Predicate Calculus

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2019

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Predicate Calculus

- We have spoken earlier about Truth Valued Statements.
- A Truth Valued Statement is a statement whose value is either True or False.
- The idea of these Truth Valued Statements is central to how we write specifications.
- We describe the effect of an operation by a a series of Truth Valued Statements, all of whom are true, when the operation is successfully completed.

Predicates and Propositions

Using mathematical terminology, we have two forms of Truth Valued Statements:

- Predicates
- Propositions

Predicates

Two commonly-used definitions of what a predicate is, exist:

1. A predicate is a truth-valued statement which may contain variables;
2. A predicate is a statement whose truth-value depends on a value or values.

When a predicate is applied to a definite value it becomes a proposition.

Examples from Mathematics - predicates

$x < 6$
$Fraction(k)$
$FourLegged(A)$
$oddNumber(x)$

These are **true** or **false**, depending on the values of x, k, and A, respectively.

Examples from Mathematics - propositions

When we apply these predicates to actual values, we have propositions:

$10 < 6$	is a false proposition.
$Fraction(0.7)$	is a true proposition.
$FourLegged(whale)$	is a false proposition.
$oddNumber(4)$	is a false proposition.

A Predicate is a parameterised Proposition

Some predicates - the Z way

Whereas in mathematics (as seen above), we write

$oddNumber(4)$ is a false proposition.

The style we use in Z is as follows:

1. We construct a set of entities with a particular property, e.g. **"this number is odd"**

   $$\begin{array}{|l} oddNumbers : \mathbb{P}\,\mathbb{Z} \\ \hline oddNumbers = \{x : \mathbb{Z} \bullet 2x + 1\} \end{array}$$

   So, *oddNumbers* is the set of all possible odd numbers.

2. We then construct a predicate that is true if our element (e.g. $ourNum?$) is an element of this set:

   $$ourNum? \in oddNumbers$$

Some predicates - the Z way

We use this style extensively when modelling data.

- In modelling a systems with employees,
- an employee is either a manager or in personnel (etc.).
- The style in programming might be to have a field (enumerated type?) in the employee class which would have one of the possible values to indicate the employee's role.
- Not so in formal specification and specifically, Z.

Some predicates - the Z way

In Z, we construct (or just declare) a set for each, so

- We have a set containing all the employees who are managers;
- We have a set containing all the employers who are in personnel.
- We may then model some relationships between these sets (e.g. no employee is in both).. more later.
- Note that because this is modelling there is no extra storage to worry about. Just clarity and elegance.

The Existential Quantifier

The quantifier $\exists$, read as **'there exists'** is used in the construction of

$$\exists \text{ declaration} \mid \text{constraint (or ``predicate'')} \bullet \text{predicate}$$

- The declaration introduces a variable which is then constrained (optional).
- The predicate(constraint) applies to this variable.
- The constraint acts like a filter through which all candidates for the variable are "screened" on their way to taking part in the final predicate.
- The final predicate is "what we need to state about this filtered down selection of entities".

The Existential Quantifier

For example, the statement:

- “A man wrote beautiful Music.” will be written:
- “There exists a man and we can state (about this man) that he wrote beautiful music.”
- Rewrite this using our new terminology:

$$\exists\, m : MAN \bullet m \text{ “wrote beautiful music”}$$

Who wrote beautiful music?

- This statement is true if I can choose such an m.
- I choose Beethoven as my m.
- This means that this statement is true, as I now state (!) that Beethoven wrote beautiful music.
- Of course, when we write using mathematical terms, there will be no personal judgements as to what is beautiful music, etc.

Someone up there loves me

Another example (with a constraint):

- “Someone up there loves me” will be written:
- “There exists a person.. not any person, a person who is “up there”, and we can state (about this person) that that person loves me.”
- Rewrite this using our new terminology:

$$\exists\, y : PERSON \mid y \text{ is “up there”} \bullet y \text{ “loves me”}$$

More (mathematical) example

- There is a positive number less that 10 whose square is between 30 and 40 (inclusive). Firstly write it as:
- There exists a number n, and n is between 1 and 9, and we that state about this number n, that n^2 is between 30 and 40(inclusive).
- Rewrite this using our new terminology:

  $$\exists\, n : \mathbb{Z} \mid 0 < n < 10 \bullet 30 \leq n^2 \leq 40$$

  This could equivalently be written:

  $$\exists\, n : \mathbb{Z} \mid 1 \leq n \leq 9 \bullet 30 \leq n^2 \leq 40$$

- $n = 6$ makes it true. We call this number a witness.

Existential Quantification and Disjunction - the link

If the set in the declaration, suitably constrained, is finite, then $\exists$ can be replaced by **disjunction** $\vee$ (logical or) as in :

$$\exists\, n : \mathbb{Z} \mid 1 \leq n \leq 9 \bullet 30 \leq n^2 \leq 40$$

can be replaced by

$$30 < 1^2 < 40 \;\vee\; 30 < 2^2 < 40 \;\vee\; 30 < 3^2 < 40 \\ \vee \ldots \vee 30 < 9^2 < 40$$

In both cases, as long as one value of n make it true, then the entire statement is true.

$n = 6$ or
$30 < 6^2 < 40$

Unique Existential Quantifier

Figure 1: A very special Irish Hare

- The Unique Quantifier symbolises “There exists one and only one”.
- If we wish to model the following:

  > *There exists one and only one hare type which is a Native Irish Species.*

- we can use:

$$\exists_1 \, h : \text{Types of hare} \bullet \text{h is a native Irish species}$$

Careful - Use with care

- This should only be used in very specific circumstances.

Figure 2: U2 - The only fantastic Irish Band?

- Specifically, the statement

  "There is a fantastic Irish music band" (!).

  is not the same as:

  "There is one and only one fantastic Irish music band."

Careful - Use with care

- The first statement allows for there to be more than one and states that, essentially,

  "There is at least one fantastic Irish band".

Figure 3: Another fantastic Irish Band - The Gloaming

Universal Quantifier

- It is used in the construction of

  $\forall$ declaration | constraint (or "predicate") • predicate

- $\forall$ is read as "For all" or "for every" and its definition has a similar structure to $\exists$:

- It is used in the construction of

  $\forall$ declaration | constraint (or "predicate") • predicate

Squares Example

- “All numbers over 10 have squares over 100” will be written:

- “For all numbers n, where $>$ 10, the squares of this n, $n^2 > 100$.”

- Rewrite this using our new terminology:

$$\forall\, n : \mathbb{N} \mid n > 10 \bullet n^2 > 100$$

Snow White Example

- “All the dwarfs in the little house in the wood love Snow White.” will be written:
- “For all dwarfs *d*, where *d* “lives in the little house in the wood” , it is true to state that “ *d* loves Snow White”.
- Rewrite this using our new terminology:

$$\forall\, d : DWARF \mid d \text{ “lives in the little house in the wood”} \bullet d \ lovesSnowWhite$$

Universal Quantification and Conjunction - the link

- If the set in the declaration, suitably constrained, is finite, then $\forall$ can be replaced by **conjunction** $\wedge$ (logical and) as in :

$$\forall\, d : \mathit{DWARF} \mid d \text{ “lives in the little house in the wood”} \\ \bullet\, d\ \mathit{lovesSnowWhite}$$

can be replaced by

$$\mathit{Sneezy}\ \mathit{lovesSnowWhite}\ \wedge\ \mathit{Dopey}\ \mathit{lovesSnowWhite} \\ \wedge\ \mathit{Doc}\ \mathit{lovesSnowWhite} \wedge\ \ldots\ \wedge\ \mathit{Grumpy}\ \mathit{lovesSnowWhite}$$

- In both cases, all dwarfs must love Snow White to make the full statement true.

Using Universal and Existential Quantification together

In general, anywhere there is a predicate (sometimes we call them constraints), we can use ∃ or ∀ constructs. Sometimes we can use them together in a nested fashion for predicate construction.

We look at the “old saying”

Every cloud has a silver lining.

- “Every cloud has a silver lining” will be written:
- “For all clouds c, there is a silver lining s, and we can state that c has s.
- Rewrite this using our new terminology:

$$\forall c : \mathit{CLOUD} \bullet (\exists s : \mathit{SilverLining} \bullet c \text{ has } s)$$

(In this case, there are no constraints, but of course we may need to have constraints in other examples.)

Any questions?