Relations

Mairead Meagher

September, 2017

Contents

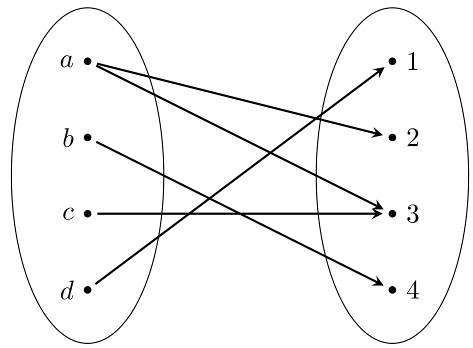

Figure 1: Relation

1 Relations

The study of sets so far has been limited by an inability to relate sets to one another. (Merely forming their union or intersection is not relating them).

The concept of a relation is fundamental to set theory and to Z.

An ordered pair is a couple of objects, say (Jim, Joe), in which the order of the objects is significant (Jim, Joe) (Joe, Jim).

1.1 What is a Relation

A relation is a set of ordered pairs. It may be finite or infinite but its most important feature is that it is a set.
In particular one relation can be a subset of another.
The set

$$R = \{(a, 2), (a, 3), (b, 4), (c, 3), (d, 1)\}$$

is a relation

1.1.1 Venn Diagram

Arrows may be used with Venn diagrams to denote relations. With the above relation R. It maps elements of the set containing letters to elements of the number set:

1.1.2 Maplet

Each ordered pair can be represented by a maplet, for which the symbol is$\mapsto$. Thus (x, y) and x $\mapsto$ y are the same thing.

1.1.3 Cartesian Product

The cartesian product of two sets X and Y, written X $\times$ Y is the set of all ordered pairs whose first components come from X and second component comes from Y. The infinite set of (x, y) coordinates in a plane is a cartesian product.

$$X \times Y = \{x : Xy : Y \bullet (x, y)\}$$

Clearly a relation R from X to Y is a subset of X $\times$ Y.

A cartesian product may have more than two components. For example, the cartesian product

$$X \times Y \times Z = \{x : Xy : Yz : Z \bullet (x, y, z)\}$$

is a set of ordered triples. These can also be called 3-tuples

Sets of ordered n-tuples are defined similarly.

1.1.4 Declaring a Relation

If

$R \subseteq X \times Y$
then
$R \in \mathbb{P}(X \times Y)$

Since every powerset in Z is by convention a type:

$$R : \mathbb{P}(X \times Y)$$

or, we say:

R has type $\mathbb{P}(X \times Y)$

Another notation(*syntactic sugar*[1]*)* for $\mathbb{P}(X \times Y)$ is X $\leftrightarrow$ Y. Hence,we say

$$R : X \leftrightarrow Y$$

Note: This is the notation we usually use.

[1]Syntactic sugar is a nicer ('sweeter') way of describing something

1.1.5 Infix Notation

Using infix notation, operators are written in-between their operands, e.g. $A+B$. Any relation name may be used as our infix symbol provided it is underlined.

9 $\underline{\text{isSquareof}}$ 3
fish $\underline{\text{menuForDay}}$ Tuesday

Mathematical symbols are not underlined.

$$(50, x) \in >$$

may be written as

$$50 > x$$

but not as

$$50 \leq x$$

1.1.6 Naming Relations

It is good practice to use a concatenated word suggesting the kinds of entities involved in the relation. Example:

$$isSquareOf : \mathbb{N} \leftrightarrow \mathbb{N}$$
$$isSquareOf = \{(1, 1), (1, -1), (4, 2), (9, 3), (400, -20)\}$$

Note : In checking is the name a good one for a relation, try using it in an infix manner, so as well as saying

$$(1, 1) \in isSquareOf$$

We can say (using *infix*)

$$1 \ \underline{isSquareOf} \ 1$$

If this describes the relationship clearly, then the name is good.

1.2 Constant Relations

Some relations such as $\leq$ have constant effect, which can be described symbolically. This makes possible an axiomatic definition of the relation, using low-line characters _ for the components, for example

$$\begin{array}{|l}
_ \leq _ : \mathbb{N} \leftrightarrow \mathbb{N} \\
\hline
\forall m, n : \mathbb{N} \bullet \\
\quad m \leq n \Leftrightarrow \exists k : \mathbb{N} \bullet m + k = n
\end{array}$$

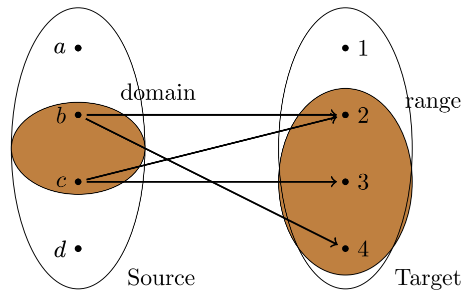

Figure 2: Source and Target,domain and range

1.3 Target and Source, Domain and Range

Diagrammatically the above four terms mean for a relation as shown in Figure 2 where $R : X \leftrightarrow Y$.

X is the source set or *from-set*, and Y is the target or *to-set*. R acts only on a subset (the Domain) of the Source and maps it to a subset (the Range) of the Target.

1.3.1 Domain

The domain is the set of first components in the ordered pairs of R. Consider the relation as shown in Figure 2:
The domain of the relation, written *dom R* is defined as follows:

$$dom\ R = \{x : X;\ y : Y \mid (x, y) \in R \bullet x\}$$

1.3.2 Range

Analogously, the range of a relation is the set of second components in the ordered pairs of R. Again, consider the relation as shown in figure 2:
The range of the relation, written *ran R* is defined as follows:

$$ran\ R = \{x : X;\ y : Y \mid (x, y) \in R \bullet y\}$$

1.4 Relational Image

We frequently need to know what a certain subset of the domain of R is mapped to. Given that $S : \mathbb{P}\, X$, the image of such a subset S is called its relational image, written

$$R \,(\!|\, S \,|\!)$$

By definition,

$$R \,(\!|\, S \,|\!) = \{x : X;\ y : Y \mid (x, y) \in R \land x \in S \bullet y\}$$

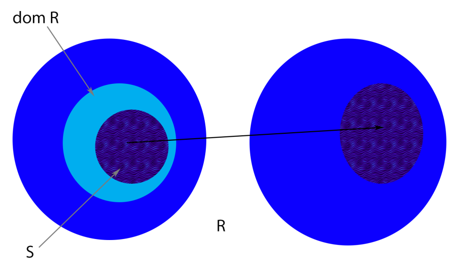

Figure 3: Relational Image

Diagrammatically, where $R : X \leftrightarrow Y$:
This is an extremely useful operation

1.5 Restrictions and Anti-restrictions

There are four important operators on a relation in Z which restrict its sphere of influence. For the definitions, we will use the following

$R : X \leftrightarrow Y$ and
$S : \mathbb{P}\, X$ and
$T : \mathbb{P}\, Y$

1.5.1 Domain Restriction

The definition of domain restriction is:

$$S \lhd R = \{x : X;\ y : Y \mid (x, y) \in R \land x \in S \bullet (x, y)\}$$

1.5.2 Domain Anti-Restriction

If S ⊆ dom R, then S⩤R is the complement of S◁ R. The definition of domain anti-restriction (Also called domain subtraction) is:

$$S \ntriangleleft R = \{x : X;\ y : Y \mid (x, y) \in R \land x \notin S \bullet (x, y)\}$$

1.5.3 Range Restriction

The definition of range restriction is:

$$R \rhd S = \{x : X;\ y : Y \mid (x, y) \in R \land y \in T \bullet (x, y)\}$$

1.5.4 Range Anti-Restriction

The definition of range anti-restriction is:

$$R \ntriangleright S = \{x : X;\ y : Y \mid (x, y) \in R \land y \notin T \bullet (x, y)\}$$

1.6 Inverse of a Relation

If $R : X \leftrightarrow Y$
then $R^{\sim}$ is the set of inverse ordered pairs

$$R^{\sim} = \{x : X;\ y : Y \mid (x, y) \in R \bullet (y, x)\}$$

Example
R = { (a,1), (b,2), (a, 4) }
then
$R^{\sim}$ = { (1, a), (2, b), (4, a) }

1.7 Identity of a Relation

$$\mathrm{id}\, R == \{x : X \bullet (x, x)\}$$

We can apply Id to any type,

Example

$$\mathrm{id}\, PERSON == \{x : PERSON \bullet (x, x)\}$$

1.8 Composition

To compose two relations R and T means to activate one, and then the other using the result of the first. Certain interfacing conditions must be met. See Figure 4.

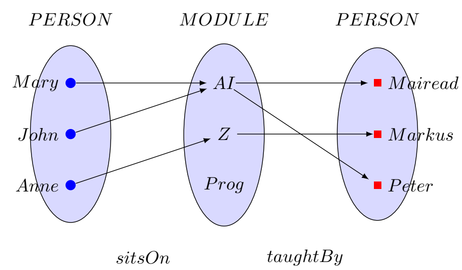

Figure 4: Relational Composition

We are interested in which lecturers teach which students. We use composition to do this. For example, from Figure 4:

$(Mary \mapsto AI) \in sitsOn$ and$(AI \mapsto Peter) \in taughtBy$
so we can say
$(Mary \mapsto Peter) \in sitsOn \fatsemi taughtBy$

In general

$R \fatsemi T$

means : 'Do R first and then do T'.
In order for this to be allowed, the Target of R must be the same as the Source of T (As in Figure 4)

1.8.1 Repeated Composition

A homogeneous relation is one where the Source and Target have the same type. It may be possible to compose such a function with itself.

$R : X \leftrightarrow X$
$R \fatsemi R : X \leftrightarrow X$

For example if the relation is

$addOne : \mathbb{Z} \leftrightarrow \mathbb{Z}$
$addOne == \{x : \mathbb{Z} \bullet (x, x + 1)\}$

so that

$6 \ \underline{addOne} \ 7$ and
$7 \ \underline{addOne} \ 8$ etc.
then
$6 \ \underline{addOne \fatsemi addOne} \ 8$ and
$6 \ \underline{addOne \fatsemi addOne \fatsemi addOne} \ 9$ etc

We have a shorthand way of writing these repeated compositions:

$addOne^2 == addOne \fatsemi addOne$ so
$addOne^3 == addOne^2 \fatsemi addOne$
and so on

So,

$6 \ \underline{addOne} \ 7$ and
$6 \ \underline{addOne}^2 \ 8$ and

and so on..

1.9 Transitive Closure

In general

$R^+ = R \ \cup \ R^2 \ \cup R^3 \ \cup \ R^4 \ \ldots \ R^n \ldots$
These may be empty for$n \geq 2$.

$addOne^+ = addOne \ \cup \ addOne^2 \ \cup \ addOne^3 \ \cup \ addOne^4 \ \ldots \ addOne^n \ldots$

So :

$x \ \underline{addOne^+ y}$

means that 'there exists a repeated composition of addOne which maps x to y'.
This means that $addOne^+$ is another name for the relation $<$.

1.9.1 Reflexive Relation

A relation R is reflexive if $(a, a) \in R$ for every $a \in R$. See Figure 5.

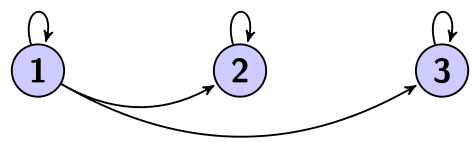

Figure 5: Reflexive Composition

1.9.2 Reflexive Relation

The reflexive-transistive closure of R is the smallest extension of R which is both transitive and reflexive. Clearly if we add a loop to each element in $domR^{+}$, we have the reflexive-transitive closure R^{*}. (We can also call id R^{0}).

Since id maps a to a,

$$R^{*} = \mathrm{id} \cup R^{+}$$

For example and returning to *addOne*,

$$addOne^{*} = addOne^{+} \cup \mathrm{id}\,\mathbb{Z}$$

so this is another name for $\leq$

1.10 Examples using Relations and Relational Composition

We will use the examples below during our exercises.

1.10.1 Modules

A student 'studies' (possibly many)modules. A module is 'studied' by (possibly many) students:

$[PERSON, MODULE]$ The set of all persons, modules.

$$
\begin{array}{|l}
\textit{ModuleSys} \\
\hline
studies : PERSON \leftrightarrow MODULE \\
students : \mathbb{P}\, PERSON \quad \text{The set of all registered students.} \\
degModules : \mathbb{P}\, MODULE \quad \text{The set of all degree modules.} \\
\hline
\operatorname{dom} studies \subseteq students \\
\operatorname{ran} studies \subseteq degModules \\
\hline
\end{array}
$$

The invariants state that :

1. Only registered students can 'take' a module;
2. Only degree modules can be 'studied'.

1.10.2 Borders

Countries are related by the relation borders if they share a border:

$[COUNTRY]$
$borders : COUNTRY \leftrightarrow COUNTRY$

e.g.

$$borders =$$
$$\{\ldots\ (france, switzerland), (switzerland, austria), (austria, hungary),$$
$$(hungary, romania), (romania, bulgaria), (bulgaria, turkey), (turkey, iran) \ldots,$$
$$(peru, brazil), (brazil, paraguay), (paraguay, chile) \ldots\}$$

So, you can see that

$france\ \underline{borders^{+}}\ iran$

This is another way of stating that france is directly bordering iran or is bordering a country which is bordering a country ... which is bordering iran. So, we can reach iran from france (i.e. it is on the same landmass).

We can also see that $peru \ \underline{borders^+} \ chile$
so peru and chile are on the same landmass.

but $(france, peru) \notin borders^+$
which means that they are not on the same landmass.

This concept of connectivity is central to the idea of transitive closure. We can model clusters of connected elements by using transitive closure.

1.10.3 Airports

Given

$$[AIRPORT]$$
$$connected : AIRPORT \leftrightarrow AIRPORT$$
$$\ldots$$
$$connected = \{ \ \ldots \ (lhr, dublin), (dublin, jfk), (jfk, rome) \ \ldots \}$$

If we wish to state that we can get to rome from lhr, then we state that

$$lhr \ \underline{connected^+} \ rome$$

This does not give us any hint as to how many trips we need, just that it is possible to get there.

We will revisit this example later when we look at building a route from one airport to another.

1.10.4 Citing Papers

In this example, we model how we cite academic research papers. Given

$$[PAPER]$$
$$cites : PAPER \leftrightarrow PAPER \qquad \text{and}$$
$$(paper1, paper2) \in cites$$

This has the meaning that $paper1 \ \underline{cites} \ paper2$.

1.10.5 Family Relations

Given

$$[PERSON]$$
$$parent : PERSON \leftrightarrow PERSON \qquad \text{and}$$
$$male, female : \mathbb{P}\, PERSON$$

and that $(abe, homer) \in parent$ means that abe is homer's parent. We do much modelling of these kinds of relationships.