# 2nd Report, Computer Simulations

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May 15, 2023

# 1 Multivariate stable distributions

We will analyze the Multivariate stable vectors. At first we will show a method used to simulate such vectors with illustrations to show their behavior. Next we will look closer at the 2-dimensional empirical densities. At the end we will shortly discuss a method of estimation of spectral measure with few examples.

## 1.1 Simulation of stable vectors

Let's begin with a definition of multivariate stable distribution. A d-dimensional stable vector is defined using a spectral measure $\Gamma$ and a shift vector $\mu^0 \in \mathbb{R}^d$. We will denote it as

$$X \sim S_{\alpha,d}(\Gamma, \mu^0). \tag{1}$$

The characteristic function of $X$ is then

$$\phi_X(t) = \mathbf{E}(\exp\{i\langle X, t\rangle\}). \tag{2}$$

We will examine the case with a discrete spectral measure

$$\Gamma(\cdot) = \sum_{j=0}^{n} \gamma_j \delta_{s_j}(\cdot). \tag{3}$$

Then the characteristic function takes the form

$$\phi(t) = \exp\left(-\sum_{j=1}^{n} \psi_\alpha(\langle t, s_j\rangle)\gamma_j\right), \tag{4}$$

where

$$\psi_\alpha(u) = \begin{cases} |u|^\alpha(1 - ßsign(u))\tan\frac{\pi\alpha}{2}, & \alpha \neq 1, \\ |u|(1 + ß\frac{2}{\pi}sign(u))\log|u|, & \alpha = 1. \end{cases} \tag{5}$$

Then $X$ can be expressed as

$$X = \begin{cases} \sum_{j=1}^{n} \gamma_j^{1/\alpha} Z_j s_j + \mu^0, & \alpha \neq 1, \\ \sum_{j=1}^{n} \gamma_j^{1/\alpha}(Z_j + \frac{2}{\pi}\log\gamma_j)s_j + \mu^0, & \alpha \neq 1, \end{cases} \tag{6}$$

where $Z_1, \ldots, Z_n$ are one dimensional $\alpha$-stable random variables. We will use $Z_i \sim S_\alpha(\beta = 1, \gamma = 1, \delta = 0)$. So the only thing needed to simulate multivariate stable distribution is the ability to generate a vector of univariate variables. Now let's take a look on how the vectors look depending on $\alpha$ and the spectral measure. For visualization simplicity we will only look at 2-dimensional case and present it with a scatter plot, 3D histogram and a density heatmap. All three were generated using `python` with `plt.scatter`, `np.histogram2d` with `bar3d` functionality and `plotly.express.density_contour` respectively. Here we analyze the case where

$$\begin{aligned} \gamma &= [0.25, 0.125, 0.25, 0.25, 0.125, 0.25] \\ \mathbf{s} &= [(1,0), (\frac{1}{2}, \frac{\sqrt(3)}{2}), (-\frac{1}{2}, \frac{\sqrt(3)}{2}), (-1,0), (-\frac{1}{2}, -\frac{\sqrt(3)}{2}), (\frac{1}{2}, -\frac{\sqrt(3)}{2})] \end{aligned} \tag{7}$$

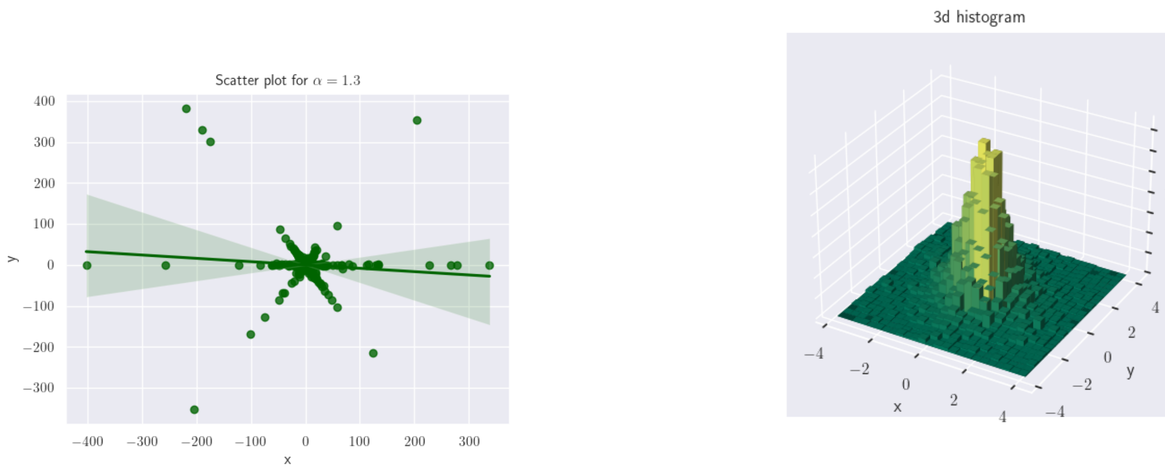



Figure 1: Plots for $\alpha = 1.3$

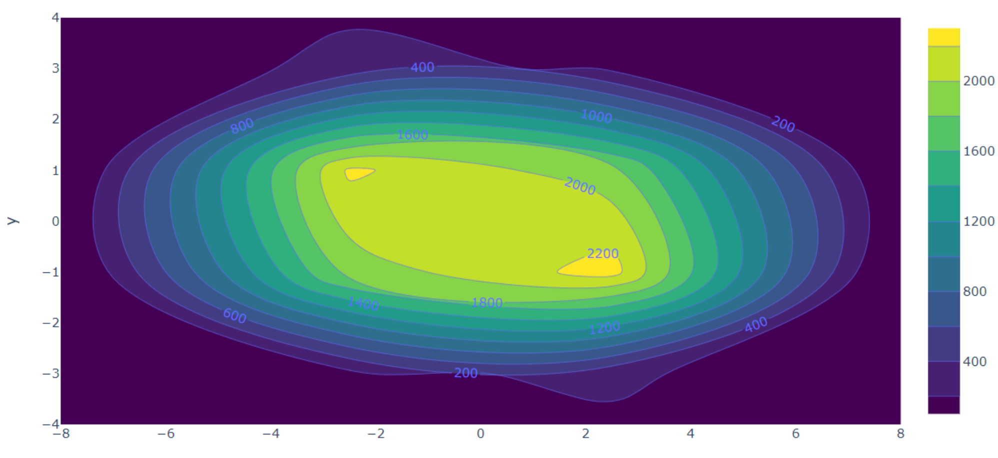



Figure 2: Heatmap for $\alpha = 1.3$

Now we will try to change the parameters to observe vectors with different spectral measure. We take the same values, but in different orders. From now on we take vector $s$ as

$$s_j = \left(\cos(2\pi\frac{j-1}{m}), \sin(2\pi\frac{j-1}{m})\right), \tag{8}$$

where $m$ is the number of masses in the spectral measure. So for $m = 6$ and $\Gamma = [0.25, 0.25, 0.125, 0.125, 0.25, 0.25]$ we have

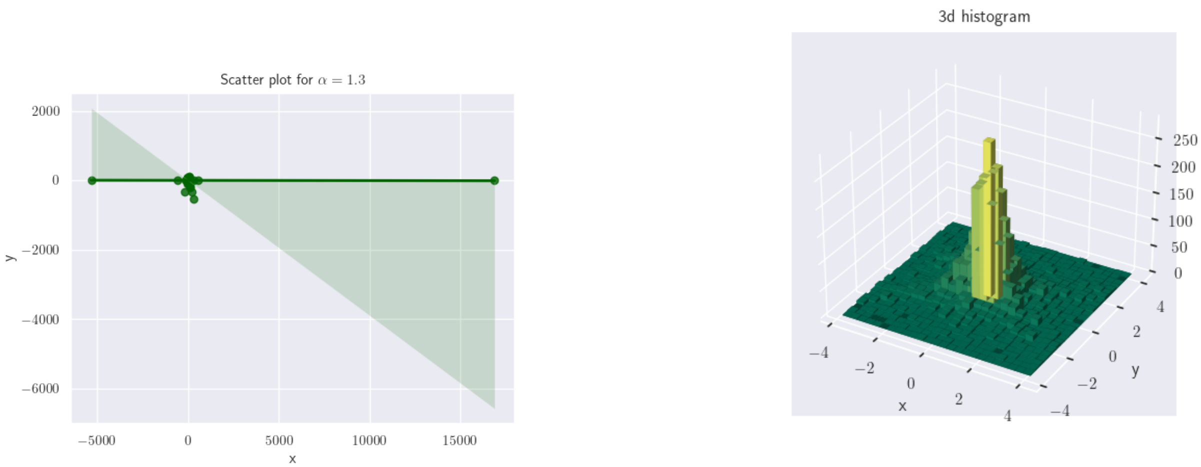



Figure 3: Plots for $\alpha = 1.3$

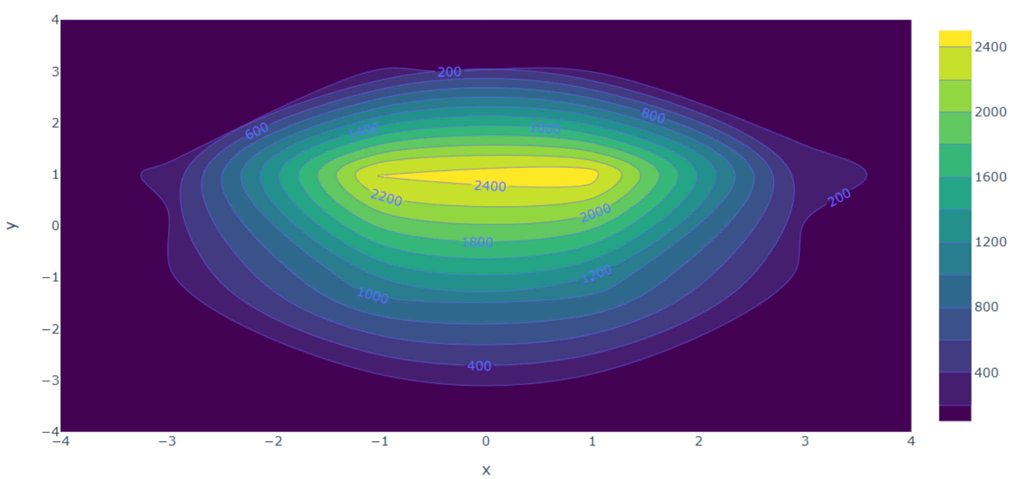


Figure 4: Heatmap for $\alpha = 1.3$

We will present some more results for different spectral measures and values of $alpha$ this time using only density map. Here we will also present the unsymmetrical case with $m = 5$ -point masses.

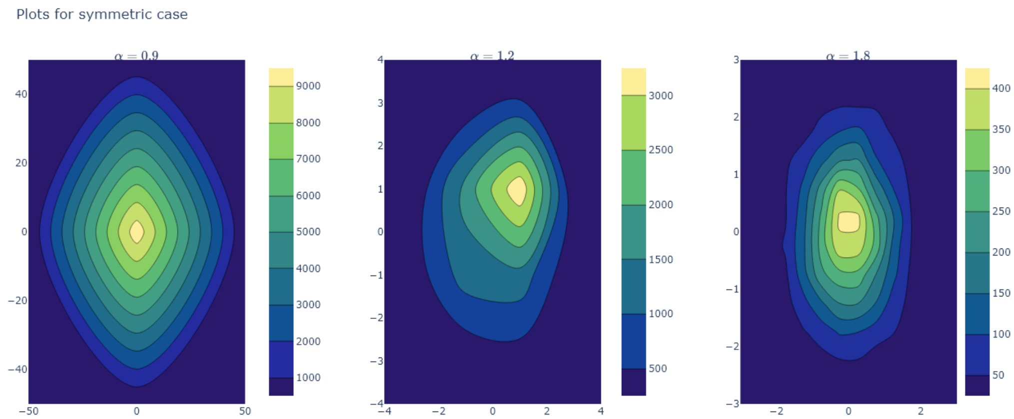



Figure 5

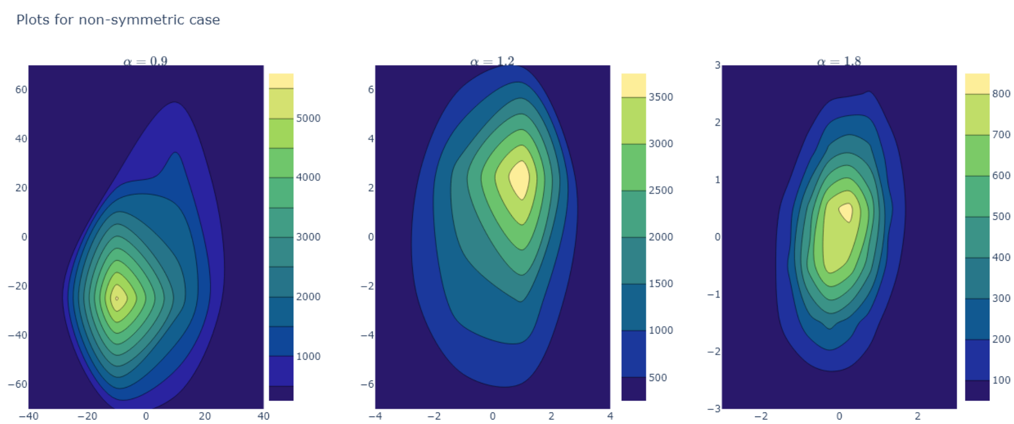
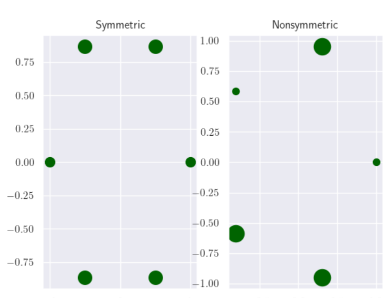



Figure 6

We can easily observe that all the parameters have a big influence on the resulting 2-dimensional vector.




Also it's visible that we get far more regular histogram shapes in the symmetric cases. Additionally if we drew mass points with specific spectral measure we could observe that the plots reflect that parameter. When comparing earlier presented histogram contours with scatter of mass points, we can see that they strictly correspond to each other, showing us how big role plays the spectral measure. We can also observe that the dependency seems higher for greater values of $\alpha$.

## 1.2 Spectral measure estimation

This section will focus on estimation of the discrete spectral measure, given by (3). We will use the Rachev-Xin-Chen method, described below.

First, for a set $A \subset S_d$, let's define a cone generated by $A$

$$Cone(A) = \{x \in \mathbb{R}^d : |x| > 0, \frac{x}{|x|} \in A\} = \{ra : r > 0, a \in A\}. \tag{9}$$

With such defined cone, we can estimate the spectral measure, using the following formula

$$\hat{\Gamma}(A) = \frac{\#\{X_i : X_i \in Cone(A), |X_i| > r\}}{\#\{X_i : |X_i| > r\}}, \tag{10}$$

where $X_i$, $i = 1, 2, \ldots, n$ is an $i$-th observation of a multivariate stable vector of length $n$, and $r$ is an arbitrarily chosen radius.

To simulate the random stable vector, we will use the method described in the previous section. Let us start with symmetric case of $\alpha = 1.3$ and $m = 6$ point masses:

$$\gamma = [0.25, 0.125, 0.25, 0.25, 0.125, 0.25],$$

$$s = \left[\left(1, 0\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-1, 0\right), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\right].$$

We will estimate the spectral measure four times, with four different values of the radius, $r = 50, 75, 100, 125$.

On Figure 10 we can see the scatter plots of this symmetric vector for the subsequent values of $r$, divided into six cones. The dashed lines represent the borders of cones and the purple-shaded circle is the area generated by $r$. Looking at those plots and the point mass plot (Figure 8), we would expect that a big proportion of mass will be assigned to the third, fourth and sixth cone, as we can observe quite heavy tails in those directions. Mass point plot shows a big propotion in the first cone as well, however outliers observed in that area are not that significant. Second and fifth cone also don't show a heavy tail behavior (especially the second one). Indeed, as Table 1 shows, our guess is correct. The results of estimation are pretty similar for $r = 50$ and $r = 75$ — the biggest proportion of mass is observed for $\hat{\Gamma}_3$, while $\hat{\Gamma}_4$ and $\hat{\Gamma}_6$ fall right behind. The situation shifts slightly for $r = 100$. Here we can see that the domination of the third cone is no longer so explicit. The propotions seem more even, with exception of $\hat{\Gamma}_2$, which has definiely smaller value assigned compared to the rest. This tendency persists with $r = 125$, but is now even more obvious. Proportion assigned to $\hat{\Gamma}_2$ is now equal to zero, and the rest of the mass is distributed evenly. Overall, it seems that choosing too big value for $r$ can lead to misleading results. For the first three values of $r$, the estimation corresponded well with the initial guess, as well as the point mass plot, but for $r = 125$ the results seem to be less accurate.

Table 1: Spectral measure estimation for symmetric case

| | $\hat{\Gamma}_1$ | $\hat{\Gamma}_2$ | $\hat{\Gamma}_3$ | $\hat{\Gamma}_4$ | $\hat{\Gamma}_5$ | $\hat{\Gamma}_6$ |
|---|---|---|---|---|---|---|
| $r = 50$ | 0.115385 | 0.115385 | 0.326923 | 0.153846 | 0.134615 | 0.153846 |
| $r = 75$ | 0.076923 | 0.115385 | 0.269231 | 0.192308 | 0.153846 | 0.192308 |
| $r = 100$ | 0.133333 | 0.066667 | 0.266667 | 0.133333 | 0.133333 | 0.266667 |
| $r = 125$ | 0.200000 | 0.000000 | 0.200000 | 0.200000 | 0.200000 | 0.200000 |

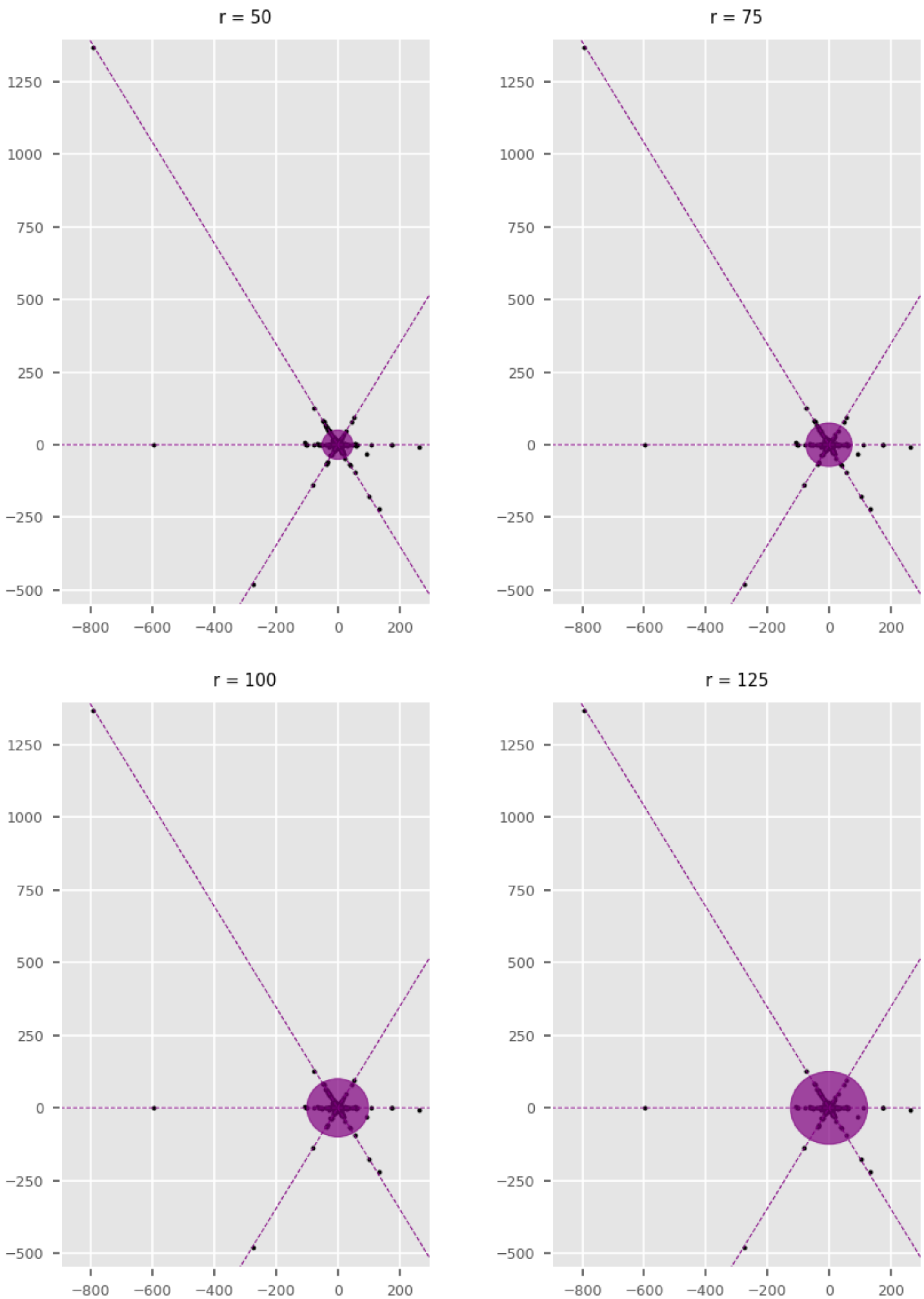



Figure 7: Symmetric stable vector $\alpha = 1.3$ divided into six cones.

Now, we will look at a nonsymmetric case of $\alpha = 1.7$ and $m = 5$ point masses:

$$\gamma = [0.3, 0.3, 0.1, 0.1, 0.1],$$

$$s = \left[ \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right), \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right), \left( 0, -1 \right), \left( -1, 0 \right), \left( 1, 0 \right) \right].$$

Again, we will estimate the spectral measure four times, this time for radius values $r = 10, 20, 30, 40$. Let's first take a look at the point mass plot (Figure 9) and the scatter plots of the vector for the subsequent values of $r$. As before, the dashed lines represent the borders of cones and the purple-shaded circle is the area generated by $r$. We see that more mass is concentrated in the first and second cone, where we can observe quite heavy tails compared to the rest of the cones. There are also some outliers visible in the third and fourth cone, however much smaller. With that considered, we can guess that our spectral measure estimation will result in a big proportion of mass assigned to the first and second cone.

Table 2: Spectral measure estimation for nonsymmetric case

| | $\hat{\Gamma}_1$ | $\hat{\Gamma}_2$ | $\hat{\Gamma}_3$ | $\hat{\Gamma}_4$ | $\hat{\Gamma}_5$ |
|---|---|---|---|---|---|
| $r = 10$ | 0.380952 | 0.380952 | 0.095238 | 0.095238 | 0.047619 |
| $r = 20$ | 0.461538 | 0.384615 | 0.076923 | 0.076923 | 0.000000 |
| $r = 30$ | 0.500000 | 0.500000 | 0.000000 | 0.000000 | 0.000000 |
| $r = 40$ | 0.500000 | 0.500000 | 0.000000 | 0.000000 | 0.000000 |

Let's now analyze the results visible in Table 2. We see that, in fact, $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ hold the biggest proportion of mass for all values of $r$. In the case of $r = 10$, we can also observe some mass assigned to the remaining cones (smallest proportion at $\hat{\Gamma}_5$), however for $r = 20$ we can already see a zero assigned to $\hat{\Gamma}_5$. For remaining values of $r$, the superiority of the first and second cone is even more explicit. Here, $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ hold equally 50% of mass each, whereas $\hat{\Gamma}_3$, $\hat{\Gamma}_4$ and $\hat{\Gamma}_5$ are left with zeros, which doesn't seem very accurate. Again then, we can observe that choosing $r$ too big can disturb the accuracy of spectral measure estimation.

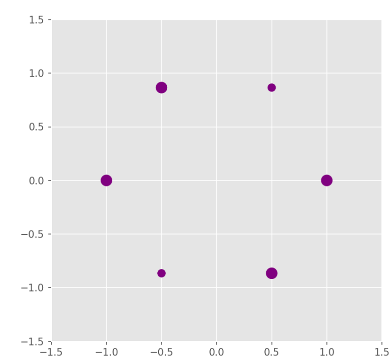



Figure 8: Point mass plot for symmetric case

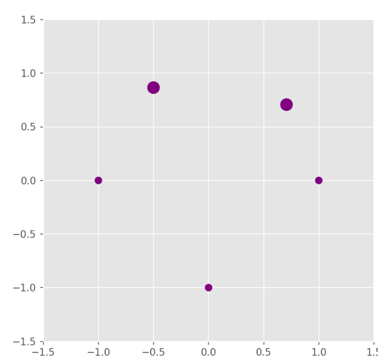



Figure 9: Point mass plot for nonsymmetric case

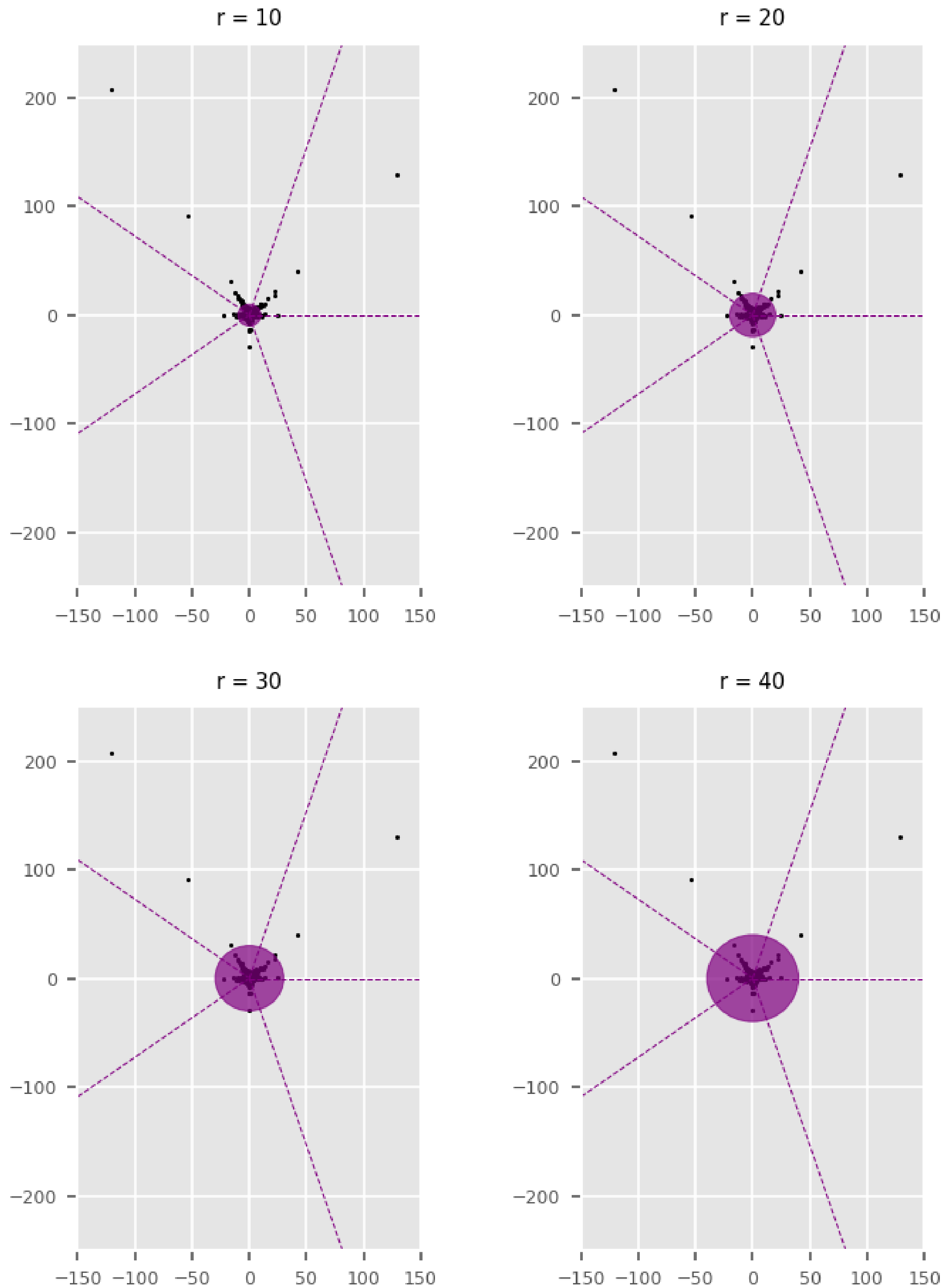



Figure 10: Nonsymmetric stable vector $\alpha = 1.7$ divided into five cones.