Basis and Rank in Linear Algebra
With NumPy Examples

Basis: Theoretical Foundation

Definition:
A *basis* of a vector space V is a set of vectors that is both **linearly independent** and **spans** the space. Every vector in V can be uniquely written as a linear combination of the basis vectors.

Key Properties:

- **Linear Independence:** No vector in the basis can be written as a linear combination of the others.
- **Spanning:** Any vector in the space can be constructed from the basis vectors.
- The number of vectors in any basis for V is called the **dimension** of V.

Example:
The standard basis for $\mathbb{R}^3$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

NumPy Example: Checking for a Basis

```python
import numpy as np

# Example vectors
v1 = np.array([1, 0, 0])
v2 = np.array([0, 1, 0])
v3 = np.array([0, 0, 1])

# Stack as columns to form a matrix
A = np.column_stack((v1, v2, v3))

# Check if the matrix is full rank (i.e., vectors are linearly
    independent)
print(np.linalg.matrix_rank(A))  # Output: 3
```

Listing 1: Check if vectors form a basis by verifying full rank

If the rank equals the number of vectors, they form a basis for $\mathbb{R}^3$.

Example: Non-Standard Basis

```python
import numpy as np

v1 = np.array([1, 0])
v2 = np.array([1, 1])

A = np.column_stack((v1, v2))
print(np.linalg.matrix_rank(A))  # Output: 2
```

Listing 2: Check linear independence of non-standard vectors

These vectors are linearly independent and span $\mathbb{R}^2$, so they form a basis.

Rank: Theoretical Foundation

Definition:
The **rank** of a matrix is the dimension of the vector space spanned by its columns (column rank) or rows (row rank). For any matrix, row rank equals column rank.

Key Points:

- Rank = Maximum number of linearly independent columns (or rows).
- If a matrix is $m \times n$, then
$$\text{rank}(A) \leq \min(m, n).$$
- A matrix is **full rank** if its rank equals the smaller of the number of rows or columns.
- Rank determines the solvability of linear systems and the invertibility of square matrices.

NumPy Example: Computing Rank

```python
import numpy as np

A = np.array([[1, 2, 3],
              [2, 4, 6],
              [1, 1, 1]])

print(np.linalg.matrix_rank(A))  # Output: 2
```

Listing 3: Computing the rank of a matrix

The rank is 2 because only two columns are linearly independent.

Example: Full Rank Matrix

```python
B = np.array([[1, 2],
              [3, 4]])

print(np.linalg.matrix_rank(B))  # Output: 2
```

Listing 4: Full rank matrix example

Here, the matrix is full rank since rank equals the number of columns (and rows).

Example: Rank of a 1D Array (Ranking Values)

```python
import numpy as np

array = np.array([24, 27, 30, 29, 18, 14])
argsort_array = array.argsort()
ranks_array = np.empty_like(argsort_array)
```

```
ranks_array[argsort_array] = np.arange(len(array))
print(ranks_array)  # Output: [2 3 5 4 1 0]
```

Listing 5: Ranking elements in a 1D array

This ranks each element in the array, with 0 for the smallest value.

Summary Table

Concept	Definition	NumPy Function
Basis	Set of linearly independent vectors that span a space	`np.linalg.matrix_rank` (check if rank = number of
Rank	Dimension of the span of the columns (or rows) of a matrix	`np.linalg.matrix_rank`

Additional Notes

- Any set of n linearly independent vectors in an n-dimensional space forms a basis.
- If the rank of a matrix is less than its number of columns, the columns are linearly dependent.
- NumPy's `np.linalg.matrix_rank` is a practical tool for both theoretical and applied linear algebra.

These concepts are fundamental in understanding vector spaces, solutions to linear systems, and the structure of matrices in linear algebra.