Exercises and Solutions
Relations

Exercise 1.
Given the following sets:

$A = \{1\}$
$B = \{2, 3\}$

Write down the sets:

1. $A \times B$
2. $\mathbb{P}(A) \times B$
3. $\mathbb{P}(A \times B)$
4. $(A \times B) \times (A \times B)$

Exercise 2.
Given the relation

$R = \{1 \mapsto 1, 2 \mapsto 4, 3 \mapsto 9, 4 \mapsto 16, 5 \mapsto 25\}$
and the set
$S = \{1, 4, 5\}$

Simplify the value of each of the following expressions :

1. $S \lhd R$
2. $R \rhd S$
3. $(S \lhd R) \rhd S$
4. $(R \rhd \operatorname{dom} R)^{\sim} \rhd S$

Exercise 3.
Citing Papers
Given the following:

$[PAPER]$
$cites : PAPER \leftrightarrow PAPER$
and that
$(paper1, paper2) \in cites$ has the meaning that paper1 cites paper2 .

Write down the following:

1. Write a Z expression for the set of all papers cited directly or indirectly by paper x.
2. Write a Z expression for the set of all papers which cite other papers(directly or indirectly) but themselves are not cites (directly or indirectly).
3. Write a Z expression which states that if any paper cites another (directly or indirectly) the second one must not cite the first (directly or indirectly).
4. Write a Z expression for the number of papers cited directly by paper x.

Exercise 4.

Family Relations

Given

$[PERSON]$
and $parent : PERSON \leftrightarrow PERSON$
$male, female : \mathbb{P}\, PERSON$
and that
$(abe, homer) \in parent$ has the meaning that abe is homer's parent.

1. Write a Z expression for each of the following:
   (a) The parents of person x.
   (b) The grandparents of person x.
   (c) The grandchildren of person x.
   (d) The descendants of person x.
   (e) The siblings of person x.
   (f) The aunts of person x.
2. Give a Z expression for the set of all people in the database who have no relatives in the database.
3. Write an invariant to say that no person can have more than 2 parents.

Solutions

Solution 1.

1. $A \times B$

   $\{(1,2),(1,3)\}$

2. $\mathbb{P}(A) \times B$

   $\{(\{1\},2),(\{1\},3),(\{\},2),(\{\},3)\}$

3. $\mathbb{P}(A \times B)$

   $\{\{(1,2),(1,3)\},\{(1,2)\},\{(1,3)\},\{\}\}$

4. $(A \times B) \times (A \times B)$

   $\{((1,2),(1,2)),((1,2),(1,3)),((1,3),(1,2)),((1,3),(1,3))\}$

Solution 2.

1. $S \lhd R$

   $\{1 \mapsto 1, 4 \mapsto 16, 5 \mapsto 25\}$

2. $R \rhd S$

   $\{3 \mapsto 9, 4 \mapsto 16, 5 \mapsto 25\}$

3. $(S \lhd R) \rhd S$

   $\{1 \mapsto 1\}$

4. $(R \rhd \operatorname{dom} R)^{\sim} \rhd S$

   $\{9 \mapsto 3\}$

Solution 3.

1.

   $cites^{+}(\!|\{x\}|\!)$

2.

   $\operatorname{dom} cites^{+} \setminus \operatorname{ran} cites^{+}$

3.

$$\forall p, q : PAPER \mid (p, q) \in cites^+ \bullet (q, p) \notin cites^+$$

4.

$$\#(cites \, (\!| \{x\} |\!))$$

Solution 4.

1. Write a Z expression for each of the following:
   (a) The parents of person x.

   $$parent^\sim \, (\!| \{x\} |\!)$$

   (b) The grandparents of person x.

   $$(parent^\sim)^2 \, (\!| \{x\} |\!)$$

   (c) The grandchildren of person x.

   $$parent^2 \, (\!| \{x\} |\!)$$

   (d) The descendants of person x.

   $$parent^+ \, (\!| \{x\} |\!)$$

   (e) The siblings of person x.
   Let

   $$sibling \quad == (parent^\sim \fatsemi parent) \setminus \{p : PERSON \bullet (p, p)\}$$

   then
   $sibling \, (\!| \{x\} |\!)$

   (f) The aunts of person x.

   $$(\, (parent^\sim \fatsemi sibling) \rhd female) \, (\!| \{x\} |\!)$$

2. Give a Z expression for the set of all people in the database who have no relatives in the database.

   $$PERSON \setminus \mathrm{dom}\, parents \setminus \mathrm{ran}\, parents$$

3. Write an invariant to say that no person can have more than 2 parents.

   $$\forall p : PERSON \bullet \# parent^\sim \, (\!| \{p\} |\!) \leq 2$$