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Pair Susceptibility and Mode Propagation in Superconductors: A Microscopic Approach*

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We derive the full microscopic set of equations governing small oscillations: (1) in the magnitude of the superconducting order parameter (the Schmid mode), (2) the phase of the order parameter in a neutral superfluid (the Anderson–Bogoliubov mode), and (3) the coupled oscillations in the phase of the order parameter and in the electric field (the transverse, or Carlson–Goldman mode). The derivation is not limited by the restrictions of previous papers. No limitations are required for the magnitude of the frequency, the concentration of impurities, or the magnitude of the temperature. Special attention is given to the Carlson–Goldman (CG) mode, whose dispersion law frequency (ω) vs. wave vector ($\mathbf{k}$) and damping is calculated. The velocity of the CG mode in the propagation region $\omega/|\mathbf{k}|$ is found to equal $c = [2\Delta D\chi(2\pi T\tau)]$, where D is the diffusion constant and χ is the function appearing in the theory of superconducting alloys. In the dirty ($l \ll \xi_0$) and clean ($l \gg \xi_0$) limits, this expression reduces to those previously derived by Schmid and Schön, and by Artemenko and Volkov, respectively. At large values of $\mathbf{k}$, the frequency of the CG mode approaches a limiting value of 2Δ. The damping is small in this limit and tends to zero as $|\mathbf{k}|$ increases.

Our results are obtained by calculating the linear response of a superconductor to a perturbation in the magnitude and phase of the order parameter, and the electromagnetic potentials. The response of the superconductor to these perturbations is calculated by properly continuing the thermodynamic perturbation function of linear response from imaginary frequencies to the real ones, then inserting into the self-consistency BCS equation and Poisson's

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equation. The derivation is based on the self-consistent BCS scheme. No kinetic equations are introduced at any stage of the calculation.

1. INTRODUCTION

The dynamical properties of superconductors have proven to be quite complex, requiring more than a decade of development to elaborate those theoretical techniques necessary to treat nonlinear, nonstationary problems. The techniques are generally quite complex, and are effectively limited to specific ranges of temperature, frequency, or impurity concentration. These approaches have clarified the processes of order parameter relaxation, charge imbalance (or branch mixing) relaxation, characteristic penetration length for an electric field in a superconductor, and the possibility of unequal quasiparticle and pair chemical potentials.

The most illuminating and, fundamentally, simplest experimental method for studying the dynamical properties of the order parameter is the pair-susceptibility technique developed by the Goldman group.[1] This method was originally developed theoretically by Ferrell,[2] Scalapino,[3] and Kulik.[4] It permits measurement of the dispersion law of collective waves which propagate in the superconductor. The wave vector is selected by the magnetic field of a given strength transverse to a tunneling junction between two superconductors, while the frequency is determined by the voltage across the junction using the Josephson relation. The method was first applied[5] to superconductors above the transition temperature, where the order parameter dynamics is simple and can be described by the time-dependent Ginzburg–Landau (GL) equation.[6–8] Below T_c, as shown by Gorkov and Eliashberg,[9] the GL equation is valid only in the case of gapless superconductors (e.g , for sufficiently strong pair-breaking scattering by magnetic impurities). Carlson and Goldman[1] discovered the existence of a collective mode of oscillation when the gap is finite but small ($T \lesssim T_c$), which we shall later refer to as the Carlson–Goldman (CG) model. Their discovery overcame the more or less widespread opinion that collective oscillations predicted for a neutral superfluid[10,11] should be forced to frequencies of the order of the plasma frequency ($\omega_p \sim 10^{16}$ sec^{-1}) by virtue of the Coulomb interaction in a real metal. The Goldman group's experiments[1,5] stimulated a number of theoretical papers[12–18] predicting low-frequency propagating modes in the vicinity of T_c. The technique of the present paper generates most of these results and allows one to extend many of the previous calculations to arbitrary electron mean free paths l, higher frequencies, and to all temperatures.

The methods currently in use for studying collective oscillations in superconductors all lead to kinetic equations governing the time dependence of the quasiparticle distribution and the BCS self-consistency

equation for the order parameter. We find it more natural for the purpose of studying small oscillations of the superconducting order parameter not to invoke kinetic equations of any type, but rather to use a microscopic approach based on linear response theory. We perturb the symmetry-breaking state (here, the superconductor) by a small deviation in the magnitude and phase of the order parameter, followed by small perturbations in the electromagnetic field. This allows the appearance of oscillating modes (collective modes) of different types. We shall classify them as follows, following historical precedent.

1.1. The Anderson–Bogoliubov Mode (AB Mode)

This is a collective oscillation of the phase of the order parameter in a neutral superfluid Fermi liquid (we neglect Landau Fermi liquid corrections in what follows, treating a Fermi gas rather than a Fermi liquid). The AB mode is just the Goldstone mode allowed by the degeneracy of the ground state of the Fermi superfluid. At $T = 0$ and in the clean limit ($l = \infty$) the frequency of the mode is given by $\omega = kv_F/\sqrt{3}$, corresponding to the compressibility of the Fermi gas. The mode is not heavily damped, at least at $T = 0$, a direct consequence of the existence of an energy gap.

1.2. The Schmid Mode (Longitudinal or Energy Mode)

This type of excitation is composed of a collective oscillation in the magnitude of the order parameter. As discussed in Ref. 19, if a Cooper pair were to spontaneously dissociate, it would require the binding energy 2Δ (Δ being the energy gap) and so violate energy conservation. This is possible, however, due to the quantum mechanical energy–time uncertainty condition, at times short compared to $(2\Delta)^{-1}$. We shall show that the magnitude of the order parameter, if perturbed at time $t = 0$, oscillates at frequency $\omega = 2\Delta$, rather than relaxing directly to an equilibrium value. Inelastic processes, like those connected with electron–phonon or electron–electron collisions, will cause relaxation of this mode. One can argue that the Schmid mode, as it is introduced here, is not a true collective mode because there does exist[20,21] damping resembling Landau damping of collective oscillations in plasmas. The oscillating perturbation in the magnitude of the order parameter thereby decreases in magnitude as a function of time as some power of $1/t$. This type of damping does not require any collision mechanism, and is strong in the sense that the characteristic rate of the nonexponential damping is of the order of Δ.

1.3. The Carlson–Goldman Mode (CG or Transverse Mode)

This type of oscillation is a coupled perturbation in the phase of the order parameter and electromagnetic potential. In a two-fluid picture,

supercurrent and normal current oscillate in such a way that there is no net charge produced in the superconductor. This provides for the existence of oscillations having small frequencies compared to the plasma frequency. The discussion of the properties of this mode at all impurity concentrations and temperatures is the main purpose of the paper.

In Sections 2–4 the necessary techniques are developed to treat the general impure case and to properly take into account the electromagnetic effects of the oscillations. The primary physical content of the paper is concentrated in Section 5, in which the dispersion law of the CG mode is derived for arbitrary mean free paths and frequencies near the critical temperature T_c. We summarize our results in Section 6, indicating future extensions of the theory. We show how, even at this stage, one can include the effects of pair-breakers (inelastic phonon scattering) in the theory, obtaining as an example the frequency-dependent penetration length for an electric field in a superconductor (the Waldram[22] length). The calculation is valid for arbitrary concentration of impurities, reducing to the results of Artemenko *et al.*[23] in the dirty limit. Some lengthy calculations concerning the analytical continuation procedure encountered in the linear response theory are included in the Appendix.

2. BASIC EQUATIONS

In the pseudo-isospin (or Nambu) notation, the superconductor can be described by the Hamiltonian

$$H = \sum_{\mathbf{pq}} A^{\dagger}_{\mathbf{p}+\mathbf{q}/2}\left\{(\xi_{\mathbf{p}}\delta_{\mathbf{q}0} + e\phi_{\mathbf{q}})\hat{\tau}_3 - \frac{e}{mc}(\mathbf{A}_{\mathbf{q}}\cdot\mathbf{p})\hat{\tau}_0 - \Delta_{\mathbf{q}}\hat{\tau}_- - \Delta^*_{-\mathbf{q}}\hat{\tau}_+\right\}A_{\mathbf{p}-\mathbf{q}/2} \quad (1)$$

where $A_{\mathbf{p}}$, $A^{\dagger}_{\mathbf{p}}$ are the vectors

$$A_{\mathbf{p}} = \begin{bmatrix} a_{\mathbf{p}\uparrow} \\ a^{\dagger}_{-\mathbf{p}\downarrow} \end{bmatrix}, \qquad A^{\dagger}_{\mathbf{p}} = [a^{\dagger}_{\mathbf{p}\uparrow} \quad a_{-\mathbf{p}\downarrow}]$$

and $\hat{\tau}_i$ are the Pauli matrices:

$$\hat{\tau}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \hat{\tau}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \hat{\tau}_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \hat{\tau}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\hat{\tau}_+ = \tfrac{1}{2}(\hat{\tau}_1 - i\hat{\tau}_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \hat{\tau}_- = \tfrac{1}{2}(\hat{\tau}_1 + i\hat{\tau}_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (2)$$

In Eq. (1), $\xi_{\mathbf{p}} = (\mathbf{p}^2/2m) - \varepsilon_{\mathrm{F}}$, ε_{F} is the Fermi energy; $\phi_{\mathbf{q}}(t)$, $\mathbf{A}_{\mathbf{q}}(t)$ stand for the electromagnetic scalar and vector potentials, respectively; and $\Delta_{\mathbf{q}}(t)$ is the Fourier transform of the pair potential of the superconductor.

The quantities $\Delta_{\mathbf{q}}$, $\Delta_{\mathbf{q}}^*$ and $\phi_{\mathbf{q}}$, $\mathbf{A}_{\mathbf{q}}$ should be considered as the self-consistent off-diagonal and diagonal potentials, respectively. To obtain a closed set of equations, it is necessary to take advantage of the BCS self-consistency condition for Δ:

$$\Delta_{\mathbf{q}}(t) = g \operatorname{Tr} \sum_{\mathbf{p}} \langle A^{\dagger}_{\mathbf{p}-\mathbf{q}/2}(t)\hat{\tau}_+ A_{\mathbf{p}+\mathbf{q}/2}(t)\rangle \tag{3}$$

and Maxwell's equations for the diagonal potentials ϕ, $\mathbf{A}$. In what follows, we drop the vector potential without loss of generality, because of the gauge-invariant nature of the theory {i.e., only the quantity $(1/2m)$ $[\nabla\phi - (2e/c)\mathbf{A}]$, the gauge-invariant superfluid velocity, will enter in all physical quantities; see Section 6 for an interesting use of this idea}. The diagonal potential satisfies Poisson's equation,

$$q^2\phi_{\mathbf{q}}(t) = 4\pi e \operatorname{Tr} \sum_{\mathbf{p}} \langle A^{\dagger}_{\mathbf{p}-\mathbf{q}/2}(t)\hat{\tau}_3 A_{\mathbf{p}+\mathbf{q}/2}(t)\rangle \tag{4}$$

In Eqs. (3) and (4), $\langle \cdots \rangle$ denotes the Gibbs average, and Tr the trace over the components of the vectors $A_{\mathbf{p}}$, $A^{\dagger}_{\mathbf{p}}$.

The Hamiltonian (1) is appropriate for a pure superconductor. To include impurity scattering, we add to (1) the scattering term

$$H_i = \sum_{\mathbf{pq}} A^{\dagger}_{\mathbf{p}+\mathbf{q}/2}(V^{(n)}_{\mathbf{q}}\hat{\tau}_3 + V^{(s)}_{\mathbf{q}}\hat{\tau}_0)A_{\mathbf{p}-\mathbf{q}/2} \tag{5}$$

where

$$V^{(n,s)}_{\mathbf{q}} = \sum_i \exp(-i\mathbf{q}\cdot\mathbf{R}_i)\, V_{n,s}(\mathbf{R}_i) \tag{6}$$

Here, $V^{(n,s)}_{\mathbf{q}}$ is the Fourier transform of the potential (n) and spin-flip (s) scattering amplitudes, and $\mathbf{R}_i$ denotes the impurity coordinates.

It is important to note that Eqs. (3) and (4) satisfy the continuity condition

$$\partial\rho/\partial t + \operatorname{div}\mathbf{j} = 0 \tag{7}$$

To see this, note that the charge density ρ is the right-hand side of Eq. (4) divided by 4π. The current density $\mathbf{j}$ is given by

$$\mathbf{j}_{\mathbf{q}} = e \operatorname{Tr} \sum_{\mathbf{p}} \langle A^{\dagger}_{\mathbf{p}-\mathbf{q}/2}\mathbf{v}A_{\mathbf{p}+\mathbf{q}/2}\rangle, \qquad \mathbf{v} = \mathbf{p}/m \tag{8}$$

To prove Eq. (7), we calculate the derivative

$$\partial\rho_{\mathbf{q}}/\partial t = ie \operatorname{Tr} \sum_{\mathbf{p}} \langle [H, A^{\dagger}_{\mathbf{p}-\mathbf{q}/2}\hat{\tau}_3 A_{\mathbf{p}+\mathbf{q}/2}]\rangle$$

We then find

$$\frac{\partial\rho_{\mathbf{q}}}{\partial t} + i\mathbf{q}\cdot\mathbf{j}_{\mathbf{q}} = 2ie \operatorname{Tr} \sum_{\mathbf{pq}'} \langle A^{\dagger}_{\mathbf{p}-\mathbf{q}'/2}(\Delta_{\mathbf{q}-\mathbf{q}'}\tau_- - \Delta^*_{-\mathbf{q}+\mathbf{q}'}\tau_+)A_{\mathbf{p}+\mathbf{q}'/2}\rangle$$

The right-hand side of this equality, from the self-consistency condition (3) and its complex conjugate, vanishes, and our conjecture is proven. In most Boltzmann equation schemes (but not in the case of IOK[21] and for the Green's function method) the continuity equation (7) is introduced as an independent condition. In these schemes, the supercurrent cannot be introduced in the same manner as the quasiparticle current. Nevertheless, in a proper kinetic scheme, with the phase of the order parameter taken into account self-consistently, the continuity equation must be a consequence of the kinetic equation itself, in the same manner as it would be for a nonsuperfluid system. As shown above, this condition is satisfied in the general nonequilibrium case, for arbitrarily strong order parameter perturbations.

We now consider the case of small variations of the order parameter near its equilibrium value Δ_0. In equilibrium, $\Delta(\mathbf{r}, t) = \Delta_0$ and $\phi = 0$ (ϕ is the phase of the order parameter). Later, we shall denote Δ_0 by Δ and suppose it to be real (without loss of generality). The perturbation of the spatial Fourier transform of $\Delta(\mathbf{r}, t)$ will be denoted by $\Delta_{\mathbf{q}}(t)$. In this manner, the Hamiltonian splits into two parts, an equilibrium expression and a part containing the departures from equilibrium:

$$H = H_0 + H' \tag{9}$$

where (including impurity scattering) H_0 is

$$H_0 = \sum_{\mathbf{p}} A_{\mathbf{p}}^{\dagger} \hat{\varepsilon}_{\mathbf{p}} A_{\mathbf{p}} + \sum_{\mathbf{pq}} A_{\mathbf{p}+\mathbf{q}/2}^{\dagger} (V_{\mathbf{q}}^{(s)} \hat{\tau}_0 + V_{\mathbf{q}}^{(n)} \hat{\tau}_3) A_{\mathbf{p}-\mathbf{q}/2} \tag{10}$$

and H' is

$$H' = \sum_{\mathbf{pk}} A_{\mathbf{p}+\mathbf{k}}^{\dagger} (\Delta_1 \hat{\tau}_1 + \Delta_2 \hat{\tau}_2 + e\phi \hat{\tau}_3) A_{\mathbf{p}} \, e^{-i\omega t} + \text{H.c.} \tag{11}$$

Here, $\hat{\varepsilon}_{\mathbf{p}}$ is the single-particle Hamiltonian

$$\hat{\varepsilon}_{\mathbf{p}} = \xi_{\mathbf{p}} \hat{\tau}_3 - \Delta \hat{\tau}_1 \tag{12}$$

and Δ_1, Δ_2 are the real and imaginary components of the perturbed order parameter. We have selected out the (k, ω) components of the perturbation in (11) so that Δ_1, Δ_2, and ϕ are amplitudes of expressions like $\Delta_1 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$. The matrix Hamiltonian (11) may be considered as a scalar product $\sum_i \gamma_i \hat{\tau}_i$, where γ_i is a 4-vector,

$$\gamma_i = (0, \Delta_1, \Delta_2, e\phi)$$

To calculate the response of the superconductor to the perturbation (11), we introduce the response functions η_i:

$$\eta_i = \text{Tr} \sum_{\mathbf{p}} \langle A_{\mathbf{p}}^{\dagger}(t) \hat{\tau}_i A_{\mathbf{p}+\mathbf{k}}(t) \rangle \, e^{i\omega t} \tag{13}$$

where η_3 denotes the density perturbation, and $\eta_\pm = \eta_1 \mp i\eta_2$ are the perturbations of the order parameter. These quantities shall now be calculated, using standard linear response theory.

The linear response can be written, using usual techniques (see Ref. 24), as

$$\eta_i(\mathbf{k}, \Omega) = \sum_{j=0}^{3} Q_{ij}(\mathbf{k}, \Omega)\gamma_j(\mathbf{k}, \Omega) \tag{14}$$

at the imaginary frequencies $\Omega = 2\pi mT$. It will then be necessary to perform an analytical continuation of Q_{ij} to the upper half (physical) complex frequency plane. The matrix Q_{ij} may be written explicitly as

$$Q_{ij}(\mathbf{k}, \Omega) = -\sum_{\mathbf{p}\mathbf{p}'} \langle \hat{T}[\bar{A}_{\mathbf{p}}(\tau)\hat{\tau}_i A_{\mathbf{p}+\mathbf{k}}(\tau)\bar{A}_{\mathbf{p}'+\mathbf{k}}(\tau')\hat{\tau}_j A_{\mathbf{p}'}(\tau')]\rangle_\Omega \tag{15}$$

where $\hat{T}$ is the ordering operator along the axis of the imaginary time τ, and

$$\bar{A}_{\mathbf{p}}(\tau) = e^{H\tau}A_{\mathbf{p}}^{\dagger}\, e^{-H\tau}, \qquad A_{\mathbf{p}}(\tau) = e^{H\tau}A_{\mathbf{p}}\, e^{-H\tau}$$

Introducing the Matsubara Green's functions

$$\hat{G}_{\mathbf{p},\mathbf{p}'}(\tau - \tau') = -\langle \hat{T}A_{\mathbf{p}}(\tau)\bar{A}_{\mathbf{p}'}(\tau')\rangle$$

and making use of Wick's theorem, we obtain

$$Q_{ij}(\mathbf{k}, \Omega) = \mathrm{Av\ Tr}\ T\sum_{\omega}\sum_{\mathbf{p}\mathbf{p}'} \hat{\tau}_i\hat{G}_{\mathbf{p},\mathbf{p}'}(\omega)\hat{\tau}_j\hat{G}_{\mathbf{p}'-\mathbf{k},\mathbf{p}-\mathbf{k}}(\omega - \Omega) \tag{16}$$

where Av denotes averaging over the positions of the impurities and $\hat{G}_{\mathbf{p},\mathbf{p}'}(\omega)$ is the Fourier transform of $\hat{G}_{\mathbf{p},\mathbf{p}'}(\tau)$ at the odd frequencies, $\omega = (2n+1)\pi T$.

At real frequencies, the linear response (14) leads $\eta_i = \sum_j Q_{ij}(\mathbf{k}, \omega)\gamma_j$. Substituting into the self-consistency equations (3) and (4) and using (13), we arrive at the set of equations

$$\begin{bmatrix} 2/\lambda + q_{11} & q_{12} & q_{13} \\ q_{21} & 2/\lambda + q_{22} & q_{23} \\ q_{31} & q_{32} & -k^2/4\pi e^2 \mathrm{N}(0) + q_{33} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ e\phi \end{bmatrix} = 0 \tag{17}$$

Here, $q_{ij} = Q_{ij}/N(0)$, where $N(0)$ is the density of states at the Fermi level, $N(0) = mp_{\mathrm{F}}/2\pi^2$, and λ is the dimensionless electron–phonon coupling constant, $\lambda = N(0)g$. As will be shown later, the matrix elements q_{12}, q_{13}, q_{21}, and q_{31} are identically zero in the absence of a supercurrent,[15] so that the 11 mode (for Δ_1) is decoupled from the 22 and 33 modes. We identify the 11 mode as the Schmid (or longitudinal) mode, with a dispersion relation

$$2/\lambda + q_{11}(\mathbf{k}, \omega) = 0 \tag{18}$$

The remainder of (17) yields the dispersion relation for the Carlson–Goldman, or transverse mode (coupled phase of the order parameter and electric potential):

$$\begin{bmatrix} 2/\lambda + q_{22} & q_{23} \\ q_{32} & -k^2/4\pi e^2 N(0) + q_{33} \end{bmatrix} \begin{bmatrix} \Delta_2 \\ e\phi \end{bmatrix} = 0 \tag{19}$$

For an uncharged superfluid ($e = 0$) the second component of the vector in (19) vanishes identically, leaving only the coefficient of the first component to vanish. The resulting equation is the Anderson–Bogoliubov (AB) mode relationship,

$$2/\lambda + q_{22}(\mathbf{k}, \omega) = 0 \tag{20}$$

In real superconducting metals, because of the strong Coulomb interactions, Eq. (19) yields the dispersion relation for the Carlson–Goldman (CG) mode

$$\begin{bmatrix} 2/\lambda + q_{22} & q_{23} \\ q_{32} & q_{33} \end{bmatrix} = 0 \tag{21}$$

where Eq. (21) is obtained from Eq. (19) by taking the limit $e \to \infty$. We now proceed to analyze these results for the pure (clean) and general impure superconductor.

3. THE CLEAN LIMIT

We consider the case of a pure superconductor, so that no impurity scattering is included in the Hamiltonian. The Green's function in this case is $\hat{G}_{\mathbf{p},\mathbf{p}'} = \hat{G}_{\mathbf{p}} \delta_{\mathbf{p},\mathbf{p}'}$, where[24]

$$\hat{G}_{\mathbf{p}}(\omega) = \frac{1}{i\omega - \hat{\varepsilon}_p} = \frac{-i\omega\hat{\tau}_0 + \Delta\hat{\tau}_1 - \xi_{\mathbf{p}}\hat{\tau}_3}{\omega^2 + \Delta^2 + \xi_p^2} \tag{22}$$

Inserting this expression into Eq. (16), we find for the linear response function

$$Q_{ij}(\mathbf{k}, \Omega) = T \sum_{\omega} \sum_{\mathbf{p}} \mathrm{Tr} \left(\hat{\tau}_i \frac{1}{i\omega - \hat{\varepsilon}_{\mathbf{p}}^+} \hat{\tau}_j \frac{1}{i(\omega - \Omega) - \hat{\varepsilon}_{\mathbf{p}}^-} \right) \tag{23}$$

where

$$\hat{\varepsilon}_{\mathbf{p}}^{\pm} = \hat{\varepsilon}_{\mathbf{p} \pm \mathbf{k}/2} = (\xi_{\mathbf{p}} \pm \mathbf{k} \cdot \mathbf{v}/2)\hat{\tau}_3 - \Delta\hat{\tau}_1 \tag{24}$$

Making use of the identity

$$T \sum_{\omega} \frac{1}{i\omega - \varepsilon_1} \frac{1}{i(\omega - \Omega) - \varepsilon_2} = \frac{n(\varepsilon_1) - n(\varepsilon_2)}{\varepsilon_1 - \varepsilon_2 - i\Omega} \tag{25}$$

where $n(\varepsilon) = (e^{\varepsilon/T}+1)^{-1}$, we can express Eq. (23) as

$$Q_{ij}(\mathbf{k}, \Omega) = \mathrm{Tr} \sum_{\mathbf{p}} \int d\varepsilon_1 \int d\varepsilon_2 \frac{n(\varepsilon_1)-n(\varepsilon_2)}{\varepsilon_1-\varepsilon_2-i\Omega} \hat{\tau}_i \delta(\varepsilon_1-\hat{\varepsilon}_{\mathbf{p}}^{+}) \hat{\tau}_j \delta(\varepsilon_2-\hat{\varepsilon}_{\mathbf{p}}^{-}) \quad (26)$$

The δ-function of the operator is

$$\delta(\varepsilon-\hat{\varepsilon}_{\mathbf{p}}) = \hat{u}_{\mathbf{p}} \delta(\varepsilon-\varepsilon_{\mathbf{p}}) + \hat{v}_{\mathbf{p}} \delta(\varepsilon+\varepsilon_{\mathbf{p}})$$

where

$$\hat{u}_{\mathbf{p}} = \tfrac{1}{2}[1+(\hat{\varepsilon}_{\mathbf{p}}/\varepsilon_{\mathbf{p}})], \qquad \hat{\mathrm{v}}_{\mathbf{p}} = \tfrac{1}{2}[1-(\hat{\varepsilon}_{\mathbf{p}}/\varepsilon_{\mathbf{p}})] \quad (27)$$

Equation (27) follows from taking $\delta(\varepsilon-\hat{\varepsilon}_{\mathbf{p}})$ as a Fourier transform of $\exp(-i\hat{\varepsilon}_{\mathbf{p}}t)$ and using the commutation relations for the Pauli matrices.

The analytical continuation of (26) may be achieved by simply putting $i\Omega$ equal to $\omega + i\delta$, where δ is an infinitesimal small positive quantity, $\delta = +0$. In this manner, the function (26) will have simple poles in the lower half-plane of the complex variable ω, and no singularities in the upper half-plane. At the same time, for $\omega = i\Omega$, $\Omega = 2\pi mT$, this function coincides with (26). Thus, the matrix $q_{ij}(\mathbf{k}, \omega)$ which generates the dispersion relations of Section 2 is $[q_{ij} = Q_{ij}/N(0)]$

$$\begin{aligned} q_{ij}(\mathbf{k}, \omega) = \int d\xi_{\mathbf{p}} \int \frac{d\Omega_p}{4\pi} \Bigg\{ & -[1-n(\varepsilon_{\mathbf{p}}^{+})-n(\varepsilon_{\mathbf{p}}^{-})] \\ & \times \left[\frac{(uv)_{ij}}{\varepsilon_{\mathbf{p}}^{+}+\varepsilon_{\mathbf{p}}^{-}-\omega-i\delta} + \frac{(vu)_{ij}}{\varepsilon_{\mathbf{p}}^{+}+\varepsilon_{\mathbf{p}}^{-}+\omega+i\delta} \right] + [n(\varepsilon_{\mathbf{p}}^{+})-n(\varepsilon_{\mathbf{p}}^{-})] \\ & \times \left[\frac{(uu)_{ij}}{\varepsilon_{\mathbf{p}}^{+}-\varepsilon_{\mathbf{p}}^{-}-\omega-i\delta} + \frac{(vv)_{ij}}{\varepsilon_{\mathbf{p}}^{+}-\varepsilon_{\mathbf{p}}^{-}+\omega+i\delta} \right] \Bigg\} \end{aligned} \quad (28)$$

Here, $\int d\Omega_p/4\pi$ denotes the integration over the solid angle of the vector $\mathbf{p}$ on the Fermi surface, and the coefficients $(uu)_{ij}$, etc., are the coherence factors

$$\begin{aligned} (uu)_{ij} &= \mathrm{Tr}\{\hat{\tau}_i \hat{u}_{\mathbf{p}}^{+} \hat{\tau}_j \hat{u}_{\mathbf{p}}^{-}\} \\ (vv)_{ij} &= \mathrm{Tr}\{\hat{\tau}_i \hat{v}_{\mathbf{p}}^{+} \hat{\tau}_j \hat{v}_{\mathbf{p}}^{-}\} \\ (uv)_{ij} &= \mathrm{Tr}\{\hat{\tau}_i \hat{u}_{\mathbf{p}}^{+} \hat{\tau}_j \hat{v}_{\mathbf{p}}^{-}\} \\ (vu)_{ij} &= \mathrm{Tr}\{\hat{\tau}_i \hat{v}_{\mathbf{p}}^{+} \hat{\tau}_j \hat{u}_{\mathbf{p}}^{-}\} \end{aligned} \quad (29)$$

where $\hat{u}$, $\hat{v}$ are given by Eq. (27).

An explicit calculation of Eq. (28) shows that the components $ij = 12$, 13, 21, and 31 vanish. Thus, the 11 mode is left pure, while the 22 and 33 modes are coupled together, as discussed in Section 2.

Making use of the BCS self-consistency equation for the unperturbed order parameter,

$$1=\lambda\int_{-\omega_D}^{\omega_D} d\xi_{\mathbf{p}}\frac{1-2n(\varepsilon_{\mathbf{p}})}{2\varepsilon_{\mathbf{p}}},\qquad \varepsilon_{\mathbf{p}}=(\xi_{\mathbf{p}}^2+\Delta^2)^{1/2} \tag{30}$$

we now inspect individually the three modes.

3.1. The Mode (Schmid or Longitudinal Mode)

From (29) we obtain

$$\begin{aligned}(uu)_{11}&=(vv)_{11}=\frac{1}{2}\left(1-\frac{\xi_{\mathbf{p}}^{+}\xi_{\mathbf{p}}^{-}-\Delta^2}{\varepsilon_{\mathbf{p}}^{+}\varepsilon_{\mathbf{p}}^{-}}\right)\\ (uv)_{11}&=(vu)_{11}=\frac{1}{2}\left(1+\frac{\xi_{\mathbf{p}}^{+}\xi_{\mathbf{p}}^{-}-\Delta^2}{\varepsilon_{\mathbf{p}}^{+}\varepsilon_{\mathbf{p}}^{-}}\right)\end{aligned} \tag{31}$$

Consider for simplicity the case $\mathbf{k}=0$. The dispersion relation (18), when we use (30), takes the form

$$\int d\xi_p \tanh\frac{\varepsilon_{\mathbf{p}}}{2T}\left\{\frac{\xi_{\mathbf{p}}^2}{\varepsilon_{\mathbf{p}}^2}\left(\frac{1}{2\varepsilon_{\mathbf{p}}+\omega+i\delta}+\frac{1}{2\varepsilon_{\mathbf{p}}-\omega-i\delta}\right)-\frac{1}{\varepsilon_{\mathbf{p}}}\right\}=0 \tag{32}$$

The solution of (32) is $\omega=2\Delta$ at arbitrary temperature, $\Delta=\Delta(T)$. This is the frequency of the Schmid mode at $\mathbf{k}=0$. There is, however, an important point to notice. The left-hand side of Eq. (32) behaves as $(\omega^2-4\Delta^2)^{1/2}$ as $\omega\to 2\Delta$, rather than as the first power of $(\omega-2\Delta)$. Therefore, if one puts an external driving source on the right-hand side of (32), the perturbation of the order parameter will be determined not by a simple pole, but by a more complicated singularity of the branch-point type. The consequences of branch-point behavior were analyzed in detail in Refs. 20 and 21. The perturbed portion of the order parameter oscillates as $\cos(2\Delta t+\delta)$, and at the same time decreases as a function of time like $1/\sqrt{t}$ as $t\to\infty$. To our knowledge, this class of behavior has never been observed. In real metals, electron–phonon scattering or pair-breaking mechanisms (small traces of magnetic impurities, stray magnetic fields, etc.) will alter the above picture and lead to a more conventional exponential relaxation of the order parameter.

3.2. The Mode (AB Mode for Neutral Superfluid)

For an uncharged superfluid ($e=0$) we have only the 22 mode [the Anderson–Bogoliubov (AB) mode]. The coefficients $(uu)_{22}$, etc., in Eq. (28)

are

$$(uu)_{22} = (vv)_{22} = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{p}}^{+}\xi_{\mathbf{p}}^{-} + \Delta^2}{\varepsilon_{\mathbf{p}}^{+}\varepsilon_{\mathbf{p}}^{-}}\right)$$
$$(uv)_{22} = (vu)_{22} = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{p}}^{+}\xi_{\mathbf{p}}^{-} + \Delta^2}{\varepsilon_{\mathbf{p}}^{+}\varepsilon_{\mathbf{p}}^{-}}\right) \tag{33}$$

Inserting Eqs. (28) and (30) in Eq. (20), we obtain the dispersion relation for the AB mode at $T = 0$ (only this limit is considered explicitly, for simplicity)

$$\int d\xi_{\mathbf{p}} \int \frac{d\Omega_p}{4\pi} \frac{(\varepsilon_{\mathbf{p}}^{+} + \varepsilon_{\mathbf{p}}^{-})[\omega^2 - (\xi_{\mathbf{p}}^{+} - \xi_{\mathbf{p}}^{-})^2]}{[\omega^2 - (\varepsilon_{\mathbf{p}}^{+} + \varepsilon_{\mathbf{p}}^{-})^2]\varepsilon_{\mathbf{p}}^{+}\varepsilon_{\mathbf{p}}^{-}} = 0 \tag{34}$$

At small wave vectors, the solution of (34) is

$$\omega = (1/\sqrt{3})kv_{\mathrm{F}} \tag{35}$$

which is a well-known result. At small wave vectors and at finite but small temperatures ($T \ll T_c$), one obtains the result (see Ref. 25)

$$\omega = \frac{kv_{\mathrm{F}}}{\sqrt{3}}\left\{1 + \frac{2}{5}\left(\frac{2\pi T}{\Delta}\right)^{1/2} \exp\left(-\frac{\Delta}{T}\right) - \frac{2\pi i}{\sqrt{3}} \exp\left[-\frac{\Delta}{T}\left(\frac{3}{2}\right)^{1/2}\right]\right\} \tag{36}$$

3.3. Carlson–Goldman (CG) Mode

The discussion of the CG mode is not meaningful for the pure neutral superconductor because of an identity for the (2, 3) part of the matrix q_{ij} (see below). We postpone discussion of this mode until Section 5, in which the general impure case will be considered.

In general, the matrix q_{ij}, where $i, j = 0, 1, 2, 3$ [see discussion after Eq. (12)] can be split into (0, 1) and (2, 3) boxes, which we label q_A and q_B:

$$q_{ij} = \begin{bmatrix} q_A & 0 \\ 0 & q_B \end{bmatrix} \tag{37}$$

Note that q_A and q_B are 2×2 matrices. We have already calculated the coefficients $(uu)_{ij}$, etc., entering q_A [see Eq. (31)]. To complete the calculation of the coefficients entering q_B, which will be used in what follows, we

note that from (29)

$$
\begin{aligned}
(uu)_{33} &= (vv)_{33} = \frac{1}{2}\left(1+\frac{\xi_{\mathbf{p}}^{+}\xi_{\mathbf{p}}^{-}-\Delta^2}{\varepsilon_{\mathbf{p}}^{+}\varepsilon_{\mathbf{p}}^{-}}\right)\\
(uv)_{33} &= (vu)_{33} = \frac{1}{2}\left(1-\frac{\xi_{\mathbf{p}}^{+}\xi_{\mathbf{p}}^{-}-\Delta^2}{\varepsilon_{\mathbf{p}}^{+}\varepsilon_{\mathbf{p}}^{-}}\right)\\
(uu)_{23} &= -(vv)_{23} = -(uu)_{32} = (vv)_{32} = \tfrac{1}{2}i\Delta[(1/\varepsilon_{\mathbf{p}}^{+})-(1/\varepsilon_{\mathbf{p}}^{-})]\\
(uv)_{23} &= -(vu)_{23} = -(uv)_{32} = (vu)_{32} = \tfrac{1}{2}i\Delta[(1/\varepsilon_{\mathbf{p}}^{+})+(1/\varepsilon_{\mathbf{p}}^{-})]
\end{aligned}
\tag{38}
$$

At $k=0$, we obtain from q_B of (37) and Eq. (21)

$$
\frac{1}{\lambda}(\tau_0+\tau_3)+q_B(\omega) = \left[-\left(1+\frac{\omega^2}{4\Delta^2}\right)\tau_0-\frac{\omega}{\Delta}\tau_2+\left(1-\frac{\omega^2}{4\Delta^2}\right)\tau_3\right]I(\omega) \tag{39}
$$

where

$$
I(\omega) = \int d\xi_{\mathbf{p}}\,[1-2n(\varepsilon_{\mathbf{p}})]\frac{2\Delta^2}{\varepsilon_{\mathbf{p}}(4\varepsilon_{\mathbf{p}}^2-\omega^2)} \tag{40}
$$

In (39), τ_i are again the Pauli matrices, but this time without "hats" because they operate on the subspace (2, 3). As will be shown below, the function (39) will not change in the presence of nonmagnetic impurities. This property will be of use in the discussion of collective modes in Section 5.

4. THE GENERAL IMPURE CASE

In an impure superconductor, the function $Q_{ij}(\mathbf{k}, \Omega)$, Eq. (16), can be expressed as a product of the averaged Green's functions,

$$
\mathrm{Av}\,\hat{G}_{\mathbf{p},\mathbf{p}'} = \hat{G}_{\mathbf{p}}(\omega)\delta_{\mathbf{p},\mathbf{p}'}
$$

and the vertex function $\hat{\Gamma}(\mathbf{p}, \omega; \mathbf{p}-\mathbf{k}, \omega-\Omega)$. In the ladder approximation,[24] the equation for the impurity-averaged Green's function and for the vertex function are represented by Feynman diagrams exhibited in Fig. 1. Assuming isotropic scattering and neglecting spin-flip effects (e.g., magnetic impurity scattering) one obtains

$$
\hat{G}_{\mathbf{p}}(\omega) = \frac{1}{(i\omega+\Delta\hat{\tau}_1)\eta_\omega-\xi_{\mathbf{p}}\hat{\tau}_3}, \qquad \eta_\omega = 1+\frac{\nu}{2(\Delta^2+\omega^2)^{1/2}} \tag{41}
$$

Here, $\nu=\tau^{-1}=2\pi n_i N(0)|V_n|^2$ is the impurity scattering rate.

Equations (2) and (3) in Fig. 1, written analytically, read

$$
Q_{ij}(\mathbf{k}, \Omega) = T\sum_{\omega}\sum_{\mathbf{p}}\mathrm{Tr}\,\{\hat{\tau}_i\hat{G}_{\mathbf{p}}(\omega)\hat{\Gamma}_j(\mathbf{p}, \omega; \mathbf{p}-\mathbf{k}, \omega-\Omega)\hat{G}_{\mathbf{p}-\mathbf{k}}(\omega-\Omega)\} \tag{42}
$$

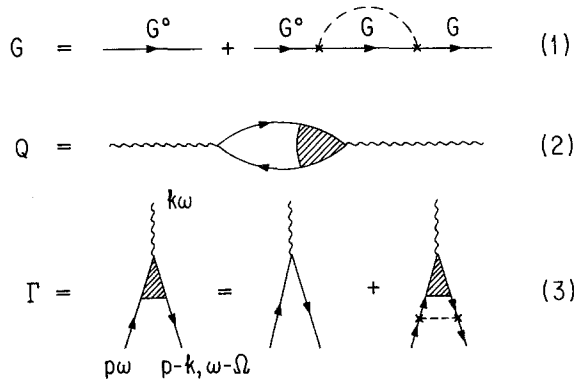

Fig. 1. Feynman diagrams representing the effect of impurity scattering on (1) the Green's function, (2) the linear response function, and (3) the vertex function. The wavy lines represent coupling to the external sources and do not enter the equations.

and

$$\hat{\Gamma}_j(\mathbf{p}, \omega; \mathbf{p}-\mathbf{k}, \omega-\Omega) = \hat{\tau}_j + n_i \sum_{\mathbf{p}'} |V_n(\mathbf{p}-\mathbf{p}')|^2 \hat{\tau}_3 \hat{G}_{\mathbf{p}'}(\omega) \hat{\Gamma}_j(\mathbf{p}', \omega; \mathbf{p}'-\mathbf{k}, \omega-\Omega) \times \hat{G}_{\mathbf{p}'-\mathbf{k}}(\omega-\Omega)\hat{\tau}_3 \tag{43}$$

To solve the latter equation, we expand $\hat{\Gamma}_j$ in Pauli matrices

$$\hat{\Gamma}_j = \sum_{l=0}^{3} T_{jl}\hat{\tau}_l \tag{44}$$

and introduce the polarization function

$$\hat{\Pi}_l(\mathbf{k}; \omega, \omega-\Omega) = \frac{1}{N(0)} \sum_{\mathbf{p}} \hat{G}_{\mathbf{p}}(\omega)\hat{\tau}_l \hat{G}_{\mathbf{p}-\mathbf{k}}(\omega-\Omega). \tag{45}$$

Then, Eqs. (42) and (43) becomes

$$q_{ij} = T \sum_{\omega} \sum_{l} \mathrm{Tr}\,(\hat{\tau}_i T_{jl} \hat{\Pi}_l) \tag{46}$$

and

$$\hat{\Gamma}_j = \hat{\tau}_j + \frac{\nu}{2\pi} \sum_{l} T_{jl}\hat{\tau}_3 \hat{\Pi}_l \hat{\tau}_3 \tag{47}$$

Expanding the polarization function $\hat{\Pi}_l$ in Pauli matrices

$$\hat{\Pi}_l = \sum_{m} S_{lm}\hat{\tau}_m \tag{48}$$

where the coefficients S_{lm} can be found from Eqs. (41) and (45), we find that the equation for the vertex function takes the form

$$\sum_l T_{jl}\left(\hat{\tau}_l - \frac{\nu}{2\pi}\sum_m \hat{\tau}_3\hat{\tau}_m\hat{\tau}_3 S_{lm}\right) = \hat{\tau}_j \tag{49}$$

Now, regarding T_{jl} as the elements of a matrix T, this matrix is the inverse of the matrix:

$$\delta_{jl} - (\nu/2\pi)S_{jm}E^{(33)}_{ml}$$

where

$$E^{(ij)}_{ml} = \tfrac{1}{2}\,\mathrm{Tr}\,(\hat{\tau}_i\hat{\tau}_l\hat{\tau}_j\hat{\tau}_m) \tag{50}$$

From these results we obtain the following expression for the response function $q(\mathbf{k}, \Omega)$:

$$q(\mathbf{k}, \Omega) = 2T\sum_\omega \left(1 - \frac{\nu}{2\pi} SE^{(33)}\right)^{-1} S \tag{51}$$

written in matrix notation. The matrix S, from Eq. (45), is

$$S_{lm} = \frac{1}{2}\,\mathrm{Tr}\int d\xi_{\mathbf{p}} \int \frac{d\Omega_p}{4\pi}\,\hat{G}_{\mathbf{p}+\mathbf{k}/2}(\omega)\hat{\tau}_l\hat{G}_{\mathbf{p}-\mathbf{k}/2}(\omega - \Omega)\hat{\tau}_m \tag{52}$$

Inserting the expression for $\hat{G}_{\mathbf{p}}$, Eq. (41), into Eq. (52) yields a combination of the matrices $E^{(ij)}$ [see Eq. (50)]. These are readily calculated, and found to be of the quasidiagonal structure involving blocks (0, 1) and (2, 3):

$$E^{(00)} = \begin{bmatrix} \tau_0 & 0 \\ 0 & \tau_0 \end{bmatrix}, \quad E^{(11)} = \begin{bmatrix} \tau_0 & 0 \\ 0 & -\tau_0 \end{bmatrix}, \quad E^{(33)} = \begin{bmatrix} \tau_3 & 0 \\ 0 & -\tau_3 \end{bmatrix}$$
$$E^{(01)} = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix}, \quad E^{(10)} = \begin{bmatrix} \tau_1 & 0 \\ 0 & -\tau_2 \end{bmatrix} \tag{53}$$

Thus, for a general concentration of impurities, the (0, 1) block of the perturbation matrix is decoupled from the (2, 3) block. The latter describes the AB and CG modes, as we have discussed in Section 2.

Using Eqs. (51) and (53), we find that the (2, 3) part, denoted by q_B [see Eq. (37)] is

$$q_B(\mathbf{k}, \Omega) = 2T\sum_\omega \left(1 + \frac{\nu}{2\pi} S_B\tau_3\right)^{-1} S_B \tag{54}$$

where, from Eqs. (52) and (53),

$$S_B = -\pi\left\langle \frac{R + R' + \nu}{(R + R' + \nu)^2 + (\mathbf{k}\cdot\mathbf{v}_F)^2}\right\rangle \left(\frac{\Delta^2 + \omega\omega'}{RR'}\tau_0 + \frac{i\Delta\Omega}{RR'}\tau_2 + \tau_3\right) \tag{55}$$

Here the following notations have been introduced:

$$R=(\Delta^2+\omega^2)^{1/2}, \qquad R'=(\Delta^2+\omega'^2)^{1/2}, \qquad \omega'=\omega-\Omega$$

and $\langle\cdots\rangle$ denotes average over the Fermi surface. The inverse of the matrix $[1+(\nu/2\pi)S_B\tau_3]$ is calculated using the identity

$$\frac{1}{A+B\tau_2+C\tau_3}=\frac{A-B\tau_2-C\tau_3}{A^2-B^2-C^2} \tag{56}$$

The dispersion relation for the AB mode (neutral superfluid, only the 22 matrix element is considered) is given by

$$(1/\lambda)(\tau_0+\tau_3)+q_B(\mathbf{k},\omega)|_{22}=0 \tag{57}$$

whereas for the CG mode we have

$$\det[(1/\lambda)(\tau_0+\tau_3)+q_B(\mathbf{k},\omega)]=0 \tag{58}$$

In Eqs. (57) and (58), $q_B(\mathbf{k},\omega)$ means the analytically continued quantity $q_B(\mathbf{k},\Omega)$. The problem of analytical continuation is far from trivial and is solved for the expression (55) in the Appendix.

We now expand $q_B(\mathbf{k},\Omega)$ in powers of $\mathbf{k}$:

$$q_B(\mathbf{k},\Omega)=a(\Omega)+\langle(\mathbf{k}\cdot\mathbf{v}_F)^2\rangle b(\Omega)+\text{higher orders in } k^2 \tag{59}$$

Using Eqs. (54)–(56) we obtain

$$a(\Omega)=-2\pi T\sum_\omega\frac{1}{R+R'}\left(\frac{\Delta^2+\omega\omega'}{RR'}\tau_0+\frac{i\Delta\Omega}{RR'}\tau_2+\tau_3\right) \tag{60}$$

$$b(\Omega)=2\pi T\sum_\omega\frac{1}{(R+R')^2(R+R'+\nu)}\left(\frac{\Delta^2+\omega\omega'}{RR'}\tau_0+\frac{i\Delta\Omega}{RR'}\tau_2+\tau_3\right) \tag{61}$$

Only the $\mathbf{k}$-dependent part of $q_B(\mathbf{k},\Omega)$ depends on the scattering [ν does not appear in (60)]. This may be considered as a manifestation of the Anderson theorem (which states that time-reversal-invariant scattering does not affect the superconducting properties in the ground state). We may therefore expect that the analytical continuation of $a(\Omega)$ should coincide with the quantity $q_B(\omega)$, Eq. (39), calculated in the previous section (in the clean limit). In fact, this is the case, as shown in the Appendix by directly performing the procedure of analytical continuation.

The next term in the expansion (59) is impurity scattering-dependent. The analytical continuation of $b(\Omega)$ is accomplished in the Appendix. Close to the critical temperature T_c, the problem can be solved in a much simpler

way. Putting $\Delta = 0$ in $b(\Omega)$, we obtain

$$b_0(\Omega) = 2\pi T \sum_{\omega} \frac{\tau_3 + \tau_0[\omega\omega'/(|\omega|\,|\omega'|)]}{(|\omega| + |\omega'|)^2(|\omega| + |\omega'| + \nu)}, \qquad \omega' = \omega - \Omega \tag{62}$$

This can be rewritten in the form ($\Omega = 2\pi m T$)

$$b_0(\Omega) = \frac{1}{\pi T} \sum_{n=0}^{\infty} \frac{\tau_0 + \tau_3}{(2n+m+1)^2[\nu + 2\pi T(2n+m+1)]} - \frac{(\tau_0 - \tau_3)(1 - \delta m, 0)}{2\pi m T(\nu + 2\pi m T)}, \qquad m \geqslant 0 \tag{63}$$

The function b_0 should be analytical in the upper half-space. A function satisfying this condition is

$$b_0(\omega) = \frac{1}{\pi\nu T} \sum_{n=0}^{\infty} \frac{\tau_0 + \tau_3}{(2n+1-i\omega/2\pi T)^2[1 + (2\pi T/\nu)(2n+1-i\omega/2\pi T)]} - \frac{i\omega(\tau_0 - \tau_3)}{\omega(\omega + i\gamma)(\nu - i\omega)} \tag{64}$$

where γ is an infinitesimal positive quantity (see the Appendix). At frequencies such that

$$\omega \ll T_c \tag{65}$$

which will be supposed to be the case in the coefficient of the k^2 term, Eq. (64) becomes

$$b_0(\omega) = \begin{bmatrix} (\pi/4\nu T)\chi(2\pi T/\nu) & 0 \\ 0 & 2/[\omega(\omega + i\nu)] \end{bmatrix} \tag{66}$$

Here,

$$\chi(x) = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2[1 + (2n+1)x]}, \qquad x = \frac{2\pi T}{\nu}, \quad \nu = \frac{1}{\tau}$$

$$\chi(x) \to 1, \qquad x \ll 1 \quad \text{(dirty limit)} \tag{67}$$

$$\chi(x) \to \frac{7\zeta(3)}{2\pi^3 T\tau}, \qquad x \gg 1 \quad \text{(clean limit)}$$

is the Gorkov function[26] introduced in the theory of superconducting alloys. A plot of this function is given in Fig. 2.

In the opposite limiting case, near $T = 0$, the expansion of the analytical function $b(\omega)$ in powers of ω corresponds to the expansion of $b(\Omega)$ in Ω (in the sense that the coefficients of both expansions coincide). Putting $\Omega = 0$

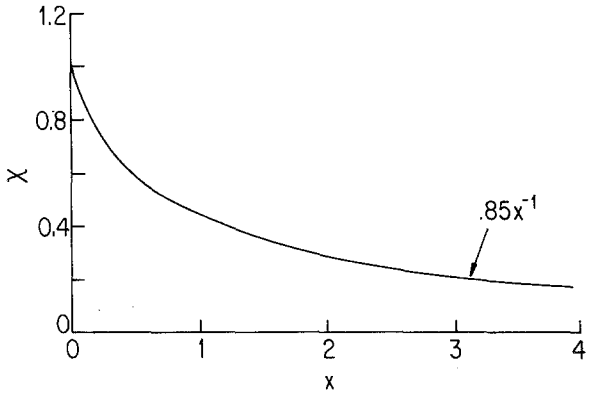

Fig. 2. The Gorkov impurity function $\chi(x)$, Eq. (67). At large x, the asymptotic value of χ is $7\zeta(3)/\pi^2 x$.

in Eq. (61), we obtain

$$b(0)=\frac{1}{2}\int_0^\infty \frac{d\omega}{(\omega^2+\Delta^2)[(\omega^2+\Delta^2)^{1/2}+\nu/2]}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \tag{68}$$

The integral is proportional to the density of superelectrons n_s. We set

$$b(0)=\frac{1}{4\Delta^2}\frac{n_s}{n}(\tau_0+\tau_3) \tag{69}$$

where $n_s/n\to 1$, $2\Delta\tau\gg 1$ (clean limit), and $n_s/n\to \pi\Delta/\nu=\pi\Delta\tau$, $2\Delta\tau\ll 1$ (dirty limit).

Finally, we estimate the next term in the expansion (59) for the limit $T\to T_c$. The fourth-power term in $\mathbf{k}$ is

$$-\langle(\mathbf{k}\cdot\mathbf{v}_F)^2\rangle^2 2\pi T\sum_\omega \frac{9/5+\nu/(R+R')}{(R+R')^2(R+R'+\nu)^3}\left(\frac{\Delta^2+\omega\omega'}{RR'}\tau_0+\frac{i\Delta\Omega}{RR'}\tau_2+\tau_3\right) \tag{70}$$

Consider first the dirty limit. Close to T_c, the contribution to the matrix $q_B(\mathbf{k},\omega)$ from (50) is

$$-(\tfrac{1}{3}k^2v_F^2)^2\begin{bmatrix}7\zeta(3)/2(2\pi T\nu)^2 & 0\\ 0 & 2/\omega^2\nu^2\end{bmatrix} \tag{71}$$

Comparing this with the second-order term, Eq. (66), we see that the

fourth-order term is much smaller than the second-order one if the condition $Dk^2 \ll T$ holds ($D = \frac{1}{3}v_F^2\tau$). This holds even for "high" values of $\mathbf{k}$, corresponding now to $Dk^2 \gtrsim \Delta$, considered in the next section. In the extreme clean limit ($\nu = 0$), and at the vicinity of T_c, the fourth-order term becomes

$$(\tfrac{1}{3}k^2v_F^2)^2\,\frac{18}{5}\begin{bmatrix}(4\pi T)^{-4}\sum_{n=0}^{\infty}(n+1/2)^{-5} & 0\\ 0 & -1/\omega^4\end{bmatrix} \tag{72}$$

The corresponding second-order term is [see Eqs. (66) and (67)]

$$\tfrac{1}{3}k^2v_F^2 2\begin{bmatrix}(4\pi T)^{-2}\sum_{n=0}^{\infty}(n+1/2)^{-3} & 0\\ 0 & 1/\omega^2\end{bmatrix} \tag{73}$$

If we compare (73) term by term with (72), we see the 11 terms are in the approximate ratio $k^2v_F^2/T^2$ while the 22 terms are in the ratio $k^2v_F^2/\omega^2$. We work in the regime where $\omega \ll T$, so only the 22 term in the fourth order needs to be examined. A calculation of the CG dispersion relation (see next section) in the extreme clean limit shows that the corrections due to this term enter the equation in the form $a(\Delta/T)^2/(1+k^2v_F^2/\omega^2)$, where a is a number of order unity. Anticipating the results of the next section, we put $k^2v_F^2/\omega^2 \sim T/\Delta$, so that the correction term (near T_c) enters as $a(\Delta/T)^3$ and therefore can be neglected.

5. FREQUENCIES OF THE COLLECTIVE MODES

The formal results of the previous sections will now be applied to derive the frequency–wave vector dependence of the AB and CG modes. We shall exhibit results, for simplicity, at zero temperature and at temperatures close to T_c, but we stress again that our calculation holds over the full temperature range.

Collecting the results (17), (19), (37), (39), and (59), we obtain at all temperatures

$$\left\{\left[-\left(1+\frac{\omega^2}{4\Delta^2}\right)\tau_0-\frac{\omega}{\Delta}\tau_2+\left(1-\frac{\omega^2}{4\Delta^2}\right)\tau_3\right]I(\omega)\right.$$
$$\left.-\frac{k^2}{8\pi e^2N(0)}(\tau_0-\tau_3)+\frac{1}{3}v_F^2k^2b(\omega)\right\}\begin{bmatrix}\Delta_2\\ e\phi\end{bmatrix}=0 \tag{74}$$

where $I(\omega)$ is given by Eq. (40). The matrix $b(\omega)$ is given by Eq. (66) for T close to T_c and by Eq. (68) for zero temperature. Equation (74) is our central result and now will be analyzed in the two temperature regimes.

5.1. Zero Temperature

The function $I(\omega)$ in this limit is denoted by $I_0(\omega)$ and is [see Eq. (40)],

$$I_0(\omega) = \int_0^\infty d\xi \frac{\Delta^2}{\varepsilon[\varepsilon^2 - (\omega/2 + i\delta)^2]}$$

$$= \frac{1}{iz(1-z^2)^{1/2}} \ln[(1-z^2)^{1/2} + iz], \qquad z = \frac{\omega + i\delta}{2\Delta} \tag{75}$$

where $I_0(\omega) \to 1$ for $\omega \ll 2\Delta$.

We consider first the AB mode, which pertains to a neutral superfluid. In this case we need only the 22 matrix element of (74). Making use of Eqs. (68) and (69) for the matrix b, we find

$$-\tfrac{1}{2}\omega^2 I_0(\omega) + \tfrac{1}{6}k^2 v_F^2 n_s/n = 0 \tag{76}$$

In the limit $\omega \ll 2\Delta$, the dispersion law of the collective mode becomes[12,13]

$$\omega = ck, \qquad c = (v_F/\sqrt{3})(n_s/n)^{1/2} \tag{77}$$

Using Eq. (69), we see that the velocity of the AB mode at $T = 0$ reduces to $v_F/\sqrt{3}$ in the clean limit ($\Delta\tau \gg 1$) and to

$$c = (\pi D \Delta)^{1/2} \tag{78}$$

in the dirty limit ($\Delta\tau \ll 1$). Here D is the diffusion coefficient, $D = \frac{1}{3}v_F^2\tau$, where $\tau = 1/\nu$. It is curious to note that in the dirty limit the $T = 0$ velocity of the AB mode for a neutral Fermi gas is [with the replacement of π in Eq. (78) by 2] the same as for the CG mode appropriate to a charged Fermi gas near T_c.

We investigate next the CG mode at $T = 0$. We require that the determinant of the full matrix in Eq. (74) vanishes (rather than just the 11 term for the AB mode). Investigation of this determinant shows that without the Coulomb term $-k^2(\tau_0 - \tau_3)/8\pi e^2 N(0)$ [compare Eq. (17) to Eq. (21)] this requirement can be satisfied only at $\mathbf{k} = 0$. Including the Coulomb term, we find, using the approximation $I_0(\omega) = 1$ valid for $\omega \ll 2\Delta$,

$$\frac{\omega}{\omega_p^2} - \frac{n_s}{n} - \frac{n_2}{2n}\frac{v_F^2 k^2}{3\omega_p^2} = 0 \tag{79}$$

where ω_p is the plasma frequency, $\omega_p^2 = 4\pi n e^2/m$. In the limit $\mathbf{k} \to 0$, this gives

$$\omega = (4\pi n_s e^2/m)^{1/2} = \omega_p (n_s/n)^{1/2} \tag{80}$$

Our limit $\omega \ll 2\Delta$ puts a severe limitation on the applicability of this result. For a dirty superconductor (attempting to get as small a ratio n_s/n as possible), this limit requires $\omega_p(\tau/T)^{1/2} < 1$,' or very dirty systems indeed.

It is conceivable that this condition may be satisfied in some granular superconductors.

5.2. Temperatures Close to T_c

At this range of temperatures, the function $I(\omega)$, from Eq. (40), becomes

$$I(\omega)=\frac{\Delta^2}{2T}\int_0^\infty \frac{d\xi}{\varepsilon^2-(\omega/2+i\delta)^2}$$

$$=\frac{\pi i\Delta^2}{2T(\omega^2-4\Delta^2)^{1/2}}, \qquad \mathrm{Im}\,(\omega^2-4\Delta^2)^{1/2}>0 \tag{81}$$

The limiting values of $I(\omega)$ are

$$I(\omega)=\begin{cases}\pi\Delta/4T, & \omega\ll 2\Delta\\ \pi i\Delta^2/2\omega T, & \omega\gg 2\Delta\end{cases} \tag{82}$$

The equation for the AB mode [the 22 matrix element of Eq. (74)] is

$$-\frac{\omega^2}{2\Delta^2}I(\omega)+\frac{\pi}{12\nu T}\chi(2\pi T\tau)k^2v_F^2=0 \tag{83}$$

where χ is given by Eq. (67). The small-frequency limit of $I(\omega)$ results in

$$\omega=ck, \qquad c=(2\Delta D\chi)^{1/2} \tag{84}$$

Solving (83) for arbitrary ω, we obtain the following dispersion relation:

$$\omega^2=-\tfrac{1}{2}(Dk^2\chi)^2+\tfrac{1}{2}Dk^2\chi[(Dk^2\chi)^2+16\Delta^2]^{1/2} \tag{85}$$

The frequency of the mode changes linearly at small k, whereas at large k (such that $Dk^2\chi\geq 4\Delta$) it saturates at 2Δ. This behavior is depicted in Fig. 3.

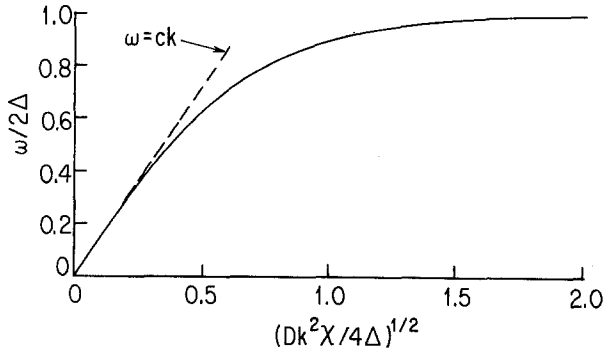

Fig. 3. The dependence of the frequency of the Anderson–Bogoliubov mode on the wave vector.

We now turn to the CG model In this case we may neglect the Coulomb term $-k^2(\tau_0-\tau_3)/8\pi e^2 N(0)$ because of the lack of net charge associated with this mode (see the discussion in Section 2). The matrix in Eq. (74) becomes

$$\begin{bmatrix} -\omega^2/2\Delta^2 & i(\omega/\Delta) \\ -i(\omega/\Delta) & -2 \end{bmatrix} I(\omega) + Dk^2 \begin{bmatrix} \pi\chi/4T & 0 \\ 0 & 2\nu/\omega(\omega+i\nu) \end{bmatrix} \tag{86}$$

where the functions $I(\omega)$ and $\chi(2\pi T/\nu)$ are given by Eqs. (67) and (81), respectively. The requirement that the determinant of (86) vanishes leads to the dispersion relation for the CG mode at arbitrary frequencies and impurity concentrations. We investigate the solution for small frequencies, such that $\omega < 2\Delta$ and $\omega\tau < 1$. In this regime we obtain

$$\omega^2 + \frac{\pi i \Delta^2 \chi}{2T}\omega - 2\Delta D k^2 \chi = 0 \tag{87}$$

which yields

$$\omega = -\frac{\pi i \Delta^2 \chi}{4T} \pm \left[-\left(\frac{\pi\Delta^2}{4T}\chi\right)^2 + 2\Delta D k^2 \chi \right]^{1/2} \tag{88}$$

At $\mathbf{k} = 0$, we have a relaxation mode,

$$\omega = -i\Gamma, \qquad \Gamma = (\pi\Delta^2/2T)\chi(2\pi T\chi) \tag{89}$$

The relaxation rate (i.e., the damping of the mode) is

$$\Gamma = \begin{cases} \dfrac{\pi\Delta^2}{2T} = \dfrac{4\pi^3}{7\zeta(3)}(T_c - \mathrm{T}), & T\tau \ll 1 \quad \text{(dirty limit)} \\ \dfrac{7\zeta(3)\Delta^2}{4\pi^3 T^2 \tau} = \dfrac{2}{T\tau}(T_c - T), & T\tau \gg 1 \quad \text{(clean limit)} \end{cases} \tag{90}$$

The relaxation time Γ^{-1} diverges as $1/(T_c - T)$, as observed experimentally.[27] For $2\Delta D\chi k^2 \gg 2\Gamma^2$, we obtain from (88) a soundlike propagating mode, with a velocity c given by

$$c = [2\Delta D\chi(2\pi T\tau)]^{1/2} \tag{91}$$

which coincides with the velocity of the AB mode near T_c, Eq. (84). Equation (91) reduces to the result of Schmid and Schön[14] in the dirty limit, where [see Eq. (67)]

$$c = (2\Delta D)^{1/2} \tag{92}$$

In the clean limit, it yields the result of Artemenko and Volkov[17]

$$c = v_{\mathrm{F}}\left[\frac{7\zeta(3)}{3\pi^3}\frac{\Delta}{T}\right]^{1/2} \tag{93}$$

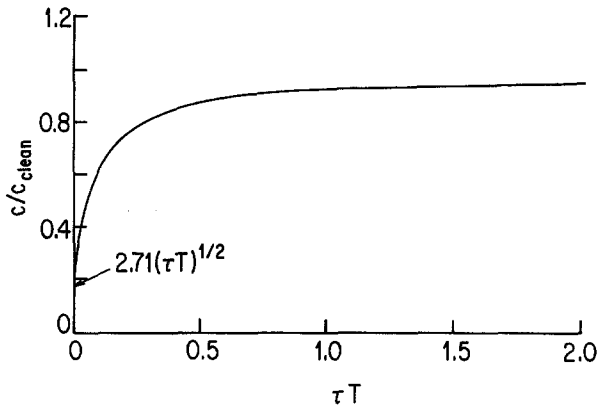

Fig. 4. The dependence of the velocity of the propagating mode on the scattering rate $\nu = \tau^{-1}$.

The velocity of the mode as a function of the scattering rate ν is exhibited in Fig. 4. It is seen that the velocity is smaller as the superconductor becomes dirtier.

From Eqs. (88), (89), and (91) we see that the condition for propagation is

$$\Gamma \ll ck \tag{94}$$

In the dirty limit this becomes

$$\pi\Delta^2/2T < (2\Delta D)^{1/2}k \qquad \text{or} \qquad \omega T > \Delta^2 \qquad \text{(dirty limit)}$$

In the clean limit

$$\frac{7\zeta(3)\Delta^2}{4\pi^2 T^2 \tau} < v_F\left[\frac{7\zeta(3)}{3\pi^3}\,\frac{\Delta}{T}\right]^{1/2} k$$

or

$$v_F \tau k > (\Delta/T)^{3/2} \qquad \text{(clean limit).}$$

The last results mean that lk must be greater than $(\Delta/T)^{3/2}$, which is small in the vicinity of T_c (l is the mean free path). The clean-limit requirement can also be written in the form

$$\omega\tau > \Delta^2/T^2$$

These conditions agree with those of Artemenko and Volkov.[17] In deriving (87) we have assumed $\omega\tau < 1$ and $\omega < 2\Delta$. Therefore, those two conditions must be added to the condition (94). We stress, however, that one can solve

for the dispersion relation of the CG mode of T close to T_c for arbitrary frequencies by using the full expression (86).

Keeping only the assumption $\omega\tau < 1$, and using in (86) the result (81) for $I(\omega)$, we obtain

$$\Omega = \frac{i\omega}{(\omega^2 - 4\Delta^2)^{1/2}}(\omega + i\Gamma) \tag{95}$$

where Γ is given by Eq. (89) and

$$\Omega = Dk^2\chi(2\pi T\tau) \tag{96}$$

The right-hand side of Eq. (95) is plotted in Fig. 5 as a function of ω. There exists a solution of (95) corresponding to ω less than 2Δ with a small imaginary part due to Γ. There is no solution, real or imaginary, for larger values of ω.

For frequencies much higher than Γ (the damping rate) one can solve (95) by first dropping Γ and then including it as a small perturbation. The

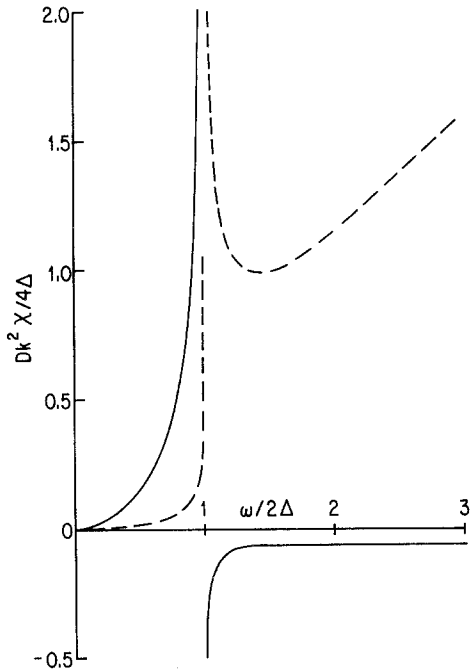

Fig. 5. Graphic representation of the dispersion relation of the Carlson–Goldman mode, Eq. (95), for $\Gamma = 0.1$, and $\Gamma = \pi\Delta\chi/4T$. Solid line Re $(Dk^2\chi/4\Delta)$, dashed line Im $(Dk^2\chi/4\Delta)$.

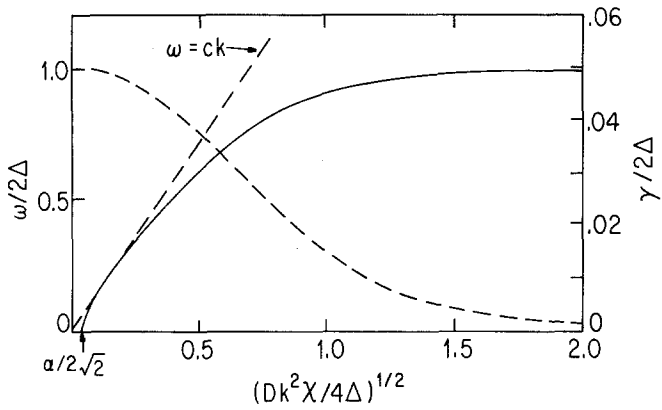

Fig. 6. Frequency (solid line) and damping (dashed line) of the Carlson–Goldman mode as a function of the wave vector k, for $\alpha = \pi\Delta\chi/4T = 0.1$.

resulting frequency–wave vector dependence is

$$\omega = \omega(k) - i\gamma(k) \tag{97}$$

where

$$\omega(k) = [-\tfrac{1}{2}\Omega^2 + \tfrac{1}{2}\Omega(\Omega^2 + 16\Delta^2)^{1/2}]^{1/2} \tag{98}$$

$$\gamma(k) = \Gamma \frac{1 - \omega^2(k)/4\Delta^2}{2 - \omega^2(k)/4\Delta^2} \tag{99}$$

The dependence of the frequency and damping of the CG mode on the wave vector is depicted in Fig. 6.

6. DISCUSSION AND SUMMARY

We have presented a linear response approach to the problem of a superconductor subjected to small perturbations. Our treatment allows a direct calculation of the pair-field susceptibilities (the longitudinal susceptibility, related to changes in the magnitude of the order parameter, and the transverse susceptibility, related to the coupled changes in the phase of the order parameter and the scalar potential). The main advantage of the procedure is its applicability to the full range of frequencies, temperatures, and impurity concentration.

We can summarize the range of validity of our results by putting the AB and CG mode equations in the following form:

$$A + Bk^2 = 0 \tag{100}$$

The impurity scattering rate enters only in B. We have found the function A for arbitrary frequencies and temperatures. As for the function B, we have calculated it explicitly for an arbitrary scattering rate in two different temperature regimes: (i) at T close to T_c, where we have assumed that $\omega \gg T_c$; and (ii) at $T = 0$, where we set $\omega = 0$. We point out in the Appendix how B can be calculated in the dirty limit for any value of frequency and temperature. It should be stressed that the limits on the validity of our results affect only the k^2 term, and that at temperatures close to T_c the only limitation is that $\omega \ll T_c$. Our results are not limited to the dirty or clean limit, and may therefore be appropriate to experiments which are conducted in the intermediate regime where no applicable theory is available in the literature.

The calculation presented here has not included electron–phonon inelastic scattering or pair-breaking effects (e.g., magnetic impurity scattering, constant supercurrent). The electron–phonon scattering is a crucial factor for low-frequency phenomena, e.g., relaxation of branch imbalance. It is clearly necessary to take the next step and calculate its influence using the approach we have presented in this paper. In fact, we can obtain some results even at this stage by resorting to a well-known trick. Consider, for example, the characteristic penetration length for an electric field inside a superconductor (the so-called Waldram length[22,28]).

Were we to have included electron–phonon scattering, it would have entered our formalism in two ways: (i) In the equilibrium Green's functions. The expression (41) would be modified because of the inclusion of self-energy terms arising from the electron–phonon interaction (see, for example, Ref. 18). (ii) In the vertex corrections. They will appear in the linear response function Q [Eq. (42)]. With these conditions in mind we return to Eq. (74), obtaining the following set of equations valid in the vicinity of T_c:

$$\begin{aligned} &i\omega\left(e\phi + i\frac{\omega}{2\Delta}\Delta_2\right) + \chi D k^2 \Delta_2 = 0, \qquad \chi = \chi(2\pi T\tau) \\ &\frac{\pi\Delta}{4T}\left(e\phi + i\frac{\omega}{2\Delta}\Delta_2\right) + Dk^2\left(\frac{1}{8\pi e^2 DN(0)} + \frac{i}{\omega + i\gamma}\right) e\phi = 0 \end{aligned} \tag{101}$$

We have used Eqs. (64), (66), and (82) to derive (101).

We investigate the first equation in (101). From our definition of Δ_2 [see Eq. (11)] we have $\Delta_2 = -\Delta\theta$, where θ is the phase of the order parameter. The combination

$$e\phi + i\frac{\omega}{2\Delta}\Delta_2 = e\phi - \frac{1}{2}\frac{\partial\theta}{\partial t}$$

is just the gauge-invariant form of the scalar potential. On the other hand,

$$\chi Dk^2\Delta_2 = \chi D\Delta\nabla^2\phi = \frac{2T}{\pi\Delta}\frac{1}{eN(0)}(-\nabla\cdot\mathbf{j}_s) \tag{102}$$

where $\mathbf{j}_s$ is the supercurrent density. Therefore, the first equation of (101) reads

$$\nabla\cdot\mathbf{j}_s = \frac{\pi\Delta}{2T}\frac{\partial}{\partial t}\left[eN(0)\left(e\phi - \frac{1}{2}\frac{\partial\theta}{\partial t}\right)\right] \tag{103}$$

This equation has been derived, of course, in the absence of electron–phonon inelastic scattering. However, it is known (see Refs. 29, 30) that when electron–phonon inelastic scattering is included, one should change $\partial/\partial t \to \partial/\partial t + i/\tau_E$, where τ_E denotes the electron–phonon inelastic scattering time. This replacement enters in the first partial time derivative in (103), but not in $\partial\theta/\partial t$. The latter is part of the gauge-invariant form of the scalar potential and remains inviolate in the presence of pair-breakers.

Next, consider the second equation of (101). The combination of $\omega + i\gamma$ appears explicitly. In our formalism, γ is a small, positive quantity. Physical considerations suggest we replace γ with $1/\tau_E$.

We make the changes discussed above in Eqs. (101). When we do this, and neglect $1/8\pi e^2 DN(0) = 1/2\omega_p^2\tau$ compared to $1/\omega + i\gamma$ (ω_p is the plasma frequency), we find

$$\left[Dk^2 + \frac{\pi\Delta}{4T\tau_E}(1 - i\omega\tau_E)\left(1 - \frac{2T}{\pi\Delta^2\chi}i\omega\right)\right]\left(e\phi + i\frac{\omega}{2\Delta}\Delta_2\right) = 0 \tag{104}$$

This equation is our final expression "corrected" for inelastic electron–phonon scattering. It generates the frequency-dependent Waldram length:

$$\lambda_s^{-2} = \frac{\pi\Delta}{4T\tau_E}\frac{1}{D}(1 - i\omega\tau_E)\left(1 - \frac{2T}{\pi\Delta^2\chi}i\omega\right) \tag{105}$$

In the dirty limit, where $\chi \to 1$, Eq. (105) reduces to the well-known result.[23,30] In the clean limit, from Eq. (67), $\chi \to 1/T\tau$. As a result, the term in the second parentheses become more prominent as the superconductor becomes cleaner. In both limits, λ_s diverges as $(T_c - T)^{-1/4}$ as T approaches T_c. This has been verified experimentally.[31] Our expression for the Waldram length, Eq. (105), is valid over the full impurity concentration regime, and is not restricted to the dirty limit.

APPENDIX: ANALYTICAL CONTINUATION OF EQ. (59)

We first consider the analytical continuation of $a(\Omega)$, Eqs. (59) and (60). To this end, we use the identities

$$\int_{-\infty}^{\infty} \frac{d\xi}{(\xi^2+R^2)(\xi^2+R'^2)} = \frac{1}{RR'}\frac{\pi}{R+R'}$$
$$\int_{-\infty}^{\infty} \frac{\xi^2\, d\xi}{(\xi^2+R^2)(\xi^2+R'^2)} = \frac{\pi}{R+R'} \tag{A1}$$

where

$$R=(\Delta^2+\omega^2)^{1/2}, \qquad R'=(\Delta^2+R'^2)^{1/2}, \qquad \omega'=\omega-\Omega \tag{A2}$$

With these identities, the function $a(\Omega)$ is

$$a(\Omega) = -2T\sum_{\omega}\int d\xi \frac{(\Delta^2+\omega\omega')\tau_0 + i\Delta(\omega-\omega')\tau_2 + \xi^2\tau_3}{(\xi^2+\Delta^2+\omega^2)(\xi^2+\Delta^2+\omega'^2)} \tag{A3}$$

The summation over the frequencies ω must be accomplished prior to the integration over ξ, as discussed by Abrikosov *et al.*[24] We first decompose the integrand in (A3) as follows:

$$\begin{aligned} a(\Omega) = -\frac{1}{2}T\sum_{\omega}\int d\xi \Bigg\{ & \frac{(1+\Delta^2/\varepsilon^2)\tau_0 + (2\Delta/\varepsilon)\tau_2 + (\xi^2/\varepsilon^2)\tau_3}{(\varepsilon-i\omega)(\varepsilon+i\omega')} \\ & + \frac{(1+\Delta^2/\varepsilon^2)\tau_0 - (2\Delta/\varepsilon)\tau_2 + (\xi^2/\varepsilon^2)\tau_3}{(\varepsilon+i\omega)(\varepsilon-i\omega')} \\ & + \frac{\xi^2}{\varepsilon^2}(\tau_3-\tau_0)\left[\frac{1}{(\varepsilon+i\omega)(\varepsilon+i\omega')} + \frac{1}{(\varepsilon-i\omega)(\varepsilon-i\omega')}\right]\Bigg\} \\ \varepsilon = (\xi^2+\Delta^2)^{1/2} & \end{aligned} \tag{A4}$$

The summation over ω is carried out using the identities

$$T\sum_{\omega}\frac{1}{(\varepsilon-i\omega)(\varepsilon+i\omega')} = \frac{\tanh(\varepsilon/2T)}{2\varepsilon-i\Omega}$$
$$T\sum_{\omega}\frac{1}{(\varepsilon-i\omega)(\varepsilon-i\omega')} = -\frac{1}{4T\cosh^2(\varepsilon/2T)}\delta_{\Omega,0} \tag{A5}$$

The analytical continuation of the quantity $1/(2\varepsilon - i\Omega)$ is achieved by simply putting $i\Omega = \omega + i\delta$, $\delta = +0$, where now ω is the physical (real) frequency. Obviously, $1/(2\varepsilon - \omega - i\delta)$ is an analytical function in the upper half-space, $\mathrm{Im}\,\omega > 0$, which coincides at the points $\omega = i\Omega = 2\pi i m T$ with $1/(2\varepsilon - i\Omega)$. The analytical continuation of the delta function $\delta_{\Omega,0} = \delta_{m,0}$ is

obtained as the limit

$$\lim_{\gamma \to +0} \frac{i\gamma}{\omega + i\gamma}$$

This quantity equals 1 at $\omega = 0$, and equals 0 at $\omega = 2\pi i m T$, $m \neq 0$. At the same time, it is an analytic function in the half-plane $\mathrm{Im}\,\omega > 0$. From physical considerations, one may regard γ as a finite (but small) positive quantity, as discussed in Section 6. At a finite frequency, much larger than γ, the term coming from $\delta_{\Omega,0}$ may be dropped. Then, using Eq. (30), we obtain the result (39), which was derived by a different method in Section 3.

The analytic continuation of $b(\Omega)$ [Eqs. (59) and (61)] is more complicated. For simplicity, we consider the dirty limit, in which

$$b(\Omega) = \frac{2\pi T}{\nu} \sum_{\omega} \frac{1}{(R+R')^2} \left(\frac{\Delta^2 + \omega\omega'}{RR'} \tau_0 + \frac{i\Delta(\omega - \omega')}{RR'} \tau_2 + \tau_3 \right) \tag{A6}$$

This expression differs from the expression for $a(\Omega)$ [Eq. (60)] in that $(R+R')^{-1}$ is replaced by $(R+R')^{-2}$. Unfortunately, there are no identities analogous to (A1) for this type of expression. The analytic continuation of (A6) may be achieved by rewriting it in the form of a double integral:

$$b(\Omega) = \frac{T}{\pi\nu} \sum_{\omega} \int d\xi_1 \int d\xi_2 \times \frac{(\xi_1^2 + \xi_2^2)[(\Delta^2 + \omega\omega')\tau_0 + i\Delta(\omega - \omega')\tau_2] + 2\xi_1^2\xi_2^2\tau_3}{(\varepsilon_1^2 + \omega^2)(\varepsilon_1^2 + \omega'^2)(\varepsilon_2^2 + \omega^2)(\varepsilon_2^2 + \omega'^2)} \tag{A7}$$

The order of summation and integration is irrelevant here, unlike in (A3), because of the convergence properties of this expression. The next step is to present the integrand of (A7) as a sum of simple fractions, $1/(\varepsilon_1 \pm i\omega) \times (\varepsilon_2 \pm i\omega) \ldots$, and to perform the summation over ω using a generalized version of (A5):

$$T \sum_{\omega} \prod_{s=1}^{n} \frac{1}{\omega + i\lambda_s} = \sum_{s=1}^{n} \prod_{m \neq s} \frac{n(\lambda_s)}{\lambda_m - \lambda_s} \tag{A8}$$

where $n(\lambda) = 1/(e^{\lambda/T} + 1)$. The analytic continuation of the fraction $1/(\lambda_s - \lambda_m)$, which reduces to $1/(\varepsilon_1 \pm \varepsilon_2 \pm i\Omega)$, is achieved by putting $i\Omega = \omega + i\delta$, as discussed above.

This procedure is in principle elementary, but practically results in an extremely complicated expression. We shall not present the result, because of its length. It is worthwhile to mention, however, that the procedure

outlined above enables one to extend the calculations, at least in the dirty limit, to arbitrary temperatures.

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