Predicate Calculus

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1 Predicate Calculus

We have spoken earlier about Truth Valued Statements.[1] The idea of these Truth Valued Statements is central to how we write specifications. We describe the effect of an operation by a a series of Truth Valued Statements, all of whom are true, when the operation is successfully completed.

1.1 Predicates and Propositions

Using mathematical terminology, we have two forms of Truth Valued Statements:

- Predicates
- Propositions

Two commonly-used definitions of what a predicate is, exist:

1. A predicate is a truth-valued statement which may contain variables;
2. A predicate is a statement whose truth-value depends on a value or values.

Note that the variable(s) in definition 1 takes on the value(s) referred to in definition 2, so the two definitions intend the same thing.

When a predicate is applied to a definite value it becomes a proposition.

1.1.1 Examples from Mathematics - predicates

$$x < 6$$
$$Fraction(k)$$
$$FourLegged(A)$$
$$oddNumber(x)$$

These are **true** or **false**, depending on the values of x, k, and A, respectively.

Examples from Mathematics - propositions When we apply these predicates to actual values, we have propositions:

$10 < 6$	is a false proposition.
$Fraction(0.7)$	is a true proposition.
$FourLegged(whale)$	is a false proposition.
$oddNumber(4)$	is a false proposition.

We can use the idea that

A Predicate is a parameterised Proposition

[1] A Truth Valued Statement is a statement whose value is either True or False.

1.1.2 Some predicates - the Z way

Whereas in mathematics (as seen above in section 1.1.1), we write

$oddNumber(4)$	is a false proposition.
$oddNumber(5)$	is a true proposition.

The style we use in Z is as follows:

1. We construct a set of entities with a particular property, e.g. **"this number is odd"**

   $$\begin{array}{|l} oddNumbers : \mathbb{P}\,\mathbb{Z} \\ \hline oddNumbers = \{x : \mathbb{Z} \bullet 2x + 1\} \end{array}$$

   So, $oddNumbers$ is the set of all possible odd numbers.

2. We then construct a predicate that is true if our element (e.g. $ourNum?$) is an element of this set:

   $$ourNum? \in oddNumbers$$

We use this style extensively when modelling data.

For instance, if we are modelling a systems with employees. In this system, an employee is either a manager or in personnel (etc.). The style in programming might be to have a field (enumerated type?) in the employee class which would have one of the possible values to indicate the employee's role. Not so in formal specification and specifically, Z.

In Z, we construct (or just declare) a set for each, so

- We have a set containing all the employees who are managers;
- We have a set containing all the employers who are in personnel.
- We may then model some relationships between these sets (e.g. no employee is in both).. more later.
- Note that because this is modelling there is no extra storage to worry about. Just clarity and elegance.

1.2 The Existential Quantifier

The quantifier $\exists$, read as **'there exists'** is used in the construction of

$$\exists \text{ declaration} \mid \text{constraint (or "predicate")} \bullet \text{predicate}$$

The declaration introduces a variable which is then constrained (optional). The predicate (constraint) applies to this variable. The constraint acts like a filter through which all candidates for the variable are "screened" on their way to taking part in the final predicate. The "constraint" is a truth valued statement. The final predicate is "what we need to state about this filtered down selection of entities".

For example, the statement:

- "A man wrote beautiful Music." will be written:
- "There exists a man and we can state (about this man) that he wrote beautiful music."
- Rewirite this using our new terminology:

$$\exists\, m : MAN \bullet m \text{ "wrote beautiful music"}$$

This statement is true if I can choose such an m. I choose Beethoven as my m. This means that this statement is true, as I now state (!) that Beethoven wrote beautiful music. Of course, when we write using mathematical terms, there will be no personal judgements as to what is beautiful music, etc.

Another example (with a constraint):

- "Someone up there loves me" will be written:
- "There exists a person.. not any person, a person who is "up there", and we can state (about this person) that that person loves me."
- Rewrite this using our new terminology:

$$\exists\, y : PERSON \mid y \text{ is "up there"} \bullet\, y \text{ "loves me"}$$

More (mathematical) example

- There is a positive number less that 10 whose square is between 30 and 40 (inclusive). Firstly write it as:
- There exists a number n, and n is between 1 and 9, and we that state about this number n, that n^2 is between 30 and 40(inclusive).
- Rewrite this using our new terminology:

$$\exists\, n : \mathbb{Z} \mid 0 < n < 10 \bullet 20 \leq n^2 \leq 40$$

This could equivalently be written:

$$\exists\, n : \mathbb{Z} \mid 1 \leq n \leq 9 \bullet 20 \leq n^2 \leq 40$$

Is this true or false? It is true - a value that makes it true is $n = 6$. We call this number a witness.

1.2.1 Existential Quantification and Disjunction - the link

If the set in the declaration, suitably constrained, is finite, then $\exists$ can be replaced by **disjunction** $\vee$ (logical or) as in :

$$\exists\, n : \mathbb{Z} \mid 1 \leq n \leq 9 \bullet 20 \leq n^2 \leq 40$$

can be replaced by

$$30 < 1^2 < 40 \;\vee\; 30 < 2^2 < 40 \;\vee\; 30 < 3^2 < 40 \vee \ldots \vee 30 < 9^2 < 40$$

In both cases, as long as one value of n make it true, then the entire statement is true.

$n = 6$ or
$30 < 6^2 < 40$

$n = 6$ is the witness.

1.2.2 Unique Existential Quantifier

Figure 1: A very special Irish Hare

The Unique Quantifier symbolises "There exists one and only one". If we wish to model the following:

> There exists one and only one hare type which is a Native Irish Species.

we can use:

$\exists_1\, h$: Types of hare $\bullet$ h is a native Irish species

Careful: This should only be used in very specific circumstances. Specifically, the statement

> "There is a fantastic Irish music band" (!).

Figure 2: U2 - The only fantastic Irish Band?

is not the same as:

> "There is one and only one fantastic Irish music band."

The first statement allows for there to be more than one and states that, essentially,

> "There is at least one fantastic Irish band".

Figure 3: Another Irish Band - The Gloaming

1.3 Universal Quantifier

$\forall$ is read as "For all" or "for every" and its definition has a similar structure to $\exists$: It is used in the construction of

$$\forall \text{ declaration} \mid \text{constraint (or "predicate")} \bullet \text{predicate}$$

Squares Example

- "All numbers over 10 have squares over 100" will be written:
- "For all numbers n, where > 10, the squares of this n, $n^2 > 100$."
- Rewrite this using our new terminology:

$$\forall\, n : \mathbb{N} \mid n > 10 \bullet n^2 > 100$$

Snow White Example

- "All the dwarfs in the little house in the wood love Snow White." will be written:
- "For all dwarfs d, where d "lives in the little house in the wood" , it is true to state that " d loves Snow White".
- Rewrite this using our new terminology:

$$\forall\, d : DWARF \mid d \text{ “lives in the little house in the wood”} \\ \bullet\ d\ lovesSnowWhite$$

Figure 4: Snow White is much loved

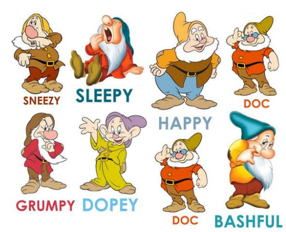

Figure 5: The seven Dwarfs

1.3.1 Universal Quantification and Conjunction - the link

If the set in the declaration, suitably constrained, is finite, then $\forall$ can be replaced by **conjunction** $\wedge$ (logical or) as in :

$$\forall\, d : DWARF \mid d \text{ “lives in the little house in the wood”} \\ \bullet\ d\ lovesSnowWhite$$

can be replaced by

$$Sneezy\ lovesSnowWhite \ \wedge\ Dopey\ lovesSnowWhite \ \wedge\ Doc\ lovesSnowWhite \\ \wedge \quad \ldots \quad \wedge\ Grumpy\ lovesSnowWhite$$

In both cases, all dwarfs must love Snow White to make the full statement true.

1.4 Using Universal and Existential Quantification together

In general, anywhere there is a predicate (sometimes we call them constraints), we can use $\exists$ or $\forall$ constructs. Sometimes we can use them together in a nested fashion for predicate construction.

Example using quantifications together We look at the "old saying"

> Every cloud has a silver lining.

- "Every cloud has a silver lining" will be written:
- "For all clouds c, there is a silver lining s, and we can state that c has s.
- Rewrite this using our new terminology:

  $$\forall c : CLOUD \bullet (\exists s : SilverLining \bullet c \text{ has } s)$$

  (In this case, there are no constraints, but of course we may need to have constraints in other examples.)