

Predicate Calculus

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- We have spoken earlier about Truth Valued Statements.
- A Truth Valued Statement is a statement whose value is either True or False.
- The idea of these Truth Valued Statements is central to how we write specifications.
- We describe the effect of an operation by a series of Truth Valued Statements, all of whom are true, when the operation is successfully completed.

Using mathematical terminology, we have two forms of Truth Valued Statements:

- Predicates
- Propositions

Two commonly-used definitions of what a predicate is, exist:

- ① A predicate is a truth-valued statement which may contain variables;
- ② A predicate is a statement whose truth-value depends on a value or values.

When a predicate is applied to a definite value it becomes a proposition.

Examples from Mathematics - predicates

$x < 6$

Fraction(k)

FourLegged(A)

oddNumber(x)

These are **true** or **false**, depending on the values of x , k , and A , respectively.

When we apply these predicates to actual values, we have propositions:

$10 < 6$

is a false proposition.

$Fraction(0.7)$

is a true proposition.

$FourLegged(whale)$

is a false proposition.

$oddNumber(4)$

is a false proposition.

A Predicate is a parameterised Proposition

Some predicates - the \mathbb{Z} way

Whereas in mathematics (as seen above), we write

oddNumber(4) is a false proposition.

The style we use in \mathbb{Z} is as follows:

- 1 We construct a set of entities with a particular property, e.g.
“this number is odd”

$$\begin{array}{|l} \textit{oddNumbers} : \mathbb{P} \mathbb{Z} \\ \hline \textit{oddNumbers} = \{x : \mathbb{Z} \bullet 2x + 1\} \end{array}$$

So, *oddNumbers* is the set of all possible odd numbers.

- 2 We then construct a predicate that is true if our element (e.g. *ourNum?*) is an element of this set:

$$\textit{ourNum?} \in \textit{oddNumbers}$$

We use this style extensively when modelling data.

- In modelling a systems with employees,
- an employee is either a manager or in personnel (etc.).
- The style in programming might be to have a field (enumerated type?) in the employee class which would have one of the possible values to indicate the employee's role.
- Not so in formal specification and specifically, Z.

Some predicates - the Z way

In Z, we construct (or just declare) a set for each, so

- We have a set containing all the employees who are managers;
- We have a set containing all the employers who are in personnel.
- We may then model some relationships between these sets (e.g. no employee is in both).. more later.
- Note that because this is modelling there is no extra storage to worry about. Just clarity and elegance.

The Existential Quantifier

The quantifier \exists , read as '**there exists**' is used in the construction of

\exists declaration | constraint (or "predicate") • predicate

- The declaration introduces a variable which is then constrained (optional).
- The predicate(constraint) applies to this variable.
- The constraint acts like a filter through which all candidates for the variable are "screened" on their way to taking part in the final predicate.
- The final predicate is "what we need to state about this filtered down selection of entities".

The Existential Quantifier

For example, the statement:

- “A man wrote beautiful Music.” will be written:
- “There exists a man and we can state (about this man) that he wrote beautiful music.”
- Rewrite this using our new terminology:

$\exists m : \text{MAN} \bullet m$ “wrote beautiful music”

Who wrote beautiful music?

- This statement is true if I can choose such an m .
- I choose Beethoven as my m .
- This means that this statement is true, as I now state (!) that Beethoven wrote beautiful music.
- Of course, when we write using mathematical terms, there will be no personal judgements as to what is beautiful music, etc.

Someone up there loves me

Another example (with a constraint):

- “Someone up there loves me” will be written:
- “There exists a person.. not any person, a person who is “up there”, and we can state (about this person) that that person loves me.”
- Rewrite this using our new terminology:

$\exists y : PERSON \mid y \text{ is “up there”} \bullet y \text{ “loves me”}$

More (mathematical) example

- There is a positive number less than 10 whose square is between 30 and 40 (inclusive). Firstly write it as:
- There exists a number n , and n is between 1 and 9, and we state about this number n , that n^2 is between 30 and 40(inclusive).
- Rewrite this using our new terminology:

$$\exists n : \mathbb{Z} \mid 0 < n < 10 \bullet 20 \leq n^2 \leq 40$$

This could equivalently be written:

$$\exists n : \mathbb{Z} \mid 1 \leq n \leq 9 \bullet 20 \leq n^2 \leq 40$$

- $n = 6$ makes it true. We call this number a witness.

Existential Quantification and Disjunction - the link

If the set in the declaration, suitably constrained, is finite, then \exists can be replaced by **disjunction** \vee (logical or) as in :

$$\exists n : \mathbb{Z} \mid 1 \leq n \leq 9 \bullet 20 \leq n^2 \leq 40$$

can be replaced by

$$30 < 1^2 < 40 \vee 30 < 2^2 < 40 \vee 30 < 3^2 < 40 \vee \dots \vee 30 < 9^2 < 40$$

In both cases, as long as one value of n make it true, then the entire statement is true.

$$\begin{array}{ll} n = 6 & \text{or} \\ 30 < 6^2 < 40 & \end{array}$$

Unique Existential Quantifier



Figure 1: A very special Irish Hare

- The Unique Quantifier symbolises “There exists one and only one”.
- If we wish to model the following:

There exists one and only one hare type which is a Native Irish Species.

- we can use:

$\exists_1 h$: Types of hare • h is a native Irish species

Careful - Use with care

- This should only be used in very specific circumstances.



Figure 2: U2 - The only fantastic Irish Band?

- Specifically, the statement

“There is a fantastic Irish music band” (!).

is not the same as:

“There is one and only one fantastic Irish music band.”

- The first statement allows for there to be more than one and states that, essentially, *"There is at least one fantastic Irish band"*.



Figure 3: Another fantastic Irish Band - The Gloaming

- It is used in the construction of

\forall declaration | constraint (or “predicate”) • predicate

- \forall is read as “For all” or “for every” and its definition has a similar structure to \exists :

- It is used in the construction of

\forall declaration | constraint (or “predicate”) • predicate

Squares Example

- “All numbers over 10 have squares over 100” will be written:
- “For all numbers n , where $n > 10$, the squares of this n , $n^2 > 100$.”
- Rewrite this using our new terminology:

$$\forall n : \mathbb{N} \mid n > 10 \bullet n^2 > 100$$

Snow White Example



- “All the dwarfs in the little house in the wood love Snow White.” will be written:
- “For all dwarfs d , where d “lives in the little house in the wood” , it is true to state that “ d loves Snow White” .
- Rewrite this using our new terminology:

$\forall d : DWARF \mid d$ “lives in the little house in the wood”
• d loves Snow White

- If the set in the declaration, suitably constrained, is finite, then \forall can be replaced by **conjunction** \wedge (logical or) as in :

$\forall d : DWARF \mid d$ “lives in the little house in the wood”
• d lovesSnowWhite

can be replaced by

$Sneezy$ lovesSnowWhite \wedge $Dopey$ lovesSnowWhite \wedge Doc lo
 $\wedge \dots \wedge$ $Grumpy$ lovesSnowWhite

- In both cases, all dwarfs must love Snow White to make the full statement true.

Using Universal and Existential Quantification together

In general, anywhere there is a predicate (sometimes we call them constraints), we can use \exists or \forall constructs. Sometimes we can use them together in a nested fashion for predicate construction.

We look at the “old saying”

Every cloud has a silver lining.

- “Every cloud has a silver lining” will be written:
- “For all clouds c , there is a silver lining s , and we can state that c has s .”
- Rewrite this using our new terminology:

$\forall c : \text{CLOUD} \bullet (\exists s : \text{SilverLining} \bullet c \text{ has } s)$

(In this case, there are no constraints, but of course we may need to have constraints in other examples.)

Any questions?

