Solutions - Chapter 6

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Problem 6.1

a) Base case:

$$[\hat{x}, \hat{p}_x] = i\hbar = i\hbar(1)\hat{x}^{1-1}$$

Assume:

$$[\hat{x}^k, \hat{p}_x] = i\hbar k\hat{x}^{k-1}$$

Induction:

$$[\hat{x}^{k+1}, \hat{p}_x] = \hat{x}^{k+1}\hat{p}_x - \hat{p}_x\hat{x}^{k+1} = \hat{x}(\hat{x}^k\hat{p}_x) - (\hat{p}_x\hat{x})\hat{x}^k$$

$$= \hat{x}([\hat{x}^k, \hat{p}_x] + \hat{p}_x\hat{x}^k) + ([\hat{x}, \hat{p}_x] - \hat{x}\hat{p}_x)\hat{x}^k = \hat{x}[\hat{x}^k, \hat{p}_x] + [\hat{x}, \hat{p}_x]\hat{x}^k$$

$$= \hat{x}i\hbar k\hat{x}^{k-1} + i\hbar\hat{x}^k = i\hbar(k+1)\hat{x}^k$$

$$[\hat{x}^n, \hat{p}_x] = i\hbar n\hat{x}^{n-1}$$

b)

$$F(x) = F(a) + F'(a)(x-a) + \frac{F''(a)}{2!}(x-a)^2 + ... = \sum_0^\infty \frac{F^{(n)}(a)}{n!}(x-a)^n$$

$$F'(x) = F'(a) + F''(a)(x-a) + \frac{F'''(a)}{2!}(x-a)^2 + ... = \sum_1^\infty \frac{F^{(n)}(a)}{(n-1)!}(x-a)^{(n-1)}$$

$$\left[\sum_0^\infty \frac{F^{(n)}(0)}{n!}\hat{x}^n, \hat{p}_x\right] = \sum_0^\infty \frac{F^{(n)}(0)}{n!}[\hat{x}^n, \hat{p}_x] = i\hbar\sum_1^\infty \frac{F^{(n)}(0)}{(n-1)!}\hat{x}^{n-1} = i\hbar\frac{\partial F}{\partial x}(\hat{x})$$

$$[F(\hat{x}), \hat{p}_x] = i\hbar\frac{\partial F}{\partial x}(\hat{x})$$

c)

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + V(\hat{x})$$

$$[\hat{H}, \hat{p}_x] = [V(\hat{x}), \hat{p}_x] = i\hbar\frac{dV}{dx}(\hat{x})$$

$$\frac{d\langle p_x\rangle}{dt} = \frac{i}{\hbar}\langle\psi|[\hat{H}, \hat{p}_x]|\psi\rangle = \frac{i}{\hbar}\langle\psi|i\hbar\frac{dV}{dx}(\hat{x})|\psi\rangle = \left\langle -\frac{dV}{dx}\right\rangle$$

Problem 6.2

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}$$

$$i\hbar \frac{\partial}{\partial p} \langle p|x\rangle = x \langle p|x\rangle$$

$$\langle x|\hat{x}|\psi\rangle = \int dx' x' \langle x|x'\rangle \langle x'|\psi\rangle = x \langle x|\psi\rangle$$

$$\langle p|\hat{x}|\psi\rangle = \int dx \langle p|x\rangle \langle x|\hat{x}|\psi\rangle = \int dx\, x \langle p|x\rangle \langle x|\psi\rangle = \int dx\, i\hbar \frac{\partial}{\partial p} \langle p|x\rangle \langle x|\psi\rangle$$

$$\langle p|\hat{x}|\psi\rangle = i\hbar \frac{\partial}{\partial p} \langle p|\psi\rangle$$

$$\langle \varphi|\hat{x}|\psi\rangle = \int dp\ \langle \varphi|p\rangle \langle p|\hat{x}\psi\rangle = \int dp\ \langle p|\varphi\rangle^* i\hbar \frac{\partial}{\partial p} \langle p|\varphi\rangle$$

$$\hat{x} \underset{p-basis}{\rightarrow} i\hbar \frac{\partial}{\partial p}$$

Problem 6.3

$$|\psi\rangle = \int dx\, |x\rangle \langle x|\psi\rangle$$

$$|\psi'\rangle = \hat{T}(\delta x)\, |\psi\rangle = \int dx\, \hat{T}(\delta x)\, |x\rangle \langle x|\psi\rangle = \int dx\ |x + \delta x\rangle \langle x|\psi\rangle$$

$$\psi'(x) = \langle x|\psi'\rangle = \int dx'\ \langle x|x' + \delta x\rangle \langle x'|\psi\rangle = \psi(x - \delta x)$$

$$\langle x\rangle = \langle \psi|x|\psi\rangle = \int dx\, \psi^*(x) x \psi(x)$$

$$\langle x\rangle' = \langle \psi'|x|\psi'\rangle = \int dx\, \psi'^*(x) x \psi'(x) = \int dx\, \psi^*(x - \delta x) x \psi(x - \delta x)$$

$$= \int dx\ \left[\psi^*(x) - \frac{\partial \psi^*}{\partial x}\delta x\right] x \left[\psi(x) - \frac{\partial \psi}{\partial x}\delta x\right]$$

Keep first-order terms and integrate by parts (boundary term goes to zero due to normalization constraint):

$$\langle x\rangle' = \int dx\, \psi^*(x) x \psi(x) - \delta x \int dx\, x \left(\psi \frac{\partial \psi^*}{\partial x} + \frac{\partial \psi}{\partial x}\psi^*\right)$$

$$= \langle x\rangle - \delta x \int dx\, x \frac{\partial}{\partial x}(\psi^* \psi) = \langle x\rangle + \delta x \int dx\, |\psi|^2$$

$$\langle x\rangle' = \langle x\rangle + \delta x$$

$$\langle p_x\rangle = \int dx\,\psi^*(x)\frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x)$$

$$\langle p_x\rangle' = \int dx\,\left[\psi^*(x) - \frac{\partial\psi^*}{\partial x}\delta x\right]\frac{\hbar}{i}\frac{\partial}{\partial x}\left[\psi(x) - \frac{\partial\psi}{\partial x}\delta x\right]$$

Keep first-order terms:

$$= \int dx\,\psi^*(x)\frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x) - \frac{(\delta x)\hbar}{i}\int dx\,\left[\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x} + \psi^*(x)\frac{\partial^2\psi}{\partial x^2}\right]$$

Integrate by parts:

$$= \langle p_x\rangle - \frac{(\delta x)\hbar}{i}\int dx\,\frac{\partial}{\partial x}\left[\psi^*(x)\frac{\partial\psi}{\partial x}\right]$$

$$\langle p_x\rangle' = \langle p_x\rangle$$

Problem 6.4

a) Free particle:

$$\hat{H} = \frac{\hat{p}_x^2}{2m}$$

Gaussian wave-packet:

$$\langle x|\psi\rangle = \frac{1}{\sqrt{\sqrt{\pi}a}}e^{-x^2/2a^2}$$

$$\langle p|\psi\rangle = \sqrt{\frac{a}{\hbar\sqrt{\pi}}}e^{-p^2a^2/2\hbar^2}$$

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\int dp\,|p\rangle\langle p|\psi\rangle = \int dp\,e^{-ip^2t/2m\hbar}\,|p\rangle\langle p|\psi\rangle$$

$$\psi(x,t) = \int dp\,e^{-ip^2t/2m\hbar}\,\langle x|p\rangle\langle p|\psi\rangle = \int dp\,e^{-ip^2t/2m\hbar}\frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}\sqrt{\frac{a}{\hbar\sqrt{\pi}}}e^{-p^2a^2/2\hbar^2}$$

$$I(a,b) = \int_{-\infty}^{+\infty} dx\,e^{-ax^2+bx} = e^{b^2/4a}\sqrt{\frac{\pi}{a}}$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{a}{\hbar\sqrt{\pi}}}\int dp\,e^{-ip^2t/2m\hbar}e^{ipx/\hbar}e^{-p^2a^2/2\hbar^2}$$

$$A = \frac{it}{2m\hbar} + \frac{a^2}{2\hbar^2} = \frac{i\hbar t + ma^2}{2m\hbar^2}$$

$$B = \frac{ix}{\hbar}$$

$$\frac{B^2}{4A} = \frac{-x^2}{\hbar^2}\frac{m\hbar^2}{2(i\hbar t + ma^2)} = -\frac{x^2}{2a^2(1+i\hbar t/ma^2)}$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{a}{\hbar\sqrt{\pi}}}\sqrt{\frac{2\pi m\hbar^2}{i\hbar t + ma^2}}e^{-x^2/2a^2[1+(i\hbar t/ma^2)]}$$

$$\psi(x,t) = \frac{1}{\sqrt{\sqrt{\pi}[a+(i\hbar t/ma)]}}e^{-x^2/2a^2[1+(i\hbar t/ma^2)]}$$

$$|\psi(x,t)|^2 = \frac{1}{\sqrt{\pi[a^2+(\hbar t/ma)^2]}}e^{-x^2/a^2[1+(\hbar t/ma^2)^2]}$$

$$\Delta x = \frac{a}{\sqrt{2}}\sqrt{1+\left(\frac{\hbar t}{ma^2}\right)^2}$$

b)

$$\langle p|\psi(t)\rangle = \int dp'\, e^{-ip'^2 t/2m\hbar}\,\langle p|p'\rangle\,\langle p'|\psi\rangle = e^{-ip^2 t/2m\hbar}\sqrt{\frac{a}{\hbar\sqrt{\pi}}}e^{-p^2a^2/2\hbar^2}$$

$$\psi(p,t) = \sqrt{\frac{a}{\hbar\sqrt{\pi}}}e^{-p^2(i\hbar t+ma^2)/2m\hbar^2}$$

$$|\psi(p,t)|^2 = \frac{a}{\hbar\sqrt{\pi}}e^{-p^2a^2/\hbar^2}$$

Odd function:

$$\langle p\rangle = \int dp\,|\psi(p,t)|^2 p = \frac{a}{\hbar\sqrt{\pi}}\int dp\, e^{-p^2a^2/\hbar^2}p = 0$$

$$\int dx\, e^{-x^2/a^2}x^2 = \frac{a^3\sqrt{\pi}}{2}$$

$$\langle p^2\rangle = \int dp\,|\psi(p,t)|^2 p^2 = \frac{a}{\hbar\sqrt{\pi}}\int dp\, e^{-p^2a^2/\hbar^2}p^2 = \frac{a}{\hbar\sqrt{\pi}}\left(\frac{\hbar}{a}\right)^3\frac{\sqrt{\pi}}{2} = \frac{\hbar^2}{2a^2}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \frac{\hbar}{a\sqrt{2}}$$

Problem 6.5

$$\langle p|\psi\rangle = \begin{cases} 0 & p < -P/2 \\ N & -P/2 < P/2 \\ 0 & p > P/2 \end{cases}$$

a) Normalization:

$$\int dp\,|\,\langle p|\psi\rangle\,|^2 = 1$$

$$N^2 P = 1$$

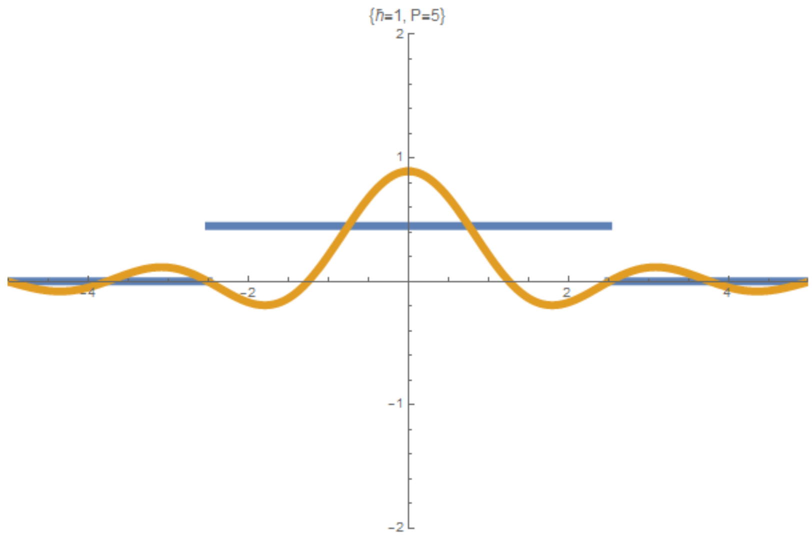
$$N = \frac{1}{\sqrt{P}}$$

b)

$$\psi(x) = \int dp\ \langle x|p\rangle\,\psi(p) = \int_{-P/2}^{P/2} dp\,\frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} N = \frac{N}{\sqrt{2\pi\hbar}}\frac{\hbar}{ix} e^{ipx/\hbar}\Big|_{-P/2}^{P/2}$$

$$\psi(x) = \frac{1}{ix}\sqrt{\frac{\hbar}{2\pi P}}(e^{iPx/2\hbar} - e^{-iPx/2\hbar}) = \frac{1}{x}\sqrt{\frac{2\hbar}{\pi P}}\sin\left(\frac{Px}{2\hbar}\right)$$

c) Plot of $\psi(p)$ (blue) and $\psi(x)$ (orange) for $\hbar = 1, P = 5$



Estimation:

$$\Delta p \approx \frac{P}{4}$$

$$\frac{P\Delta x}{2\hbar} = \pi$$

$$\Delta x \approx \frac{2\pi\hbar}{P}$$

Independent of P:

$$\Delta x \Delta p \approx \frac{\pi\hbar}{2}$$

Problem 6.6

a) Real wave function:

$$\langle x|\psi\rangle = \langle \psi|x\rangle$$

$$\langle p\rangle = \int dx\ \langle\psi|x\rangle\,\frac{\hbar}{i}\frac{\partial}{\partial x}\,\langle x|\psi\rangle = \frac{\hbar}{i}\int dx\,\psi(x)\frac{\partial\psi}{\partial x}$$

$$\int dx\,\psi(x)\frac{\partial\psi}{\partial x} = \int dx\,\frac{\partial\psi}{\partial x}\psi(x)$$

Normalization constraint: $\psi \to 0, x \to \pm\infty$

$$\langle p \rangle = \frac{\hbar}{2i} \int dx \, \frac{\partial}{\partial x}(\psi^2) = 0$$

b)

$$\psi(x)' = e^{ip_0 x/\hbar}\psi(x)$$

$$\langle x \rangle' = \int dx \, \psi'^* x \psi' = \int dx \, e^{-ip_0 x/\hbar}\psi^*(x) x e^{ip_0 x/\hbar}\psi(x) = \langle x \rangle$$

$$\langle p \rangle' = \int dx \, \psi'^* \frac{\hbar}{i}\frac{\partial}{\partial x}\psi' = \frac{\hbar}{i}\int dx \, e^{-p_0 x/\hbar}\psi^*(x)\frac{\partial}{\partial x}[e^{ip_o x/\hbar}\psi(x)]$$

$$= \frac{\hbar}{i}\int dx \, e^{-p_o x/\hbar}\psi^*(x)\left[\frac{ip_0}{\hbar}e^{ip_o x/\hbar}\psi(x) + e^{ip_0 x/\hbar}\frac{\partial}{\partial x}\psi(x)\right]$$

$$= p_0 \int dx \, \psi^*(x)\psi(x) + \int dx \, \psi^*(x)\frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x)$$

$$\langle p \rangle' = \langle p \rangle + p_0$$

Problem 6.7

a)

$$\langle \varphi|\hat{x}|\psi\rangle = \int dx \, \langle \varphi|x\rangle \, x \, \langle x|\psi\rangle = \int dx \, \langle x|\varphi\rangle^* \, x \, \langle \psi|x\rangle^* = \left(\int dx \, \langle \psi|x\rangle \, x \, \langle x|\varphi\rangle\right)^* = \langle \psi|\hat{x}|\varphi\rangle^*$$

b)

$$-[\hat{x}, \hat{p}_x] = \hat{p}_x\hat{x} - \hat{x}\hat{p}_x = -i\hbar$$

$$[\hat{x}, \hat{p}_x]^\dagger = (\hat{x}\hat{p}_x)^\dagger - (\hat{p}_x\hat{x})^\dagger = \hat{p}_x^\dagger\hat{x}^\dagger - \hat{x}^\dagger\hat{p}_x^\dagger = \hat{p}_x\hat{x}^\dagger - \hat{x}^\dagger\hat{p}_x = -i\hbar$$

$$\hat{x}^\dagger = \hat{x}$$

Problem 6.8 - SKIPPED

Problem 6.9

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - a|x|)\psi = 0$$

$$E = \epsilon(\hbar^2 a^2/m)^{1/3}$$
$$x = z(\hbar^2/ma)^{1/3}$$

a)

$$[E] = \frac{ML^2}{T^2}, [x] = L, [\hbar] = \frac{ML^2}{T}, [a] = \frac{ML}{T^2}, [m] = M$$

$$\frac{ML^2}{T^2} = [\epsilon]\left(\frac{M^2L^4}{T^2}\frac{M^2L^2}{T^4}\frac{1}{M}\right)^{1/3} = [\epsilon]\frac{ML^2}{T^2}$$

$$L = [z]\left(\frac{M^2L^4}{T^2}\frac{1}{M}\frac{T^2}{ML}\right)^{1/3} = [z]L$$

$$[\epsilon] = [z] = [\ \ ]$$

b)

$$\frac{d\psi}{dx} = \frac{d\psi}{dz}\frac{dz}{dx}$$

$$\frac{dz}{dx} = (\hbar^2/ma)^{-1/3}$$

$$\frac{d\psi}{dx} = \frac{d\psi}{dz}(\hbar^2/ma)^{-1/3}$$

$$\frac{d^2\psi}{dx^2} = \frac{d}{dz}\left(\frac{d\psi}{dx}\right)\frac{dz}{dx} = \frac{d^2\psi}{dz^2}(\hbar^2/ma)^{-2/3}$$

$$\frac{d^2\psi}{dz^2}(\hbar^2/ma)^{-2/3} + \frac{2m}{\hbar^2}[\epsilon(\hbar^2a^2/m)^{1/3} - a|z|(\hbar^2/ma)^{1/3}]\psi = 0$$

$$\frac{d^2\psi}{dz^2} + \frac{2m}{\hbar^2}\left(\frac{\hbar^2a^2}{m}\right)^{1/3}\left(\frac{\hbar^2}{ma}\right)^{2/3}(\epsilon - |z|)\psi = 0$$

$$\frac{d^2\psi}{dz^2} + 2(\epsilon - |z|)\psi = 0$$

c) - SKIPPED

Problem 6.10

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{elsewhere} \end{cases}$$

Outside the potential well:

$$\psi(x) = 0$$

Inside the potential well:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi(x)$$

$$\frac{d^2\psi}{dx^2} = -\omega^2\psi(x), \quad \omega = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\psi(x) = A\sin\omega x + B\cos\omega x$$

Boundary conditions:

$$\psi(0) = B = 0$$

$$\psi(L) = A \sin \omega L = 0$$

Energy eigenvalues:

$$\omega L = n\pi, \quad n = 1, 2, 3, ...$$

$$\sqrt{\frac{2mE_n}{\hbar^2}} = \frac{n\pi}{L}$$

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}, \quad n = 1, 2, 3, ...$$

Normalization:

$$A^2 \int_0^L dx \, \sin^2 \frac{n\pi x}{L} = 1$$

$$A = \sqrt{\frac{2}{L}}$$

Energy eigenfunctions:

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} & 0 < x < L \\ 0 & \text{elsewhere} \end{cases} \qquad n = 1, 2, 3, ...$$

Problem 6.11

Ground state:

$$\psi(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L} & 0 < x < L \\ 0 & \text{elsewhere} \end{cases}$$

$$\langle x \rangle = \int dx \, \psi^*(x) x \psi(x) = \frac{2}{L} \int_0^L dx \, x \sin^2 \frac{\pi x}{L} = \frac{L}{2}$$

$$\langle x^2 \rangle = \int dx \, \psi^*(x) x^2 \psi(x) = \frac{2}{L} \int_0^L dx \, x^2 \sin^2 \frac{\pi x}{L} = \frac{1}{6} \left(2 - \frac{3}{\pi^2} \right) L^2$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{1}{6} \left(2 - \frac{3}{\pi^2} \right) L^2 - \frac{L^2}{4}} = L \sqrt{\frac{1}{12} - \frac{1}{2\pi^2}}$$

$$\langle p \rangle = \int dx \, \psi^*(x) \frac{\hbar}{i} \frac{d}{dx} \psi(x) = \frac{2\hbar\pi}{iL^2} \int_0^L dx \, \sin \frac{\pi x}{L} \cos \frac{\pi x}{L} = 0$$

$$\langle p^2 \rangle = \int dx \, \psi^*(x) \left(\frac{\hbar}{i} \right)^2 \frac{d^2}{dx^2} \psi(x) = \frac{2\hbar^2 \pi^2}{L^3} \int_0^L dx \, \sin^2 \frac{\pi x}{L} = \frac{\pi^2 \hbar^2}{L^2}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\pi \hbar}{L}$$

$$\Delta x \Delta p = \pi\hbar\sqrt{\frac{1}{12} - \frac{1}{2\pi^2}} = \hbar\sqrt{\frac{\pi^2 - 6}{12}} > \frac{\hbar}{2}$$

Problem 6.12

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{elsewhere} \end{cases}$$

$$\psi(x) = \begin{cases} \left(\frac{1+i}{2}\right)\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L} + \frac{1}{\sqrt{2}}\sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L} & 0 < x < L \\ 0 & \text{elsewhere} \end{cases}$$

a)

$$\psi(x) = \left(\frac{1+i}{2}\right)\psi_1(x) + \frac{1}{\sqrt{2}}\psi_2(x)$$

$$\psi(x,t) = \left(\frac{1+i}{2}\right)e^{-iE_1t/\hbar}\psi_1(x) + \frac{1}{\sqrt{2}}e^{-iE_2t/\hbar}\psi_2(x)$$

b)

$$\langle E \rangle = \left|\frac{1+i}{2}\right|^2 E_1 + \left(\frac{1}{\sqrt{2}}\right)^2 E_2 = \frac{E_1 + E_2}{2}$$

$$\langle E \rangle = \langle \psi | \hat{H} | \psi \rangle = \left[\left(\frac{1-i}{2}\right)\psi_1^*(x) + \frac{1}{\sqrt{2}}\psi_2^*(x)\right]\left[\left(\frac{1+i}{2}\right)E_1\psi_1(x) + \frac{1}{\sqrt{2}}E_2\psi_2(x)\right] = \frac{E_1 + E_2}{2}$$

c)

$$|\psi(t)\rangle = \left(\frac{1+i}{2}\right)e^{-iE_1t/\hbar}|E_1\rangle + \frac{1}{\sqrt{2}}e^{-iE_2t/\hbar}|E_2\rangle$$

$$|\langle E_1|\psi\rangle|^2 = \frac{1}{2}$$

d) $\langle x \rangle$ is time-dependent due to the relative phase difference between the two energy eigenstates. Time enters through the cross terms between the wave functions.

Problem 6.13

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{elsewhere} \end{cases}$$

$$|\psi(0)\rangle = |E_1^L\rangle$$

$t = 0$: Wall pulled back to $x = 2L$ instantaneously

a)

$$\langle E_1^{2L}|E_1^L\rangle = \int_0^L dx\, \psi_1^{2L*}(x)\psi_1^L(x) = \int_0^L dx\,\sqrt{\frac{1}{L}}\sin\left(\frac{\pi x}{2L}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right) = \frac{4\sqrt{2}}{3\pi}$$

$$|\langle E_1^{2L}|E_1^L\rangle|^2 = \frac{32}{9\pi^2} = 0.36$$

b)

$$|\psi(0)\rangle = \sum_n |E_n^{2L}\rangle\langle E_n^{2L}|E_1^L\rangle$$

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\sum_n |E_n^{2L}\rangle\langle E_n^{2L}|E_1^L\rangle = \sum_n e^{-iE_n^{2L}t/\hbar}|E_n^{2L}\rangle\langle E_n^{2L}|E_1^L\rangle$$

The system is no longer is an energy eigenstate. Calculate amplitudes $\langle E_n^{2L}|E_1^L\rangle$ and let computer carry out the sum numerically.

Problem 6.14

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{elsewhere} \end{cases}$$

$$\psi(x) = \begin{cases} Nx(x-L) & 0 < x < L \\ 0 & \text{elsewhere} \end{cases}$$

a) Normalization:

$$\int dx\,|\psi(x)|^2 = N^2\int_0^L dx\,x^2(x-L)^2 = \frac{N^2L^5}{30} = 1$$

$$N = \sqrt{\frac{30}{L^5}}$$

b)

$$\langle E_1|\psi\rangle = \int dx\,\langle E_1|x\rangle\langle x|\psi\rangle = \int_0^L dx\,\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)Nx(x-L) = \sqrt{\frac{60}{L^6}}\left(-\frac{4L^3}{\pi^3}\right) = -\frac{8\sqrt{15}}{\pi^3}$$

$$|\langle E_1|\psi\rangle|^2 = \frac{960}{\pi^6} = 0.998555$$

c)

$$\langle E\rangle = \sum_n |\langle E_n|\psi\rangle|^2 E_n = \left(\frac{960}{\pi^6}\right)\frac{\hbar^2\pi^2}{2mL^2} + \left(\frac{320}{243\pi^6}\right)\frac{9\hbar^2\pi^2}{2mL^2} + ... \approx \frac{13120\hbar^2}{27\pi^4 mL^2}$$

Problem 6.15 - SKIPPED

Problem 6.16

$$\frac{2m}{\hbar^2}V(x) = -\frac{\lambda}{b}\delta(x)$$

$$\left(\frac{d\psi}{dx}\right)_{x+\epsilon} - \left(\frac{d\psi}{dx}\right)_{x-\epsilon} = \int_{x-\epsilon}^{x+\epsilon} dx\,\frac{d}{dx}\frac{d\psi}{dx} = \int_{x-\epsilon}^{x+\epsilon} dx\,\frac{2m}{\hbar^2}(V-E)\psi$$

$x = 0$:

$$\left(\frac{d\psi}{dx}\right)_{0^+} - \left(\frac{d\psi}{dx}\right)_{0^-} = \lim_{\epsilon\to 0}\int_{0-\epsilon}^{0+\epsilon} dx\,\frac{d}{dx}\frac{d\psi}{dx} = \lim_{\epsilon\to 0}\int_{0-\epsilon}^{0+\epsilon} dx\,\frac{2m}{\hbar^2}\left[-\frac{\lambda\hbar^2}{2bm}\delta(x) - E\right]\psi$$

$$= \lim_{\epsilon\to 0}\int_{0-\epsilon}^{0+\epsilon} dx\,\left[-\frac{\lambda}{b}\delta(x) - \frac{2mE}{\hbar^2}\right]\psi = -\frac{\lambda}{b}\psi(0)$$

b)

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

$E < 0$:

$$\frac{d^2\psi}{dx^2} = -k^2\psi$$

$x < 0$ $(B = 0)$:

$$\psi = Ae^{ikx}$$

$$\psi' = ikAe^{ikx}$$

$x > 0$ $(C = 0)$:

$$\varphi = De^{-ikx}$$

$$\varphi' = -ikDe^{-ikx}$$

Boundary conditions: $\psi(0) = \varphi(0)$

$$A = D$$

$\varphi'(0) - \psi'(0) = -\frac{\lambda}{b}\varphi(0) = -\frac{\lambda}{b}\psi(0)$

$$-ikD - ikA = -2ikA = -\frac{\lambda}{b}A$$

$$2ik = \frac{\lambda}{b}$$

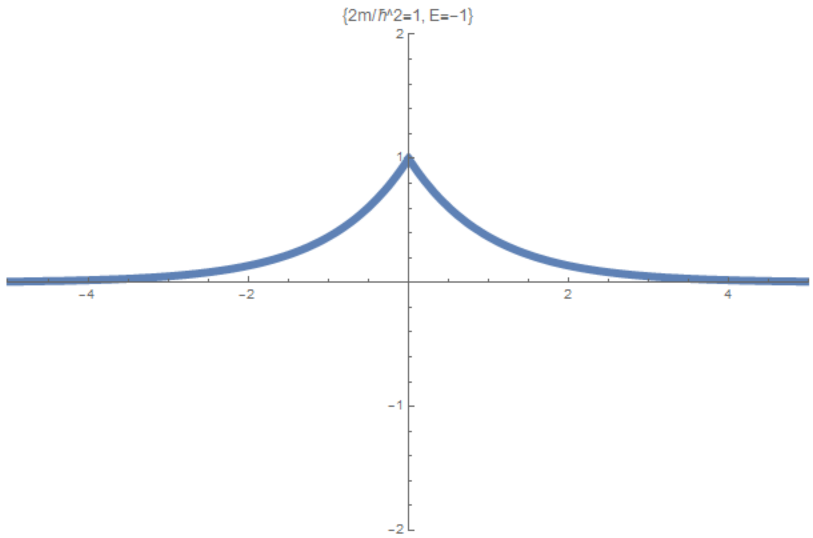
$$2\sqrt{\frac{2m|E|}{\hbar^2}} = \frac{\lambda}{b}$$

$$E_e = -\frac{\hbar^2\lambda^2}{8mb^2}$$

Normalization:

$$A = \left(\frac{2m|E|}{\hbar^2}\right)^{1/4}$$

Plot of $\psi(x)$ for $\frac{2m}{\hbar^2} = 1, E = -1$



Problem 6.17

$$\frac{2m}{\hbar^2}V(x) = -\frac{\lambda}{b}\delta(x)$$

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

$E > 0$:

$$\frac{d^2\psi}{dx^2} = -k^2\psi$$

$x < 0$:

$$\psi = Ae^{ikx} + Be^{-ikx}$$

$$\psi' = ikAe^{ikx} - ikBe^{-ikx}$$

$x > 0$:

$$\varphi = Ce^{ikx} + De^{-ikx}$$

$$\varphi' = ikCe^{ikx} - ikDe^{-ikx}$$

Boundary conditions: $\psi(0) = \varphi(0)$

$$A + B = C + D$$

$\varphi'(0) - \psi'(0) = -\frac{\lambda}{b}\varphi(0) = -\frac{\lambda}{b}\psi(0)$

$$(ikC - ikD) - (ikA - ikB) = -\frac{\lambda}{b}(A + B)$$

Scattering:

$$D = 0$$

$$A + B = C$$

$$ikC - ikA + ikB = -\frac{\lambda}{b}(A+B)$$

$$ikC - ikA + ik(C-A) = -\frac{\lambda}{b}C$$

$$\left(2ik + \frac{\lambda}{b}\right) C = 2ikA$$

$$T = \frac{|C|^2}{|A|^2} = \left(\frac{2ik}{2ik + \frac{\lambda}{b}}\right)\left(\frac{-2ik}{-2ik + \frac{\lambda}{b}}\right) = \frac{4k^2}{4k^2 + (\lambda/b)^2}$$

$$ik(A+B) - ikA + ikB = -\frac{\lambda}{b}(A+B)$$

$$\left(2ik + \frac{\lambda}{b}\right) B = -\frac{\lambda}{b}A$$

$$R = \frac{|B|^2}{|A|^2} = \left(\frac{-\lambda/b}{2ik + \lambda/b}\right)\left(\frac{-\lambda/b}{-2ik + \lambda/b}\right) = \frac{(\lambda/b)^2}{4k^2 + (\lambda/b)^2}$$

$$T + R = 1$$

Probability current:

$$j = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)$$

$x < 0$:

$$\psi^* = Ae^{-ikx} + Be^{ikx}$$

$$j = \frac{\hbar}{2mi}\left[\left(Ae^{-ikx} + Be^{ikx}\right)\left(ikAe^{ikx} - ikBe^{-ikx}\right) - (Ae^{ikx} + Be^{-ikx})(-ikAe^{-ikx} + ikBe^{ikx})\right]$$

$$j = \frac{\hbar}{2mi}(2ik|A|^2 - 2ik|B|^2) = \frac{\hbar k}{m}(|A|^2 - |B|^2)$$

$$j_{inc} = \frac{\hbar k}{m}|A|^2$$

$$j_{ref} = \frac{\hbar k}{m}|B|^2$$

$$R = \frac{j_{ref}}{j_{inc}} = \frac{|B|^2}{|A|^2}$$

$x > 0$ $(D = 0)$:

$$\psi^* = Ce^{-ikx}$$

$$j = \frac{\hbar}{2mi}[Ce^{-ikx}(ikCe^{ikx}) - Ce^{ikx}(-ikCe^{-ikx})] = \frac{\hbar k}{m}(|C|^2)$$

$$j_{trans} = \frac{\hbar k}{m}|C|^2$$

$$T = \frac{j_{trans}}{j_{inc}} = \frac{|C|^2}{|A|^2}$$

Problem 6.18

$E > V_0$

$$k\sqrt{\frac{2mE}{\hbar^2}}$$

$$Q = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

$x < 0$:

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

$x > 0$:

$$\psi(x) = Ce^{iQx} + De^{-iQx}$$

Boundary conditions:

$$A + B = C + D$$

$$ik(A - B) = iQ(C - D)$$

Particles are incident from the left:

$$R = \frac{(k - Q)^2}{(k + Q)^2}$$

$$T = \frac{4kQ}{(k + Q)^2}$$

Particles are incident from the right:

$$B = C + D$$

$$-ikB = iQ(C - D)$$

$$-ik(C + D) = iQ(C - D)$$

$$(-k + Q)D = (k + Q)C$$

$$R = \frac{|C|^2}{|D|^2} = \frac{(k - Q)^2}{(k + Q)^2}$$

$$-ikB = iQ(B - 2D)$$

$$(k + Q)B = (2Q)D$$

$$T = \frac{k}{Q}\frac{|B|^2}{|D|^2} = \frac{k}{Q}\frac{4Q^2}{(k+Q)^2} = \frac{4kQ}{(k+Q)^2}$$

Problem 6.19

Square potential barrier:

$$V = \begin{cases} 0 & x < 0 \\ V_0 & 0 < x < a \\ 0 & x > a \end{cases}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$q = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ Fe^{qx} + Ge^{-qx} & 0 < x < a \\ Ce^{ikx} & x > a \end{cases}$$

Boundary conditions:

$$A + B = F + G$$

$$ik(A - B) = q(F - G)$$

$$Fe^{qa} + Ge^{-qa} = Ce^{ika}$$

$$q(Fe^{qa} - Ge^{-qa}) = ikCe^{ika}$$

$$F = \frac{1}{2}\left(1 + \frac{ik}{q}\right)A + \frac{1}{2}\left(1 - \frac{ik}{q}\right)B$$

$$G = \frac{1}{2}\left(1 - \frac{ik}{q}\right)A + \frac{1}{2}\left(1 + \frac{ik}{q}\right)B$$

$$2Fe^{qa} = \left(1 + \frac{ik}{q}\right)Ce^{ika}$$

$$2Ge^{-qa} = \left(1 - \frac{ik}{q}\right)Ce^{ika}$$

$$\left(1 + \frac{ik}{q}\right)Ae^{qa} + \left(1 - \frac{ik}{q}\right)Be^{qa} = \left(1 + \frac{ik}{q}\right)Ce^{ika}$$

$$\left(1 - \frac{ik}{q}\right)Ae^{-qa} + \left(1 + \frac{ik}{q}\right)Be^{-qa} = \left(1 - \frac{ik}{q}\right)Ce^{ika}$$

$$\left(1-\frac{ik}{q}\right)Ae^{-qa}+\left(\frac{1+ik/q}{1-ik/q}\right)\left[\left(1+\frac{ik}{q}\right)Ce^{ika}-\left(1+\frac{ik}{q}\right)Ae^{qa}\right]e^{-2qa}=\left(1-\frac{ik}{q}\right)Ce^{ika}$$

$$\left(1-\frac{ik}{q}\right)^2Ae^{-qa}+\left[\left(1+\frac{ik}{q}\right)^2Ce^{(ik-2q)a}-\left(1+\frac{ik}{q}\right)^2Ae^{-qa}\right]=\left(1-\frac{ik}{q}\right)^2Ce^{ika}$$

$$\left(\frac{4ik}{q}\right)Ae^{-qa}=\left[\left(1+\frac{ik}{q}\right)^2e^{-2qa}-\left(1-\frac{ik}{q}\right)^2\right]Ce^{ika}$$

$$\frac{C}{A}=\frac{\left(\frac{4ik}{q}\right)e^{-(q+ik)a}}{\left[\left(1+\frac{ik}{q}\right)^2e^{-2qa}-\left(1-\frac{ik}{q}\right)^2\right]}$$

$$T=\frac{j_{x>a}}{j_{inc}}=\frac{|C|^2}{|A|^2}=\frac{\left(\frac{4ik}{q}\right)e^{-(q+ik)a}}{\left[\left(1+\frac{ik}{q}\right)^2e^{-2qa}-\left(1-\frac{ik}{q}\right)^2\right]}\frac{\left(\frac{-4ik}{q}\right)e^{-(q-ik)a}}{\left[\left(1-\frac{ik}{q}\right)^2e^{-2qa}-\left(1+\frac{ik}{q}\right)^2\right]}$$

$$=\frac{(4k/q)^2e^{-2qa}}{\left(1+\frac{k^2}{q^2}\right)^2e^{-4qa}-\left[\left(1+\frac{ik}{q}\right)^4+\left(1-\frac{ik}{q}\right)^4\right]e^{-2qa}+\left(1+\frac{k^2}{q^2}\right)^2}$$

$$=\frac{(4k/q)^2}{\left(1+\frac{k^2}{q^2}\right)^2(e^{2qa}+e^{-2qa})-2\left(1-\frac{6k^2}{q^2}+\frac{k^4}{q^4}\right)}$$

$$\sinh x=\frac{e^x-e^{-x}}{2}$$

$$\sinh^2 x=\frac{e^{2x}-2+e^{-2x}}{4}$$

$$T=\frac{(4k/q)^2}{\left(1+\frac{k^2}{q^2}\right)^2(4\sinh^2 qa+2)-2\left(1-\frac{6k^2}{q^2}+\frac{k^4}{q^4}\right)}$$

$$=\frac{(4k/q)^2}{4\left(1+\frac{k^2}{q^2}\right)^2\sinh^2(qa)+(16k^2/q^2)}=\frac{1}{\left(\frac{q}{2k}+\frac{k}{2q}\right)^2\sinh^2(qa)+1}$$

$$T=\frac{1}{1+\left(\frac{k^2+q^2}{2kq}\right)^2\sinh^2(qa)}$$

Problem 6.20

a)

$$V = \begin{cases} 0 & x < 0 \\ -V_0 & 0 < x < a \\ 0 & x > a \end{cases}$$

$$q' \to i\sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

$$T = \frac{1}{1 + \left(\frac{k^2+q'^2}{2kq'}\right)^2 \sinh^2(q'a)}$$

$$\sinh(ix) = i\sin(x)$$
$$\sinh^2(ix) = -\sin^2(x)$$

$$\left(\frac{k^2+q'^2}{2kq'}\right)^2 = \left[\frac{\frac{2mE}{\hbar^2} - \frac{2m(E+V_0)}{\hbar^2}}{2i\sqrt{\frac{2mE}{\hbar^2}}\sqrt{\frac{2m(E+V_0)}{\hbar^2}}}\right]^2 = -\frac{(2m/\hbar^2)^2(V_0)^2}{4(2m/\hbar^2)^2 E(E+V_0)}$$

$$\left(\frac{k^2+q'^2}{2kq'}\right)^2 \sinh^2(q'a) = \frac{{V_0}^2}{4E(E+V_0)} \sin^2\sqrt{\frac{2m(E+V_0)}{\hbar^2}}a$$

$$T = \left[1 + \frac{\sin^2\sqrt{\frac{2m}{\hbar^2}(E+V_0)}a}{4\frac{E}{V_0}\frac{(E+V_0)}{V_0}}\right]^{-1}$$

b) - SKIPPED

Problem 6.21

Approximation:

$$T \simeq \exp\left(-2\int dx\sqrt{\frac{2m}{\hbar^2}[V(x)-E]}\right)$$

$$V(x) = -e|E|x + (E_f + W)$$
$$L = \frac{W}{e|E|}$$

$$-2\int dx\sqrt{\frac{2m}{\hbar^2}[V(x)-E]} = -2\sqrt{\frac{2m}{\hbar^2}}\int_0^L dx\sqrt{-e|E|x+W} = -2\sqrt{\frac{2m}{\hbar^2}}\frac{2}{3e|E|}W^{3/2}$$

$$T \simeq \exp\left(\frac{-4\sqrt{2m}W^{3/2}}{3e|E|\hbar}\right)$$