Introduction to the Discrete Fourier Transform (DFT)

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1 Prior knowledge

Euler's formula:

$$e^{ix} = \cos x + i \sin x \tag{1}$$

Cross-correlation at lag 0:

$$f * g\ (0) = \int_{-\infty}^{\infty} f^*(t) g(t) dt \tag{2}$$

$$f * g\ [0] = \sum_{n=-\infty}^{\infty} f^*[n] g[n] \tag{3}$$

2 Fourier series (FS)

$$S_{FS}[k] = \int_{-N/2}^{N/2} s(t) \cdot e^{-i2\pi \frac{k}{N} t} dt, \qquad \forall\ k \in \mathbb{Z}, t \in \mathbb{R} \tag{4}$$

$$s_{FS}(t) = \sum_{k=-\infty}^{\infty} S_{FS}[k] \cdot e^{i2\pi \frac{k}{N} t}, \qquad \forall\ k \in \mathbb{Z}, t \in \mathbb{R} \tag{5}$$

$$s_{FS}(t) = A_0 + \sum_{k=1}^{\infty} \left(A_n \cos(2\pi \frac{k}{N} t) + B_n \sin(2\pi \frac{k}{N} t) \right), \qquad \forall\ t \in \mathbb{R}, k \in \mathbb{Z} \tag{6}$$

3 Fourier transform (FT)

$$S_{FT}(f) = \int_{-\infty}^{\infty} s(t) \cdot e^{-i2\pi f t} dt, \qquad t, f, \in \mathbb{R} \tag{7}$$

$$s(t) = \int_{-\infty}^{\infty} S_{FT}(f) \cdot e^{i2\pi f t} dt, \qquad t, f, \in \mathbb{R} \tag{8}$$

4 Discrete Time Fourier Transform (DTFT)

Sampling of s(t) with finite time interval T:

$$dt \approx T \quad \Rightarrow \quad t_n = Tn \quad \Rightarrow \quad s(t) \cdot dt = s(Tn) \cdot T, \qquad n \in \mathbb{Z} \tag{9}$$

Let $s[n] = s(Tn) \cdot T$:

$$S_{DTFT}(f) = \sum_{n=-\infty}^{\infty} s[n] \cdot e^{-i2\pi fTn} \tag{10}$$

5 Discrete Fourier Transform (DFT)

Assumption: Signal is periodic with period N:

$$s[n + mN] = s[n] \qquad m \in \mathbb{Z} \tag{11}$$

The evenly sampled frequency range is given by $fT = \frac{k}{N}$

$$S_{DFT}[k] = \sum_{n=0}^{N-1} s[n] \cdot e^{-i2\pi \frac{k}{N} n}, \qquad k \in 0, 1, \ldots N-1 \tag{12}$$

6 Matrix form

In Equation 12 we see two running indices: $n \in 0, 1, \ldots N-1$ (time-domain) and $k \in 0, 1, \ldots, N-1$ (frequency domain). This suggests that we can formulate the computation in matrix form:

$$\begin{bmatrix} S_{DFT}[k=0] \\ S_{DFT}[k=1] \\ \ldots \\ S_{DFT}[k=N-1] \end{bmatrix} = \tag{13}$$

$$\begin{bmatrix} s[0] \cdot e^{-i2\pi \frac{0}{N} 0}, & s[1] \cdot e^{-i2\pi \frac{0}{N} 1}, & \ldots, & s[N-1] \cdot e^{-i2\pi \frac{0}{N}(N-1)} \\ s[0] \cdot e^{-i2\pi \frac{1}{N} 0}, & s[1] \cdot e^{-i2\pi \frac{1}{N} 1}, & \ldots, & s[N-1] \cdot e^{-i2\pi \frac{1}{N}(N-1)} \\ \ldots, & \ldots, & \ldots, & \ldots \\ s[0] \cdot e^{-i2\pi \frac{N-1}{N} 0}, & s[1] \cdot e^{-i2\pi \frac{N-1}{N} 1}, & \ldots, & s[N-1] \cdot e^{-i2\pi \frac{N-1}{N}(N-1)} \end{bmatrix} \tag{14}$$

$$= \begin{bmatrix} e^{-i2\pi \frac{0}{N} 0}, & e^{-i2\pi \frac{0}{N} 1}, & \ldots, & e^{-i2\pi \frac{0}{N}(N-1)} \\ e^{-i2\pi \frac{1}{N} 0}, & e^{-i2\pi \frac{1}{N} 1}, & \ldots, & e^{-i2\pi \frac{1}{N}(N-1)} \\ \ldots, & \ldots, & \ldots, & \ldots \\ e^{-i2\pi \frac{N-1}{N} 0}, & e^{-i2\pi \frac{N-1}{N} 1}, & \ldots, & e^{-i2\pi \frac{N-1}{N}(N-1)} \end{bmatrix} \cdot \begin{bmatrix} s[0] \\ s[1] \\ \ldots \\ s[N-1] \end{bmatrix} \tag{15}$$

$$\overline{\mathbf{S}}_{\mathbf{DFT}} = \mathbf{D}\overline{\mathbf{s}}\ , \tag{16}$$

where $\overline{\mathbf{S}}_{\mathbf{DFT}} \in [\mathbb{C}^N \times 1], \quad \overline{\mathbf{s}} \in [\mathbb{R}^N \times 1], \quad \mathbf{D} \in [\mathbb{C}^N \times \mathbb{C}^N]$

Inverse transform matrix:

$$\mathbf{D}^{-1} = \frac{1}{N}\mathbf{D}^* \tag{17}$$

$$\mathbf{D}^{-1}\mathbf{D} = \mathbf{I} \tag{18}$$

7 Properties

Lossless transform:

$$\mathcal{F}^{-1}(\mathcal{F}(s_t)) = s_t \tag{19}$$

$$\mathcal{F}(\mathcal{F}^{-1}(S_f)) = S_f \tag{20}$$

Linearity:

$$a + b = \mathcal{F}(a) + \mathcal{F}(b) \tag{21}$$

Convolution theorem:

$$a * b = \mathcal{F}(a) \cdot \mathcal{F}(b) \tag{22}$$

$$a \cdot b = \mathcal{F}(a) * \mathcal{F}(b) \tag{23}$$

Direct form representation:

$$S_{FT}(f) = A_f + iB_f, \qquad \forall \ \ A_f, B_f \in \mathbb{R} \tag{24}$$

Magnitude-phase form representation:

$$S_{FT}(f) = M_f \cdot e^{i\phi_f}, \qquad \forall \ \ M_f, \phi_f \in \mathbb{R} \tag{25}$$