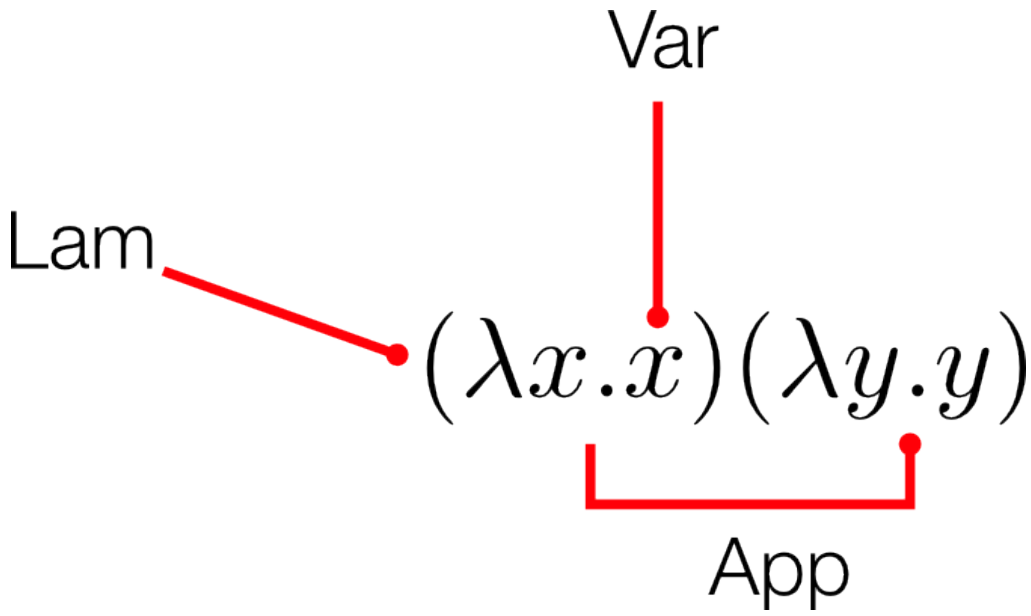
PROGRAMMING IN HASKELL

Chapter 4.2 The Lambda Calculus

Lambda Calculus

- lambda calculus is a model of computation devised in the 1930s by Alonzo Church.
- Functional programming languages all based on the lambda calculus
- Haskell is a *pure* functional language because all its features are translatable into lambda expressions
- allows higher degree of abstraction and composability

The structure of lambda terms



Alpha Equivalence

We have seen the function

$$\lambda x.x$$

The variable x here is not semantically meaningful except in its role in that single expression. Because of this, there's a form of equivalence between lambda terms called alpha equivalence. In other words,

$$\lambda x.x$$
$$\lambda apple.apple$$
$$\lambda orange.orange$$

all mean the same thing.

Beta Reduction

When we apply a function to an argument, we

- substitute the input expression for all instances of bound variables within the body of the abstraction
- eliminate the head of the abstraction (its only purpose was to bind a variable)

This process is called beta reduction.

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The only bound variable is the single x, so applying this function to 2 returns 2. This function is the *identity* function.

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In this case, we substitute the entire abstraction in for x. We use a new syntax here, $[x := z]$, to indicate that z will be substituted for all occurrences of x (here z is the function $(\lambda y.y)$). We reduce this application like this:

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$$\begin{aligned}(\lambda x.x)(\lambda y.y)\\ [x := (\lambda y.y)]\\ (\lambda y.y)\end{aligned}$$

Our final result is another identity function. There is no argument to apply it to, so we have nothing to reduce.

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We can't reduce this any further as there is nothing left to apply, and we know nothing about z.

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x is bound by λ over the body $x * y$, but y is a free variable.When we apply this function to an argument, nothing can be done with the y. It remains irreducible.

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The head has been applied away, and there are no more heads or bound variables. Since we know nothing about z or y, we can reduce this no further.

Multiple Arguments

Each lambda can only bind one parameter and can only accept one argument. Functions that require multiple arguments have multiple, nested heads. When you apply it once and eliminate the first (leftmost) head, the next one is applied and so on. This means that the following

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is simply syntactic sugar for

$$\lambda x(\lambda y.xy)$$

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Named Function	Lambda Calculus (maths)	Lambda Calculus (Haskell)	Result
$f\ x = x + 1$	$(\ \lambda x.x + 1)\ 2$	$(\backslash x \rightarrow x + 1)\ 2$	3
$f\ x\ y = x * y$	$(\ \lambda x\ y.x * y)\ 2\ 3$	$(\backslash x\ y \rightarrow x * y)\ 2\ \ 3$	6
$f\ xs = `c` : xs$	$(\ \lambda xs.`c` : xs)\ \ \text{"at"}$	$(\backslash xs \rightarrow `c` : xs)\ \ \text{"at"}$	"cat"

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Lambda functions are used extensively in Haskell, notably with Higher Order Functions.

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- Church encodings were developed by Alonzo Church
- Church encodings are a very interesting development arising from lambda calculus,
- Church found out that every concept in programming languages can be represented using functions!
- Everything from boolean logic, conditional statements, numbers (natural, integer, real, complex, imaginary), and even loops (infinite loops also)!

Encoding Lambda Calculus.. contd.

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Encoding Lambda Calculus.. contd.

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- **Idea:** Encode the ***behavior*** of values and not their structure

Encoding Booleans in λ Calculus

- What can we do with a boolean?
- We can make a binary choice (***if*** expression)
- A boolean is a function that, given two choices, selects one of them:

$$TRUE = \lambda x.\lambda y.x$$

$$FALSE = \lambda x.\lambda y.y$$

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if b then FALSE;
else TRUE

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if we think of the structure :

$$\textit{if } b \textit{ then } \textit{FALSE};$$
$$\textit{else } \textit{TRUE}$$

$$\textit{NOT} = \lambda b.b\ \textit{FALSE}\ \textit{TRUE}$$

Encoding Booleans in λ Calculus

$$NOT = \lambda b.b\ FALSE\ TRUE$$

Let's test this on *NOT TRUE*:

$$NOT = \lambda b.b\ FALSE\ TRUE$$

$$NOT\ \ TRUE$$

$$(\lambda b.b\ FALSE\ TRUE)\ TRUE$$

$$(TRUE)\ FALSE\ TRUE$$

$$= FALSE$$

Encoding Booleans in λ Calculus

Next, let's test this on *NOT FALSE*:

$$NOT\ \ FALSE$$

$$(\lambda b.b\ FALSE\ TRUE)\ FALSE$$

$$(FALSE)\ FALSE\ TRUE$$

$$= TRUE$$

Encoding Lambda Calculus.. Exercise

Using the techniques above, encode the following using lambda calculus:

1. AND
2. OR

Encoding Lambda Calculus..video

For a very good introduction to lambda calculus and encoding boolean logic using lambdas, see link to Graham Hutton's video (Computerphile) here

(You'll need to view this pdf through Adobe Reader to use the link. Otherwise Google 'youtube Lambda Calculus Computerphile Graham Hutton')