(2.2) Show that this method $\left[x_{n+1} = \frac{x_n + a/x_n}{2}\right]$ is equivalent to the Newton's method $\left[x_{n+1} = x_n - \frac{f(x)}{f'(x)}\right]$

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

Our Guess

We want an f, such that $f(x) = x^2 - a = 0$

$$\therefore x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n}$$

$$x_{n+1} = \frac{2x_n^2 - x_n^2 + a}{2x_n}$$

$$\therefore \boxed{x_{n+1} = \frac{x_n^2 + a}{2x_n}}$$

(4.1) Taylor Series:

$$f(x+2h) = f(x) + (2h)f'(x) + \left(\frac{(2h)^2}{2!}\right)f''(x) + \left(\frac{(2h)^3}{3!}\right)f'''(\xi_{2h})$$

$$= f(x) + 2hf'(x) + 2h^2 f''(x) + \tfrac{4}{3}h^3 f'''(\xi_{2h}) //$$

$$f(x+h) = f(x) + hf'(x) + \left(\frac{h^2}{2}\right)f''(x) + \left(\frac{h^3}{6}\right)f'''(\xi_h) //$$

$$4f(x+h) - f(x+2h) = 3f(x) + 2hf'(x) + \cancel{(0)f''(x)} + \frac{4h^3}{6}f'''(\xi_h) - \frac{4h^3}{3}f'''(\xi_{2h})$$

$$f'(x) = \frac{4f(x+h) - f(x+2h) - 3f(x)}{2h} + \underbrace{\frac{h^2}{3}\left[2f'''(\xi_{2h}) - f'''(\xi_h)\right]}_{O(h^2)}$$

(4.2) $$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f''''(\xi_{-h})$$

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + \frac{h^4}{24}\left[f''''(\xi_h) + f''''(\xi_{-h})\right]$$

$$f''(x) = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} - \underbrace{\frac{h^2}{24}\left[f''''(\xi_h) + f''''(\xi_{-h})\right]}_{O(h^2)}$$