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A review of structural shape optimization

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Abstract

Shape optimization attempts to integrate geometrical modeling, structural analysis, and optimization into one complete and automated computer-aided design process. It determines the shape of the boundary of a two- or three-dimensional structural component of minimum mass under constraints on geometry and structural responses such as stress, displacements and natural frequencies. This paper presents a general review of structural shape optimization, with emphasis on the recent developments in this field. First the general concepts are introduced. Different approaches of the process of shape optimization, which consists of geometrical representation, structural analysis, sensitivity analysis, and optimization algorithms, are reviewed. Then the difficulties in expanding from two- to three-dimensional shape optimization are discussed. Finally the paper concludes that more attention should be paid to research of zero-order optimization algorithms which are better suitable for shape optimization problems, and mechanical design optimization problems in general.

Keywords: Shape optimization; Computer-aided design

1. Introduction

Shape optimization attempts to integrate geometrical modeling, structural analysis, and optimization into one complete and automated computer-aided design process. It determines the shape of the boundary of a two- or three-dimensional structural component of minimum mass under constraints on geometry and structural responses such as stress, displacements and natural frequencies.

As faster computers and more powerful structural analysis and optimization programs become widely available to designers, integrating analysis and optimization into the mechanical design process becomes a realistic goal. Optimization plays a crucial role in using the results from analysis programs to improve a design systematically and efficiently.

This paper presents a general review of shape optimization. The purpose of this paper is to introduce the general concepts, discuss the different methods currently used in shape optimization, and finally point out the need for designing new zero-order optimization algorithms for three-dimensional shape optimization, and mechanical design optimization in general. This paper focuses on the recent developments in the field of shape optimization from 1987 to 1993. Detail reviews of the early developments of shape optimization can be found in articles by Haftka and Grandhi [1] and Ding [2].

Section 2 describes the general process of shape optimization, which consists of geometrical

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representation, structural analysis, and optimization algorithms. Section 3 discusses several geometrical representations that have been used to describe the design model. Specific structural analysis techniques that have been applied in shape optimization are reviewed in Section 4.1. Sensitivity analysis techniques for calculating the gradients of the structural responses with respect to shape design variables are discussed in Section 4.2. Section 5 presents the optimization algorithms that have been applied in shape optimization. Section 6 presents two design examples that are commonly used in the literature of shape optimization. Section 7 discusses the difficulties in the research of three-dimensional shape optimization, points out the need for designing new zero-order optimization algorithms, and discusses their possible advantages.

2. The general process of shape optimization

The process of shape optimization consists of three modules: geometrical representation, structural analysis, and optimization algorithms. To construct the design model, first the geometrical

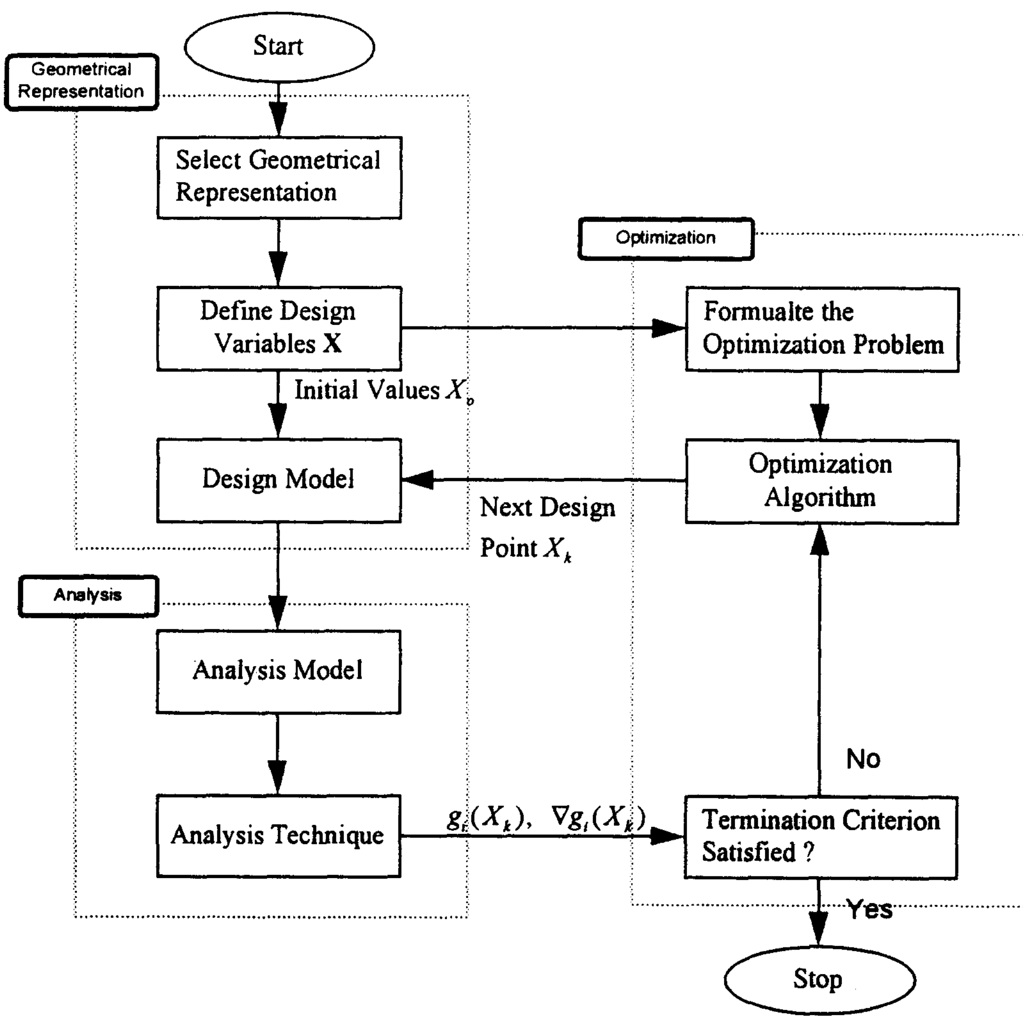


Fig. 1. The general process of shape optimization.

representation for the boundary shape is chosen. Then design variables are defined based on this representation. These design variables are also used to formulate the shape optimization problem as below:

$$\begin{aligned} &\text{minimize} && f(\boldsymbol{x}), \\ &\text{subject to} && \boldsymbol{g}_e(\boldsymbol{x}) \leqslant \mathbf{0}, \\ &&& \boldsymbol{g}_i(\boldsymbol{x}) \leqslant \mathbf{0}, \end{aligned} \tag{1}$$

where $\boldsymbol{x}$ is the vector of design variables, $f(\boldsymbol{x})$ is the objective function, and $\boldsymbol{g}_e(\boldsymbol{x}) \leqslant \mathbf{0}$ and $\boldsymbol{g}_i(\boldsymbol{x}) \leqslant \mathbf{0}$ are the constraints. The objective function can be defined in many ways. In many cases, it is the weight or volume of the design. There are two kinds of constraints in the formulation: $\boldsymbol{g}_e(\boldsymbol{x}) \leqslant \mathbf{0}$ are the explicit constraints which can be expressed explicitly in terms of design variables (e.g., the upper and lower bounds of the design variables); $\boldsymbol{g}_i(\boldsymbol{x}) \leqslant \mathbf{0}$ are the implicit constraints which cannot be expressed explicitly in terms of design variables. Typical examples of implicit constraints are those on structural responses such as stress, displacement and natural frequency.

Given the initial values of the variables, the initial design model is constructed. In the analysis module, this design model is converted into an analysis model. A structural analysis technique such as finite element analysis or the boundary element method is then used to evaluate the implicit constraints of this particular design. The analysis module must provide the necessary information on the implicit constraints required by the optimization module. For example, a typical first-order optimization algorithm requires function values of the constraints and the sensitivity of the constraints with respect to each design variable.

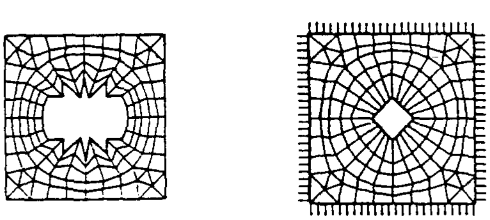
Using the information provided by the analysis module, the optimization algorithm finds a new set of values of design variables. A new design model is constructed and fed into the analysis module again. The analysis module generates function values and sensitivity information of the implicit constraints for the new design. If the prescribed termination criterion is satisfied, this procedure stops; if not, the optimization module finds the next design point. Fig. 1 summarizes the process of shape optimization. The individual modules are discussed in detail in the following sections.

3. Geometrical representation

The first step in the shape optimization process is to select a geometrical representation. In one of the earliest works on shape optimization by Zienkiewicz and Campbell [3], the finite element method was used to analyze the designs, and the finite element mesh was used as the geometrical representation. The coordinates of the boundary nodes were used as design variables. This representation then became a common practice in early works on shape optimization.

Using nodal coordinates as design variables is very intuitive and directly relates to the finite element method. However, this idea was soon discarded because experience showed that it was very difficult to maintain a smooth boundary shape using this representation. An example is shown in Fig. 2 [4]. This representation often generates impractical designs as in Fig. 2(b). Moreover, the stress data cannot be expected to be accurate using the finite element mesh of Fig. 2(b).

This experience showed that the design model has to be separated from the analysis model because boundary smoothness is a basic requirement when choosing a geometrical representation for shape optimization. To construct design models with smooth boundary shapes, researchers started to use polynomials to describe the boundary (e.g., [5–7]). The coefficients of the polyno-



(a) initial design. (b) final design

Fig. 2. Geometrical representation by finite element mesh [4].

mial were used as design variables even though they do not relate directly to mechanical properties.

Although this polynomial boundary representation guarantees the smoothness of the boundary, it can also give an impractical oscillatory boundary shape when the polynomial order is too high. Moreover, polynomial functions do not represent general shapes; they can only represent a family of curves (quadratic, cubic, etc.). From a designer's point of view, another problem with this representation is lack of local control of the curve. Changing the value of one polynomial coefficient will alter the shape of the whole curve.

Spline curves [8, pp. 98–112] eliminate the problem of shape oscillation because they are composed of lower-order polynomial pieces which are combined to maximize smoothness. This representation guarantees smoothness of the boundary shape. With enough control points, a spline curve is also general and has good local control. Thus the spline representation became the most popular geometrical representation in shape optimization (e.g., [4,9–12] and almost all recent papers in shape optimization used spline representation). The control point coordinates are the design variables. Note that these variables do not relate directly to mechanical properties either. Non uniform rational B-splines (NURBs) were also used to describe the shape of the structure in shape optimization [13].

Design variables other than those describing the boundary shape have also been considered. The design element concept was introduced for the purpose of achieving an adequate finite element mesh throughout the iterations [4,14,15]. Rajan and Belegundu [16] used the magnitudes of a set of fictitious loads as the design variables and the deformation produced by those loads was used to update the shape.

4. The analysis module

4.1. Structural analysis techniques

Given a set of design variable values, the geometry of a design is defined. As described in Section 2, this design model is then converted into an analysis model for structural analysis evaluating the constraints on stress, displacement and frequency for this design. The major concerns when choosing an analysis technique for shape optimization are the cost of converting the design model into the analysis model, and the capability and accuracy of the analysis technique.

Finite element analysis has been the most popular analysis technique used in shape optimization. Applicable to a broad range of engineering problems, robust commercial finite element programs with friendly user interfaces are widely available to designers. One important issue in

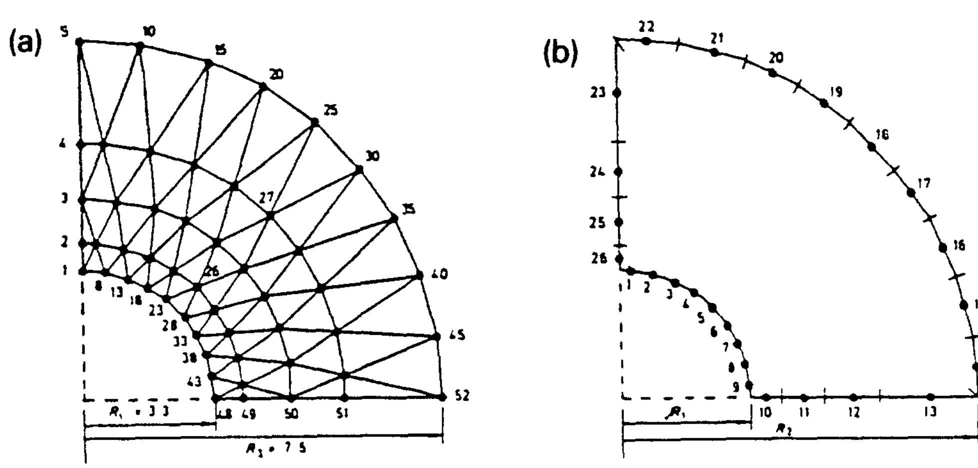


Fig. 3. (a) Finite element mesh versus (b) boundary element mesh [27, p. 163].

shape optimization using finite element analysis is the mesh generation needed to convert the design model into the analysis model. The capability of automatic remeshing throughout the iterations is necessary to automate the shape optimization process. Furthermore, if the mesh is distorted during the iterations, it has to be refined to assure accuracy. Therefore, much attention has been paid to adaptive mesh generation, that is, how to refine efficiently the mesh or remesh for the next design based on the analysis results obtained from the current design [17–19]. Higher-order finite elements (p-version finite elements) have also been used in shape optimization, taking advantage of the fact that higher-order elements are less sensitive to shape distortion [20].

In recent years the boundary element method has attracted much attention as an alternative to finite element analysis in shape optimization [12,21–26]. This is largely because its meshes are relatively easy to generate and its input data is simple. Fig. 3 compares the boundary element model to the finite element model for a 90° segment of a pipe under internal pressure [27, pp. 162–164]. Unlike the finite element method, discretization for the boundary element technique is required only on the boundary of the components. The finite element mesh in Fig. 3(a) consists of 52 nodes and 76 three-noded elements, whereas the boundary element discretization in Fig. 3(b) has only 26 constant segments. Much smaller systems of equations are required for the boundary element method, and it was reported that the numerical accuracy for boundary elements was greater than that for finite elements in this case [27, p. 164]. The major restriction on the boundary element method is its inability at this stage of development to handle analysis problems like buckling and modal analysis.

Experimental techniques, especially photoelasticity, have also been used in shape optimization problems for evaluating the implicit constraints. Durelli and Rajaiah [28,29] developed a step-by-step procedure for modifying hole boundaries in a two-dimensional photoelastic model until the tensile and compressive boundary stresses were approximately constant. The hole shapes were changed by removing material from low stress regions to obtain uniform stress conditions as well as significant reductions in stress concentration. Experimental analysis techniques are usually costly because they call for a series of experiments for each particular problem. Moreover, the optimization process cannot be automated because the experiments have to be set up manually. However, experiments are necessary for mechanical analyses which do not have corresponding numerical simulation methods, or to validate numerical simulation.

4.2. Sensitivity analysis

As discussed in Section 1, the analysis module must provide the necessary information about the implicit constraints required by the optimization module. A typical first-order optimization algorithm requires function values and sensitivity of the implicit constraints. Therefore, sensitivity analysis has been an important research topic in the field of shape optimization.

A simple and very popular technique for calculating gradients of structural responses with respect to design variables is the finite difference approximation. Given a function $u(x)$ of a design variable x, the forward difference approximation $\Delta u / \Delta x$ to the derivative $\mathrm{d}u / \mathrm{d}x$ is given as

$$\frac{\Delta u}{\Delta x} = \frac{u(x + \Delta x) - u(x)}{\Delta x}. \tag{2}$$

From this formula, we can see that the calculation of $\Delta u / \Delta x$ is independent of the function, which makes the finite difference technique easy to implement. This technique is also general to all structural analysis methods (e.g., finite element or boundary element method) and analysis types (e.g., static or modal analysis), since no analytic knowledge of function is required. However, the finite difference technique is computationally expensive if the number of design variables is large. If we need to find the derivatives of the structural response with respect to n design variables, the forward difference approximation requires n additional analyses. Moreover, the finite difference approach often has accuracy problems.

Analytical sensitivity analysis methods are more efficient. There are two basic approaches, the implicit differentiation approach and the variational approach. Implicit differentiation is based on the discretized formulation of the structural system. For example, the equation of equilibrium from the finite element formulation is

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}, \tag{3}$$

where $\boldsymbol{K}$ is the stiffness matrix, $\boldsymbol{u}$ the nodal displacement vector, and $\boldsymbol{f}$ the force vector. Differentiating Eq. (3) with respect to a design variable x_i and assuming the force vector is independent of x_i, we obtain

$$\boldsymbol{K}\frac{\partial \boldsymbol{u}}{\partial x_i} + \frac{\partial \boldsymbol{K}}{\partial x_i}\boldsymbol{u} = \boldsymbol{0}. \tag{4}$$

Thus $\partial \boldsymbol{u}/\partial x_i$ can be calculated if $\partial \boldsymbol{K}/\partial x_i$ is known.

The derivative of the stiffness matrix $\partial \boldsymbol{K}/\partial x_i$ can be derived analytically for sizing variables such as thickness, cross-sectional area or moment of inertia. But it is very difficult to calculate $\partial \boldsymbol{K}/\partial x_i$ analytically for general shape design variables because perturbations in shape design variables change the element geometry and thus the whole stiffness matrix $\boldsymbol{K}$. Therefore, traditionally $\partial \boldsymbol{K}/\partial x_i$ is calculated from finite differences; this is called the semi-analytical method. Although it saves the computation of solving the matrix equation $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$, the semi-analytical method is still rather expensive since it has to reassemble the stiffness matrix for every perturbation of each variable. Moreover, Haftka et al. reported that the accuracy of the semi-analytical method is even poorer than that of the overall finite difference method. Another problem with the implicit differentiation approach is the difficulty of implementing into a general-purpose finite element program [30, pp. 223–225].

The variational approach is based upon the total derivative of the variational state equation. Haftka et al. reviewed the variational approach [30, pp. 251–281]. The implementation of this approach is less difficult because the analytical expression for the sensitivity depends only on the boundary solution quantities, which are usually available from general-purpose analysis programs. The primary disadvantage is that these quantities are difficult to obtain accurately [31].

5. Optimization algorithms

Various optimization algorithms have been applied to solve the shape optimization problems. They can be divided into two categories, the sequential approximation methods and the direct search methods.

The basic idea of sequential approximation is to use a simple subproblem to approximate the hard, exact problem at each iteration. Sequential linear programming is one such method widely used in shape optimization (e.g., [3,4,7]. Using the gradient information at a design point, the objective function and the constraints are linearized and a linear programming subproblem is formed. To compensate for the fact that the linear approximation is good only near the current design point, and to prevent the linear programming subproblem from becoming unbounded, limits on the amount of allowable movement (move limits) are imposed on each variable as additional constraints. This linear programming subproblem can be easily solved. The solution of this subproblem is then used as the design point for the next iteration. The optimization algorithm CONLIN [32] is a further generalization of the sequential linear programming method. Besides linear approximation, this algorithm also uses reciprocal transformation to approximate the nonlinear behaviors of the objective function and constraints.

In contrast with the sequential approximation methods, direct search methods work directly in the feasible domain of the problem. One example of a direct search method is the FORTRAN program CONMIN [33], one of the most popular optimization programs applied to shape optimization (e.g., [4,17,22,34]). CONMIN uses a feasible direction algorithm. A typical iteration of the feasible direction algorithm starts at the boundary of the feasible domain. A feasible descent direction is defined using the gradient information at this starting point. CONMIN then moves in this direction till it hits a constraint or the objective function starts increasing. This new design point is

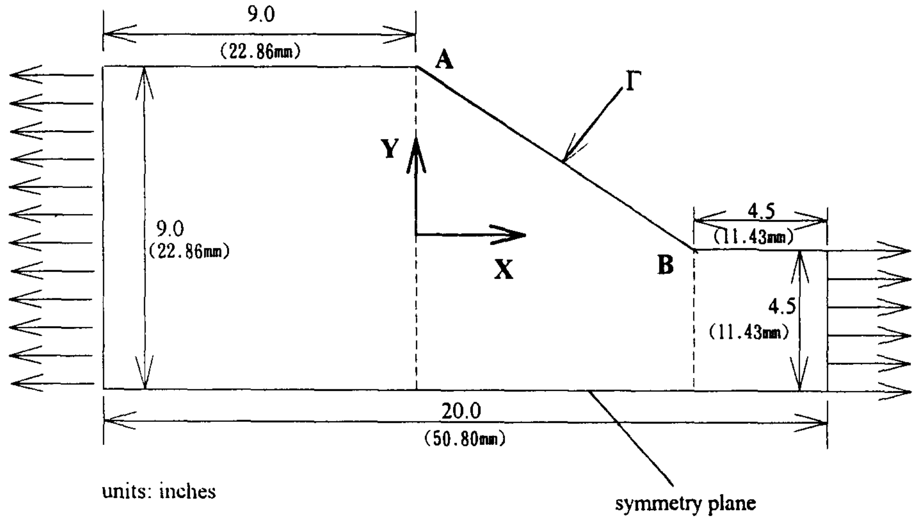


Fig. 4. Geometric configuration of a fillet.

then used as the starting point of the next iteration. This procedure continues until no feasible descent direction can be found.

The growth strain method proposed by Arutyunyan and Drozdov [35], Mattheck [36], and Azegami [37,38] does not use any mathematical programming techniques. In this method, the shape simply swells or contracts according to the current stress–strain state, simulating the growth behavior of living organisms which change their own shapes to adapt themselves to the mechanical living environment.

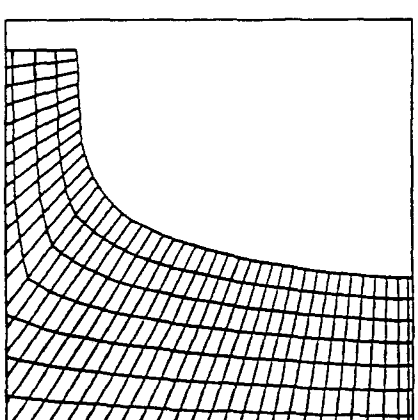


Fig. 5. Fillet example: finite element model of the final design for $K_t = 1.05$ [41].

6. Examples of shape optimization

Two examples that are commonly used in the literature of shape optimization are presented in this section. First a fillet that tapers a tension bar from one section size to another is shown in Fig. 4. Only the upper half of the bar is considered because of symmetry about the $y = -4.5''$ axis. The boundary Γ between points A and B is to be optimized. This problem was first proposed by Yang, Choi and Haug [39]. It was also studied in the articles by Shyy and Fleury [20], Yang [22], and Bendsoe and Rodrigues [40]. The task was to find the minimum area fillet with a maximum stress concentration factor $K_t = 1.20$. Fig. 5 shows the finite element model of the final design by Hsu [41] of the same problem for $K_t = 1.05$, using a zero-order optimization method.

Fig. 6 shows the initial dimensions and loading conditions of a torque arm. The arm is constrained around the circumference of the hole located at $x = 416$ mm. Only the boundary Γ between point A and B was to be varied to minimize the area of the arm. The shaded regions were held to their original dimensions throughout.

The original problem which was proposed by Botkin [34] had a stress and a vertical displacement constraint. This problem was also studied in

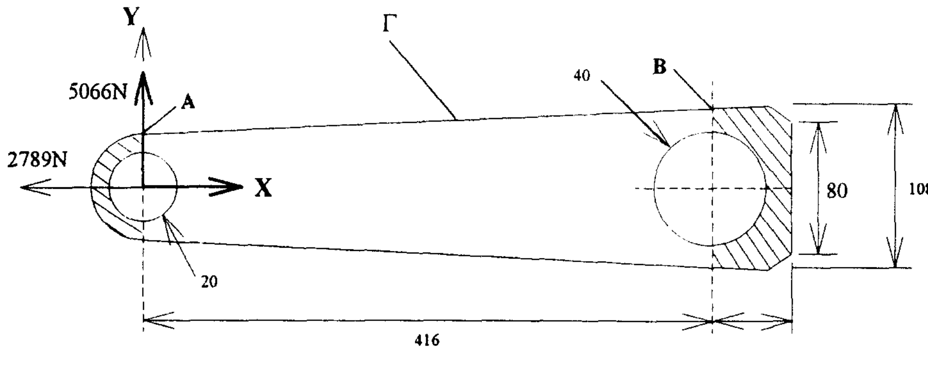


Fig. 6. The dimensions and loading conditions of a torque arm.

the articles by Bennett and Botkin [17], Braibant and Fleury [15], Yang et al. [39], and Rajan and Belegundu [16]. Fig. 7 shows the iteration history of the optimal design of the arm by Hsu [41], with only stress constraint, using a zero-order optimization method.

Besides optimizing the structural shapes of mechanical components, shape optimization techniques also find industrial application in the design of bonded joints [42,43], helical springs [21,44], gas turbine disks [45], and arch dams [46], to name just a few.

7. The need for zero-order optimization algorithms

Only a limited amount of work has been published on three-dimensional shape optimization. The earliest publication on three-dimensional shape optimization is by Iman [14]. Botkin and Yang have been very active in this field in recent years [47–49]. Kodiyalam et al. also proposed an approximation method for three-dimensional shape optimization [50,51]. Mattheck [52] optimized the shape of a three-dimensional bar with a rectangular hole using the biological growth method.

The natural expansion of the design of two-dimensional curves should be three-dimensional

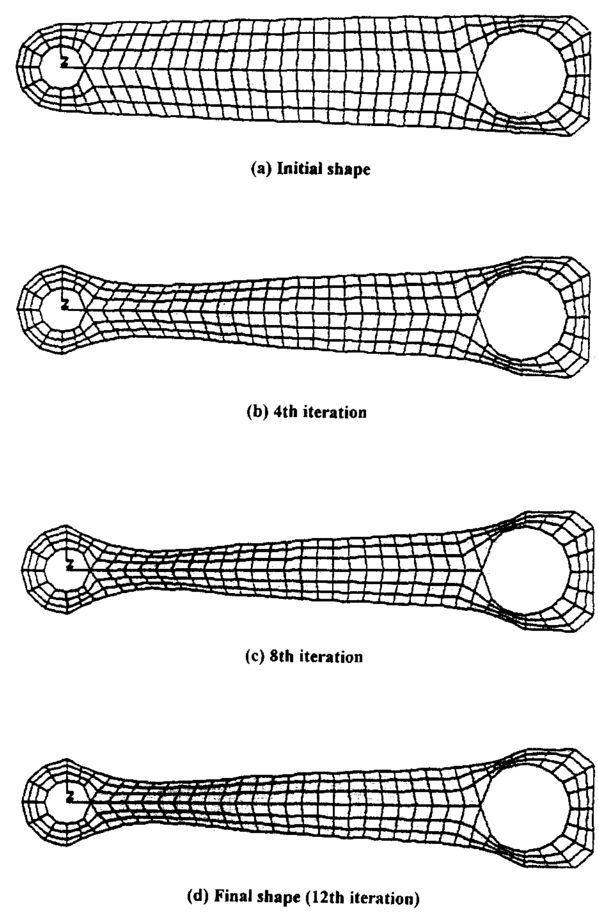


Fig. 7. Torque arm example: finite element models during the iterations [41].

surfaces, know as sculptured or free form surfaces [53]. They arise extensively in automotive die and mold making, aerospace, and appliance industries—to name a few. But the current publications on three-dimensional shape optimization are not on free form surface design. Instead, some of their examples are simply two-dimensional shape optimization problems using three-dimensional finite element models, and some are combined sizing and two-dimensional shape optimization problems, that is, only a sizing variable (e.g. thickness, moment of inertia) was used in the third dimension. It is safe to say, in the opinion of the author, that as of 1994 successful results yet have to be published on three-dimensional free form surface shape optimal design.

Shape optimization problems are different from general optimization problems in that in the former the constraints on stresses, displacements and natural frequencies often cannot be expressed explicitly in terms of shape design variables. They have to be evaluated by structural analysis techniques such as finite element or boundary element methods which usually dominate the computational load in the whole optimization process. Moreover, although many commercially available structural analysis programs (e.g. ANSYS, NASTRAN) provide the gradients of structural responses with respect to sizing variables (e.g., thickness, cross-sectional area, moment of inertia), gradient calculation for general shape design variables is still under research.

The gradients of the structural responses with respect to shape design variables can be calculated by finite difference schemes or by analytical methods. Finite difference schemes are computationally very expensive when the number of design variables is large. Analytical methods are computationally more efficient, but implementing them into a general-purpose analysis program usually means major modification of the program, and the available design variables are still limited. This is the major obstacle for expanding the work on two-dimensional shape optimization into three-dimensional free form surface design.

For example, spline curve is the most popular boundary representation method in two-dimensional shape optimization, as mentioned in Section 3. At least four control points are required to define a cubic B-spline. The natural expansion of a two-dimensional B-spline is a three-dimensional B-spline surface. At least 16 control points are required to defined a bicubic B-spline surface, that is, a total of 48 design variables are required for a bicubic B-spline surface. To calculate the gradients of these 48 design variables will be very tedious, if at all possible, whether using finite difference or analytical methods. And in most applications, one bicubic surface is far from enough to describe a three-dimensional shape.

Most research in shape optimization is focused on interfacing structural analysis programs directly with existing, general-purpose optimization algorithms. However, most general-purpose optimization algorithms require at least first-order information: the gradients of the structural responses with respect to shape design variables. The difficulties in obtaining first-order information in shape optimization problems motivates the need for zero-order optimization algorithms, that is, optimization algorithms that require only function values of the implicit constraints.

Besides solving the difficulties of obtaining gradient information in the first-order methods, another important advantage of zero-order methods is that the implementation of zero-order methods will be completely external to the analysis program, no modification of the analysis program is needed. This is ideal for industry where there is usually no access to the source code of the commercial analysis programs. Moreover, unlike the analytical methods for sensitivity calculation, which are usually developed for one specific formulation, a zero-order method will be equally applicable to various analysis techniques, be they finite element methods, boundary element methods, or even physical experiments using photoelasticity or strain gages.

There exist some general-purpose zero-order optimization algorithms, for example, function comparison methods, and the polytope algorithm [54]. But these methods are so limited that they cannot even deal with constraints. Therefore in shape optimization research, more attention should be paid to the research of zero-order optimization algorithms which are better suitable for shape optimization problems, and mechanical design optimization problems in general.

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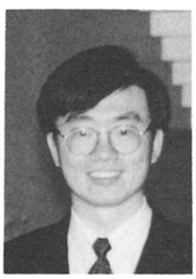

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