Exercises and Solutions
Predicate Calculus

Exercise 1.
Rewrite the following using quantifiers:

1. (10 $>$ 1) $\wedge$ (11>1) $\wedge$ (12>1) $\wedge$ (13>1) $\wedge$(14>1) $\wedge$ (15>1)
2. (20>22) $\vee$ (21>22) $\vee$ (22>22) $\vee$ (23>22) $\vee$(24>22)
3. . . . (-2 $<$ 0) $\wedge$ (-1<0) $\wedge$ (0<0) $\wedge$ (1<0) $\wedge$ . . .
4. (0^3= 125) $\vee$ (1^3 = 125)$\vee$ (2^3 = 125) $\vee$ . . .

Exercise 2.
State whether the following predicates evaluate to true or false.

1. $\exists n : \mathbb{N} \bullet n < 0$
2. $\exists n : \mathbb{N} \bullet n \geq 0$
3. $\forall n : \mathbb{N} \bullet n \geq 0$

Exercise 3.
Assume that the sets EU and Scandinavia are defined as follows:
EU = {Belgium, France, Germany, Italy, Luxembourg, Holland, Denmark, Greece, Ireland, Spain, Portugal, UK}

Scandinavia = {Denmark, Finland, Norway, Sweden, Iceland}
Are the following true or false?

1. $\exists c : EU \bullet c \in Scandinavia$
2. $\neg(\forall c : EU \bullet c \in Scandinavia)$
3. $\exists c : EU \bullet \neg(c \in Scandinavia)$

Exercise 4.
Express the following in logic notation.

1. All that glisters is not gold
   You can assume the existence of the following:

   $[THING]$
   $glisterThings, goldThings : \mathbb{P}\, THING$

2. All the nice girls love a sailor.
   You can assume the existence of the following:

   $[PERSON]$
   $niceGirls, sailors : \mathbb{P}\, PERSON$
   $loves : PERSON \leftrightarrow PERSON$ where $x \mapsto y \in$ loves means that x loves y.

Exercise 5.

1. $\{i : \mathbb{Z} \mid i \in \{1, 3, 5\} \bullet i - 1\}$
2. $\{i : \mathbb{Z} \mid i^2 \in \{4, 9\} \bullet i\}$
3. $\{x, y :\mid x \geq 0 \land y \geq 0 \land x + y = 3 \bullet x\}$
4. $\{a, b : 0..3 \mid a + b = 3 \bullet (a, b)\}$
5. $\{z : \{7, 8, 9, 10\} \bullet (z, z)\}$

Exercise 6.
Define the set of whole numbers divisible by 4 but not by 100 (**mod** is the Z remainder operator: e.g. 9 **mod** 4 = 1).

Exercise 7.
Using the following sets:
$[PERSON]$of all people,

$prog : \mathbb{P}\, PERSON$ of people who are programmers
$code : \mathbb{P}\, PERSON$ of people who write code
$spec : \mathbb{P}\, PERSON$ of people who write specifications
$read : \mathbb{P}\, PERSON$ of people who read specifications

Express the following rules using the quantifiers $\forall$ and $\exists$:

1. All specifiers read specifications.
2. Some programmers write specifications.
3. All programmers who write code read specifications.
4. Only one programmer writes specifications
5. No more than 10 programmers write code.

Exercise 8.
Give logic expressions to define formally the meaning of:

1. set intersection;
2. set difference;
3. generalised union.

Exercise 9.
Describe the following situation using Z notation already covered. Assume that you have the following:

$[PERSON[$	the set of all people.
$men, women : \mathbb{P}\, PERSON$	
$employees : \mathbb{P}\, PERSON$	the set of all employees in the company.
$personnel : \mathbb{P}\, PERSON$	the set of all employees in the personnel department of the company.
$marketing : \mathbb{P}\, PERSON$	the set of all employees in the marketing department of the company.
$production : \mathbb{P}\, PERSON$	the set of all employees in the marketing department of the company.

1. People are either women or men, but not both.
2. A company employs people in three departments: marketing, personnel and production. Each employee is in precisely one of these departments.
3. Each department has a maximum of 10 staff.
4. All the staff in marketing are women.
5. The company employs more men than women.

Exercise 10.
Now, assume that each employee in the previous question can be in more than one department. Write down expressions for:

1. The number of women who work in all three departments.
2. The number of men who work in marketing and personnel but not in production.

Solutions

Solution 1.

1. $$\forall n : \mathbb{N} \mid 10 \leq n \leq 15 \bullet n > 1$$

2. $$\exists n : \mathbb{N} \mid 20 \leq n \leq 24 \bullet n > 22$$

3. $$\forall x : \mathbb{Z} \bullet x^2 \geq 0$$

4. $$\exists x : \mathbb{N} \bullet x^3 = 125$$

Solution 2.

1. false.
2. true - witness is n = 1
3. true

Solution 3.

1. true (witness - Denmark)
2. true (witness - Ireland)
3. true (witness - Ireland)

Solution 4.

1. All that glisters is not gold

   $$[THING]$$
   $$glisterThings, goldThings : \mathbb{P}\, THING$$

   We can look at this in two ways:

   (a) If we take the usual meaning, i.e. 'not everything that glisters is gold', we have

   $$\exists t : THING \mid t \in glisterThings \bullet t \notin goldThings \qquad \text{or}$$
   $$\neg\, (\forall t : THING \mid t \in glisterThings \bullet t \in goldThings)$$

(b) If we take a different than usual (but valid) meaning, i.e. ‘nothing that glisters is gold’, we have

$$\forall t : THING \mid t \in glisterThings \bullet t \notin goldThings$$

2. All the nice girls love a sailor.
You can assume the existence of the following:

$[PERSON]$
$niceGirls, sailors : \mathbb{P}\, PERSON$
$loves : PERSON \leftrightarrow PERSON$ where $x \mapsto y \in$ loves means that x loves y.

Again, we can look at this in two ways:

(a) If we take the usual meaning, i.e. ’It can be said about all nice girls that they each love a (possibly different) sailor’, we have

$$\forall g : niceGirls \bullet \exists s : sailors \bullet g \mapsto s \in loves$$

(b) If we take a different than usual meaning, i.e. ‘there is a sailor that all the nice girls love’, we have

$$\exists s : sailors \bullet \forall g : niceGirls \bullet g \mapsto s \in loves$$

Solution 5.
List the elements of these sets:

1. $\{0, 2, 4\}$
2. $\{2, -2, 3, -3\}$
3. $\{0, 1, 2, 3\}$
4. $\{(0, 3), (1, 2), (2, 1), (3, 1)\}$
5. $\{(7, 7), (8, 8), (9, 9), (10, 10)\}$

Solution 6.

$$somewhatleap == \{n : \mathbb{Z} \mid n \bmod 4 = 0 \land n \bmod 100 \neq 0 \bullet n\}$$

Solution 7.

1. $\forall\, s : spec \bullet s \in read$
2. $\exists\, p : prog \bullet p \in spec$
3. $\forall\, p : PERSON \mid p \in prog \land p \in code \bullet p \in read$
4. $\exists_1\, p : prog \bullet p \in spec$
5. This is non-trivial and is left as an exercise for the reader.

Solution 8.

1. set intersection

   $$\forall\, x : X;\; A, B : \mathbb{P}\, X \bullet x \in A \cup B \Leftrightarrow x \in A \land x \in B$$

2. set difference:

   $$\forall\, x : X;\; A, B : \mathbb{P}\, X \bullet x \in A \setminus B \Leftrightarrow x \in A \land x \notin B$$

3. generalised union.:

   $$\begin{aligned}&\forall\, x : X;\; genUnionSet : \mathbb{P}(\mathbb{P}(X)) \bullet x \in genUnionSet \Leftrightarrow \\ &\qquad \exists\, setA : \mathbb{P}\, X \mid setA \in genUnionSet \bullet x \in setA\end{aligned}$$

Solution 9.

1. People are either women or men, but not both:

   $$\langle women, men \rangle \text{ partition } PERSON$$

2. A company employs people in three departments: marketing, personnel and production. Each employee is in precisely one of these departments.

   $$\langle marketing, personnelandproduction \rangle \text{ partition } employees$$

3. Each department has a maximum of 10 staff.

   $$\begin{aligned}&\#\, marketing \leq 10 \land \\ &\#\, personnel \leq 10 \land \\ &\#\, production \leq 10\end{aligned}$$

4. All the staff in marketing are women.

   $$marketing \subset women$$

5. The company employs more men than women.

   $$\#(men \cap employees) > \#(women \cap employees)$$

Solution 10.

1. The number of women who work in all three departments.

$$\#(\bigcap\{marketing, personnel, production, women\})$$

2. The number of men who work in marketing and personnel but not in production.

$$\#(\bigcap\{marketing, personnel, men\} \setminus production)$$