2.10 Phase and Group Velocity

QUESTION 1

Consider the Klein-Gordon equation,

$$(1)\ u(x,t): \quad u_{tt} - {c_0}^2 u_{xx} = -q^2 u$$

, where q and c_0 are general constants, with initial conditions

$$(2)\ u(x,0) = f(x), \qquad u_t(x,0) = 0$$

, where $f(x)$ is some specified real even function with $f(x) \to 0$ as $|x| \to \infty$. We take $c_0 = 1$ without loss of generality, as physically this is just equivalent to rescaling time such that the wave speed becomes 1; rescale $t' = c_0 t$:

- $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t\prime}\frac{\partial t\prime}{\partial t} = c_0 u_{t\prime}$, by chain rule
- Repeating chain rule, we obtain $u_{tt} = {c_0}^2 u_{t\prime t\prime}$

Substitute into the general equation $u_{tt} - {c_0}^2 u_{xx} = -q^2 u$:

$${c_0}^2 u_{tt} - {c_0}^2 u_{xx} = -q^2 u$$

$$\Rightarrow u_{tt} - u_{xx} = -\left(\frac{q}{c_0}\right)^2 u$$

So, by rescaling time and redefining $q \to q / c_0$, we show that we can take $c_0 = 1$ without loss of generality.

Hence equations (1) and (2) become:

$$(1)\ u_{tt} - u_{xx} = -q^2 u$$

$$(2)\ u(x,0) = f(x), \qquad u_t(x,0) = 0$$

Define the Fourier transform of $u(x,t)$ in x:

$$\tilde{u}(k,t) = \int_{-\infty}^{\infty} u(x,t) e^{-ikx}\, dx$$

Recall the Fourier transform rules:

- $\widetilde{u_{xx}} = -k^2 \tilde{u}$
- $\widetilde{u_{tt}} = \frac{\partial^2 \tilde{u}}{\partial t^2}$

Using these, apply the Fourier transform to equation (1):

$$\frac{\partial^2 \tilde{u}}{\partial t^2} + k^2 \tilde{u} = -q^2 \tilde{u}$$

$$\Rightarrow \frac{\partial^2 \tilde{u}}{\partial t^2} + (q^2 + k^2)\tilde{u} = 0$$

Note, as given in the question, we can write $\Omega^2(k) = q^2 + k^2$.

$$\Rightarrow \frac{\partial^2 \tilde{u}}{\partial t^2} + \Omega^2(k)\tilde{u} = 0$$

This is a second-order ordinary differential equation. We can treat $\Omega^2(k)$ as a constant because it is a differential equation in t.

CHARACTERISTIC EQUATION: $\lambda^2 + \Omega^2 = 0 \;\Rightarrow\; \lambda = \pm i\Omega$

So, the general solution is $\tilde{u}(k,t) = A(k)\cos(\Omega(k)t) + B(k)\sin(\Omega(k)t)$

Initial conditions: $\tilde{u}(k,0) = \tilde{f}(k),\ \widetilde{u_t}(k,0) = 0$

$$\Rightarrow \tilde{u}(k,0) = A(k)\cos(0) + B(k)\sin(0) = A(k) = \tilde{f}(k)$$

$$\Rightarrow \boldsymbol{A(k) = \tilde{f}(k)}$$

Differentiate the general solution with respect to t:

$$\widetilde{u_t}(k,t) = -\tilde{f}(k)\Omega(k)\sin(\Omega(k)t) + B(k)\,\Omega(k)\cos(\Omega(k)t)$$

$$\Rightarrow \widetilde{u_t}(k,0) = -\tilde{f}(k)\Omega(k)\sin(0) + B(k)\,\Omega(k)\cos(0) = B(k) = 0$$

$$\Rightarrow \boldsymbol{B(k) = 0}$$

Hence $\tilde{u}(k,t) = \tilde{f}(k)\cos(\Omega(k)t)$.

Now do the inverse Fourier transform on $\tilde{u}(k,t)$:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{u}(k,t)e^{ikx}\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\cos(\Omega(k)t)\,e^{ikx}\,dk$$

Rewrite $\cos(\Omega(k)t)$ in exponential form:

$$\cos(\Omega(k)t) = \frac{e^{i\Omega(k)t} + e^{-i\Omega(k)t}}{2}$$

$$\Rightarrow u(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \tilde{f}(k)(e^{i\Omega(k)t} + e^{-i\Omega(k)t})e^{ikx}\,dk = \frac{1}{4\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx - i\Omega(k)t}\,dk + c.c. \qquad (4)$$

, $c.c.$ standing for complex conjugate, as required!

Next, consider the behaviour of phase velocity and group velocity:

- Phase Velocity $v_p(k) = \frac{\Omega(k)}{k} = \frac{\sqrt{q^2+k^2}}{k}$
- Group Velocity $v_g(k) = \frac{d\Omega(k)}{dk} = \frac{1}{2}(q^2+k^2)^{-\frac{1}{2}} \times 2k = \frac{k}{\sqrt{q^2+k^2}}$

Phase Velocity:

- As k → ∞, $\frac{\sqrt{q^2+k^2}}{k} \to \frac{\sqrt{k^2}}{k} = \frac{k}{k} = 1$
- As k → −∞, $\frac{\sqrt{q^2+k^2}}{k} \to -1$
- As k → 0^+, $\frac{\sqrt{q^2+k^2}}{k} \to \infty$
- As k → 0^-, $\frac{\sqrt{q^2+k^2}}{k} \to -\infty$
- $\boldsymbol{\frac{dv_p}{dk} = \frac{-q^2}{k^2\sqrt{q^2+k^2}}}$, which is never equal to zero for non-zero q, so there are no stationary points.

Group Velocity:

- As k → ∞, $\frac{k}{\sqrt{q^2+k^2}} \to 1$
- As k → −∞, $\frac{k}{\sqrt{q^2+k^2}} \to -1$
- As k → 0, $\frac{k}{\sqrt{q^2+k^2}} \to 0$
- $\frac{\boldsymbol{dv_p}}{\boldsymbol{dk}} = \frac{q^2}{(q^2+k^2)^{3/2}}$, which is never equal to zero for non-zero q, so there are no stationary points.

Hence the graphs for phase velocity and group velocity against time on the same set of axes are as follows:

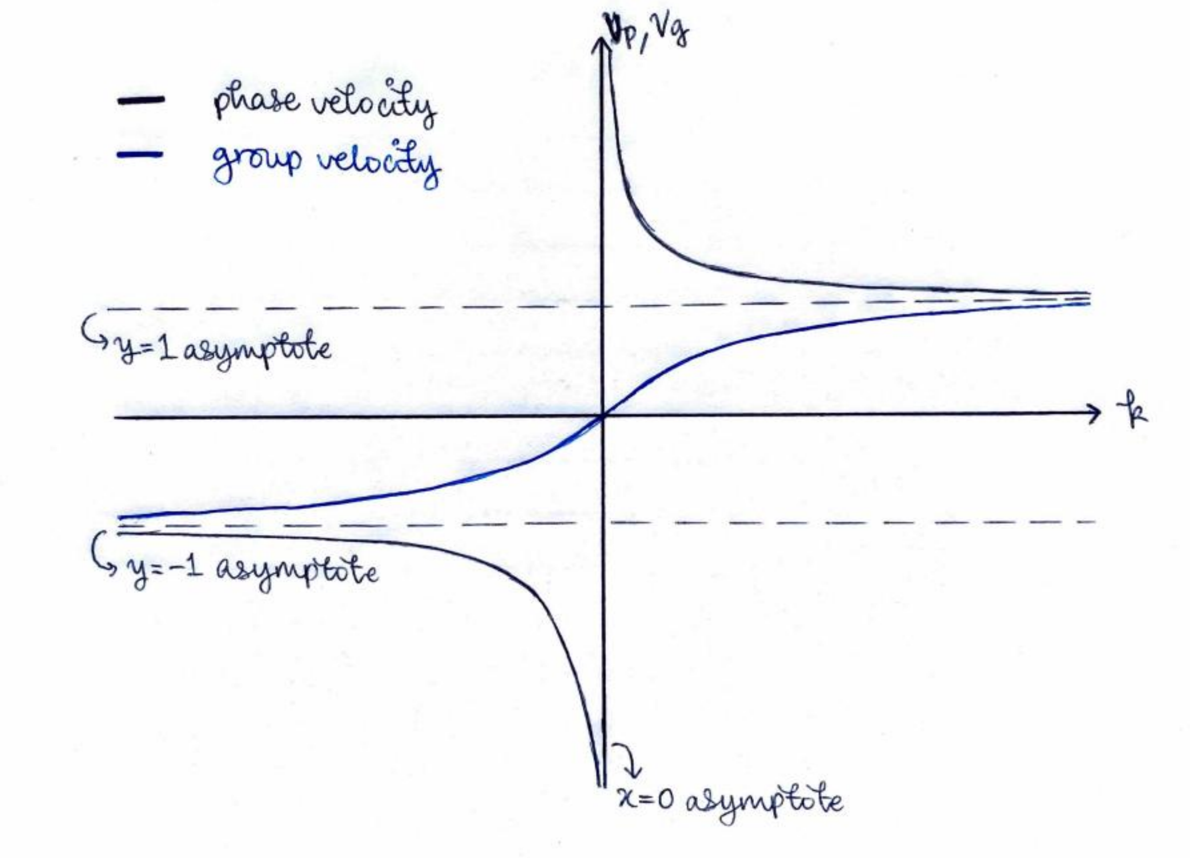

Figure 1: *A graph sketch of phase velocity and group velocity against wavenumber k.*

Note that as q approaches zero, the graphs become closer and closer to the asymptotes.

Phase velocity tells us the speed and direction in which individual wave crests propagate. No energy or information is carried by this individual wave crest – it's pure oscillation.

QUESTION 2

Recall integral (4):

$$u(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx-i\Omega(k)t}\, dk + c.c.$$

Define $\theta(k) = kx - \Omega(k)t$ as the 'phase'; then $\theta'(k) = x - \Omega'(k)t = [V - \theta'(k)]t$.

At the stationary phase point k_0, $\theta'(k_0) = 0 \Rightarrow \theta'(k_0) = V$.

Taylor expand $\theta(k)$ near k_0, noting that $\theta'(k_0) = 0$:

$$\theta(k) = [k_0V - \Omega(k_0)]t - \frac{1}{2}(k-k_0)^2\Omega''(k_0)t + \cdots$$

Therefore, we can conclude that θ changes rapidly with k when

$$|k-k_0| \gg (|\Omega''(k_0)t|)^{-\frac{1}{2}},$$

And so only contributions for wavenumbers in the range

$$|k-k_0| = O\left(\frac{1}{\sqrt{\Omega''(k_0)t}}\right)$$

are significant.

As only this range is significant, the dominant asymptotic contribution, after substituting the expansion into the integral, is as follows:

$$u(x,t) \sim \frac{1}{4\pi}\tilde{f}(k_0)e^{i[k_0V-\Omega(k_0)]t}\int_{-\infty}^{\infty} exp\left\{-\frac{i\Omega''(k_0)(k-k_0)^2t}{2}\right\} d(k-k_0) \; + \; c.c. \qquad (*)$$

Now evaluate the integral; recognise that it takes the form of a general Gaussian integral

$$\int_{-\infty}^{\infty} exp\left\{-\frac{ax^2}{2}\right\} dx = \sqrt{\frac{\pi}{a}}$$

, with $a = \frac{i\Omega''(k_0)t}{2}$; hence the integral's equal to

$$\int_{-\infty}^{\infty} exp\left\{-\frac{i\Omega''(k_0)(k-k_0)^2t}{2}\right\} d(k-k_0) = \sqrt{\frac{\pi}{\frac{i\Omega''(k_0)t}{2}}} = \sqrt{\frac{2\pi}{i\Omega''(k_0)t}}$$

Substitute this back into $(*)$:

$$\frac{1}{4\pi}\tilde{f}(k_0)e^{i[k_0V-\Omega(k_0)]t}\sqrt{\frac{2\pi}{i\Omega''(k_0)t}} \; + \; c.c. = \frac{1}{2}(2\pi i\Omega''(k_0)t)^{-\frac{1}{2}}\tilde{f}(k_0)e^{i[k_0V-\Omega(k_0)]t}$$

Write $\tilde{f}(k_0) = \left|\tilde{f}(k_0)\right|e^{iarg(\tilde{f}(k_0))}$ and substitute:

$$\frac{1}{2}(2\pi i\Omega''(k_0)t)^{-\frac{1}{2}}\tilde{f}(k_0)e^{i[k_0V-\Omega(k_0)]t} = \frac{1}{2}(2\pi i\Omega''(k_0)t)^{-\frac{1}{2}}\left|\tilde{f}(k_0)\right|e^{iarg(\tilde{f}(k_0))}e^{i[k_0V-\Omega(k_0)]t}$$
$$= \frac{1}{2}(2\pi i\Omega''(k_0)t)^{-\frac{1}{2}}\left|\tilde{f}(k_0)\right|e^{i([k_0V-\Omega(k_0)]t+arg(\tilde{f}(k_0)))}$$

Also write $i^{-\frac{1}{2}}$ in exponential form:

$$i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = e^{\frac{i\pi}{2}}$$

$$i^{-\frac{1}{2}} = \left(e^{\frac{i\pi}{2}}\right)^{-\frac{1}{2}} = e^{-\frac{i\pi}{4}}$$

Substitute back in:

$$\frac{1}{2}(2\pi i\Omega''(k_0)t)^{-\frac{1}{2}}\left|\tilde{f}(k_0)\right|e^{i([k_0V-\Omega(k_0)]t+arg(\tilde{f}(k_0)))}$$
$$= \frac{1}{2}(2\pi\Omega''(k_0)t)^{-\frac{1}{2}}\left|\tilde{f}(k_0)\right|e^{i\left([k_0V-\Omega(k_0)]t+arg\left(\tilde{f}(k_0)\right)-\frac{\pi}{4}\right)}$$

After adding the complex conjugate, $\frac{1}{2}(2\pi\Omega''(k_0)t)^{-\frac{1}{2}}\left|\tilde{f}(k_0)\right|e^{-i\left([k_0V-\Omega(k_0)]t+arg\left(\tilde{f}(k_0)\right)-\frac{\pi}{4}\right)}$, and recalling that

$$\frac{1}{2}\left(e^{i\left([k_0V-\Omega(k_0)]t+arg\left(\tilde{f}(k_0)\right)-\frac{\pi}{4}\right)} + e^{-i\left([k_0V-\Omega(k_0)]t+arg\left(\tilde{f}(k_0)\right)-\frac{\pi}{4}\right)}\right)$$

$$= \cos\left(k_0x - \Omega(k_0)t + arg\left(\tilde{f}(k_0)\right) - \frac{\pi}{4}\right)$$

, we get the final result of

$$u(x,t) \sim \frac{1}{4\pi}\tilde{f}(k_0)e^{i[k_0V-\Omega(k_0)]t}\int_{-\infty}^{\infty} exp\left\{-\frac{i\Omega''(k_0)(k-k_0)^2t}{2}\right\} d(k-k_0) \; + \; c.c.$$
$$= (2\pi\Omega''(k_0)t)^{-\frac{1}{2}}\left|\tilde{f}(k_0)\right|\cos\left(k_0x - \Omega(k_0)t + arg\left(\tilde{f}(k_0)\right) - \frac{\pi}{4}\right) \qquad (8)$$

, as required.

Next, find expressions for k_0, $\Omega(k_0)$ & $\Omega''(k_0)$ to obtain the second result; recall the definition of $\Omega(k)$:

$$\Omega(k) = \sqrt{q^2+k^2} = (q^2+k^2)^{\frac{1}{2}}$$

Differentiate:

$$\Omega'(k) = \frac{k}{\sqrt{q^2+k^2}}$$

$$\Omega''(k) = \frac{q^2}{(q^2+k^2)^{\frac{3}{2}}}$$

To find an expression for k_0, use the fact that it is specified by $\Omega'(k_0) = V$, a.k.a

$$\frac{k_0}{\sqrt{q^2+{k_0}^2}} = V$$

$$\Rightarrow \frac{{k_0}^2}{q^2+{k_0}^2} = V^2 \;\Rightarrow {k_0}^2 = V^2q^2 + V^2{k_0}^2 \Rightarrow (1-V^2){k_0}^2 = V^2q^2$$

$$\Rightarrow {k_0}^2 = \frac{V^2q^2}{(1-V^2)} \quad \& \, k_0 = \frac{qV}{\sqrt{1-V^2}}$$

Recall $V \equiv \frac{x}{t}$, and substitute this into the equations above:

$${k_0}^2 = \frac{x^2q^2}{(t^2-x^2)} \quad \& \, k_0 = \frac{qx}{\sqrt{t^2-x^2}}$$

Now find $\Omega(k_0)$ & $\Omega''(k_0)$ using this:

$$\Omega(k_0) = \left(q^2 + {k_0}^2\right)^{\frac{1}{2}} = \left(q^2 + \frac{x^2q^2}{(t^2-x^2)}\right)^{\frac{1}{2}} = \left(\frac{q^2(t^2-x^2)+x^2q^2}{(t^2-x^2)}\right)^{\frac{1}{2}} = \left(\frac{q^2t^2-q^2x^2+x^2q^2}{(t^2-x^2)}\right)^{\frac{1}{2}}$$

$$= \left(\frac{q^2t^2}{(t^2-x^2)}\right)^{\frac{1}{2}} = \frac{qt}{\sqrt{t^2-x^2}}$$

$$\Omega''(k_0) = \frac{q^2}{\left(q^2+{k_0}^2\right)^{\frac{3}{2}}} = \frac{q^2}{\left(q^2+\frac{x^2q^2}{(t^2-x^2)}\right)^{\frac{3}{2}}} = \frac{q^2}{\left(\frac{q^2t^2}{t^2-x^2}\right)^{\frac{3}{2}}} = q^2 \times \left(\frac{t^2-x^2}{q^2t^2}\right)^{\frac{3}{2}} = q^2 \times \frac{(t^2-x^2)^{\frac{3}{2}}}{q^3t^3}$$

$$= \frac{(t^2-x^2)^{\frac{3}{2}}}{qt^3}$$

So,

$$(2\pi\Omega''(k_0)t)^{-\frac{1}{2}} = \left(2\pi \times \frac{(t^2-x^2)^{\frac{3}{2}}}{qt^3} \times t\right)^{-\frac{1}{2}} = \left(2\pi \times \frac{(t^2-x^2)^{\frac{3}{2}}}{qt^2}\right)^{-\frac{1}{2}} = \frac{q^{\frac{1}{2}}t}{(2\pi)^{\frac{1}{2}}(t^2-x^2)^{\frac{3}{4}}}$$

$$\tilde{f}(k_0) = \tilde{f}\left(\frac{qx}{\sqrt{t^2-x^2}}\right)$$

$$k_0x - \Omega(k_0)t + arg\left(\tilde{f}(k_0)\right) - \frac{\pi}{4} = \frac{qx^2}{\sqrt{t^2-x^2}} - \frac{qt^2}{\sqrt{t^2-x^2}} + arg\left(\tilde{f}(k_0)\right) - \frac{\pi}{4}$$

$$= \frac{qx^2-qt^2}{\sqrt{t^2-x^2}} + arg\left(\tilde{f}(k_0)\right) - \frac{\pi}{4} = -q\sqrt{t^2-x^2} + arg\left(\tilde{f}(k_0)\right) - \frac{\pi}{4}$$

If $\tilde{f}(k)$ is real for all k, we have two cases: either $arg\left(\tilde{f}(k_0)\right) = 0$ if $\tilde{f}(k_0)$ is positive, or $arg\left(\tilde{f}(k_0)\right) = \pi$ if $\tilde{f}(k_0)$ is negative.

Consider the first case and take $arg\left(\tilde{f}(k_0)\right) = 0$. Then

$$k_0x - \Omega(k_0)t + arg\left(\tilde{f}(k_0)\right) - \frac{\pi}{4} = -q\sqrt{t^2-x^2} - \frac{\pi}{4}$$

Also, $\left|\tilde{f}(k_0)\right| = \tilde{f}(k_0)$, as $\tilde{f}(k_0)$ is positive.

Therefore,

$$\left|\tilde{f}(k_0)\right|\cos\left(k_0x - \Omega(k_0)t + arg\left(\tilde{f}(k_0)\right) - \frac{\pi}{4}\right) = \tilde{f}(k_0)\cos\left(-q\sqrt{t^2 - x^2} - \frac{\pi}{4}\right)$$
$$= \tilde{f}(k_0)\cos\left(q\sqrt{t^2 - x^2} + \frac{\pi}{4}\right)$$

, as $\cos(x)$ is an even function.

Next, consider the second case and take $arg\left(\tilde{f}(k_0)\right) = \pi$. Then

$$\cos\left(k_0x - \Omega(k_0)t + arg\left(\tilde{f}(k_0)\right) - \frac{\pi}{4}\right) = \cos\left(-q\sqrt{t^2 - x^2} + \pi - \frac{\pi}{4}\right) = -\cos\left(-q\sqrt{t^2 - x^2} - \frac{\pi}{4}\right)$$
$$= \cos\left(q\sqrt{t^2 - x^2} + \frac{\pi}{4}\right)$$

, as $\cos(x + \pi) = -\cos(x)$.

Also, $\left|\tilde{f}(k_0)\right| = -\tilde{f}(k_0)$, as $\tilde{f}(k_0)$ is negative.

Therefore,

$$\left|\tilde{f}(k_0)\right|\cos\left(k_0x - \Omega(k_0)t + arg\left(\tilde{f}(k_0)\right) - \frac{\pi}{4}\right) = -\tilde{f}(k_0) \times -\cos\left(q\sqrt{t^2 - x^2} + \frac{\pi}{4}\right)$$
$$= \tilde{f}(k_0)\cos\left(q\sqrt{t^2 - x^2} + \frac{\pi}{4}\right)$$

, the same result as the positive case.

Substituting everything back into the first result, we get that:

$$u(x,t) \sim \frac{q^{\frac{1}{2}}t}{(2\pi)^{\frac{1}{2}}(t^2 - x^2)^{\frac{3}{4}}}\tilde{f}\left(\frac{qx}{\sqrt{t^2 - x^2}}\right)\cos\left(q\sqrt{t^2 - x^2} + \frac{\pi}{4}\right) \tag{10}$$

for $|x| < t$, as required!

The stationary phase approximation provides a physical understanding of group velocity. A general wave packet can be written as:

$$u(x,t) = \int_{-\infty}^{\infty} A(k)e^{i(kx - \Omega(k)t)}\,dk$$

When t is large, the integral becomes rapidly oscillatory, and dominant contributions come from the point where the phase is stationary, i.e. where

$$\frac{d}{dk}(kx - \Omega(k)t) = 0 \Rightarrow x = \frac{d\Omega(k)}{dk}t$$

This implies that the peak of the wave packet travels at the group velocity $v_g(k) = \frac{d\Omega(k)}{dk}$.

Physically, this means that even though each frequency component travels at its own phase velocity, the overall envelope of the packet, carrying energy and information, propagates at the group velocity. An observer moving with speed V will eventually only see those waves in the initial spectrum $A(k)$ with wave number k_0 such that $v_g(k) = V$ – the dominant contribution comes from when the observer's travelling at the group velocity.

QUESTION 3

So, we are given the following finite-difference equation using the centred-difference approximation for u_{tt} and u_{xx}:

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{(\Delta t)^2} - \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2} = -q^2\left(\frac{u_{i+1}^j + u_{i-1}^j}{2}\right)$$

Rearrange this to compute u_i^{j+1} from j and $j - 1$, i.e. the solution at the next time step.

Multiply the equation by $(\Delta t)^2$:

$$u_i^{j+1} - 2u_i^j + u_i^{j-1} - \left(\frac{\Delta t}{\Delta x}\right)^2 \left(u_{i+1}^j - 2u_i^j + u_{i-1}^j\right) = -\frac{(q\Delta t)^2}{2}\left(u_{i+1}^j + u_{i-1}^j\right)$$

Then rearrange the equation to make u_i^{j+1} the subject:

$$u_i^{j+1} = 2u_i^j - u_i^{j-1} + \left(\frac{\Delta t}{\Delta x}\right)^2 \left(u_{i+1}^j - 2u_i^j + u_{i-1}^j\right) - \frac{(q\Delta t)^2}{2}\left(u_{i+1}^j + u_{i-1}^j\right)$$

This is the equation implemented in the code to carry out the iteration scheme. However, the first step for both x and t needs to be separately considered.

For the first step in t, use $u_i^{-1} = u_i^1$. Then, substituting this into the equation above and making u_i^1 the subject, we get

$$u_i^1 = u_i^0 + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 \left(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0\right) - \frac{(q\Delta t)^2}{4}\left(u_{i+1}^0 + u_{i-1}^0\right)$$

, which is implemented into the program.

For the first step in x, use a similar condition, $u_{-1}^j = u_1^j$. No extra rearranging is required here, but $u_{i+1}^j + u_{i-1}^j = u_1^j + u_{-1}^j = u_1^j + u_1^j = 2u_1^j$ in the q term on the right-hand side. Furthermore, $u_{i+1}^j - 2u_i^j + u_{i-1}^j = u_1^j - 2u_0^j + u_{-1}^j = u_1^j - 2u_0^j + u_1^j = 2u_1^j - 2u_0^j$ in the $\left(\frac{\Delta t}{\Delta x}\right)^2$ term on the right-hand side, so the equation becomes

$$u_0^1 = u_0^0 + \left(\frac{\Delta t}{\Delta x}\right)^2 \left(u_1^0 - u_0^0\right) - \frac{(q\Delta t)^2}{2}\left(u_1^0\right)$$

for $j = 0$, and

$$u_0^{j+1} = 2u_0^j - u_0^{j-1} + 2\left(\frac{\Delta t}{\Delta x}\right)^2 \left(u_1^j - u_0^j\right) - (q\Delta t)^2\left(u_1^j\right)$$

for general j.

We wish to compute the numerical solution using the finite-difference scheme for $q = 0$ and $q = 1$.

$\boldsymbol{q = 0}$:

The program used to plot the graphs for q = 0 is **Code 1** on page 19, labelled as

```
centred_difference(dx,lambda,t_finals,q)
```

The program plots the solutions for $t = 0, 10, 20, 30, 40, 50$ seconds on the same set of axes, and the x-range is set to be up to $x = 60$, so that the boundary value at $x = 60$ can be comfortably set to 0 for each t-value, as done so in the program. Furthermore 'lambda' is defined in the program to be the ratio between the time step and spatial step, i.e. $\frac{\Delta t}{\Delta x}$.

The parameters entered into the program are:

```
(dx, lambda, t_finals, q) = (0.1, 1, [0,10,20,30,40,50], 0)
```

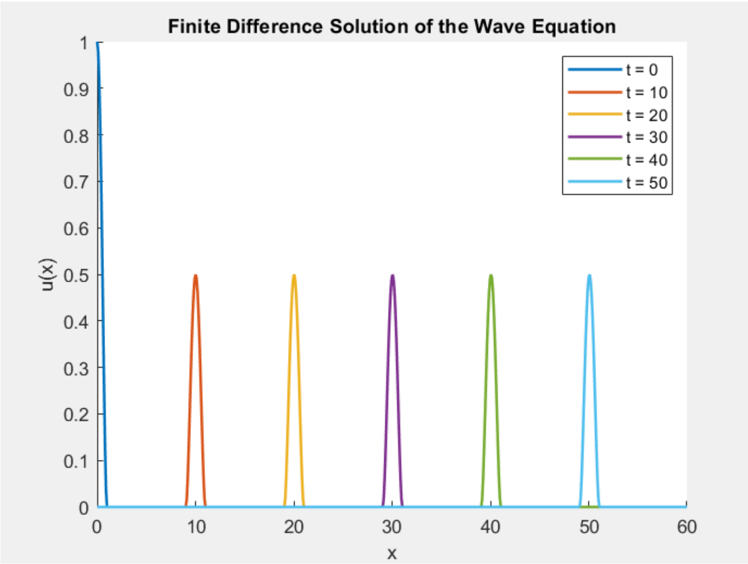

Figure 2: *A plot of the solution to u(x,t) obtained using the finite-difference method, with parameters* $\Delta x = \Delta t = 0.1$, *t_finals* = $[0, 10, 20, 30, 40, 50]$ *& q = 0.*

Comparing this alongside the exact solution $u(x,t) = \frac{1}{2}(f(x-t) + f(x+t))$, it aligns exactly, showing that this program is highly accurate for these parameters.

Now experiment with different values of Δx and $\frac{\Delta t}{\Delta x}$; begin by changing Δx to 0.01 but keeping $\frac{\Delta t}{\Delta x}$ as the same value of 1. We get the following graph:

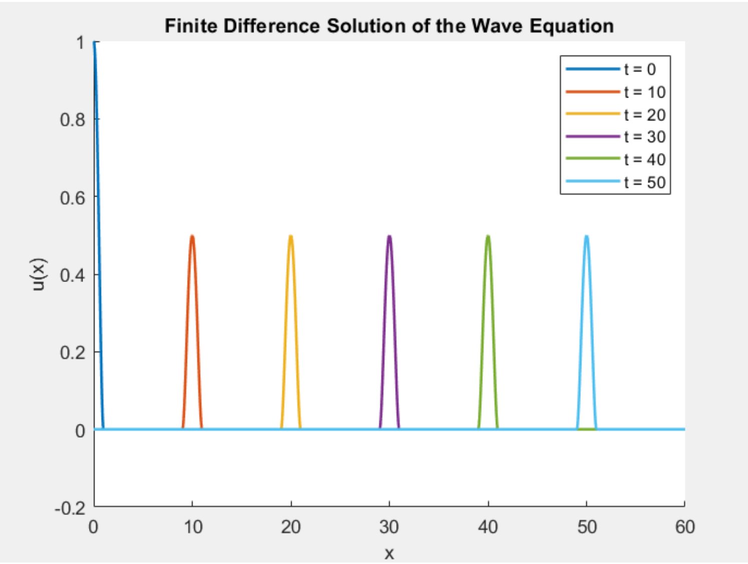

Figure 3: *A plot of the solution to u(x,t) obtained using the finite-difference method, with parameters* $\Delta x = \Delta t = 0.01$, *t_finals* = $[0, 10, 20, 30, 40, 50]$ *& q = 0.*

After trying other values of Δx, like 0.05, 0.001 and more, but keeping $\frac{\Delta t}{\Delta x}$ as the same value of 1, it can be seen that the value of Δx does not seem to have an impact on the shape of the numerical solution obtained using the finite-difference scheme.

However, if we change the value of $\frac{\Delta t}{\Delta x}$ used, there is a slight difference to be noticed: see below the graphs for $\frac{\Delta t}{\Delta x} = 0.5$ and $\frac{\Delta t}{\Delta x} = 0.01$:

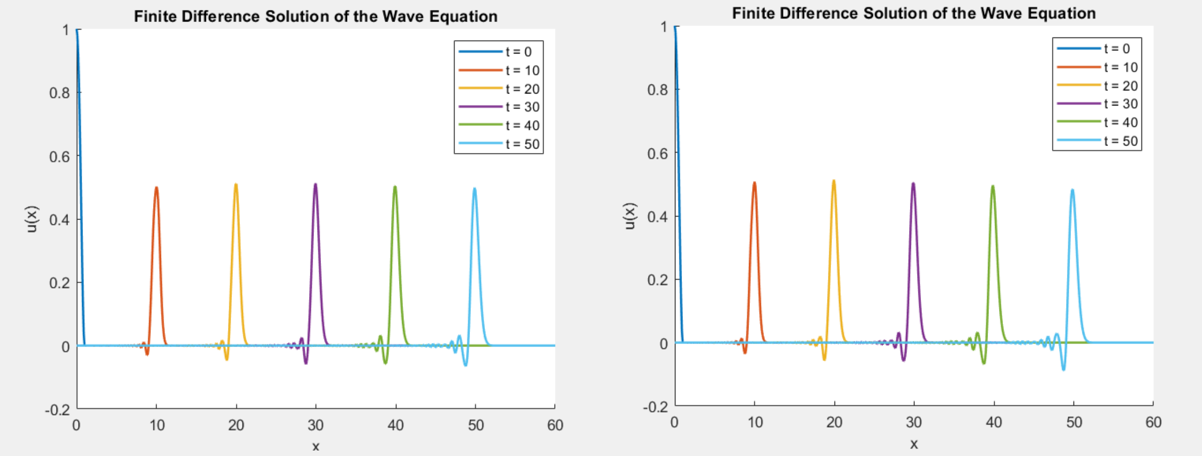

Figure 4a: $\frac{\Delta t}{\Delta x} = 0.5$ Figure 4b: $\frac{\Delta t}{\Delta x} = 0.01$

Figure 4: *A plot of the solution to u(x,t) obtained using the finite-difference method, with parameters* $\Delta x = 0.1$, $\frac{\Delta t}{\Delta x} = 0.5$ & 0.01, *t_finals* = $[0, 10, 20, 30, 40, 50]$ & *q* = 0.

As the value of $\frac{\Delta t}{\Delta x}$ decreases from 1, but stays between 0 and 1, slight inaccuracies in the finite-difference scheme as seem, as shown on the diagram, the size of the inaccuracies increasing as $\frac{\Delta t}{\Delta x}$ deviates further from 1. However, if $\frac{\Delta t}{\Delta x}$ exceeds 1 even a little bit, the solution completely changes shape and becomes unstable, as depicted below:

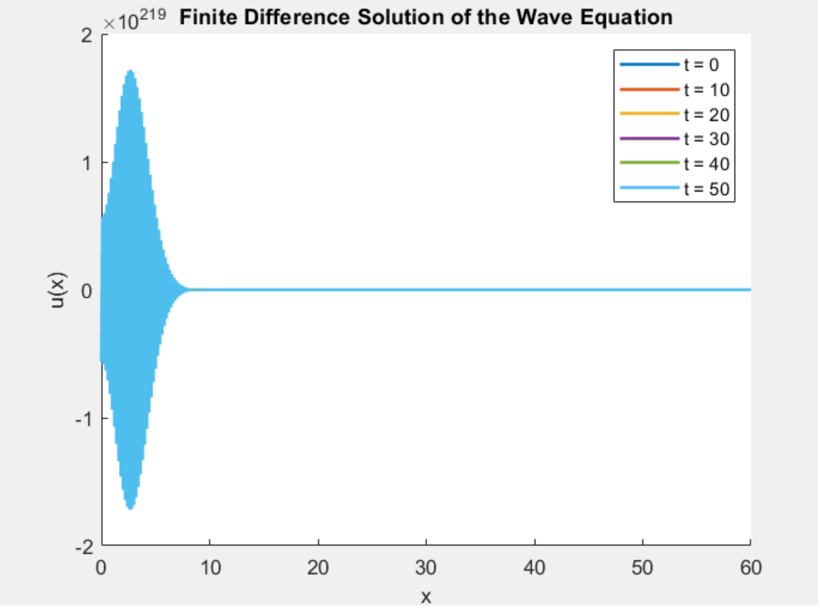

Figure 5: *A plot of the solution to u(x,t) obtained using the finite-difference method, with parameters* $\Delta x = 0.1$, $\frac{\Delta t}{\Delta x} = 1.2$, *t_finals* = $[0, 10, 20, 30, 40, 50]$ & *q* = 0.

Now we wish to find the Fourier transform of $f(x)$:

$$f(x) = \begin{cases} (1-x^2)^2, & |x| \le 1 \\ \quad 0 \quad , & |x| \ge 1 \end{cases}$$

Then

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx = \int_{-1}^{1}(1-x^2)^2 e^{-ikx}\,dx = \int_{-1}^{1}(x^2-1)^2 e^{-ikx}\,dx$$

Integrate by parts:

- $u = (x^2-1)^2,\ u' = 4x(x^2-1)$
- $v' = e^{-ikx},\ v = \frac{i}{k}e^{-ikx}$

$$\Rightarrow \tilde{f}(k) = \left[\frac{i}{k}(x^2-1)^2 e^{-ikx}\right]_{-1}^{1} - \frac{4i}{k}\int_{-1}^{1}(x^3-x)e^{-ikx}\,dx = -\frac{4i}{k}\int_{-1}^{1}(x^3-x)e^{-ikx}\,dx$$

Use integration by parts again:

- $u = x^3 - x,\ u' = 3x^2 - 1$
- $v' = e^{-ikx},\ v = \frac{i}{k}e^{-ikx}$

$$\Rightarrow \tilde{f}(k) = \left[\frac{4}{k^2}(x^3-x)e^{-ikx}\right]_{-1}^{1} - \frac{4}{k^2}\int_{-1}^{1}(3x^2-1)e^{-ikx}\,dx = -\frac{4}{k^2}\int_{-1}^{1}(3x^2-1)e^{-ikx}\,dx$$

Use integration by parts again:

- $u = 3x^2 - 1,\ u' = 6x$
- $v' = e^{-ikx},\ v = \frac{i}{k}e^{-ikx}$

$$\Rightarrow \tilde{f}(k) = \left[-\frac{4i}{k^3}(3x^2-1)e^{-ikx}\right]_{-1}^{1} + \frac{24i}{k^3}\int_{-1}^{1}xe^{-ikx}\,dx$$

Note that

$$\left[-\frac{4i}{k^3}(3x^2-1)e^{-ikx}\right]_{-1}^{1} = -\frac{4i}{k^3}\left(2e^{ik} - 2e^{ik}\right) = -\frac{16}{k^3}\sin k$$

$$\Rightarrow \tilde{f}(k) = -\frac{16}{k^3}\sin k + \frac{24i}{k^3}\int_{-1}^{1}xe^{-ikx}\,dx$$

Use integration by parts again:

- $u = x,\ u' = 1$
- $v' = e^{-ikx},\ v = \frac{i}{k}e^{-ikx}$

$$\Rightarrow \tilde{f}(k) = -\frac{16}{k^3}\sin k - \frac{24}{k^4}\left[xe^{-ikx}\right]_{-1}^{1} + \frac{24}{k^4}\int_{-1}^{1}e^{-ikx}\,dx$$

Note that

$$\frac{24}{k^4}\left[xe^{-ikx}\right]_{-1}^{1} = \frac{24}{k^4}\left(e^{-ik} + e^{ik}\right) = \frac{48}{k^4}\cos k$$

And

$$\frac{24}{k^4}\int_{-1}^{1} e^{-ikx}\,dx = \frac{24i}{k^5}\left(e^{-ik} - e^{ik}\right) = \frac{48}{k^5}\sin k$$

$$\Rightarrow \tilde{f}(k) = -\frac{16}{k^3}\sin k - \frac{48}{k^4}\cos k + \frac{48}{k^5}\sin k = -\frac{16[(k^2-3)\sin k + 3k\cos k]}{k^5}$$

The programs used to plot the graph of $\tilde{f}(k)$ are **Code 2** and **Code 3** on page 20, labelled as

```
fourier_f(k)
```

, which defines the Fourier transform of $f(x)$ and is inclusively used in **Code 4**, and

```
plot_fourier_function
```

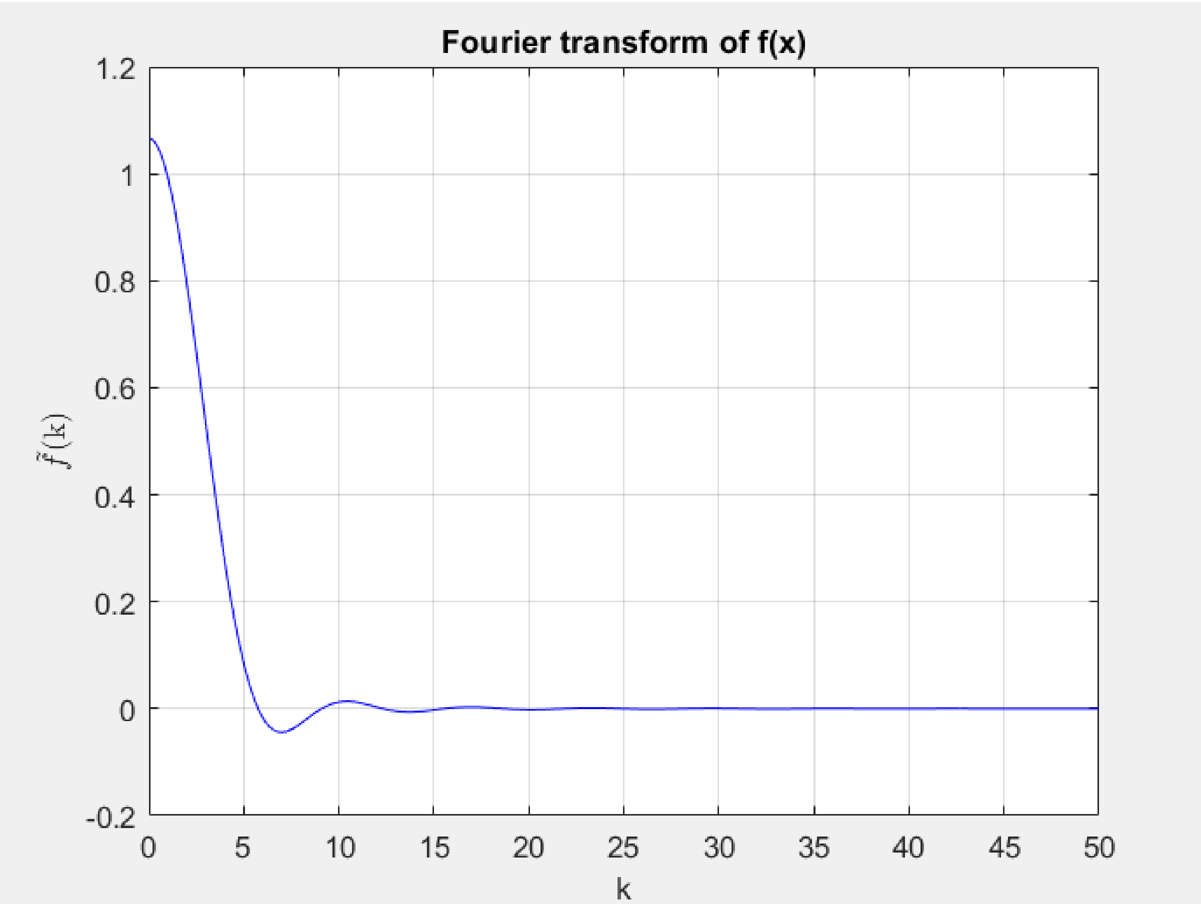

Figure 6: *A plot of* $\tilde{f}(k)$ *against k.*

***q* = 1**:

The program used to plot the graphs for q = 1 is **Code 4** on page 21, labelled as

```
centred_difference_2(dx,lambda,t_final,q)
```

This time, it doesn't plot the graphs for different values of t_final on the same set of axes, but for each value of t_final, the stationary phase approximation (10) is inclusively plotted alongside the graph from the finite-difference scheme. The x-range is set to be up to $x = t\_final + 10$, so that the boundary value can be comfortably set to 0 for each t-value, as done so for $q = 0$ as well. For the stationary phase approximation, the program takes into the account that for $x > t\_final$, the graph is zero.

Find the graphs for different values of t below:

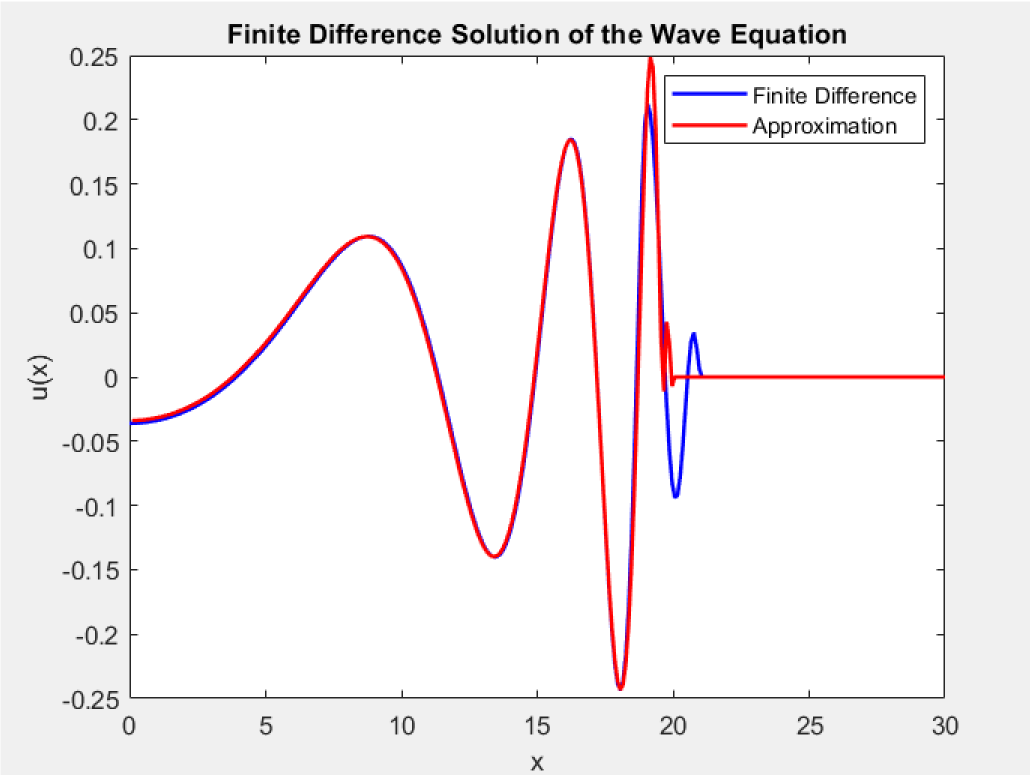

Figure 7: *A plot of the solution to u(x,t) obtained using the finite-difference method, with parameters* $\Delta x = 0.1$, $\frac{\Delta t}{\Delta x} = 1$, *t_final* = 20 & *q* = 1.

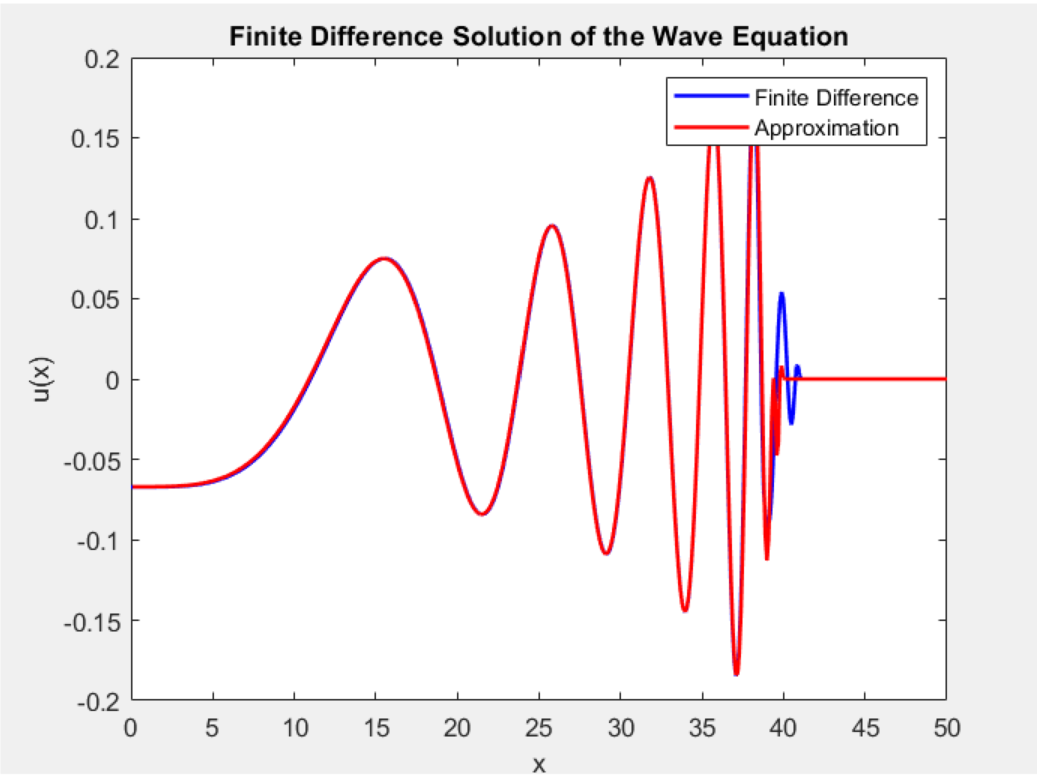

Figure 8: *A plot of the solution to u(x,t) obtained using the finite-difference method, with parameters* $\Delta x = 0.1$, $\frac{\Delta t}{\Delta x} = 1$, *t_final* = 40 & *q* = 1.

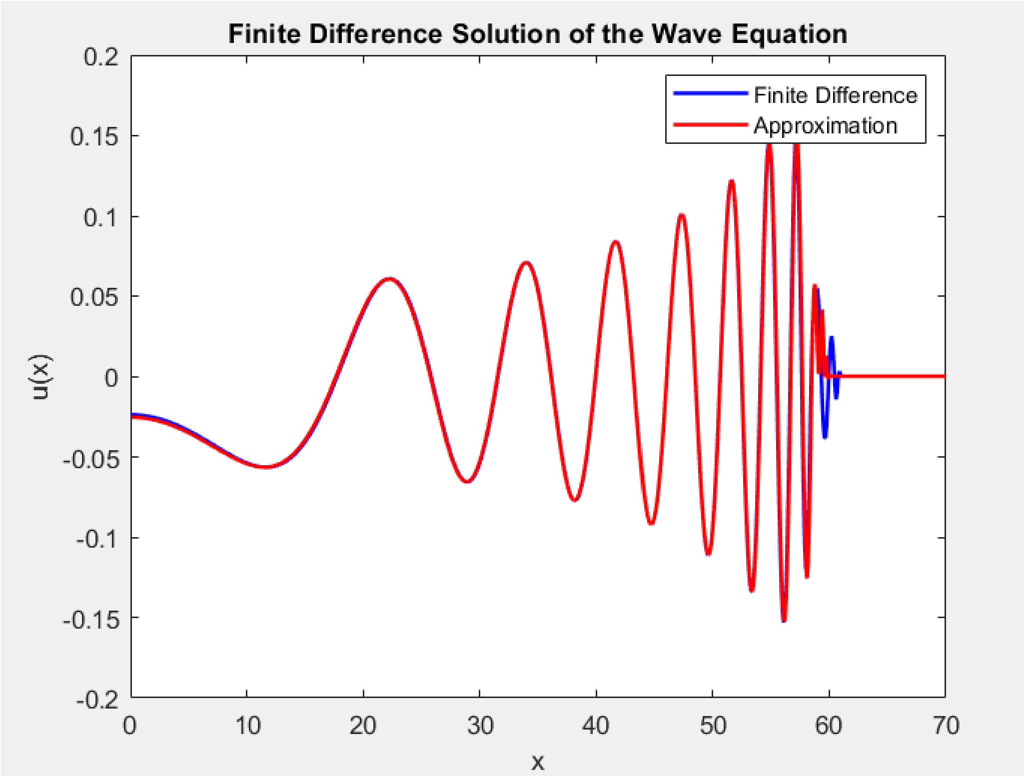

Figure 9: *A plot of the solution to u(x,t) obtained using the finite-difference method, with parameters* $\Delta x = 0.1$, $\frac{\Delta t}{\Delta x} = 1$, *t_final* = 60 *&* *q* = 1.

The graph from the finite-difference scheme matches really well with the graph of the stationary-phase approximation for each value of t, until at $x = t$, where there are slight differences. The graphs for $q = 0$ and $q = 1$ have distinct differences. For $q = 0$, the graphs take on a soliton-like shape, with the peak of the soliton changing depending on the value of t: as t gets bigger, the peak of the soliton moves more to the right. However, for $q = 1$, the graph takes on a more sinusoidal shape, with the width of the peaks and troughs decreasing as x increases, and the amplitude of each peak and trough generally increasing with x; to put it shortly, the peaks and troughs become sharper with x. There aren't any distinct differences that come with changing the value of t.

If q takes on a very small non-zero value instead, we will still expect to see the main wave propagating like in the case of $q = 0$, but the wave will no longer be perfectly preserved as the wave propagates. There will be dispersive spreading – instead of a sharp pulse, the wave will develop 'ripples' behind the main wave, subtle at first, but then more noticeable for larger times.

QUESTION 4

Now consider the Klein-Gordon equation for $x, t > 0$ with initial conditions $u(x, 0) = u_t(x, 0) = 0$ and boundary condition $u(0, t) = \sin(\omega_0 t)$. For the same reason as before, we can set $c_0 = 1$ without loss of generality.

To find the analytic solution of the equation with q = 0, firstly note what the equation becomes:

$$u_{tt} - u_{xx} = 0$$

Start with D'Alembert's solution:

$$u(x, t) = f(x - t) + g(x + t)$$

$$\Longrightarrow u(0, t) = f(-t) + g(t) = \sin(\omega_0 t)$$

$$\Longrightarrow f(-t) = \sin(\omega_0 t) - g(t)$$

Send $t \to t - x$:

$$\Longrightarrow f(x - t) = \sin(\omega_0 (t - x)) - g(t - x)$$

Substitute back into D'Alembert's solution:

$$u(x, t) = \sin(\omega_0 (t - x)) - g(t - x) + g(x + t)$$

We aim to work in the range x ≤ t; when in x > t, the influence of the boundary's not 'felt' and so the solution's zero due to the initial conditions being zero. For better understanding, we can think of it as such: a disturbance at $x = 0$ at time $t = 0$ can only travel as far as $x = t$ at time t.

So:

- If you are standing at a point x, the earliest you can feel anything from the boundary forcing is at time $t = x$.
- Before that, i.e., when $t < x$, nothing has reached you yet, so $u(x, t) = 0$ for all $x > t$.

Hence plugging in t = 0 and making use of the initial conditions in this case will cause issues. So we substitute x = t instead.

By continuity, at x = t, set $u(x, t) = 0$:

$$\sin(\omega_0(0)) - g(0) + g(2x) = 0$$

$$\Rightarrow -g(0) + g(2x) = 0$$

$$\Rightarrow g(0) = g(2x)$$

Hence $g(x)$ must be some constant k; without loss of generality, we can set $g(x) = 0$.

$$\Longrightarrow u(x, t) = f(x - t) = \sin(\omega_0 (t - x))$$

Hence the analytical solution when q = 0 is as follows:

$$u(x, t) = f(x) = \begin{cases} \sin(\omega_0 (t - x)), & x \le t \\ x \quad , & x > t \end{cases}$$

The program used to solve the above system is **Code 5** on page 22, labelled as

```
centred_difference_3(dx,lambda,t_final,q,w0)
```

The parameters entered into the program are:

`(dx, lambda, t_final, q, w0) = (0.1, 1, 150, 0, [0.9]).`

For each of the graphs below, the domain is chosen to be from 0 to the value of t_final, because, as suggested by the analytic solution, the graph should be zero for $x > t$.

For $q = 0$, the analytic solution is inclusively plotted alongside the numerical solution; see the graph below:

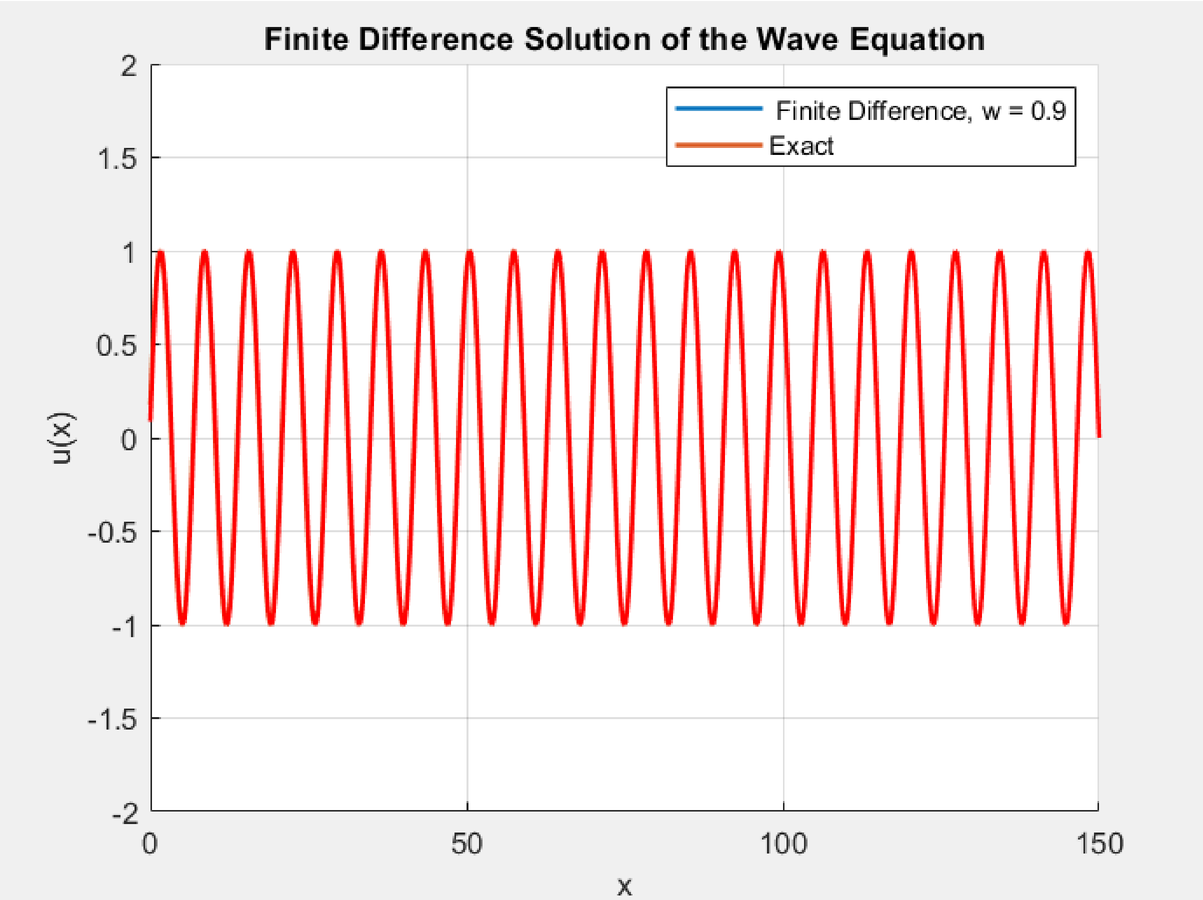

Figure 10: *A plot of the solution to u(x,t) obtained using the finite-difference method, with parameters* $\Delta x = 0.1$, $\frac{\Delta t}{\Delta x} = 1$, *t_final* = 150, $q = 0$ & $\omega_0 = 0.9$.

As the graph evidently shows, the two solutions clearly coincide, confirming that the program is highly accurate.

For $q = 1$, see below the graphs plotted for each value of w_0 on the same set of axes to clearly depict how the solution from the finite difference scheme alters with w_0, for t_final = 50, 100 and 150. As shown by the graphs, the value of t doesn't have much of an impact on how the graph comes out. However, we can see that as ω_0 increases, the peaks and troughs of the sinusoidal-shaped graph increase in amplitude and inclusively decrease in thickness, meaning the peaks and troughs are a lot shaper for larger ω_0.

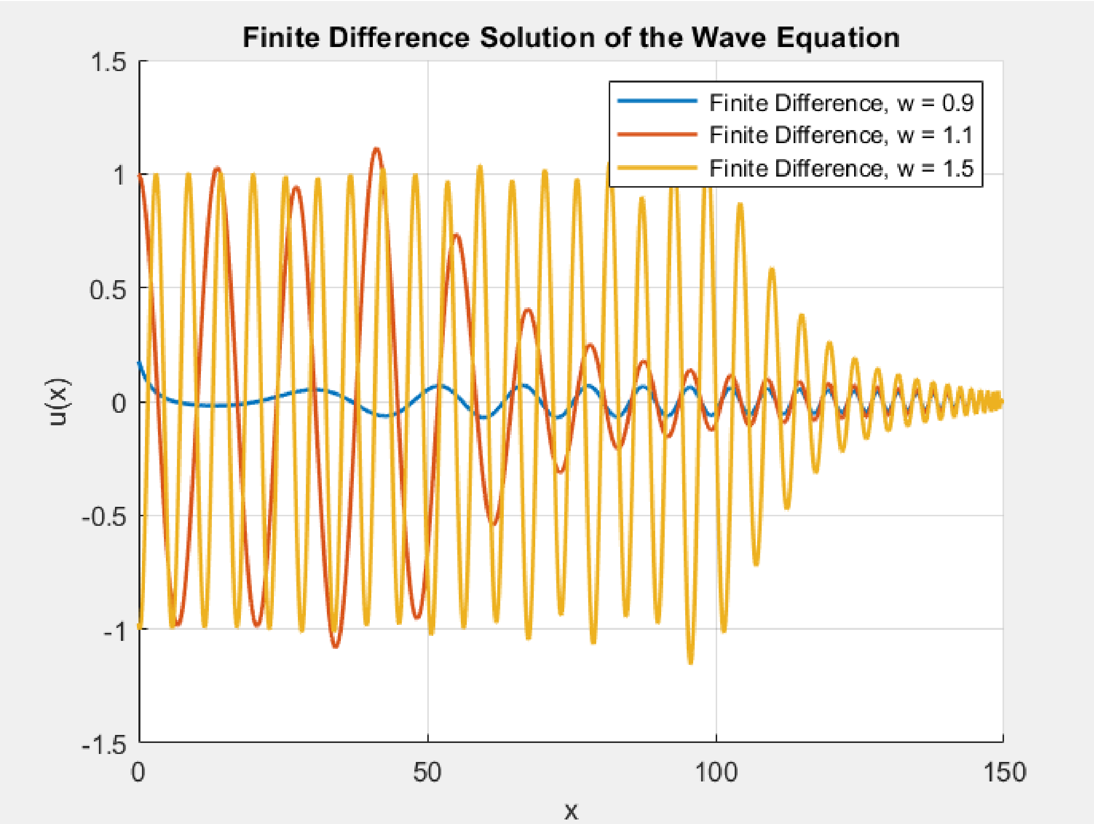

Figure 11: *A plot of the solution to u(x,t) obtained using the finite-difference method, with parameters* $\Delta x = 0.1$, $\frac{\Delta t}{\Delta x} = 1$, $t\_final = 150$, $q = 1$ & $\omega_0 = 0.9, 1.1, 1.5$.

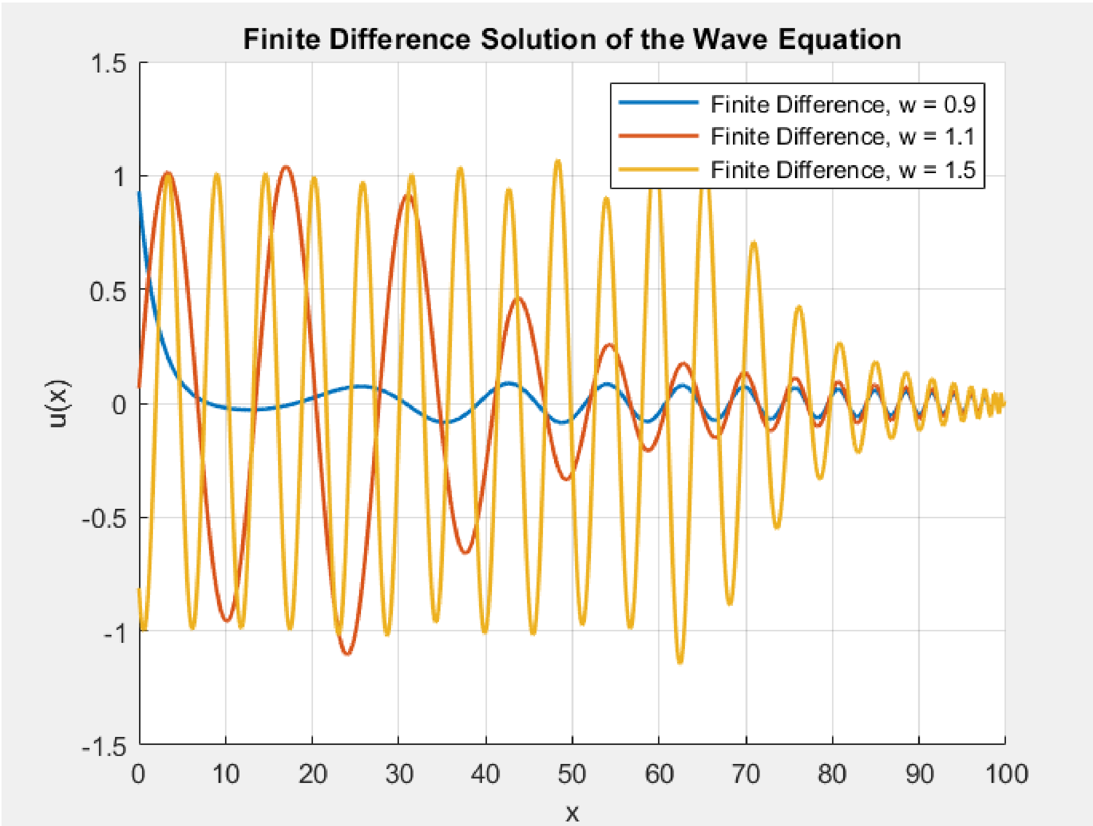

Figure 12: *A plot of the solution to u(x,t) obtained using the finite-difference method, with parameters* $\Delta x = 0.1$, $\frac{\Delta t}{\Delta x} = 1$, $t\_final = 100$, $q = 1$ & $\omega_0 = 0.9, 1.1, 1.5$

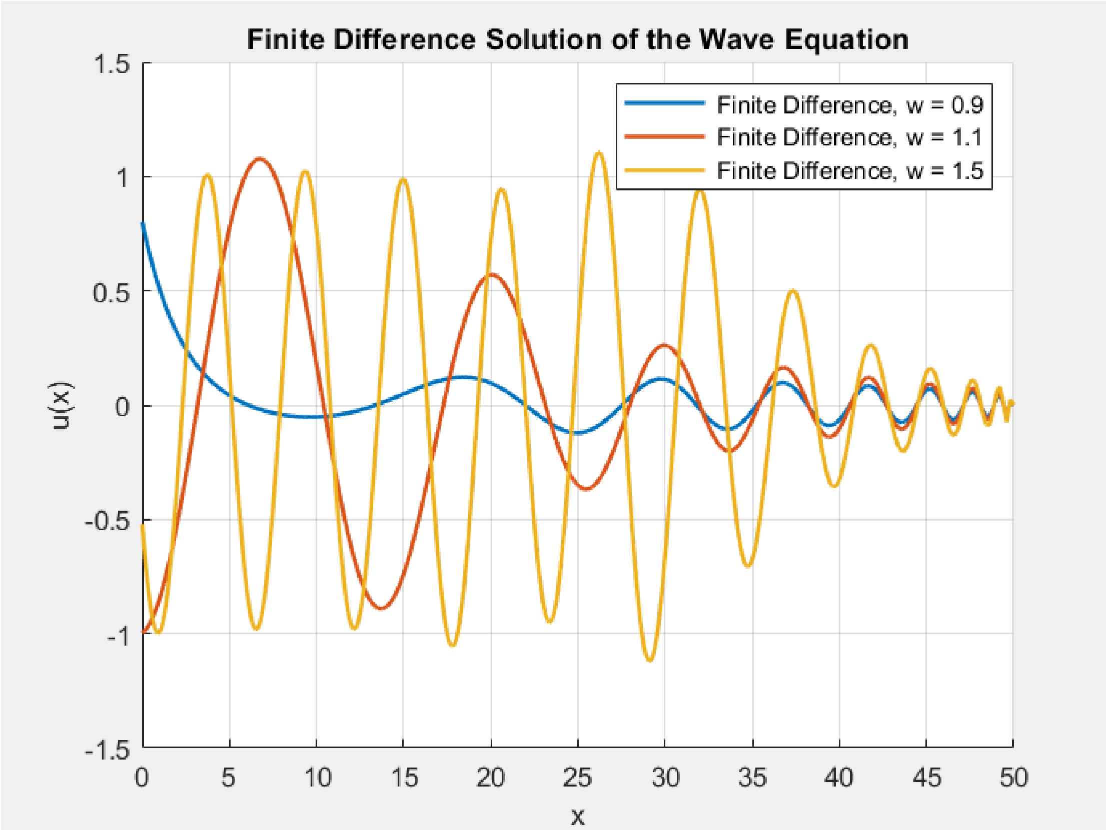

Figure 13: *A plot of the solution to u(x,t) obtained using the finite-difference method, with parameters* $\Delta x = 0.1$, $\frac{\Delta t}{\Delta x} = 1$, *t_final* = 50, q = 1 & $\omega_0 = 0.9, 1.1, 1.5$.

As ω_0 increases, the magnitude of the graph tends closer and closer to 1 – this trend follows through if we inclusively enter larger values of ω_0 like 5 and 10. This aligns with what we know about phase velocity and group velocity; when ω_0 is larger, the frequency's also larger. From the dispersion relation, it can be understood that this means the value of k would be larger for bigger ω_0. Looking back at the graph sketch in **Question 1** of phase velocity and group velocity against k, it can be seen that as k tends to infinity, both phase and group velocity tend to 1, which is the same behaviour exhibited by the graphs above.

Programs

CODE 1

```
function centred_difference(dx,lambda,t_finals,q)
%lambda is the ratio of the time step and the spatial step
dt = dx*lambda;
x_final = 60;
N = x_final/dx;
x = linspace(0,x_final,N);

u_0 = zeros(size(x));

%set initial condition u(x,0) = f(x)
for n = 1:length(x)
    if x(n) < 1
        u_0(n) = (1 - x(n)^2).^2;
    else
        u_0(n) = 0;
    end
end

colors = lines(length(t_finals));
legend_entries = cell(1, length(t_finals));

figure;
hold on;

u_new = zeros(size(x));

for k = 1:length(t_finals)
    t_final = t_finals(k);
    if t_final == 0
        u_new = u_0;
    else
        %first time step using the given symmetric condition
        u = zeros(size(x));
        for i = 1:N-1
            if i == 1
                u(i) = u_0(i) + ((lambda)^2)*(u_0(i+1)-u_0(i)) -
0.5*((q*dt)^2)*(u_0(i+1));
            else
                u(i) = u_0(i) + 0.5*((lambda)^2)*(u_0(i+1)-2*u_0(i)+u_0(i-1)) -
0.25*((q*dt)^2)*(u_0(i+1)+u_0(i-1));
            end
        end

        u_prev = u_0;

        for j = 2:t_final/dt
            for i = 1:N-1
                if i == 1
                    u_new(i) = 2*u(i) - u_prev(i) + 2*((lambda)^2)*(u(i+1)-u(i)) -
((q*dt)^2)*(u(i+1));
```

```
                else
                    u_new(i) = 2*u(i) - u_prev(i) + ((lambda)^2)*(u(i+1)-
2*u(i)+u(i-1)) - 0.5*((q*dt)^2)*(u(i+1)+u(i-1));
                end
            end
            u_prev = u;
            u = u_new;
        end
    end
    plot(x, u_new,'LineWidth', 1.5, 'Color', colors(k,:));
    legend_entries{k} = sprintf('t = %d', t_final);
end

xlabel('x'), ylabel('u(x)')
title('Finite Difference Solution of the Wave Equation');
legend(legend_entries);

end
```

CODE 2

```
function[f] = fourier_f(k)
f = -((16/(k^3))*sin(k)) - ((48/(k^4))*cos(k)) + ((48/(k^5))*sin(k));
end
```

CODE 3

```
k_vector = linspace(0,50,500);
fourier_f_vector = zeros(size(k_vector));
for k = 1:length(k_vector)
    fourier_f_vector(k) = fourier_f(k_vector(k));
end

plot(k_vector, fourier_f_vector, 'b-');
xlabel('k'), ylabel('$\tilde{f}$(k)', 'Interpreter', 'latex');
title('Fourier transform of f(x)')
grid on;
```

CODE 4

```
function centred_difference_2(dx,lambda,t_final,q)
%lambda is the ratio of the time step and the spatial step
x_final = t_final + 10;
dt = dx*lambda;
N = x_final/dx;
x = linspace(0,x_final,N);
u_0 = zeros(size(x));

%set initial condition u(x,0) = f(x)
for n = 1:length(x)
    if x(n) < 1
        u_0(n) = (1 - x(n)^2).^2;
    else
        u_0(n) = 0;
    end
end

u_new = zeros(size(x));

if t_final == 0
    u_new = u_0;
else
    %first time step using the given symmetric condition
    u = zeros(size(x));
    for i = 1:N-1
        if i == 1
            u(i) = u_0(i) + ((lambda)^2)*(u_0(i+1)-u_0(i)) -
0.5*((q*dt)^2)*(u_0(i+1));
        else
            u(i) = u_0(i) + 0.5*((lambda)^2)*(u_0(i+1)-2*u_0(i)+u_0(i-1)) -
0.25*((q*dt)^2)*(u_0(i+1)+u_0(i-1));
        end
    end

    u_prev = u_0;

    for j = 2:t_final/dt
        for i = 1:N-1
            if i == 1
                u_new(i) = 2*u(i) - u_prev(i) + 2*((lambda)^2)*(u(i+1)-u(i)) -
((q*dt)^2)*(u(i+1));
            else
                u_new(i) = 2*u(i) - u_prev(i) + ((lambda)^2)*(u(i+1)-2*u(i)+u(i-
1)) - 0.5*((q*dt)^2)*(u(i+1)+u(i-1));
            end
        end
        u_prev = u;
        u = u_new;
    end
end

u_approx = zeros(size(x));
```

```
for n = 1:length(x)
    x_n = x(n);
    if x_n < t_final
        u_approx_1 = (q^0.5 * t_final)/((2*pi)^0.5 * (t_final^2 - x_n^2).^0.75);
        u_approx_2 = fourier_f((x_n*q)/sqrt(t_final^2 - x_n^2));
        u_approx_3 = cos(q*sqrt(t_final^2 - x_n^2) + pi/4);
        u_approx(n) = u_approx_1*u_approx_2*u_approx_3;
    else
        u_approx(n) = 0;
    end
end

plot(x, u_new, 'b-','LineWidth', 1.5); hold on;
plot(x, u_approx, 'r-','LineWidth', 1.5)
legend('Finite Difference', 'Approximation')
xlabel('x'), ylabel('u(x)')
title('Finite Difference Solution of the Wave Equation');

end
```

CODE 5

```
function centred_difference_3(dx,lambda,t_final,q,w0)
%lambda is the ratio of the time step and the spatial step
x_final = t_final;
dt = dx*lambda;
N = x_final/dx;
x = linspace(0,x_final,N);
u_0 = zeros(size(x));

colors = lines(length(w0));
legend_entries = cell(1, length(w0));

figure;
hold on;
grid on;

u_new = zeros(size(x));

for k = 1:length(w0)
    w_0 = w0(k);
    if t_final == 0
        u_new = u_0;
    else
        %first time step using the given symmetric condition
        u = zeros(size(x));
        for i = 1:N-1
            if i == 1
                u(i) = 0;
            else
                u(i) = u_0(i) + 0.5*((lambda)^2)*(u_0(i+1)-2*u_0(i)+u_0(i-1)) -
0.25*((q*dt)^2)*(u_0(i+1)+u_0(i-1));
            end
        end
```

```
        u_prev = u_0;

        for j = 2:t_final/dt
            for i = 1:N-1
                if i == 1
                %this time if i==1, it's u(0,t) so use the initial condition
                    u_new(i) = sin(w_0*(j-1)*dt);
                else
                    u_new(i) = 2*u(i) - u_prev(i) + ((lambda)^2)*(u(i+1)-
2*u(i)+u(i-1)) - 0.5*((q*dt)^2)*(u(i+1)+u(i-1));
                end
            end
            u_prev = u;
            u = u_new;
        end
    end

    plot(x, u_new,'LineWidth', 1.5,'Color', colors(k,:));
    legend_entries{k} = sprintf(' Finite Difference, w = %.1f', w_0);

    if q == 0
        u_exact = sin(w_0*(t_final-x));
        plot(x, u_new,'LineWidth', 1.5)
        plot(x, u_exact, 'r-', 'LineWidth', 1.5)
        ylim([-2 2]);
        legend_entries{end+1} = 'Exact';
    end
end

xlabel('x'), ylabel('u(x)');
title('Finite Difference Solution of the Wave Equation');
legend(legend_entries);

end
```