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REVIEW ARTICLE

Truss optimization with discrete design variables: a critical review

Mathias Stolpe[1]

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Abstract This review presents developed models, theory, and numerical methods for structural optimization of trusses with discrete design variables in the period 1968 – 2014. The comprehensive reference list collects, for the first time, the articles in the field presenting deterministic optimization methods and meta heuristics. The field has experienced a shift in focus from deterministic methods to meta heuristics, i.e. stochastic search methods. Based on the reported numerical results it is however not possible to conclude that this shift has improved the competences to solve application relevant problems. This, and other, observations lead to a set of recommended research tasks and objectives to bring the field forward. The development of a publicly available benchmark library is urgently needed to support development and assessment of existing and new heuristics and methods. Combined with this effort, it is recommended that the field begins to use modern methods such as performance profiles for fair and accurate comparison of optimization methods. Finally, theoretical results are rare in this field. This means that most recent methods and heuristics are not supported by mathematical theory. The field should therefore re-focus on theoretical issues such as problem analysis and convergence properties of new methods.

✉ Mathias Stolpe
matst@dtu.dk

[1] Department of Wind Energy, Technical University of Denmark (DTU), Frederiksborgvej 399, 4000, Roskilde, Denmark

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1 Introduction

Numerical optimization of truss structures with *continuous design variables* first appeared in the 1960s in Dorn et al. (1964). The models were based on the ground structure approach in which nodes are first distributed over the design domain and then connected by potential bars. A (continuous) design variable describing the cross-section area is then associated with each potential bar. The first problems were single load minimum weight problems modelled as linear programs, for details see e.g. the textbook Hemp (1973). This field has, since then, experienced tremendous activity both in modelling and development of theory and optimization methods for solving large-scale problems. These developments are largely covered in the textbooks Bendsøe and Sigmund (2003) and Ohsaki (2011). The history of numerical truss topology optimization with *discrete design variables* is almost as long. To the best knowledge of the author the first journal article using discrete design variables for trusses and suggesting a method for the considered problems appeared in the late 1960s in Toakley (1968). Developments of heuristics and methods continued in the early 1970s and some early numerical results can be found in e.g. Lipson and Agrawal (1974) and Lipson and Gwin (1977).

The complexity of solving minimum weight truss design problems with discrete design variables has been studied already in Yates et al. (1982). The theoretical results strongly suggest that the problems are very difficult to solve. Perhaps the results from Yates et al. (1982) are best described by the authors' own words: *"The problem of minimising the weight of structural trusses subject to constraints on joint deflection and member stress and gauge has been considered. In particular, the case DMT*[1] *where member sizes are available from a discrete set has been examined and shown to be NP-hard. As an intermediate step, the problem DMTD*[2] *where only joint deflection constraints apply, was also shown to be of equivalent difficulty. Consequent upon these results, the problem of devising an approximation algorithm for DMT that guaranteed a bound on the absolute error was considered, but unfortunately, this too proved to be NP-hard."*

Although a seemingly narrow sub-field of structural optimization, both in terms of research and engineering applications, there is a vast number of research articles in a wide variety of journals and conference proceedings.

Optimal design of truss structures is an important field within structural optimization. Perhaps not so much because of the actual engineering application since the mechanical assumptions are rarely met in real design situations. More because truss topology and sizing optimization can be used to quickly find good and new conceptual and preliminary designs for further detailed analysis and design.

From an optimization point of view, truss design problems experience all the difficulties which arise for optimal design of more advanced structures. The problems are (or at least, can be) large-scale both in terms of number of variables and constraints. The problems are, besides the discrete requirements on the variables, generally non-convex and their continuous relaxations normally do not satisfy standard constraint qualifications. In the case of truss topology optimization the structural analysis also becomes troublesome since the stiffness matrix in general is only positive semidefinite. The nodal displacements are, for a fixed design, no longer uniquely determined which may introduce non-differentiability in the objective and constraint functions.

One general difference between optimal design of truss structures and other structures when applying the finite element method for structural analysis is that truss problems often have many design variables but normally much fewer state variables, i.e. nodal displacements.

One of the major advantages of truss structures in linear elasticity is that they do not require advanced finite elements for the analysis. The analysis equations are easily stated and implemented in any high level programming language. This includes optimization modelling languages such as AMPL (Fourer et al. 2003) and GAMS (Brooke et al. 1992) which are commonly used in operations research and mathematical programming.

Optimal design of truss structures is thus the ideal candidate to communicate structural optimization problems to other research communities such as mathematical programming, operations research, and integer and combinatorial optimization. These fields have extensive experience in advanced modelling and development of theory and heuristics and methods. In the light of the complexity results from Yates et al. (1982) it would be beneficial if the structural optimization field could obtain assistance from other fields.

Optimal design of truss structures thus provides an excellent opportunity for development and assessment of new optimization heuristics and methods. If these methods can be successfully applied to optimal design of trusses they are also good candidates to be modified to solve other optimal structural design problems with discrete design variables.

1.1 Stochastic versus deterministic

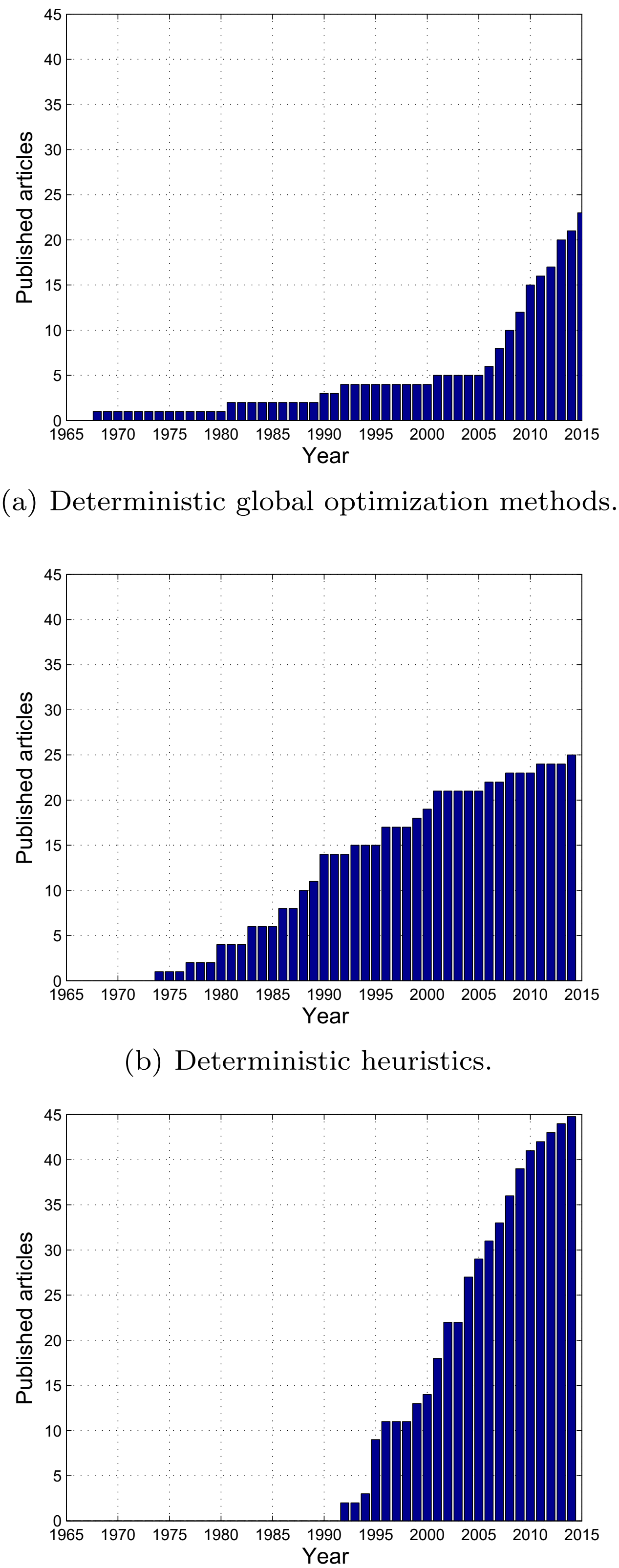
Two main classifications of the references are used in this article. The references are presented according to the type of problems, i.e. the choices of design parametrization, objective, and constraint functions. They are also categorized according to the class of optimization method or heuristic used to solve[3] the problem. This means that many references are cited more than once. The classification of the references based on the optimization techniques revealed that the development of methods and heuristics has two essentially disjunct main directions. Truss optimization problems are either solved by meta heuristics such as genetic algorithms, or ant colony optimization methods or they are solved by deterministic optimization methods and heuristics. The deterministic methods are often based on branch-and-bound algorithms and the deterministic heuristics are often based on solving (a sequence of) continuous problems combined with rounding techniques. Notably, very few cross-citations between these two fields can be found. Furthermore, the two fields make use of different benchmark problems and different ways of comparing two methods.

Figure 1 shows the accumulated number of published research articles from the reference list on truss optimization with discrete design variables. The figure is organized based on the choice of optimization algorithms in the articles. Figure 1 shows that the field has experienced a shift in focus from deterministic heuristics and methods to meta heuristics. The figure also shows that developments of deterministic heuristics is levelling out whereas the number of articles on deterministic optimization is increasing since 2005. The latter observation can possibly be explained by the extensive developments in methods and commercial

[1] Discrete member-size Minimum weight Truss problem.

[2] Discrete member-size Minimum weight Truss problem with Deflection constraints.

[3] Note that this word is often used in a very broad sense in this article.



(a) Deterministic global optimization methods.

(b) Deterministic heuristics.

(c) Meta heuristics.

Fig. 1 The accumulated number of journal articles based on the choice of optimization method in the field of truss optimization with discrete design variables

software for both general mixed integer linear programming and mixed integer nonlinear programming over the past ten years.

1.2 Purposes of this review

The purposes of this review article are to

- collect an extensive list of references within the field and additionally, provide references to the relevant background literature on optimization theory and optimization methods,
- present current state-of-the-art in terms of theory and numerical methods and heuristics,
- propose future research areas and targets, and
- present some (deserved) critique of the field.

1.3 Limitations of this review

For the purpose of easy search and access the reference list mainly contains journal articles and textbooks. The reference list does, for the same reason, not contain almost any MSc or PhD theses and only a few conference proceeding articles. This means that certain developments may be missing. The description of the order in which new models and methods have been introduced is potentially also inaccurate.

This review focuses entirely on optimization techniques which utilize the discrete nature of the design variables. All methods and heuristics either produce iterates which satisfy the discreteness requirements or the final result does with certainty. Approaches that do not satisfy one, or both, of these two properties are considered out of scope. For example, optimal design of truss structures with discrete design variables share many mathematical properties with multi-material topology optimization, see e.g. Bendsøe and Sigmund (1999), and with discrete material optimization of laminated composite structures, see e.g. Lund and Stegmann (2005) and Stegmann and Lund (2005). These kinds of problems are often modelled as problems with continuous design variables combined with generalization of material interpolation schemes (Bendsøe and Sigmund 1999). This approach does however not have theoretical results claiming convergence to a discrete design and the computational results in the literature generally contain designs that do not satisfy the discreteness requirements on the design variables. These techniques are therefore not mentioned further in this review.

1.4 Organization

This review is organized in the following manner. The three most commonly encountered problem formulations in the field together with several possible reformulations are reviewed in Section 2. The articles that modify or extend the basic problem formulations are outlined in Section 3. The three following sections present the heuristics and methods proposed for optimal truss design with discrete design variables. The heuristics and methods are categorized into meta heuristics (Section 4), deterministic global optimization methods (Section 5), and deterministic heuristics (Section 6). The corresponding articles are presented in chronological order for each class of method. Relatively extensive critique of the field and a set of research tasks and objectives are proposed in Section 7. The review ends with conclusions in Section 8.

2 Problem formulations

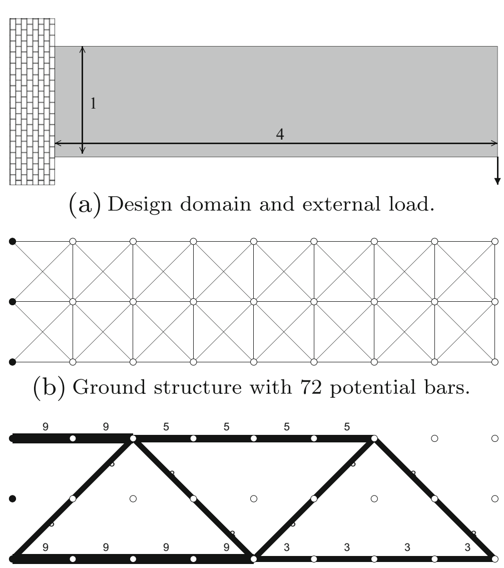
This section presents the three most commonly encountered classes of problems in the literature together with some of the many possible reformulation techniques.

2.1 The ground structure approach

The problem formulations in this article and in all listed references follow the popular ground structure approach for truss topology optimization problems, see e.g. Dorn et al. (1964), Bendsøe and Sigmund (2003), and Ohsaki (2011). The starting point is a given two- or three-dimensional design domain supplied with appropriate boundary conditions and external static loads. An example of a design domain is illustrated in Fig. 2a. Frictionless nodes are distributed, often evenly, over the design domain. The nodes are then connected by n potential straight bars resulting in a ground structure. An example of a ground structure corresponding to the design domain in Fig. 2a is illustrated in Fig. 2b. The optimal design to a minimum compliance problem is illustrated in Fig. 2c.

The use of the ground structure approach is not entirely unproblematic. Designs with undesired properties may result because of the particular choices in the generation of the ground structure. A list of issues with the ground structure approach is provided in Ohsaki (2011).

Some ground structures and associated problems are well-represented in the literature and have received names based on the number of potential bars. These are further mentioned numerous times in this review. The most well-known ground structures are the 10-bar truss in Fig. 3a, the spatial 25-bar truss in Fig. 3b, the planar 47-bar in Fig. 3c, and the spatial 72-bar truss in Fig. 3d. The dimensions and load conditions are intentionally left out from the figures since they vary (in units, magnitude, and directions) in the literature.



(a) Design domain and external load.
(b) Ground structure with 72 potential bars.
(c) Optimal design.

Fig. 2 A cantilever topology optimization problem from Achtziger and Stolpe (2009). The number of available bar areas is six including zero

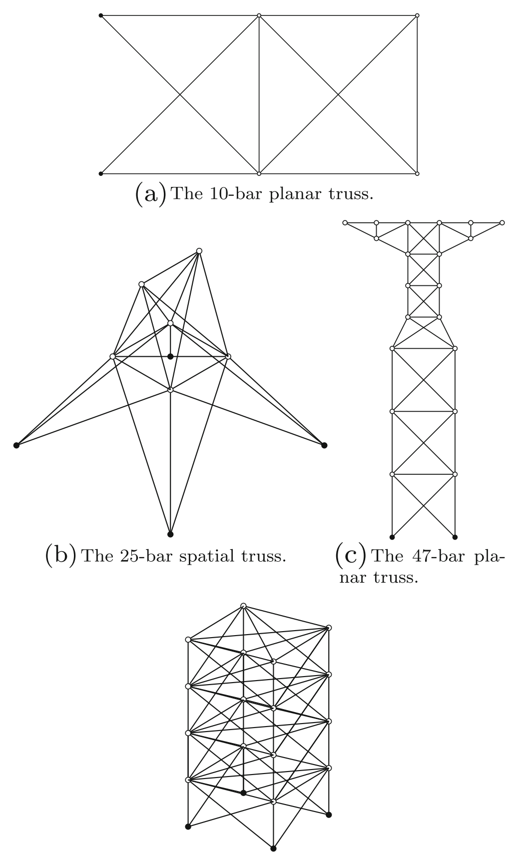

(a) The 10-bar planar truss.
(b) The 25-bar spatial truss.
(c) The 47-bar planar truss.
(d) The 72-bar spatial truss.

Fig. 3 Some of the ground structures which frequently are mentioned in the literature. Note that the dimensions are not to scale

2.2 Structural analysis and assumptions

The truss structure is normally described by a vector of design variables $\mathbf{a} \in \mathbb{R}^n$ where a_j represents the area of the j-th bar in the ground structure. Associated with the external loads $\mathbf{f}_1, \ldots, \mathbf{f}_\ell \in \mathbb{R}^d$ are displacement vectors $\mathbf{u}_1, \ldots, \mathbf{u}_\ell \in \mathbb{R}^d$ satisfying the linear elasticity equilibrium equations $\mathbf{K}(\mathbf{a})\mathbf{u}_l = \mathbf{f}_l$ for all l, where $\mathbf{K}(\mathbf{a}) : \mathbb{R}^n \to \mathbb{R}^{d\times d}$ denotes the stiffness matrix and d is the number of non fixed degrees of freedom.

For the structural analysis, the following assumptions are common.

(A1) The stiffness matrix is affine in the design variables and given by

$$\mathbf{K}(\mathbf{a}) = \mathbf{K}_0 + \sum_{j=1}^{n} a_j \mathbf{K}_j$$

where $\mathbf{K}_0 = \mathbf{K}_0^T \succeq \mathbf{0}$ is a given constant matrix[4]. The scaled local stiffness matrix $\mathbf{K}_j = \mathbf{K}_j^T$ is also symmetric positive semidefinite and given by the rank-1 matrix

$$\mathbf{K}_j = a_j \frac{E_j}{l_j} \mathbf{r}_j \mathbf{r}_j^T$$

where $l_j > 0$ is the length of the j-th bar, $E_j > 0$ is Young's modulus of the material, and $\mathbf{r}_j \in \mathbb{R}^d$ contains the direction cosines of the j-th bar, see e.g. Cook et al. (1989).

(A2) The stiffness matrix is positive definite for all $\mathbf{a} > \mathbf{0}$.

2.3 Minimum compliance problems

The first problem is the single load[5] minimum compliance (maximum overall stiffness) problem with a volume constraint

$$\begin{aligned}
&\underset{\mathbf{a}\in\mathbb{R}^n,\mathbf{u}\in\mathbb{R}^d}{\text{minimize}} \ \mathbf{f}^T\mathbf{u} && (\mathrm{P}_v^c)\\
&\text{subject to } \mathbf{K}(\mathbf{a})\mathbf{u} = f,\\
&\qquad \sum_{j=1}^{n} a_j l_j \le V,\\
&\qquad a_j \in \{a_j^1, \ldots, a_j^{m_j}\},\ j = 1, \ldots, n,
\end{aligned}$$

where $V > 0$ is the volume limit and m_j is the number of available areas for the jth bar. Variants of this problem have been extensively studied in e.g. Achtziger and Stolpe (2007), Achtziger and Stolpe (2008), and Achtziger and Stolpe (2009), Cerveira et al. (2009), and Kočvara (2010). Problem (P_v^c) is stated using the concept of Simultaneous ANalysis and Design (SAND), see e.g. Haftka (1985). The discrete design variables $\mathbf{a}$ are accompanied by continuous state variables $\mathbf{u} \in \mathbb{R}^d$, which are nodal displacements. The equilibrium equations are explicitly stated as nonlinear equality constraints. Problem (P_v^c) is, due to the equilibrium equations and the mixture of continuous and discrete variables, classified as a mixed integer[6] nonlinear problem with bilinear equality constraints. If $a_j^1 = 0$ for some, or all, j then it is possible to change the topology compared to the ground structure. In this situation (P_v^c) is called a *topology* optimization problem. If, on the other hand, $a_j^1 > 0$ for all j then the optimal design has the same topology as the ground structure and (P_v^c) is called a *sizing* problem.

The following assumptions are often explicitly or implicitly stated for the minimum compliance problem (P_v^c).

(A3) The resource bound V satisfies

$$0 < V < \sum_{j=1}^{n} a_j^{m_j} l_j.$$

(A4) The external load $\mathbf{f} \neq 0$ and $\mathbf{f}$ is independent of the design variables.

(A5) The entries in the set of possible values on the design variables satisfy

$$0 \le a_j^1 < a_j^2 < \cdots < a_j^{m_j} < +\infty.$$

Similar assumptions have been proposed also for minimum compliance problems with continuous variables in Achtziger et al. (1992). Assumption (A3) is included to avoid trivial infeasibility (the first inequality) and to avoid trivial optimal solutions (the second inequality). Assumption (A4) is stated to avoid the uninteresting situation that any design satisfying the two last constraints in (P_v^c) together with $\mathbf{u} = \mathbf{0}$ produces a feasible and optimal solution. The first assumption (A1) guarantees, among other things, certain convexity properties of reformulations and relaxations of (P_v^c). One could also add an assumption stating that the feasible set of (P_v^c) should be non-empty. This assumption should not be necessary since a good optimization algorithm should be able to detect infeasibility.

The natural continuous relaxation of the minimum compliance problem (P_v^c) is given if the constraints $a_j \in \{a_j^1, \ldots, a_j^{m_j}\}$ are replaced by $a_j \in [a_j^1, a_j^{m_j}]$ for all j. This continuous approximation of (P_v^c) is

[4] The notation $\mathbf{A} \succeq \mathbf{B}$ means that the matrix $\mathbf{A} - \mathbf{B}$ is positive semidefinite.

[5] In this article only single load problems are stated to simplify the notation. All problem formulations, and reformulations, and most methods and heuristics can be generalized to multiple load cases.

[6] The notation mixed integer is herein used for optimization problems with both continuous and discrete/integer/0-1 variables. The term mixed discrete is often used in the structural optimization community.

$$\begin{aligned}
&\underset{\mathbf{a}\in\mathbb{R}^n,\mathbf{u}\in\mathbb{R}^d}{\text{minimize}} && \mathbf{f}^T\mathbf{u} && (\mathrm{R}_v^c)\\
&\text{subject to} && \mathbf{K}(\mathbf{a})\mathbf{u} = \mathbf{f},\\
& && \sum_{j=1}^n a_j l_j \le V,\\
& && a_j^1 \le a_j \le a_j^{m_j},\ j = 1,\dots,n.
\end{aligned}$$

Problem (R_v^c) is, by its own merit, an important structural optimization problem. It can be reformulated as a convex problem in a number of ways. Problem (R_v^c) has therefore been extensively studied by researchers from the mathematical programming community, see e.g. Achtziger et al. (1992), Achtziger (1998), Ben-Tal and Nemirovski (1994) and Ben-Tal and Nemirovski (1995), and Achtziger and Stolpe (2008).

2.4 Reformulations in the design variables only

Some of the convex reformulations of the continuous problem (R_v^c) are also valid for the problem with discrete design variables. These reformulated problems can have properties which are beneficial for the development of optimization methods and heuristics. The mixed integer nonlinear problem (P_v^c) can be reformulated as a mixed integer Semidefinite Program (SDP)[7]. The following well-known theorem from e.g. Ben-Tal and Nemirovski (1997) or Achtziger and Kočvara (2007) is the basis for this reformulation.

Proposition 1 *Let* $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a} \ge 0$, *and* $\tau \in \mathbb{R}$ *be fixed. There exists* $\mathbf{u} \in \mathbb{R}^d$ *satisfying*

$$\mathbf{K}(\mathbf{a})\mathbf{u} = \mathbf{f} \quad \text{and} \quad \mathbf{f}^T\mathbf{u} \le \tau,$$

if and only if

$$\begin{pmatrix} \tau & -\mathbf{f}^T \\ -\mathbf{f} & \mathbf{K}(\mathbf{a}) \end{pmatrix} \succeq 0.$$

The minimum compliance problem (P_v^c) can, by Proposition 1, be reformulated as the mixed integer linear SDP

$$\begin{aligned}
&\underset{\mathbf{a}\in\mathbb{R}^n,\tau\in\mathbb{R}}{\text{minimize}} && \tau\\
&\text{subject to} && \begin{pmatrix} \tau & -\mathbf{f}^T \\ -\mathbf{f} & \mathbf{K}(\mathbf{a}) \end{pmatrix} \succeq 0,\\
& && \sum_{j=1}^n a_j l_j \le V, && (\mathrm{P}_{\mathrm{SDP}}^c)\\
& && a_j \in \{a_j^1,\dots,a_j^{m_j}\},\ j = 1,\dots,n.
\end{aligned}$$

The continuous relaxation of ($\mathrm{P}_{\mathrm{SDP}}^c$) is a linear SDP and hence a convex problem. It can be solved to global optimality using standard methods for SDP, see e.g. Vandenberghe and Boyd (1996). One possibility to solve (P_v^c) is to apply a branch-and-bound method based on solving a sequence of continuous approximations of ($\mathrm{P}_{\mathrm{SDP}}^c$). This approach is suggested and numerically tested in e.g. Kočvara (2010).

Using the Projection Theorem, see e.g. Geoffrion (1970), the minimum compliance problem (P_v^c) is equivalent to the projected problem

$$\begin{aligned}
&\underset{\mathbf{a}\in\mathbb{R}^n}{\text{minimize}} && c(\mathbf{a})\\
&\text{subject to} && \sum_{j=1}^n a_j l_j \le V, && (1)\\
& && a_j \in \left\{a_j^1,\dots,a_j^{m_j}\right\},\ j = 1,\dots,n,
\end{aligned}$$

where $c(\mathbf{a})$ is the optimal value to the analysis problem

$$\begin{aligned}
c(\mathbf{a}) = &\inf_{\mathbf{u}\in\mathbb{R}^d} && \mathbf{f}^T\mathbf{u} && (2)\\
&\text{subject to} && \mathbf{K}(\mathbf{a})\mathbf{u} = \mathbf{f}.
\end{aligned}$$

The analysis problem (2) is a linear program with some special duality properties.

For sizing problems the stiffness matrix is positive definite, according to assumption (A2), for all design variables satisfying the bound constraints $a_j \in \left[a_j^1, a_j^{m_j}\right]$. For this situation the minimum compliance problem (P_v^c) can be reformulated as a problem in standard form in the design variables only

$$\begin{aligned}
&\underset{\mathbf{a}\in\mathbb{R}^n}{\text{minimize}} && \mathbf{f}^T\mathbf{K}(\mathbf{a})^{-1}\mathbf{f} && (\mathrm{P}_N^c)\\
&\text{subject to} && \sum_{j=1}^n a_j l_j \le V,\\
& && a_j \in \{a_j^1,\dots,a_j^{m_j}\},\ j = 1,\dots,n.
\end{aligned}$$

The continuous approximation of (P_N^c) is given by the linearly constrained nonlinear problem

$$\begin{aligned}
&\underset{\mathbf{a}\in\mathbb{R}^n}{\text{minimize}} && \mathbf{f}^T\mathbf{K}(\mathbf{a})^{-1}\mathbf{f} && (\mathrm{R}_N^c)\\
&\text{subject to} && \sum_{j=1}^n a_j l_j \le V,\\
& && a_j^1 \le a_j \le a_j^{m_j},\ j = 1,\dots,n.
\end{aligned}$$

The function $\mathbf{f}^T\mathbf{K}^{-1}(\mathbf{a})\mathbf{f}$ is convex under assumption (A1) see e.g. Svanberg (1994). It is therefore possible to design a convergent branch-and-bound method for (P_N^c) based on solving continuous approximations in the form (R_N^c).

2.5 Reformulation as nonlinear 0-1 problems

The problems (P_v^c) and (P_N^c) can be reformulated as 0-1 nonlinear problems by introducing the binary variables $x_{ij} \in \{0, 1\}$ with the interpretation

$$x_{ij} = \begin{cases} 1 & \text{if area } i \text{ is chosen in bar } j \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

[7] Semidefinite Programming deals with optimization problems with matrix variables that must be symmetric and positive semidefinite or problems with linear matrix inequalities.

The bar areas are then given by

$$a_j(\mathbf{x}) = \sum_{i=1}^{m_j} x_{ij} a_j^i \text{ for all } j.$$

Similar changes of variables were introduced already in Toakley (1968). Problem (P_v^c) can, with this choice of parametrization, be written as

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^m,\mathbf{u}\in\mathbb{R}^d}{\text{minimize}} \quad & \mathbf{f}^T\mathbf{u} \\ \text{subject to} \quad & \mathbf{K}(\mathbf{a}(\mathbf{x}))\mathbf{u} = \mathbf{f}, \\ & \sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij} a_j^i l_j \le V, \\ & \sum_{i=1}^{m_j} x_{ij} = 1, \qquad \forall\, j, \\ & x_{ij} \in \{0, 1\}, \qquad \forall\, (i, j). \end{aligned} \tag{3}$$

Note that problem (3) is a mixed 0-1 nonlinear problem in standard form with bilinear equality constraints. Similarly, problem (P_N^c) can be written as

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^m}{\text{minimize}} \quad & \mathbf{f}^T\mathbf{K}(\mathbf{a}(\mathbf{x}))^{-1}\mathbf{f} \\ \text{subject to} \quad & \sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij} a_j^i l_j \le V, \\ & \sum_{i=1}^{m_j} x_{ij} = 1, \qquad \forall\, j, \\ & x_{ij} \in \{0, 1\}, \qquad \forall\, (i, j). \end{aligned} \tag{4}$$

Note that problem (4) is a 0-1 nonlinear problem in standard form with a convex objective function.

2.6 Minimum weight problems

The second class of problems, often encountered in the literature, are minimum weight problems with a constraint on the compliance. The problem can be written as

$$\begin{aligned} \underset{\mathbf{a}\in\mathbb{R}^n,\mathbf{u}\in\mathbb{R}^d}{\text{minimize}} \quad & \sum_{j=1}^{n} a_j l_j \rho_j \\ \text{subject to} \quad & \mathbf{K}(\mathbf{a})\mathbf{u} = \mathbf{f}, \\ & \mathbf{f}^T\mathbf{u} \le c^{\max}, \\ & a_j \in \{a_j^1, \ldots, a_j^{m_j}\},\ j = 1, \ldots, n, \end{aligned} \tag{P_c^w}$$

where $\rho_j > 0$ is the density of the material in the jth bar, and $c^{\max} > 0$ is a given limit on the compliance. Just like the minimum compliance problem (P_v^c), this problem can be reformulated as a mixed integer convex problem in a number of ways. Heuristics and methods for the minimum weight problem (P_c^w), or one of the many equivalent reformulations, are proposed in e.g. Yates et al. (1983) and Groenwold et al. (1999).

The educational article Stolpe (2010) lists some fundamental mathematical properties of structural topology optimization problems such as the minimum compliance problem (P_v^c) and the minimum weight problem (P_c^w) and their respective continuous relaxations. The article shows, by truss topology optimization examples, which can be solved by hand calculations, that the optimal solutions to these problems in general are not unique and that the discrete problems, in contrast to the continuous problems, possibly have inactive volume or compliance constraints. Furthermore, Stolpe (2010) illustrates that optimal designs to the considered problems in general are not symmetric even if the design domain, the external loads, and the boundary conditions are symmetric around an axis.

The third, and by far most common, class of problems are minimum weight optimization problems with constraints on the nodal displacements and on the local bar stresses. The problem is

$$\begin{aligned} \underset{\mathbf{a}\in\mathbb{R}^n,\mathbf{u}\in\mathbb{R}^d}{\text{minimize}} \quad & \sum_{j=1}^{n} a_j l_j \rho_j \\ \text{subject to} \quad & \mathbf{K}(\mathbf{a})\mathbf{u} = \mathbf{f}, \\ & \mathbf{C}\mathbf{u} \le \mathbf{c}, \\ & \sigma_j^{\min} \le \sigma_j(\mathbf{u}) \le \sigma_j^{\max},\ \text{if } a_j > 0, \\ & a_j \in \{a_j^1, \ldots, a_j^{m_j}\}, \quad j = 1, \ldots, n, \end{aligned} \tag{P_s^w}$$

where $\sigma_j(\mathbf{u})$ is the stress in the j-th bar, $\mathbf{C} \in \mathbb{R}^{m\times d}$ is a given matrix while $\mathbf{c} \in \mathbb{R}^m$ is a given vector, both of appropriate size. The displacement constraints are general enough to model nodal displacement bounds such as $u_i^{\min} \le u_i \le u_i^{\max}$. The given lower and upper bounds on the local stress in the j-th element are denoted by $\sigma_j^{\min}$ and $\sigma_j^{\max}$, respectively.

Besides assumptions (A1), (A4), and (A5) which are also relevant to problem (P_s^w) the following assumptions are normally imposed.

(A6) The lower and upper stress bounds satisfy $-\infty < \sigma_j^{\min} < \sigma_j^{\max} < +\infty$ for all j.

(A7) If lower and upper nodal displacement limits are present they satisfy $u_i^{\min} < u_i^{\max}$ for all i.

Problem (P_s^w) is not in standard from due to the "if" statement following the stress constraints. The stress constraints should be removed if the corresponding bar vanishes. The stress constraints are thus modelled as *vanishing constraints*. These kinds of constraints are extensively studied in Achtziger and Kanzow (2008) for problems with continuous design variables. These constraints are also referred to as *design dependent constraints* in e.g. Rozvany (2001). Note that if the stress constraints are not modelled as vanishing constraints the feasible set can become too restrictive, i.e. stress constraints for bars that vanish become design driving. This was illustrated already in the late 1960s in Sved and Ginos (1968) for problems with continuous design variables. Modelling the design-dependent stress constraints as standard constraints is not difficult in the case of discrete design variables. However, for methods which rely on solving sequence of continuous relaxations such as branch-and-bound methods difficult issues arise. The continuous relaxation of (P_s^w) is non-convex due to the equilibrium and stress constraints. Furthermore, it does not satisfy standard constraint qualifications (Achtziger and Kanzow 2008). It can therefore not immediately be used as a basis for a convergent branch-and-bound method for solving (P_s^w).

3 Variations of the main theme

The problems presented in Section 2 cover the major part of the problems presented in the literature. The alternative problems which are suggested in the literature are covered in this section. Variations are often achieved by adding some engineering relevant constraint functions or by changing the mechanical analysis models, e.g. by including geometric non linearities. The choice of design parametrizations is also an important factor which can greatly change the properties of the problems. Several alternatives, normally generalizations, to the design parametrizations from Section 2 are also presented.

3.1 Alternative design parametrizations

Several alternative design parametrizations have been suggested in the literature. In e.g. Bennage and Dhingra (1995b) and Chai et al. (1999) additional binary variables $\mathbf{t} \in \mathbb{B}^n$ that represent the topology of the structure are introduced, i.e. $t_j \in \{0, 1\}$ represents presence or absence of the j-th bar in the final design. These topology optimization variables are accompanied by continuous area sizing variables. This parametrization has, of course, redundancies in the sense that continuous design variables with a lower bound of zero are sufficient to model both topology and sizing. The design parametrization has certain advantages since design dependence in the stress constraints can easily be correctly modelled. If the binary topology variables are fixed, for example as part of a two-stage algorithm, the resulting problem is a standard nonlinear optimization problem rather than a problem with vanishing constraints. The problem is however in general still non-convex.

Several articles also generalize sizing/topology problem by introducing additional continuous variables representing the node positions in the ground structure. This results in a *combined topology/sizing and geometry* problem. This approach is followed in e.g. Lipson and Agrawal (1974), Lipson and Gwin (1977), Salajegheh and Vanderplaats (1993), Galante (1996), Deb and Gulati (2001), Kawamura et al. (2002), Kaveh and Kalatjari (2004), Stolpe and Kawamoto (2005), Rahami et al. (2008), Shojaee et al. (2013), Shallan et al. (2014). Dimou and Koumousis (2009) also allow certain changes to the truss geometry by introducing certain overall height and length variables. This parametrization is not as general as the ones mentioned above since the positions of the nodes are coupled.

Templeman and Yates (1983) propose a novel design parametrization which does not involve discrete area variables. Instead each bar in the ground structure is divided into segments which are assigned the different available areas. The lengths of each of the segments is then used as (essentially discrete) design variables. For statically determinate structure the minimum weight problem with displacement and stress constraints can be well-approximated by removing the discrete requirements. The resulting problem is a linear program providing tight lower bounds on the optimal objective function value.

Several articles also propose to introduce 0-1 variables indicating presence or absence of nodes in the ground structure. In Cerveira et al. (2009) these variables are used to enforce that at least two bars meet at nodes which are present in the structure. In Mela (2014) truss topology design problems with the objective to minimize weight with stress and local buckling constraints are modelled as mixed 0-1 linear problems. The node variables are used to ensure kinematic stability of the structure and to correctly detect chains of bars in compression to determine the correct buckling length.

In a series of articles Ehara and Kanno (2010) and Kanno (2012), Kanno (2013a), and Kanno (2013b) truss topology optimization with discrete design variables is used to design tensegrity structures, i.e. structures consisting of continuously connected cables and disjoint struts. Two 0-1 design variables are assigned to each potential member such that the members can be classified into struts, cables, and vanishing members.

3.2 Alternative mechanical models

Most of the optimal design articles in the reference list assume linear elasticity and static and design independent loads. Only two articles in the reference list include design dependent loads, such as self-weight, which should be straight forward for most heuristics and methods. Kanno (2012) proposes to use truss topology optimization for design of tensegrity structures under self-weight. Kanno and Guo (2010) use design dependent loads to model uncertainty in the external loads. Less surprising is however the complete lack of time-dependent loads and transient analysis in this field.

Pyrz (1990) introduces geometric nonlinear behaviour. The objective is to minimize the structural weight with stress, local stability, and global stability constraints. An enumeration method is used to solve the problems. The model and method are applied to sizing optimization of shallow space trusses. Csébfalvi (1999) proposes a branch-and-bound algorithm for discrete minimum weight design of geometrically nonlinear truss structure subject to constraints on local stresses and nodal displacements.

In Stolpe and Kawamoto (2005) the ground structure approach is used for design of planar articulated mechanisms. The parametrization allows for simultaneous topology and geometry optimization and the model allows for large displacements and guarantees elastic stability.

3.3 Alternative objective and constraint functions

In a large majority of the listed references the structural weight is minimized with constraints on the bar stresses and the nodal displacements. Some references also include other types of objectives and constraints which are important in engineering applications.

Stress and frequency constraints are combined in John and Ramakrishnan (1990) together with a heuristic technique involving both continuous and discrete optimization. Local and global buckling constraints are introduced in Pyrz (1990) and Bennage and Dhingra (1995a). In Mela (2014) truss topology design problems with the objective to minimize weight with stress and local buckling constraints are modelled a solved.

Several articles explicitly or implicitly include constraints on kinematic stability of the truss. Examples include Hajela and Lee (1995), Deb and Gulati (2001), Faustino et al. (2006), and Mela (2014).

Hajela and Lin (1992) propose multicriteria objective function minimizing the conflicting goals of minimum structural mass and minimum displacement in the load direction. The objective in the computations is a weighted combination of the two objectives. Ohsaki (1995) proposes an objective function which combines the member and node costs. Galante (1996) suggest that the objective should be a combination of the structural weight and the number of different cross-section areas. This is one way to (implicitly) control the overall cost of the structure. In Kanno (2015) the cost of the structure is also (implicitly) limited by imposing constraints on the number of different cross-sections in the structure.

Tong and Liu (2001) study problems with constraints on stresses, natural frequencies, and frequency responses.

In Dimou and Koumousis (2009) reliability-based optimization of trusses with discrete design variables is proposed.

This concludes the presentations of problem formulations proposed in the literature. The next three sections present the developments of heuristics and methods in the field. The next section covers the meta heuristics and the following two sections present the deterministic heuristics and methods, respectively.

4 Meta heuristics

Since the early 1990s a large number of research articles proposing meta heuristics for truss design with discrete design variables have appeared. The developed and implemented heuristics are, among others, based on evolutionary algorithms such as genetic algorithms (Holland 1992), swarm algorithms such as particle swarm optimization (Kennedy and Eberhart 1995; 2001), ant colony optimization (Dorigo and Gambardella 1997), and artificial bee colony Optimization, see e.g. Karaboga and Basturk (2008). Tabu search, originally proposed in Glover (1989) and Glover (1990), and simulated annealing, see e.g. Kirkpatrick et al. (1983), have also been modified to solve optimal truss design problems with discrete design variables.

A review of optimization methods for nonlinear problems with discrete variables from applications in structural optimization is presented in Arora et al. (1994). The review contains descriptions of several of the meta heuristics presented in this section.

Detailed descriptions of implementations of several meta heuristics and numerical experiments for optimal design of truss structures with discrete design variables are presented in Hasançebi et al. (2009). The methods included in the benchmarking study are simulated annealing, evolution strategies, genetic algorithms, tabu search, ant colony optimization, and harmony search.

4.1 Genetic Algorithms (GA)

The, by far, most popular class of meta heuristic for optimal design of truss structures with discrete variables is genetic algorithms (Holland 1992).

4.1.1 The 1990s

The earliest developments of genetic algorithms for truss design with discrete variables are described in Rajeev and Krishnamoorthy (1992) and Hajela and Lin (1992). The algorithm in Rajeev and Krishnamoorthy (1992) is intended for minimum weight problems with displacement and local stress constraints, i.e. problem (P_s^w). The constraints are grouped into a penalty function and moved to the objective function.[8] A binary string of length four is used to represent the area in the individual bars resulting in the

[8]This is, in fact, the predominant way of dealing with constraints in meta heuristics for optimal truss design.

possibility to chose among 16 different area values. The proposed method is applied to the 10-bar truss, the 25-bar truss, and a 160-bar transmission tower. The algorithm in Hajela and Lin (1992) is developed for multicriteria optimization with combinations of continuous and discrete design variables. The objective in the considered truss design examples is a weighted combination of the two objectives to minimize structural mass and minimize a nodal displacement. The algorithm is applied to the statically loaded 10-bar truss ground structure with nodal displacement constraints. Three variants of the problem are given by different combinations of continuous and discrete variables and variable linking. The algorithm is also used on several problem instances in which the ground structure is subjected to a harmonically varying load and the steady state displacements are constrained. All problems in both Rajeev and Krishnamoorthy (1992) and Hajela and Lin (1992) are sizing problems.

A general purpose implementation of a genetic algorithm with certain advanced enhancements (directed crossover and multi-stage search) is described in Lin and Hajela (1994). Constraints are handled through a penalty function which is added to the objective function. The 25-bar truss sizing problems is reported as one of the examples. Rather than grouping the bars into eight members, which is the common choice, each bar area is assumed to be a design variable.

The first development of genetic algorithms for truss *topology* optimization is found in Hajela and Lee (1995). The method is a two-stage algorithm. The first stage generates a number of kinematically stable topologies while ignoring the structural response constraints. The representation of the truss in the first stage involves a bit-string of length one per potential bar indicating presence or absence of the bar in the topology. In the second stage the objective is to minimize weight with constraints on nodal displacements, local stresses, and buckling. The structure is represented by a bit-string of sufficient length to model all available areas. The algorithm is applied to a 14-bar truss and a bridge like structure with 61 bars in the ground structure.

Rajan (1995) develops a genetic algorithm for combined geometry, sizing, and topology optimization of trusses. This is the first time these three disciplines are coupled using a genetic algorithm. Minimum weight problems with stress, displacement, and Euler buckling constraints are modelled with three types of variables. The design dependent stress constraints are, to the best of the author's understanding, not modelled correctly. The algorithm is applied to the 10-bar truss problem, a 14-node planar truss with 31 bars, and a modified version of the 14-node problem with 42 bars.

Wu and Chow (1995) present a steady-state genetic algorithm for minimum weight problems with discrete area variables. The problems include bounds on stresses, nodal displacements, and bounds on Euler buckling stresses. The GA is applied to sizing problems of the 10-bar truss, the 25-bar space truss, the 52-bar planar truss, and the 72-bar space truss. The steady-state genetic algorithm from Wu and Chow (1995) and the problem instances and results from the article are frequently used for comparisons with other meta heuristics.

Galante (1996) suggests an improved genetic algorithm for combined sizing and geometry optimization of trusses. Constraints on member stresses, nodal displacements, and local buckling are included in the problem. The objective function is either the structural weight or a combination of weight and the number of different cross-section areas. The main improvement in the algorithm is the introduction of what the author calls re-birth. This means that a new population is formed around the incumbent once the algorithm stalls. The algorithm then continues with the new population and searches a smaller set around the incumbent. The 10-bar truss is extensively studied for different combinations of objective and constraints functions as well as different algorithmic parameters. The results indicate that the re-birth process improves the results. The algorithm is also applied to a 160-bar transmission tower.

Ghasemi et al. (1999) suggest one standard and one modified genetic algorithm for minimum weight problems with stress and displacement constraints, for both continuous and discrete design variables. The modified version is obtained by a high mutation cross-over parameter and different scheme for selecting mates. The discrete sizing problem instances include the 10-bar truss and the results are compared to the results in e.g. Galante (1996). The planar 200-bar truss, for which the variables are linked into 96 groups, is also considered by the GAs.

A regional genetic algorithm, is proposed in Groenwold et al. (1999). The algorithm first solves the continuous relaxation of the problem at hand and then applies a genetic algorithm searching a neighbourhood of the continuous optimum. The authors either minimize weight under (essentially) compliance constraints or minimize weight with displacement and stress constraints. The authors explicitly assume that the continuous problem has a solution and can be solved and that the discrete optimum is close to the continuous optimum. Additionally, variables in the optimum of the continuous problem which are in the list of available areas are fixed in the genetic algorithm. These three assumptions may, or may not be valid, depending on the problem type and the problem data. The algorithm is applied to various versions of the 10-bar, the 25-bar, the 36-bar, the 160-bar, and the 200-bar trusses. All benchmark instances are sizing problems and the issues with design-dependent stress constraints are thus avoided.

4.1.2 After 2000

An implementation of a genetic algorithm especially intended for handling optimal design of structures made of one-dimensional elements is described in Erbatur et al. (2000). The method contains multilevel capability in which the feasible set is divided into smaller subsets which are searched. The method is applied to minimum weight sizing problems with stress and displacement limits, notably the 25-bar and the 72-bar trusses and a 112-bar dome.

Deb and Gulati (2001) propose a genetic algorithm for minimum weight problem with stress, displacement, and kinematic stability constraints. The algorithm allows for combined topology and geometry optimization with both continuous and discrete area design variables. It is not possible to conclude from the problem formulation in Deb and Gulati (2001) if the stress constraints are design-dependent or not. Instead of using a two stage approach the fixed-length variable representation scheme different variables are determined simultaneously. The algorithm is mostly described and assessed with continuous design variables. One example, with 11 bars in the ground structure, has discrete area variables in Deb and Gulati (2001).

A genetic algorithm with enhancements called "annealing perturbation" and "adaptive reduction of the design space" are presented in Hasançebi and Erbatur (2001) for combined geometry, sizing, and topology optimization for minimum weight problems with stress, displacement, and Euler buckling constraints. The algorithm is applied to the 47-bar truss tower and the results are compared to those in Salajegheh and Vanderplaats (1993). A 224-bar space pyramid is also considered. Due to symmetry the sizing variables are linked into 32 groups and the size of the problem is thus reduced significantly.

A genetic algorithm for minimum weight truss topology optimization with displacement and stress limits is presented in Kawamura et al. (2002). The algorithm codes the truss topology using connected triangular substructures while avoiding useless members, bar overlaps, and unstable structures. The proposed genetic algorithm is also capable of handling geometry optimization as well as (continuous) sizing optimization. The capabilities of the algorithm are illustrated by applying it to topology and size optimization of a 15-bar planar truss, combined topology and geometry of a planar 21-bar truss, and a spatial 33-bar truss.

Jenkins (2002) proposes a genetic algorithm based on integer decimal coding, instead of the standard binary coding. The algorithm is applied to minimum weight problems with stress and displacement constraints. The targeted sizing problem instances are based on the 10-bar truss, and a 640 member space truss deck. The size of the latter problem is reduced to five design variable groups.

Kaveh and Kalatjari (2002) consider the discrete-sizing problem of minimizing the weight with stress and displacement constraints. The force method is used for the structural analysis. The algorithm is modified to include a contraction method reducing the size of the feasible set. The algorithm is applied to the 10-bar and the 25-bar benchmark problems and the results are compared to those in Rajeev and Krishnamoorthy (1992) and Erbatur et al. (2000).

Lemonge and Barbosa (2004) propose a genetic algorithm which is based on binary coding and standard mutation and cross-over operators. The need for parameter tuning is avoided through the use of adaptive penalty parameters that are determined automatically, i.e. without any user input. The algorithm is applied to the minimum weight problem for the 10-bar, the 25-bar, and the 52-bar trusses. The obtained designs are compared to the results in the literature, e.g. the genetic algorithms in Wu and Chow (1995) and Galante (1996) and the improved Templeman's algorithm in Ming-Zhu (1986).

In previous work, for example Lin and Hajela (1994) and Hajela and Lee (1995) the length of the representation of the structure is fixed and generally long for ground structures with many potential bars and problems with many available areas. In Ryoo and Hajela (2004) variable bit-string lengths are used to represent the structures in truss topology optimization by genetic algorithms, i.e. different topologies require different length bit-strings. This change to the Genetic Algorithm requires modifications of the crossover process. The new algorithm is applied to the 10-bar truss problem.

Kaveh and Kalatjari (2004) propose a genetic algorithm for combined sizing and geometry optimization based on a dynamic penalty function. A two-phase approach is described. The first phase has fixed geometry and cross-sections are found using GA. Then upper and lower bounds on the area variables are introduced squeezing the design space before the second phase (using a modified version of the approach in Kaveh and Kalatjari (2002)). In the second phase the geometry and the (reduced) sizing optimization is done simultaneously. The considered problems are minimum weight with stress, Euler buckling, and nodal displacement constraints. The algorithm is applied to an 18-bar planar cantilever truss and the 25-bar spatial truss and the results are compared to those in Wu and Chow (1995) and Hasançebi and Erbatur (2001).

Tang et al. (2005) propose improvements of genetic algorithms for minimum weight problems with nodal displacement and local stress constraints. The article correctly models the design dependence in the stress constraints. Three types of variables are included in the problem formulation. The topology is described by 0-1 variables and the member areas are chosen from a discrete set. The geometry, i.e. the nodal positions, is modelled using continuous variables. Tang et al. (2005) suggest new coding schemes to deal with the mixture of variables. A compliance constraint is included to avoid the undesired situation, which is a result of the modelling, that all bars are removed from the structure while satisfying all stress constraints. The algorithm is applied to a 15-bar planar truss, the 25-bar spatial truss, and the 10-bar truss. The results are compared to those presented in Wu and Chow (1995) and Rajan (1995).

A genetic algorithm for combined sizing and geometry optimization is proposed in Rahami et al. (2008). The analysis equations are included in the problem formulation by combining energy and force methods. The numerical examples include minimum weight problems with various combinations of stress, Euler buckling, and nodal displacement constraints. The considered ground structures consist of the 10-bar truss, a 15-bar truss, an 18-bar truss, and the 25-bar truss.

Toğan and Daloğlu (2006) discusses the adaptive penalty and mutation and crossover operations for genetic algorithms. The article additionally proposes adaptive member grouping, i.e. the variable linking is not applied before the problem is solved but based on the analysis of an initial design. Toğan and Daloğlu (2006) consider sizing optimization for minimum volume with stress constraints. For the 112-bar steel dome the variables are linked into two or three groups and the results are compared to those presented in Saka (1990). Two additional ground structures are presented, a 200-bar space truss and a 244-bar transmission tower. The latter problem also includes displacement constraints. For both problems the bars are grouped to a handful of design variables. The work on self-adaptive member grouping strategies is continued in Toğan and Daloğlu (2008).

An untraditional approach is chosen in Shallan et al. (2014) for minimum mass problems. Instead of using the area as design variables the authors suggest using the nodal coordinates and nodal displacements as variables. The corresponding designs, i.e. the bar areas, are then computed from the strength criteria. The approach thus allows both for geometry and sizing. The main advantage brought forward in Shallan et al. (2014) is the reduction in the number of variables. The algorithm is applied to the sizing problems of the 10-bar and the 25-bar trusses and numerically compared to a wide range of meta-heuristics such as genetic algorithms in Rajeev and Krishnamoorthy (1992), Erbatur et al. (2000), Toğan and Daloğlu (2008), big bang–big crunch from Camp (2007), and harmony search from Lee et al. (2005), among others.

4.2 Particle Swarm Optimization (PSO) methods

Recently several particle swarm optimization methods have been proposed for optimal truss design.

Three versions of particle swarm optimization methods for discrete variables are proposed in Li et al. (2009) for minimum weight truss design problems with displacement and stress constraints. The three PSO methods are applied to two variants of the 10-bar truss, a multiple load planar 15-bar truss, the spatial 25-bar truss, and two variants of the 72-bar truss. All problem instances in Li et al. (2009) are sizing problems. The PSO algorithms are compared to the steady-state genetic algorithm from Wu and Chow (1995) and the harmony search algorithm from Lee et al. (2005).

Kaveh and Talatahari (2009a) propose a combination of PSO and Ant Colony Optimization (ACO) together with elements from harmony search for minimum weight problems with stress and displacement constraints. The algorithm is assessed by considering sizing of the 25-bar, the planar 52-bar, and the spatial 72-bar ground structures. The results are compared to the designs obtained by genetic algorithms in Wu and Chow (1995), harmony search from Lee et al. (2005), and the PSO algorithms from Li et al. (2009). The final example is a spatial tower with 582 bars in the ground structure. The members are linked into 16 groups.

In Dimou and Koumousis (2009) reliability-based optimal design of statically determinate trusses is proposed. The design variables, besides discrete area variables, also include some possibilities to change the geometry of the truss. The objective function is a combination of the structural cost and the cost of potential structural failure whereas the constraints limit the probability of structural failure and probability of failure of each of the individual members. The random variables of the problem are the load magnitude, the stress limits, the buckling factor and the bar area. A binary PSO algorithm is applied to a planar 25-bar truss and a planar 30-bar arch and the effect of varying certain parameters in the algorithm are presented.

Shojaee et al. (2013) propose a hybrid method for minimum weight sizing and geometry optimization with stress and displacement constraints. The algorithm combines the Method of Moving Asymptotes (MMA), Svanberg (1987), with a discrete PSO algorithm. In the two-stage procedure the area variables are kept fixed for all particles and the resulting problems in only geometry variables are optimized using MMA. The area variables are the updated using PSO while keeping the geometry variables fixed. The hybrid algorithm is applied to several planar problems such as the 15-bar truss, the 18-bar truss, and the Michell arch. It is also used for solving the 25-bar spatial truss and the 39-bar spatial truss. The algorithm is compared to e.g. the genetic algorithms in e.g. Wu and Chow (1995), Hasançebi and Erbatur (2001), Wang et al. (2002), Tang et al. (2005), Rahami et al. (2008), and Kaveh and Kalatjari (2004).

4.3 Ant Colony Optimization (ACO)

The first attempt to solve optimal truss design problems by ant colony optimization is reported in Bland (2001). The minimum weight problem (P_s^w) is considered for the 25-bar ground structure. The ACO method in Bland (2001) is complemented with a tabu search for local search. The next use of ACO for truss design problems with discrete variables is presented in Camp and Bichon (2004) (surprisingly without referencing Bland (2001)). This article also considers the minimum weight problem (P_s^w). The numerical experiments additionally contain the 10-bar and the 72-bar truss ground structures.

Rajendran et al. (2006) suggests coupling ant colony optimization with tabu search for local improvement and elitist ant strategy. The algorithm is applied to minimum weight truss sizing problems with nodal displacement, local stress, and critical buckling load constraints. The algorithm is numerically compared to a genetic algorithm and found to produce better designs in general. Unfortunately, no statistics regarding computation time or number of function evaluations are reported.

Capriles et al. (2007) present a variant of ACO called rank-based ant system for the minimum weight problem (P_s^w). The set of benchmark problems is increased compared to the previous articles on ACO and includes spatial ground structures with up to 160 bars. Results compared to the Lagrangian-based search heuristic in Juang and Chang (2006), the ACO in Camp and Bichon (2004), and the genetic algorithms in Wu and Chow (1995) and Lemonge and Barbosa (2004).

Two mechanisms to improve ACO for truss design were proposed in Kaveh et al. (2008). The first improvement divides the search space into smaller parts whereas the other reduces the number of ants. The article does surprisingly not cite the articles Bland (2001), Capriles et al. (2007), or Rajendran et al. (2006).

It is noteworthy that none of the articles developing ACO for truss design problems consider the challenging topology optimization problem and instead focus on sizing.

4.4 Big bang–big crunch

Erol and Eksin (2006) propose a new optimization algorithm based on a theory of the evolution of the universe called big bang–big crunch. The first implementation and numerical testing of big bang–big crunch for optimal design of trusses with discrete design variables is presented in Camp (2007). The minimum weight problem (P_s^w) coupled to the 10-bar and the 25-bar ground structures are considered for benchmarking. The big bang–big crunch algorithm is compared to the genetic algorithm from Rajeev and Krishnamoorthy (1992) and the ACO from Camp and Bichon (2004).

Kaveh and Talatahari (2010b) propose a discrete version of the hybrid big bang–big crunch method suggested for sizing optimization of trusses in Kaveh and Talatahari (2009b). The hybrid method introduces several improvements to the big bang–big crunch algorithm by introducing recent ideas from particle swarm methods and division of the feasible set into sub-domains. The hybrid algorithm is applied to minimum weight problems with displacement and local stress constraints. The test examples include a 354-bar braced dome (with 22 design variables after element grouping) and a 582-bar tower (with 32 design variables). The hybrid algorithm is numerically compared to simulated annealing from Hasançebi et al. (2009), and particle swarm optimization from Kaveh and Talatahari (2009a) and Hasançebi et al. (2009).

4.5 Tabu search

The first attempt to use tabu search for optimal truss design with discrete design variables is reported in Bennage and Dhingra (1995a). Several choices of objective functions, such as minimization of weight, minimization of the norm of the nodal displacements, maximization of fundamental frequency, are provided in the described implementation. The constraints include limitations on stresses, nodal displacements, and local and global buckling. The problem formulations include topological 0-1 variables indicating presence or absence of the bars in the ground structure, and continuous area variables. The algorithm works by solving an outer topology optimization by tabu search and an inner problem for sizing. The inner problem is solved either by tabu search or a gradient based optimizer.

Tabu search is, as previously mentioned, also used for local improvement in the ant colony optimization developed in Rajendran et al. (2006).

4.6 Simulated annealing

Several variants of simulated annealing for truss design with stress and nodal displacement constraints and both discrete and continuous design variables are described in detail in Bennage and Dhingra (1995b). The method is, among others, applied to the 10-bar and the 25-bar minimum weight problem (P_s^w).

A parallel and combined simulated annealing and simulated quenching algorithm for optimization of steel structures is described in Park and Sung (2002). It is (among others) applied to the minimum weight 25-bar truss problem with stress and displacement constraints and compared to the genetic algorithm in Rajeev and Krishnamoorthy (1992), the neural network approach in Adeli and Park (1996), and the improved Templeman's algorithm from Ming-Zhu (1986).

Kripka (2004) develops a simulated annealing algorithm for minimum weight truss sizing design with constraints on stresses and nodal displacements. The algorithm is applied to the single load 10-bar and the single load 25-bar trusses. The article contains comparisons with designs found by other methods and heuristics, for example the genetic algorithms from Rajeev and Krishnamoorthy (1992).

4.7 Other meta heuristics

Besides the meta heuristics already mentioned several others have been suggested in the literature and new algorithms are published on a regular basis.

A parallel modified evolution strategy is presented in Cai and Thierauf (1996) for minimum weight problems with stress and nodal displacement constraints. The algorithm is applied to the 10-bar and the planar 200-bar ground structures.

A random search method is proposed in Ohsaki (2001) for minimum volume truss topology optimization with stress constraints. Artificial nodal displacement constraints are included to avoid trivial and undesired optimal designs. The design dependency in the stress constraints is dealt with in a heuristic way by enforcing a relatively small lower bound on the area variables and allow violation of the stress constraint for bars which reach the lower bound. The focus is however on the use of exact reanalysis to decrease the computational cost in the optimization process.

A specialization of harmony search (Geem et al. 2001) for truss optimization with discrete variables is described in Lee et al. (2005). The algorithm is intended for minimum weight problems with stress and local buckling constraints as well as nodal displacement constraints. The algorithm is applied to sizing problems of two versions of the 25-bar truss, the planar 52-bar truss, the spatial 72-bar truss, and the 47-bar planar truss. The results are compared to those from the meta heuristics in e.g. Rajeev and Krishnamoorthy (1992), Wu and Chow (1995), Erbatur et al. (2000), Adeli and Park (1996), Park and Sung (2002), and the deterministic heuristics in Ringertz (1988) and Schmit and Fleury (1980).

The Charged System Search (CSS), which is a particle method developed in Kaveh and Talatahari (2010c), is modified in Kaveh and Talatahari (2010a) for minimum weight optimization of trusses with stress and displacement limits. The movements of the particles are largely based on classical laws from physics and mechanics as each particle is assigned a radius and a charge. The algorithm is applied to sizing problems of the planar 52-bar, the spatial 72-bar truss, and a 354-bar truss dome. After variable linking the number of variables is actually 22 for the dome problem.

The mine blast algorithm suggested in Sadollah et al. (2012) models how the pieces of shrapnel from a mine explosion set off other mines in the vicinity of the explosion. This idea of searching the feasible set is compared to e.g. the genetic algorithm from Rajeev and Krishnamoorthy (1992), harmony search from Lee et al. (2005), and various versions of particle swarm optimization (Li et al. 2009) on minimum weight sizing problems of the planar 10-bar, 15-bar, and 52-bar trusses and the spatial 25-bar and 72-bar trusses.

A generalization of the Colliding Bodies Optimization (CBO) method for problems with discrete variables is described in Kaveh and Mandavi (2014). The algorithm is applied to truss sizing optimization with stress and nodal displacement constraints. The problem instances include the 52-bar planar truss, the 72-bar spatial truss, the 582-bar spatial tower, and the 47-bar power line tower. The number of design variables is kept relatively low using variable grouping but the number of available areas is relatively large, with up to 137 values. The method is numerically compared to a range of other meta heuristics including particle swarm optimization from Li et al. (2009) and Kaveh and Talatahari (2009a), harmony search from Lee et al. (2005), and the genetic algorithm from Wu and Chow (1995),

Teaching-learning based optimization, a population-based evolutionary algorithm, is developed in Camp and Farshchin (2014) for minimum weight design of trusses with stress and displacement constraints. The idea is applied to sizing optimization of the planar 10-bar and the spatial 25-bar and 72-bar trusses. The algorithm is numerically compared to e.g. genetic algorithm (Rajeev and Krishnamoorthy 1992), ant colony optimization (Camp and Bichon 2004), and big bang–big crunch (Camp 2007).

An Artificial Bee Colony (ABC) optimization method is proposed in Sonmez (2011) for truss design with discrete design variables and constraints on stresses, displacements, and bar slenderness. The algorithm is applied to the planar 10-bar, the spatial 25-bar and 72-bar trusses, and finally the 582-bar tower, which after variable grouping has 32 design variables.

4.8 Discussion about meta heuristics for optimal design of trusses

The main motivations brought forward in the literature for applying meta heuristics to optimization problems are that these methods can be applied to essentially all types of problems. This includes problems with noisy and non-differentiable objective and/or constraint functions, problems for which derivatives are not available or too expensive, and problems which require black-box function evaluations. For the optimal truss design problems encountered in the research literature none of these conditions are normally satisfied. The objective and constraint functions in this field are by no means noisy. In the minimum compliance problem (P_v^c) and the minimum weight problems (P_c^w) and (P_s^w) the functions are polynomials in the design and state variables. The structural analysis in truss design is, due to its simplicity in both modelling and solution, generally not done through call to black-box software. This implies that analytical gradients of the objective and constraint functions functions are readily available and easily implementable. In fact, for the problems stated using simultaneous analysis and design second order information is also available at essentially no additional computational cost.

Structural optimization problems are generally nonlinearly constrained and this is also the situation for most optimal truss design problems. For stress constrained problems there is at least one nonlinear constraint per design variable and load condition. Meta heuristics generally handle nonlinear constraints through aggregation in a penalty function which is added to the objective function. Aggregation of the constraints combined with derivative free methods result in a significant loss of information which could be utilized to promote convergence to feasible and optimal points. The numerical results in the literature indicate that meta heuristics applied to truss design problems require a large number of structural analysis. Normally meta heuristics require thousands, if not tens of thousands, of function evaluations per run. Since meta heuristics are stochastic multiple runs are normally required. This is perhaps acceptable for trusses with only a few degrees of freedom but it will be computationally too demanding for problems which require expensive and large-scale structural analyses. Similar critique of non gradient methods for topology optimization has also been presented in Sigmund (2011).

The artificial bee colony optimization method proposed in Sonmez (2011) for discrete design variables is challenged in Stolpe (2011) by a heuristics based on mathematical programming followed by rounding and greedy search or branch-and-bound. These heuristics are in fact similar to the early heuristics presented in e.g. Ringertz (1988). For the considered benchmark problems the heuristics suggest designs which are as good as the best designs proposed by artificial bee colony optimization using several orders of magnitude fewer structural analyses.

5 Global optimization methods

Global optimization is an established branch of mathematical programming. Theory, methods, and applications for integer and combinatorial optimization are extensively covered in Nemhauser and Wolsey (1999). Deterministic global optimization methods, such as branch-and-bound, for general problems are presented and analysed in e.g. Horst and Tuy (1993). A collection of applications and global optimization methods is presented in Horst and Pardalos (1995) and Pardalos and Romeiin (2002). A review of optimization methods for applications in structural optimization with discrete design variables is presented in Arora et al. (1994). A review of methods for general nonlinear mixed-integer programs, including branch-and-bound, outer approximation, and generalized Benders' decomposition is provided in Grossmann (2002).

This section presents a literature survey of the models and deterministic methods used to solve optimal truss design problems with discrete design variables to proven global optimality.[9] Truss topology and sizing optimization problems are, as already mentioned, intrinsically nonconvex. Many of the global optimization approaches suggested in this field therefore rely on equivalent problem re-formulations with certain desirable mathematical properties in combination with global optimization methods, often branch-and-bound algorithms.

5.1 Mixed 0-1 linear and mixed 0-1 quadratic programs

Several classes of relevant truss topology optimization problems can be formulated or re-formulated as mixed 0-1 linear programs (MILPs) or mixed 0-1 quadratic programs (MIQPs). The main advantage of these formulations is that the problems can be solved by the robust and efficient branch-and-cut solvers which today are available off-the-shelf.

Toakley (1968) considers optimal sizing design of statically determinate planar trusses for minimum weight with displacement and member stress constraints. The problem is formulated as a 0-1 linear problem and solved by Gomory's cutting plane method, see e.g. Nemhauser and Wolsey (1999). Toakley (1968) reports convergence difficulties with the method but the approach must be classified as a global optimization method none the less. These convergence issues are, by now, well-understood and Gomory cuts are today frequently used for enhancing the convergence in branch-and-cut methods.

Bauer et al. (1981) consider minimum weight optimization of space trusses with a single displacement constraint. The constraint is written explicitly using a Galerkin procedure and the final formulation is a 0-1 linear problem. A problem with many bars, but only two design variables, is solved by Balas Additive Algorithm (Balas 1965) which is a branch-and-bound type method.

[9]This section is an extension of parts of the Introduction in the doctoral thesis Stolpe (2013). Parts of this section are therefore identical to the corresponding paragraphs in Stolpe (2013).

Even when the problems cannot immediately be written as mixed 0-1 linear programs there are many situations in which the topology optimization problems can be reformulated as mixed integer nonlinear problems with convex objectives and constraint functions, see e.g. Stolpe (2007) for an overview of the possibilities. The reformulation uses so-called "big-M" techniques which are prone to give rise to weak continuous relaxations. When the variables are 0-1 then the different formulations are equivalent. However, certain parts of the problem, normally the compatibility equations, are not satisfied at the optimum of the natural continuous relaxation of the reformulated problem. This partly explains why only problems of modest size can be solved by standard branch-and-bound type algorithms applied to the reformulated problem. A number of articles which rely on these reformulation techniques are listed below.

Mixed 0-1 linear programming formulations of truss topology design problems with stress and displacement constraints are presented in e.g. Grossman et al. (1992) and Bollapragada et al. (2001). In Bollapragada et al. (2001) a special purpose branch-and-cut method based on solving a sequence of quasi-relaxations is also proposed and numerically investigated.

Reformulation techniques are also used in Faustino et al. (2006) to find kinematically stable truss structures for minimum volume problems. The problem formulation contain boundedness of the state variables as well as stress constraints. The problem instances in Faustino et al. (2006) are solved by a commercial branch-and-cut algorithm.

In Stolpe (2007) both mixed 0-1 linear or convex quadratic reformulations of maximum stiffness truss topology optimization problems are proposed and numerical experiments with a commercial branch-and-cut method are presented. Problems with up to 200 bars in the ground structure are solved to proven global optimality.

The reformulation techniques from the above mentioned articles are also used in Rasmussen and Stolpe (2008) for stress and displacement constrained truss topology optimization problems. The problem formulations are strengthened by adding several types of valid inequalities based on mechanical reasoning and projected Chvátal-Gomory cuts as proposed in Bonami et al. (2008). The strengthened problems are then solved by an open source parallel branch-and-bound method, rendering a cut-and-branch method. An L-shaped two-dimensional design domain with a ground structure consisting of 54 bars and a three-dimensional cantilever ground structure with 40 bars are solved to global optimality.

Mixed 0-1 linear programming formulations of robust truss topology optimization problems with stress constraints are presented in Kanno and Guo (2010) together with numerical experiments based on a commercial branch-and-cut solver. Optimal design of tensegrity structures by mixed integer linear programming is proposed in Ehara and Kanno (2010) and Kanno (2012), Kanno (2013a), and Kanno (2013b).

In Mela (2014) truss topology design problems with discrete design variables and the objective to minimize weight with stress and local buckling constraints are modelled as a mixed 0-1 linear problem. The problem instances in Mela (2014) are solved by a commercial branch-and-cut algorithm.

5.2 Mixed 0-1 Conic Programs

Several classes of structural topology optimization problems with continuous design variables can, as briefly described in Section 2.4, be reformulated as linear Semidefinite Programs (SDPs). This result applies to minimum compliance problem with a volume constraint, see Ben-Tal and Nemirovski (1994), Ben-Tal and Nemirovski (1995), and Ben-Tal and Nemirovski (2000), and Vandenberghe and Boyd (1996). Certain types of robust maximum stiffness optimization of truss structures can also be modelled as SDPs as described in Ben-Tal and Nemirovski (1997). The minimum volume problems with compliance constraints and a lower bound on the first fundamental frequency can also be cast as SDPs, see e.g. Ohsaki et al. (1999) and Achtziger and Kočvara (2007). These reformulations can also be used when the design variables are discrete. The main advantage of these reformulations is that no "big-M" techniques are used. The implication is that the equilibrium and compatibility equations are satisfied at the optimum of the continuous relaxation of the mixed 0-1 SDP. The major disadvantages are that solvers for mixed 0-1 semidefinite programs are not readily available and currently in early stages of development.

Maximum stiffness problems are formulated as mixed integer linear semidefinite programs using the reformulation techniques from e.g. Ben-Tal and Nemirovski (1995) and solved to global optimality by special purpose branch-and-bound methods in Cerveira et al. (2009).

Minimum weight problems with stiffness and vibration constraints are formulated as mixed integer linear semidefinite programs in Kočvara (2010). A set of single and multiple load problems are solved by an off-the-shelf branch-and-bound method in which the continuous relaxations are solved by SeDuMi (Sturm 1999).

Kanno (2015) proposes mixed integer second order cone formulations of minimum compliance truss topology design problems with constraints limiting the structural volume and the number of different cross-section in the structure. The formulations are based on the reformulation techniques for problems with continuous design variables presented in e.g. Ben-Tal and Nemirovski (2001). In Kanno (2015) a large set of problem instances based on planar ground structures are solved by a commercial branch-and-cut algorithm with functionality for mixed-integer second order cone programs. The numerical results suggest that the proposed formulations outperform the MILP and MIQP formulations for the considered benchmark problems.

5.3 Mixed 0-1 nonlinear programs

A third alternative is to model and solve structural sizing and topology optimization problem as mixed 0-1 nonlinear programs (MINLPs) such as the minimum compliance problem (P_v^c). The main advantage is that the equilibrium equations are satisfied at the optimum to the continuous relaxation. One disadvantage is that the natural continuous relaxation generally is non-convex and therefore possibly difficult to solve to global optimality. Another disadvantage is that special purpose global optimization methods must be developed and implemented. Here, a list of articles that follow this and similar approaches is presented.

Pyrz (1990) includes geometric nonlinear modelling in the structural analysis. The objective is to minimize the structural weight with stress, local stability, and global stability constraints. An enumeration method is used to solve the problems with the result that only small-scale problems can be solved. The largest problem has four design variables and seven available areas. The model and method are applied to sizing optimization of shallow space trusses.

In Stolpe and Kawamoto (2005) a special purpose branch-and-reduce method is developed for optimal design of articulated mechanisms. The mechanisms are modelled as truss structures allowing for large displacements. The method makes extensive use of the available reformulation techniques and matrix inequalities.

Maximum stiffness truss topology design problems are solved to global optimality by nonlinear branch-and-bound methods in Achtziger and Stolpe (2008), Achtziger and Stolpe (2009), and Achtziger and Stolpe (2007). These approaches differ from the previously mentioned references since the problem is not reformulated before relaxing the discrete variables. Instead the continuous non-convex relaxation is reformulated as a convex problem such that it can be solved to global optimality. In Achtziger and Stolpe (2007) topology optimization problems with over 600 bars in the ground structure are solved without problem size reductions using variable grouping. In Achtziger and Stolpe (2009) the nonlinear branch-and-bound method is computationally compared to a commercial branch-and-cut method applied to a mixed 0-1 quadratic formulation of the problem and shown to be superior in performance.

A generalized Benders' decomposition method (see Geoffrion (1972), Lazimy (1986), and Sahinidis and Grossmann (1991)) for maximum stiffness problems is suggested and analyzed in Muñoz and Stolpe (2011). The approach is also theoretically compared to an outer approximation (see e.g. Duran and Grossmann (1986) and Fletcher and Leyffer (1994)) approach applied to a perturbed nested version of the considered problem. In Stolpe (2015) several variants of outer approximation are developed, implemented and numerically tested on a set of minimum volume truss topology optimization problems. The relaxed master problems are solved by a commercial and parallel branch-and-cut method. The numerical results suggest that outer approximation is competitive to other global optimization methods for the considered class of problems.

A nonlinear branch-and-bound algorithm is proposed in Stolpe (2013) for global optimization minimum compliance truss topology design problems with a volume constraint. The lower bounds are computed through variable splitting, i.e. Lagrangean decomposition, combined with a bundle method. This advanced relaxation provides stronger lower bounds than the natural continuous relaxation.

5.4 Discussion about global optimization methods for optimal design of trusses

Based on the theoretical and numerical results in the literature it is fair to conclude that the deterministic global optimization methods which have been developed for optimal truss design with discrete design variables are limited in the size and type of problems which can be solved. All approaches described in the section above make extensive use of the mathematical properties of the given problem. This implies that extending the methods to other problems, even after changes to the problem formulation which from an engineering perspective are both natural and minor, normally is a challenging task. The global optimization methods proposed thus far are therefore not particularly versatile. Furthermore, they also require large computational efforts. Most of the computational time is however spent on proving optimization of a design which has been found early on by some heuristic. This observation implies that these methods can, after slight modifications, potentially be applied as heuristics for large-scale problems.

Global optimization methods which generally require at least gradients of objective and constraint functions are not suitable for problems which require black-box structural analysis, for problems with noisy functions, and problems for which no gradient information is available.

6 Deterministic heuristics

A number of deterministic heuristic, i.e. optimization algorithms with no guarantee of success in finding an optimal (or even feasible) design, have been proposed for optimal truss design with discrete design variables. A critical discussion about the difficulties in solving truss design problems with discrete variables is presented in Templeman (1988). The discussion also includes examples where several common heuristics, mainly based on rounding of continuous solutions, have difficulties in finding good designs.

6.1 The 1970s

Lipson and Agrawal (1974) propose heuristics for minimum weight design of planar trusses under multiple loads and stress constraints. The variables model selection of topology, changes to the geometry, and choice of the sectional properties (areas). The derivative free method from Box (1965) is applied to a 3-bar truss, a 10-bar truss (not the 10-bar truss shown in Fig. 3), an 11-bar truss, and the 47-bar planar tower.

The work in Lipson and Agrawal (1974) is extended to spatial trusses in Lipson and Gwin (1977). The article also introduces another objective function which models structural cost. The algorithm is applied to several versions of the 25-bar truss.

6.2 The 1980s

A heuristic for structural optimization with discrete design variables based on separable approximations of objective and constraint functions coupled with dual methods is presented in Schmit and Fleury (1980) and Fleury and Schmit (1980). The algorithm is applied to minimum weight problems with stress and displacement constraints for several truss ground structures including the 10-bar, the 25-bar, and the 72-bar truss in Fleury and Schmit (1980).

Yates et al. (1983) propose a heuristic for sizing optimization of statically determinate trusses with a compliance constraint under a single load. A Lagrangian function is used for finding a lower bound on the optimal weight. The heuristic is based on a scheme for choosing the Lagrange multiplier for the compliance constraint. Theoretical results guaranteeing error bounds are also presented. The capability of the heuristics is illustrated on a planar 38-bar truss example. Generalizations of the heuristic to include stress constraints and several displacement constraints are briefly described but no numerical results are reported.

Templeman and Yates (1983) consider minimum weight design of statically determinate trusses with stress and displacement bounds. The problems are first formulated as 0-1 linear programs using the (known) member forces. Instead of using discrete area variables each bar in the ground structure is divided into segments with the different available areas. The lengths of each of the segments is then used as (essentially discrete) design variables. The considered problem can be well-approximated by removing the discrete requirements. The resulting problem is a linear program (LP) providing tight lower bounds. For the bars with more than one segment in the optimal solution to the LP rounding is used to get feasible discrete designs. A planar 38-bar truss example is sued to demonstrate the algorithm.

Sepulveda and Cassis (1986) propose a heuristic for minimum weight problems under displacement and stress constraints. The heuristic is capable of handling problems with both discrete and continuous design variables. The heuristic resembles the ones in Schmit and Fleury (1980) and is based on separable approximations of objective and constraint functions coupled with dual methods. The algorithm is applied to the 10-bar truss with a mixture of discrete and continuous sizing variables, and to the 25-bar truss with all variables discrete.

Ming-Zhu (1986) considers minimum weight sizing problems with stress and displacement constraints. A formulation in which the displacement constraint are stated in member forces and virtual forces is used. The forces are, in the case of statically indeterminate structures, updated between iterations. In each iteration a neighbourhood search limited to three different cross-sections is performed using a revised version of the simplex method for linear programming. Several versions of the algorithm are applied to the 25-bar truss problem.

Ringertz (1988) proposes several algorithms for minimum weight design with displacement constraints. One of the numerical examples additionally has stress constraints. The first algorithm is a standard branch-and-bound method based on solving a sequence of nonlinear continuous problems (relaxations) with variable dichotomy branching. The method produces global optimal designs for convex problems and is otherwise an heuristics. The second heuristics is based on a generalized Lagrangean function which provides a dual problem which is solved by a sub-gradient method. The third heuristic is based on dynamic rounding of solutions to the continuous relaxation. The algorithms are applied to several versions of the 10-bar truss and the 25-bar truss, a 36-bar spatial truss, and a 63-bar spatial truss. The results are compared to those in e.g. Schmit and Fleury (1980).

Olsen and Vanderplaats (1989) propose a heuristic for general nonlinear discrete optimization problems. The heuristic begins by solving the continuous relaxation of the considered problem followed by rounding. The found design is then the starting point for linearization of objective and constraint functions. This results in a MILP which is solved by a branch-and-bound method. The optimal design is then used as the new linearization point resulting in sequential linear discrete programming. The strategy also uses a kind of trust-region strategy to promote convergence. The algorithm generally terminates when two consecutive iterations are identical and does neither guarantee optimality nor feasibility. The heuristic is tested on the 10-bar and the 25-bar minimum weight truss problems with stress and nodal displacement constraints.

6.3 The 1990s

Bremicker et al. (1990) propose a heuristic for general inequality constrained mixed discrete problems. It is based on solving a sequence of linear mixed discrete problems by branch-and-bound. The problems locally approximate the objective and constraint functions. A move limit strategy is included to promote convergence. The algorithm shares similarities to the heuristic in Olsen and Vanderplaats (1989). It also solves a nonlinear problem per iteration with fixed discrete variable as in outer approximation, see e.g. Fletcher and Leyffer (1994). The heuristic does however not accumulate the linearizations of the objective and constraints as in outer approximation. The heuristic is applied to several combined sizing and geometry minimum weight problems with stress and displacement constraints such as a 3-bar truss, the 10-bar truss, and a 39-bar spatial tower.

John and Ramakrishnan (1990) consider minimum weight sizing problems with displacement, stress, and eigenfrequency constraints. The suggested heuristic combines sequential linear programming with move limits and a branch-and-bound method. For the reported problems with discrete design variables two ground structures are used. The first is a 3-bar truss and the second is the 25-bar truss.

Shin et al. (1990) modify the sequential unconstrained minimization technique (normally called SUMT) to problems with discrete variables. The resulting heuristic is applied to several minimum weight problems with stress and displacement constraints, including the 10-bar truss. The results are compared to the results in e.g. Ringertz (1988).

Salajegheh and Vanderplaats (1993) consider combined sizing and geometry optimization for minimum weight with stress and displacement constraints. The heuristic alternates between solving a continuous problem locally approximating the continuous relaxation of the original problem and solving the discrete version of the approximate problem by a branch-and-bound method. The heuristic is applied to the 18-bar, the 25-bar, and the 47-bar groundstructures.

A heuristic based on sequential rounding of the design variables, similar to the dynamic rounding heuristic presented in Ringertz (1988), is proposed in Groenwold et al. (1996). The main difference is that the chosen variable is allowed to be rounded both upwards and downwards. This implies that more continuous problems are solved but it increases the chances to find good designs. The heuristic is applied to a set of minimum weight sizing problems (such as the 10-bar, the 25-bar, the 36-bar, and the 200-bar truss) with either compliance constraints or stress and displacement constraints. The results from the heuristic is compared to the results in Ringertz (1988).

Adeli and Park (1996) propose a neural dynamics model and a neural network topology for minimum weight sizing problems with stress and displacement constraints. The 25-bar truss is used for comparisons with the genetic algorithm from Rajeev and Krishnamoorthy (1992) and the heuristic in Ming-Zhu (1986). The technique is then used for sizing of two large steel structures. The largest problem has almost 9000 bars in the ground structure. Variable linking is used to significantly reduce the number of design variables.

Chai et al. (1999) combines topological 0-1 variables and continuous area sizing variables for minimum weight problems with nodal displacement and stress constraints. The design-dependence in the stress constraints is modelled. A heuristic based on the relative difference quotient algorithm is suggested and applied on a 4-bar, a 10-bar, a 12-bar, a 15-bar, and a 25-bar ground structure.

6.4 After 2000

A very simple derivative-free heuristic for single static load minimum weight problems with stress constraints is proposed in Gutkowski et al. (2000). The heuristic starts from the heaviest structure and recursively removes redundant material from low stressed elements until no more reductions are possible. The heuristic is evaluated by applying it to the 10-bar truss.

A heuristic similar to the one in Gutkowski et al. (2000) is proposed for design of steel structures under the European Code 3 in Guerlement et al. (2001) and using member sizes listed in steel catalogues.

Tong and Liu (2001) propose problems with constraints on stresses, natural frequencies, and frequency responses. A two-stage heuristics is proposed for these problems. First a point which satisfies all the constraints using continuous design variables is found. Secondly, the discrete values of the design variables are determined by approximating the structural dynamic optimization process into a linear 0-1 problem which is treated by a combinatorial algorithm. Several variants of the 10-bar and 25-bar trusses are considered in the numerical examples.

Juang and Chang (2006) propose a discrete Lagrangian based search procedure for minimum weight problems with stress and displacement constraints. The heuristic is applied to sizing optimization of the 10-bar truss, the 22-bar truss, and the 160-bar spatial truss with 38 design variables. The results are compared to those obtained by the genetic algorithms in e.g. Rajeev and Krishnamoorthy (1992) and Groenwold et al. (1999).

Blachowski and Gutkowski (2008) present a generalization of the heuristic from Gutkowski et al. (2000) to multiple loads minimum weight problems with stress, nodal displacement constraints, and eigenfrequency constraints. The algorithm is illustrated by applying it to the 10-bar and 25-bar ground structures.

Several simple heuristics for stress constrained minimum weight problems are presented in Stolpe (2011) as a response to the ACO method presented in Sonmez (2011). They are all based on solving a sequence of continuous relaxations of the considered problem coupled with rounding heuristics and additionally local greedy search. The numerical experiments indicate that even these simple deterministic heuristics outperform ACO for the benchmark problems from Sonmez (2011).

In Zhang et al. (2014) the minimum weight problem with stress and displacement constraints is first converted to a problem with variables in the set $\{-1, 1\}$ using a parametrization based on generalized shape functions. This problem is then equivalently reformulated as a continuous problem with a quadratic equality constraint forcing the variables to -1 or 1. This constraint is the added to the objective by Lagrangian relaxation and the Lagrange multiplier is estimated. A mathematical programming methods is the applied to this continuous (and non-convex) problem. The obtained variables are then rounded to -1 or 1. This approach is applied to the 10-bar, the 25-bar, the 72-bar, and the 200-bar ground structures. The obtained designs are then compared to the results obtained in e.g. Ming-Zhu (1986), Rajeev and Krishnamoorthy (1992), Cai and Thierauf (1996), Wu and Chow (1995), Erbatur et al. (2000), Camp and Bichon (2004), and Lee et al. (2005), among others.

This concludes the presentation of the heuristics and methods proposed in the literature. The next section presents some critique on problems formulations, the development of heuristics and methods, and numerical experiments reported in the literature.

7 Critique

It is absolutely necessary to critically examine the field and someone unfortunately has to do it.[10] Together with the intentionally blunt criticism in this section, some research and development objectives are suggested. Thoughts about improvements to the way the numerical experiments are conducted and reported in the field are also presented.

[10]It can perhaps provide some consolation to those that feel targeted in this section that the author is occasionally throwing bricks in glass houses.

7.1 Learning from history is so last year

Models, theory, and methods for optimal truss design problems with discrete design variables have been documented since the end of the 1960s. While reading recent articles it became apparent that the theoretical results and the best heuristics and methods from the 1960s – 1990s are (seemingly) largely forgotten or ignored. The heuristics from e.g. Ringertz (1988) and Groenwold et al. (1996) have not been re-implemented using modern state-of-the-art methods for continuous optimization. They are furthermore not included in the benchmarking of new heuristics and methods although they would perform very well. These heuristics would, most likely, outperform most modern meta heuristics in terms of function evaluations and objective function value (this statement is, of course, speculative).

7.2 Sizing problems are too popular

Many of the numerical methods and heuristics proposed in the literature for truss design with discrete design variables are only applied to *sizing* problems, i.e. problems with non zero lower bounds on the area variables. Meta heuristics such as ant colony optimization, big bang–big crunch, the colliding bodies optimization method, the artificial bee colony algorithm, harmony search, etc. have yet to prove their capabilities in truss *topology* optimization. The sizing assumption simplifies the problem considerably compared to the situation that the topology is allowed to change, i.e. bars in the ground structure are allowed to vanish. First of all, the theoretical and computational issues with the design-dependent stress constraints disappear completely in sizing problems. Secondly, there is no risk that nodes in the ground structure become “floating”, i.e. the stiffness matrix is positive definite for all design variables satisfying the lower bound constraints. Thirdly, if the ground structure is chosen to represent a stable structure, this property is inherited by all feasible designs. These listed challenges potentially do not pose any issues for the mentioned meta heuristics but this should be carefully investigated both from theoretical and numerical points of view. In summary, new methods and heuristics should always be benchmarked on both sizing and topology optimization problems.

7.3 New problem formulations are often old news

Most articles listed in this review consider the minimum weight problem with bar stress and nodal displacement constraints. This is essentially the same problem as studied in Toakley (1968) and Lipson and Agrawal (1974). For this problem more than twenty[11] heuristics and methods have been developed, implemented, and numerically tested with various levels of success. Either this is the ultimate problem in the field and we are done. Alternatively, we need to propose new application relevant combinations of objective and constraint functions to spur the development of new theory, methods, and heuristics. Examples include development of problems involving design- and time-dependent loads. Additional work can also, with advantage, be directed towards problems with buckling and stability constraints, and robust and reliability based optimization, but the list is by no means complete.

7.4 Benchmark problems

The literature contains a small number of frequently reoccurring benchmark problems which are used to illustrate the efficiency and/or robustness of new methods and heuristics. For future development of methods and heuristics, the field is in need of a publicly available and well-documented library of benchmark problems.[12] A good start is presented in e.g. Hasançebi et al. (2009). Benchmark libraries are commonly used in the mathematical programming, numerical optimization, and integer optimization societies. Examples of optimization benchmark libraries include MIPLIB with real-world mixed integer programs (Koch et al. 2011), NETLIB LP with linear programming problems (Gay 1985), and CUTEr with nonlinear constrained and unconstrained problems (Gould et al. 2003).

Certain requirements should be imposed on a truss optimization benchmark library. The problem instances should be distributed in easily readable formats for defining the grounds structure, boundary conditions, and loads, etc. Furthermore, the objective and constraint functions and the problem data (such as material properties, stress and displacement limits, frequency limits, variable grouping, etc.) should be completely defined. The library must be large enough and preferably contain hundreds of problem instances. The problem instances should represent a wide diversity in design domain geometries, number of design variables and degrees of freedom, and lists of available areas. The library should also contain infeasible problems. Finally, the library should be dynamic and always keep a balance between problems from the literature for which good or even optimal designs are known and problems which are unsolved.

[11] This number does not include different variants and flavours within each class of heuristic or method.

[12] Development and dissemination of the benchmark library together with approval of new problem instances could be a task for a working group under the International Society for Structural and Multidisciplinary Optimization (ISSMO).

7.5 Statistics

A large number of articles present numerical results which are insufficient in the sense that it is impossible for the reader to judge if the proposed method/heuristic is efficient and/or robust. Often only a handful of problem instances are reported which by no means is enough to claim that the method is robust, i.e. capable of obtaining good designs for many problems without running into numerical difficulties.

Many articles in the literature emphasize the number of bars in the ground structure in the abstract and conclusions. This number is used as a measure of the problem size. The actual number of design variables, after variable linking (grouping of design variables), is often not highlighted. This tendency is unfortunate. For these combinatorial problems, the number of design variables and the number of available areas generally have much larger influence on the solution time than the number of bars in the ground structure.

Many articles completely ignore to report the total number of function evaluations, the computational time or other means to justify that the method/heuristic is computationally efficient. Occasionally it is possible to reverse engineer the number of function evaluations. This exercise often leads to other conclusions about the heuristic than promoted by the authors.

There is a consensus in numerical optimization to use performance profiles as suggested in Dolan and Moré (2002) applied to large sets of benchmark problems as a tool for comparing different optimization methods, or the effects of parameter changes in one particular method. The field of structural optimization should embrace these benchmarking techniques.

7.6 Theoretical results

Only a handful of the articles in this review that present new methods or heuristics are accompanied by theoretical convergence results. What is perhaps even worse, most articles do not even try to motivate (and even less prove) that the steps in the algorithms are actually well-defined and can be performed without theoretical or numerical difficulties. Comments pointing out theoretical shortcomings and numerical pitfalls are also largely lacking in the literature.

All new heuristics and methods should at least be accompanied by certain basic theoretical results. These results do not need to be stated as formally as lemmas or theorems but can be incorporated in the text. The individual steps of the algorithms should, as an absolute minimum requirement, be well-defined. The rationale of the algorithms to be optimization methods should also be motivated. It is, for example, very difficult to comprehend that modelling the behaviour of shrapnel from mine blasts (Sadollah et al. 2012) in any way helps in solving optimization problems. In these situations convergence results should be required by reviewers and journal editors.

7.7 Parallel computing

Despite the complexity results from Yates et al. (1982) and the limited increase in problem sizes seen over the past 50 years it is surprising that only a few articles document the use of parallel methods in this field. Cai and Thierauf (1996) propose a parallel modified evolution strategy for the minimum weight problem with stress and displacement constraints. Park and Sung (2002) propose a distributed simulated annealing algorithm for optimal design of steel structures and applied it to the minimum weight 25-bar truss problem. Rasmussen and Stolpe (2008) formulate the same problem as a MILP and apply an open source parallel branch-and-bound method to solve the problems to global optimality. Reformulating the problems as MILPs or MIQPs has an advantage in this respect. Modern implementations of global optimization methods such as branch-and-cut and associated advanced heuristics become parallel with a simple parameter change. For mixed 0-1 nonlinear problems the level of (parallel) maturity is much lower and investments in research and software development are necessary.

In order to increase the problem sizes which are solvable in this field development and implementation of parallel optimization methods should be prioritized.

7.8 Some thoughts about the use of the words *solve*, *solution*, and *optimal*

Three of the most frequently occurring words in the articles in the reference list are solve, solution, and optimal. They are often used inaccurately and sometimes even incorrectly. Authors and also reviewers and journal editors should enforce a much more carefully use of these words. Authors should at least provide a clear description of what the intended definition of the words are in the given context. Since essentially all optimal design problems mentioned in this review are non-convex, potentially infeasible, and have continuous relaxations which generally do not satisfy standard constraint qualifications the word solve, solution, and optimal must be clearly defined.

An often encountered example is algorithms which contain the step "Solve the continuous optimization problem ...". Normally the problem which is to be solved is non-convex and has not been previously analysed and the intended meaning of the word solve is nowhere to be found. The lack of analysis means that existence of solutions is not guaranteed or the problem may be infeasible or unbounded. Even if neither of these situations occur the problem remains non-convex. The word solve should therefore, in the strict sense, be interpreted as finding a global optimum to the problem. Since this is generally not possible, questions immediately arise. For example "What conditions should be satisfied in order for the problem to be considered as solved?" and "Is it enough to satisfy some first-order necessary optimality conditions or should second-order sufficient conditions be satisfied?" and "What happens to the overall algorithm if not the correct local minimizer is found?" and "What should be done if the problem is infeasible or unbounded?".

The words solution and optimum are likewise also very often inaccurately used. A design is generally not a solution to an optimal design problem just because it is the outcome of an algorithm which has stopped because some user defined iteration/time/function evaluations limit has been reached. This is especially true for algorithms for which there are no convergence proofs and this applies to almost all heuristics and methods mentioned in this review. In fact, the word optimal has been used for designs found by methods applied to well-documented benchmark problems for which better feasible designs are known!

8 Conclusions

Many problem formulations, heuristics, and methods have been proposed in recent years within optimal truss design with discrete design variables. The field is currently also experiencing a shift from the deterministic heuristics and methods that dominated the literature from the 1960s to the meta heuristics that have gained increasing popularity since the mid 1990s. Many of these meta heuristics have not been proven in truss topology optimization. It is also uncertain if they actually improve the computational competences in the field since they are generally not numerically compared to existing deterministic methods and heuristics. Moreover, numerical results for these new algorithms are normally only reported on small sets of benchmark problems. These observations motivate the urgent need for a large and publicly available benchmark library combined with changes in the ways numerical results comparing different methods are done and reported.

Stochastic meta heuristic and deterministic numerical optimization methods and heuristics both have their respective advantages and disadvantages. The two lists of advantages however indicate that these methods should generally be applied to different classes of problems. Meta heuristics have advantages for problems where gradients cannot be computed or are not available. In this field they are however applied to the same classes of problems. For these problems both first- and second-order sensitivity information can be computed. The choice of problem classes used when benchmarking meta heuristics makes them compete directly with gradient based methods. Unfortunately, very limited numerical comparisons between the two classes of methods are reported. Meta heuristics and deterministic optimization methods and heuristics should however not be viewed as competitors in this field. Instead they should be regarded as complements since they are, with advantage, applicable to different classes of problems. Combining the two classes of methods might thus produce very positive synergies.

The final part of the conclusion is a quote from Templeman (1988) which is still valid: *"In fact, the problem of practical discrete optimum structural design contains some very meaty problems which research has not yet touched, yet which seem amenable to solution. Further research is required into discrete optimum design, with particular attention to the requirement of structural design practice."*

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