Computer Simulations of Stochastic Processes — Report I

Katarzyna Macioszek, Ada Majchrzak

April 4, 2023

1 Alpha stable parameters estimation methods comparison

This section will focus on checking the speed and accuracy for different methods of alpha stable parameters estimation. We chose to compare three algorithms — Maximum Likelihood, Kogon regression and Iterative Koutrouvelis regression. In all cases, we will consider following parametrization of an alpha stable distribution:

$$X \sim S(\alpha, \beta, \gamma, \delta).$$

Let us start with short desription for each of the methods.

Description of methods

Maximum Likelihood method

Maximum Likelihood is also called EM algorithm (*Expectation-Maximization*), as it consists of two steps, Expectation and Maximization, described below.

Let us denote $\overline{x} = (x_1, x_2, \ldots, x_n)$ — alpha stable sample of length n, $f(\overline{x};\theta)$ — probability distibution function for considered sample. The likelihood function is given by

$$L(\overline{x};\theta) = \prod_{i=1}^{n}(\overline{x};\theta), \tag{1}$$

where θ is a vector of distibution parameters. We aim to maximize the conditional expectation of (1) for given data and current guess of θ — θ_t. The maximum likelihood algorithm works as follows.

Algorithm 1 EM algorithm [1]

1. E-step: given $\overline{x}$ and θ_t, it computes $\mathbb{E}\left[L(\overline{x};\theta)|\overline{x},\theta_t\right]$.
2. M-step: it finds such θ that $\mathbb{E}\left[L(\overline{x};\theta)|\overline{x},\theta_t\right]$ is maximized.
3. Steps 2. and 3. are repeated until the value of (1) doesn't change significantly.

Iterative Koutrouvelis regression

Iterative Koutrouvelis regression is a regression-type method which starts with an initial estimate of the parameters and proceeds iteratively until some prespecified convergence criterion is satisfied.

Let $\phi(t)$ be a characteristic function of considered alpha stable distribution, $\mathcal{R}\{\phi(t)\}$ its real part, and $\mathcal{I}\{\phi(t)\}$ its imaginary part. We have

$$\ln(-\ln|\phi(t)|^2) = \ln(2\delta^\alpha) + \alpha \ln|t|, \tag{2}$$

and

$$\arctan\left(\frac{\mathcal{I}\{\phi(t)\}}{\mathcal{R}\{\phi(t)\}}\right) = \gamma t + \beta\delta^\alpha \tan\left(\frac{\pi\alpha}{2}\right)\operatorname{sign}(t)|t|^\alpha. \tag{3}$$

From (2) we derive regression for α and δ parameters

$$y_k = \ln(2\delta^\alpha) + \alpha ln|t_k| + \epsilon_k,$$

where ϵ_k is an error and $t_k = \frac{\pi k}{25}$, $k = 1, 2, \ldots, K$, and K depends on sample size. Once we obtian estimates $\hat{\alpha}$ and $\hat{\delta}$, we fix α and δ on these values and we compute $\hat{\beta}$ and $\hat{\gamma}$ from (3). As a result, we obtain a full set of initial parameters. Now, the iteration needs to be continued until a prespecified convergence criterion is satisfied [3].

Kogon regression

This method uses a McCulloch method to provide initial estimates of parameters. McCulloch method is based on quantiles. Again, let us denote $\overline{x} = (x_1, x_2, \ldots, x_n)$ — alpha stable sample of length n and q_k — kth quantile of considered alpha stable distribution. Let us define statistics

$$\nu_\alpha = \frac{q_{0.95} - q_{0.05}}{q_{0.75} - q_{0.25}} \tag{4}$$

and

$$\nu_\beta = \frac{q_{0.95} + q_{0.05} - 2q_{0.50}}{q_{0.95} - q_{0.05}}. \tag{5}$$

Above statistics are functions of α and β. This relationship can be inverted and we can look at α and β as functions of (4) and (5), respectively.

$$\alpha = f_1(\nu_\alpha, \nu_\beta), \tag{6}$$

$$\beta = f_2(\nu_\alpha, \nu_\beta). \tag{7}$$

Now, to obtain $\hat{\alpha}$ and $\hat{\beta}$, we need to substitute ν_α and ν_β for their sample values and perform a linear interpolation between the values found in tables provided by McCulloch [2]. The procedure for γ and δ is similar, however much more complicated and thus will not be described here. It can be found in [2].

After the initial parameters are obtain, Kogon method uses simplified Koutrouvelis procedure. The countinuous representation of characteristic funcion is used instead of the classical one and t_k is a fixed set of 10 equally spaced points [4].

Accuracy comparison

To check the accuracy of above methods, we first computed 100 estimators for each parameters for each method, and stored them in vectors, following below procedure.

Algorithm 2 Estimator sample generation

1. Create empty vector of length 100 for each parameter.
2. Generate random alpha stable sample with predefined parameters.
3. Estimate sample parameters.
4. Add each paramater to corresponding vector.
5. Repeat steps 2.-4. 100 times.

Random alpha samples were drawn from $S(1.8, -0.3, 2, 5)$. To estimate the sample paramaters, we used following functions from *StableEstim* library in R environment:

- *MLParametersEstim* for Maximum Likelihood,
- *IGParametersEstim* for Kogon regression,
- *KoutParametersEstim* for Koutrouvelis regression.

For such computed vectors of estimators, we calculated the empirical bias and variance. Let's first take a look at the bias.

Table 1: Bias for different alpha stable parameters estimation methods

	ML	Kogon	Koutrouvelis
α	0.0008064183	0.004799012	0.007214648
β	-0.0341880912	0.002840303	-0.025354638
γ	-0.0041155419	-0.025622263	0.003573691
δ	-0.0035664033	-0.099953427	-0.016958822

Looking at the Table 1 it is clear that non of the used method is unbiased. However, we can see that from all three, the Kogon regression has the biggest bias for all parameters except β, for which it is actually the smallest. Maximum Likelihood, unexpectedly, doesn't seem to stand out much in this comparison. What about variance? The results shown in Table 2 tell us that in terms of variance,

Table 2: Variance for different alpha stable parameters estimation methods

	ML	Kogon	Koutrouvelis
α	0.002066670	0.002614959	0.001795733
β	0.030451613	0.067988983	0.045338841
γ	0.002751521	0.024001103	0.003748864
δ	0.012798250	0.514571504	0.027156664

Maximum Likelihood method generally fares better compared to the other two methods, whereas Kogon method definitely looks much worse. Although, if we compare only Maximum Likelihood with Koutrouvelis, there is no huge discrepancy between them.

The next thing we want to look at while comparing the accuracy of used methods are boxplots of estimated paramaters. Boxplot actually provide a lot of useful information — looking at it we can easily identify the minimum and maximum value fo our data, the median, first and third quartile, also if the sample contains a lot of outliers or not.

Figure 1 shows the boxplots for α estimators. We can see from here that the median of the estimators obtained by the Maximum Likelihood is definitely the closest to the real value of α. The range of the data is, unsuprisingly, the biggest for the Kogon method, and ML and Koutrouvelis fall pretty close to each other in this regard. As for the outliers, we can observe them in similar quantities for all methods, but for the Koutrouvelis they seem to be less distant from the median. As for the estimators of β parameter, Figure 2 shows that in this case we observe a definite maximum Likelihood supremacy. Here, the range of the data (excluding outliers) is significantly smaller than for the other two methods, and the trird and the first quartiles are much closer to the median, which falls pretty close to the real value of β (marked by dashed line). What's suprising here, Kogon method resulted in even better median than the Maximum Likelihood method.

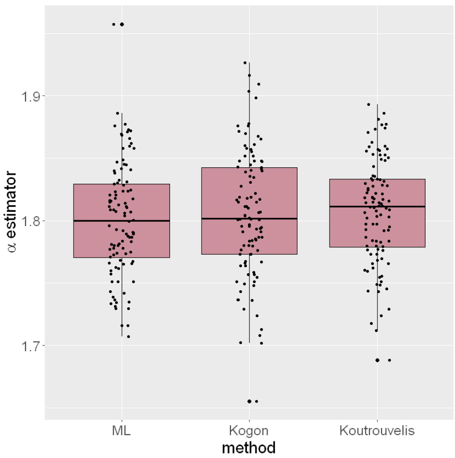

Figure 1: Boxplots of α parameter estimators for different methods

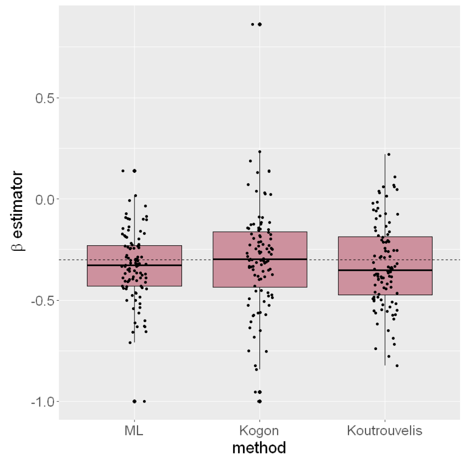

Figure 2: Boxplots of β parameter estimators for different methods

Now, let's take a look at the boxplots for γ estimators. From Figure 3 we can see that the Kogan method produced some huge outliers, so to see the boxplots better, we plotted them again on the smaller range (Figure 4). Again, similarly to previous case, the Maximum Likelihood method seems the most accurate and the Koutrouvelis doesn't fall far behind. Biggest discrepancy is again observed for the Kogon regression.

Similar ovservations can be derived from Figure 5 and Figure 6. Same as before, we can see the big outliers coming from the Kogon method. Also the data range for this method seems to be the biggest again. As expected, Maximum Likelihood looks like the most accurate method here.

To summarize above accuracy analysis, we think it is safe to say that the Maximum Likelihood method scored best in the overall comaprison. Koutrouvelis regression showed a slightly worse performance in this regard, and the Kogon regression is the least accurate of all three. Next, we will compare the speed of analysed methods.

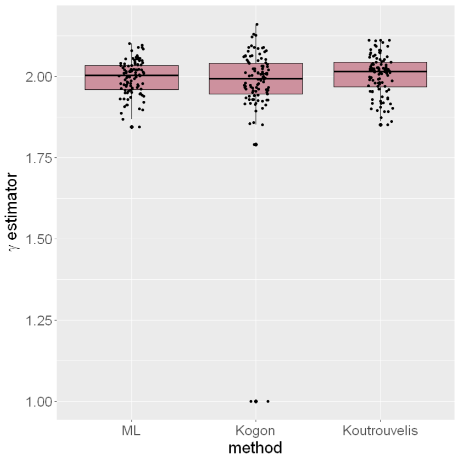

Figure 3: Boxplots of γ parameter estimators for different methods

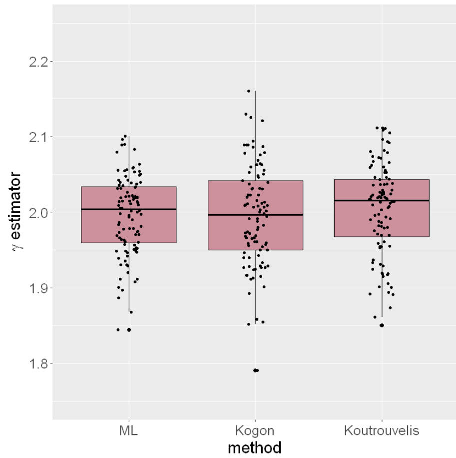

Figure 4: Boxplots of γ parameter estimators for different methods on decreased range

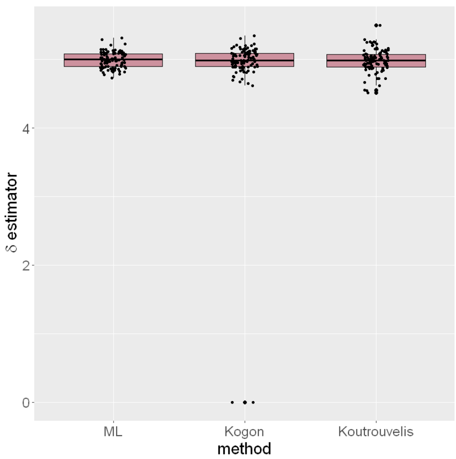

Figure 5: Boxplots of δ parameter estimators for different methods on decreased range

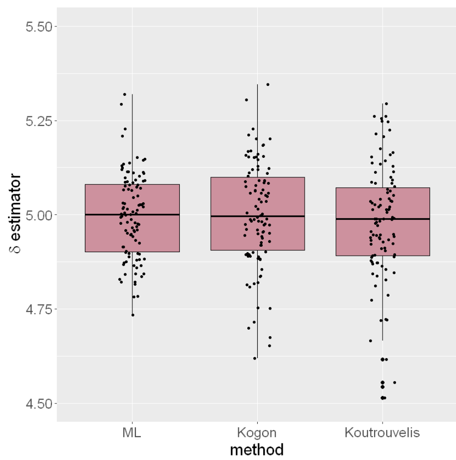

Figure 6: Boxplots of δ parameter estimators for different methods

Speed comparison

To compare the performance of the three methods, we used *benchmark* function from *rbenchmark* library in R environment. It computes the code execution time in seconds. First, we measured the time for one iteration. The results can be observed in Table 3. As we can see, the single run of Koutrouvelis and Kogon methods takek approximately 0.02s, whereas for Maximum Likelihood the duration is almost 7 thousands time longer! Let's see what happens for 100 iterations.

Table 3: Execution time for single run of different alpha stable parameters estimation methods

method	time [s]	replications
Koutrouvelis	0.02	1
Kogon	0.02	1
ML	138.35	1

Table 4 shows that for 100 runs, execution time for Maximum Likelihood method amounts to roughly 14 thousand seconds (which is almost 4 hours). Meanwhile, the Kogon method doesn't even exceed one second, and Koutrouvelis is only slightly slower, summing up to 1.57 seconds of execution time.

Table 4: Execution time for 100 runs of different alpha stable parameters estimation methods

method	time [s]	replications
Koutrouvelis	1.57	100
Kogon	0.90	100
ML	13904.73	100

Considering all above analysis both for speed and accuracy, Koutrouvelis method seems like the best choice if we don't want much loss on either and Maximum Likelihood should be chosen if we aim for the highest accuracy. For the fastest estimation, we can of course choose the Kogon method, although the Koutrouvelis doesn't execute much longer and is by far more accurate, so it can still be considered a better choice in that case.

2 Analysis of the tails of α-stable random variable

We represent the α-stable random variable X in the following way

$$X \sim S_\alpha(\sigma, \beta, \mu),$$

where $\alpha \in (0, 2]$ is the stability parameter, sometimes also called tail index, $\beta \in [-1, 1]$ is skewness parameter, $\sigma > 0$ is a scale parameter and $\mu \in \mathbb{R}$ is a location parameter.

As we want to analyze the tails and as a result, try to estimate α parameter using them, we need to analyze the tail index. On Figure 7 we present a plot of probability density function for different values of α and other parameters set with rather standard values for symmetric α-stable distribution

$$\beta = \mu = 0, \quad \sigma = 1$$

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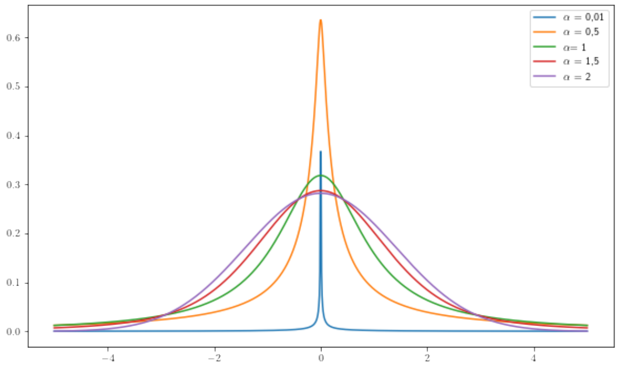

Figure 7: Probability density function for different values of α.

Power law behavior of the tails

When $\alpha = 2$ we get the Gaussian distribution and for $\alpha = 1$ - Cauchy distribution. Additionally for $0 < \alpha \leq 2$ we can observe power law behavior for the tails, which can be represented as

$$\begin{cases} \lim_{t\to\infty} t^\alpha P(X > t) = c_\alpha(1+\beta)\sigma^\alpha, \\ \lim_{t\to-\infty} t^\alpha P(X < -t) = c_\alpha(1+\beta)\sigma^\alpha, \end{cases} \tag{8}$$

where

$$c_\alpha = \frac{1}{\pi}\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right).$$

We can observe the power law behavior of α-stable random variable using the fact that when $t \to \infty$

$$\log(1 - F_\alpha(t)) = -\alpha\log(t) + B,$$

where $B = \log(c_\alpha(1+\beta)\sigma^\alpha)$. So if we plot this on the double logarythmic scale, e.g. for $\alpha = 1.5$ we should obtain a linear function, which is represented on Figure 8.

We can easily observe, that the dependence is indeed linear and the fitted curve should have a slope approximately equal to $-\alpha$ parameter of the given sample. In our case for samples of size 10^4 and 10^6 respectively we get $\alpha_1 = 1.72$ and $\alpha_2 = 1.63$. It's relatively close, but in the next section we will try to examine this method and find the efficient way to get the best results possible.

Estimation of α parameter using tail behavior of α-stable random variables

As seen in previous section by analyzing the tail of α-stable random variables sample we can obtain the α parameter. To show that the method can lead to accurate estimation on Figure 9 we show the boxplots of estimated αs for different size samples (n) and different number of iterations (N). This time we will use $\alpha = 1.6$ and the following algorithm

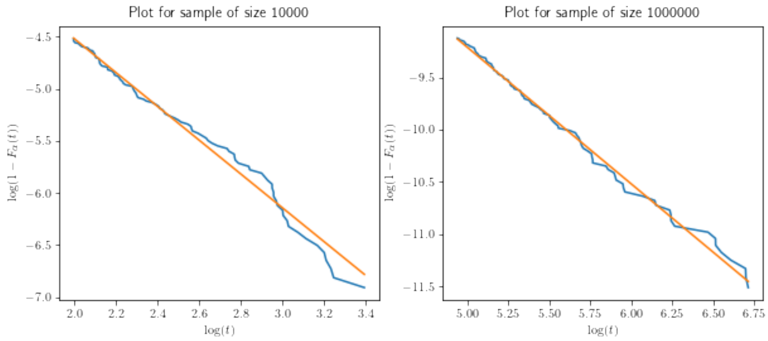

Figure 8: Right tail (last 100 observations) of an empirical distribution function on a double logarythmic scale for samples of different sizes and $\alpha = 1.5$.

Algorithm 3 Estimation of α using tail behavior

Given the sample it obtains the empirical distribution function.
It calculates $\log(t)$ and $\log(1 - F_\alpha(t))$ for the right tail.
Linear regression is used to obtain α using the fact that $\log(1 - F_\alpha(t)) = -\alpha \log(t) + B$.
All steps are repeated N times to provide better accuracy.

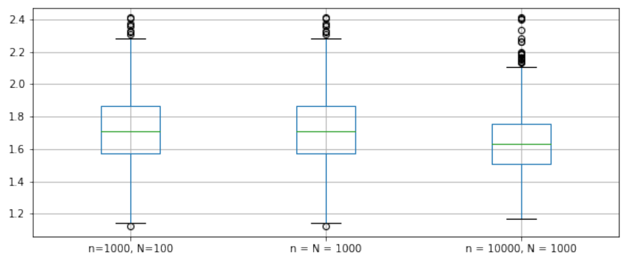

Figure 9: Boxplot of estimated values of alpha.

The boxplot shows, that the method is not very accurate for smaller samples. However with boxplots shown on Figure 9 we expect that the sample size is more relevant when it comes to estimation accuracy. On the next plot we will try to examine the accuracy for different values of α, n and N .

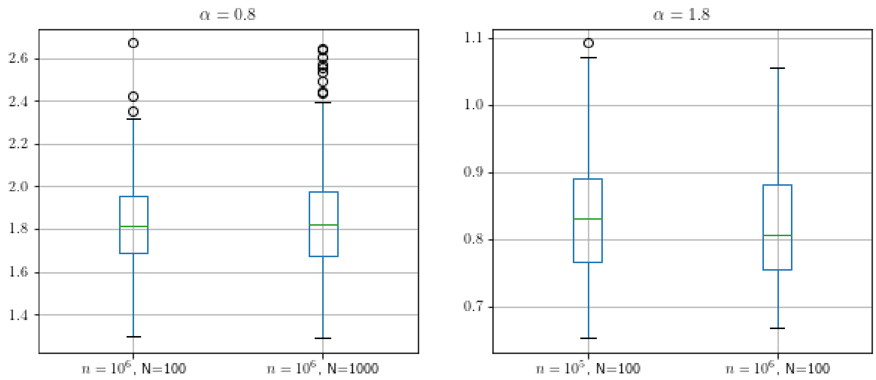

Figure 10: Boxplot of estimated values of alpha

Again we observe that with only $N = 100$ iterations we can obtain good results when the sample is long enough. That can be also determined from the previos section when we showed that the power law behavior is observed at $t \to \infty$. Next we will calculate the variance and bias of presented estimators to show that observations based on Figure 10 were correct. In Table 2 we present the results for discussed estimators.

$\alpha = 1.8$

	Variance	Bias
$N = 1000, n = 10^6$	0.047137	0.034701
$N = 100, n = 10^6$	0.056292	0.043853

$\alpha = 0.8$

$N = 100$	Variance	Bias
$n = 10^5$	0.009296	0.033892
$n = 10^6$	0.006714	0.018673

We can observe that variance is smaller for smaller value of α. Additionally we can compare the results with methods presented in section 1. We see that tail based estimator performs a little worse than earlier described estimators. Yet it shows that the Central Limit Theorem can yield to a rather good estimation of α parameter with variance comparable to other methods such as maximum likelihood method, iterative Koutrouvelis regression or Kogon regression.

Additionally on Figure 11 we present a scatter plot of biase and variance for different sample sizes and different numbers of iterations to show how presented algorithm performs. We again see that length of a sample is extremely important when it comes to α simulation.

We managed to show that the power-law behavior of tails itself is sufficient to estimate α-parameter of an α stable distribution sample with remarkable accuracy.

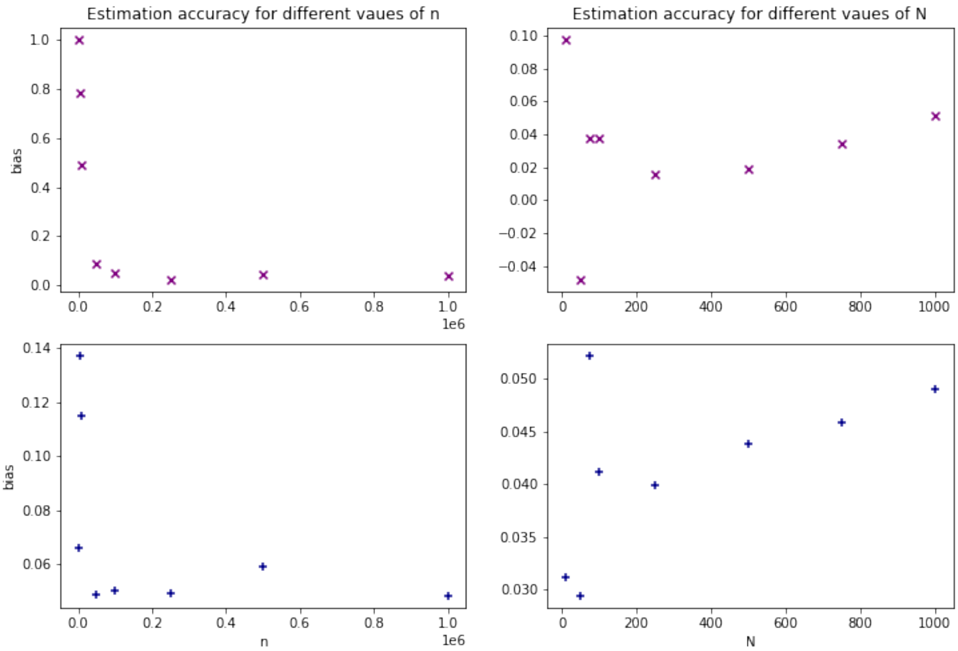
Estimation accuracy for different vaues of n
Estimation accuracy for different vaues of N
bias
bias
n
N
1e6
1e6

Figure 11:

References

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