NETWORK SYSTEMS: MODELLING AND ANALYSIS

Finite-Time Influence Systems and the Wisdom Crowd Effect

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Introduction

- The phenomenon of "wisdom of crowds" is the idea that large groups of people can be collectively smarter than individuals.
- Golub and Jackson's naive learning model explains that individual opinions, influenced by independent noise, result in a wise crowd when averaged.
- The given paper contributes to Golub and Jackson's model by investigating finite-time settings of influence systems.
- It aims to define and characterize wisdom notions, studying how influence systems affect the wisdom of crowds within a finite time frame.

Background

- Some of the concepts learnt in the course are necessary for understanding and the same are summarized in the beginning of the research paper:
  - Norms
  - Row Stochastic Matrices
  - Irreducible Matrices
  - Primitive Matrices
  - Left Dominant Eigenvector
  - Perron-Frobenius Theorem
  - Equal Neighbor Matrix
- Apart from these, some additional concepts are also defined which are presented later.

French-DeGroot Model of Opinion Dynamics

- French-DeGroot Model of Opinion Dynamics: $x(k+1) = \boldsymbol{P}x(k)$

  where x represents vector of opinions and matrix P is a row stochastic matrix and defines a graph G.

  $x_i(k)$ denotes opinion of individual i and P_{ij} is the weight that individual i assigns to the opinion of individual j.

- We assume $x_i(0) = \mu + \xi_i(0)$ where $\mu \in \mathbb{R}$ is unknown parameter and the noisy terms $\xi_i(0)$ are a family of independent Gaussian-distributed variables such that $E[\xi_i(0)] = 0$ and $\mathrm{Var}[\xi_i(0)] = \sigma^2 < \infty$ $\forall$i $\in \{1, \dots, n\}$.

- Considering sequence of DeGroot models with increasing dimensions n, the state of n-dimensional model is $x^{[n]}$.

  Assuming μ is kept constant, the law of large numbers implies that, almost surely, $\lim_{n\to\infty} avg\left(x^{[n]}(0)\right) = \mu$.

Wisdom Notions

- Given a sequence of row stochastic matrices of increasing dimensions $\{P^{[n]} \in \mathbb{R}^{n\times n}\}_{n\in N}$ define a sequence of opinion dynamics problems with initial state $\{x^{[n]}(0) \in \mathbb{R}^{n}\}_{n\in N}$ satisfying $x_i(0) = \mu + \xi_i(0)$ and evolution $\{x^{[n]}(k) \in \mathbb{R}^{n}\}_{n\in N}$ at times k ∈ N. The sequence $\{P^{[n]} \in \mathbb{R}^{n\times n}\}_{n\in N}$ is

  i. One time wise: if $\lim_{n\to\infty} avg\left(x^{[n]}(1)\right) = \mu$

  ii. Finite time wise: if $\lim_{n\to\infty} avg\left(x^{[n]}(k)\right) = \mu$ ∀ k ∈N

  iii. Wise: if $\lim_{n\to\infty}\lim_{k\to\infty} avg\left(x^{[n]}(k)\right) = \mu$

  iv. Uniformly wise: if $\lim_{n\to\infty}\sup_{k\in N}\left|avg\left(x^{[n]}(k)\right) - \mu\right| = 0*$

*sup ("supremum"):refers to the largest value of the term inside modulus gets as k varies

Necessary and Sufficient Conditions for Finite-time Wisdom and Wisdom

- Consider a sequence of row stochastic matrices of increasing dimensions $\{P^{[n]} \in \mathbb{R}^{n\times n}\}_{n\in N}$. The sequence is

i. One time wise: if and only if $\lim_{n\to\infty} \left|\left|\frac{1}{n} P^{[n]}\right|\right|_1 = 0$

ii. Finite time wise: if and only $\lim_{n\to\infty} \left|\left|\frac{1}{n} \left(P^{[n]}\right)^k\right|\right|_1 = 0 \; \forall$ k ∈N

iii. Wise: if and only if $\lim_{n\to\infty} \left|\left|\pi^{[n]}\right|\right|_\infty = 0$ where $\pi^{[n]} \in \Delta_n$* is the left dominant eigenvector of $P^{[n]}$ for n ∈N (assuming $\{P^{[n]}\}_{n\in N}$ is primitive)

* $\Delta_n = \{x \in \mathbb{R}^n | x \geq 0, 1_n^T x = 1$

If x ∈ Δ_n the given inequality holds: $||x||^2{}_\infty \leq ||x||_\infty$

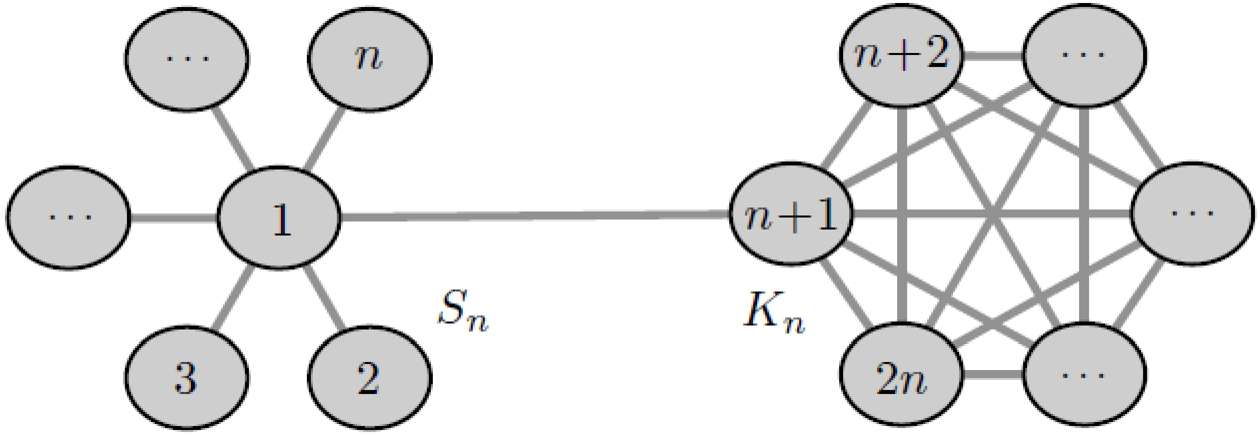
Pre-uniform Wisdom and Sufficient Condition for Uniform Wisdom

- Pre-uniform wisdom: A sequence of stochastic matrices of increasing dimensions $\{P^{[n]} \in \mathbb{R}^{n\times n}\}_{n\in N}$ is pre-uniformly wisdom wise if $\lim_{n\to\infty} \sup_{\mathrm{k}\in\mathrm{N}} \left\|\frac{1}{n}\left(P^{[n]}\right)^k\right\|_1 = 0$

- A sufficient condition for wisdom: A sequence of primitively row stochastic matrices of increasing dimensions $\{P^{[n]} \in \mathbb{R}^{n\times n}\}_{n\in N}$ is uniformly wise if $\lim_{n\to\infty} \sup_{\mathrm{k}\in\mathrm{N}} \left\|\frac{1}{n}\left(P^{[n]}\right)^k\right\|_1 \tau_{mix}\left(P^{[n]}\right) = 0.*$

* For a primitive row stochastic matrix, mixing time $\tau_{mix}(P) \coloneqq \inf\left\{t \in N \middle| \max_{i,j} \sum_k \left|(P)^t{}_{ik} - (P)^t{}_{jk}\right| \leq \frac{1}{e}\right\}$; here, inf("Infinium") is the greatest lower bound.

Example: The union/contraction of star and complete graph is wise, but not one-time wise



$\{W^{[n]}\}_{n \in N}$ sequence of adjacency matrices $\{P^{[n]}\}_{n \in N}$ corresponding sequence of equal neighbour matrices

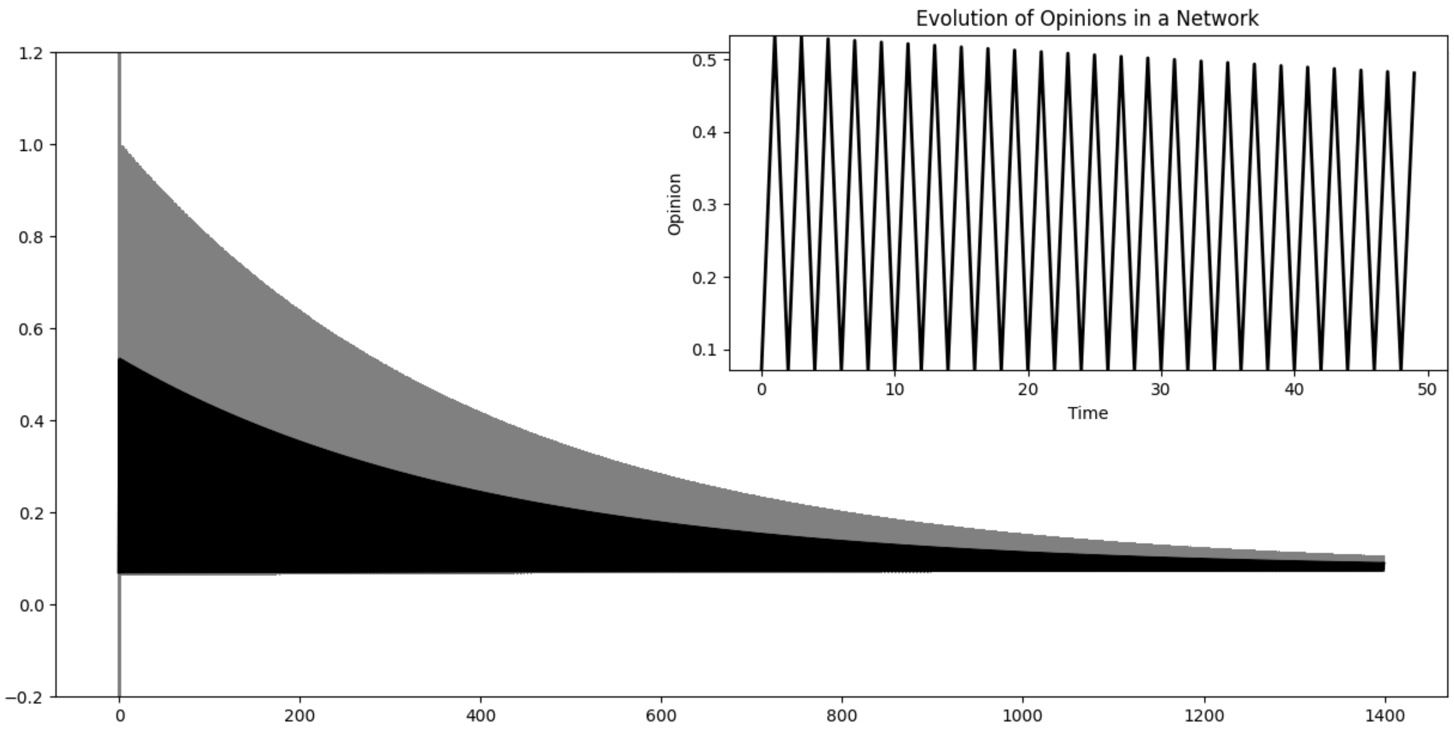
The graph $S_n \cup K_n$, for $n = 6$ where node 1 is always the center of the star graph and node n+1 belongs to both star graph and complete graph.

Note: The matrices are primitive as K_n is aperiodic.

Simulation Approach

- Construct $2n \times 2n$ adjacency matrix for the union graph of star graph and complete graph: (W = np.zeros((2*n, 2*n)))
- Connect all the leaf nodes(1 to n-1) of star graph to the central node '0': W[0, j] = W[j, 0] = 1 for j in range(1, n)
- Connect each node to every other node (n to 2n-1) in the complete graph: W[i, j] = 1 for i, j in range(n, 2n)
- Connect the leaf node n-1 of star graph to node n of the complete graph: W[n, 0] = W[0, n] = 1
- Construct the equal-neighbor matrix P[n] where each row i is W[i] / sum(W[i]): (P = np.diag(1 / W.sum(axis=1)) @ W)
- Initialize opinions: opinions = np.random.normal(mu, sigma, 2*n); opinions[0]=1 with mu=0 and sigma=1
- Update opinions for T iterations: opinions = P @ opinions

Result



- Union/contraction of a star and a complete graph with n = 200 nodes.
- The time horizon is T = 1400.
- Individual opinions are in gray, the average opinion is in black.
- The right upper-corner figure illustrates the first 50 steps of the evolution.

Inferences

- The graph is wise but not one-time wise.
- The graph of the initial steps right upper-corner shows that average of the opinions move away from zero.
- Theoretically, $\lim_{n \to \infty} avg\left(x^{[n]}(1)\right) = \mu$

References

1. Bullo, F., Fagnani, F. and Franci, B., 2020. Finite-time influence systems and the wisdom of crowd effect. *SIAM Journal on Control and Optimization*, *58*(2), pp.636-659.

- Code: Google Colab Link