2.2 Parallel-Plate Capacitor: Laplace's Equation

QUESTION 1

$$\Phi\big((i-1)h\,,jh\big)+\Phi\big((i+1)h\,,jh\big)+\Phi(ih\,,(j-1)h)+\Phi(ih\,,(j+1)h)-4\Phi(ih\,,jh)=0$$

Rearranging the terms on the left-hand side and then dividing both sides by h, we get the following:

$$\frac{\Phi\big((i-1)h\,,jh\big)-\Phi(ih\,,jh)}{h}+\frac{\Phi\big((i+1)h\,,jh\big)-\Phi(ih\,,jh)}{h}+\frac{\Phi(ih\,,(j-1)h)-\Phi(ih\,,jh)}{h}+\frac{\Phi(ih\,,(j+1)h)-\Phi(ih\,,jh)}{h}=0$$

Taking the limit $h\to 0$, we get

$$-\frac{\partial\Phi}{\partial X}\big((i-1)h\big)+\frac{\partial\Phi}{\partial X}(ih)-\frac{\partial\Phi}{\partial Y}\big((j-1)h\big)+\frac{\partial\Phi}{\partial Y}(jh)=0$$

Rearranging the terms on the left-hand side and then dividing both sides by h once again, we get:

$$\frac{\frac{\partial\Phi}{\partial X}(ih)-\frac{\partial\Phi}{\partial X}\big((i-1)h\big)}{h}+\frac{\frac{\partial\Phi}{\partial Y}(jh)-\frac{\partial\Phi}{\partial Y}\big((j-1)h\big)}{h}=0$$

Taking the limit $h\to 0$, we get

$$\frac{\partial^2\Phi}{\partial X^2}\big((i-1)h\big)+\frac{\partial^2\Phi}{\partial Y^2}\big((j-1)h\big)=0$$

As i and j are arbitrary, we get that as h → 0, this numerical solution approximates a solution of the Laplace equation.

Programming Task:

The program used to carry out the successive over-relaxation (SOR) method to determine Φ in the positive quadrant of D is **Code 1** on page 20, labelled as

```
SOR(L, DX, DY, h, omega, tol)
```

QUESTION 2

Using **Code 1,** we get the following table of values for Φ and 3D plot:

Table 1: Φ(X, Y) when L=1, D_X=D_Y=2, h=0.5, ω=1 and $\varepsilon_{tol}=10^{-6}$

		y				
		0	0.5	1	1.5	2
x	0	0.0000	0.2440	0.5000	0.2440	0.0000
	0.5	0.0000	0.2381	0.5000	0.2381	0.0000
	1	0.0000	0.2083	0.5000	0.2083	0.0000
	1.5	0.0000	0.0952	0.1726	0.0952	0.0000
	2	0.0000	0.0000	0.0000	0.0000	0.0000

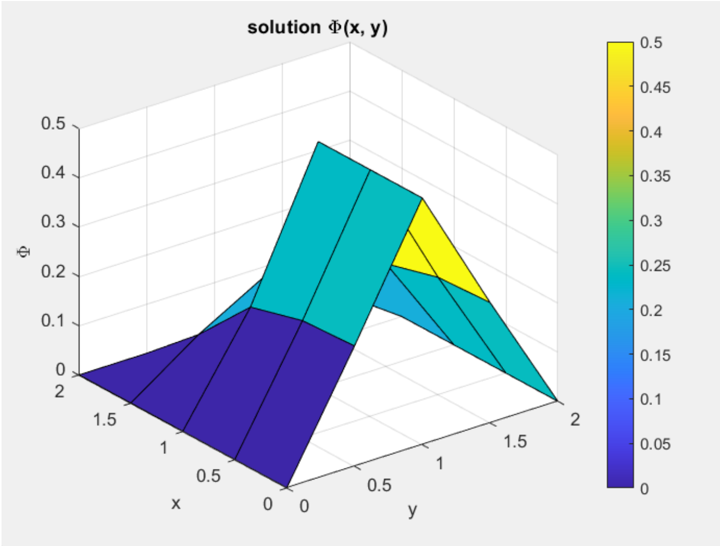

Figure 1: *A 3D plot of the solution to Φ obtained using the SOR method, with input parameters L=1, D_X=D_Y=2, h=0.5, ω=1 and ε_{tol}=10^{-6}.*

When setting the error tolerance to be 10^{-6}, we get that $\Phi(0.5, 0.5) = \Phi_{1,1} = \Phi_{1,3} = \Phi(0.5, 1.5) =$ 0.2381 to four decimal places, which is 0.238 to 3 decimal places. To verify that the validity of the method does not depend on the initial guess $\Phi^{(0)}$, just try the iteration with two other initial guesses, e.g. a matrix of just ones and a matrix with each row being 1, 2, 3, 4, 5 respectively. When we try both of these, we get the exact same table of values and 3D plot, verifying our belief.

QUESTION 3

The program used to plot the numerical approximations to Φ(0, Y) and Φ(2, Y) for $0 < Y < D_Y$ when L=2, D_X=D_Y=4, h=0.25, ω=1 and ε_{tol}=10^{-6} is **Code 2** on page 21, labelled as

```
SOR_2(L, DX, DY, h, omega, tol)
```

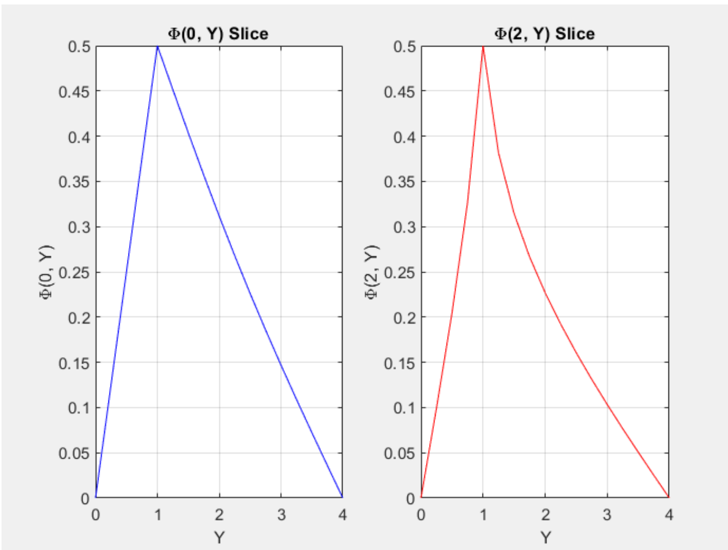

Figure 2: *2D plot of the solution to Φ(0,Y) and Φ(2,Y) obtained using the SOR method, with input parameters L=2, D_X=D_Y=4, h=0.25, ω=1 and ε_{tol}=10^{-6}.*

We are given that $\varepsilon_Y \approx \pm\frac{1}{h}[\Phi(X, Y \mp h) - \Phi(X, Y)]$, so substituting Y=0 gives:

$$\varepsilon_Y \approx \pm\frac{1}{h}[\Phi(X, \mp h) - \Phi(X, 0)]$$

Because Φ is odd in Y,

- $\Phi(X, 0) = 0$
- $\frac{1}{h}[\Phi(X, -h)] = -\frac{1}{h}[\Phi(X, h)]$

So, we can just say that:

$$\boldsymbol{\varepsilon_Y(X, 0) \approx -\frac{1}{h}[\Phi(X, h)]}$$

The program used to plot the estimate of $\varepsilon_Y(X, 0)$ for L=2, $D_X=D_Y=4$, h=0.25, $\omega=1$ and $\varepsilon_{tol}=10^{-6}$ is **Code 3** on page 22, labelled as

```
SOR_3_efield(L, DX, DY, h, omega, tol)
```

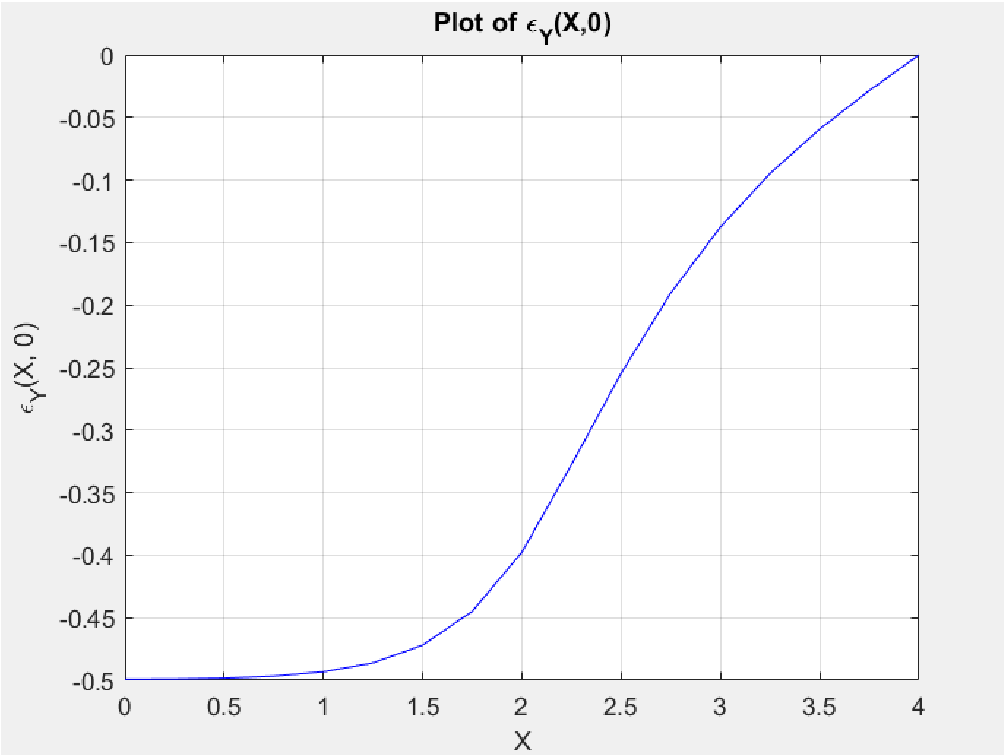

Figure 3: *2D plot of the solution to $\varepsilon_Y(X,0)$ obtained using the SOR method, with input parameters L=2, $D_X=D_Y=4$, h=0.25, $\omega=1$ and $\varepsilon_{tol}=10^{-6}$.*

The program used to plot the estimate of $\varepsilon_Y(X, 1)$ for L=2, $D_X=D_Y=4$, h=0.25, $\omega=1$ and $\varepsilon_{tol}=10^{-6}$ on both the lower surface and upper surface is **Code 4** on page 23, labelled as

```
SOR_3_efield_plate(L, DX, DY, h, omega, tol)
```

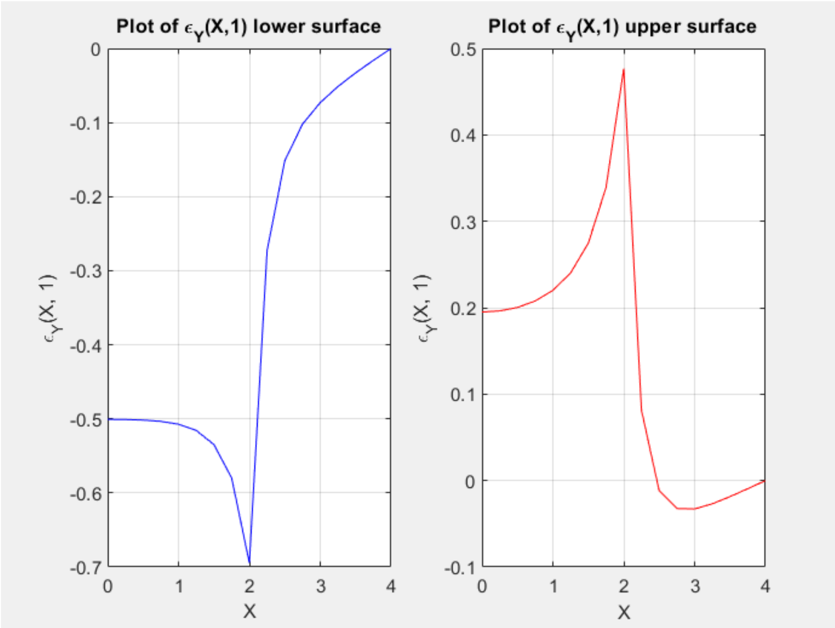

Figure 4: *2D plot of the solution to $\varepsilon_Y(X,1)$ on the lower and upper surfaces of the plate obtained using the SOR method, with input parameters $L=2$, $D_X=D_Y=4$, $h=0.25$, $\omega=1$ and $\varepsilon_{tol}=10^{-6}$.*

To check the accuracy of this program, I inclusively outputted the table with the values of Φ for smaller tolerances, like 10^{-15}, and saw that they all outputted the same values of Φ to four significant figures.

QUESTION 4

Firstly look at how the plots of the numerical approximations to Φ(0, Y) and Φ(2, Y) change with h, with all other parameters remaining the same.

h=1/8:

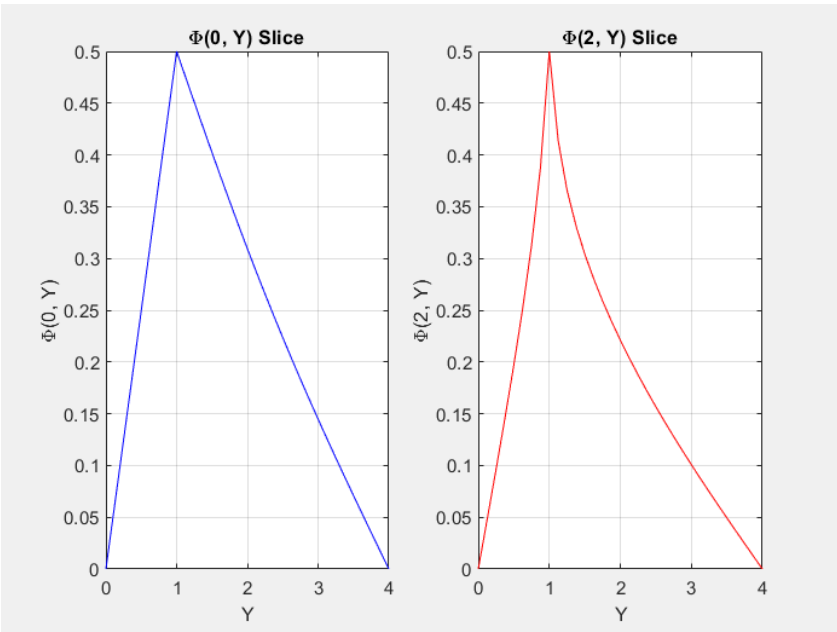

Figure 5: *2D plot of the solution to Φ(0,Y) and Φ(2,Y) obtained using the SOR method, with input parameters $L=2$, $D_X=D_Y=4$, $h=1/8$, $\omega=1$ and $\varepsilon_{tol}=10^{-6}$.*

h=1/12:

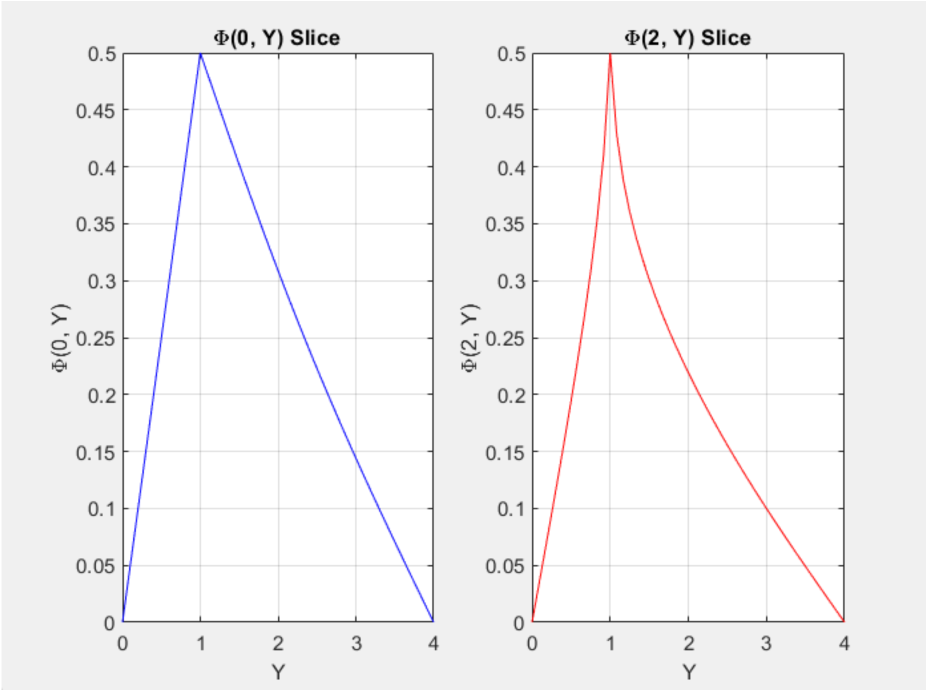

Figure 6: *2D plot of the solution to $\Phi(0,Y)$ and $\Phi(2,Y)$ obtained using the SOR method, with input parameters $L=2$, $D_X=D_Y=4$, $h=1/12$, $\omega=1$ and $\varepsilon_{tol}=10^{-6}$.*

h=1/20:

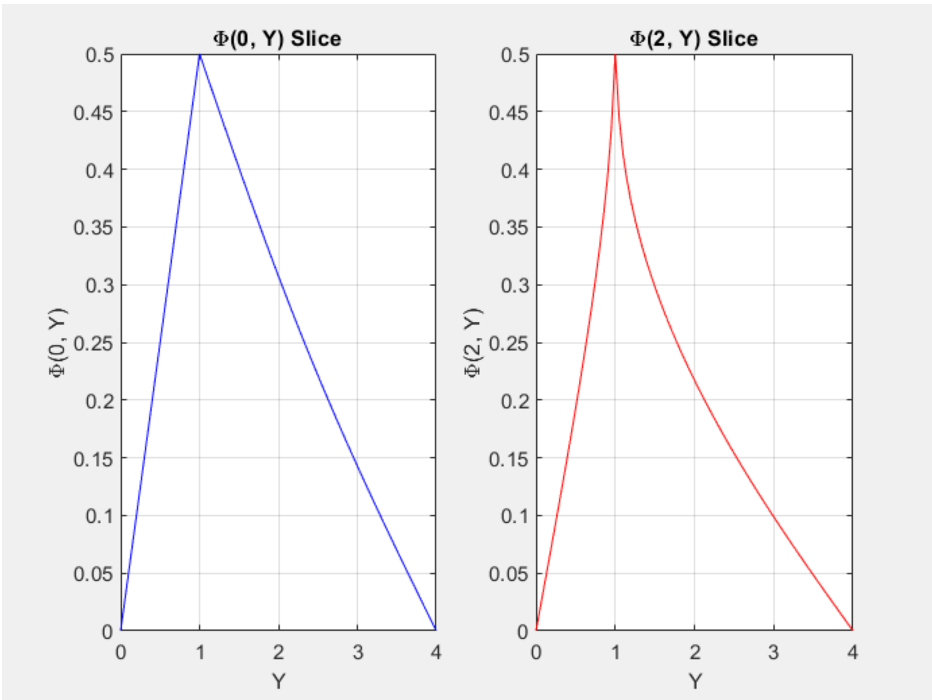

Figure 7: *2D plot of the solution to $\Phi(0,Y)$ and $\Phi(2,Y)$ obtained using the SOR method, with input parameters $L=2$, $D_X=D_Y=4$, $h=1/20$, $\omega=1$ and $\varepsilon_{tol}=10^{-6}$.*

h=1/50:

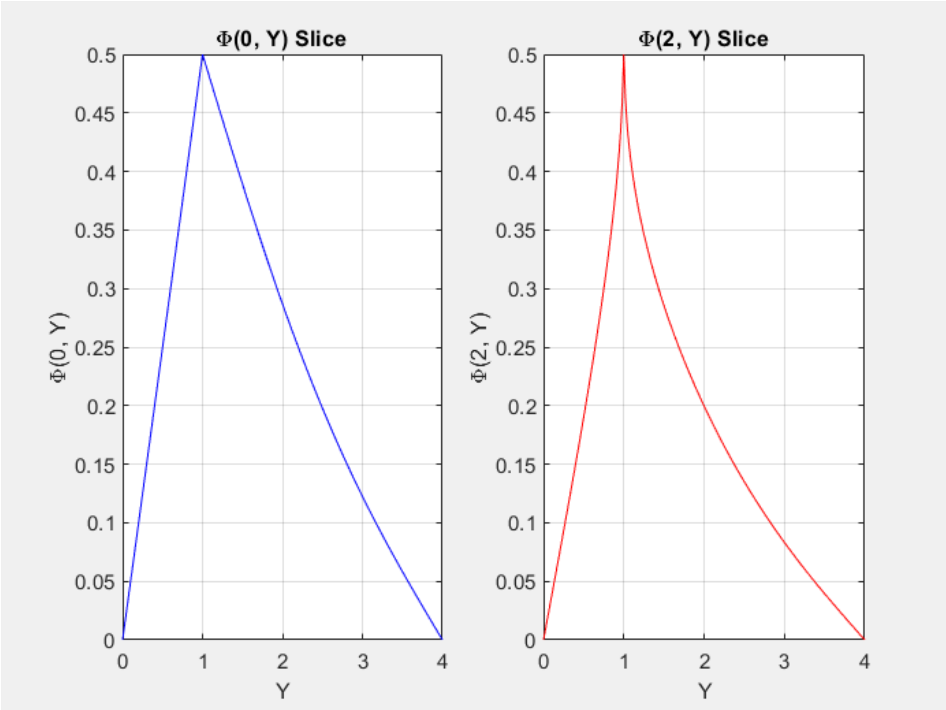

Figure 8: *2D plot of the solution to $\Phi(0,Y)$ and $\Phi(2,Y)$ obtained using the SOR method, with input parameters $L=2$, $D_X=D_Y=4$, $h=1/50$, $\omega=1$ and $\varepsilon_{tol}=10^{-6}$.*

h=1/100:

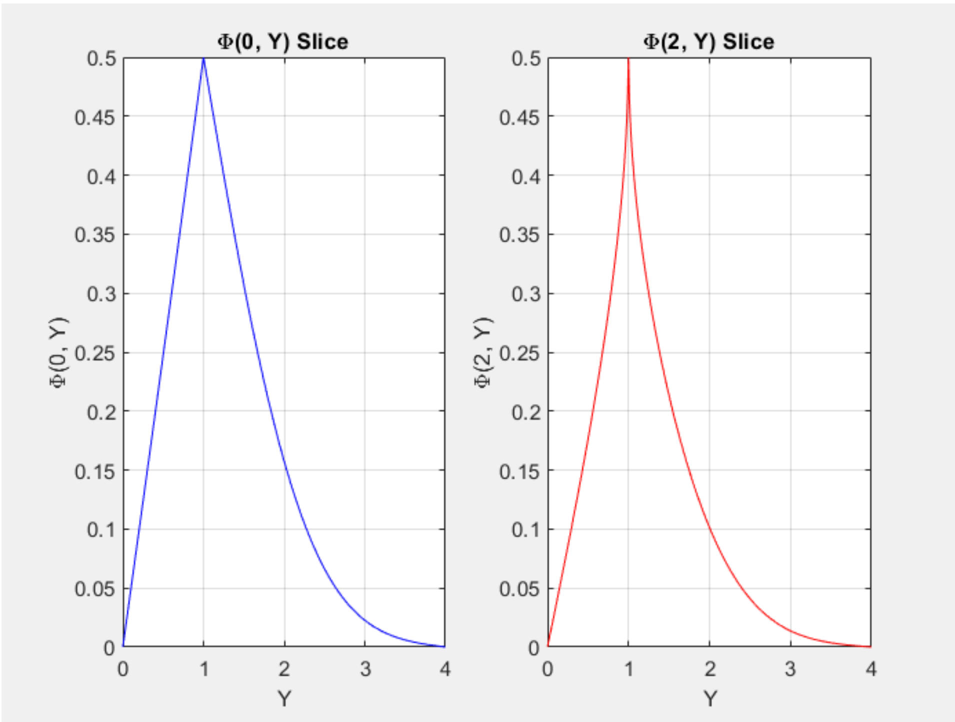

Figure 9: *2D plot of the solution to $\Phi(0,Y)$ and $\Phi(2,Y)$ obtained using the SOR method, with input parameters $L=2$, $D_X=D_Y=4$, $h=1/100$, $\omega=1$ and $\varepsilon_{tol}=10^{-6}$.*

Then look at how the plot of the numerical approximation to $\varepsilon_Y(X,0)$ changes with h, with all other parameters remaining the same.

h=1/8:

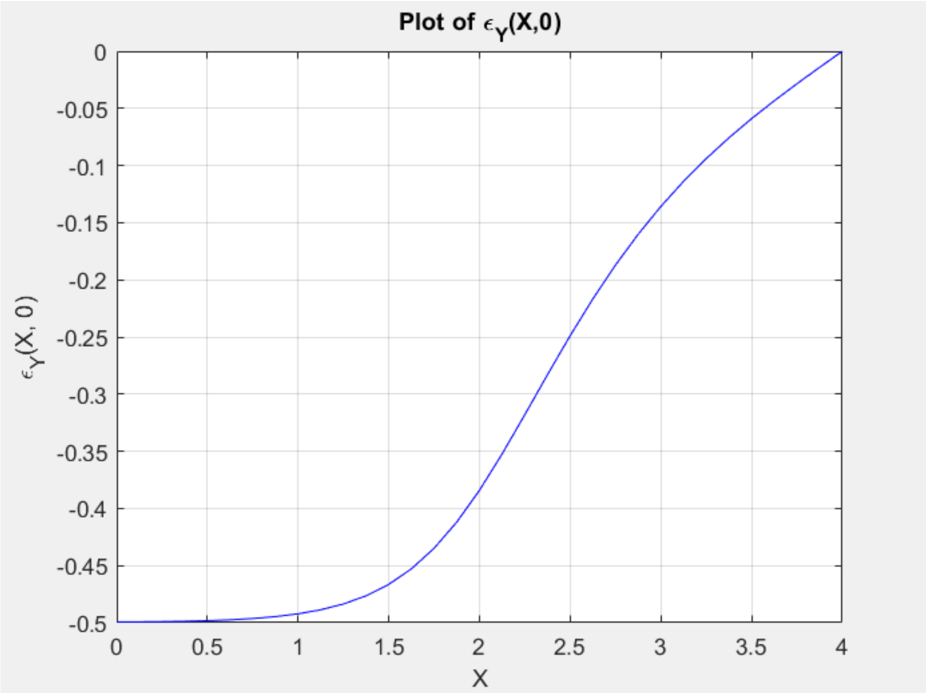

Figure 10: *2D plot of the solution to $\varepsilon_Y(X,0)$ obtained using the SOR method, with input parameters $L=2$, $D_X=D_Y=4$, $h=1/8$, $\omega=1$ and $\varepsilon_{tol}=10^{-6}$.*

h=1/12:

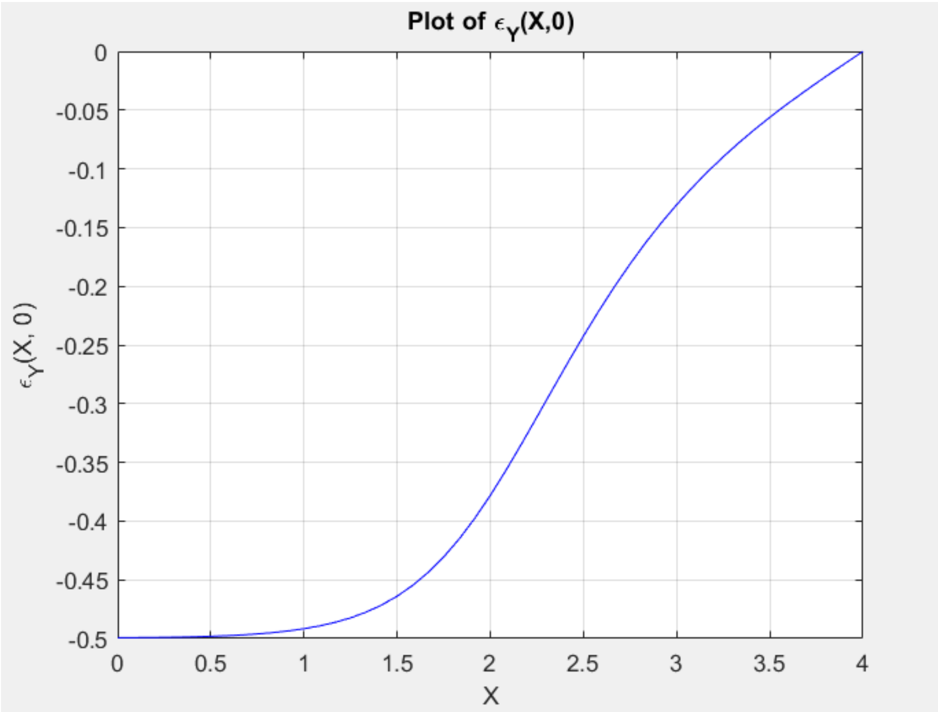

Figure 11: *2D plot of the solution to $\varepsilon_Y(X,0)$ obtained using the SOR method, with input parameters $L=2$, $D_X=D_Y=4$, $h=1/12$, $\omega=1$ and $\varepsilon_{tol}=10^{-6}$.*

h=1/20:

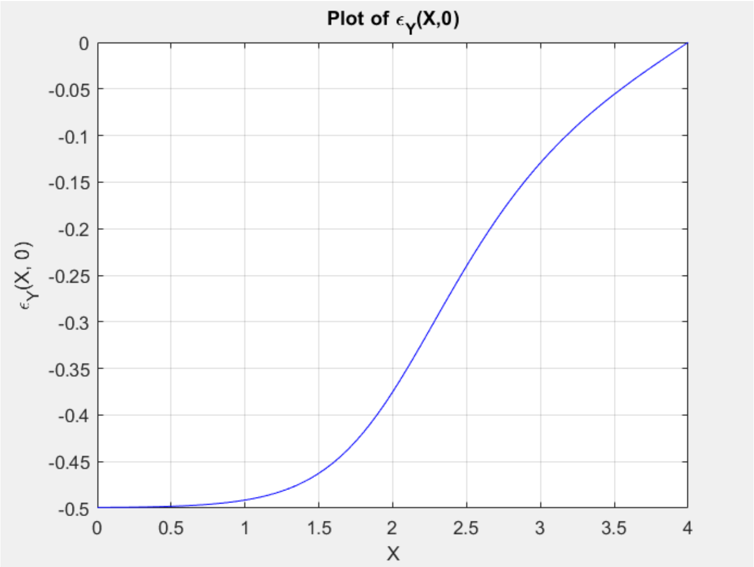

Figure 12: *2D plot of the solution to $\varepsilon_Y(X,0)$ obtained using the SOR method, with input parameters L=2, D_X=D_Y=4, h=1/20, ω=1 and $\varepsilon_{tol}=10^{-6}$.*

h=1/50:

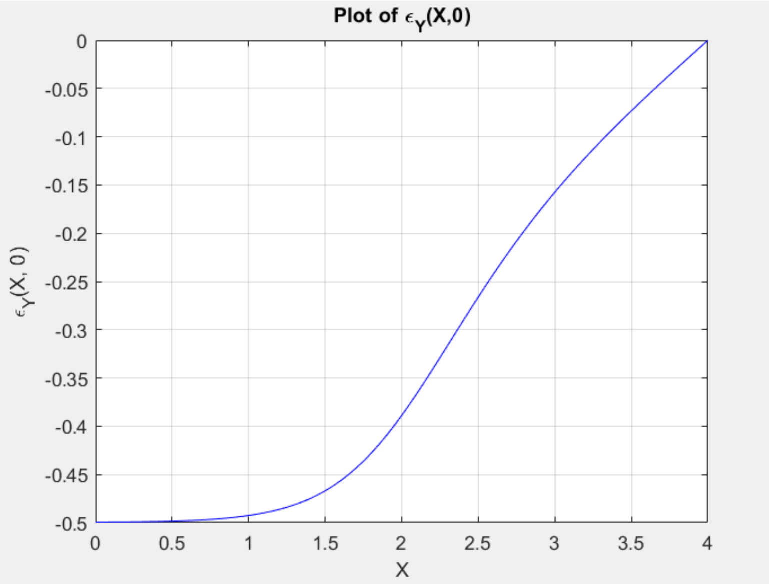

Figure 13: *2D plot of the solution to $\varepsilon_Y(X,0)$ obtained using the SOR method, with input parameters L=2, D_X=D_Y=4, h=1/50, ω=1 and $\varepsilon_{tol}=10^{-6}$.*

As the value of h decreases, the peaks for $\Phi(0,Y)$ and $\Phi(2,Y)$ become sharper, which is expected and consistent with convergence to a suitable solution to the Laplace equation. Furthermore, we would expect it to be a straight line because in the analytic case, as stated in the project booklet, for large L the expected physical behaviour in the gap between the plates is that $\nabla\Phi$ is (almost) aligned with the Y – direction, which implies little to no X dependence. Hence to solve Laplace's equation, we just have to solve:

$$\frac{\partial^2 \Phi}{\partial Y^2} = 0 \quad \Rightarrow \quad \Phi = AY + B$$

Using boundary conditions $\Phi = \pm 0.5$ at $Y = \pm 1$, we get that A=0.5, B=0, meaning that we get:

$$\Phi = \frac{1}{2}Y$$

, which aligns with what is shown in the graphs.

QUESTION 5

The number of iterations required for convergence of the SOR code, using the same parameters as in **Question 3** (L=2, $D_X=D_Y=4$, h=1/4, $\omega=1$ and $\varepsilon_{tol}=10^{-6}$) is 213.

To investigate how this depends on ω, I added the whole iteration within another FOR loop that runs through different values of ω within the range $1 \leq \omega < 2$ with 0.01 intervals and plots a graph of ω against the number of iterations. The program used to carry this out is **Code 5** on page 24, labelled as

```
SOR_4_changing_omega(L, DX, DY, h, tol)
```

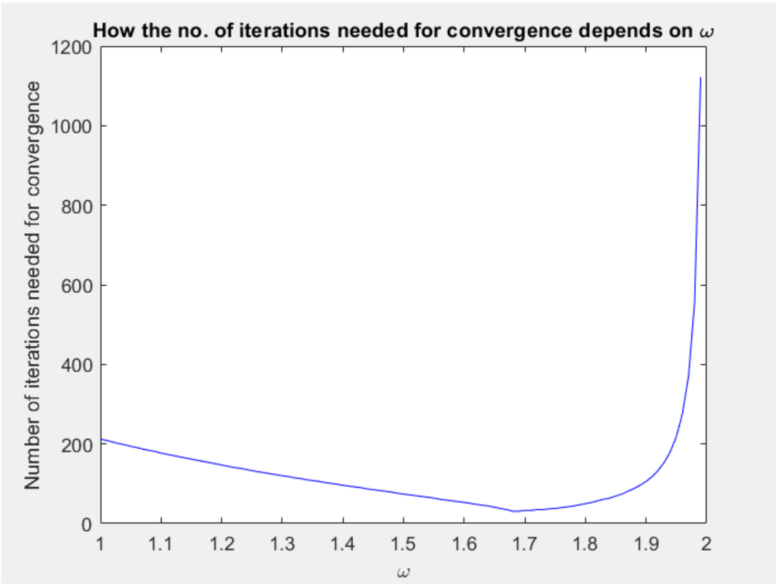

Figure 14: *Graph to indicate how the number of iterations changes with ω, with input parameters L=2, $D_X=D_Y=4$, h=0.25, $\omega=1$ and $\varepsilon_{tol}=10^{-6}$.*

From this, we can see that the optimum value of ω is approximately 1.7.

QUESTION 6

Try out values (D_X, D_Y) = (10, 10), (20, 20), (20, 25), (30, 30), (50, 60) and (80, 100) (testing out cases where $D_X = D_Y$ and also $D_X \neq D_Y$). I've plotted some graphs at the end to display the behaviour, using **Code 1** to do so.

From these, we can see that as D_X and D_Y increase in value, due to the boundary conditions of $\Phi = 0$, the right side of the peak spreads out more. Intuitively thinking, this makes sense because if the boundary value of Φ is zero, but the boundary is made to be further from the peak, the function can be non-zero over a larger area. The right side of the peak looks to decrease in a way that indicates proportionality to $1/r$, which aligns with what we know about electric potential generally.

As for the left side of the peak, this remains a straight line for all values of D_X and D_Y, which does make sense as the distance till the boundary on that side is not increasing, and it is a requirement that $\Phi(X, 0) = 0$.

Furthermore, as proven in **Question 4,** the analytic solution for the left side of the peak is $\Phi = \frac{1}{2}Y$, meaning that we would expect a straight line for the left side of the peak.

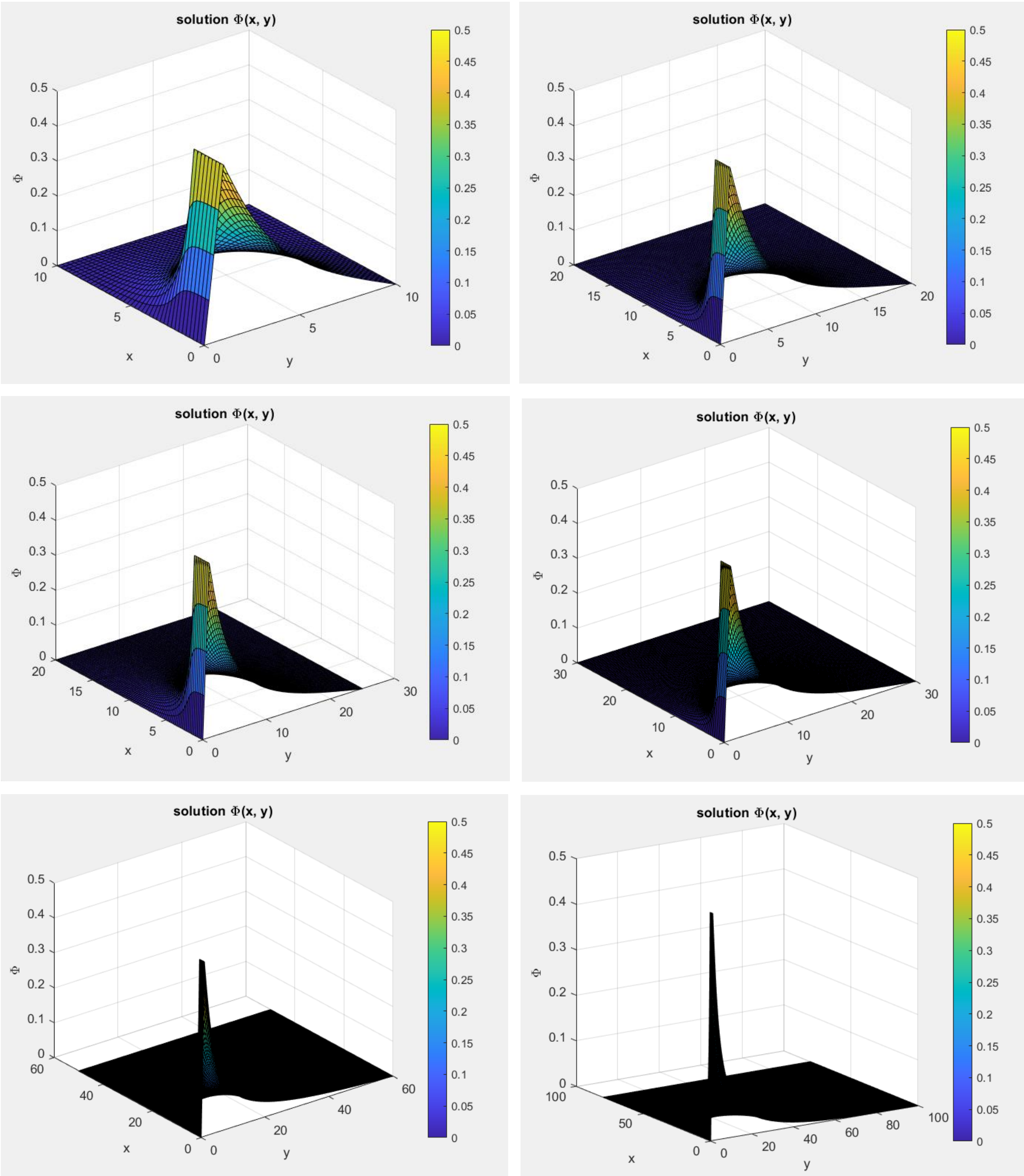

Figure 15: *Six 3D graphs of $\Phi(X, Y)$ for various D_X and D_Y to indicate how much the numerical solutions near the plates are affected by the boundary condition that $\Phi = 0$ on the boundary of D.*

QUESTION 7

We are given the following definition: $W = -\Phi + i\Psi$. To interpret this parametrically, take Φ=c and Ψ=k, where c and k are both real constants, with c ∈ [− 1/2, 1/2], so now W = −c + ik.

Also recall that we are given the following equation relating X, Y, L and W:

$$(X - L) + iY = \frac{1 + e^{-2\pi i W}}{\pi} - 2iW$$

Substitute the formula we have for W into this, and L=0:

$$X + iY = \frac{1 + e^{-2\pi i(-c+ik)}}{\pi} - 2i(-c + ik)$$

$$X + iY = \frac{1 + e^{2\pi i c + 2\pi k}}{\pi} + 2ic + 2k$$

$$X + iY = \frac{1 + e^{2\pi k}(\cos(2\pi c) + \mathrm{i}\sin(2\pi c))}{\pi} + 2ic + 2k$$

$$X + iY = \frac{1 + e^{2\pi k}\cos(2\pi c)}{\pi} + 2k + 2ic + \frac{\mathrm{i}e^{2\pi k}\sin(2\pi c)}{\pi}$$

Equating real and imaginary parts, we obtain the following parametric equations:

$$X = \frac{1 + e^{2\pi k}\cos(2\pi c)}{\pi} + 2k \qquad\qquad Y = \frac{e^{2\pi k}\sin(2\pi c)}{\pi} + 2c$$

To plot each equipotential line, we need to keep c fixed and vary k.

The program used to plot the equipotential lines when L=0 is **Code 6** on page 25, labelled as

`equipotential_lines`

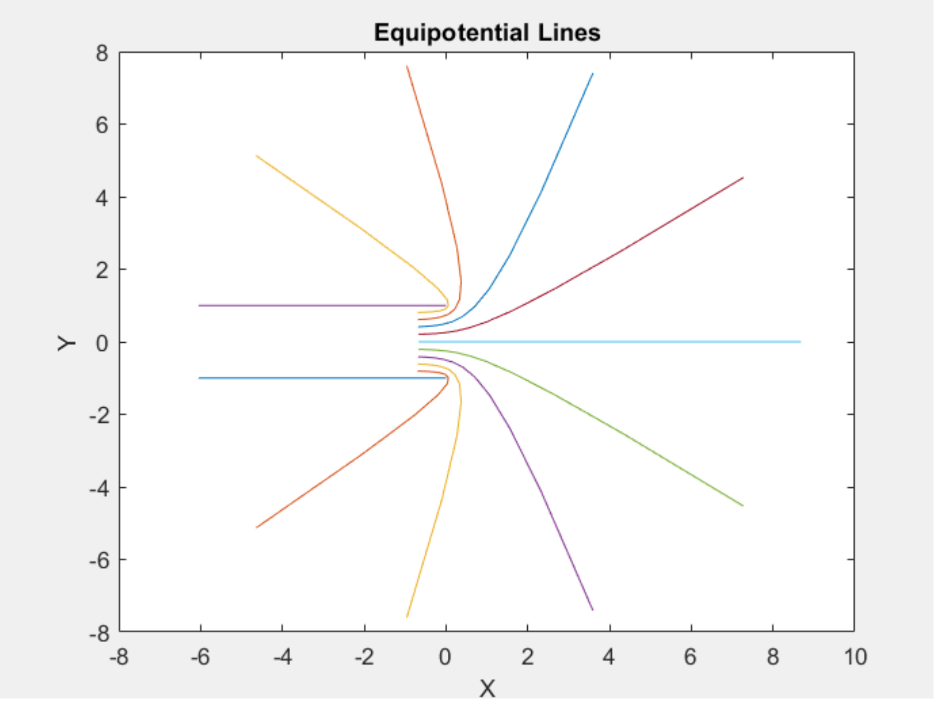

Figure 16: *Plot of the equipotential lines for the semi-infinite case with L = 0.*

The program used to plot the field lines when L=0 is **Code 7** on page 25, labelled as

```
field_lines
```

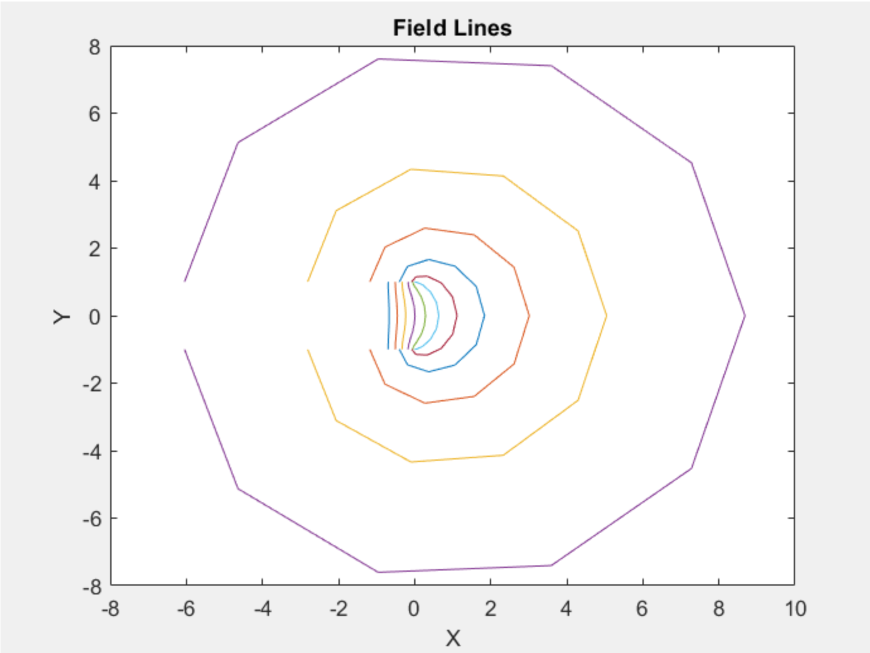

Figure 17: *Plot of the field lines for the semi-infinite case with L = 0.*

To see the behaviour of the equipotentials near the plate (below and above), I have written two codes – one to show the equipotentials right below the plate, and the other to show the equipotentials right above the plate. For the lower plate, I used negative Ψ, and positive Ψ for the upper plate.

The program used to plot the equipotential lines when L=0 right below the plate is **Code 8** on page 26, labelled as

```
equipotential_at_plate_lower
```

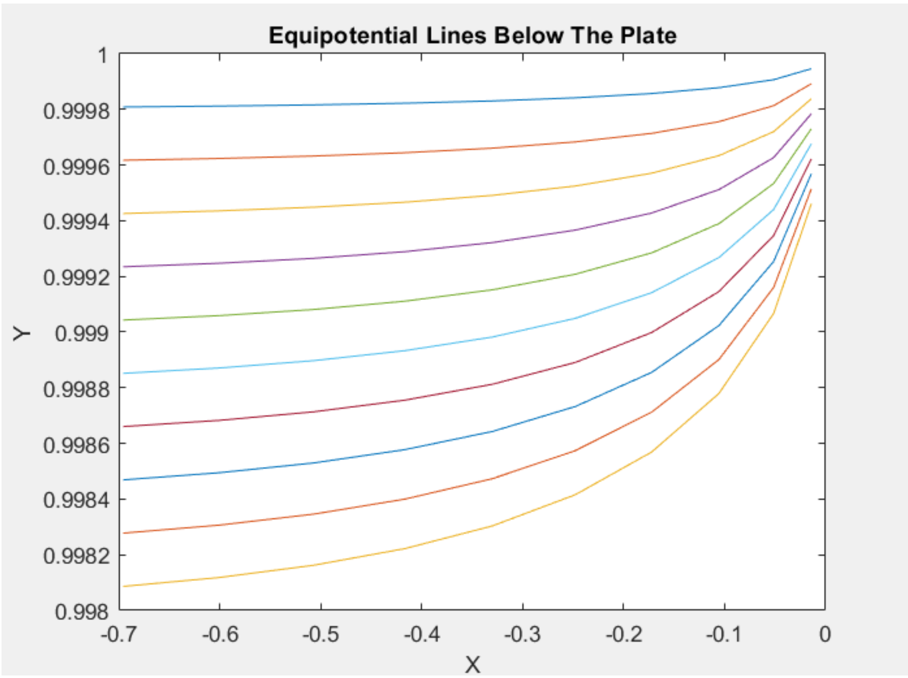

Figure 18: *Plot of the equipotential lines for the semi-infinite case with L = 0 at the lower surface of the plate.*

The program used to plot the equipotential lines when L=0 right above the plate is **Code 9** on page 26, labelled as

`equipotential_at_plate_upper`

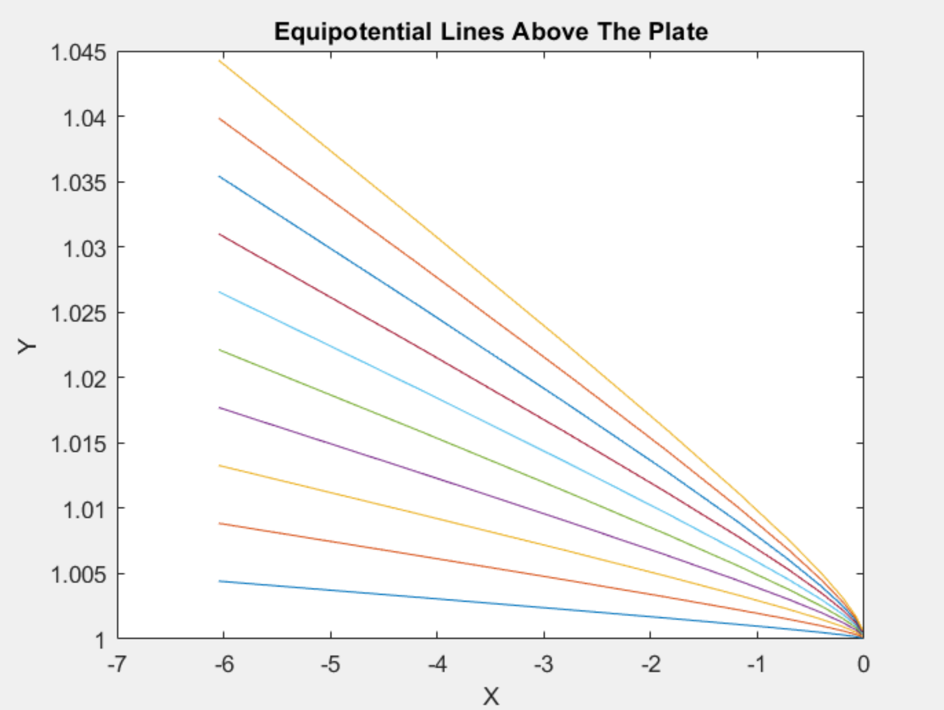

Figure 19: *Plot of the equipotential lines for the semi-infinite case with L = 0 at the upper surface of the plate.*

QUESTION 8

Recall equation (12):

$$-\frac{\partial\Phi}{\partial X}+i\frac{\partial\Phi}{\partial Y}=\frac{i}{2(e^{-2\pi iW}+1)}$$

Once again, take W = −c + ik, but this time, as we are looking at the behaviour on the plate, c = 1/2:

$$-\frac{\partial\Phi}{\partial X}+i\frac{\partial\Phi}{\partial Y}=\frac{i}{2\left(e^{-2\pi i\left(-\frac{1}{2}+ik\right)}+1\right)}$$

$$-\frac{\partial\Phi}{\partial X}+i\frac{\partial\Phi}{\partial Y}=\frac{i}{2(e^{i\pi}e^{2\pi k}+1)}$$

Recall that $e^{i\pi}=-1$, so:

$$-\frac{\partial\Phi}{\partial X}+i\frac{\partial\Phi}{\partial Y}=\frac{i}{2(-e^{2\pi k}+1)}$$

, which is purely imaginary, so when we compare real parts, we get that $\frac{\partial\Phi}{\partial X}=0$ hence the electric field will just be in the Y direction.

Equating imaginary components, the magnitude of the electric field at the plate is:

$$\frac{\partial \Phi}{\partial Y} = \frac{1}{2(1 - e^{2\pi k})}$$

, where k is the magnitude of Ψ.

<u>Asymptotic behaviour of Equation (11)</u>

At the plate, see what the equations for X and Y become:

$$X - L = \frac{1 + e^{2\pi k}\cos(2\pi c)}{\pi} + 2k \qquad Y = \frac{e^{2\pi k}\sin(2\pi c)}{\pi} + 2c$$

$$X - L = \frac{1 + e^{2\pi k}\cos(2\pi(0.5))}{\pi} + 2k \qquad Y = \frac{e^{2\pi k}\sin(2\pi(0.5))}{\pi} + 2(0.5)$$

$$X - L = \frac{1 + e^{2\pi k}\cos(\pi)}{\pi} + 2k \qquad Y = \frac{e^{2\pi k}\sin(\pi)}{\pi} + 1$$

$$\boldsymbol{X - L = \frac{1 - e^{2\pi k}}{\pi} + 2k} \qquad \boldsymbol{Y = 1}$$

, where k is the magnitude of Ψ.

Ψ→0⁺:

- For k>0, $e^{2\pi k} > 1$, so the numerator of the first term is negative.
- However, as k approaches zero from above, $e^{2\pi k} \to 1$, so $1 - e^{2\pi k} \to 0^-$.
- $1 - e^{2\pi k}$ becomes negative faster than 2k becomes positive, so X−L remains negative, tending to zero from below.
- Hence $X - L \to 0^-$.

Ψ→∞:

- For k→∞, $e^{2\pi k} \to \infty$, so the numerator of the first term becomes very large and negative.
- 2k→∞ as k→∞ too, but $\frac{1-e^{2\pi k}}{\pi}$ becomes large negative much faster than 2k becomes large positive, so X−L→−∞ as Ψ→∞.

<u>Asymptotic behaviour of Equation (12)</u>

At the plate, we've already shown that the equation just becomes:

$$\boldsymbol{\frac{\partial \Phi}{\partial Y} = \frac{1}{2(1 - e^{2\pi k})}}$$

, where k is the magnitude of Ψ.

Ψ→0⁺:

- For k>0, $e^{2\pi k} > 1$, so the denominator is negative.
- However, as k approaches zero from above, $e^{2\pi k} \to 1$, so the denominator tends to zero.
- Hence $\frac{\partial \Phi}{\partial Y} \to -\infty$.

Ψ→∞:

- For k→∞, $e^{2\pi k} \to \infty$, so the denominator becomes very large and negative.
- Hence $\frac{\partial \Phi}{\partial Y} \to 0^-$.
- So, the electric field tends to zero as Ψ→∞.

To get the first formula for $X \to L^-$, firstly recall that this is equivalent to $k \to 0^+$.

Now focus on the formula $X - L = \frac{1-e^{2\pi k}}{\pi} + 2k$. From this, we firstly get that:

$$L - X = \frac{e^{2\pi k}-1}{\pi} - 2k.$$

Taylor expand $e^{2\pi k}$ and simplify the expression on the right-hand side:

$$L - X = \frac{1 + 2\pi k + \frac{(2\pi k)^2}{2!} + \cdots - 1}{\pi} - 2k$$

$$L - X = \frac{1 + 2\pi k + \frac{(2\pi k)^2}{2!} + \cdots - 1 - 2\pi k}{\pi}$$

$$L - X \approx \frac{\frac{(2\pi k)^2}{2!}}{\pi}$$

We ignore higher-order terms this time, as for $k \to 0^+$, higher-order terms become extremely small and therefore negligible. So:

$$L - X \approx \frac{4\pi^2 k^2}{2\pi} = 2\pi k^2$$

$$k \approx \left(\frac{L - X}{2\pi}\right)^{1/2}$$

Now focus on the equation for the Y-component of the electric field:

$$\varepsilon_Y(X, 1^+) = -\frac{\partial \Phi}{\partial Y} = \frac{1}{2(e^{2\pi k} - 1)}$$

Taylor expand $e^{2\pi k}$ again and simplify the expression on the right-hand side:

$$\varepsilon_Y(X, 1^+) \approx \frac{1}{2(1 + 2\pi k + \cdots - 1)} = \frac{1}{4\pi k}$$

Substitute the expression for k derived:

$$\frac{1}{4\pi k} \approx \frac{1}{4\pi}\left(\frac{2\pi}{L - X}\right)^{1/2} = \frac{1}{\sqrt{8\pi}}(L - X)^{-1/2}$$

Hence, we get that:

$$\varepsilon_Y(X, 1^+) \approx a(L - X)^{-1/2}$$

for $X \to L^-$, where $a = \frac{1}{\sqrt{8\pi}}$.

To get the second formula for $X \to -\infty$ ($X - L \to -\infty$), recall that this is equivalent to $k \to \infty$.

Go back to $X - L = \frac{1-e^{2\pi k}}{\pi} + 2k$; we will ignore the second 2k term as its positive growth is trivial compared to the negative growth of the first term.

Now rearrange $X - L \approx \frac{1-e^{2\pi k}}{\pi}$:

$$e^{2\pi k} - 1 \approx \pi(L - X)$$

Substituting this into the expression for $\varepsilon_Y(X, 1^+)$,

$$\varepsilon_Y(X, 1^+) \approx \frac{1}{2\big(\pi(L - X)\big)} = \frac{1}{2\pi}(L - X)^{-1}$$

Hence, we get that:

$$\varepsilon_Y(X, 1^+) \approx b(L - X)^{-1}$$

for X→−∞, where $b = \frac{1}{2\pi}$.

On the lower surface of the plate, instead consider Ψ→0⁻ and Ψ→−∞.

<u>Asymptotic behaviour of Equation (11)</u>

$$\boldsymbol{X - L = \frac{1 - e^{2\pi k}}{\pi} + 2k \qquad\qquad Y = 1}$$

, where k is the magnitude of Ψ.

Ψ→0⁻:

- For k<0, $0 < e^{2\pi k} < 1$, so the numerator of the first term is positive.
- However, as k approaches zero from below, $e^{2\pi k} \to 1$, so $1 - e^{2\pi k} \to 0^+$.
- $1 - e^{2\pi k}$ becomes positive slower than 2k becomes negative, so X−L remains negative, tending to zero from negative. Hence $X - L \to 0^-$.

Ψ→−∞:

- For k→−∞, $e^{2\pi k} \to 0$, so the first term just tends to 1/π.
- 2k→−∞ as k→−∞ too, so the 1/π term becomes trivial and X−L→−∞ as Ψ→−∞.

<u>Asymptotic behaviour of Equation (12)</u>

$$\boldsymbol{\frac{\partial \Phi}{\partial Y} = \frac{1}{2(1 - e^{2\pi k})}}$$

Ψ→0⁻:

- For k<0, $0 < e^{2\pi k} < 1$, so the denominator is positive.
- However, as k approaches zero from below, $e^{2\pi k} \to 1$, so the denominator tends to zero.
- Hence $\frac{\partial \Phi}{\partial Y} \to \infty$.

Ψ→−∞:

- For k→−∞, $e^{2\pi k} \to 0$, so $\frac{\partial \Phi}{\partial Y} \to \frac{1}{2}$

For X→L⁻, by using the same method as before but taking the negative square root for k in terms of L and X, we can derive the same result as we did for the upper surface of the plate, but $a = -1/\sqrt{8\pi}$.

For X→−∞,

$$\varepsilon_Y(X, 1^+) = -\frac{\partial \Phi}{\partial Y} \approx -\frac{1}{2}$$

QUESTION 9

The program used to plot the solutions for $\varepsilon_Y(X,1)$ on both the lower surface and upper surface for the finite and semi-finite case is **Code 10** on page 27, labelled as

```
SOR_3_efield_plate_q9(L, DX, DY, h, omega, tol)
```

(L, D_X, D_Y, h, ω, ε_{tol}) = (20,100,100,0.25,1.7,10^{-6}):

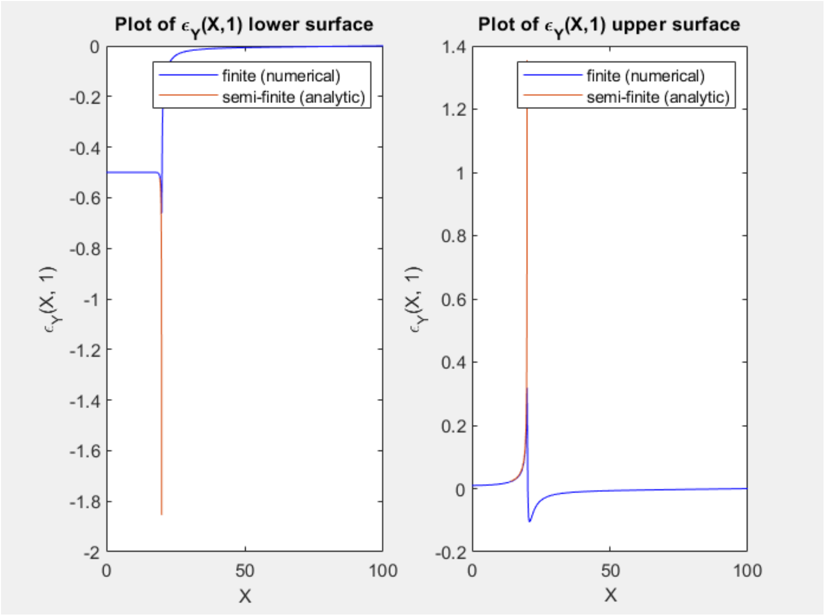

Figure 20: 2D *plot of the numerical and analytic solutions to $\varepsilon_Y(X,1)$ on the lower and upper surfaces of the plate for L=20.*

(L, D_X, D_Y, h, ω, ε_{tol}) = (40,200,200,0.25,1.7,10^{-6}):

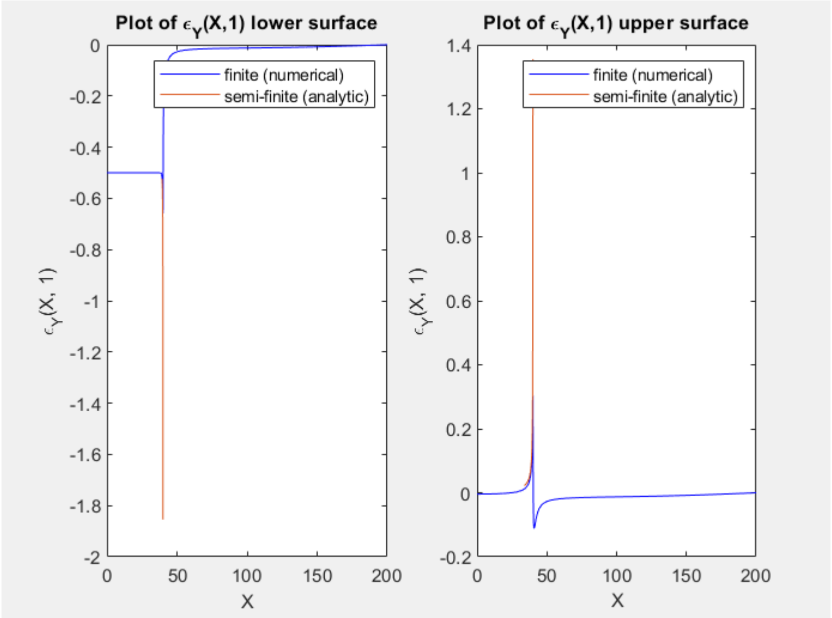

Figure 21: 2D *plot of the numerical and analytic solutions to $\varepsilon_Y(X,1)$ on the lower and upper surfaces of the plate L=40.*

(L, D_X, D_Y, h, ω, ε_{tol}) = (50,250,250,0.25,1.7,10^{-6}):

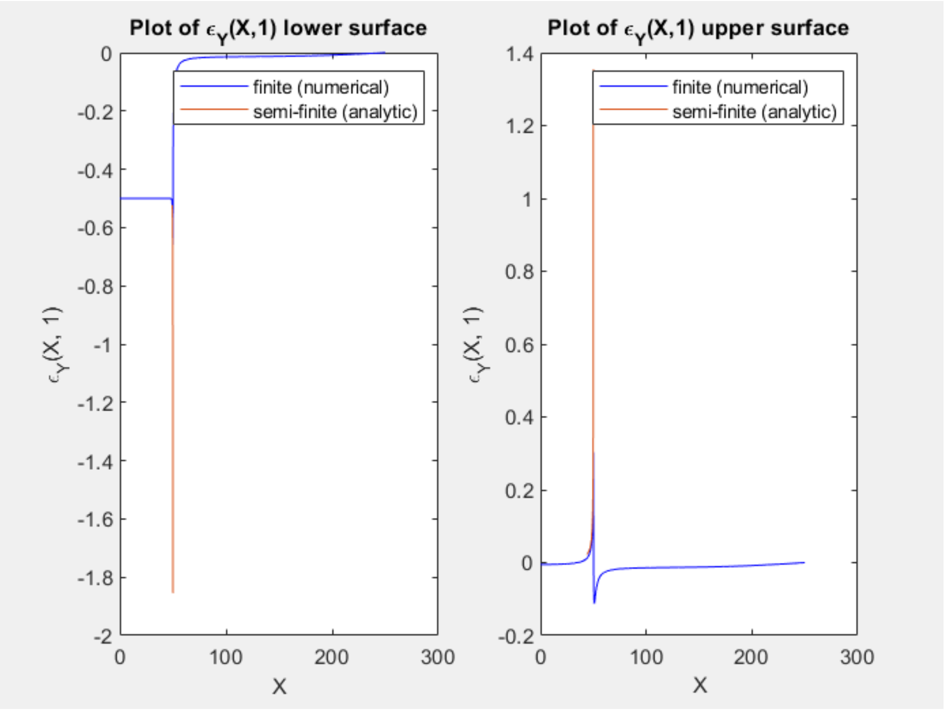

Figure 22: 2D *plot of the numerical and analytic solutions to $\varepsilon_Y(X,1)$ on the lower and upper surfaces of the plate for L=50.*

Firstly, look into the effect of changing h whilst keeping other parameters the same; I will take parameters (L, D_X, D_Y, h, ω, ε_{tol})=(10, 40, 40, h, 1.7, 10^{-6}), using ω=1.7 throughout this question as we showed in **Question 5** that this is the optimum value. I will be comparing h=0.1 and h=0.25.

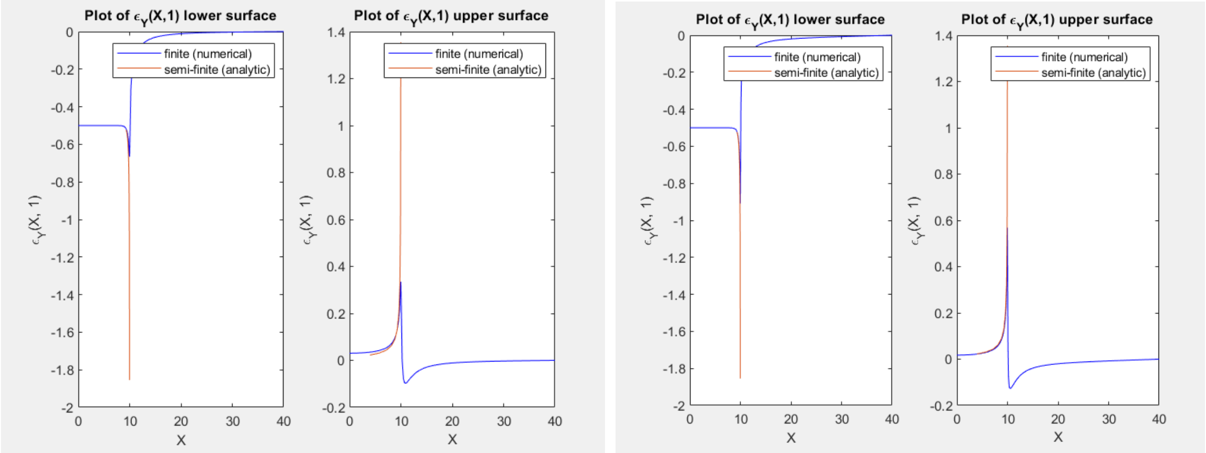

Figure 23: 2D *plot of the numerical and analytic solutions to $\varepsilon_Y(X,1)$ on the lower and upper surfaces of the plate for (L, D_X, D_Y, h, ω, ε_{tol})=(10, 40, 40, h, 1.7, 10^{-6}), with h=0.1 and h=0.25.*

As we can see from these graphs, the peaks are higher at X=L when h is smaller. Furthermore, the numerical solution looks to be more accurate for smaller h (as it aligns more with the analytic solution), which makes sense. However, we use 0.25 for most of this question as the computational time for h=0.1 is incredibly high.

We can inclusively see the effect of changing D_X and D_Y whilst keeping other parameters the same; I will take parameters (L, D_X, D_Y, h, ω, ε_{tol})=(10, D_X, D_Y, 0.25, 1.7, 10^{-6}), and test D_X=D_Y=20 and D_X=D_Y=40.

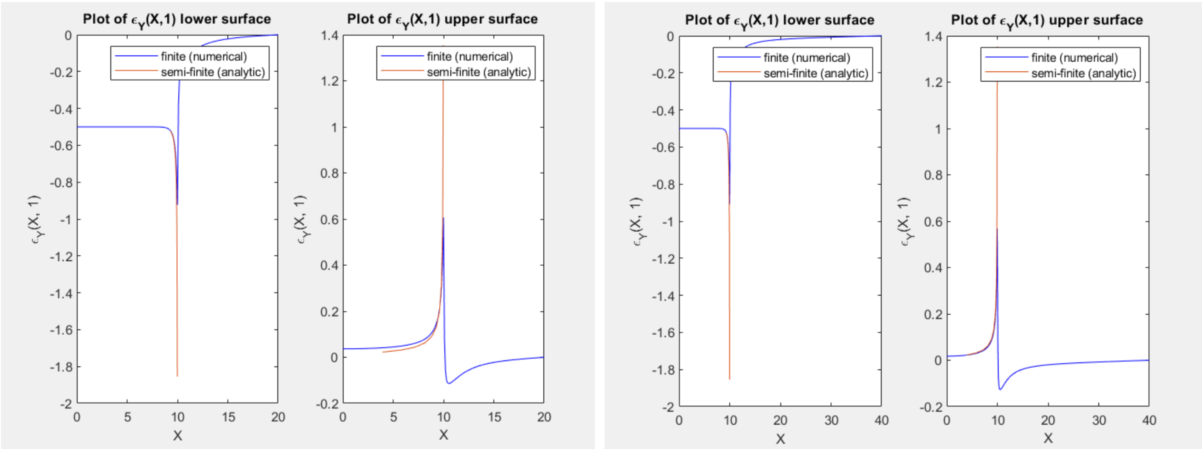

Figure 24: 2D *plot of the numerical and analytic solutions to $\varepsilon_Y(X,1)$ on the lower and upper surfaces of the plate for $(L, D_X, D_Y, h, \omega, \varepsilon_{tol})=(10, D_X, D_Y, 0.25, 1.7, 10^{-6})$, with $D_X=D_Y=20$ and $D_X=D_Y=40$.*

As Figure 24 shows, the numerical solution aligns with the analytic solution at X=L better for higher D_X and D_Y, suggesting that a higher D_X and D_Y outputs a more accurate solution.

For all of these, the shape of the numerical graph depends on L, as this is where the peaks occur. Furthermore, we can see that the part of the domain for which the numerical solution depends the most on the other parameters is at X=L itself.

The solutions for the finite and semi-finite cases seem to match between X=0 and X=L; the semi-finite solutions show a higher peak, but considering the fact that peak height of the finite case increases when a smaller h is used, as shown before, this is probably just due to the use of a higher h (we used h=0.25 for the numerical solutions generally because otherwise the computational time is far too high).

Programs

CODE 1

```
function[phi] = SOR(L, DX, DY, h, omega, tol)
Nx = DX / h;
Ny = DY / h;
N=(Nx+1)*(Ny+1);
x = linspace(0, DX, Nx+1);
y = linspace(0, DY, Ny+1);

phi = zeros(Nx+1, Ny+1);
phi(Nx+1,:) = 0;
phi(:,1) = 0;
phi(:,Ny+1) = 0;

max_iter = 10000;

for k=1:max_iter
    residual=0;
    phi_old = phi;
    for i = 1:Nx
        for j = 2:Ny
            if i==1
                phi(i,j) = (1 - omega) * phi(i,j) + (omega/4) * (phi(i+1,j) +
phi(i+1,j) + phi(i,j-1) + phi(i,j+1));
            else
                phi(i,j) = (1 - omega) * phi(i,j) + (omega/4) * (phi(i-1,j) +
phi(i+1,j) + phi(i,j-1) + phi(i,j+1));
            end
            if j==(1/h)+1 && i<=(L/h)+1
                phi(i,j)=1/2;
            end
            residual = residual+abs(phi(i,j) - phi_old(i,j));
        end
    end
    residual=residual/N;
    if residual < tol
        break;
    end
end

fprintf(num2str(k))

[Y, X] = meshgrid(y, x);
surf(Y, X,phi);
title('solution \Phi(x, y)');
xlabel('y');
ylabel('x');
zlabel('\Phi');
colorbar;
end
```

CODE 2

```
function[phi] = SOR_2(L, DX, DY, h, omega, tol)
Nx = DX / h;
Ny = DY / h;
N=(Nx+1)*(Ny+1);
x = linspace(0, DX, Nx+1);
y = linspace(0, DY, Ny+1);

phi = zeros(Nx+1, Ny+1);
phi(Nx+1,:) = 0;
phi(:,1) = 0;
phi(:,Ny+1) = 0;

max_iter = 10000;

for k=1:max_iter
    residual=0;
    phi_old = phi;
    for i = 1:Nx
        for j = 2:Ny
            if i==1
                phi(i,j) = (1 - omega) * phi(i,j) + (omega/4) * (phi(i+1,j) +
phi(i+1,j) + phi(i,j-1) + phi(i,j+1));
            else
                phi(i,j) = (1 - omega) * phi(i,j) + (omega/4) * (phi(i-1,j) +
phi(i+1,j) + phi(i,j-1) + phi(i,j+1));
            end
            if j==(1/h)+1 && i<=(L/h)+1
                phi(i,j)=1/2;
            end
            residual=residual+abs(phi(i,j) - phi_old(i,j));
        end
    end
    residual=residual/N;
    if residual < tol
        break;
    end
end

subplot(1, 2, 1);
plot(y, phi(1, :), 'b-');
title('\Phi(0, Y) Slice');
xlabel('Y');
ylabel('\Phi(0, Y)');
grid on;

subplot(1, 2, 2);
plot(y, phi((L/h)+1,:), 'r-');
title(['\Phi(', num2str(L), ', Y) Slice']);
xlabel('Y');
ylabel(['\Phi(', num2str(L), ', Y)']);
grid on;
end
```

CODE 3

```
function[phi] = SOR_3_efield(L, DX, DY, h, omega, tol)
Nx = DX / h;
Ny = DY / h;
N=(Nx+1)*(Ny+1);
x = linspace(0, DX, Nx+1);
y = linspace(0, DY, Ny+1);

phi = ones(Nx+1, Ny+1);
phi(Nx+1,:) = 0;
phi(:,1) = 0;
phi(:,Ny+1) = 0;

max_iter = 10000;

for k=1:max_iter
    residual=0;
    phi_old = phi;
    for i = 1:Nx
        for j = 2:Ny
            if i==1
                phi(i,j) = (1 - omega) * phi(i,j) + (omega/4) * (phi(i+1,j) +
phi(i+1,j) + phi(i,j-1) + phi(i,j+1));
            else
                phi(i,j) = (1 - omega) * phi(i,j) + (omega/4) * (phi(i-1,j) +
phi(i+1,j) + phi(i,j-1) + phi(i,j+1));
            end
            if j==(1/h)+1 && i<=(L/h)+1
                phi(i,j)=1/2;
            end
            residual = residual+abs(phi(i,j) - phi_old(i,j));
        end
    end
    residual=residual/N;
    if residual < tol
        break;
    end
end

e_field_Y0=-(1/h)*(phi);
plot(y, e_field_Y0(:,2), 'b-');
title('Plot of \epsilon_Y(X,0)');
xlabel('X');
ylabel('\epsilon_Y(X, 0)');
grid on;
end
```

CODE 4

```
function[phi] = SOR_3_efield_plate(L, DX, DY, h, omega, tol)
Nx = DX / h;
Ny = DY / h;
N=(Nx+1)*(Ny+1);
x = linspace(0, DX, Nx+1);
y = linspace(0, DY, Ny+1);

phi = ones(Nx+1, Ny+1);
phi(Nx+1,:) = 0;
phi(:,1) = 0;
phi(:,Ny+1) = 0;

max_iter = 10000;

for k=1:max_iter
    residual=0;
    phi_old = phi;
    for i = 1:Nx
        for j = 2:Ny
            if i==1
                phi(i,j) = (1 - omega) * phi(i,j) + (omega/4) * (phi(i+1,j) +
phi(i+1,j) + phi(i,j-1) + phi(i,j+1));
            else
                phi(i,j) = (1 - omega) * phi(i,j) + (omega/4) * (phi(i-1,j) +
phi(i+1,j) + phi(i,j-1) + phi(i,j+1));
            end
            if j==(1/h)+1 && i<=(L/h)+1
                phi(i,j)=1/2;
            end
            residual = residual+abs(phi(i,j) - phi_old(i,j));
        end
    end
    residual=residual/N;
    if residual < tol
        break;
    end
end

e_field_Y1_lower=(1/h)*(phi);
e_field_Y1_lower_2=e_field_Y1_lower(:,((1-h)/h)+1)-e_field_Y1_lower(:,(1/h)+1);
e_field_Y1_higher=-(1/h)*(phi);
e_field_Y1_higher_2=e_field_Y1_higher(:,((1+h)/h)+1)-e_field_Y1_higher(:,(1/h)+1);

subplot(1, 2, 1);
plot(y, e_field_Y1_lower_2, 'b-');
title('Plot of \epsilon_Y(X,1) lower surface');
xlabel('X');
ylabel('\epsilon_Y(X, 1)');
grid on;

subplot(1, 2, 2);
plot(y, e_field_Y1_higher_2, 'r-');
title('Plot of \epsilon_Y(X,1) upper surface');
xlabel('X');
ylabel('\epsilon_Y(X, 1)');
grid on;
```

CODE 5

```
function[phi] = SOR_4_changing_omega(L, DX, DY, h, tol)
Nx = DX / h;
Ny = DY / h;
N=(Nx+1)*(Ny+1);
x = linspace(0, DX, Nx+1);
y = linspace(0, DY, Ny+1);
k_vector=zeros(1,100);
omega_vector=linspace(1,1.99,100);

for omega_index=1:length(omega_vector)
    omega = omega_vector(omega_index);
    phi = zeros(Nx+1, Ny+1);
    phi(Nx+1,:) = 0;
    phi(:,1) = 0;
    phi(:,Ny+1) = 0;
    phi(1,(L/h)+1)=0.5;

    max_iter = 10000;

    for k=1:max_iter
        residual=0;
        phi_old = phi;
        for i = 1:Nx
            for j = 2:Ny
                if i==1
                    phi(i,j) = (1 - omega) * phi(i,j) + (omega/4) * (phi(i+1,j) +
phi(i+1,j) + phi(i,j-1) + phi(i,j+1));
                else
                    phi(i,j) = (1 - omega) * phi(i,j) + (omega/4) * (phi(i-1,j) +
phi(i+1,j) + phi(i,j-1) + phi(i,j+1));
                end
                if j==(1/h)+1 && i<=(L/h)+1
                    phi(i,j)=1/2;
                end
                residual = residual+abs(phi(i,j) - phi_old(i,j));
            end
        end
        residual=residual/N;
        if residual < tol
            break;
        end
    end
    k_vector(omega_index)=k;
end
disp(k_vector)
disp(omega_vector)
plot(omega_vector,k_vector,'-b')
title('How the no. of iterations needed for convergence depends on \omega' )
xlabel('\omega');
ylabel('Number of iterations needed for convergence');
end
```

CODE 6

equipotential_lines:

```
psi_values = linspace(-1/2, 1/2, 11);
phi_values = linspace(-1/2,1/2,11);
for i = 1:11
    X_vector=zeros(1,11);
    Y_vector=zeros(1,11);
    phi = phi_values(i);
    for j = 1:11
        psi = psi_values(j);
        X=(1/pi)*(1+(exp(2*pi*psi)*cos(2*pi*phi)))+2*psi;
        Y=(1/pi)*(exp(2*pi*psi)*sin(2*pi*phi))+2*phi;
        X_vector(j)=X;
        Y_vector(j)=Y;
    end
    plot(X_vector,Y_vector)
    hold on;
end
xlabel('X');
ylabel('Y');
title('Equipotential Lines');
```

CODE 7

field_lines:

```
psi_values = linspace(-1/2, 1/2, 11);
phi_values = linspace(-1/2,1/2,11);
for i = 1:11
    X_vector=zeros(1,11);
    Y_vector=zeros(1,11);
    psi = psi_values(i);
    for j = 1:11
        phi = phi_values(j);
        X=(1/pi)*(1+(exp(2*pi*psi)*cos(2*pi*phi)))+2*psi;
        Y=(1/pi)*(exp(2*pi*psi)*sin(2*pi*phi))+2*phi;
        X_vector(j)=X;
        Y_vector(j)=Y;
    end
    plot(X_vector,Y_vector)
    hold on;
end
xlabel('X');
ylabel('Y');
title('Field Lines');
```

CODE 8

equipotential_at_plate_lower:

```
psi_values_1 = linspace(-0.5, -0.05, 10);
delta_values = linspace(0.0001,0.001,10);
phi_original=1/2;

for m = 1:10
    X_vector=zeros(1,10);
    Y_vector=zeros(1,10);
    phi = phi_original-delta_values(m);
    for n = 1:10
        psi_2 = psi_values_1(n);
        X=(1/pi)*(1+(exp(2*pi*psi_2)*cos(2*pi*phi)))+2*psi_2;
        Y=(1/pi)*(exp(2*pi*psi_2)*sin(2*pi*phi))+2*phi;
        X_vector(n)=X;
        Y_vector(n)=Y;
    end
    plot(X_vector,Y_vector)
    hold on;
end

xlabel('X');
ylabel('Y');
title('Equipotential Lines Below The Plate');
```

CODE 9

equipotential_at_plate_upper:

```
psi_values_2 = linspace(0.05, 0.5, 10);
delta_values = linspace(0.0001,0.001,10);
phi_original=1/2;

for p = 1:10
    X_vector=zeros(1,10);
    Y_vector=zeros(1,10);
    phi = phi_original-delta_values(p);
    for q = 1:10
        psi_2 = psi_values_2(q);
        X=(1/pi)*(1+(exp(2*pi*psi_2)*cos(2*pi*phi)))+2*psi_2;
        Y=(1/pi)*(exp(2*pi*psi_2)*sin(2*pi*phi))+2*phi;
        X_vector(q)=X;
        Y_vector(q)=Y;
    end
    plot(X_vector,Y_vector)
    hold on;
end

xlabel('X');
ylabel('Y');
title('Equipotential Lines Above The Plate');
```

CODE 10

```
function[phi] = SOR_3_efield_plate_q9(L, DX, DY, h, omega, tol)
Nx = DX / h;
Ny = DY / h;
N=(Nx+1)*(Ny+1);
x = linspace(0, DX, Nx+1);
y = linspace(0, DY, Ny+1);

phi = ones(Nx+1, Ny+1);
phi(Nx+1,:) = 0;
phi(:,1) = 0;
phi(:,Ny+1) = 0;

max_iter = 10000;

for k=1:max_iter
    residual=0;
    phi_old = phi;
    for i = 1:Nx
        for j = 2:Ny
            if i==1
                phi(i,j) = (1 - omega) * phi(i,j) + (omega/4) * (phi(i+1,j) +
phi(i+1,j) + phi(i,j-1) + phi(i,j+1));
            else
                phi(i,j) = (1 - omega) * phi(i,j) + (omega/4) * (phi(i-1,j) +
phi(i+1,j) + phi(i,j-1) + phi(i,j+1));
            end
            if j==(1/h)+1 && i<=(L/h)+1
                phi(i,j)=1/2;
            end
            residual = residual+abs(phi(i,j) - phi_old(i,j));
        end
    end
    residual=residual/N;
    if residual < tol
        break;
    end
end

psi_values = linspace(-1/2, 0, 11);
X_vector=zeros(1,11);
e_field_vector=zeros(1,11);
for j = 1:11
    psi = psi_values(j);
    X=(1/pi)*(1-exp(2*pi*psi))+2*psi+L;
    X_vector(j)=X;
    e_field=-0.5*(1-exp(2*pi*psi))^-1;
    e_field_vector(j)=e_field;
end

psi_values_2 = linspace(0, 1/2, 11);
X_vector_2=zeros(1,11);
e_field_vector_2=zeros(1,11);
for m = 1:11
    psi_2 = psi_values_2(m);
    X_2=(1/pi)*(1-exp(2*pi*psi_2))+2*psi_2+L;
    X_vector_2(m)=X_2;
    e_field_2=-0.5*(1-exp(2*pi*psi_2))^-1;
```

```
    e_field_vector_2(m)=e_field_2;
end

e_field_Y1_lower=(1/h)*(phi);
e_field_Y1_lower_2=e_field_Y1_lower(:,((1-h)/h)+1)-e_field_Y1_lower(:,(1/h)+1);
e_field_Y1_higher=-(1/h)*(phi);
e_field_Y1_higher_2=e_field_Y1_higher(:,((1+h)/h)+1)-e_field_Y1_higher(:,(1/h)+1);

subplot(1, 2, 1);
plot(y, e_field_Y1_lower_2, 'b-');
hold on;
plot(X_vector,e_field_vector)
legend('finite (numerical)','semi-finite (analytic)')
title('Plot of \epsilon_Y(X,1) lower surface');
xlabel('X');
ylabel('\epsilon_Y(X, 1)');

subplot(1, 2, 2);
plot(y, e_field_Y1_higher_2, 'b-');
hold on;
plot(X_vector_2,e_field_vector_2)
legend('finite (numerical)','semi-finite (analytic)')
title('Plot of \epsilon_Y(X,1) upper surface');
xlabel('X');
ylabel('\epsilon_Y(X, 1)');
```