinear coordinate system and shape optimization of shell structures. Struct Multidisc Optim 2015;52(6):1069–88.
[225] Fubeder D, Simeon B, Vuong A-V. Fundamental aspects of shape optimization in the context of isogeometric analysis. Comput Meth Appl Mech Eng 2015;286: 313–31.
[226] Kostas KV, Ginnis AI, Politis CG, Kaklis PD. Ship-hull shape optimization with a T-spline based BEM-isogeometric solver. Comput Meth Appl Mech Eng 2015;284: 611–22.
[227] Wang Z-P, Turteltaub S. Isogeometric shape optimization for quasi-static processes. Int J Numer Methods Eng 2015;104(5):347–71.
[228] He G, Pan Z, Zou Z, Zhu D. The isogeometric shape optimization method based on finite cell method. Key Eng Mater 2014;621:655–62.
[229] Kiendl J, Schmidt R, Wuchner R, Bletzinger K-U. Isogeometric shape optimization of shells using semi-analytical sensitivity analysis and sensitivity weighting. Comput Meth Appl Mech Eng 2014;274:148–67.
[230] Koo B, Ha S-H, Kim H-S, Cho S. Isogeometric shape design optimization of geometrically nonlinear structures. Mech Based Des Struct Mach 2013;41(3): 337–58.
[231] Azegami H, Fukumoto S, Aoyama T. Shape optimization of continua using NURBS as basis functions. Struct Multidisc Optim 2013;47(2):247–58.
[232] Manh ND, Evgrafov A, Gersborg AR, Gravesen J. Isogeometric shape optimization of vibrating membranes. Comput Meth Appl Mech Eng 2011;200(13-16): 1343–53.
[233] Qian X. Full analytical sensitivities in NURBS based isogeometric shape optimization. Comput Meth Appl Mech Eng 2010;199(29-32):2059–71.
[234] Nagy AP, Abdalla MM, Gurdal Z. Isogeometric sizing and shape optimisation of beam structures. Comput Meth Appl Mech Eng 2010;199(17-20):1216–30.
[235] Cho S, Ha S-H. Isogeometric shape design optimization: exact geometry and enhanced sensitivity. Struct Multidisc Optim 2009;38(1):53–70.
[236] Bobaru F, Mukherjee S. Meshless approach to shape optimization of linear thermoelastic solids. Int J Numer Methods Eng 2002;53(4):765–96.
[237] Belytschko T, Lu YY, Gu L. Element-free Galerkin methods. Int J Numer Methods Eng 1994;37(2):229–56.
[238] Zhu T, Atluri SN. A modified collocation method and a penalty formulation for enforcing the essential boundary conditions in the element free Galerkin method. Comput Mech 1998;21:211–22.
[239] Dolbow J, Belytschko T. An introduction to programming the meshless element free Galerkin method. Arch Comput Methods Eng 1998;5(3):207–41.
[240] Krongauz Y, Belytschko T. Enforcement of essential boundary conditions in meshless approximations using finite elements. Comput Meth Appl Mech Eng 1996;131(1-2):133–45.
[241] Lu YY, Belytschko T, Gu L. A new implementation of the element free Galerkin method. Comput Meth Appl Mech Eng 1994;113(3-4):397–414.
[242] Liu WK, Jun S, Zhang Y. Reproducing kernel particle methods. Int J Numer Methods Fluids 1995;20:1081–106.
[243] Gong SG, Xie GL, Zhang JP, Nie SH, Li YM. Sensitivity analysis and shape optimization based on FE-EFG coupled method. Res Eng Des 2009;20(2):117–28.
[244] Rabczuk T, Xiao SP, Sauer M. Coupling of mesh-free methods with finite elements: basic concepts and test results. Commun Numer Methods Eng 2006;22 (10):1031–65.
[245] Rohit G, Prajapati J, Patel V. Structural shape optimization by coupled FE-MM and swarm intelligence based algorithm. Int J Comput Methods 2020. https://doi.org/10.1142/s0219876220500486.
[246] Daxini SD, Prajapati JM. Numerical shape optimization based on meshless method and stochastic optimization technique. Eng Comput 2020;36(2):565–86.
[247] Daxini SD, Prajapati JM. Structural shape optimization with meshless method and swarm-intelligence based optimization. Int J Mech Mater Des 2020;16(1):167–90.
[248] Zhang J, Gong S, Huang Y, Qiu A, Chen R. Structural dynamic shape optimization and sensitivity analysis based on RKPM. Struct Multidisc Optim 2008;36(3): 307–17.
[249] Zou W, Zhou JX, Zhang ZQ, Li Q. A truly meshless method based on partition of unity qudarature for shape optimization of continua. Comput Mech 2007;39(4): 357–65.
[250] Bobaru F, Rachakonda SE. FG)2: A new fixed-grid shape optimization method based on the element-free Galerkin mesh-free analysis: taking large steps in shape optimization. Struct Multidisc Optim 2006;32(3):215–28.
[251] Zhang ZQ, Zhou JX, Zhou N, Wang XM, Zhang L. Shape optimization using reproducing kernel particle method and an enriched genetic algorithm. Comput Meth Appl Mech Eng 2005;194(39-41):4048–70.
[252] Bobaru F, Rachakonda S. Optimal shape profiles for cooling fins of high and low conductivity. Int J Heat Mass Transfer 2004;47(23):4953–66.

[253] Bobaru F, Rachakonda S. Boundary layer in shape optimization of convective fins using a meshfree approach. Int J Numer Methods Eng 2004;60(7):1215–36.

[254] Kim NH, Choi KK, Botkin ME. Numerical method for shape optimization using meshfree method. Struct Multidisc Optim 2003;24(6):418–29.

[255] Grindeanu I, Kim NH, Choi KK, Chen JS. CAD-based shape optimization using a meshfree method. Concurr Eng 2002;10(1):55–66.

[256] Kim NH, Choi KK, Chen JS. Die shape design optimization of sheet metal stamping process using meshfree method. Int J Numer Methods Eng 2001;51(12):1385–405.

[257] Kim NH, Choi KK. Design sensitivity analysis and optimization of nonlinear transient dynamics. Mech Struct Mach 2001;29(3):351–71.

[258] Kim NH, Choi KK, Chen JS, Park YH. Meshless shape design sensitivity analysis and optimization for contact problem with friction. Comput Mech 2000;25:157–68.

[259] Kim NH, Choi KK, Chen JS. Shape design sensitivity analysis and optimization of elasto–plasticity with frictional contact. AIAA J 2000;38(9):1742–53.

[260] Kim NH, Park YH, Choi KK. Optimization of a hyper-elastic structure with multibody contact using continuum-based shape design sensitivity analysis. Struct Multidisc Optim 2000;21:196–208.

[261] Bischof C, Carle A, Khademi P, Mauer A. The ADIFOR 2.0 system for the automatic differentiation of Fortran 77 programs. IEEE Comput Sci Eng 1996;3:18–32.

[262] Griewank A, Juedes D, Utke J. ADOL-C, a package for the automatic differentiation of algorithms written in C/C++,. TOMS 1996;22(2):131–67.

[263] Shiriaev D, Griewank A. ADOL-F: automatic differentiation of Fortran codes. Comput Differentiat Tech Appl Tools 1996;1:375–84.

[264] Wujek BA, Renaud JE. Automatic differentiation for more efficient system analysis and optimization. Eng Optim 1998;31:101–39.

[265] Hardee E, Chang KH, Tu J, Choi KK, Grindeanu I, Yu X. A CAD based design parameterization of elastic solids. Adv Eng Softw 1999;30:185–99.

[266] Agarwal D, Robinson TT, Armstrong CG, Marques S, Vasilopoulos I, Meyer M. Parametric design velocity computation for CAD-based design optimization using adjoint methods. Eng Comput 2018;34(2):225–39.

[267] Li X, Sederberg TW. S-splines: a simple surface solution for IGA and CAD. Comput Meth Appl Mech Eng 2019;350:664–78.