Solutions - Chapter 5

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Problem 5.1

$$\hat{H} = \frac{2A}{\hbar^2}\hat{S}_1 \cdot \hat{S}_2 \underset{1,2,3,4}{\longrightarrow} \begin{pmatrix} A/2 & 0 & 0 & 0 \\ 0 & -A/2 & A & 0 \\ 0 & A & -A/2 & 0 \\ 0 & 0 & 0 & A/2 \end{pmatrix}$$

$$\langle 1|\omega_0 \hat{S}_{1z}|1\rangle = \frac{\hbar\omega_0}{2}$$

$$\langle 2|\omega_0 \hat{S}_{1z}|2\rangle = \frac{\hbar\omega_0}{2}$$

$$\langle 3|\omega_0 \hat{S}_{1z}|3\rangle = \frac{-\hbar\omega_0}{2}$$

$$\langle 4|\omega_0 \hat{S}_{1z}|4\rangle = \frac{-\hbar\omega_0}{2}$$

Off-diagonal terms are 0:

$$\hat{H} = \frac{2A}{\hbar^2}\hat{S}_1 \cdot \hat{S}_2 + \omega_0 \hat{S}_{1z} \underset{1,2,3,4}{\longrightarrow} \begin{pmatrix} \frac{A+\hbar\omega_0}{2} & 0 & 0 & 0 \\ 0 & \frac{-A+\hbar\omega_0}{2} & A & 0 \\ 0 & A & \frac{-A-\hbar\omega_0}{2} & 0 \\ 0 & 0 & 0 & \frac{A-\hbar\omega_0}{2} \end{pmatrix}$$

Energy eigenvalues:

$$\begin{vmatrix} \frac{A+\hbar\omega_0}{2} - E & 0 & 0 & 0 \\ 0 & \frac{-A+\hbar\omega_0}{2} - E & A & 0 \\ 0 & A & \frac{-A-\hbar\omega_0}{2} - E & 0 \\ 0 & 0 & 0 & \frac{A-\hbar\omega_0}{2} - E \end{vmatrix} = 0$$

$$\left(\frac{A+\hbar\omega_0}{2} - E\right)\left[\left(\frac{-A+\hbar\omega_0}{2} - E\right)\left(\frac{-A-\hbar\omega_0}{2} - E\right)\left(\frac{A-\hbar\omega_0}{2} - E\right) - A^2\left(\frac{A-\hbar\omega_0}{2} - E\right)\right] = 0$$

$$E_e = \frac{A+\hbar\omega_0}{2}$$

$$\left(\frac{-A+\hbar\omega_0}{2} - E\right)\left(\frac{-A-\hbar\omega_0}{2} - E\right)\left(\frac{A-\hbar\omega_0}{2} - E\right) - A^2\left(\frac{A-\hbar\omega_0}{2} - E\right) = 0$$

$$E_e = \frac{A - \hbar\omega_0}{2}$$

$$\left(\frac{-A + \hbar\omega_0}{2} - E\right)\left(\frac{-A - \hbar\omega_0}{2} - E\right) - A^2 = 0$$

$$\left(E + \frac{A}{2} + \frac{\hbar\omega_0}{2}\right)\left(E + \frac{A}{2} - \frac{\hbar\omega_0}{2}\right) = A^2$$

$$\left(E + \frac{A}{2}\right)^2 - \left(\frac{\hbar\omega_0}{2}\right)^2 = A^2$$

$$E_e = -\frac{A}{2} \pm \sqrt{A^2 + \left(\frac{\hbar\omega_0}{2}\right)^2}$$

Energy eigenvalues:

$$E_e = \frac{A \pm \hbar\omega_0}{2}, -\frac{A}{2} \pm \sqrt{A^2 + \left(\frac{\hbar\omega_0}{2}\right)^2}$$

Approximation ($x \ll 1$): $(1+x)^n \approx 1 + nx$

Limiting Case: $A \gg \hbar\omega_0$

$$E_e = \frac{A \pm \hbar\omega_0}{2}, -\frac{A}{2} \pm \left[A + \frac{(\hbar\omega_0)^2}{8A}\right]$$

Limiting Case: $A \ll \hbar\omega_0$

$$E_e = \frac{\pm\hbar\omega_0 + A}{2}, \frac{\pm\hbar\omega_0 - A}{2}$$

Problem 5.2

$$|1,1\rangle = |+\mathbf{z},+\mathbf{z}\rangle = \left[\frac{1}{\sqrt{2}}(|+\mathbf{x}\rangle_1 + |-\mathbf{x}\rangle_1)\right]\left[\frac{1}{\sqrt{2}}(|+\mathbf{x}\rangle_2 + |-\mathbf{x}\rangle_2)\right]$$

$$= \frac{1}{2}(|+\mathbf{x},+\mathbf{x}\rangle + |-\mathbf{x},+\mathbf{x}\rangle + |+\mathbf{x},-\mathbf{x}\rangle + |-\mathbf{x},-\mathbf{x}\rangle)$$

$$|1,-1\rangle = |-\mathbf{z},-\mathbf{z}\rangle = \left[\frac{1}{\sqrt{2}}(|+\mathbf{x}\rangle_1 - |-\mathbf{x}\rangle_1)\right]\left[\frac{1}{\sqrt{2}}(|+\mathbf{x}\rangle_2 - |-\mathbf{x}\rangle_2)\right]$$

$$= \frac{1}{2}(|+\mathbf{x},+\mathbf{x}\rangle - |-\mathbf{x},+\mathbf{x}\rangle - |+\mathbf{x},-\mathbf{x}\rangle + |-\mathbf{x},-\mathbf{x}\rangle)$$

Problem 5.3

$$|0,0\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z},-\mathbf{z}\rangle - \frac{1}{\sqrt{2}}|-\mathbf{z},+\mathbf{z}\rangle$$

$$|+\mathbf{z}\rangle = \cos\frac{\theta}{2}|+\mathbf{n}\rangle + \sin\frac{\theta}{2}|-\mathbf{n}\rangle$$

$$|-\mathbf{z}\rangle = e^{-i\phi}\sin\frac{\theta}{2}|+\mathbf{n}\rangle - e^{-i\phi}\cos\frac{\theta}{2}|-\mathbf{n}\rangle$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2}|+\mathbf{n}\rangle + \sin\frac{\theta}{2}|-\mathbf{n}\rangle\right)\left(e^{-i\phi}\sin\frac{\theta}{2}|+\mathbf{n}\rangle - e^{-i\phi}\cos\frac{\theta}{2}|-\mathbf{n}\rangle\right)$$
$$-\frac{1}{\sqrt{2}}\left(e^{-i\phi}\sin\frac{\theta}{2}|+\mathbf{n}\rangle - e^{-i\phi}\cos\frac{\theta}{2}|-\mathbf{n}\rangle\right)\left(\cos\frac{\theta}{2}|+\mathbf{n}\rangle + \sin\frac{\theta}{2}|-\mathbf{n}\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{e^{-i\phi}}{2}\sin\theta\,|+\mathbf{n},+\mathbf{n}\rangle + e^{-i\phi}\sin^2\frac{\theta}{2}|-\mathbf{n},+\mathbf{n}\rangle - e^{-i\phi}\cos^2\frac{\theta}{2}|+\mathbf{n},-\mathbf{n}\rangle - \frac{e^{-i\phi}}{2}\sin\theta\,|-\mathbf{n},-\mathbf{n}\rangle\right)$$

$$-\frac{1}{\sqrt{2}}\left(\frac{e^{-i\phi}}{2}\sin\theta\,|+\mathbf{n},+\mathbf{n}\rangle - e^{-i\phi}\cos^2\frac{\theta}{2}|-\mathbf{n},+\mathbf{n}\rangle + e^{-i\phi}\sin^2\frac{\theta}{2}|+\mathbf{n},-\mathbf{n}\rangle - \frac{e^{-i\phi}}{2}\sin\theta\,|-\mathbf{n},-\mathbf{n}\rangle\right)$$

$$|0,0\rangle = \frac{e^{-i\phi}}{\sqrt{2}}|-\mathbf{n},+\mathbf{n}\rangle - \frac{e^{-i\phi}}{\sqrt{2}}|+\mathbf{n},-\mathbf{n}\rangle$$

Problem 5.4

$$\langle+\mathbf{n},+\mathbf{z}|0,0\rangle = \frac{1}{\sqrt{2}}(\langle+\mathbf{n},+\mathbf{z}|+\mathbf{z},-\mathbf{z}\rangle - \langle+\mathbf{n},+\mathbf{z}|-\mathbf{z},+\mathbf{z}\rangle) = -\frac{1}{\sqrt{2}}\langle+\mathbf{n}|-\mathbf{z}\rangle$$

$$|\langle+\mathbf{n},+\mathbf{z}|0,0\rangle|^2 = \frac{1}{2}\sin^2\frac{\theta}{2}$$

$$\langle+\mathbf{n},-\mathbf{z}|0,0\rangle = \frac{1}{\sqrt{2}}(\langle+\mathbf{n},-\mathbf{z}|+\mathbf{z},-\mathbf{z}\rangle - \langle+\mathbf{n},-\mathbf{z}|-\mathbf{z},+\mathbf{z}\rangle) = \frac{1}{\sqrt{2}}\langle+\mathbf{n}|+\mathbf{z}\rangle$$

$$|\langle+\mathbf{n},-\mathbf{z}|0,0\rangle|^2 = \frac{1}{2}\cos^2\frac{\theta}{2}$$

$$\langle-\mathbf{n},+\mathbf{z}|0,0\rangle = \frac{1}{\sqrt{2}}(\langle-\mathbf{n},+\mathbf{z}|+\mathbf{z},-\mathbf{z}\rangle - \langle-\mathbf{n},+\mathbf{z}|-\mathbf{z},+\mathbf{z}\rangle) = -\frac{1}{\sqrt{2}}\langle-\mathbf{n}|-\mathbf{z}\rangle$$

$$|\langle-\mathbf{n},+\mathbf{z}|0,0\rangle|^2 = \frac{1}{2}\cos^2\frac{\theta}{2}$$

$$\langle-\mathbf{n},-\mathbf{z}|0,0\rangle = \frac{1}{\sqrt{2}}(\langle-\mathbf{n},-\mathbf{z}|+\mathbf{z},-\mathbf{z}\rangle - \langle-\mathbf{n},-\mathbf{z}|-\mathbf{z},+\mathbf{z}\rangle) = \frac{1}{\sqrt{2}}\langle-\mathbf{n}|+\mathbf{z}\rangle$$

$$|\langle-\mathbf{n},-\mathbf{z}|0,0\rangle|^2 = \frac{1}{2}\sin^2\frac{\theta}{2}$$

$$P(S_{2z} = \hbar/2) = |\langle+\mathbf{n},+\mathbf{z}|0,0\rangle|^2 + |\langle-\mathbf{n},+\mathbf{z}|0,0\rangle|^2 = \frac{1}{2}$$

$$P(S_{2z} = -\hbar/2) = |\langle+\mathbf{n},-\mathbf{z}|0,0\rangle|^2 + |\langle-\mathbf{n},-\mathbf{z}|0,0\rangle|^2 = \frac{1}{2}$$

Problem 5.5

a) Interaction between electron and magnetic field:

$$\hat{H} = \omega_0 \hat{S}_{1z}$$

Interaction between positron and magnetic field (opposite charge):

$$\hat{H} = -\omega_0 \hat{S}_{2z}$$

Neglecting electron-positron interaction, spin Hamiltonian:

$$\hat{H} = \omega_0 (\hat{S}_{1z} - \hat{S}_{2z})$$

b)

$$\hat{H} \underset{1,2,3,4}{\longrightarrow} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \hbar\omega_0 & 0 & 0 \\ 0 & 0 & -\hbar\omega_0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Energy eigenvalues:

$$\begin{vmatrix} -E & 0 & 0 & 0 \\ 0 & \hbar\omega_0 - E & 0 & 0 \\ 0 & 0 & -\hbar\omega_0 - E & 0 \\ 0 & 0 & 0 & -E \end{vmatrix} = 0$$

$$(-E)(\hbar\omega_0 - E)(-\hbar\omega_0 - E)(-E) = 0$$

$$E_e = -\hbar\omega_0, 0, \hbar\omega_0$$

Energy eigenstates:

$E = 0$:

$$\begin{pmatrix} -0 & 0 & 0 & 0 \\ 0 & \hbar\omega_0 - 0 & 0 & 0 \\ 0 & 0 & -\hbar\omega_0 - 0 & 0 \\ 0 & 0 & 0 & -0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$$

$$b = c = 0$$

$$|E = 0\rangle = \frac{1}{\sqrt{2}} |+\mathbf{z}, +\mathbf{z}\rangle \pm \frac{1}{\sqrt{2}} |+\mathbf{z}, -\mathbf{z}\rangle = \frac{1}{\sqrt{2}} (|1, 1\rangle \pm |1, -1\rangle)$$

$E = -\hbar\omega_0$:

$$\begin{pmatrix} \hbar\omega_0 & 0 & 0 & 0 \\ 0 & \hbar\omega_0 + \hbar\omega_0 & 0 & 0 \\ 0 & 0 & -\hbar\omega_0 + \hbar\omega_0 & 0 \\ 0 & 0 & 0 & \hbar\omega_0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$$

$$a = b = d = 0$$

$$|E = -\hbar\omega_0\rangle = |-\mathbf{z}, +\mathbf{z}\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle - |0,0\rangle)$$

$E = \hbar\omega_0$:

$$\begin{pmatrix} -\hbar\omega_0 & 0 & 0 & 0 \\ 0 & \hbar\omega_0 - \hbar\omega_0 & 0 & 0 \\ 0 & 0 & -\hbar\omega_0 - \hbar\omega_0 & 0 \\ 0 & 0 & 0 & -\hbar\omega_0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$$

$$a = c = d = 0$$

$$|E = \hbar\omega_0\rangle = |+\mathbf{z}, -\mathbf{z}\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + |0,0\rangle)$$

$$|\psi(0)\rangle = |0,0\rangle = \frac{1}{\sqrt{2}}|E = \hbar\omega_0\rangle - \frac{1}{\sqrt{2}}|E = -\hbar\omega_0\rangle$$

$$|\psi(t)\rangle = \frac{e^{-i\omega_0 t}}{\sqrt{2}}|E = \hbar\omega_0\rangle - \frac{e^{i\omega_0 t}}{\sqrt{2}}|E = -\hbar\omega_0\rangle$$

$$= \frac{e^{-i\omega_0 t} - e^{i\omega_0 t}}{2}|1,0\rangle + \frac{e^{-i\omega_0 t} + e^{i\omega_0 t}}{2}|0,0\rangle$$

$$|\psi(t)\rangle = \cos(\omega_0 t)|0,0\rangle - i\sin(\omega_0 t)|1,0\rangle$$

Period:

$$T = \frac{2\pi}{\omega_0}$$

c)

$$P(S_{1x} = S_{2x} = \hbar/2) = |\langle +\mathbf{x}, +\mathbf{x}|\psi(t)\rangle|^2 = \frac{\sin^2(\omega_0 t)}{2}$$

$$\langle +\mathbf{x}, +\mathbf{x}|\psi(t)\rangle = \cos(\omega_0 t)\langle +\mathbf{x}, +\mathbf{x}|0,0\rangle - i\sin(\omega_0 t)\langle +\mathbf{x}, +\mathbf{x}|1,0\rangle$$

$$= \cos(\omega_0 t)(0) - i\sin(\omega_0 t)\left(\frac{1}{\sqrt{2}}\right)$$

Problem 5.6

$$\hat{H} = \frac{2A}{\hbar^2}\hat{S}_1 \cdot \hat{S}_2 + \omega_0(\hat{S}_{1z} - \hat{S}_{2z}) \underset{1,2,3,4}{\rightarrow} \begin{pmatrix} \frac{A}{2} & 0 & 0 & 0 \\ 0 & -\frac{A}{2} + \hbar\omega_0 & A & 0 \\ 0 & A & -\frac{A}{2} - \hbar\omega_0 & 0 \\ 0 & 0 & 0 & \frac{A}{2} \end{pmatrix}$$

Energy eigenvalues:

$$\begin{vmatrix} \frac{A}{2}-E & 0 & 0 & 0 \\ 0 & -\frac{A}{2}+\hbar\omega_0-E & A & 0 \\ 0 & A & -\frac{A}{2}-\hbar\omega_0-E & 0 \\ 0 & 0 & 0 & \frac{A}{2}-E \end{vmatrix} = 0$$

$$\left(\frac{A}{2}-E\right)\left[\left(-\frac{A}{2}+\hbar\omega_0-E\right)\left(-\frac{A}{2}-\hbar\omega_0-E\right)\left(\frac{A}{2}-E\right)-A^2\left(\frac{A}{2}-E\right)\right]=0$$

$$E_e = \frac{A}{2}$$

$$\left(E+\frac{A}{2}+\hbar\omega_0\right)\left(E+\frac{A}{2}-\hbar\omega_0\right)-A^2=0$$

$$E = \pm A - \frac{A}{2} - \hbar\omega_0$$

Energy eigenvalues:

$$E_e = -\frac{3A}{2}-\hbar\omega_0, \frac{A}{2}-\hbar\omega_0, \frac{A}{2}$$

Problem 5.7

$$|\psi\rangle = \frac{1}{\sqrt{2}}|R,R\rangle - \frac{1}{\sqrt{2}}|L,L\rangle$$

a)

$$|\langle R,R|\psi\rangle|^2 = \frac{1}{2}$$

$$|\langle L,L|\psi\rangle|^2 = \frac{1}{2}$$

b)

$$|\langle x,y|\psi\rangle|^2 = \frac{1}{2}$$

$$\langle x,y|\psi\rangle = \frac{1}{\sqrt{2}}\langle x,y|R,R\rangle - \frac{1}{\sqrt{2}}\langle x,y|L,L\rangle = \frac{i}{2\sqrt{2}}+\frac{i}{2\sqrt{2}}=\frac{i}{\sqrt{2}}$$

$$|\langle y,x|\psi\rangle|^2 = \frac{1}{2}$$

$$\langle x,y|\psi\rangle = \frac{1}{\sqrt{2}}\langle y,x|R,R\rangle - \frac{1}{\sqrt{2}}\langle y,x|L,L\rangle = \frac{i}{2\sqrt{2}}+\frac{i}{2\sqrt{2}}=\frac{i}{\sqrt{2}}$$

c)

$$\langle x,x|\psi\rangle = \frac{1}{\sqrt{2}}\langle x,x|R,R\rangle - \frac{1}{\sqrt{2}}\langle x,x|L,L\rangle = 0$$

$$\langle y,y|\psi\rangle = \frac{1}{\sqrt{2}}\langle y,y|R,R\rangle - \frac{1}{\sqrt{2}}\langle y,y|L,L\rangle = 0$$

$$|\langle x,x|R,R\rangle|^2 = \frac{1}{4}$$

$$|\langle y,y|R,R\rangle|^2 = \frac{1}{4}$$

$$|\langle x,x|L,L\rangle|^2 = \frac{1}{4}$$

$$|\langle y,y|L,L\rangle|^2 = \frac{1}{4}$$