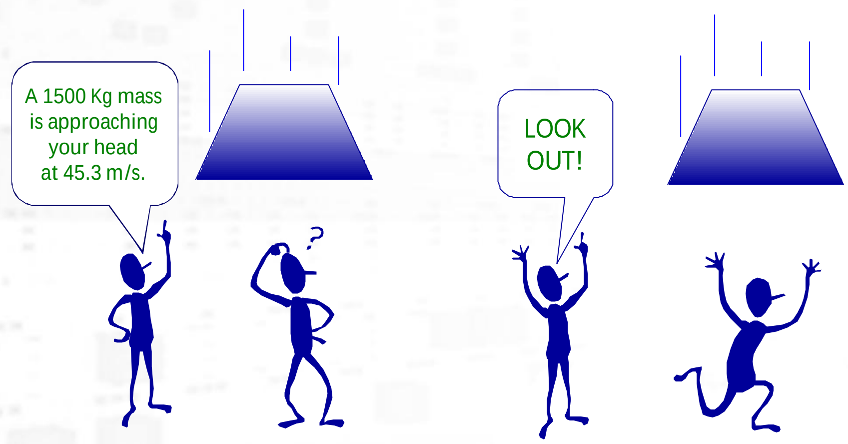
FUZZY SETS

Precision vs. Relevancy



Introduction

- How to simplify very complex systems?
  - *Allow some degree of uncertainty in their description!*
- How to deal mathematically with uncertainty?
  - Using probabilistic theory (*stochastic*).
  - Using the theory of fuzzy sets (*non-stochastic*).
- Proposed in 1965 by Lotfi Zadeh (Fuzzy Sets, *Information Control*, 8, pp. 338-353).
- Imprecision or vagueness in natural language does not imply a loss of accuracy or meaningfulness!

Examples

- *Give travel directions in terms of city blocks* OR *in meters?*
- *The day is sunny* OR *the sky is covered by 5% of clouds?*
  - If the sky is covered by 10% of clouds is still *sunny*?
  - And 25%?
  - And 50%?
  - Where to draw the line from *sunny* to *not sunny*?
  - Member and not member or membership degree?

Probability vs. Possibility

- Event u: Hans ate X eggs for breakfast.
- Probability distribution: $P_X(u)$
- Possibility distribution: $\pi_X(u)$

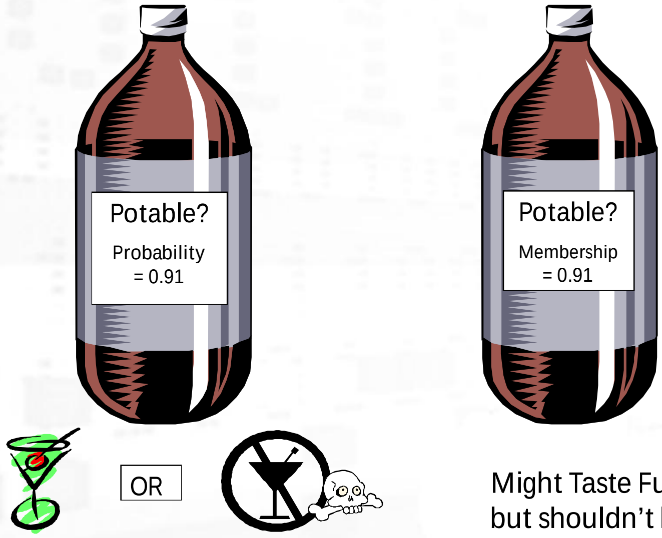
u	1	2	3	4	5	6	7	8
$P_X(u)$	0.1	0.8	0.1	0	0	0	0	0
$\pi_X(u)$	1	1	1	1	0.8	0.6	0.4	0.2

Probability vs. Fuzzy Set Membership

- You're lost in the Outback; Dying of Thirst
- You Come Upon Two Bottles Containing Liquid
- Which One Will You Choose?

  How Will You Process the Information?

Probability vs. Fuzzy Set Membership



Might Taste Funky,
but shouldn't kill you

Applications of fuzzy sets

- Fuzzy mathematics (measures, relations, topology, etc.)
- Fuzzy logic and AI (approximate reasoning, expert systems, etc.)
- Fuzzy systems
  - Fuzzy modeling
  - Fuzzy control, etc.
- Fuzzy decision making
  - Multi-criteria optimization
  - Optimization techniques

Classical set theory

- Set: collection of objects with a common property.
- Examples:
  - Set of basic colors:

    A = {red, green, blue}

  - Set of positive integers:

    $A = \{x \in \mathbb{Z} \mid x \geq 0\}$

  - A line in $\mathbb{R}^3$:

    $A = \{(x,y,z) \mid ax + by + cz + d = 0\}$

Representation of sets

- Enumeration of elements: $A = \{x_1, x_2, \ldots, x_n\}$
- Definition by property P: $A = \{x \in X \mid P(x)\}$
- *Characteristic function* $\mu_{A(x)}: X \to \{0,1\}$

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \text{ is member of } A \\ 0, & \text{if } x \text{ is not member of } A \end{cases}$$

- Example:
  - Set of odd numbers: $\mu_A(x) = x \bmod 2$

Set operations

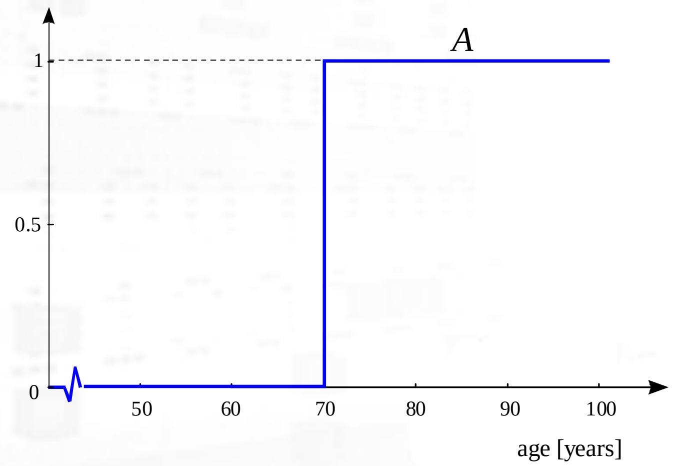
- Intersection: $C = A \cap B$
  - C contains elements that belong to A and B
  - Characteristic function: $\mu_C = \min(\mu_A , \mu_B) = \mu_A \cdot \mu_B$
- Union: $C = A \cup B$
  - C contains elements that belong to A or to B
  - Characteristic function: $\mu_C = \max(\mu_A , \mu_B)$
- Complement: $C = \bar{A}$
  - C contains elements that do not belong to A
  - Characteristic function: $\mu_C = 1 - \mu_A$

Fuzzy sets

- Represent *uncertain* (vague, ambiguous, etc.) knowledge in the form of propositions, rules, etc.
- Propositions:
  - expensive cars,
  - cloudy sky,...
- Rules (decisions):
  - Want to buy a big and new house for a low price.
  - If the temperature is *low*, then *increase* the heating.
  - ...

Classical set

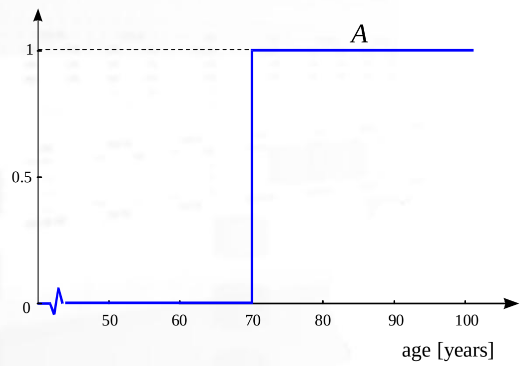
- Example: set of *old people* $A = \{age \mid age \geq 70\}$



42

Logic propositions

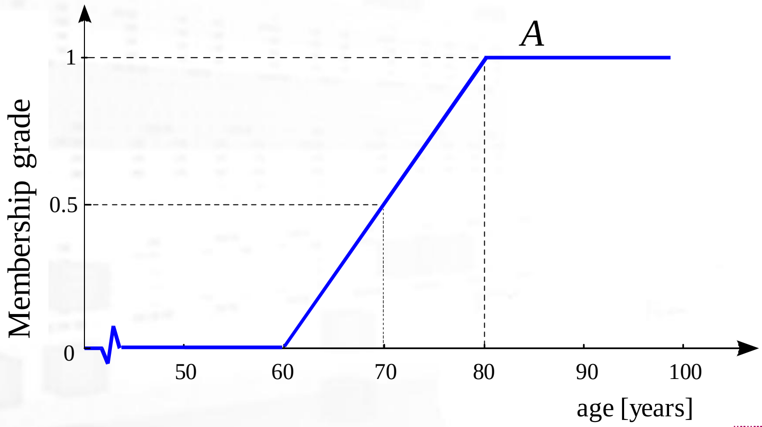
- "Nick is old" ... true or false
- Nick's age:
  - $age_{\text{Nick}} = 70$, $\mu_A(70) = 1$ (true)
  - $age_{\text{Nick}} = 69.9$, $\mu_A(69.9) = 0$ (false)



43

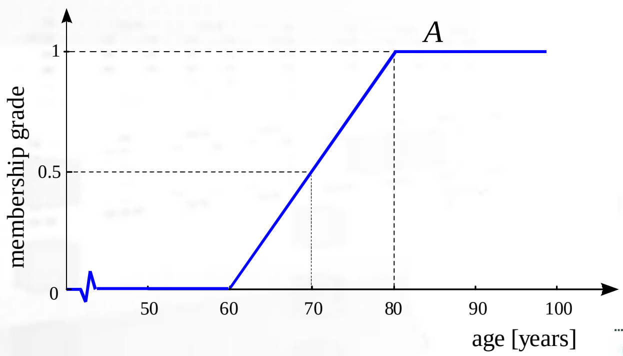
Fuzzy set

- *Graded membership*, element belongs to a set to a certain degree.

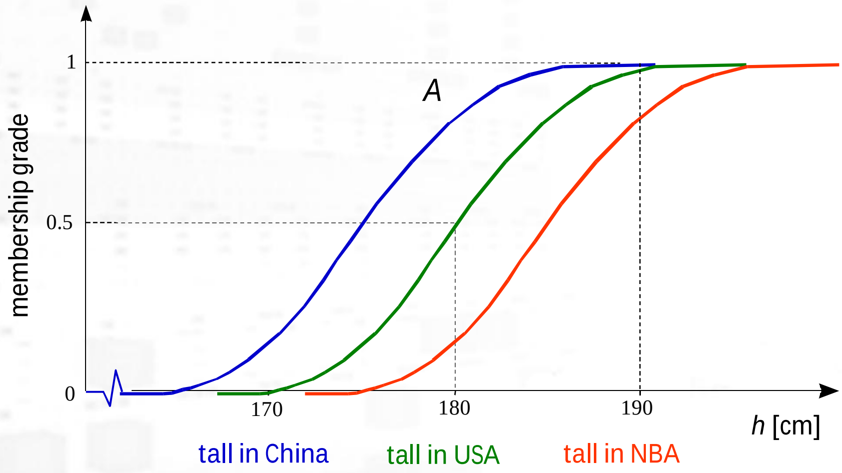


Fuzzy proposition

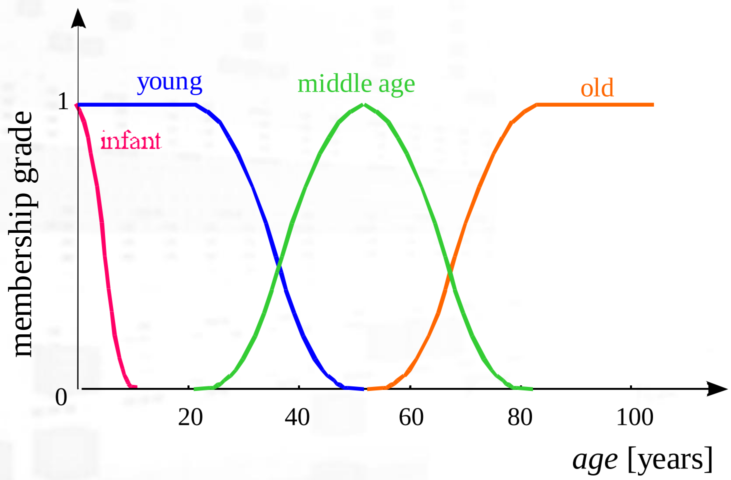
- "Nick is old"... degree of truth
  - $age_{\text{Nick}} = 70$, $\mu_A(70) = 0.5$
  - $age_{\text{Nick}} = 69.9$, $\mu_A(69.9) = 0.49$
  - $age_{\text{Nick}} = 90$, $\mu_A(90) = 1$



Context dependent

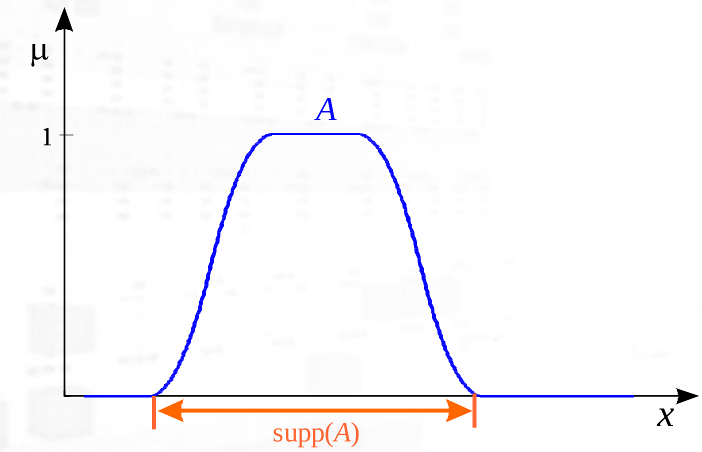


Typical linguistic values



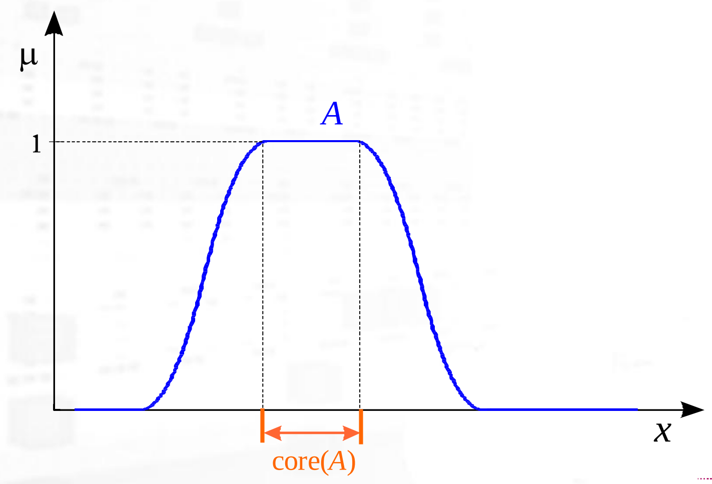
Support of a fuzzy set

- $\mathrm{supp}(A) = \{x \in X \mid \mu_A(x) > 0\}$



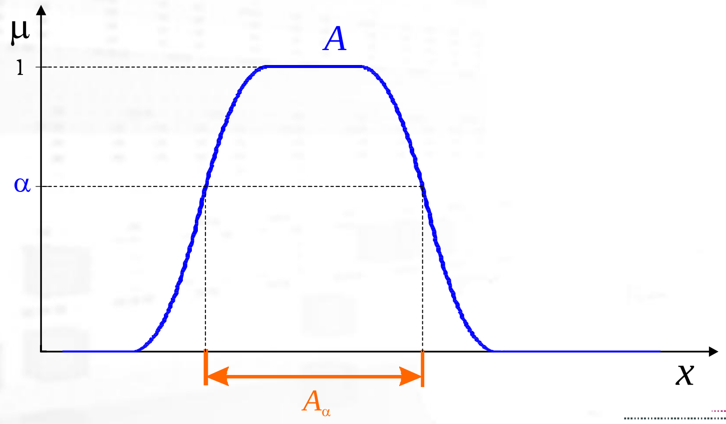
Core (nucleous, kernel)

- $\mathrm{core}(A) = \{x \in X \mid \mu_A(x) = 1\}$



α-cut of a fuzzy set

- Crisp set: $A_\alpha = \{\, x \in X \mid \mu_A(x) \geq \alpha \}$
- *Strong* α-cut: $A_\alpha = \{\, x \in X \mid \mu_A(x) > \alpha \}$



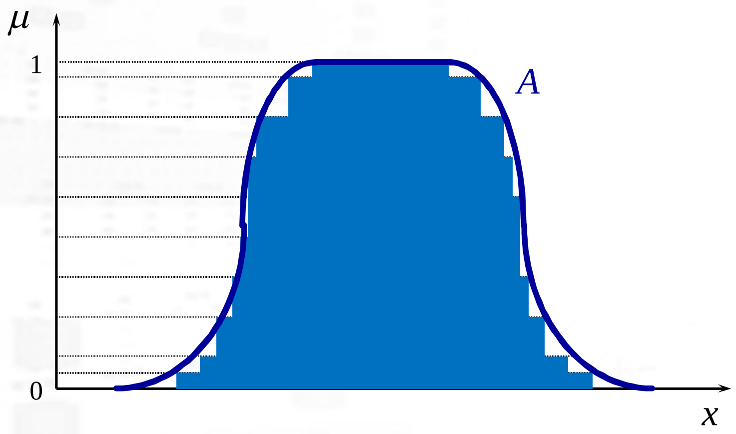
Resolution principle

- Every fuzzy set A can be uniquely represented as a collection of α-level sets according to

$$\mu_A(x) = \sup_{\alpha \in [0,1]} [\alpha \mu_{A_\alpha}(x)]$$

- Resolution principle implies that fuzzy set theory is a generalization of classical set theory, and that its results can be represented in terms of classical set theory.

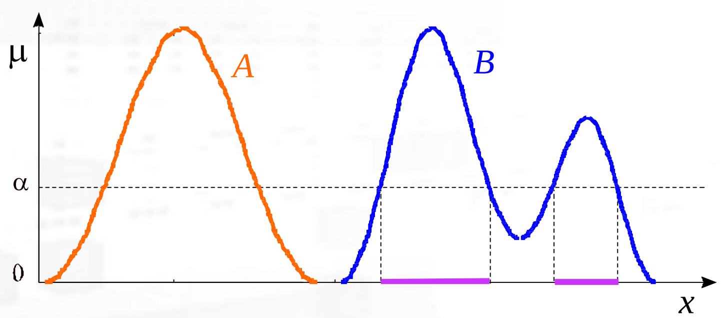
Resolution principle



Other properties

- Height of a fuzzy set: hgt(A) = sup $\mu_A(x)$, $x \in X$
- Fuzzy set is normal(ized) when hgt(A) = 1.
- A fuzzy set A is convex iff $\forall\ x,y \in X$ and $\lambda \in [0,1]$:

$$\mu_A(\lambda\ x + (1 - \lambda)\ y) \geq \min(\mu_A(x),\ \mu_A(y))$$



Other properties (2)

- **Fuzzy singleton:** single point $x \in X$ where $\mu_A(x) = 1$.
- **Fuzzy number:** fuzzy set in $\mathbb{R}$ that is *normal* and *convex*.
- Two fuzzy sets are *equal* ($A = B$) iff:

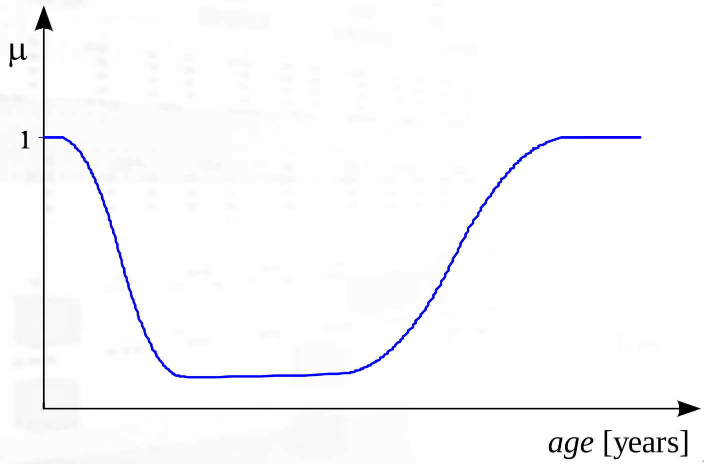
$$\forall x \in X,\ \mu_A(x) = \mu_B(x)$$

- A is a *subset* of B iff:

$$\forall x \in X,\ \mu_A(x) \le \mu_B(x)$$

Non-convex fuzzy sets

- **Example**: car insurance risk



Representation of fuzzy sets

Discrete Universe of Discourse:

- Point-wise as a list of membership/element pairs:
  - $A = \mu_A(x_1)/x_1 + \ldots + \mu_A(x_n)/x_n = \Sigma_i \, \mu_A(x_i)/x_i$
  - $A = \{\mu_A(x_1)/x_1, \ldots, \mu_A(x_n)/x_n\} = \{\mu_A(x_i)/x_i \mid x_i \in X\}$
- As a list of α-level/ α-cut pairs:
  - $A = \{\alpha_1/A_{\alpha_1}, \ldots, \alpha_n/A_{\alpha_n}\} = \{\alpha_i/A_{\alpha_i} \mid \alpha_i \in [0,1]\}$

Representation of fuzzy sets

Continuous Universe of Discourse:

- $A = \int_X \mu_A(x)/x$
- Analytical formula: $\mu_A(x) = \dfrac{1}{1+x^2}, \quad x \in \mathbb{R}$
- Various possible notations:
  - $\mu_A(x)$, $A(x)$, A, a, etc.

Examples

Discrete universe

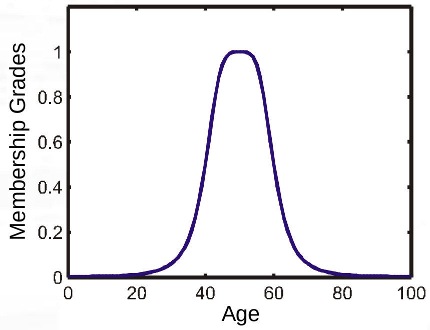
- Fuzzy set A ="sensible number of children".
  - number of children: $X = \{0, 1, 2, 3, 4, 5, 6\}$
  - $A = 0.1/0 + 0.3/1 + 0.7/2 + 1/3 + 0.6/4 + 0.2/5 + 0.1/6$
- Fuzzy set C = "desirable city to live in"
  - X = {SF, Boston, LA} (discrete and non-ordered)
  - C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}

Examples

Continuous universe

- Fuzzy set B = "about 50 years old"
  - $X = \mathbb{R}^+$ (set of positive real numbers)
  - $B = \{(x, \mu_B(x)) \mid x \in X\}$

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^4}$$



Complement of a fuzzy set

$$c: [0,1] \rightarrow [0,1]; \quad \mu_A(x) \rightarrow c(\mu_A(x))$$

- Fundamental axioms

1. *Boundary conditions* - c behaves as the ordinary complement

$$c(0) = 1; \quad c(1) = 0$$

2. *Monotonic non-increasing*

$$\forall\, a,b \in [0,1], \text{ if } a < b, \text{ then } c(a) \geq c(b)$$

Complement of a fuzzy set

Other axioms:

- c is a *continuous* function.
- c is *involutive*, which means that

$$c(c(a)) = a, \quad \forall a \in [0,1]$$

Complement of a fuzzy set

Equilibrium point

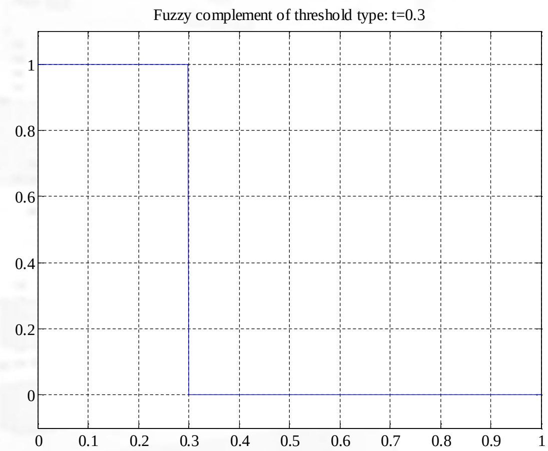
$$c(a) = a = e_c, \quad \forall a \in [0,1]$$

- Each complement has at most one equilibrium.
- If c is a continuous fuzzy complement, it has one equilibrium point.

Examples of fuzzy complements

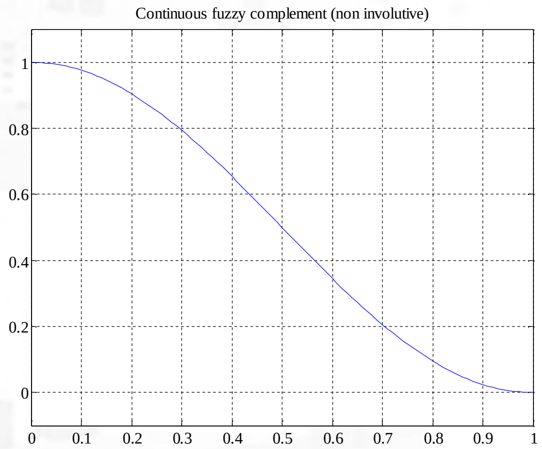
- Satisfying only fundamental axioms:

$$c(a) = \begin{cases} 1, & \text{if } a \le t \\ 0, & \text{if } a > t \end{cases}$$



Examples of fuzzy complements

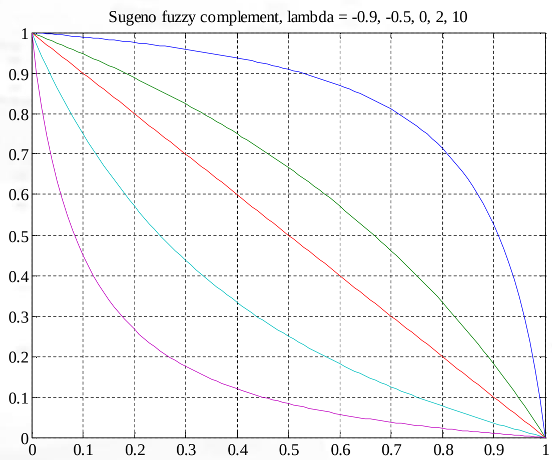
- Satisfying fundamental axioms and continuity:



$$c(a)=\frac{1}{2}(1+\cos\pi a)$$

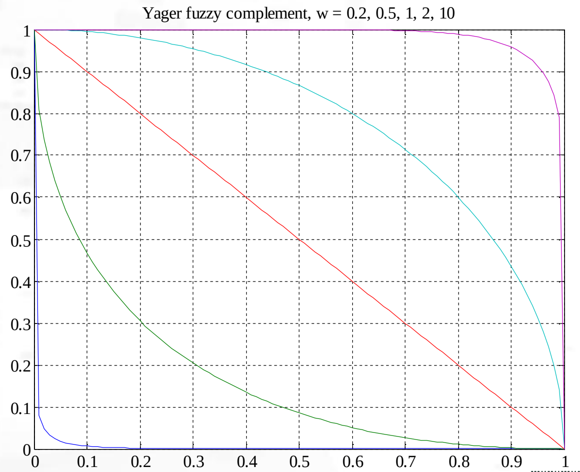
Examples of fuzzy complements

- Sugeno complement: $$c_\lambda(a)=\frac{1-a}{1+\lambda a},\ \lambda\in]-1,\infty]$$



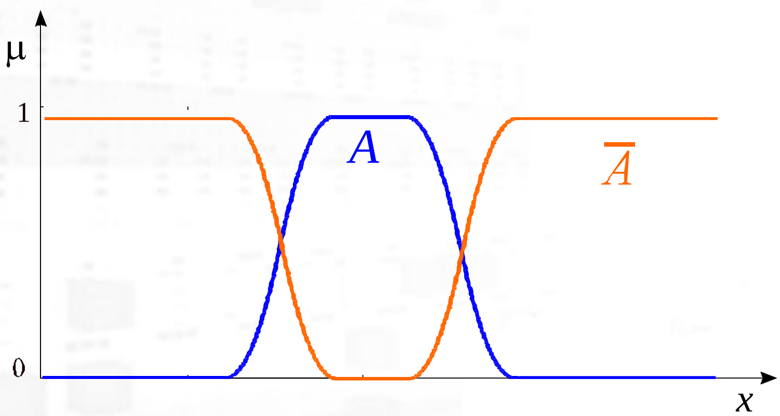
Examples of fuzzy complement

- Yager complement: $c_w(a) = \left(1 - a^w\right)^{1/w}, \quad w \in]0, \infty]$

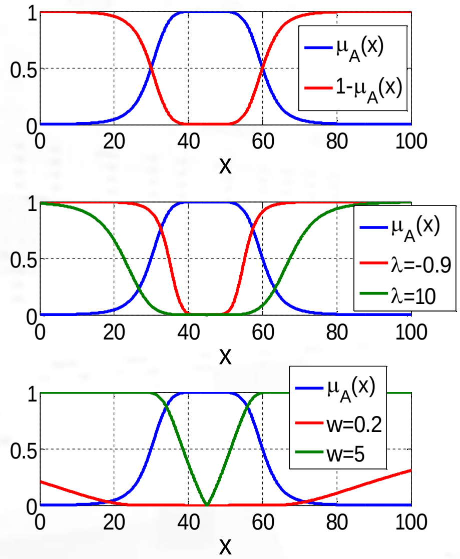


Representation of complement

- $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$



Representation of complement



Intersection of fuzzy sets

$$i: [0,1] \times [0,1] \to [0,1];$$
$$\mu_{A \cap B}(x) \to i(\mu_A(x),\ \mu_B(x))$$

- Fundamental axioms: *triangular norm* or *t*-norm

1. *Boundary conditions* - i behaves as the classical intersection

$$i(1,1) = 1;$$
$$i(0,1) = i(1,0) = i(0,0) = 0$$

2. *Commutativity*

$$i(a,b) = i(b,a)$$

Intersection of fuzzy sets

3. *Monotonicity*

   If $a \le a'$ and $b \le b'$, then $i(a,b) \le i(a',b')$

4. *Associativity*

   $i(i(a,b),c) = i(a,i(b,c))$

- Other axioms:
  - i is a *continuous* function.
  - $i(a,a) = a$ (idempotent).

Examples of fuzzy conjunctions

- Zadeh

  $$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

- Probabilistic

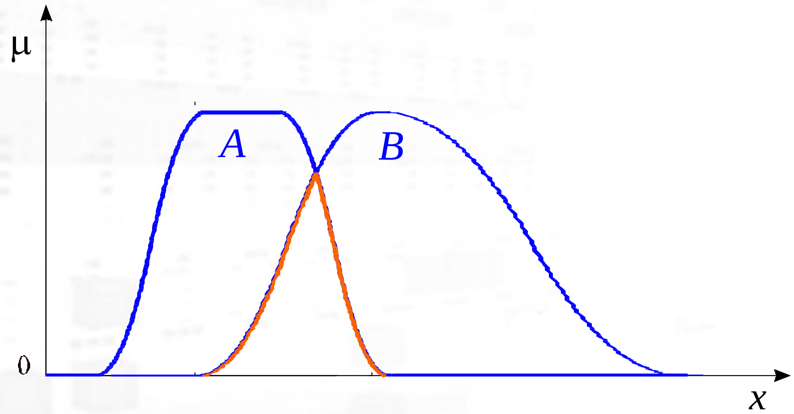
  $$\mu_{A\cap B}(x) = \mu_A(x)\ .\ \mu_B(x)$$

- Lukaziewicz

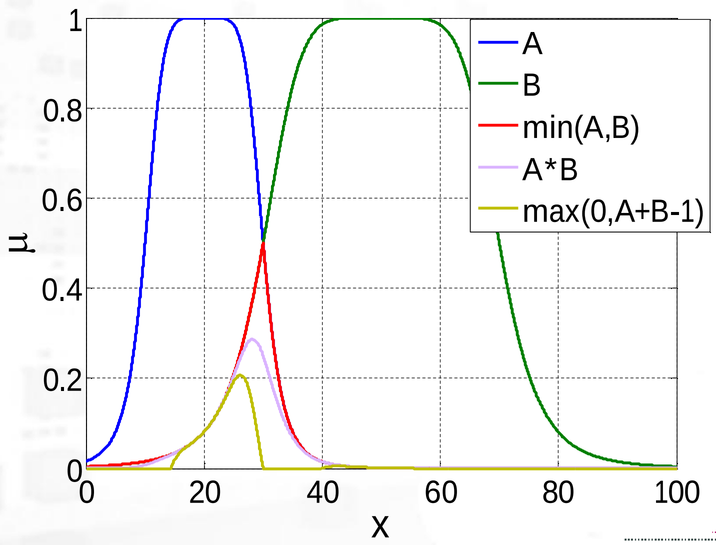
  $$\mu_{A\cap B}(x) = \max(0, \mu_A(x) + \mu_B(x) - 1)$$

Intersection of fuzzy sets

- $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$



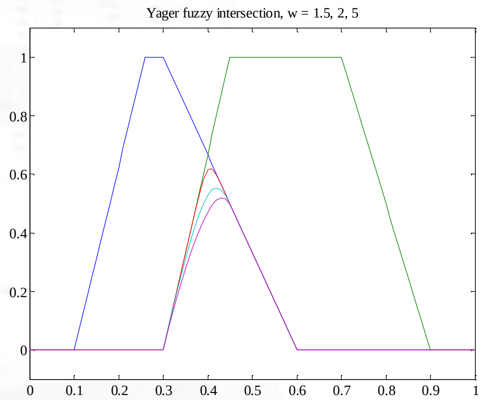
Intersection of fuzzy sets



Yager *t*-norm

- Example of *week* and *strong* intersections:

$$i_w(a,b) = 1 - \min\left[1, \left((1-a)^w + (1-b)^w\right)^{1/w}\right], \quad w \in]0,\infty]$$



Union of fuzzy sets

$$u: [0,1] \times [0,1] \to [0,1];$$

$$\mu_{A\cup B}(x) \to u(\mu_A(x), \mu_B(x))$$

- Fundamental axioms: *triangular co-norm* or *s*-norm

1. *Boundary conditions* - u behaves as the classical union

$$u(0,0) = 0;$$

$$u(0,1) = u(1,0) = u(1,1) = 1$$

2. *Commutativity*

$$u(a,b) = u(b,a)$$

Union of fuzzy sets

3. *Monotonicity*

   If $a \le a'$ and $b \le b'$, then $u(a,b) \le u(a',b')$

4. *Associativity*

   $u(u(a,b),c) = u(a,u(b,c))$

- Other axioms:
  - u is a *continuous* function.
  - $u(a,a) = a$ (idempotent).

Examples of fuzzy disjunctions

- Zadeh

  $$\mu_{A\cup B}(x) = \max(\mu_A(x),\mu_B(x))$$

- Probabilistic

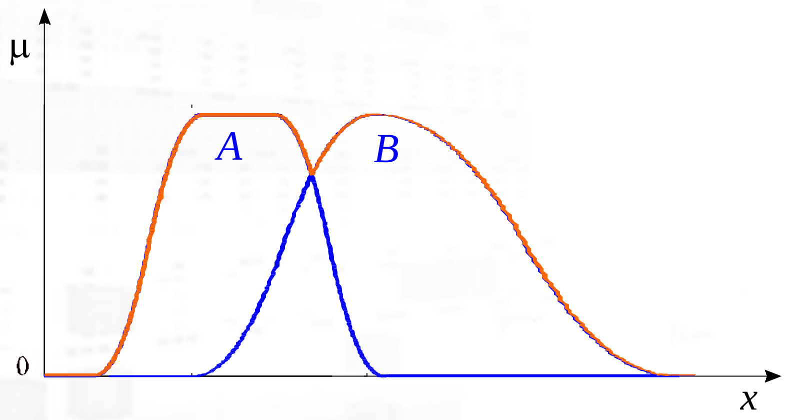
  $$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \,.\, \mu_B(x)$$

- Lukasiewicz

  $$\mu_{A\cup B}(x) = \min(1, \mu_A(x) + \mu_B(x))$$

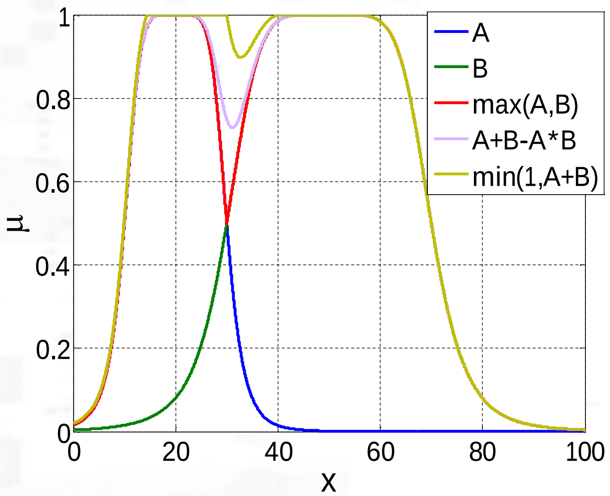
Union of fuzzy sets

- $\mu_{A\cup B}(x) = \max(\mu_A(x),\ \mu_B(x))$



78

Union of fuzzy sets

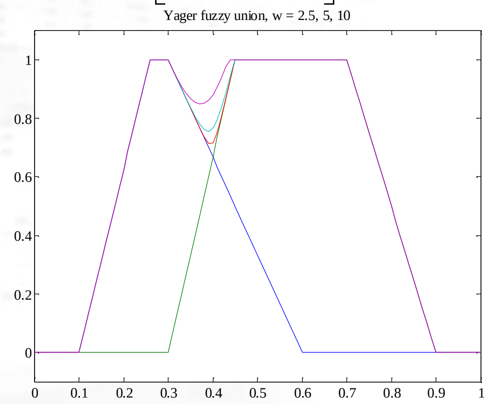


79

Yager s-norm

- Example of *week* and *strong* disjunctions:

$$u_w(a,b) = \min\left[1, \left(a^w + b^w\right)^{1/w}\right], \; w \in]0, \infty]$$



General aggregation operations

$$h: [0,1]^n \to [0,1];$$

$$\mu_A(x) \to h(\mu_{A_1}(x), \dots, \mu_{A_n}(x))$$

- Axioms

1. *Boundary conditions*

   $h(0,\dots,0) = 0$

   $h(1,\dots,1) = 1$

2. *Monotonic non-decreasing*

   For any pair $a_i, b_i \in [0,1], i \in \mathbb{N}$

   If $a_i \geq b_i$ then $h(a_i) \geq h(b_i)$

General aggregation operations

- Other axioms:
  - h is a *continuous* function.
  - h is a *symmetric* function in all its arguments:

    $$h(a_i) = h(a_{p(i)})$$

    for any permutation p on $\mathbb{N}$

Averaging operations

- When all the four axioms hold:

$$\min(a_1,\ldots,a_n) \le h(a_1,\ldots,a_n) \le \max(a_1,\ldots,a_n)$$

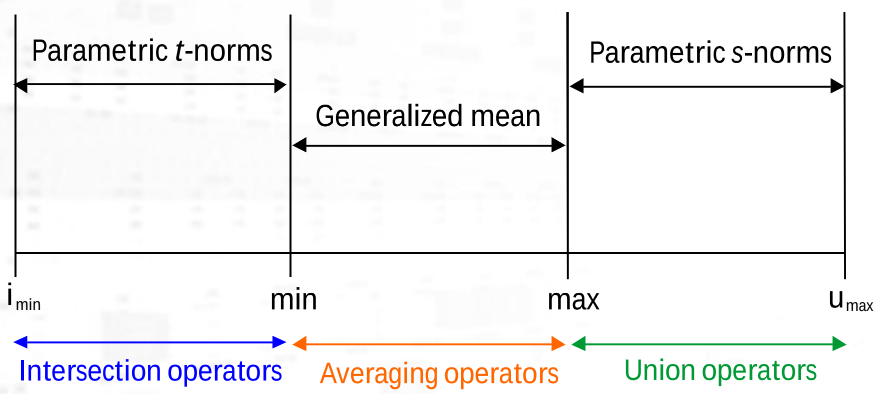
- Operator covering this range: Generalized mean

$$h_\alpha(a_1,\ldots,a_n) = \left(\frac{\left(a_1^\alpha + \ldots + a_n^\alpha\right)}{n} \right)^{1/\alpha}$$

Generalized mean

- Typical cases:
  - Lower bound: $h_{-\infty} = \min(a_1, \ldots, a_n)$
  - *Geometric mean*: $h_0 = (a_1 \cdot a_2 \cdot \ldots \cdot a_n)^n$
  - *Harmonic mean*: $h_{-1} = \dfrac{n}{\dfrac{1}{a_1} + \ldots + \dfrac{1}{a_n}}$
  - *Arithmetic mean*: $h_1 = \dfrac{a_1 + \ldots + a_n}{n}$
  - Upper bound: $h_{\infty} = \max(a_1, \ldots, a_n)$

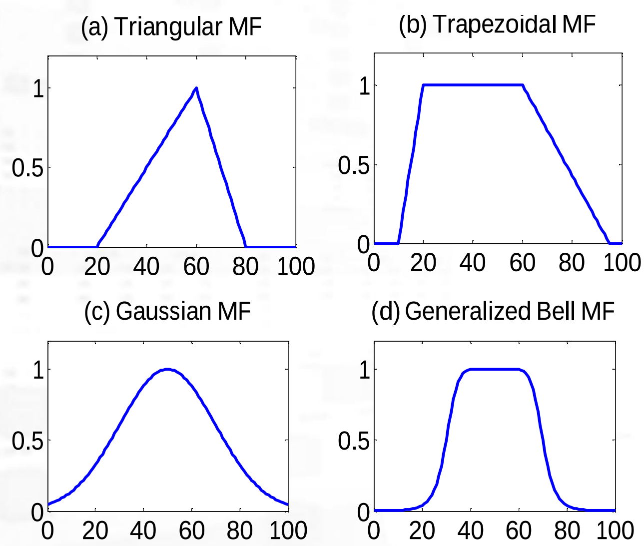
Fuzzy aggregation operations



Membership functions (MF)

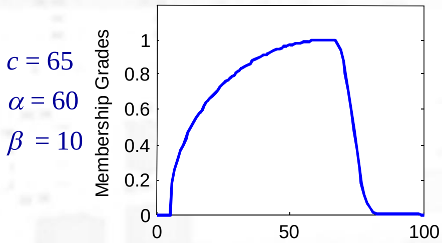
- Triangular MF: $Tr(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$
- Trapezoidal MF: $Tp(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$
- Gaussian MF: $Gs(x;a,b,c) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$
- Generalized bell MF: $Bell(x;a,b,c) = \dfrac{1}{1+\left|\frac{x-c}{b}\right|^{2a}}$

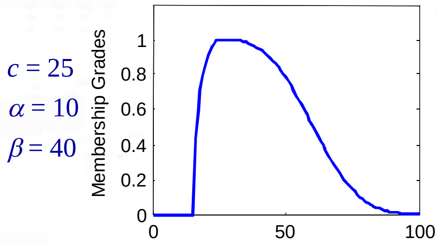
Membership functions



Left-right MF

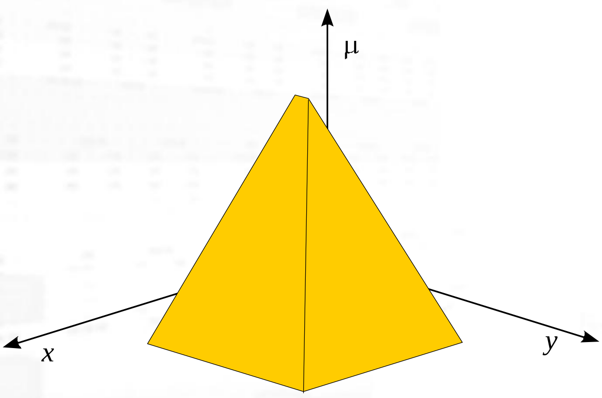
$$LR(x;c,\alpha,\beta) = \begin{cases} F_L\left(\dfrac{c-x}{\alpha}\right), x < c \\ F_R\left(\dfrac{x-c}{\beta}\right), x \geq c \end{cases}$$



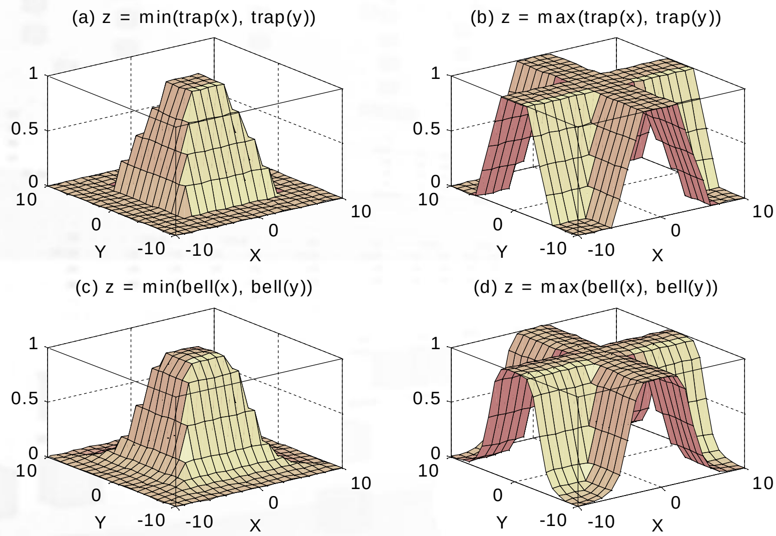


Two-dimensional fuzzy sets

$$A = \int_{X\times Y} \mu_A(x,y) = \{\mu_A(x,y) | (x,y) \in X \times Y\}$$



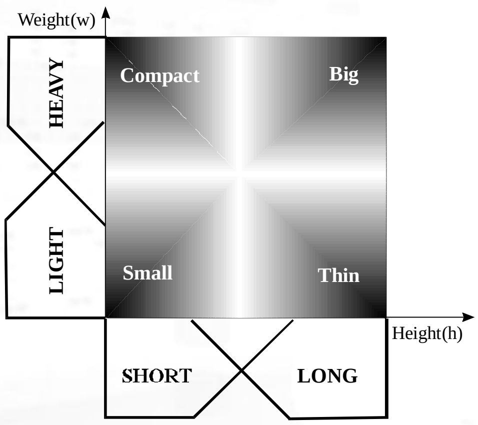
2-D membership functions



Compound fuzzy propositions

- *Small* = *Short* and *Light* (conjunction)

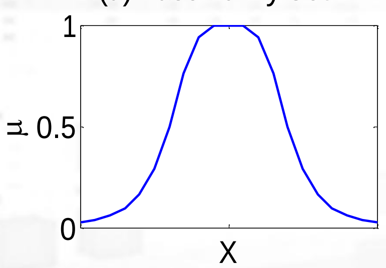
$$\mu_{Small}(h,w) = \mu_{Short}(h) \cap \mu_{Light}(w)$$

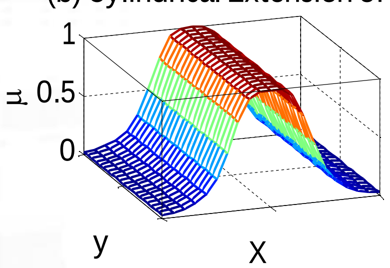


Cylindrical extension

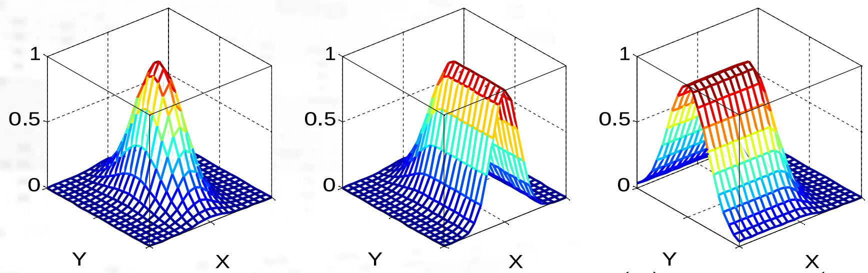
- Cylindrical extension of fuzzy set A in X into Y results in a two-dimensional fuzzy set in $X \times Y$, given by

$$\text{cext}_y(A) = \int_{X\times Y} \mu_A(x)/(x,y) = \left\{\mu_A(x)/(x,y) \middle| (x,y) \in X \times Y\right\}$$





Projection



$\mu_R(x,y)$ $\quad \mu_A(x) = \max_y \mu_R(x,y)$ $\quad \mu_B(y) = \max_x \mu_R(x,y)$

Cartesian product

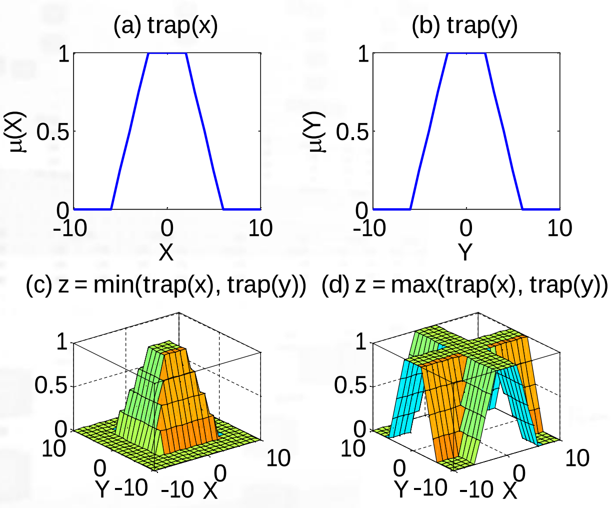
- Cartesian product of fuzzy sets A and B is a fuzzy set in the product space $X \times Y$ with membership

$$\mu_{A\times B}(x,y) = \min(\mu_A(x), \mu_B(y))$$

- Cartesian co-product of fuzzy sets A and B is a fuzzy set in the product space $X \times Y$ with membership

$$\mu_{A+B}(x,y) = \max(\mu_A(x), \mu_B(y))$$

Cartesian product



Classical relations

- Classical relation R(X_1, X_2,..., X_n) is a subset of the Cartesian product:

$$\mathrm{R}(X_1, X_2, \ldots, X_n) \subset X_1 \times X_2 \times \ldots \times X_n$$

- Characteristic function:

$$\mu_{\mathrm{R}}(x_1, x_2, \ldots, x_n) = \begin{cases} 1, & \text{iff } (x_1, x_2, \ldots, x_n) \in \mathrm{R} \\ 0, & \text{otherwise} \end{cases}$$

Example

- X = {English, French}
- Y = {dollar, pound, euro}
- Z = {USA, France, Canada, Britain, Germany}
- R(X, Y, Z) = {(English, dollar, USA),
  (French, euro, France), (English, dollar,Canada),
  (French, dollar, Canada), (English, pound,Britain)}

Matrix representation

$$
\begin{array}{c|ccccc}
 & \text{USA} & \text{Fra} & \text{Can} & \text{Brit} & \text{Ger} \\
\text{Dollar} & 1 & 0 & 1 & 0 & 0 \\
\text{Pound} & 0 & 0 & 0 & 1 & 0 \\
\text{Euro} & 0 & 0 & 0 & 0 & 0
\end{array}
$$

English

$$
\begin{array}{c|ccccc}
 & \text{USA} & \text{Fra} & \text{Can} & \text{Brit} & \text{Ger} \\
\text{Dollar} & 0 & 0 & 1 & 0 & 0 \\
\text{Pound} & 0 & 0 & 0 & 0 & 0 \\
\text{Euro} & 0 & 1 & 0 & 0 & 0
\end{array}
$$

French

Fuzzy relation

- Fuzzy relation:

$$\text{R}: X_1 \times X_2 \times \ldots \times X_n \rightarrow [0,1]$$

- Each tuple $(x_1, x_2, \ldots, x_n)$ has a *degree of membership*.
- Fuzzy relation can be represented by an *n*-dimensional *membership function* (continuous space) or a *matrix* (discrete space).
- Examples:
  - x is close to y
  - x and y are similar
  - x and y are related (dependent)

Discrete examples

- Relation R "*very far*" between X = {New York, Lisbon} and Y = {New York, Beijing, London}:

  R(x,y) = 0/(NY,NY) + 1/(NY,Beijing) +
  0.6/(NY,London) + 0.5/(Lisbon,NY) +
  0.8/(Lisbon,Beijing) + 0.1/(Lisbon,London)

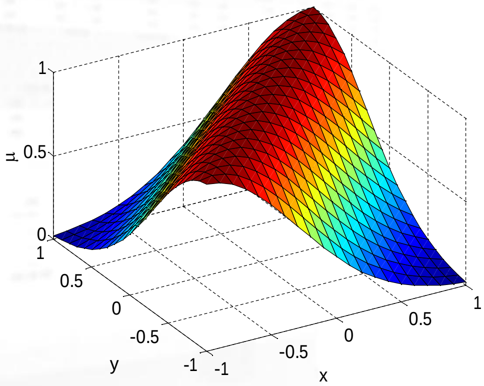
- Relation: "*is an important trade partner of*"

	Holland	Germany	USA	Japan
Holland	1	0,9	0,5	0,2
Germany	0,3	1	0,4	0,2
USA	0,3	0,4	1	0,7
Japan	0,6	0,8	0,9	1

Continuous example

- R: $x \approx y$ ("x is approximately equal to y")

$$\mu_{\Re}(x,y) = e^{-(x-y)^2}$$



Composition of relations

- R(X,Z) = P(X,Y) $\circ$ Q(Y,Z)

Conditions:

- (x,z) $\in$ R iff exists $y \in Y$ such that
- (x,y) $\in$ P and (y,z) $\in$ Q.

- Max-min composition

$$\mu_{\mathrm{P}\circ\mathrm{Q}}(x,z) = \max_{y\in Y} \min\left[\mu_{\mathrm{P}}(x,y), \mu_{\mathrm{Q}}(y,z)\right]$$

Properties

- Associativity:

$$\mathrm{R}\circ(\mathrm{S}\circ\mathrm{T}) = (\mathrm{R}\circ\mathrm{S})\circ\mathrm{T}$$

- Distributivity over union:

$$\mathrm{R}\circ(\mathrm{S}\cup\mathrm{T}) = (\mathrm{R}\circ\mathrm{S})\cup(\mathrm{R}\circ\mathrm{T})$$

- Weak distributivity over intersection:

$$\mathrm{R}\circ(\mathrm{S}\cap\mathrm{T}) \subseteq (\mathrm{R}\circ\mathrm{S})\cap(\mathrm{R}\circ\mathrm{T})$$

- Monotonicity:

$$\mathrm{S}\subseteq\mathrm{T} \Rightarrow (\mathrm{R}\circ\mathrm{S}) \subseteq (\mathrm{R}\circ\mathrm{T})$$

Other compositions

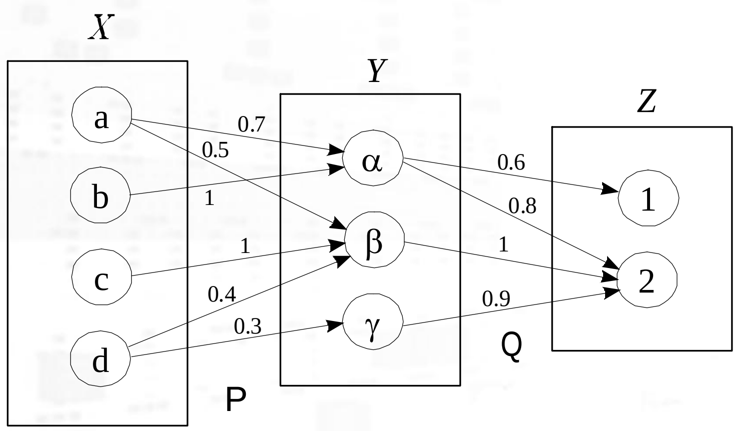
- Max-prod composition

$$\mu_{P \circ Q}(x,z) = \max_{y \in Y} \left(\mu_P(x,y) \cdot \mu_Q(y,z) \right)$$

- Max-*t* composition

$$\mu_{P \circ Q}(x,z) = \max_{y \in Y} t\left(\mu_P(x,y), \mu_Q(y,z) \right)$$

Example



Example

- Composition R = P ∘ Q

x	z	$\mu_R(x,z)$
a	1	0.6
a	2	0.7
b	1	0.6
b	2	0.8
c	2	1
d	2	0.4

- Composition R = P ⊗ Q?

Matrix notation examples

$$\begin{bmatrix} 0.3 & 0.5 & 0.8 \\ 0 & 0.7 & 1 \\ 0.4 & 0.6 & 0.5 \end{bmatrix} \circ \begin{bmatrix} 0.9 & 0.5 & 0.7 & 0.7 \\ 0.3 & 0.2 & 0 & 0.9 \\ 1 & 0 & 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.3 & 0.5 & 0.5 \\ 1 & 0.2 & 0.5 & 0.7 \\ 0.5 & 0.4 & 0.5 & 0.6 \end{bmatrix}$$

$$\begin{bmatrix} 0.3 & 0.5 & 0.8 \\ 0 & 0.7 & 1 \\ 0.4 & 0.6 & 0.5 \end{bmatrix} \otimes \begin{bmatrix} 0.9 & 0.5 & 0.7 & 0.7 \\ 0.3 & 0.2 & 0 & 0.9 \\ 1 & 0 & 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.15 & 0.4 & 0.45 \\ 1 & 0.14 & 0.5 & 0.63 \\ 0.5 & 0.2 & 0.28 & 0.54 \end{bmatrix}$$

Relations on the same universe

- Let R be a relation defined on $U \times U$, then it is called:
  - Reflexive, if $\forall\ u \in U$, the pair $(u,u) \in$ R
  - Anti-reflexive, if $\forall\ u \in U$, $(u,u) \notin$ R
  - Symmetric, if $\forall\ u,v \in U$, if $(u,v) \in$ R, then $(v,u) \in$ R too
  - Anti-symmetric, if $\forall\ u,v \in U$, if (u,v) and $(v,u) \in$ R, then $u = v$
  - Transitive, if $\forall\ u,v,w \in U$, if (u,v) and $(v,w) \in$ R, then $(u,w) \in$R too.

Examples

- R is an *equivalence relation* if it is reflexive, symmetric and transitive.
- R is a *partial order relation* if it is reflexive, anti-symmetric and transitive.
- R is a *total order relation* if R is a partial order relation, and $\forall\ u, v \in U$, either (u,v) or $(v,u) \in$ R.
  - Examples:
    - The subset relation on sets ($\subseteq$) is a partial order relation.
    - The relation $\leq$ on $\mathbb{N}$ is a total order relation.