Bernoulli's Equation

Bernoulli's equation is simply the law of conservation of energy applied to an ideal fluid moving through a tube (discussed in volume flow rate). An ideal fluid in reality cannot exist because it is incompressible and experiences no internal friction which means it has no viscosity. However, using the idea can give us a fairly accurate guess as to how the dynamics of regular fluids work but keep in mind that the equation applies only to ideal fluids. The equation also demonstrated a very important principle to assist in lift which is Bernoulli's principle.

To understand this, we will begin with a tube with an ideal fluid flowing through it. The tube starts narrow and low and ends wide and higher than its beginning as shown in the diagram.

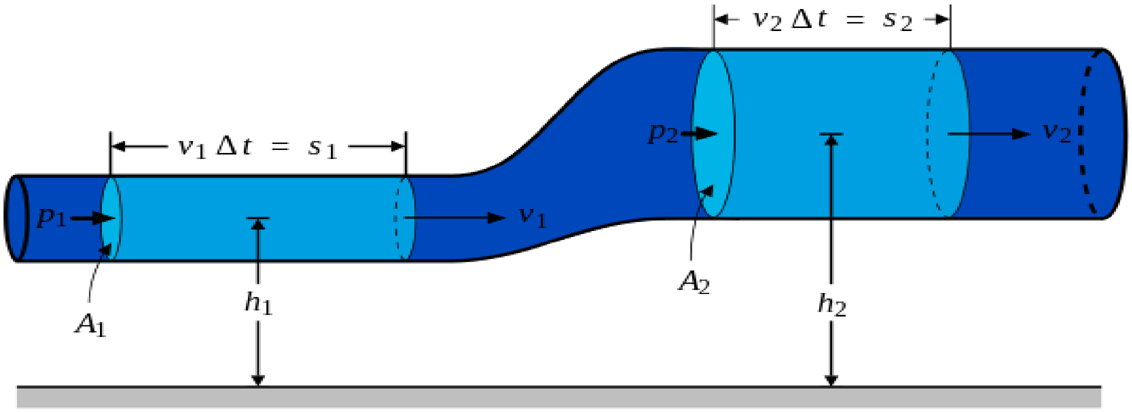

Fig 1.1

The equation applies to all pipes but this example is given to show what changes may take place. When applying the law of conservation of energy to each change we find that the sum of the total energy must be conserved. This means that the initial pressure(p), kinetic energy($\frac{1}{2}\rho v_1^2$) and potential(ρgh) energy all need to add up to be equal to the sum of the final pressure, kinetic energy and potential energy.

By doing this we get the equation;

$$p_1 + \tfrac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \tfrac{1}{2}\rho v_2^2 + \rho g h_2 \quad \text{(Bernoulli's equation)}$$

This means that $p + \frac{1}{2}\rho v^2 + \rho gh = \text{constant}$

The equation is not meant to demonstrate a principle but is more a reformulation of a principle. To see this we can apply it to a fluid at rest where v_1 and v_2 are equal to 0 you will then be able to see that $p_2 = p_1 + \rho g(h_1 - h_2)$ which demonstrates the pressure of fluids at rest. Another thing that the equation predicts when y is a constant (we can therefore set it = 0) is $p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$. This means that if the speed of a fluid particle increases the pressure of that fluid must decrease. So when looking at airflow diagrams the dense streamlines mean that that is an area of low pressure and high velocity because they are inversely proportional to each other. This makes sense if you think about the fact that when a fluid particle enters a narrow part of a tube, the high pressure from the back of the tube accelerates it giving it a higher velocity and a wide part of the tube in front of it will have a high pressure that decelerates it.

One can see many examples of this in real life. When in a moving car if you open the window just enough where as not to disturb the airflow outside the car, if there is visible smoke in the car you will observe that it drifts towards the moving stream of air outside the car. When you stick your hand out, it is not a pressure difference that you feel but simply the stream of fast moving air outside the car.

Proof of Bernoulli's Equation

As shown in the previous diagram, we will take the entire volume of the ideal fluid to be our system. Applying the law of conservation of energy to it will tell us that the total work done will be equal to the change in Kinetic energy.

$$dK = \frac{1}{2} m(v_2^2 - v_1^2)$$

If we take m to be the density times the change in volume of the fluid we get that:

$$dK = \frac{1}{2} \rho dV(v_2^2 - v_1^2)$$

For the work done by weight which will be = -mgh (it is negative because the force is in the opposite direction of the displacement as it is being lifted upwards)

$$W_g = -dmg(h_2 - h_1)$$

And again because m is equal to density into the volume of the fluid we get:

$$W_g = -\rho dV\, g(h_2 - h_1)$$

We must also find the work done by W_p which is the work done by a force at the input end of the tube which is what drives the fluid through the tube. This is given by the following equation using dx to be the change in horizontal displacement. The product of the pressure and the cross-sectional area into the change in horizontal displacement gives us the work done.

$$F\, dx = (pA)(dx) = p(dV)$$

So the work done is:

$$W_p = p_1 dV - p_2 dV = -\rho dV\, g(p_2 - p_1)$$

Using the work kinetic energy theorem (W = dK) we can say that $W = W_p + W_g = dK$ and substituting the equations we found earlier into this theorem will give us:

$$-\rho dV\, g(h_2 - h_1) - \rho dV\, g(p_2 - p_1) = \frac{1}{2} \rho dV(v_2^2 - v_1^2)$$

We can finally rearrange this to get Bernoulli's Equation!