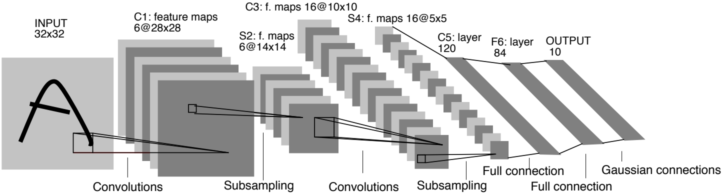
Lecture 6 - All about Convolutional Networks

DD2424

April 12, 2021

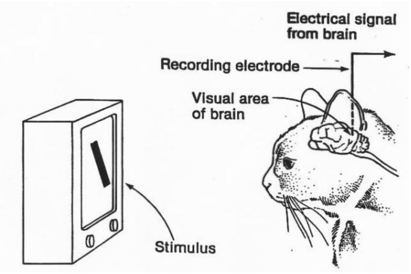
Convolutional Neural Networks (ConvNets)

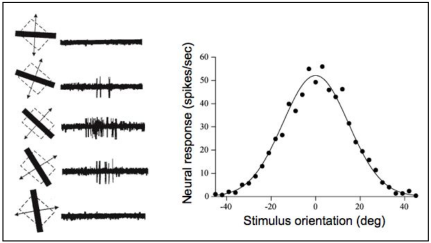


LeNet-5 (LeCun '98)

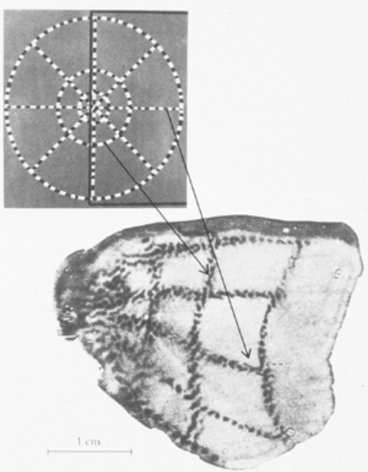
ConvNets: Some history

Hubel & Wiesel cat experiments 1968

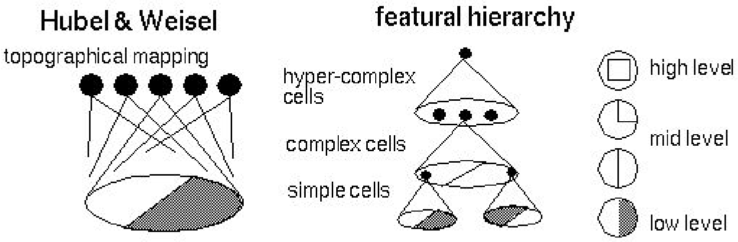




- Discovered *visual cortex* consists of a hierarchy of simple, complex, and hyper-complex cells. (Experiments in 50's & 60's)
- Hubel & Wiesel won the Nobel prize (1981).



Topographical mapping in the cortex: nearby cells in cortex represented nearby regions in the visual field.


Hubel & Weisel
topographical mapping
featural hierarchy
hyper-complex cells
complex cells
simple cells
high level
mid level
low level

Neurocognitron

Neocognitron: A Self-organizing Neural Network Model for a Mechanism of Pattern Recognition Unaffected by Shift in Position by Kunihiko Fukushima, 1980.

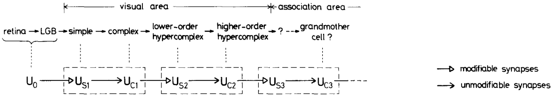
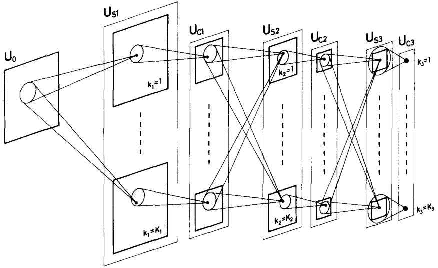
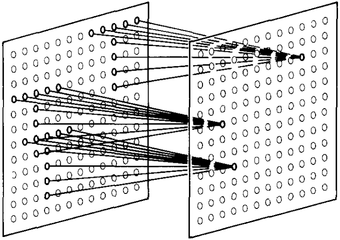


Fig. 1. Correspondence between the hierarchy model by Hubel and Wiesel, and the neural network of the neocognitron

Inspired by Hubel & Wiesel model

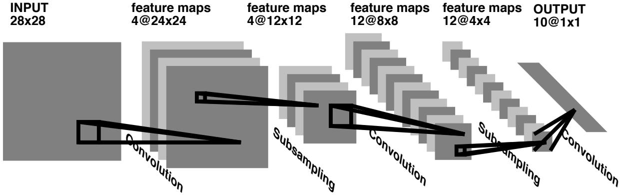




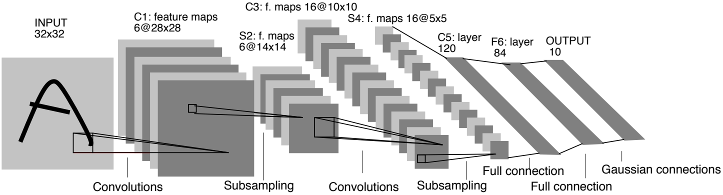
sandwich architecture (SCSCSC...)

simple cells: modifiable parameters, **complex cells**: perform pooling

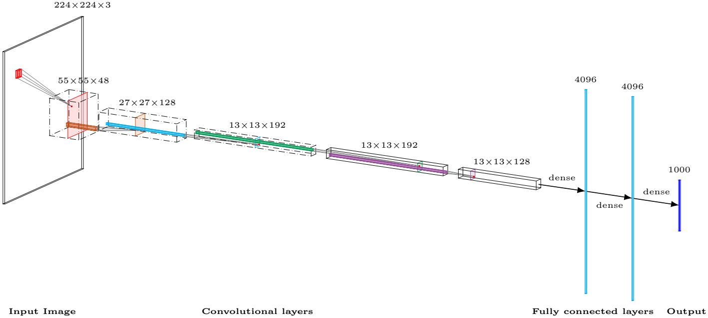
LeCun's *LeNet* ConvNets



LeNet 1 '90



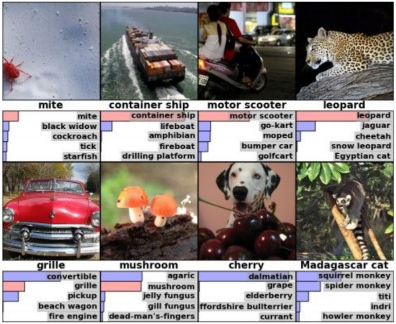
LeNet 5 '95



ImageNet Classification with Deep Convolutional Neural Networks by Krizhevsky, Sutskever, Hinton, 2012

Fast-forward to today: ConvNets are everywhere

Classification

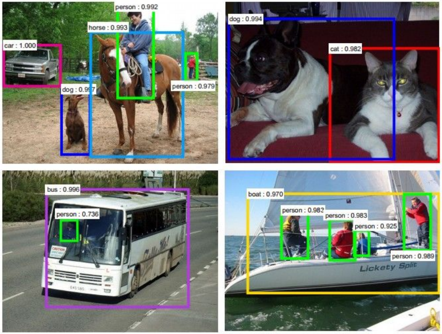


Retrieval

[Krizhevsky 2012]

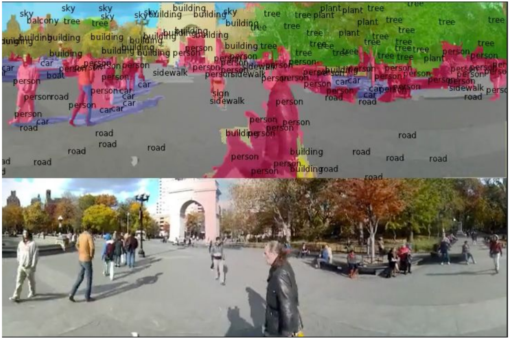
Fast-forward to today: ConvNets are everywhere

Detection



[Faster R-CNN: Ren, He, Girshick, Sun 2015]

Segmentation



[Farabet et al., 2012]

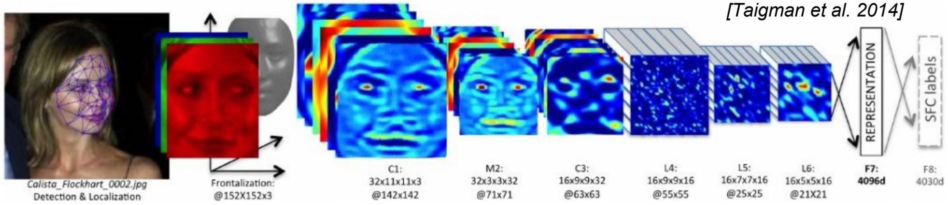
Fast-forward to today: ConvNets are everywhere

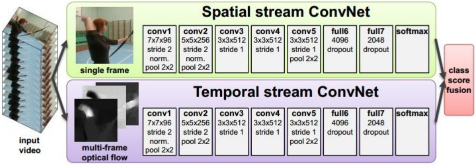
self-driving cars

NVIDIA Tegra X1

Fast-forward to today: ConvNets are everywhere

[Taigman et al. 2014]





[Simonyan et al. 2014]

[Goodfellow 2014]

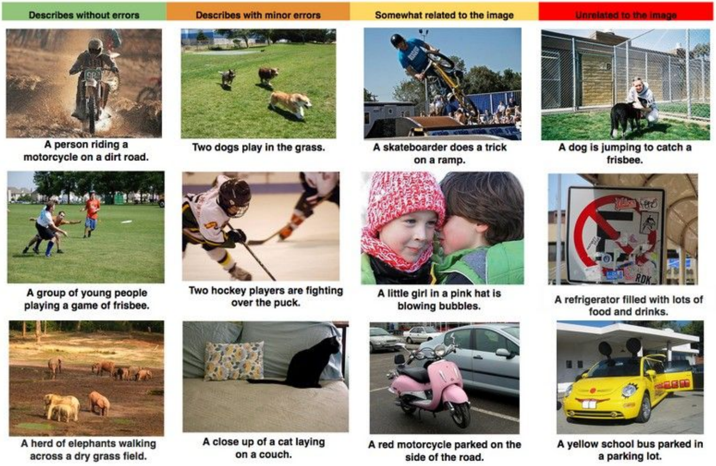
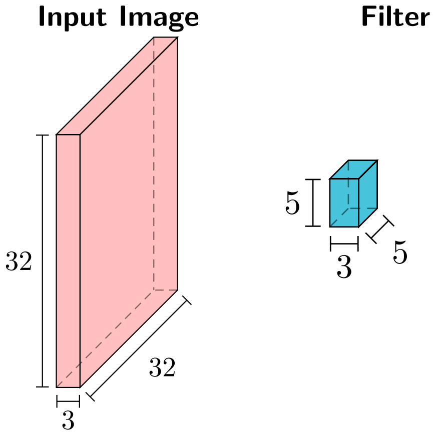


Image Captioning

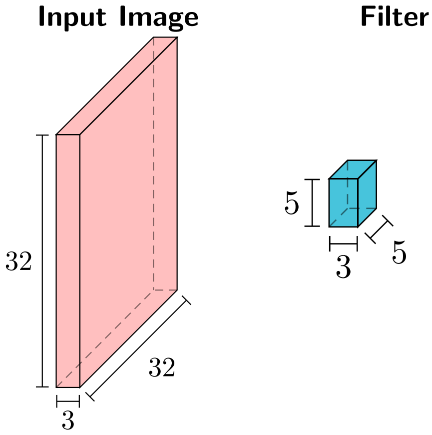
[Vinyals et al., 2015]

ConvNets for RGB Images: The Convolution Layer



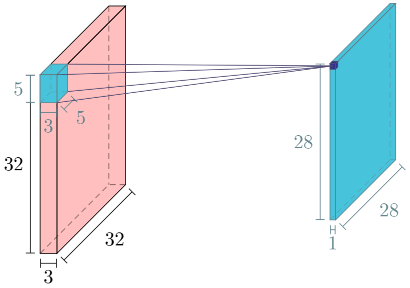
$\mathbf{X}$ is $32 \times 32 \times 3$

$\mathbf{F}$ is $5 \times 5 \times 3$



$\mathbf{X}$ is $32 \times 32 \times 3$ $\mathbf{F}$ is $5 \times 5 \times 3$

Note: Filter & input image generally have the same depth.

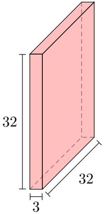


Convolve the image, $\mathbf{X}$, with the filter $\mathbf{F}$.

- Slide filter over all spatial locations in image.
- At each location output 1 number:

compute **dot product** *between* $\mathbf{F}$ *and a* $5 \times 5 \times 3$ *chunk of* $\mathbf{X}$

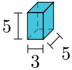
Input Image



$\mathbf{X}$

Size: $32 \times 32 \times 3$

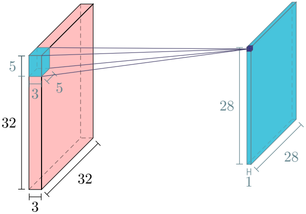
Filter



$\mathbf{F}$

Size: $5 \times 5 \times 3$

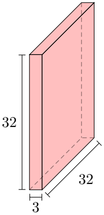
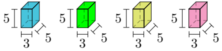
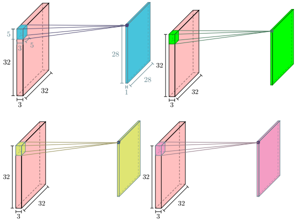
Output response map



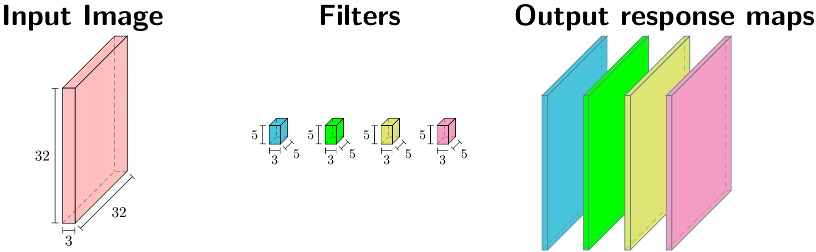
$S = \mathbf{X} * \mathbf{F}$

Size: $28 \times 28 \times 1$

Can apply multiple filters.

Input Image	Filters	Output response maps
		
$\mathbf{X}$	$\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4$	$S_i = \mathbf{X} * \mathbf{F}_i$
Size: $32 \times 32 \times 3$	Size $\mathbf{F}_i$: $5 \times 5 \times 3$	Size S_i: 28×28

Apply multiple filters and get multiple response maps



$\mathbf{X}$

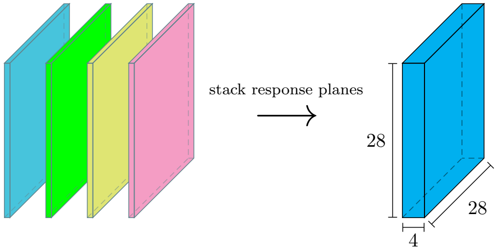
$\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4$

S_1, S_2, S_3, S_4

Each $S_i = \mathbf{X} * \mathbf{F}_i$

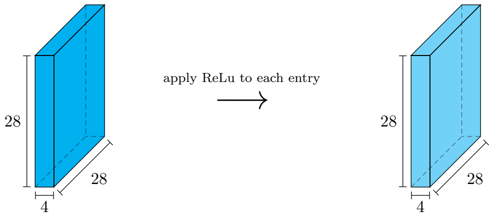
- Stack the multiple response maps to get a *new image* $\mathbf{S}$.

- In our example
  - $\mathbf{S} = \{S_1, S_2, S_3, S_4\}$ and
  - $\mathbf{S}$ has size $28 \times 28 \times 4$



- Apply the non-linear activation function to each element of $\mathbf{S}$.

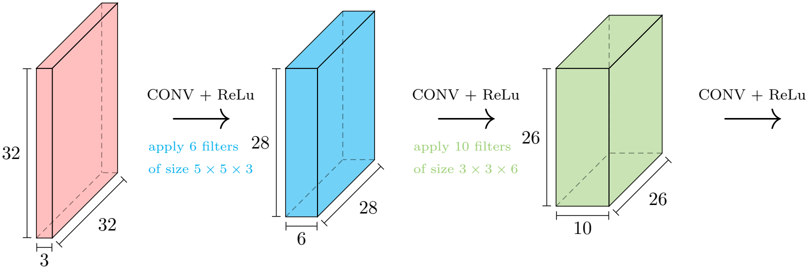
$$\mathbf{H} = \max(0, \mathbf{S})$$



Most basic Convolutional Network layers

Basic **ConvNet** is a composition of

- Convolution Layer
- Activation function



How do we produce final probs for C class labels?

- Add **fully connected layer(s)** after the convolutional layers.
- Example network:
  1 convolutional layer + 1 fully connected layer

$$S_i = \mathbf{X} * \mathbf{F}_i + b_i \quad \text{for } i = 1, \ldots, n_F \qquad \leftarrow \text{apply convolution filters}$$

$$\mathbf{S} = \{S_1, \ldots, S_{n_F}\} \qquad \leftarrow \text{stack response maps, get new 3D image}$$

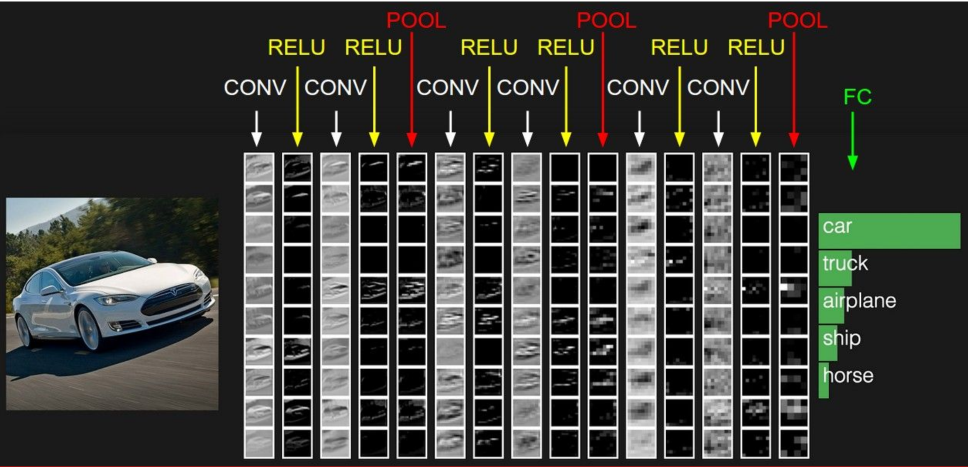
$$\mathbf{H} = \max(0, \mathbf{S}) \qquad \leftarrow \text{apply ReLu}$$

$$\mathbf{s} = W \text{vec}(\mathbf{H}) + \mathbf{b} \qquad \leftarrow \text{fully-connected layer to get } C \text{ scores}$$

$$\mathbf{p} = \text{SoftMax}(\mathbf{s}) \qquad \leftarrow \text{turn scores into probabilities}$$

- Dimensions of inputs, outputs and parameters:
  - $\mathbf{X}$ is $w \times h \times 3$
  - Each $\mathbf{F}_i$ is $f \times f \times 3$ and b_i is a scalar
  - Each S_i is $(w - f + 1) \times (h - f + 1)$
  - $\mathbf{S}$ and $\mathbf{H}$ are $(w - f + 1) \times (h - f + 1) \times n_F$
  - W is $C \times (w - f + 1)(h - f + 1)n_F$
  - $\mathbf{b}, \mathbf{s}$ and $\mathbf{p}$ are $C \times 1$

preview:


POOL
POOL
POOL
RELU
RELU
RELU
RELU
RELU
RELU
CONV
CONV
CONV
CONV
CONV
CONV
FC
car
truck
airplane
ship
horse

Questions??

How do we learn the parameters of the network?

- Add **fully connected layer(s)** after the convolutional layers.
- Example network:
  1 convolutional layer + 1 fully connected layer

$$S_i = \mathbf{X} * \mathbf{F}_i + b_i \quad \text{for } i = 1, \ldots, n_F \qquad \leftarrow \text{apply convolution filters}$$

$$\mathbf{S} = \{S_1, \ldots, S_{n_F}\} \qquad \leftarrow \text{stack response maps, get new 3D image}$$

$$\mathbf{H} = \max(0, \mathbf{S}) \qquad \leftarrow \text{apply ReLu}$$

$$\mathbf{s} = W \text{vec}(\mathbf{H}) + \mathbf{b} \qquad \leftarrow \text{fully-connected layer to get } C \text{ scores}$$

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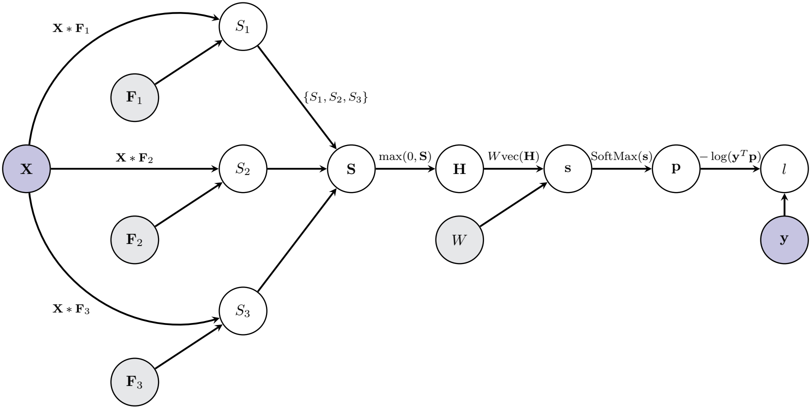
- Dimensions of inputs, outputs and parameters:
  - $\mathbf{X}$ is $w \times h \times 3$
  - Each $\mathbf{F}_i$ is $f \times f \times 3$ and b_i is a scalar
  - Each S_i is $(w - f + 1) \times (h - f + 1)$
  - $\mathbf{S}$ and $\mathbf{H}$ are $(w - f + 1) \times (h - f + 1) \times n_F$
  - W is $C \times (w - f + 1)(h - f + 1)n_F$
  - $\mathbf{b}$, $\mathbf{s}$ and $\mathbf{p}$ are $C \times 1$.

How do we learn the parameters of the network?

- Optimize the usual cross-entropy loss ($+ L_2$ regularization term) on the training data.
- Use mini-batch gradient descent to perform optimization.
- $\implies$ need to compute the gradient of the loss w.r.t. the convolutional parameters....

Gradient Computations for one Convolutional layer

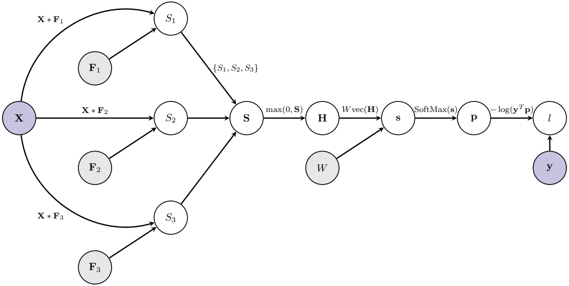
Computational Graph for our simple network



Notes about the above figure

- Apply 3 filters in the convolutional layer ($n_F = 3$).
- $\mathbf{X} = \{X_1, X_2, X_3\}$ and each X_i has size $w \times h$
- Each $\mathbf{F}_i = \{F_{i1}, F_{i2}, F_{i3}\}$ and has size $f \times f \times 3$
- Have omitted the bias weights for clarity.

Computational Graph for our simple network



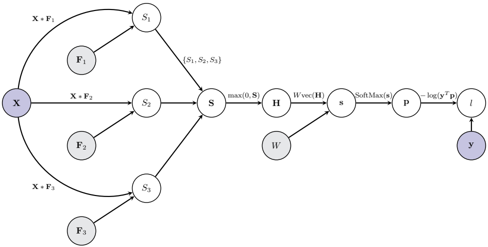
From previous lectures know that

$$\frac{\partial l}{\partial \mathbf{s}} = -(\mathbf{y} - \mathbf{p})^T$$

$$\frac{\partial l}{\partial \text{vec}(\mathbf{H})} = \frac{\partial l}{\partial \mathbf{s}} W$$

$$\frac{\partial l}{\partial \text{vec}(\mathbf{S})} = \frac{\partial l}{\partial \text{vec}(\mathbf{H})} \text{diag}\left(\text{Ind}(\text{vec}(\mathbf{S}) > 0)\right)$$

Computational Graph for our simple network



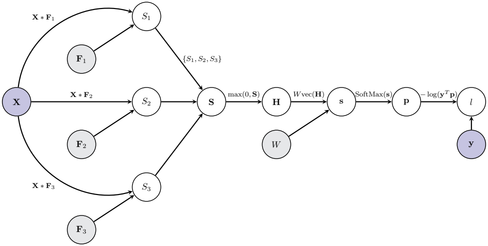
From reading the computational graph we can see that

$$\frac{\partial l}{\partial \text{vec}(\mathbf{F}_i)} = \frac{\partial l}{\partial \text{vec}(S_i)} \frac{\partial \text{vec}(S_i)}{\partial \text{vec}(\mathbf{F}_i)}$$

$$= \frac{\partial l}{\partial \text{vec}(\mathbf{S})} \frac{\partial \text{vec}(\mathbf{S})}{\partial \text{vec}(S_i)} \frac{\partial \text{vec}(S_i)}{\partial \text{vec}(\mathbf{F}_i)}$$

for $i = 1, 2, 3$.

Computational Graph for our simple network

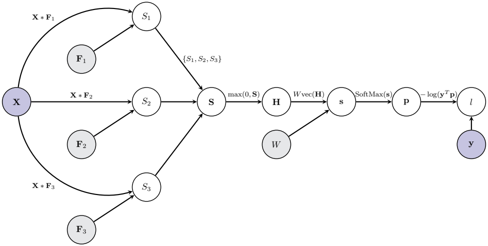


From reading the computational graph we can see that

$$\frac{\partial l}{\partial \text{vec}(F_i)} = \underbrace{\color{red}{\frac{\partial l}{\partial \text{vec}(\mathbf{S})}}}_{\color{red}{\uparrow \text{ already know}}} \frac{\partial \text{vec}(\mathbf{S})}{\partial \text{vec}(S_i)} \frac{\partial \text{vec}(S_i)}{\partial \text{vec}(\mathbf{F}_i)}$$

for $i = 1, 2, 3$.

Computational Graph for our simple network



From reading the computational graph we can see that

$$\frac{\partial l}{\partial \text{vec}(\mathbf{F}_i)} = \frac{\partial l}{\partial \text{vec}(\mathbf{S})} \underbrace{\frac{\partial \text{vec}(\mathbf{S})}{\partial \text{vec}(S_i)}}_{\substack{\uparrow \\ \text{calculate now}}} \frac{\partial \text{vec}(S_i)}{\partial \text{vec}(\mathbf{F}_i)}$$

for $i = 1, 2, 3$.

Jacobian of vec(S) w.r.t. vec(S_i)

- Have $\mathbf{S} = \{S_1, S_2, S_3\} \implies$

$$\text{vec}(\mathbf{S}) = \begin{pmatrix} \text{vec}(S_1) \\ \text{vec}(S_2) \\ \text{vec}(S_3) \end{pmatrix}$$

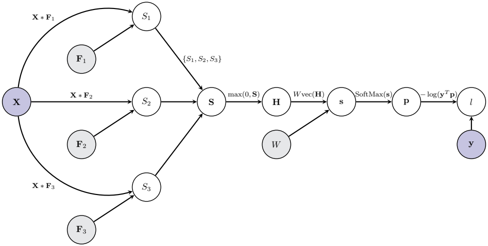
- Then

$$\frac{\partial \text{vec}(\mathbf{S})}{\partial \text{vec}(S_1)} = \begin{pmatrix} I_t \\ 0_{t\times t} \\ 0_{t\times t} \end{pmatrix}, \quad \frac{\partial \text{vec}(\mathbf{S})}{\partial \text{vec}(S_2)} = \begin{pmatrix} 0_{t\times t} \\ I_t \\ 0_{t\times t} \end{pmatrix}, \quad \frac{\partial \text{vec}(\mathbf{S})}{\partial \text{vec}(S_3)} = \begin{pmatrix} 0_{t\times t} \\ 0_{t\times t} \\ I_t \end{pmatrix}$$

where $t = (w - f + 1) \times (h - f + 1)$ and each $0_{t\times t}$ denotes a matrix of zeros of size $t \times t$.

- Each $\frac{\partial \text{vec}(\mathbf{S})}{\partial \text{vec}(S_i)}$ has size $3t \times t$

Computational Graph for our simple network



From reading the computational graph we can see that

$$\frac{\partial l}{\partial \text{vec}(\mathbf{F}_i)} = \frac{\partial l}{\partial \text{vec}(\mathbf{S})} \frac{\partial \text{vec}(\mathbf{S})}{\partial \text{vec}(S_i)} \underbrace{\frac{\partial \text{vec}(S_i)}{\partial \text{vec}(\mathbf{F}_i)}}_{\uparrow \text{ calculate now}}$$

for $i = 1, 2, 3$.

Jacobian of vec(S_i) w.r.t. vec($\mathbf{F}_i$)

- Have for $i = 1, 2, 3$:

$$S_i = \mathbf{X} * \mathbf{F}_i$$

- Can write a convolution (in a very memory in-efficient way) as a matrix multiplication

$$\text{vec}(S_i) = M_{\mathbf{X}}\, \text{vec}(\mathbf{F}_i)$$

- $M_{\mathbf{X}}$ has size $(w - f + 1)(h - f + 1) \times (3f^2)$

- What are the entries of $M_{\mathbf{X}}$?

Writing a convolution as a matrix multiplication

Simple Example

- Have an input image X of size 6×6.
- Have a filter F of size 3×3.
- Convolve X by F gives a response map of size 4×4

$$S = X * F$$

- Each entry of S can be written as

$$S_{lm} = \sum_{i=1}^{3} \sum_{j=1}^{3} X_{l+i-1,m+j-1} F_{ij}$$

Writing a convolution as a matrix multiplication

Simple Example

Want to write this convolution as a matrix multiplication:

$$S = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix} * \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

One Solution:

$$S_{11} = \begin{pmatrix} \color{red}{X_{11}} & \color{red}{X_{12}} & \color{red}{X_{13}} & \color{red}{X_{21}} & \color{red}{X_{22}} & \color{red}{X_{23}} & \color{red}{X_{31}} & \color{red}{X_{32}} & \color{red}{X_{33}} \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} \quad \text{new row corresponds to} \begin{pmatrix} \color{red}{X_{11}} & \color{red}{X_{12}} & \color{red}{X_{13}} & X_{14} & X_{15} & X_{16} \\ \color{red}{X_{21}} & \color{red}{X_{22}} & \color{red}{X_{23}} & X_{24} & X_{25} & X_{26} \\ \color{red}{X_{31}} & \color{red}{X_{32}} & \color{red}{X_{33}} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}$$

Writing a convolution as a matrix multiplication

Simple Example

Want to write this convolution as a matrix multiplication:

$$S = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix} * \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

One Solution:

$$\begin{pmatrix} S_{11} \\ S_{12} \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} \\ X_{12} & X_{13} & X_{14} & X_{22} & X_{23} & X_{24} & X_{32} & X_{33} & X_{34} \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix}$$

new row corresponds to $\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}$

Writing a convolution as a matrix multiplication

Simple Example

Want to write this convolution as a matrix multiplication:

$$S = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix} * \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

One Solution:

$$\begin{pmatrix} S_{11} \\ S_{12} \\ S_{13} \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} \\ X_{12} & X_{13} & X_{14} & X_{22} & X_{23} & X_{24} & X_{32} & X_{33} & X_{34} \\ X_{13} & X_{14} & X_{15} & X_{23} & X_{24} & X_{25} & X_{33} & X_{34} & X_{35} \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix}$$

new row corresponds to $\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}$

Simple Example

Want to write this convolution as a matrix multiplication:

$$S = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix} * \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

One Solution:

$$\begin{pmatrix} S_{11} \\ S_{12} \\ S_{13} \\ S_{14} \end{pmatrix} = \begin{pmatrix} \color{red}{X_{11}} & \color{red}{X_{12}} & \color{red}{X_{13}} & \color{red}{X_{21}} & \color{red}{X_{22}} & \color{red}{X_{23}} & \color{red}{X_{31}} & \color{red}{X_{32}} & \color{red}{X_{33}} \\ \color{cyan}{X_{12}} & \color{cyan}{X_{13}} & \color{cyan}{X_{14}} & \color{cyan}{X_{22}} & \color{cyan}{X_{23}} & \color{cyan}{X_{24}} & \color{cyan}{X_{32}} & \color{cyan}{X_{33}} & \color{cyan}{X_{34}} \\ \color{green}{X_{13}} & \color{green}{X_{14}} & \color{green}{X_{15}} & \color{green}{X_{23}} & \color{green}{X_{24}} & \color{green}{X_{25}} & \color{green}{X_{33}} & \color{green}{X_{34}} & \color{green}{X_{35}} \\ \color{purple}{X_{14}} & \color{purple}{X_{15}} & \color{purple}{X_{16}} & \color{purple}{X_{24}} & \color{purple}{X_{25}} & \color{purple}{X_{26}} & \color{purple}{X_{34}} & \color{purple}{X_{35}} & \color{purple}{X_{36}} \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} \quad \text{new row corresponds to} \begin{pmatrix} X_{11} & X_{12} & X_{13} & \color{purple}{X_{14}} & \color{purple}{X_{15}} & \color{purple}{X_{16}} \\ X_{21} & X_{22} & X_{23} & \color{purple}{X_{24}} & \color{purple}{X_{25}} & \color{purple}{X_{26}} \\ X_{31} & X_{32} & X_{33} & \color{purple}{X_{34}} & \color{purple}{X_{35}} & \color{purple}{X_{36}} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}$$

Writing a convolution as a matrix multiplication

Simple Example

Want to write this convolution as a matrix multiplication:

$$S = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix} * \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

One Solution:

$$\begin{pmatrix} S_{11} \\ S_{12} \\ S_{13} \\ S_{14} \\ S_{15} \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} \\ X_{12} & X_{13} & X_{14} & X_{22} & X_{23} & X_{24} & X_{32} & X_{33} & X_{34} \\ X_{13} & X_{14} & X_{15} & X_{23} & X_{24} & X_{25} & X_{33} & X_{34} & X_{35} \\ X_{14} & X_{15} & X_{16} & X_{24} & X_{25} & X_{26} & X_{34} & X_{35} & X_{36} \\ X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} & X_{41} & X_{42} & X_{43} \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix}$$

new row corresponds to $\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}$

Writing a convolution as a matrix multiplication

Simple Example

Want to write this convolution as a matrix multiplication:

$$S = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix} * \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

One Solution:

$$\begin{pmatrix} S_{11} \\ S_{12} \\ S_{13} \\ S_{14} \\ S_{21} \\ \vdots \\ \vdots \\ S_{44} \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} \\ X_{12} & X_{13} & X_{14} & X_{22} & X_{23} & X_{24} & X_{32} & X_{33} & X_{34} \\ X_{13} & X_{14} & X_{15} & X_{23} & X_{24} & X_{25} & X_{33} & X_{34} & X_{35} \\ X_{14} & X_{15} & X_{16} & X_{24} & X_{25} & X_{26} & X_{34} & X_{35} & X_{36} \\ X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} & X_{41} & X_{42} & X_{43} \\ & & & & \vdots & & & & \\ & & & & \vdots & & & & \\ X_{44} & X_{45} & X_{46} & X_{54} & X_{55} & X_{56} & X_{64} & X_{65} & X_{66} \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix}$$

new row corresponds to $\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}$

Writing a convolution as a matrix multiplication

Simple Example

Want to write this convolution as a matrix multiplication:

$$S = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix} * \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

One Solution:

$$\begin{pmatrix} S_{11} \\ S_{12} \\ S_{13} \\ S_{14} \\ S_{21} \\ \vdots \\ \vdots \\ S_{44} \end{pmatrix} = \underbrace{\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} \\ X_{12} & X_{13} & X_{14} & X_{22} & X_{23} & X_{24} & X_{32} & X_{33} & X_{34} \\ X_{13} & X_{14} & X_{15} & X_{23} & X_{24} & X_{25} & X_{33} & X_{34} & X_{35} \\ X_{14} & X_{15} & X_{16} & X_{24} & X_{25} & X_{26} & X_{34} & X_{35} & X_{36} \\ X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} & X_{41} & X_{42} & X_{43} \\ & & & & \vdots & & & & \\ & & & & \vdots & & & & \\ X_{44} & X_{45} & X_{46} & X_{54} & X_{55} & X_{56} & X_{64} & X_{65} & X_{66} \end{pmatrix}}_{M_X \text{ size } 16 \times 9} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix}$$

new row corresponds to $\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}$

$$\text{vec}(S) = M_X \, \text{vec}(F)$$

Questions??

Multiple planes: Convolution $\rightarrow$ Matrix multiplication

- What about when $\mathbf{X}$ and $\mathbf{F}$ have multiple planes?
- Say $\mathbf{X} = \{X_1, X_2, X_3, X_4\}$ has size $6 \times 6 \times 4$,
- $\mathbf{F} = \{F_1, F_2, F_3, F_4\}$ has size $3 \times 3 \times 4$.
- Have

$$S = \mathbf{X} * \mathbf{F} = \sum_{i=1}^{4} X_i * F_i \qquad (S \text{ has size } 4 \times 4)$$

- Then

$$\text{vec}(S) = \sum_{i=1}^{4} M_{X_i}\,\text{vec}(F_i) = M_{\mathbf{X}}\,\text{vec}(\mathbf{F})$$

where

$$M_{\mathbf{X}} = \begin{pmatrix} M_{X_1} & M_{X_2} & M_{X_3} & M_{X_4} \end{pmatrix}, \qquad \text{vec}(\mathbf{F}) = \begin{pmatrix} \text{vec}(F_1) \\ \text{vec}(F_2) \\ \text{vec}(F_3) \\ \text{vec}(F_4) \end{pmatrix}.$$

$M_{\mathbf{X}}$ has size 16×36 and $\text{vec}(\mathbf{F})$ has size 36×1.

Back to: Jacobian of $\text{vec}(S_i)$ w.r.t. $\text{vec}(\mathbf{F}_i)$

- Have for $i = 1, 2, 3$:

$$S_i = \mathbf{X} * \mathbf{F}_i$$

- Can write a convolution (in a very memory inefficient way) as a matrix multiplication

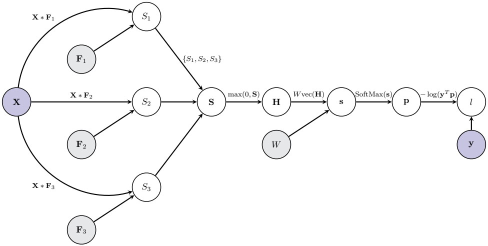
$$\text{vec}(S_i) = M_{\mathbf{X}} \text{vec}(\mathbf{F}_i)$$

- $M_{\mathbf{X}}$ has size $(w - f + 1)(h - f + 1) \times (3f^2)$

- Thus

$$\frac{\partial \text{vec}(S_i)}{\partial \text{vec}(\mathbf{F}_i)} = M_{\mathbf{X}}$$

Gradient of the loss w.r.t. $\text{vec}(\mathbf{F}_i)$



Thus

$$\frac{\partial l}{\partial \text{vec}(\mathbf{F}_1)} = \frac{\partial l}{\partial \text{vec}(\mathbf{S})} \frac{\partial \text{vec}(\mathbf{S})}{\partial \text{vec}(S_1)} \frac{\partial \text{vec}(S_1)}{\partial \text{vec}(\mathbf{F}_1)} = \frac{\partial l}{\partial \text{vec}(\mathbf{S})} \begin{pmatrix} I_t \\ 0_{t\times t} \\ 0_{t \times t} \end{pmatrix} M_{\mathbf{X}}$$

$$= \begin{pmatrix} \frac{\partial l}{\partial \text{vec}(S_1)} & \frac{\partial l}{\partial \text{vec}(S_2)} & \frac{\partial l}{\partial \text{vec}(S_3)} \end{pmatrix} \begin{pmatrix} M_{X_1} & M_{X_2} & M_{X_3} \\ 0_{t\times f^2} & 0_{t\times f^2} & 0_{t\times f^2} \\ 0_{t\times f^2} & 0_{t\times f^2} & 0_{t\times f^2} \end{pmatrix}$$

$$= \left(\frac{\partial l}{\partial \text{vec}(S_1)} M_{X_1}, \frac{\partial l}{\partial \text{vec}(S_1)} M_{X_2}, \frac{\partial l}{\partial \text{vec}(S_1)} M_{X_3} \right)$$

Gradient of the loss w.r.t. $\mathbf{F}_i$

- May want expression for $\frac{\partial l}{\partial \mathbf{F}_i}$ instead of $\frac{\partial l}{\partial \text{vec}(\mathbf{F}_i)}$.
- **Option 1**:

  Reshape $\frac{\partial l}{\partial \text{vec}(\mathbf{F}_i)}$ $\big($size $1 \times 3f^2\big)$ to $\frac{\partial l}{\partial \mathbf{F}_i}$ $\big($size $f \times f \times 3\big)$.

Gradient of the loss w.r.t. $\mathbf{F}_i$

- May want expression for $\frac{\partial l}{\partial \mathbf{F}_i}$ instead of $\frac{\partial l}{\partial \text{vec}(\mathbf{F}_i)}$.
- **Option 2**:

  Return to our simple example ...

Return to Simple Example

Consider the case

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_9 \end{pmatrix} = \begin{pmatrix} g_1 & g_2 & \cdots & g_{16} \end{pmatrix} \underbrace{\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} \\ X_{12} & X_{13} & X_{14} & X_{22} & X_{23} & X_{24} & X_{32} & X_{33} & X_{34} \\ X_{13} & X_{14} & X_{15} & X_{23} & X_{24} & X_{25} & X_{33} & X_{34} & X_{35} \\ X_{14} & X_{15} & X_{16} & X_{24} & X_{25} & X_{26} & X_{34} & X_{35} & X_{36} \\ X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} & X_{41} & X_{42} & X_{43} \\ & & & & \vdots & & & & \\ & & & & \vdots & & & & \\ X_{44} & X_{45} & X_{46} & X_{54} & X_{55} & X_{56} & X_{64} & X_{65} & X_{66} \end{pmatrix}}_{M_X \text{ size } 16 \times 9}$$

Gradient of loss w.r.t. F as opposed to vec(F)

Return to Simple Example

Consider the case

$$
\begin{pmatrix} {\color{red}v_1} & v_2 & \cdots & v_9 \end{pmatrix} = \begin{pmatrix} g_1 & g_2 & \cdots & g_{16} \end{pmatrix} \underbrace{\begin{pmatrix}
{\color{red}X_{11}} & X_{12} & X_{13} & X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} \\
{\color{red}X_{12}} & X_{13} & X_{14} & X_{22} & X_{23} & X_{24} & X_{32} & X_{33} & X_{34} \\
{\color{red}X_{13}} & X_{14} & X_{15} & X_{23} & X_{24} & X_{25} & X_{33} & X_{34} & X_{35} \\
{\color{red}X_{14}} & X_{15} & X_{16} & X_{24} & X_{25} & X_{26} & X_{34} & X_{35} & X_{36} \\
{\color{red}X_{21}} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} & X_{41} & X_{42} & X_{43} \\
 & & & & \vdots & & & & \\
 & & & & \vdots & & & & \\
{\color{red}X_{44}} & X_{45} & X_{46} & X_{54} & X_{55} & X_{56} & X_{64} & X_{65} & X_{66}
\end{pmatrix}}_{M_X \text{ size } 16 \times 9}
$$

where red column in M_X corresponds to this red block in X

$$
\begin{pmatrix}
{\color{red}X_{11}} & {\color{red}X_{12}} & {\color{red}X_{13}} & {\color{red}X_{14}} & X_{15} & X_{16} \\
{\color{red}X_{21}} & {\color{red}X_{22}} & {\color{red}X_{23}} & {\color{red}X_{24}} & X_{25} & X_{26} \\
{\color{red}X_{31}} & {\color{red}X_{32}} & {\color{red}X_{33}} & {\color{red}X_{34}} & X_{35} & X_{36} \\
{\color{red}X_{41}} & {\color{red}X_{42}} & {\color{red}X_{43}} & {\color{red}X_{44}} & X_{45} & X_{46} \\
X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\
X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66}
\end{pmatrix}
$$

Gradient of loss w.r.t. F as opposed to vec(F)

Return to Simple Example

Consider the case

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_9 \end{pmatrix} = \begin{pmatrix} g_1 & g_2 & \cdots & g_{16} \end{pmatrix} \underbrace{\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} \\ X_{12} & X_{13} & X_{14} & X_{22} & X_{23} & X_{24} & X_{32} & X_{33} & X_{34} \\ X_{13} & X_{14} & X_{15} & X_{23} & X_{24} & X_{25} & X_{33} & X_{34} & X_{35} \\ X_{14} & X_{15} & X_{16} & X_{24} & X_{25} & X_{26} & X_{34} & X_{35} & X_{36} \\ X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} & X_{41} & X_{42} & X_{43} \\ & & & & \vdots & & & & \\ & & & & \vdots & & & & \\ X_{44} & X_{45} & X_{46} & X_{54} & X_{55} & X_{56} & X_{64} & X_{65} & X_{66} \end{pmatrix}}_{M_X \text{ size } 16 \times 9}$$

where red column in M_X corresponds to this red block in X

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}$$

Gradient of loss w.r.t. F as opposed to vec(F)

Return to Simple Example

Consider the case

$$(v_1 \quad v_2 \quad {\color{red}v_3} \quad \cdots \quad v_9) = (g_1 \quad g_2 \quad \cdots \quad g_{16}) \underbrace{\begin{pmatrix} X_{11} & X_{12} & {\color{red}X_{13}} & X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} \\ X_{12} & X_{13} & {\color{red}X_{14}} & X_{22} & X_{23} & X_{24} & X_{32} & X_{33} & X_{34} \\ X_{13} & X_{14} & {\color{red}X_{15}} & X_{23} & X_{24} & X_{25} & X_{33} & X_{34} & X_{35} \\ X_{14} & X_{15} & {\color{red}X_{16}} & X_{24} & X_{25} & X_{26} & X_{34} & X_{35} & X_{36} \\ X_{21} & X_{22} & {\color{red}X_{23}} & X_{31} & X_{32} & X_{33} & X_{41} & X_{42} & X_{43} \\ & & & & \vdots & & & & \\ & & & & \vdots & & & & \\ X_{44} & X_{45} & {\color{red}X_{46}} & X_{54} & X_{55} & X_{56} & X_{64} & X_{65} & X_{66} \end{pmatrix}}_{M_X \text{ size } 16 \times 9}$$

where red column in M_X corresponds to this red block in X

$$\begin{pmatrix} X_{11} & X_{12} & {\color{red}X_{13}} & {\color{red}X_{14}} & {\color{red}X_{15}} & {\color{red}X_{16}} \\ X_{21} & X_{22} & {\color{red}X_{23}} & {\color{red}X_{24}} & {\color{red}X_{25}} & {\color{red}X_{26}} \\ X_{31} & X_{32} & {\color{red}X_{33}} & {\color{red}X_{34}} & {\color{red}X_{35}} & {\color{red}X_{36}} \\ X_{41} & X_{42} & {\color{red}X_{43}} & {\color{red}X_{44}} & {\color{red}X_{45}} & {\color{red}X_{46}} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}$$

Gradient of loss w.r.t. F as opposed to vec(F)

Return to Simple Example

Consider the case

$$\begin{pmatrix} v_1 & v_2 & v_3 & \color{red}{v_4} & \cdots & v_9 \end{pmatrix} = \begin{pmatrix} g_1 & g_2 & \cdots & g_{16} \end{pmatrix} \underbrace{\begin{pmatrix} X_{11} & X_{12} & X_{13} & \color{red}{X_{21}} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} \\ X_{12} & X_{13} & X_{14} & \color{red}{X_{22}} & X_{23} & X_{24} & X_{32} & X_{33} & X_{34} \\ X_{13} & X_{14} & X_{15} & \color{red}{X_{23}} & X_{24} & X_{25} & X_{33} & X_{34} & X_{35} \\ X_{14} & X_{15} & X_{16} & \color{red}{X_{24}} & X_{25} & X_{26} & X_{34} & X_{35} & X_{36} \\ X_{21} & X_{22} & X_{23} & \color{red}{X_{31}} & X_{32} & X_{33} & X_{41} & X_{42} & X_{43} \\ & & & & \vdots & & & & \\ & & & & \vdots & & & & \\ X_{44} & X_{45} & X_{46} & \color{red}{X_{54}} & X_{55} & X_{56} & X_{64} & X_{65} & X_{66} \end{pmatrix}}_{M_X \text{ size } 16 \times 9}$$

where red column in M_X corresponds to this red block in X

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ \color{red}{X_{21}} & \color{red}{X_{22}} & \color{red}{X_{23}} & \color{red}{X_{24}} & X_{25} & X_{26} \\ \color{red}{X_{31}} & \color{red}{X_{32}} & \color{red}{X_{33}} & \color{red}{X_{34}} & X_{35} & X_{36} \\ \color{red}{X_{41}} & \color{red}{X_{42}} & \color{red}{X_{43}} & \color{red}{X_{44}} & X_{45} & X_{46} \\ \color{red}{X_{51}} & \color{red}{X_{52}} & \color{red}{X_{53}} & \color{red}{X_{54}} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}$$

Gradient of loss w.r.t. F as opposed to vec(F)

Return to Simple Example

$$\cdots$$

Gradient of loss w.r.t. F as opposed to vec(F)

Return to Simple Example

Consider the case

$$
\begin{pmatrix} v_1 & v_2 & v_3 & v_4 & \cdots & {\color{red}v_9} \end{pmatrix} = \begin{pmatrix} g_1 & g_2 & \cdots & g_{16} \end{pmatrix} \underbrace{\begin{pmatrix}
X_{11} & X_{12} & X_{13} & X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & {\color{red}X_{33}} \\
X_{12} & X_{13} & X_{14} & X_{22} & X_{23} & X_{24} & X_{32} & X_{33} & {\color{red}X_{34}} \\
X_{13} & X_{14} & X_{15} & X_{23} & X_{24} & X_{25} & X_{33} & X_{34} & {\color{red}X_{35}} \\
X_{14} & X_{15} & X_{16} & X_{24} & X_{25} & X_{26} & X_{34} & X_{35} & {\color{red}X_{36}} \\
X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} & X_{41} & X_{42} & {\color{red}X_{43}} \\
 & & & & \vdots & & & & \\
 & & & & \vdots & & & & \\
X_{44} & X_{45} & X_{46} & X_{54} & X_{55} & X_{56} & X_{64} & X_{65} & {\color{red}X_{66}}
\end{pmatrix}}_{M_X \text{ size } 16\times 9}
$$

where red column in M_X corresponds to this red block in X

$$
\begin{pmatrix}
X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\
X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\
X_{31} & X_{32} & {\color{red}X_{33}} & {\color{red}X_{34}} & {\color{red}X_{35}} & {\color{red}X_{36}} \\
X_{41} & X_{42} & {\color{red}X_{43}} & {\color{red}X_{44}} & {\color{red}X_{45}} & {\color{red}X_{46}} \\
X_{51} & X_{52} & {\color{red}X_{53}} & {\color{red}X_{54}} & {\color{red}X_{55}} & {\color{red}X_{56}} \\
X_{61} & X_{62} & {\color{red}X_{63}} & {\color{red}X_{64}} & {\color{red}X_{65}} & {\color{red}X_{66}}
\end{pmatrix}
$$

Gradient of loss w.r.t. F as opposed to vec(F)

Return to Simple Example

Consider the case

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_4 & \cdots & \color{red}{v_9} \end{pmatrix} = \begin{pmatrix} g_1 & g_2 & \cdots & g_{16} \end{pmatrix} \underbrace{\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & \color{red}{X_{33}} \\ X_{12} & X_{13} & X_{14} & X_{22} & X_{23} & X_{24} & X_{32} & X_{33} & \color{red}{X_{34}} \\ X_{13} & X_{14} & X_{15} & X_{23} & X_{24} & X_{25} & X_{33} & X_{34} & \color{red}{X_{35}} \\ X_{14} & X_{15} & X_{16} & X_{24} & X_{25} & X_{26} & X_{34} & X_{35} & \color{red}{X_{36}} \\ X_{21} & X_{22} & X_{23} & X_{31} & X_{32} & X_{33} & X_{41} & X_{42} & \color{red}{X_{43}} \\ & & & & \vdots & & & & \\ & & & & \vdots & & & & \\ X_{44} & X_{45} & X_{46} & X_{54} & X_{55} & X_{56} & X_{64} & X_{65} & \color{red}{X_{66}} \end{pmatrix}}_{M_X \text{ size } 16 \times 9}$$

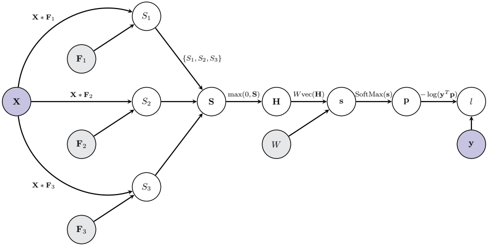
where red column in M_X corresponds to this red block in X

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & \color{red}{X_{33}} & \color{red}{X_{34}} & \color{red}{X_{35}} & \color{red}{X_{36}} \\ X_{41} & X_{42} & \color{red}{X_{43}} & \color{red}{X_{44}} & \color{red}{X_{45}} & \color{red}{X_{46}} \\ X_{51} & X_{52} & \color{red}{X_{53}} & \color{red}{X_{54}} & \color{red}{X_{55}} & \color{red}{X_{56}} \\ X_{61} & X_{62} & \color{red}{X_{63}} & \color{red}{X_{64}} & \color{red}{X_{65}} & \color{red}{X_{66}} \end{pmatrix}$$

Thus

$$\begin{pmatrix} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \\ v_7 & v_8 & v_9 \end{pmatrix} = X * \begin{pmatrix} g_1 & g_2 & g_3 & g_4 \\ g_5 & g_6 & g_7 & g_8 \\ g_9 & g_{10} & g_{11} & g_{12} \\ g_{13} & g_{14} & g_{15} & g_{16} \end{pmatrix}$$

Back to Gradient of the loss w.r.t. $\mathbf{F}_i$



Know

$$\frac{\partial l}{\partial \text{vec}(\mathbf{F}_i)} = \left(\frac{\partial l}{\partial \text{vec}(S_i)} M_{X_1}, \frac{\partial l}{\partial \text{vec}(S_i)} M_{X_2}, \frac{\partial l}{\partial \text{vec}(S_i)} M_{X_3} \right)$$

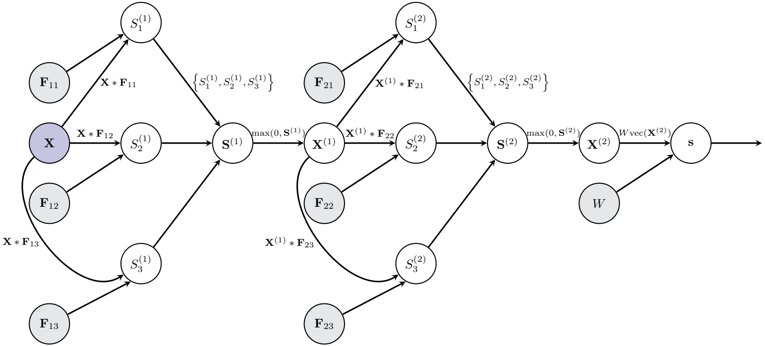
but our simple example $\implies$

$$\frac{\partial l}{\partial \mathbf{F}_i} = \left\{ X_1 * \frac{\partial l}{\partial S_i}, X_2 * \frac{\partial l}{\partial S_i}, X_3 * \frac{\partial l}{\partial S_i} \right\}$$

Questions??

Gradient Computations for two Convolutional layers

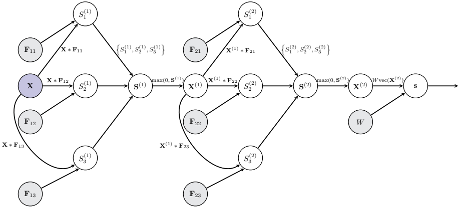
Computational Graph: two convolutional layers



Notes about the figure

- Apply 3 filters at each convolutional layer.
- Have omitted the bias weights for clarity.

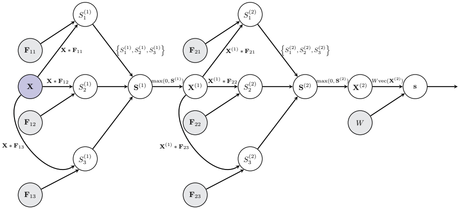
How do we back-propagate the gradient to node $\mathbf{X}^{(1)}$?



- Children of node $\mathbf{X}^{(1)}$ are $S_1^{(2)}, S_2^{(2)}$ and $S_3^{(2)}$
- Thus

$$\frac{\partial l}{\partial \text{vec}(\mathbf{X}^{(1)})} = \sum_{i=1}^{3} \frac{\partial l}{\partial \text{vec}(S_i^{(2)})} \frac{\partial \text{vec}(S_i^{(2)})}{\partial \text{vec}(\mathbf{X}^{(1)})}$$

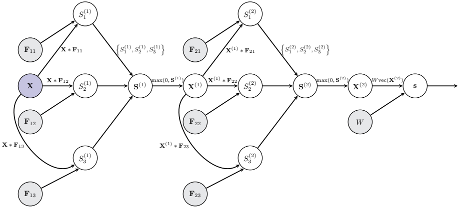
How do we back-propagate the gradient to node $\mathbf{X}^{(1)}$?



- Children of node $\mathbf{X}^{(1)}$ are $S_1^{(2)}, S_2^{(2)}$ and $S_3^{(2)}$
- Thus

$$\frac{\partial l}{\partial \text{vec}(\mathbf{X}^{(1)})} = \sum_{i=1}^{3} \frac{\partial l}{\underset{\substack{\uparrow \\ \text{already know}}}{\partial \text{vec}(S_i^{(2)})}} \frac{\partial \text{vec}(S_i^{(2)})}{\partial \text{vec}(\mathbf{X}^{(1)})}$$

How do we back-propagate the gradient to node $\mathbf{X}^{(1)}$?



- Children of node $\mathbf{X}^{(1)}$ are $S_1^{(2)}, S_2^{(2)}$ and $S_3^{(2)}$
- Thus

$$\frac{\partial l}{\partial \mathrm{vec}(\mathbf{X}^{(1)})} = \sum_{i=1}^{3} \frac{\partial l}{\partial \mathrm{vec}(S_i^{(2)})} \underbrace{\frac{\partial \mathrm{vec}(S_i^{(2)})}{\partial \mathrm{vec}(\mathbf{X}^{(1)})}}_{\uparrow \text{ calculate now}}$$

Jacobian of vec($S_i^{(2)}$) w.r.t. vec($\mathbf{X}^{(1)}$)

- Have for $i = 1, 2, 3$:

$$S_i^{(2)} = \mathbf{X}^{(1)} * \mathbf{F}_{2i}$$

- Can write a convolution (in a very memory in-efficient way) as a matrix multiplication

$$\text{vec}(S_i^{(2)}) = M^{\text{filter}}_{\mathbf{F}_{2i}} \, \text{vec}(\mathbf{X}^{(1)})$$

- $M_{F_{2i}}$ has size $(w - f + 1)(h - f + 1) \times 3wh$ (assuming $\mathbf{X}^{(1)}$ has size $w \times h \times 3$ and $\mathbf{F}_{2i}$ has size $f \times f \times 3$.)

- What are the entries of $M^{\text{filter}}_{\mathbf{F}_{2i}}$?

Writing convolution as a matrix multiplication II

Writing convolution as a matrix multiplication II

Simple Example

- Have an input image X of size 6×6.
- Have a filter F of size 3×3.
- Convolve X by F gives a response map of size 4×4

$$S = X * F$$

- Each entry of S can be written as

$$S_{lm} = \sum_{i=1}^{3} \sum_{j=1}^{3} X_{l+i-1,m+j-1} F_{ij}$$

Simple Example
Write this convolution as a matrix multiplication involving vec(X)

$$S = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix} * \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

Solution:

$$S_{11} = \big(\overbrace{F_{11} \quad F_{12} \quad F_{13} \quad 0 \quad 0 \quad 0}^{\text{entries corresponding to row 1 of } X} \quad \overbrace{F_{21} \quad F_{22} \quad F_{23} \quad 0 \quad 0 \quad 0}^{\text{row 2 of } X} \quad \overbrace{F_{31} \quad F_{32} \quad F_{33} \quad 0 \quad 0 \quad 0}^{\text{row 3 of } X} \quad \cdots \quad \big) \, \text{vec}(X)$$

S_{11} is the dot product between F and red entries of X:

$$\begin{pmatrix} \color{red}{X_{11}} & \color{red}{X_{12}} & \color{red}{X_{13}} & X_{14} & X_{15} & X_{16} \\ \color{red}{X_{21}} & \color{red}{X_{22}} & \color{red}{X_{23}} & X_{24} & X_{25} & X_{26} \\ \color{red}{X_{31}} & \color{red}{X_{32}} & \color{red}{X_{33}} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}$$

Writing convolution as a matrix multiplication

Simple Example

Write this convolution as a matrix multiplication involving vec(X)

$$S = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix} * \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

Solution:

$$\begin{pmatrix} S_{11} \\ S_{12} \end{pmatrix} = \left(\overbrace{\begin{matrix} F_{11} & F_{12} & F_{13} & 0 & 0 & 0 \\ 0 & F_{11} & F_{12} & F_{13} & 0 & 0 \end{matrix}}^{\text{entries corresponding to row 1 of } X} \; \overbrace{\begin{matrix} F_{21} & F_{22} & F_{23} & 0 & 0 & 0 \\ 0 & F_{21} & F_{22} & F_{23} & 0 & 0 \end{matrix}}^{\text{row 2 of } X} \; \overbrace{\begin{matrix} F_{31} & F_{32} & F_{33} & 0 & 0 & 0 \\ 0 & F_{31} & F_{32} & F_{33} & 0 & 0 \end{matrix}}^{\text{row 3 of } X} \; \begin{matrix} \cdots \\ \cdots \end{matrix} \right) \text{vec}(X)$$

S_{12} is the dot product between F and red entries of X:

$$\begin{pmatrix} X_{11} & \textcolor{red}{X_{12}} & \textcolor{red}{X_{13}} & \textcolor{red}{X_{14}} & X_{15} & X_{16} \\ X_{21} & \textcolor{red}{X_{22}} & \textcolor{red}{X_{23}} & \textcolor{red}{X_{24}} & X_{25} & X_{26} \\ X_{31} & \textcolor{red}{X_{32}} & \textcolor{red}{X_{33}} & \textcolor{red}{X_{34}} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}$$

Simple Example

Write this convolution as a matrix multiplication involving vec(X)

$$S = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix} * \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

Solution:

$$\begin{pmatrix} S_{11} \\ S_{12} \\ S_{13} \end{pmatrix} = \left(\begin{array}{cccccccccccccccccccc} \overbrace{F_{11} & F_{12} & F_{13} & 0 & 0 & 0}^{\text{entries corresponding to row 1 of } X} & \overbrace{F_{21} & F_{22} & F_{23} & 0 & 0 & 0}^{\text{row 2 of } X} & \overbrace{F_{31} & F_{32} & F_{33} & 0 & 0 & 0}^{\text{row 3 of } X} & \cdots \\ 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & \cdots \\ 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & \cdots \end{array} \right) \text{vec}(X)$$

S_{13} is the dot product between F and red entries of X:

$$\begin{pmatrix} X_{11} & X_{12} & \color{red}{X_{13}} & \color{red}{X_{14}} & \color{red}{X_{15}} & X_{16} \\ X_{21} & X_{22} & \color{red}{X_{23}} & \color{red}{X_{24}} & \color{red}{X_{25}} & X_{26} \\ X_{31} & X_{32} & \color{red}{X_{33}} & \color{red}{X_{34}} & \color{red}{X_{35}} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}$$

Writing convolution as a matrix multiplication

Simple Example

Write this convolution as a matrix multiplication involving vec(X)

$$S = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix} * \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

Solution:

$$\begin{pmatrix} S_{11} \\ S_{12} \\ S_{13} \\ S_{14} \end{pmatrix} = \begin{pmatrix} \overbrace{\begin{matrix} F_{11} & F_{12} & F_{13} & 0 & 0 & 0 \\ 0 & F_{11} & F_{12} & F_{13} & 0 & 0 \\ 0 & 0 & F_{11} & F_{12} & F_{13} & 0 \\ 0 & 0 & 0 & F_{11} & F_{12} & F_{13} \end{matrix}}^{\text{entries corresponding to row 1 of } X} & \overbrace{\begin{matrix} F_{21} & F_{22} & F_{23} & 0 & 0 & 0 \\ 0 & F_{21} & F_{22} & F_{23} & 0 & 0 \\ 0 & 0 & F_{21} & F_{22} & F_{23} & 0 \\ 0 & 0 & 0 & F_{21} & F_{22} & F_{23} \end{matrix}}^{\text{row 2 of } X} & \overbrace{\begin{matrix} F_{31} & F_{32} & F_{33} & 0 & 0 & 0 \\ 0 & F_{31} & F_{32} & F_{33} & 0 & 0 \\ 0 & 0 & F_{31} & F_{32} & F_{33} & 0 \\ 0 & 0 & 0 & F_{31} & F_{32} & F_{33} \end{matrix}}^{\text{row 3 of } X} & \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix} \end{pmatrix} \text{vec}(X)$$

S_{14} is the dot product between F and red entries of X:

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & \color{red}{X_{14}} & \color{red}{X_{15}} & \color{red}{X_{16}} \\ X_{21} & X_{22} & X_{23} & \color{red}{X_{24}} & \color{red}{X_{25}} & \color{red}{X_{26}} \\ X_{31} & X_{32} & X_{33} & \color{red}{X_{34}} & \color{red}{X_{35}} & \color{red}{X_{36}} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}$$

Writing convolution as a matrix multiplication

Simple Example

Write this convolution as a matrix multiplication involving vec(X)

$$S = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix} * \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

Solution:

$$\begin{pmatrix} S_{11} \\ S_{12} \\ S_{13} \\ S_{14} \\ S_{21} \end{pmatrix} = \begin{array}{c} \overbrace{\hspace{12em}}^{\text{entries corresponding to row 1 of } X} \quad \overbrace{\hspace{12em}}^{\text{row 2 of } X} \quad \overbrace{\hspace{12em}}^{\text{row 3 of } X} \\ \begin{pmatrix} F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & 0 & \cdots \\ 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & \cdots \\ 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & \cdots \\ 0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & \cdots \end{pmatrix} \end{array} \text{vec}(X)$$

S_{21} is the dot product between F and red entries of X:

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ \color{red}{X_{21}} & \color{red}{X_{22}} & \color{red}{X_{23}} & X_{24} & X_{25} & X_{26} \\ \color{red}{X_{31}} & \color{red}{X_{32}} & \color{red}{X_{33}} & X_{34} & X_{35} & X_{36} \\ \color{red}{X_{41}} & \color{red}{X_{42}} & \color{red}{X_{43}} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix}$$

Writing convolution as a matrix multiplication

Simple Example

Write this convolution as a matrix multiplication involving vec(X)

$$S = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix} * \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

Solution:

$$\begin{array}{c}
\begin{array}{cccccccccccccccccccc}
\multicolumn{6}{c}{\overbrace{\hphantom{F_{11}\;F_{12}\;F_{13}\;0\;0\;0\qquad\qquad}}^{\text{entries corresponding to row 1 of } X}} & \multicolumn{6}{c}{\overbrace{\hphantom{F_{21}\;F_{22}\;F_{23}\;0\;0\;0\qquad\qquad}}^{\text{row 2 of } X}} & \multicolumn{6}{c}{\overbrace{\hphantom{F_{31}\;F_{32}\;F_{33}\;0\;0\;0\qquad\qquad}}^{\text{row 3 of } X}} & \\
\end{array} \\
\begin{pmatrix} S_{11} \\ S_{12} \\ S_{13} \\ S_{14} \\ S_{21} \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix}
F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & 0 & \cdots \\
0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & \cdots \\
0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & \cdots \\
0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix} \mathrm{vec}(X)
\end{array}$$

Simple Example

Write this convolution as a matrix multiplication involving vec(X)

$$S = \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix} * \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

Solution:

$$\begin{pmatrix} S_{11} \\ S_{12} \\ S_{13} \\ S_{14} \\ S_{21} \\ \vdots \\ \vdots \end{pmatrix} = \begin{array}{c} \begin{array}{ccc} \overbrace{\hphantom{F_{11}\;F_{12}\;F_{13}\;0\;0\;0\;}}^{\text{entries corresponding to row 1 of } X} & \overbrace{\hphantom{F_{21}\;F_{22}\;F_{23}\;0\;0\;0\;}}^{\text{row 2 of } X} & \overbrace{\hphantom{F_{31}\;F_{32}\;F_{33}\;0\;0\;0\;}}^{\text{row 3 of } X} \end{array} \\ \underbrace{\begin{pmatrix} F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & 0 & \cdots \\ 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & \cdots \\ 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & \cdots \\ 0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}}_{M_F^{\text{filter}}} \end{array} \text{vec}(X)$$

Thus

$$\text{vec}(S) = M_F^{\text{filter}} \, \text{vec}(X)$$

Multiple planes: Convolution $\rightarrow$ Matrix multiplication

- What about when $\mathbf{X}$ and $\mathbf{F}$ have multiple planes?
- $\mathbf{X} = \{X_1, X_2, X_3, X_4\}$ has size $6 \times 6 \times 4$
- $\mathbf{F} = \{F_1, F_2, F_3, F_4\}$ has size $3 \times 3 \times 4$
- Have

$$S = \mathbf{X} * \mathbf{F} = \sum_{i=1}^{4} X_i * F_i$$

- Then

$$\text{vec}(S) = \sum_{i=1}^{4} M_{F_i}^{\text{filter}} \text{vec}(X_i) = M_{\mathbf{F}}^{\text{filter}} \text{vec}(\mathbf{X})$$

where

$$M_{\mathbf{F}}^{\text{filter}} = \begin{pmatrix} M_{F_1}^{\text{filter}} & M_{F_2}^{\text{filter}} & M_{F_3}^{\text{filter}} & M_{F_4}^{\text{filter}} \end{pmatrix}, \quad \text{vec}(\mathbf{X}) = \begin{pmatrix} \text{vec}(X_1) \\ \text{vec}(X_2) \\ \text{vec}(X_3) \\ \text{vec}(X_4) \end{pmatrix}.$$

$M_{\mathbf{F}}^{\text{filter}}$ has size 16×144 and $\text{vec}(\mathbf{X})$ size 144×1

Back to: Jacobian of vec$(S_i^{(2)})$ w.r.t. vec$(\mathbf{X}^{(1)})$

- Have for $i = 1, 2, 3$:

$$S_i^{(2)} = \mathbf{X}^{(1)} * \mathbf{F}_{2i}$$

- Can write a convolution (in a very memory in-efficient way) as a matrix multiplication

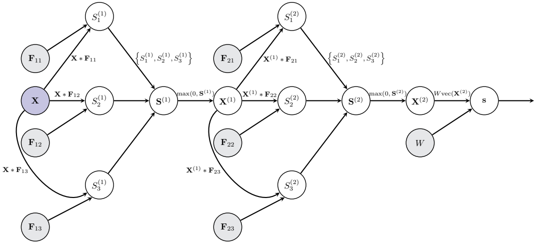
$$\text{vec}(S_i^{(2)}) = M_{\mathbf{F}_{2i}}^{\text{filter}} \text{vec}(\mathbf{X}^{(1)})$$

- $M_{\mathbf{F}_{2i}}^{\text{filter}}$ has size $(w' - f + 1)(h' - f + 1) \times 3w'h'$ (where $w' = w - f + 1$ and $h' = h - f + 1$).

- Thus

$$\frac{\partial \text{vec}(S_i^{(2)})}{\partial \text{vec}(\mathbf{X}^{(1)})} = M_{\mathbf{F}_{2i}}^{\text{filter}}$$

Gradient of the loss w.r.t. vec($\mathbf{X}^{(1)}$)



- Thus

$$\frac{\partial l}{\partial \text{vec}(\mathbf{X}^{(1)})} = \sum_{i=1}^{3} \frac{\partial l}{\partial \text{vec}(S_i^{(2)})} \frac{\partial \text{vec}(S_i^{(2)})}{\partial \text{vec}(\mathbf{X}^{(1)})}$$

$$= \sum_{i=1}^{3} \frac{\partial l}{\partial \text{vec}(S_i^{(2)})} M_{\mathbf{F}_{2i}}^{\text{filter}}$$

Gradient of the loss w.r.t. $\mathbf{X}^{(1)}$

- May want expression for $\frac{\partial l}{\partial \mathbf{X}^{(1)}}$ instead of $\frac{\partial l}{\partial \text{vec}(\mathbf{X}^{(1)})}$.
- **Option 1**:

  Reshape $\frac{\partial l}{\partial \text{vec}(\mathbf{X}^{(1)})}$ $\big($size $1 \times 3w'h'\big)$ to $\frac{\partial l}{\partial \mathbf{X}^{(1)}}$ $\big($size $w' \times h' \times 3\big)$.

Gradient of the loss w.r.t. $\mathbf{X}^{(1)}$

- May want expression for $\frac{\partial l}{\partial \mathbf{X}^{(1)}}$ instead of $\frac{\partial l}{\partial \text{vec}(\mathbf{X}^{(1)})}$.
- **Option 2**:

  Return to our simple example ...

Remember the simple example

- Have an input image X of size 6×6.
- Have a filter F of size 3×3.
- Convolve X by F gives a response map of size 4×4

$$S = X * F$$

- Each entry of S can be written as

$$S_{lm} = \sum_{i=1}^{3} \sum_{j=1}^{3} X_{l+i-1,m+j-1} F_{ij}$$

- Can re-write the convolution as a matrix multiplication

$$\text{vec}(S) = M_F^{\text{filter}} \text{vec}(X)$$

Propagate the gradient through the convolution

- Have input image X size 6×6 and filter F size 3×3
- Convolve X with F. Can write as

$$\text{vec}(S) = M_F^{\text{filter}} \text{vec}(X)$$

- If this operation part of some network, then the gradient of the loss w.r.t. $\text{vec}(X)$ is given by

$$\frac{\partial l}{\partial \text{vec}(X)} = \frac{\partial l}{\partial \text{vec}(S)} \frac{\partial \text{vec}(S)}{\partial \text{vec}(X)} = \frac{\partial l}{\partial \text{vec}(S)} M_F^{\text{filter}}$$

- Streamlining notation let

$$\mathbf{v} = \mathbf{g}\, M_F^{\text{filter}}$$

where

$$\mathbf{v} = \frac{\partial l}{\partial \text{vec}(X)}, \qquad \mathbf{g} = \frac{\partial l}{\partial \text{vec}(S)}$$

Examine back-prop eqn thru the convolution in more detail

Have

$$
(v_1 \quad v_2 \quad \dots \quad v_{36}) = (g_1 \quad g_2 \quad \dots \quad g_{16})
\begin{pmatrix}
F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & 0 & \cdots \\
0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & \cdots \\
0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & \cdots \\
0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
$$

Examine back-prop eqn thru the convolution in more detail

$$
\begin{pmatrix} v_1 & v_2 & \cdots & v_{36} \end{pmatrix} = \begin{pmatrix} g_1 & g_2 & \cdots & g_{16} \end{pmatrix}
\begin{pmatrix}
F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & 0 & \cdots \\
0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & \cdots \\
0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & \cdots \\
0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
$$

$$v_1 = g_1 F_{11}$$

Examine back-prop eqn thru the convolution in more detail

$$(v_1 \quad \color{red}{v_2} \quad \cdots \quad v_{36}) = (g_1 \quad g_2 \quad \cdots \quad g_{16}) \begin{pmatrix} F_{11} & \color{red}{F_{12}} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & 0 & \cdots \\ 0 & \color{red}{F_{11}} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & \cdots \\ 0 & \color{red}{0} & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & \cdots \\ 0 & \color{red}{0} & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & \cdots \\ 0 & \color{red}{0} & 0 & 0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & \cdots \\ \vdots & \color{red}{\vdots} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \color{red}{\vdots} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$v_1 = g_1 F_{11}$$

$$\color{red}{v_2 = g_1 F_{12} + g_2 F_{11}}$$

Examine back-prop eqn thru the convolution in more detail

$$
(v_1 \quad v_2 \quad \ldots \quad v_{36}) = (g_1 \quad g_2 \quad \ldots \quad g_{16})
\begin{pmatrix}
F_{11} & F_{12} & \color{red}{F_{13}} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & 0 & \cdots \\
0 & F_{11} & \color{red}{F_{12}} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & \cdots \\
0 & 0 & \color{red}{F_{11}} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & \cdots \\
0 & 0 & \color{red}{0} & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & \cdots \\
0 & 0 & \color{red}{0} & 0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \color{red}{\vdots} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \color{red}{\vdots} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
$$

$$
\begin{aligned}
v_1 &= g_1 F_{11} \\
v_2 &= g_1 F_{12} + g_2 F_{11} \\
\color{red}{v_3} &\color{red}{= g_1 F_{13} + g_2 F_{12} + g_3 F_{11}}
\end{aligned}
$$

Examine back-prop eqn thru the convolution in more detail

$$
(v_1 \quad v_2 \quad \cdots \quad v_{36}) = (g_1 \quad g_2 \quad \cdots \quad g_{16})
\begin{pmatrix}
F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & 0 & \cdots \\
0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & \cdots \\
0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & \cdots \\
0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
$$

$$
\begin{aligned}
v_1 &= g_1 F_{11} \\
v_2 &= g_1 F_{12} + g_2 F_{11} \\
v_3 &= g_1 F_{13} + g_2 F_{12} + g_3 F_{11} \\
v_4 &= g_2 F_{13} + g_3 F_{12} + g_4 F_{11}
\end{aligned}
$$

Examine back-prop eqn thru the convolution in more detail

$$
\begin{pmatrix} v_1 & v_2 & \cdots & v_{36} \end{pmatrix} = \begin{pmatrix} g_1 & g_2 & \cdots & g_{16} \end{pmatrix}
\begin{pmatrix}
F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & 0 & \cdots \\
0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & \cdots \\
0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & \cdots \\
0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
$$

$$
\begin{aligned}
v_1 &= g_1 F_{11} \\
v_2 &= g_1 F_{12} + g_2 F_{11} \\
v_3 &= g_1 F_{13} + g_2 F_{12} + g_3 F_{11} \\
v_4 &= g_2 F_{13} + g_3 F_{12} + g_4 F_{11} \\
v_5 &= g_3 F_{13} + g_4 F_{12} \\
v_6 &= g_4 F_{13} \\
v_7 &= g_1 F_{21} + g_5 F_{11} \\
v_8 &= g_1 F_{22} + g_2 F_{21} + g_5 F_{12} + g_6 F_{11} \\
v_9 &= g_1 F_{23} + g_2 F_{22} + g_3 F_{21} + g_5 F_{13} + g_6 F_{12} + g_7 F_{11} \\
v_{10} &= g_2 F_{23} + g_3 F_{22} + g_4 F_{21} + g_6 F_{13} + g_7 F_{12} + g_8 F_{11} \\
v_{11} &= g_3 F_{23} + g_4 F_{22} + g_7 F_{13} + g_8 F_{12} \\
v_{12} &= g_4 F_{23} + g_8 F_{13} \\
v_{13} &= g_1 F_{31} + g_5 F_{21} + g_9 F_{11} \\
v_{14} &= g_1 F_{32} + g_2 F_{31} + g_5 F_{22} + g_6 F_{21} + g_9 F_{12} + g_{10} F_{11} \\
v_{15} &= g_1 F_{33} + g_2 F_{32} + g_3 F_{31} + g_5 F_{23} + g_6 F_{22} + g_7 F_{21} + g_9 F_{13} + g_{10} F_{12} + g_{11} F_{11} \\
&\vdots
\end{aligned}
$$

Examine back-prop eqn thru the convolution in more detail

$$(v_1 \quad v_2 \quad \cdots \quad v_{36}) = (g_1 \quad g_2 \quad \cdots \quad g_{16}) \begin{pmatrix} F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & 0 & \cdots \\ 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & 0 & \cdots \\ 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & 0 & \cdots \\ 0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & F_{31} & F_{32} & F_{33} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & F_{11} & F_{12} & F_{13} & 0 & 0 & 0 & F_{21} & F_{22} & F_{23} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$\begin{aligned}
v_1 &= g_1 F_{11} \\
v_2 &= g_1 F_{12} + g_2 F_{11} \\
v_3 &= g_1 F_{13} + g_2 F_{12} + g_3 F_{11} \\
v_4 &= g_2 F_{13} + g_3 F_{12} + g_4 F_{11} \\
v_5 &= g_3 F_{13} + g_4 F_{12} \\
v_6 &= g_4 F_{13} \\
v_7 &= g_1 F_{21} + g_5 F_{11} \\
v_8 &= g_1 F_{22} + g_2 F_{21} + g_5 F_{12} + g_6 F_{11} \\
v_9 &= g_1 F_{23} + g_2 F_{22} + g_3 F_{21} + g_5 F_{13} + g_6 F_{12} + g_7 F_{11} \\
v_{10} &= g_2 F_{23} + g_3 F_{22} + g_4 F_{21} + g_6 F_{13} + g_7 F_{12} + g_8 F_{11} \\
v_{11} &= g_3 F_{23} + g_4 F_{22} + g_7 F_{13} + g_8 F_{12} \\
v_{12} &= g_4 F_{23} + g_8 F_{13} \\
v_{13} &= g_1 F_{31} + g_5 F_{21} + g_9 F_{11} \\
v_{14} &= g_1 F_{32} + g_2 F_{31} + g_5 F_{22} + g_6 F_{21} + g_9 F_{12} + g_{10} F_{11} \\
v_{15} &= g_1 F_{33} + g_2 F_{32} + g_3 F_{31} + g_5 F_{23} + g_6 F_{22} + g_7 F_{21} + g_9 F_{13} + g_{10} F_{12} + g_{11} F_{11} \\
&\vdots
\end{aligned}$$

There is a pattern here!

Write back-prop eqn through convolution as a convolution

- Reshape vectors $\mathbf{g}$ and $\mathbf{v}$ into matrices

$$G = \begin{pmatrix} g_1 & g_2 & g_3 & g_4 \\ g_5 & g_6 & g_7 & g_8 \\ g_9 & g_{10} & g_{11} & g_{12} \\ g_{13} & g_{14} & g_{15} & g_{16} \end{pmatrix}, \quad V = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} \\ v_{13} & v_{14} & v_{15} & v_{16} & v_{17} & v_{18} \\ v_{19} & v_{20} & v_{21} & v_{22} & v_{23} & v_{24} \\ v_{25} & v_{26} & v_{27} & v_{28} & v_{29} & v_{30} \\ v_{31} & v_{32} & v_{33} & v_{34} & v_{35} & v_{36} \end{pmatrix}$$

- Let

$$G_{\text{zero-pad}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_1 & g_2 & g_3 & g_4 & 0 & 0 \\ 0 & 0 & g_5 & g_6 & g_7 & g_8 & 0 & 0 \\ 0 & 0 & g_9 & g_{10} & g_{11} & g_{12} & 0 & 0 \\ 0 & 0 & g_{13} & g_{14} & g_{15} & g_{16} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad F^{\text{rot180}} = \begin{pmatrix} F_{33} & F_{32} & F_{31} \\ F_{23} & F_{22} & F_{21} \\ F_{13} & F_{12} & F_{11} \end{pmatrix}$$

- Then

$$V = G_{\text{zero-pad}} * F^{\text{rot180}} \implies \frac{\partial l}{\partial X} = \left(\frac{\partial l}{\partial S}\right)_{\text{zero-pad}} * F^{\text{rot180}}$$

Multiple planes: Write back-prop eqn as a convolution

- $\mathbf{X} = \{X_1, X_2, X_3, X_4\}$ has size $6 \times 6 \times 4$
- $\mathbf{F} = \{F_1, F_2, F_3, F_4\}$ has size $3 \times 3 \times 4$
- Have

$$S = \mathbf{X} * \mathbf{F} \qquad \Longrightarrow \quad \text{vec}(S) = M_{\mathbf{F}}^{\text{filter}} \text{vec}(\mathbf{X})$$

  where
  - S is 4×4,
  - vec(S) is 16×1,
  - $M_{\mathbf{F}}^{\text{filter}} = (M_{F_1}^{\text{filter}}, M_{F_2}^{\text{filter}}, M_{F_3}^{\text{filter}}, M_{F_4}^{\text{filter}})$ is 16×144 and
  - $\text{vec}(\mathbf{X}) = \begin{pmatrix} \text{vec}(X_1) \\ \vdots \\ \text{vec}(X_4) \end{pmatrix}$ is 144×1.

- Now

$$\frac{\partial l}{\partial \text{vec}(\mathbf{X})} = \left(\frac{\partial l}{\partial \text{vec}(X_1)}, \frac{\partial l}{\partial \text{vec}(X_2)}, \frac{\partial l}{\partial \text{vec}(X_3)}, \frac{\partial l}{\partial \text{vec}(X_4)} \right)$$

Multiple planes: Write back-prop eqn as a convolution

- Now

$$\frac{\partial l}{\partial \text{vec}(\mathbf{X})} = \left(\frac{\partial l}{\partial \text{vec}(X_1)}, \frac{\partial l}{\partial \text{vec}(X_2)}, \frac{\partial l}{\partial \text{vec}(X_3)}, \frac{\partial l}{\partial \text{vec}(X_4)}\right)$$

$$= \left(\frac{\partial l}{\partial \text{vec}(S)} M_{F_1}, \frac{\partial l}{\partial \text{vec}(S)} M_{F_2}, \frac{\partial l}{\partial \text{vec}(S)} M_{F_3}, \frac{\partial l}{\partial \text{vec}(S)} M_{F_4}\right)$$

- Remember for a single plane have

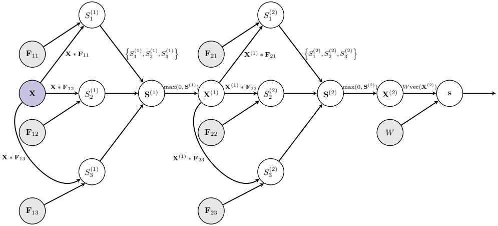
$$\frac{\partial l}{\partial X_i} = G_{\text{zero-pad}} * F_i^{\text{rot180}}$$

where $G_{\text{zero-pad}}$ is the zero-padded version of $\frac{\partial l}{\partial S}$.

- Thus

$$\frac{\partial l}{\partial \mathbf{X}} = \left\{G_{\text{zero-pad}} * F_1^{\text{rot180}}, G_{\text{zero-pad}} * F_2^{\text{rot180}}, G_{\text{zero-pad}} * F_3^{\text{rot180}}, G_{\text{zero-pad}} * F_4^{\text{rot180}}\right\}$$

Back to gradient of the loss w.r.t. $\mathbf{X}^{(1)}$



Have applied multiple filters with multiple planes and know

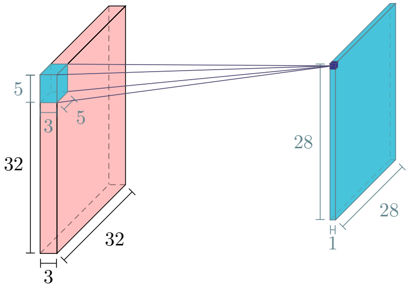
$$\frac{\partial l}{\partial \text{vec}(\mathbf{X}^{(1)})} = \sum_{i=1}^{3} \frac{\partial l}{\partial \text{vec}(S_i^{(2)})} M_{\mathbf{F}_{2i}}^{\text{filter}}$$

then

$$\frac{\partial l}{\partial X^{(1)}} = \left\{ \sum_{i=1}^{3} G_i^{\text{zero-pad}} * F_{2i,1}^{\text{rot180}}, \sum_{i=1}^{3} G_i^{\text{zero-pad}} * F_{2i,2}^{\text{rot180}}, \sum_{i=1}^{3} G_i^{\text{zero-pad}} * F_{2i,3}^{\text{rot180}} \right\}$$

where $G_i = \frac{\partial l}{\partial S_i^{(2)}}$ and $\mathbf{F}_{2i} = \{F_{2i,1}, F_{2i,2}, F_{2i,3}\}$.

More details on the Convolution layers: **Striding & Zero Padding**

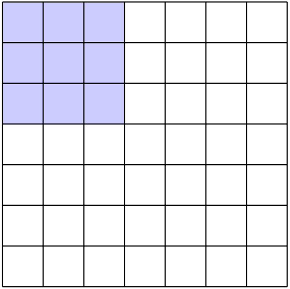


Convolve the image, $\mathbf{X}$, with the filter $\mathbf{F}$.

- Slide filter over all spatial locations in image.
- At each location output 1 number:

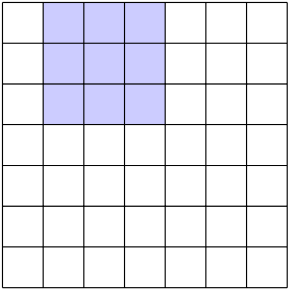
compute **dot product** *between* $\mathbf{F}$ *and a* $5 \times 5 \times 3$ *chunk of* $\mathbf{X}$

A closer look at the spatial dimensions



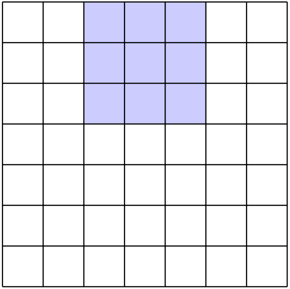
- $7 \times 7 \times D$ input (just display one plane).
- Have $3 \times 3 \times D$ filter.

A closer look at the spatial dimensions



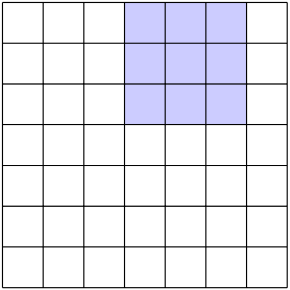
- $7 \times 7 \times D$ input (just display one plane).
- Have $3 \times 3 \times D$ filter.

A closer look at the spatial dimensions



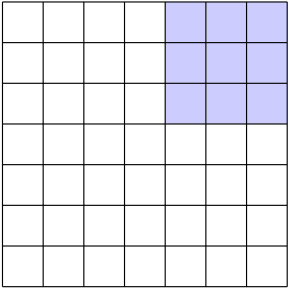
- $7 \times 7 \times D$ input (just display one plane).
- Have $3 \times 3 \times D$ filter.

A closer look at the spatial dimensions



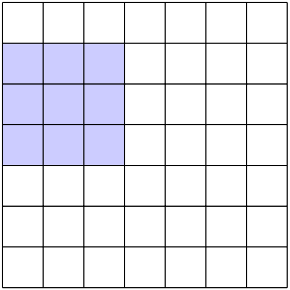
- $7 \times 7 \times D$ input (just display one plane).
- Have $3 \times 3 \times D$ filter.

A closer look at the spatial dimensions



- $7 \times 7 \times D$ input (just display one plane).
- Have $3 \times 3 \times D$ filter.

A closer look at the spatial dimensions



- $7 \times 7 \times D$ input (just display one plane).
- Have $3 \times 3 \times D$ filter.

A closer look at the spatial dimensions

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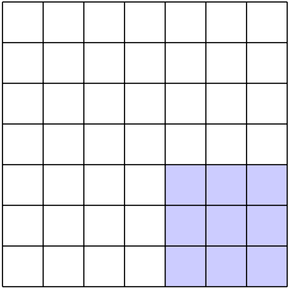
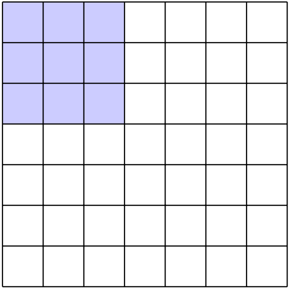
A closer look at the spatial dimensions



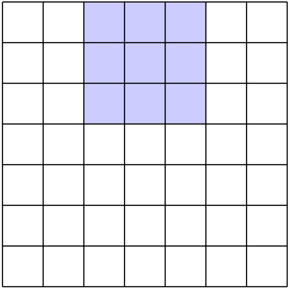
- $7 \times 7 \times D$ input (just display one plane).
- Have $3 \times 3 \times D$ filter.
- $\implies 5 \times 5$ output.

A closer look at the spatial dimensions

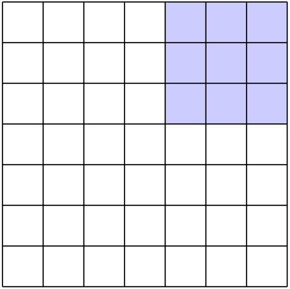


- $7 \times 7 \times D$ input (just display one plane).
- Have $3 \times 3 \times D$ filter.
- Apply with **stride** 2.

A closer look at the spatial dimensions

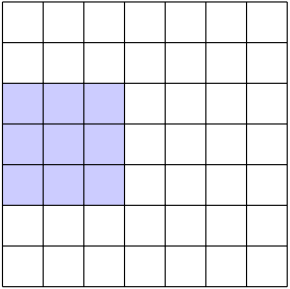


- $7 \times 7 \times D$ input (just display one plane).
- Have $3 \times 3 \times D$ filter.
- Apply with **stride** 2.



- $7 \times 7 \times D$ input (just display one plane).
- Have $3 \times 3 \times D$ filter.
- Apply with **stride** 2.

A closer look at the spatial dimensions

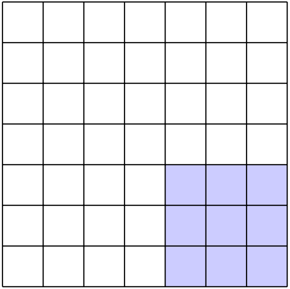


- $7 \times 7 \times D$ input (just display one plane).
- Have $3 \times 3 \times D$ filter.
- Apply with **stride** 2.

A closer look at the spatial dimensions

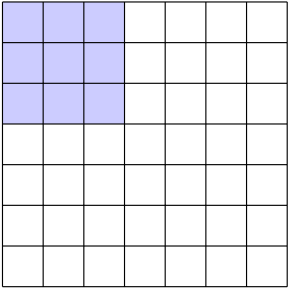
...

A closer look at the spatial dimensions



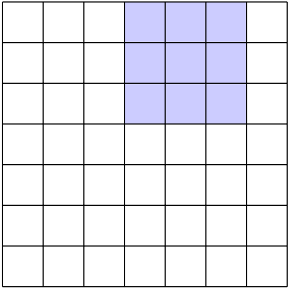
- $7 \times 7 \times D$ input (just display one plane).
- Have $3 \times 3 \times D$ filter.
- Apply with **stride** 2.
- $\implies 3 \times 3$ output.

A closer look at the spatial dimensions



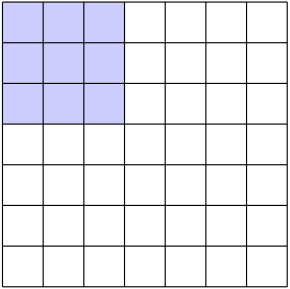
- $7 \times 7 \times D$ input (just display one plane).
- Have $3 \times 3 \times D$ filter.
- Apply with **stride** 3.

A closer look at the spatial dimensions



- $7 \times 7 \times D$ input (just display one plane).
- Have $3 \times 3 \times D$ filter.
- Apply with **stride** 3.
- **Doesn't fit nicely**! Don't include last column and row.

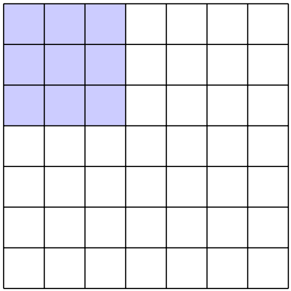
Output volume dimension



- $w \times h \times D$ input.
- Have $f \times f \times D$ filter.
- Apply with **stride** s.
- **Output dimension**: $w' \times h'$

$$w' = (w - f)/s + 1$$
$$h' = (h - f)/s + 1$$

Output volume dimension



- $w \times h \times D$ input.
- Have $f \times f \times D$ filter.
- Apply with **stride** s.
- **Output dimension**: $w' \times h'$

$$w' = (w - f)/s + 1$$
$$h' = (h - f)/s + 1$$

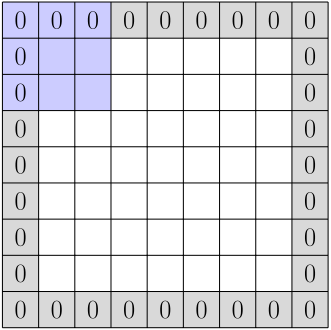
- **Our example**:
  $w = h = 7, f = 3$

  s=1 $\implies w' = (7-3)/1 + 1 = 5$

  s=2 $\implies w' = (7-3)/2 + 1 = 3$

  s=3 $\implies w' = (7-3)/3 + 1 = 2.33$

In practice: Common to zero pad the border



- $7 \times 7 \times D$ input.
- Have $3 \times 3 \times D$ filter.
- Apply with **stride** s.
- Pad input with 1 pixel border.
- **Output dimension**: ?
- Remember

$$w' = (w - f)/s + 1$$
$$h' = (h - f)/s + 1$$

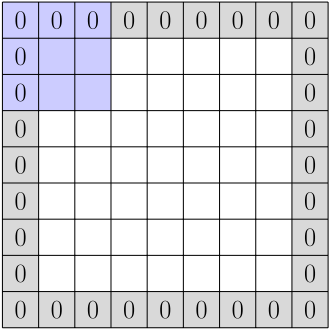
In practice: Common to zero pad the border

0	0	0	0	0	0	0	0	0
0								0
0								0
0								0
0								0
0								0
0								0
0								0
0	0	0	0	0	0	0	0	0

- $7 \times 7 \times D$ input.
- Have $3 \times 3 \times D$ filter.
- Apply with **stride** s.
- Pad input with 1 pixel border.
- **Output dimension**: ?
- Remember

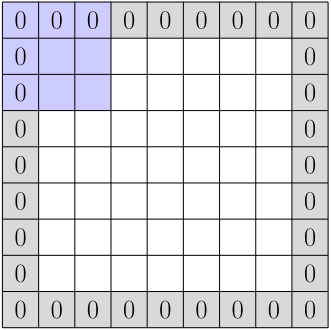
$$w' = (w - f)/s + 1$$
$$h' = (h - f)/s + 1$$

In practice: Common to zero pad the border



- $7 \times 7 \times D$ input.
- Have $3 \times 3 \times D$ filter.
- Apply with **stride** s.
- Pad input with 1 pixel border.
- **Output dimension**: 7×7

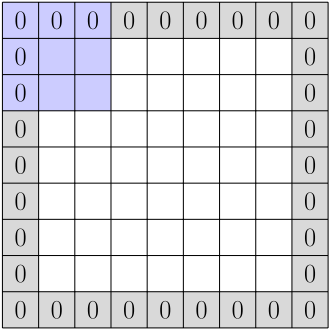
In practice: Common to zero pad the border



- $7 \times 7 \times D$ input.
- Have $3 \times 3 \times D$ filter.
- Apply with **stride** s.
- Pad input with 1 pixel border. $(P = 1)$
- **Output dimension**: 7×7
- In general

$$w' = (w + 2P - f)/s + 1$$
$$h' = (h + 2P - f)/s + 1$$

In practice: Common to zero pad the border



- $7 \times 7 \times D$ input.
- Have $3 \times 3 \times D$ filter.
- Apply with **stride** s.
- Pad input with 1 pixel border. $(P = 1)$
- **Output dimension**: 7×7
- In general, common to have convolutional layers with
  - stride $s = 1$,
  - filters of size $f \times f \times D$,
  - zero-padding $P = (f - 1)/2$

$\implies w' = w, h' = h$

Example

- **Input volume dimension**: $32 \times 32 \times 3$
- Hyper-parameters of Conv layer:
  - 10 filters of size $5 \times 5 \times 3$
  - Stride: $s = 1$
  - Pad: $P = 2$
- **Output volume dimension**: ?

Example of Dimension Counting

- **Input volume dimension**: $32 \times 32 \times 3$

- Hyper-parameters of Conv layer:
  - 10 filters of size $5 \times 5 \times 3$
  - Stride: $s = 1$
  - Pad: $P = 2$

- **Output volume dimension**: $w' \times h' \times 10$
  where

$$w' = (w + 2P - f)/s + 1 = (32 + 2 * 2 - 5)/1 + 1 = 32$$
$$h' = (h + 2P - f)/s + 1 = (32 + 2 * 2 - 5)/1 + 1 = 32$$

Example of Dimension Counting

- **Input volume dimension**: $32 \times 32 \times 3$

- Hyper-parameters of Conv layer:
  - 10 filters of size $5 \times 5 \times 3$
  - Stride $s = 1$
  - Pad $P = 2$

- **# of parameters in this layer**: ?

Example of Dimension Counting

- **Input volume dimension**: $32 \times 32 \times 3$

- Hyper-parameters of Conv layer:
  - 10 filters of size $5 \times 5 \times 3$
  - Stride: $s = 1$
  - Pad: $P = 2$

- **# of parameters in this layer**: 760
  - Each filter has $5 \times 5 \times 3 + 1 = 76$ parameters.
  - There are 10 filters $\implies$ 76×10 total parameters.

Summary of dimensions in Convolutional Layer

- **Input**:
  Volume, $\mathbf{X}^{(i)} = \{X_1^{(i)}, \ldots, X_D^{(i)}\}$, of size $w \times h \times D$

- Requires 4 hyper-parameters:
  - Number of filters: n_F
  - Spatial size of filters: f
  - Stride: s
  - Amount of zero padding: P

- **Output**:
  Volume $\mathbf{S}^{(i+1)} = \{S_1^{(i+1)}, \ldots, S_{n_F}^{(i+1)}\}$ size $w' \times h' \times n_F$ where
  - $w' = (w - f + 2P)/s + 1$
  - $h' = (h - f + 2P)/s + 1$

  where

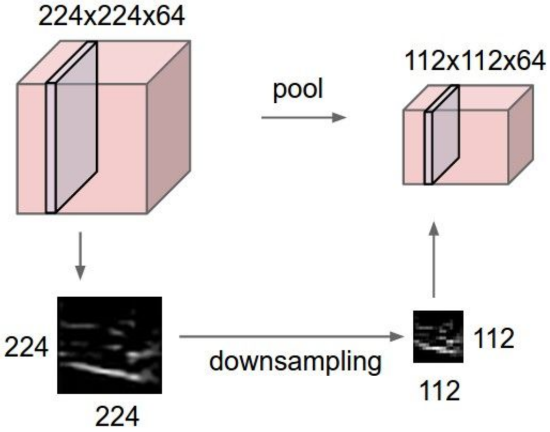
$$S_j^{(i+1)} = \mathbf{X}^{(i)} * \mathbf{F}_j + b_j$$

Common Settings

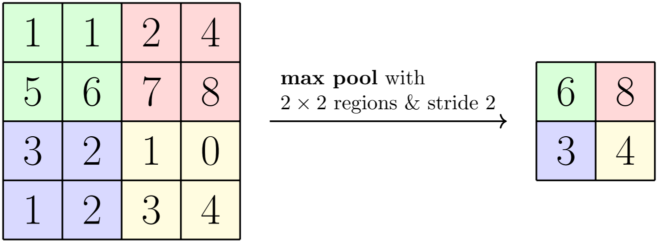
- $n_F =$ powers of 2 (e.g. 32, 64, 128, 512)
  - $f = 3, s = 1, P = 1$
  - $f = 5, s = 1, P = 2$
  - $f = 5, s = 2, P = ?$ (whatever fits)
  - $f = 1, s = 1, P = 1$

Common Operator in Convolutional Networks: **Pooling Operators**

- Make the representation smaller and more manageable
- Operates over each response/activation map independently



Simple Example: Max Pooling one response map



Max Pooling

- Denote the max pooling operation with regions of size $r \times r$ and stride s with

$$\tilde{\mathbf{H}} = \text{MaxPool}(\mathbf{H}, r, s)$$

- If $\mathbf{H}$ has size $w \times h \times D$
  $\implies$ $\tilde{\mathbf{H}}$ has size $(w - r)/s + 1 \times (h - r)/s + 1 \times D$

- Mathematical expression for each entry in $\tilde{\mathbf{H}}$:

$$\tilde{\mathbf{H}}_{ijk} = \max_{\substack{i' \le l \le i' + r - 1 \\ j' \le m \le j' + r - 1}} \mathbf{H}_{lmk} \quad \text{where } i' = (i-1)s + 1, j' = (j-1)s + 1$$

- Common settings $r = 2, s = 2$ or $r = 3, s = 2$.

- Denote the max pooling operation with regions of size $r \times r$ and stride s with

$$\tilde{\mathbf{H}} = \mathsf{MaxPool}(\mathbf{H}, r, s)$$

- If $\mathbf{H}$ has size $w \times h \times D$
  $\implies$ $\tilde{\mathbf{H}}$ has size $(w-r)/s+1 \times (h-r)/s+1 \times D$

- Mathematical expression for each entry in $\tilde{\mathbf{H}}$:

$$\tilde{\mathbf{H}}_{ijk} = \max_{\substack{i' \le l \le i'+r-1 \\ j' \le m \le j'+r-1}} \mathbf{H}_{lmk} \quad \text{where } i' = (i-1)s+1,\, j' = (j-1)s+1$$

- Common settings $r = 2, s = 2$ or $r = 3, s = 2$.

Jacobian Computations for a Max Pooling layer

- Let

$$\tilde{\mathbf{H}} = \mathsf{MaxPool}(\mathbf{H}, r, r)$$

- For backprop need to calculate:

$$\frac{\partial \mathsf{vec}(\tilde{\mathbf{H}})}{\partial \mathsf{vec}(\mathbf{H})} = ?$$

- Can write MaxPool operation as a matrix operation

$$\mathsf{vec}(\tilde{\mathbf{H}}) = A_{\mathsf{MaxPool}} \mathsf{vec}(\mathbf{H})$$

$$\Longrightarrow$$

$$\frac{\partial \mathsf{vec}(\tilde{\mathbf{H}})}{\partial \mathsf{vec}(\mathbf{H})} = A_{\mathsf{MaxPool}}$$

- What are the entries of A_{MaxPool}?

Jacobian Computations for a Max Pooling layer

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Simple Example: Max Pooling as a matrix multiplication

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 3 & 2 & 1 & 0 \\ \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \xrightarrow{\textbf{max pool}\text{ mask region 1}} \begin{array}{|c|c|c|c|}\hline 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array}$$

$$\begin{pmatrix}\tilde{H}_{11}\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} \mathsf{vec}(H)$$

Simple Example: Max Pooling as a matrix multiplication

1	1	2	4
5	6	7	8
3	2	1	0
1	2	3	4

$\xrightarrow{\textbf{max pool}\ \text{mask region 2}}$

0	0	0	0
0	0	0	1
0	0	0	0
0	0	0	0

$$\begin{pmatrix} \tilde{H}_{11} \\ \tilde{H}_{12} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mathsf{vec}(H)$$

Simple Example: Max Pooling as a matrix multiplication

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 5 & 6 & 7 & 8 \\ 3 & 2 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix} \xrightarrow{\textbf{max pool} \text{ mask region 3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \tilde{H}_{11} \\ \tilde{H}_{12} \\ \tilde{H}_{21} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mathsf{vec}(H)$$

Simple Example: Max Pooling as a matrix multiplication

1	1	2	4
5	6	7	8
3	2	1	0
1	2	3	4

$\xrightarrow{\textbf{max pool} \text{ mask region } 4}$

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	1

$$\begin{pmatrix} \tilde{H}_{11} \\ \tilde{H}_{12} \\ \tilde{H}_{21} \\ \tilde{H}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mathsf{vec}(H)$$

Simple Example: Max Pooling as a matrix multiplication

1	1	2	4
5	6	7	8
3	2	1	0
1	2	3	4

$\xrightarrow{\textbf{max pool} \text{ mask region } 4}$

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	1

$$\begin{pmatrix} \tilde{H}_{11} \\ \tilde{H}_{12} \\ \tilde{H}_{21} \\ \tilde{H}_{22} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{A_{\mathsf{MaxPool}} \text{ and has size } 4 \times 16} \mathsf{vec}(H)$$

What is A_{AvgPool}?

- Say in each region we pool by **averaging** the responses instead of choosing the max.

- What is A_{AvgPool} for our simple example?