Identify IBD regions with rare variants

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1 Implementation

The genome is divided into windows (w, may be overlapped each other), for the genotypes (G_1,G_2) of rare variants(minor allele frequency$< 0.1\%$) in w, we have

$$\begin{aligned} P(G_1, G_2|IBD, D) &= \Sigma_{h_1,h_2,h_3} P(G_1|h_1, h_2)P(G_2|h_2, h_3)P(h_1, h_2, h_3|D) \\ P(G_1, G_2|\overline{IBD}, D) &= \Sigma_{h_1,h_2,h_3,h_4} P(G_1|h_1, h_2)P(G_2|h_3, h_4)P(h_1, h_2, h_3, h_4|D) \end{aligned} \tag{1}$$

where $h_i \in \{0, 1\}$. If we consider h_i follows Bernoulli distribution ($B(n, \theta)$) and is independent from each other, then

$$\begin{aligned} P(h_1, h_2, h_3|D) &= \int_0^1 P(h_1, h_2, h_3|\theta)P(\theta|D)d\theta \\ &= \int_0^1 P(h_1|\theta)P(h_2|\theta)P(h_3|\theta)P(\theta|D)d\theta \\ &= \int_0^1 \theta^{h_1}(1-\theta)^{1-h_1}\theta^{h_2}(1-\theta)^{1-h_2}\theta^{h_3}(1-\theta)^{1-h_3}P(\theta|D)d\theta \end{aligned} \tag{2}$$

Assuming $\theta|D$ follows Beta distribution($\beta(\alpha_{update}, \beta_{update})$) $\alpha_{update} = \alpha_{prior} + 2N_{training} + 2N_{test} - t$, $\beta_{update} = \beta_{prior} + t$, where t is the number of individuals hit by the variants in test set, $N_{training}$ and N_{test} are the number of individuals in training and test set. α and β are shortened form of α_{update} and β_{update} in the following text.

$$P(\theta|D) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \tag{3}$$

Let's say $S = h_1 + h_2 + h_3$, Eq. 2 can be rewritten as

$$\begin{aligned}
P(h_1, h_2, h_3|D) &= \int_0^1 \theta^{h_1+h_2+h_3}(1-\theta)^{3-h_1-h_2-h_3}\theta^{\alpha-1}(1-\theta)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}d\theta \\
&= \int_0^1 \theta^{S+\alpha-1}(1-\theta)^{2+\beta-S}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}d\theta \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \theta^{S+\alpha-1}(1-\theta)^{2-S+\beta}d\theta \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{(2-S+\beta)!}{(S+\alpha)(S+\alpha+1)...(\alpha+\beta+2)}
\end{aligned} \tag{4}$$

2 Binomial-Beta distribution

According to the definition of Binomial-Beta distribution,

$$\begin{aligned}
k|n,\theta &\sim Binomial(\theta, n) \\
\theta|\beta_1,\theta_2 &\sim Beta(\beta_1, \beta_2) \\
then \theta|k,n,\beta_1,\beta_2 &\sim Beta(\beta_1 + k, \beta_2 + n - k)
\end{aligned} \tag{5}$$

We found Eq. **??** actually follows Binomial-Beta distribution,

$$P(h_1, h_2, h_3|D) \sim Beta(\alpha + S, \beta + 3 - S) \tag{6}$$

3 Beta distribution without sequencing error

Because $P(h_1, h_2, h_3|D)$ has included sequencing error in estimating θ, we assume k_1 and k_2 hits are observed for errors from ref$-$ >alt and alt$-$ >ref,respectively:

$$\begin{aligned}
&\Sigma_{k_1=0}^{\beta}\Sigma_{k_2=0}^{\alpha}P_{\epsilon,\beta}(k_1)P_{\epsilon,\alpha}(k_2)Beta(\alpha-k_2+k_1, \beta-k_1+k_2) \\
=&\Sigma_{k_1=0}^{\beta}\Sigma_{k_2=0}^{\alpha}\binom{\beta}{k_1}\binom{\alpha}{k_2}\epsilon^{k_1+k_2}(1-\epsilon)^{\alpha+\beta-k_1-k_2}\frac{\theta^{\alpha+k_1-k_2-1}(1-\theta)^{\beta+k_2-k_1-1}\Gamma(\alpha+\beta)}{\Gamma(\beta+k_2-k_1)\Gamma(\alpha+k_1-k_2)} \\
\propto&\Sigma_{k_1=0}^{\beta}\Sigma_{k_2=0}^{\alpha}\epsilon^{k_1+k_2}(1-\epsilon)^{\alpha+\beta-k_1-k_2}\theta^{\alpha+k_1-k_2-1}(1-\theta)^{\beta+k_2-k_1-1} \\
\propto&\Sigma_{k_1=0}^{\beta}\Sigma_{k_2=0}^{\alpha}Beta(a,b)Beta(c,d)
\end{aligned} \tag{7}$$

The Eq. **??** can be transformed to the product of independent beta variables: $\epsilon \sim Beta(a, b)$ and $\theta \sim Beta(c, d)$, where $a = k_1+k_2+1, b = \alpha+\beta-k_1-k_2+1, c = \alpha + k_1 - k_2, d = \beta + k_2 - k_1$ Based on the previous study [**?**], we can calculate

$$\begin{aligned}
M &= \frac{a}{a+b}\frac{c}{c+d} \\
N &= \frac{a(a+1)}{(a+b)(a+b+1)}\frac{c(c+1)}{(c+d)(c+d+1)}
\end{aligned} \tag{8}$$

Assume M and N can be also represented by α^* and β^*

$$
\begin{aligned}
M &= \frac{\alpha^*}{\alpha^* + \beta^*} \\
N &= \frac{\alpha^*(\alpha^* + 1)}{(\alpha^* + \beta^*)(\alpha^* + \beta^* + 1)}
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\alpha^* &= \frac{(M - N)M}{M - N^2} \\
\beta^* &= \frac{(M - N)(1 - M)}{M - N^2}
\end{aligned} \tag{10}
$$

The $\Sigma_{k_1=0}^{\beta}\Sigma_{k_2=0}^{\alpha}P_{\epsilon,\beta}(k_1)P_{\epsilon,\alpha}(k_2)Beta(\alpha - k_2 + k_1, \beta - k_1 + k_2) \sim \Sigma_{k_1=0}^{\beta}\Sigma_{k_2=0}^{\alpha}Beta(\alpha^*, \beta^*)$
Then Eq. **??** can be rewritten as

$$
\begin{aligned}
P(h_1, h_2, h_3|D) &= \int_0^1 \theta^{h_1+h_2+h_3}(1-\theta)^{3-h_1-h_2-h_3}\Sigma_{k_1=0}^{\beta}\Sigma_{k_2=0}^{\alpha}Beta(\alpha^*, \beta^*) \\
&= \Sigma_{k_1=0}^{\beta}\Sigma_{k_2=0}^{\alpha}\int_0^1 \theta^{S}(1-\theta)^{3-S}Beta(\alpha^*, \beta^*) \\
&= \Sigma_{k_1=0}^{\beta}\Sigma_{k_2=0}^{\alpha}\int_0^1 Beta(\alpha^* + S, \beta^* + 3 - S) \\
&= \Sigma_{k_1=0}^{\beta}\Sigma_{k_2=0}^{\alpha}\frac{\Gamma(\alpha^* + \beta^*)}{\Gamma(\alpha^*)\Gamma(\beta^*)}\frac{(2 - S + \beta^*)!}{(S + \alpha^*)(S + \alpha^* + 1)...(\alpha^* + \beta^* + 2)}
\end{aligned} \tag{11}
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References

[1] Da-Yin Fan (1991) The distribution of the product of independent beta variables, Communications in Statistics - Theory and Methods, 20:12, 4043-4052, DOI: 10.1080/03610929108830755