A Modern Approach to Quantum Mechanics by Townsend - Solutions

Solutions by: GT SPS

Contributors: Charles Cardot, Cody McCord, Robin Glefke, Kiernan Reising, Kevin Jin, Alexander George-Kennedy, William Benston

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5 A System of Two Spin-1/2 Particles

5.1

$$|1\rangle = |{+z},{+z}\rangle\ ,\ |2\rangle = |{+z},{-z}\rangle\ ,\ |3\rangle = |{-z},{+z}\rangle\ ,\ |4\rangle = |{-z},{-z}\rangle$$

$$\begin{aligned}
\left\langle 1|\omega_0\hat{S}_{1z}|1\middle|1|\omega_0\hat{S}_{1z}|1\right\rangle &= \frac{\omega_0\hbar}{2},\\
\left\langle 2|\omega_0\hat{S}_{1z}|2\middle|2|\omega_0\hat{S}_{1z}|2\right\rangle &= \frac{\omega_0\hbar}{2},\\
\left\langle 3|\omega_0\hat{S}_{1z}|3\middle|3|\omega_0\hat{S}_{1z}|3\right\rangle &= -\frac{\omega_0\hbar}{2},\\
\left\langle 4|\omega_0\hat{S}_{1z}|4\middle|4|\omega_0\hat{S}_{1z}|4\right\rangle &= -\frac{\omega_0\hbar}{2},
\end{aligned}$$

The Hamiltonian in this problem can be split into a spin-spin component and an external magnetic field component. The matrix representation of the spin-spin Hamiltonian is given in equation (5.14) on pg. 145. The matrix representation of the external magnetic field Hamiltonian is diagonal. The total Hamiltonian $\hat{\boldsymbol{H}}_{total} = \hat{\boldsymbol{H}}_{spin-spin} + \hat{\boldsymbol{H}}_{magnetic}$ is given by

$$\hat{\boldsymbol{H}}_{total} \rightarrow \begin{bmatrix} \frac{A+\omega_0\hbar}{2} & 0 & 0 & 0 \\ 0 & \frac{\omega_0\hbar - A}{2} & 0 & 0 \\ 0 & 0 & \frac{-\omega_0\hbar - A}{2} & 0 \\ 0 & 0 & 0 & \frac{A-\omega_0\hbar}{2} \end{bmatrix}$$

Solve $\hat{\boldsymbol{H}}_{total}$ for the eigenenergies

$$\begin{aligned}
E &= \frac{A}{2} \pm \frac{\omega_0\hbar}{2}\\
E &= \frac{-A \pm \sqrt{4A^2 + \omega_0^2\hbar^2}}{2}
\end{aligned}$$

Use binomial expansion $(1+x)^n \approx 1 + nx$ for $x \ll 1$

$A \gg \hbar\omega$ limiting case:

$$E = \frac{A}{2} \pm \frac{\omega_0\hbar}{2} \rightarrow E \approx \frac{A}{2}$$

$$E = \frac{-A \pm \sqrt{4A^2 + \omega_0^2\hbar^2}}{2} \rightarrow E = -\frac{A}{2} \pm A(1 + (\frac{\omega_0\hbar}{2A})^2)^{\frac{1}{2}} \approx -\frac{A}{2} \pm (A + \frac{1}{2}\frac{\omega_0^2\hbar^2}{4A}) \approx -\frac{A}{2} \pm A$$

$A \ll \hbar\omega$ limiting case:

$$E = \frac{A}{2} \pm \frac{\omega_0\hbar}{2} \to E \approx \pm\frac{\omega_0\hbar}{2}$$
$$E = \frac{-A \pm \sqrt{4A^2 + \omega_0^2\hbar^2}}{2} \to E = -\frac{A}{2} \pm \frac{\omega_0\hbar}{2}(1 + (\frac{2A}{\omega_0\hbar})^2)^{\frac{1}{2}} \approx -\frac{A}{2} \pm (\frac{\omega_0\hbar}{2} + \frac{A^2}{\omega_0\hbar}) \approx -\frac{A}{2} \pm \frac{\omega_0\hbar}{2}$$

5.2

5.3

$$\left|+n\right\rangle = \cos\frac{\theta}{2}\left|+z\right\rangle + e^{i\phi}\sin\frac{\theta}{2}\left|-z\right\rangle$$
$$\left|-n\right\rangle = \sin\frac{\theta}{2}\left|+z\right\rangle - e^{i\phi}\cos\frac{\theta}{2}\left|-z\right\rangle$$

$$\left|0, 0\right\rangle = \frac{1}{\sqrt{2}}(\left|+z, -z\right\rangle + \left|-z, +z\right\rangle) \tag{5.31}$$

$$\left|+z\right\rangle = \left|+n\right\rangle \left\langle +n\right| + z|+n| + z\rangle + \left|-n\right\rangle \left\langle -n\right| + z|-n| + z\rangle = \cos\frac{\theta}{2}\left|+n\right\rangle + \sin\frac{\theta}{2}\left|-n\right\rangle$$
$$\left|-z\right\rangle = \left|+n\right\rangle \left\langle +n\right| - z|+n| - z\rangle + \left|-n\right\rangle \left\langle -n\right| - z|-n| - z\rangle = e^{-i\phi}\cos\frac{\theta}{2}\left|+n\right\rangle - e^{-i\phi}\sin\frac{\theta}{2}\left|-n\right\rangle$$

$$\left|+z, -z\right\rangle = \left|+z\right\rangle \otimes \left|-z\right\rangle$$
$$\left|-z, +z\right\rangle = \left|-z\right\rangle \otimes \left|+z\right\rangle$$

Solve for $\left|+z, -z\right\rangle$ and $\left|-z, +z\right\rangle$. Then substitute them back into equation (5.31) to find

$$\left|0, 0\right\rangle = \frac{1}{\sqrt{2}}(-e^{-i\phi}\left|+n, -n\right\rangle + e^{-i\phi}\left|-n, +n\right\rangle)$$

5.4

5.5

5.6

5.7

$$
\begin{aligned}
\left|\frac{3}{2},\frac{3}{2}\right\rangle &= |{+z},{+z},{+z}\rangle \\
\hat{S}_-\left|\frac{3}{2},\frac{3}{2}\right\rangle &= (\hat{S}_{1-}+\hat{S}_{2-}+\hat{S}_{3-})\,|{+z},{+z},{+z}\rangle \\
\hbar\sqrt{3}\left|\frac{3}{2},\frac{1}{2}\right\rangle &= \hbar(|{-z},{+z},{+z}\rangle + |{+z},{-z},{+z}\rangle + |{+z},{+z},{-z}\rangle) \\
\left|\frac{3}{2},\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{3}}(|{-z},{+z},{+z}\rangle + |{+z},{-z},{+z}\rangle + |{+z},{+z},{-z}\rangle) \\
\hat{S}_-\left|\frac{3}{2},\frac{1}{2}\right\rangle &= (\hat{S}_{1-}+\hat{S}_{2-}+\hat{S}_{3-})\left|\frac{3}{2},\frac{1}{2}\right\rangle \\
2\hbar\left|\frac{3}{2},-\frac{1}{2}\right\rangle &= \frac{\hbar}{\sqrt{3}}(|{-z},{-z},{+z}\rangle + |{-z},{+z},{-z}\rangle + |{-z},{-z},{+z}\rangle + \\
&\quad |{+z},{-z},{-z}\rangle + |{-z},{+z},{-z}\rangle + |{+z},{-z},{-z}\rangle) \\
\left|\frac{3}{2},-\frac{1}{2}\right\rangle &= \frac{1}{\sqrt{3}}(|{-z},{-z},{+z}\rangle + |{-z},{+z},{-z}\rangle + |{-z},{-z},{+z}\rangle \\
\hat{S}_-\left|\frac{3}{2},-\frac{1}{2}\right\rangle &= (\hat{S}_{1-}+\hat{S}_{2-}+\hat{S}_{3-})\left|\frac{3}{2},-\frac{1}{2}\right\rangle \\
\hbar\sqrt{3}\left|\frac{3}{2},-\frac{3}{2}\right\rangle &= \frac{\hbar}{\sqrt{3}}(|{-z},{-z},{-z}\rangle + |{-z},{-z},{-z}\rangle + |{-z},{-z},{-z}\rangle) \\
\left|\frac{3}{2},-\frac{3}{2}\right\rangle &= |{-z},{-z},{-z}\rangle
\end{aligned}
$$

5.8

$$
\begin{aligned}
|{+n}\rangle &= \frac{1}{\sqrt{2}}(|{+z}\rangle + e^{i\phi}\,|{-z}\rangle) \\
|{-n}\rangle &= \frac{1}{\sqrt{2}}(|{+z}\rangle - e^{i\phi}\,|{-z}\rangle) \\
|0,0\rangle &= \frac{1}{\sqrt{2}}(|{+z},{-z}\rangle - |{-z},{+z}\rangle)
\end{aligned}
$$

To find $P_{++}(\boldsymbol{a},\boldsymbol{b})$, $P_{--}(\boldsymbol{a},\boldsymbol{b})$, $P_{+-}(\boldsymbol{a},\boldsymbol{b})$, and $P_{-+}(\boldsymbol{a},\boldsymbol{b})$ we must find the overlap between $\langle +n_a,+n_b|0,0|{+n_a},{+n_b}|0,0\rangle$, $\langle -n_a,-n_b|0,0|{-n_a},{-n_b}|0,0\rangle$, $\langle +n_a,-n_b|0,0|{+n_a},{-n_b}|0,0\rangle$, and $\langle -n_a, +n_b|0,0|-n_a, +n_b|0,0\rangle$ respectively. This is algebraically tedious, but can be expedited slightly by first looking at the general form of $|\pm n_a, \pm n_b\rangle$

$$|\pm n_a, \pm n_b\rangle = \frac{1}{2}\Bigg(|+z,+z\rangle + (\pm_b)e^{i\phi_a}|+z,-z\rangle + (\pm_a)e^{i\phi_b}|-z,+z\rangle + (\pm_a \cdot \pm_b)e^{i(\phi_a+\phi_b)}|-z,-z\rangle \Bigg)$$

and noticing that $\langle +n_a, +n_b|0,0|+n_a, +n_b|0,0\rangle = -\langle -n_a, -n_b|0,0|-n_a, -n_b|0,0\rangle$ and $\langle +n_a, -n_b|0,0|+n_a, -n_b|0,0\rangle = -\langle -n_a, +n_b|0,0|-n_a, +n_b|0,0\rangle$. Therefore $P_{++}(\boldsymbol{a},\boldsymbol{b}) = P_{--}(\boldsymbol{a},\boldsymbol{b})$ and $P_{+-}(\boldsymbol{a},\boldsymbol{b}) = P_{-+}(\boldsymbol{a},\boldsymbol{b})$, meaning we only actually have to do 2 out of the 4 calculations.

$$\begin{aligned}
\langle +n_a, +n_b|0,0|+n_a, +n_b|0,0\rangle &= \frac{1}{2}\Bigg(\langle +z,+z| + e^{i\phi_a}\langle +z,-z| + e^{i\phi_b}\langle -z,+z| \\
&\quad + e^{i(\phi_a-\phi_b)}\langle -z,-z| \Bigg) \frac{1}{\sqrt{2}}(|+z,-z\rangle - |-z,+z\rangle) \\
&= \frac{1}{2}\left(\frac{e^{-i\phi_b}}{\sqrt{2}} - \frac{e^{-i\phi_a}}{\sqrt{2}} \right) \\
P_{++}(\boldsymbol{a},\boldsymbol{b}) = |\langle +n_a, +n_b|0,0|+n_a, +n_b|0,0\rangle|^2 &= \frac{1}{4}(1-\cos(\phi_a-\phi_b)) \\
\langle +n_a, -n_b|0,0|+n_a, -n_b|0,0\rangle &= \frac{1}{2}\Bigg(\langle +z,+z| - e^{i\phi_a}\langle +z,-z| + e^{i\phi_b}\langle -z,+z| \\
&\quad - e^{i(\phi_a-\phi_b)}\langle -z,-z| \Bigg) \frac{1}{\sqrt{2}}(|+z,-z\rangle - |-z,+z\rangle) \\
&= \frac{1}{2}\left(-\frac{e^{-i\phi_b}}{\sqrt{2}} - \frac{e^{-i\phi_a}}{\sqrt{2}} \right) \\
P_{+-}(\boldsymbol{a},\boldsymbol{b}) = |\langle +n_a, -n_b|0,0|+n_a, -n_b|0,0\rangle|^2 &= \frac{1}{4}(1+\cos(\phi_a-\phi_b))
\end{aligned}$$

If you are having trouble following the algebra in the above steps, don't forget that $e^{i\phi} = \cos(\phi) + i\sin(\phi)$, $\sin(\phi) = -\sin(-\phi)$, and $\cos(\phi) = cos(-\phi)$. Now we see that

$$\begin{aligned}
E(\boldsymbol{a},\boldsymbol{b}) &= P_{++}(\boldsymbol{a},\boldsymbol{b}) + P_{--}(\boldsymbol{a},\boldsymbol{b}) - P_{+-}(\boldsymbol{a},\boldsymbol{b}) - P_{-+}(\boldsymbol{a},\boldsymbol{b}) \\
&= \frac{1}{4}(1-\cos(\phi_a-\phi_b)) + \frac{1}{4}(1-\cos(\phi_a-\phi_b)) \\
&- \frac{1}{4}(1+\cos(\phi_a-\phi_b)) - \frac{1}{4}(1+\cos(\phi_a-\phi_b)) \\
&= -\cos(\phi_a-\phi_b) = -\cos(\theta_{ab})
\end{aligned}$$

5.9

$$\hat{S}_n = \frac{\hbar}{2}\begin{bmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{bmatrix}$$

$$\hat{S}_{1a} = \frac{\hbar}{2}\begin{bmatrix} 0 & e^{-i\phi_a} \\ e^{i\phi_a} & 0 \end{bmatrix}, \text{and}; \hat{S}_{2b} = \frac{\hbar}{2}\begin{bmatrix} 0 & e^{-i\phi_b} \\ e^{i\phi_b} & 0 \end{bmatrix}$$

$$\hat{S}_n \left|+z\right\rangle = \frac{\hbar}{2}e^{i\phi}\left|-z\right\rangle$$
$$\hat{S}_n \left|-z\right\rangle = \frac{\hbar}{2}e^{-i\phi}\left|+z\right\rangle$$

$$\frac{1}{\sqrt{2}}(\langle +z,-z| - \langle -z,+z|)\hat{S}_{1a}\hat{S}_{2b}\frac{1}{\sqrt{2}}(|+z,-z\rangle - |-z,+z\rangle) =$$
$$\frac{1}{\sqrt{2}}(\langle +z,-z| - \langle -z,+z|)\frac{\hbar^2}{4}\frac{1}{\sqrt{2}}(e^{i(\phi_a-\phi_b)}\left|-z,+z\right\rangle - e^{i(\phi_b-\phi_a)}\left|+z,-z\right\rangle) =$$
$$\frac{\hbar^2}{4}\left(-\frac{e^{\phi_a-\phi_b}}{2} - \frac{e^{\phi_b-\phi_a}}{2}\right) = -\frac{\hbar^2}{4}\left(\frac{cos(\phi_a-\phi_b)}{2} + \frac{cos(\phi_b-\phi_a)}{2}\right) = -\frac{\hbar^2}{4}\cos(\theta_{ab})$$

5.10

5.11

5.12

5.13

5.14

5.15

5.16 $|\Psi\rangle = a\,|{+z}\rangle + b\,|{-z}\rangle$.

$$|{+x}\rangle = \frac{1}{\sqrt{2}}\,|{+z}\rangle + \frac{1}{\sqrt{2}}\,|{-z}\rangle \qquad |{-x}\rangle = \frac{1}{\sqrt{2}}\,|{+z}\rangle - \frac{1}{\sqrt{2}}\,|{-z}\rangle$$

a)

$$\hat{\rho} = |\Psi\rangle\,\langle\Psi| = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$$

b)

$$\hat{\rho} = \sum_{i,j} p_{ij}\,|i\rangle\,\langle j| \text{ where } i,\ j = +z,\ -z$$
$$|i\rangle = \sum_{a} |a\rangle\,\langle a|i|a|i\rangle$$
$$\hat{\rho} = \sum_{i,j} p_{ij}\,|i\rangle\,\langle j| = \sum_{i,j}\sum_{a,b} p_{ij}\,|a\rangle\,\langle a|i|a|i\rangle\,\langle j|b|j|b\rangle\,\langle b| \text{ where } a,\ b = +x,\ -z$$
$$\hat{\rho} = \frac{1}{2}\begin{bmatrix} (a+b)^2 & (a^2-b^2) \\ (a^2-b^2) & (a-b)^2 \end{bmatrix}$$

c) Note that the probability that a measurement of S_x yields $\hbar/2$ for the state $|\Psi\rangle$ is equal to $|\,\langle{+x}|\Psi|{+x}|\Psi\rangle\,|^2$. Look at the discussion on page 172 of the Townsend textbook for more information.

$$\hat{P}_{|\Phi\rangle} = |\Phi\rangle\,\langle\Phi| \tag{5.76}$$
$$tr(\hat{P}_{|\Phi\rangle}\hat{\rho}) = |\,\langle\Phi|\Psi|\Phi|\Psi\rangle\,|^2 \tag{5.77}$$
$$\hat{P}_{+x} = |{+x}\rangle\,\langle{+x}|$$
$$tr(\hat{P}_{+x}\hat{\rho}) = \frac{1}{2}(a+b)^2$$

5.17

$$\hat{\rho} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$\langle S_x|S_x\rangle = Tr(\hat{S}_x\hat{\rho}) = Tr\left(\frac{\hbar}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = -\frac{\hbar}{2}$$

Note that $Tr(\hat{S}_x\hat{\rho} = Tr(\hat{\rho}\hat{S}_x)$, see problem 5.27. This is the density operator for a pure state. This can be trivially proven by calculating $Tr(\hat{\rho}^2)$ (see page 172 equation (5.75) and page 174 equation (5.85) of the Townsend textbook). For the second justification we can use the definition of a pure state density matrix given in equation (5.70), where we see that for a pure state $\hat{\rho} = |\Psi\rangle\langle\Psi|$. We simply need to find some normalized pure state $|\Psi\rangle$ that allows us to recreate the density operator given in this problem. Indeed we see that this is satisfied by

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|+z\rangle - |-z\rangle)$$
$$|\Psi\rangle\langle\Psi| = \frac{1}{2}(|+z\rangle\langle+z| + |-z\rangle\langle-z| - |-z\rangle\langle+z| - |+z\rangle\langle-z|) = \hat{\rho}$$

5.18

$$\hat{\boldsymbol{\rho}} = \begin{bmatrix} \langle+z|\hat{\rho}| + z|+z|\hat{\rho}| + z\rangle & \langle+z|\hat{\rho}| - z|+z|\hat{\rho}| - z\rangle \\ \langle-z|\hat{\rho}| + z|-z|\hat{\rho}| + z\rangle & \langle-z|\hat{\rho}| - z|-z|\hat{\rho}| - z\rangle \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$\langle S_x|S_x\rangle = Tr(\hat{S}_x\hat{\boldsymbol{\rho}}),\ \langle S_y|S_y\rangle = Tr(\hat{S}_y\hat{\boldsymbol{\rho}}),\ \langle S_z|S_z\rangle = Tr(\hat{S}_z\hat{\boldsymbol{\rho}})$$

$$\langle S_x|S_x\rangle = \mathrm{Tr}\left(\frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\right) = 0$$
$$\langle S_y|S_y\rangle = \mathrm{Tr}\left(\frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\right) = 0$$
$$\langle S_z|S_z\rangle = \mathrm{Tr}\left(\frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\right) = \frac{\hbar}{4}$$

5.19

$$|+n\rangle = cos(\frac{\theta}{2})\,|+z\rangle + e^{i\phi}sin(\frac{\theta}{2})\,|-z\rangle$$
$$|-n\rangle = cos(\frac{\theta}{2})\,|+z\rangle - e^{i\phi}sin(\frac{\theta}{2})\,|-z\rangle$$

$$
\begin{aligned}
\left|+n\right\rangle\left\langle+n\right| &= cos(\frac{\theta}{2})^2\left|+z\right\rangle\left\langle+z\right| + e^{-i\phi}cos(\frac{\theta}{2})sin(\frac{\theta}{2})\left|+z\right\rangle\left\langle-z\right| \\
&\quad + e^{i\phi}cos(\frac{\theta}{2})sin(\frac{\theta}{2})\left|-z\right\rangle\left\langle+z\right| + sin(\frac{\theta}{2})^2\left|-z\right\rangle\left\langle-z\right| \\
\left|-n\right\rangle\left\langle-n\right| &= cos(\frac{\theta}{2})^2\left|+z\right\rangle\left\langle+z\right| - e^{-i\phi}cos(\frac{\theta}{2})sin(\frac{\theta}{2})\left|+z\right\rangle\left\langle-z\right| \\
&\quad - e^{i\phi}cos(\frac{\theta}{2})sin(\frac{\theta}{2})\left|-z\right\rangle\left\langle+z\right| + sin(\frac{\theta}{2})^2\left|-z\right\rangle\left\langle-z\right| \\
\hat{\rho} = \frac{1}{2}\left|+n\right\rangle\left\langle+n\right| &+ \frac{1}{2}\left|-n\right\rangle\left\langle-n\right| = \frac{1}{2}\left|+z\right\rangle\left\langle+z\right| + \frac{1}{2}\left|-z\right\rangle\left\langle-z\right|
\end{aligned}
$$

5.20

5.21 The density operator from Problem 5.18 is

$$
\hat{\rho} = \frac{3}{4}\left|+z\right\rangle\left\langle+Z\right| + \frac{1}{4}\left|-z\right\rangle\left\langle-Z\right|
$$

$$
\begin{aligned}
\left|\Psi_1\right\rangle &= \frac{\sqrt{3}}{2}\left|+z\right\rangle + \frac{1}{2}\left|-z\right\rangle \\
\left|\Psi_2\right\rangle &= \frac{\sqrt{3}}{2}\left|+z\right\rangle - \frac{1}{2}\left|-z\right\rangle \\
\left|\Psi_1\right\rangle\left\langle\Psi_1\right| &= \frac{3}{4}\left|+z\right\rangle\left\langle+z\right| + \frac{\sqrt{3}}{4}\left|+z\right\rangle\left\langle-z\right| + \frac{\sqrt{3}}{4}\left|-z\right\rangle\left\langle+z\right| + \frac{1}{4}\left|-z\right\rangle\left\langle-z\right| \\
\left|\Psi_2\right\rangle\left\langle\Psi_2\right| &= \frac{3}{4}\left|+z\right\rangle\left\langle+z\right| - \frac{\sqrt{3}}{4}\left|+z\right\rangle\left\langle-z\right| - \frac{\sqrt{3}}{4}\left|-z\right\rangle\left\langle+z\right| + \frac{1}{4}\left|-z\right\rangle\left\langle-z\right| \\
\hat{\rho} = \frac{1}{2}\big(\left|\Psi_1\right\rangle\left\langle\Psi_1\right| &+ \left|\Psi_2\left\langle\Psi_2\right|\right\rangle\big) = \frac{3}{4}\left|+z\right\rangle\left\langle+Z\right| + \frac{1}{4}\left|-z\right\rangle\left\langle-Z\right|
\end{aligned}
$$

5.22

$$
\hat{\boldsymbol{H}} = -\hat{\boldsymbol{\mu}}\cdot\vec{\boldsymbol{B}} = -\mu_z B \quad \text{with} \quad \hat{\mu}_z = -\frac{ge}{2mc}\hat{S}_z
$$

$$\hat{\boldsymbol{H}}\,|1\rangle = \frac{ge\hbar}{2mc}\,|1\rangle = \mu B\,|1\rangle$$
$$\hat{\boldsymbol{H}}\,|0\rangle = 0$$
$$\hat{\boldsymbol{H}}\,|{-1}\rangle = -\frac{ge\hbar}{2mc}\,|{-1}\rangle = -\mu B\,|{-1}\rangle$$

$$\hat{\rho} \xrightarrow[In\ S_z\ basis]{} \frac{1}{Z}\begin{bmatrix} e^{-\frac{\mu B}{k_b T}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\frac{\mu B}{k_b T}} \end{bmatrix}, \quad Z = e^{-\frac{\mu B}{k_b T}} + 1 + e^{\frac{\mu B}{k_b T}}$$

$$M = N\,\langle\hat{\boldsymbol{\mu}}_z|\hat{\boldsymbol{\mu}}_z\rangle = N\,Tr(\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\mu}}_z)$$
$$M = -N\left(\frac{ge\hbar}{2Zmc}\right)\,Tr\left(\begin{bmatrix} e^{-\frac{\mu B}{k_b T}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\frac{\mu B}{k_b T}} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right)$$
$$M = \frac{N\mu}{Z}(e^{\frac{\mu B}{k_b T}} - e^{-\frac{\mu B}{k_b T}}),\ \text{for}\ \frac{\mu B}{k_B T} \ll 1,\ e^{\frac{\mu B}{k_b T}} \approx 1 + \frac{\mu B}{k_b T}$$
$$M = N\mu\left(\frac{1+\frac{\mu B}{k_b T} - 1 + \frac{\mu B}{k_b T}}{1+\frac{\mu B}{k_b T} + 1 + 1 - \frac{\mu B}{k_b T}}\right) = N\mu(\frac{2\mu B}{3k_B T} = \frac{2N\mu^2}{3k_B}\cdot\frac{B}{T} = C\frac{B}{T}$$

$C_{spin\ 1} = \frac{2N\mu^2}{3k_B}$ and $C_{spin\ 1/2} = \frac{N\mu^2}{k_B}$, therefore $C_{spin\ 1}\ <\ C_{spin\ 1/2}$ from Example 5.6

5.23

5.24

5.25

$$
\begin{aligned}
Tr(\hat{A}) &= \sum_i \left\langle i|\hat{A}|i \middle| i|\hat{A}|i \right\rangle \\
Tr(\hat{P}_{|\Phi\rangle}\hat{\rho}) &= \sum_i \sum_k \langle i|\Phi|i|\Phi\rangle \left\langle \Phi|\Psi^{(k)} \middle| \Phi|\Psi^{(k)} \right\rangle p_k \left\langle \Psi^{(k)}|i \middle| \Psi^{(k)}|i \right\rangle \\
&= \sum_i \sum_k \left\langle \Psi^{(k)}|i \middle| \Psi^{(k)}|i \right\rangle p_k \langle i|\Phi|i|\Phi\rangle \left\langle \Phi|\Psi^{(k)} \middle| \Phi|\Psi^{(k)} \right\rangle \\
&= \sum_k p_k \left\langle \Psi^{(k)}|\Phi \middle| \Psi^{(k)}|\Phi \right\rangle \left\langle \Phi|\Psi^{(k)} \middle| \Phi|\Psi^{(k)} \right\rangle \\
&= \sum_k p_k | \left\langle \Phi|\Psi^{(k)} \middle| \Phi|\Psi^{(k)} \right\rangle |^2
\end{aligned}
$$

For a mixed state, p_k is the probability that the particle is in state $\left|\Psi^{(k)}\right\rangle$. To figure out the probability that a measurement on the particle yields the state $|\Phi\rangle$ we must measure the overlap between $|\Phi\rangle$ and each state $\left|\Psi^{(k)}\right\rangle$ that the particle might be in. Given that the particle is not guaranteed to be in state $\left|\Psi^{(k)}\right\rangle$, we must normalize each of these overlap measurements by the probability, p_k, that the particle ***is*** in the state $\left|\Psi^{(k)}\right\rangle$, which is exactly what the last line of math above this text describes.

5.26

5.27

$$
\hat{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \dots \\ b_{2,1} & b_{2,2} & b_{2,3} & \dots \\ b_{3,1} & b_{3,2} & b_{3,3} & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}
$$

$$
\begin{aligned}
Tr(\hat{A}\hat{B}) &= \sum_i \langle i|\, \hat{A}\hat{B} \,|i\rangle \\
&= \sum_i \sum_j a_{i,j} b_{i,j} \\
&= \sum_i \sum_j b_{i,j} a_{i,j} \\
&= \sum_j \langle j|\, \hat{B}\hat{A} \,|j\rangle \\
&= Tr(\hat{B}\hat{A})
\end{aligned}
$$

5.28

5.29

a)

$$\hat{\rho}(0) = \sum_{i,j} \rho_{i,j} \left|i(0)\right\rangle \left\langle j(0)\right|$$

$$\hat{U}(t)\hat{\rho}(0)\hat{U}^{\dagger}(t) = \sum_{i,j} \rho_{i,j}\hat{U}(t) \left|i(0)\right\rangle \left\langle j(0)\right| \hat{U}^{\dagger} = \sum_{i,j} \rho_{i,j} \left|i(t)\right\rangle \left\langle j(t)\right| = \hat{\rho}(t)$$

b) An ensemble of particles in a pure state is given by $\hat{\rho}(0) = \left|\Psi(0)\right\rangle \left\langle \Psi(0)\right|$. If an ensemble CAN evolve into a mixed state then the the density matrix will become a sum of more than one outer product of states
(ex: $\hat{\rho} = a_{1,2} \left|\Psi_1\right\rangle \left\langle \Psi_2\right| + a_{3,4} \left|\Psi_3\right\rangle \left\langle \Psi_4\right| + \ ...$). Time evolution is governed by the Schrodinger equation $i\hbar\frac{d}{dt}\hat{\boldsymbol{U}} = \hat{\boldsymbol{H}}\hat{\boldsymbol{U}} \Rightarrow i\hbar\frac{d}{dt}\left|\Psi(t)\right\rangle = \hat{\boldsymbol{H}}\left|\Psi(t)\right\rangle$. If we time evolve the density matrix of a pure state we see that $\hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^{\dagger}(t) = \left|\Psi(t)\right\rangle \left\langle \Psi(t)\right|$ as governed by the Schrodinger equation. This means that $\hat{\rho}(t)$ is still representing a pure state and does NOT take the form of a mixed state $\hat{\rho} = a_{1,2} \left|\Psi_1\right\rangle \left\langle \Psi_2\right| + a_{3,4} \left|\Psi_3\right\rangle \left\langle \Psi_4\right| + \ ....$ This implies that time evolution which is governed by the Schrodinger equation cannot evolve an ensemble of particles which begins in a pure state into a mixed state.