A Modern Approach to Quantum Mechanics by Townsend - Solutions

Solutions by: GT SPS

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Contents

12 Identical Particles

12.1

$$
\begin{aligned}
|\Psi_A\rangle = \frac{1}{2}\int d^3r_1 d^3r_2 \left(\frac{1}{\sqrt{2}}|\mathbf{r}_1,\mathbf{r}_2\rangle - \frac{1}{\sqrt{2}}|\mathbf{r}_2,\mathbf{r}_1\rangle\right) \\
\left(\frac{1}{\sqrt{2}}\langle\mathbf{r}_1,\mathbf{r}_2|\Psi_A|\mathbf{r}_1,\mathbf{r}_2|\Psi_A\rangle - \frac{1}{\sqrt{2}}\langle\mathbf{r}_2,\mathbf{r}_1|\Psi_A|\mathbf{r}_2,\mathbf{r}_1|\Psi_A\rangle\right) \\
= \frac{1}{2}\int d^3r_1 d^3r_2 \Bigg(\frac{1}{\sqrt{2}}|\mathbf{r}_1,\mathbf{r}_2\rangle \frac{1}{\sqrt{2}}\langle\mathbf{r}_1,\mathbf{r}_2|\Psi_A|\mathbf{r}_1,\mathbf{r}_2|\Psi_A\rangle - \frac{1}{\sqrt{2}}|\mathbf{r}_2,\mathbf{r}_1\rangle \frac{1}{\sqrt{2}}\langle\mathbf{r}_1,\mathbf{r}_2|\Psi_A|\mathbf{r}_1,\mathbf{r}_2|\Psi_A\rangle \\
-\frac{1}{\sqrt{2}}|\mathbf{r}_1,\mathbf{r}_2\rangle \frac{1}{\sqrt{2}}\langle\mathbf{r}_2,\mathbf{r}_1|\Psi_A|\mathbf{r}_2,\mathbf{r}_1|\Psi_A\rangle + \frac{1}{\sqrt{2}}|\mathbf{r}_2,\mathbf{r}_1\rangle \frac{1}{\sqrt{2}}\langle\mathbf{r}_2,\mathbf{r}_1|\Psi_A|\mathbf{r}_2,\mathbf{r}_1|\Psi_A\rangle\Bigg) \\
= \frac{1}{2}\int d^3r_1 d^3r_2 \Bigg(\frac{1}{2}|\mathbf{r}_1,\mathbf{r}_2\rangle \langle\mathbf{r}_1,\mathbf{r}_2|\Psi_A|\mathbf{r}_1,\mathbf{r}_2|\Psi_A\rangle - \frac{1}{2}|\mathbf{r}_2,\mathbf{r}_1\rangle \langle\mathbf{r}_1,\mathbf{r}_2|\Psi_A|\mathbf{r}_1,\mathbf{r}_2|\Psi_A\rangle \\
-\frac{1}{2}|\mathbf{r}_1,\mathbf{r}_2\rangle \langle\mathbf{r}_2,\mathbf{r}_1|\Psi_A|\mathbf{r}_2,\mathbf{r}_1|\Psi_A\rangle + \frac{1}{2}|\mathbf{r}_2,\mathbf{r}_1\rangle \langle\mathbf{r}_2,\mathbf{r}_1|\Psi_A|\mathbf{r}_2,\mathbf{r}_1|\Psi_A\rangle\Bigg) \\
= \frac{1}{4}\int d^3r_1 d^3r_2 \Bigg(|\mathbf{r}_1,\mathbf{r}_2\rangle \langle\mathbf{r}_1,\mathbf{r}_2|\Psi_A|\mathbf{r}_1,\mathbf{r}_2|\Psi_A\rangle + |\mathbf{r}_2,\mathbf{r}_1\rangle \langle\mathbf{r}_2,\mathbf{r}_1|\Psi_A|\mathbf{r}_2,\mathbf{r}_1|\Psi_A\rangle \\
+ |\mathbf{r}_1,\mathbf{r}_2\rangle \langle\mathbf{r}_1,\mathbf{r}_2|\Psi_A|\mathbf{r}_1,\mathbf{r}_2|\Psi_A\rangle + |\mathbf{r}_2,\mathbf{r}_1\rangle \langle\mathbf{r}_2,\mathbf{r}_1|\Psi_A|\mathbf{r}_2,\mathbf{r}_1|\Psi_A\rangle\Bigg) \\
= \int d^3r_1 d^3r_2\, |\mathbf{r}_1,\mathbf{r}_2\rangle \langle\mathbf{r}_1,\mathbf{r}_2|\Psi_A|\mathbf{r}_1,\mathbf{r}_2|\Psi_A\rangle
\end{aligned}
$$

12.2

a) **Determine the ground state and first excited state kets and corresponding energies when the two particles are in total spin-0 state. What are the lowest energy states and corresponding kets for the particles if they are in a total spin-1 state.**

The energy of a two particle harmonic oscillator is given by

$$E_n = (n_1 + n_2 + 1)\hbar\omega$$

The energy of one particles in a harmonic oscillator is known to be

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

The energy of the ground state and first excited state of the one particle system is

$$E_0 = \frac{\hbar\omega}{2}$$
$$E_1 = \frac{3\hbar\omega}{2}$$

For two particles, however, we must sum the possible energies of the two particles. The ground state will have both particles in their respective ground state

$$E_{00} = \hbar\omega$$

while the first excited state will have one in its ground state and the other in the first excited state

$$E_{10,01} = 2\hbar\omega$$

The spin-0 states will have anti-symmetric spin components. The ground state with energy $\hbar\omega$ is given by,

$$|n, s, m_s\rangle = |1, 0, 0\rangle = \frac{1}{\sqrt{2}} |r_1, r_2\rangle \left(|\chi_+, \chi_-\rangle - |\chi_-, \chi_+\rangle\right)$$

and the first excited state with energy $\hbar\omega$ is given by

$$|1, 1, 0\rangle = \frac{1}{2} \left(|r_1, r_2\rangle + |r_2, r_1\rangle\right) \left(|\chi_+, \chi_-\rangle - |\chi_-, \chi_+\rangle\right)$$

The spin-1 states will have symmetric spin components. The states with lowest energy $2\hbar\omega$ are given by,

$$|1, 1, 1\rangle = \frac{1}{\sqrt{2}} \left(|r_1, r_2\rangle - |r_2, r_1\rangle\right) |\chi_+, \chi_-\rangle$$

$$|1, 1, -1\rangle = \frac{1}{2} \left(|r_1, r_2\rangle - |r_2, r_1\rangle\right) \left(|\chi_+, \chi_-\rangle + |\chi_-, \chi_+\rangle\right)$$

b) **Suppose two particles interact with a potential energy of interaction**

$$V(|x_1 - x_2|) = \begin{cases} -V_0 & |x_1 - x_2| < a \\ 0 & \text{elsewhere} \end{cases}$$

Argue what the effect will be on the energies that you determined in (a), that is, whether the energy of each state moves up, moves down, or remains unchanged. ***Suggestion: Examine which spatial wave functions for the total spin-0 and total spin-1 states tend to have the particles closer together. Consider, for example, the special case of*** $x_1 = x_2$

The ground state total-spin-0 case will leave the particles closer together, as they are in the same energy level. In the total-spin-0 case, we can see that there is a higher probability of finding the system in the $|x_1 - x_2| < a$ case, therefore shifting the overall energy of the state down. For the first excited state, the particles will be in different shells, meaning that their wave functions will not overlap *as much*. This means there is a lower probability of finding the system to have $|x_1 - x_2| < a$, and while the energy of the first excited state will also decrease, it will decrease less relative to the ground state.

12.3

The energy splitting that comes from any two particles that have spin is described in section 5.2 of the the Townsend textbook. While a more detailed explanation of the hyperfine interaction can be found on Wikipedia under "Hyperfine interaction", for this problem we only need to understand that the energy splitting comes from the interaction of the magnetic moments of the two particles (which is intimately tied to their spins). The interaction depends both on the strength of the magnetic moment and the distance between the two particles. Luckily, the distance between the electron and proton in the hydrogen atom is on the same order of magnitude as the distance between the two electrons in a helium atom. However, the magnetic moment of an electron is much greater than that of a proton, by about 10^3 or 3 orders of magnitude larger. Therefore, if the splitting due to spin-spin interactions between the electron and proton of a hydrogen atom is on the order of 10^{-6} eV (see page 150 of the Townsend textbook) then we should expect the splitting due to spin-spin interactions between the electrons of a helium atom to be on the order of 10^{-3} eV.

12.4 Trial wave function we will use is the Gaussian below.

$$\phi = Ae^{-cx^2}$$

Next we will normalize the the trial wave function.

$$\int_{-\infty}^{\infty} |\phi|^2 dx = 1$$
$$A^2 \int_{-\infty}^{\infty} e^{-2cx^2} dx = A^2 \sqrt{\frac{\pi}{2c}} = 1$$
$$A^2 = \sqrt{\frac{2c}{\pi}}$$
$$A = (\frac{2c}{\pi})^{\frac{1}{4}}$$

$$\langle E|E\rangle = \left\langle \phi|\hat{H}|\phi \middle| \phi|\hat{H}|\phi \right\rangle = \left\langle \phi|\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + bx^4|\phi \middle| \phi|\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + bx^4|\phi \right\rangle$$
$$A^2 \left(\frac{-\hbar^2}{2m} \int_{-\infty}^{\infty} e^{-cx^2} \frac{\partial^2}{\partial x^2} \left(e^{-cx^2} \right) dx + b \int_{-\infty}^{\infty} x^4 e^{-2cx^2} \right)$$
$$A^2 \left(\frac{-\hbar^2}{2m} \int_{-\infty}^{\infty} e^{-cx^2} \left(-2ce^{-cx^2} + (2cx)^2 e^{-cx^2} \right) dx + b \int_{-\infty}^{\infty} x^4 e^{-2cx^2} dx \right)$$
$$A^2 \left(\frac{-\hbar^2}{2m} \int_{-\infty}^{\infty} \left(-2ce^{-2cx^2} + (2cx)^2 e^{-2cx^2} \right) dx + b \int_{-\infty}^{\infty} x^4 e^{-2cx^2} dx \right)$$
$$A^2 \left(\frac{-\hbar^2}{2m} \left(\int_{-\infty}^{\infty} -2ce^{-2cx^2} dx + \int_{-\infty}^{\infty} (2cx)^2 e^{-2cx^2} dx \right) + b \int_{-\infty}^{\infty} x^4 e^{-2cx^2} dx \right)$$
$$A^2 \left(\frac{-\hbar^2}{2m} \left((-2c)\sqrt{\frac{\pi}{2c}} + (2c)^2 \frac{1}{2} \sqrt{\frac{\pi}{(2c)^3}} \right) + b \int_{-\infty}^{\infty} x^4 e^{-2cx^2} dx \right)$$
$$A^2 \left(\frac{-\hbar^2}{2m} \left((-2c)\sqrt{\frac{\pi}{2c}} + (2c)^2 \frac{1}{2} \sqrt{\frac{\pi}{(2c)^3}} \right) + b \frac{3}{16} \frac{\sqrt{\frac{\pi}{2}}}{c^{\frac{5}{2}}} \right)$$
$$A^2 \left(\frac{-\hbar^2}{2m} \left((-2c)\frac{\pi}{2c} + (2c)^2 \frac{1}{2} \frac{1}{(2c)} \sqrt{\frac{\pi}{2c}} \right) + b \frac{3}{16} \frac{\sqrt{\frac{\pi}{2}}}{c^{\frac{5}{2}}} \right)$$
$$A^2 \left(\frac{-\hbar^2}{2m} \left(-c\sqrt{\frac{\pi}{2c}} \right) + b \frac{3}{16} \frac{\sqrt{\frac{\pi}{2}}}{c^{\frac{5}{2}}} \right)$$
$$\left(\left(\frac{2c}{\pi} \right)^{\frac{1}{2}} \frac{-\hbar^2}{2m} \left(-c\sqrt{\frac{\pi}{2c}} \right) + \left(\frac{2c}{\pi} \right)^{\frac{1}{2}} b \frac{3}{16} \frac{\sqrt{\frac{\pi}{2}}}{c^{\frac{5}{2}}} \right)$$
$$\langle E|E\rangle = \left\langle \phi|\hat{H}|\phi \middle| \phi|\hat{H}|\phi \right\rangle = \frac{\hbar^2 c}{2m} + \frac{3b}{16} \frac{1}{c^2}$$

Next we need to minimize $\langle E|E\rangle$.

$$\frac{\partial \langle E|E\rangle}{\partial c} = \frac{\hbar^2}{2m} - \frac{3b}{8}\frac{1}{c^3} = 0$$
$$\frac{\hbar^2}{2m} = \frac{3b}{8}\frac{1}{c^3}$$
$$c^3 = \frac{3b}{4}\frac{m}{\hbar^2}$$
$$c = \left(\frac{3bm}{4\hbar^2}\right)^{\frac{1}{3}}$$

Pluging c back into $\langle E|E\rangle$.

$$\langle E|E\rangle = \frac{\hbar^2}{2m}\left(\frac{3bm}{4\hbar^2}\right)^{\frac{1}{3}} + \frac{3b}{16}\left(\frac{4\hbar^2}{3bm}\right)^{frac23}$$
$$\frac{1}{2}b^{\frac{1}{3}}\left(\frac{\hbar^2}{2m}\right)^{\frac{2}{3}}3^{\frac{1}{3}} + \frac{3}{16}b^{\frac{1}{3}}\left(\frac{\hbar^2}{2m}\right)^{\frac{2}{3}}\left(\frac{8}{3}\right)^{\frac{2}{3}}$$
$$b^{\frac{1}{3}}\left(\frac{\hbar^2}{2m}\right)^{\frac{2}{3}}\left[\frac{1}{2}3^{\frac{1}{3}} + \frac{3}{16}\left(\frac{8}{3}\right)^{\frac{2}{3}}\right]$$
$$b^{\frac{1}{3}}\left(\frac{\hbar^2}{2m}\right)^{\frac{2}{3}}\left[\frac{1}{2}3^{\frac{1}{3}} + \frac{1}{4}3^{\frac{1}{3}}\right]$$
$$\frac{3\,(3)^{\frac{1}{3}}}{4}b^{\frac{1}{3}}\left(\frac{\hbar^2}{2m}\right)^{\frac{2}{3}} = 1.082b^{\frac{1}{3}}\left(\frac{2}{2m}\right)^{\frac{2}{3}}$$

This gives us about a 2.07 percent difference from the exact result.

12.5

Here we will use the same Gaussian function from the the previous problem as the trial function.

$$\phi = \left(\frac{2c}{\pi}\right)^{\frac{1}{4}} e^{-cx^2}$$

Calculating $\langle E|E\rangle$.

$$\langle E|E\rangle = A^2\left(\frac{-\hbar^2}{2m}\int_{-\infty}^{\infty} e^{-cx^2}\frac{\partial^2}{\partial x^2}\left(e^{-cx^2}\right)dx - \int_{-\infty}^{\infty}\frac{\lambda}{b}\delta(x)e^{-2cx^2}dx\right)$$
$$A^2\left(\frac{-\hbar^2}{2m}\int_{-\infty}^{\infty} e^{-cx^2}\left(-2ce^{-cx^2} + (2cx)^2e^{-cx^2}\right)dx - \int_{-\infty}^{\infty}\frac{\lambda}{b}\delta(x)e^{-2cx^2}dx\right)$$
$$\langle E|E\rangle = A^2\left(\frac{\hbar^2}{2m}\sqrt{\frac{2c}{\pi}} - \frac{\lambda}{b}\right) = \left(\frac{2c}{\pi}\right)^{\frac{1}{4}}\left(\frac{\hbar^2}{2m}\sqrt{\frac{2c}{\pi}} - \frac{\lambda}{b}\right) = \frac{\hbar^2 c}{2m} - \frac{\lambda}{b}\left(\frac{2c}{\pi}\right)^{\frac{1}{2}}$$

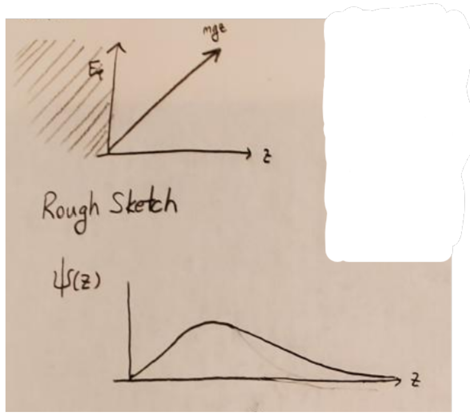

Figure 1: Graph for part a)

Next we need to minimize $\langle E|E\rangle$.

$$\frac{\partial^2 \langle E|E\rangle}{\partial c} = \frac{\hbar^2}{2m} - \frac{\lambda}{2b}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{c^{\frac{1}{2}}} = 0$$

$$\frac{\hbar^2}{2m} = \frac{\lambda}{2b}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{c^{\frac{1}{2}}}$$

$$c = \frac{2m^2\lambda^2}{\pi\hbar^4 b^2}$$

Plugging c back into $\langle E|E\rangle$.

$$\langle E|E\rangle = \frac{\hbar^2}{2m}\left(\frac{2m\lambda^2}{\pi\hbar^4 b^2}\right) - \frac{\lambda}{b}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{2m^2\lambda^2}{\pi\hbar^4 b^2}\right)^{\frac{1}{2}}$$

$$\frac{\lambda^2}{\pi b^2} - \frac{\lambda}{b}\left(\frac{4m^2\lambda^2}{\pi\hbar^4 b^2}\right)$$

$$\langle E|E\rangle = \frac{\lambda^2}{\pi b^2} - \frac{\lambda^2 2m}{b^2\pi^2}$$

12.6

a)

$$E_p(z) = \begin{cases} \infty & z \leq 0 \\ mgz & \text{elsewhere} \end{cases}$$

$$H = E_K + mgz \longrightarrow -\frac{\hbar^2}{2m}\frac{d^2\Psi}{dz^2} + mgz\Psi = E\Psi \text{ when } E_\phi \geq E \ (\phi = \text{trial})$$

$$\longrightarrow -\frac{\hbar^2}{2m}\frac{d^2\phi}{dz^2} + mgz\phi = E_\phi\phi$$

Where

$$E_\phi = \frac{\langle\phi|H|\phi|\phi|H|\phi\rangle}{\langle\phi|\phi|\phi|\phi\rangle} \text{ (Variational Theorem)}$$

b) Guess that $\phi = ce^{-z}$:

$$\langle\phi|H|\phi|\phi|H|\phi\rangle \longrightarrow \int_0^\infty c^2\bigg(-\frac{\hbar^2}{2m}e^{-2z}\bigg) + c^2mgze^{-2z}dz = \int_0^\infty E_\phi c^2e^{-2z}dz$$

$$\longrightarrow c^2\bigg(-\frac{\hbar^2}{4m}\bigg) + \frac{c^2mg}{4} = \frac{E_\phi c^2}{2} \longrightarrow -\frac{\hbar^2}{4m} + \frac{mg}{4} = \frac{E_\phi}{2}$$

$$E_\phi = \frac{mg}{2} - \frac{\hbar^2}{2m} \longrightarrow E \leq \frac{mg}{2} - \frac{\hbar^2}{2m}, \text{ because } E_\phi \text{ is an upper bound}$$

c)

Since $\int_0^\infty c^2e^{-2z} = 1 \longrightarrow c^2 = 2, \therefore c = \sqrt{2}$

$$\langle z|z\rangle \approx \int_0^\infty zc^2e^{-2z}dz \longrightarrow \langle z|z\rangle \approx \frac{1}{2}$$

12.7

We can use the variational method to solve this problem, which minimizes the energy expectation value with respect to the parameter b in order to approach the ground state energy

$$
\begin{aligned}
\langle E|E\rangle &= \langle\Psi_0|\,\hat{H}_0\,|\Psi_0\rangle \\
&= \left(\left(\frac{b}{\pi}\right)^{1/4} e^{-bx^2/2}\right)(a|\hat{x}| + \frac{\hat{p}_x^2}{2m})\left(\left(\frac{b}{\pi}\right)^{1/4} e^{-bx^2/2}\right) \\
&= \int_{-\infty}^{\infty} a|x|\left(\frac{b}{\pi}\right)^{1/2} e^{-bx^2}dx \\
&+ \int_{-\infty}^{\infty}\left(\left(\frac{b}{\pi}\right)^{1/4} e^{-bx^2/2}\right)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\left(\left(\frac{b}{\pi}\right)^{1/4} e^{-bx^2/2}\right)dx \\
&= \int_{-\infty}^{\infty} a|x|\left(\frac{b}{\pi}\right)^{1/2} e^{-bx^2}dx + \int_{-\infty}^{\infty}\left(-\frac{\hbar^2}{2m}\right)\left(\frac{b}{\pi}\right)^{1/2} e^{-bx^2}\left(bx^2-1\right)dx \\
&= \frac{a}{(\pi b)^{1/2}} + \frac{b\hbar^2}{4m}
\end{aligned}
$$

Taking the derivative of this result

$$\frac{d\langle E\rangle}{db} = \frac{\hbar}{4m} - \frac{a}{1\pi^{1/2}b^{3/2}} = 0$$

produces roots at

$$b = \left(\frac{4a^2m^2}{\pi\hbar^2}\right)^{1/3}$$

Thus the approximate ground state energy is

$$E = \frac{a}{\pi^{1/2}}\left(\frac{\pi\hbar^2}{4a^2m^2}\right)^{1/4} + \frac{\hbar^2}{4m}\left(\frac{4a^2m^2}{\pi\hbar^2}\right)^{1/2}$$

12.8

The exact solution gives $E_n = \hbar\omega(n+\frac{3}{2})$, $E_0 = \frac{3\hbar\omega}{2}$.

Using the trial wave function:

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + V(r)\right]u(r) = Eu(r) \tag{9.98}$$

$$
\begin{aligned}
\frac{u(r)}{r} &= R(r) = Ne^{-br} \\
u(r) &= Nre^{-br} \\
E &= \left\langle\Psi|\hat{H}|\Psi\middle|\Psi|\hat{H}|\Psi\right\rangle
\end{aligned}
$$

Start by normalizing:

$$\langle\Psi|\Psi|\Psi|\Psi\rangle = 1 = \int_0^\infty N^2 r^2 e^{-2br}\, dr$$
$$N = 2b^{\frac{3}{2}}$$

Then solve $\left\langle \Psi|\hat{H}|\Psi \middle| \Psi|\hat{H}|\Psi \right\rangle$, recall we are in ground state, so $l = 0$

$$\begin{aligned}
\left\langle \Psi|\hat{H}|\Psi \middle| \Psi|\hat{H}|\Psi \right\rangle &= (4b^3) \int_0^\infty re^{-br} \left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{1}{2}\mu\omega^2 r^2 \right] re^{-br}\, dr \\
&= (4b^3) \int_0^\infty -\frac{\hbar^2 b(br-2)re^{-2br}}{2\mu} + \frac{1}{2}\mu\omega^2 r^4 e^{-2br}\, dr \\
&= (4b^3)\frac{3\mu^2\omega^2 + b^4\hbar^2}{8b^5\mu}
\end{aligned}$$

$$\begin{aligned}
E &= \left\langle \Psi|\hat{H}|\Psi \middle| \Psi|\hat{H}|\Psi \right\rangle = \frac{3\mu^2\omega^2 + b^4\hbar^2}{2b^2\mu} = \frac{3\mu\omega^2}{2b^2} + \frac{\hbar^2 b^2}{2\mu} \\
\frac{dE}{db} &= \frac{\hbar^2 b}{\mu} - \frac{3\mu\omega^2}{b^3} = 0 \\
b &= (\frac{3\mu^2\omega^2}{\hbar^2})^{\frac{1}{4}} \\
E &= \frac{\hbar^2}{2\mu}\sqrt{\frac{3\mu^2\omega^2}{\hbar^2}} + \frac{3\mu\omega^2}{2}\sqrt{\frac{\hbar^2}{3\mu^2\omega^2}} = \sqrt{3}\hbar\omega
\end{aligned}$$

The variational method provides an upper bound on the energy, $\sqrt{3}\hbar\omega > \frac{3}{2}\hbar\omega$

12.9

12.10