Rank and Basis in Linear Algebra

In linear algebra, rank and basis are fundamental concepts related to vector spaces and matrices. Here's a clear explanation of both:

Basis

- A basis of a vector space is a set of vectors that satisfy two key properties:

  a. Span: The vectors in the basis can be combined (via linear combinations) to produce *any* vector in the space.

  b. Linear independence: None of the basis vectors can be written as a linear combination of the others.

- The basis essentially provides a minimal "building block" set of vectors for the entire space.

- Example: In $\mathbb{R}^3$, the standard basis is

  $$\{(1,0,0), (0,1,0), (0,0,1)\}$$

  because these three vectors are linearly independent and any vector (x, y, z) can be written as $x(1,0,0) + y(0,1,0) + z(0,0,1)$.

- The dimension of a vector space is the number of vectors in any basis of that space (all bases have the same size).

Rank

- The rank of a matrix A is the dimension of the vector space spanned by its columns (or equivalently, its rows). This means:
  - Rank = the maximum number of linearly independent columns (or rows) in the matrix.
  - It tells us how many vectors in the matrix are "essential" to span the column space.
- Rank measures the "nondegenerateness" of the system represented by the matrix. For example, if a matrix has full rank (equal to the smaller of the number of rows or columns), it means its columns are linearly independent.

- The rank-nullity theorem states:

  $$\text{rank}(A) + \text{nullity}(A) = n$$

  where n is the number of columns of A, and nullity is the dimension of the null space (solutions to $Ax = 0$).

- Example:

  Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

  The first two columns are linearly independent, but the third column is the sum of the first two. So,

$$\text{rank}(A) = 2$$

  because only two columns are linearly independent.

Relationship Between Rank and Basis

- The rank of a matrix is the dimension of the column space, which means the number of vectors in a basis for the column space.
- The basis for the column space consists of the linearly independent columns of the matrix.
- Finding the rank often involves reducing the matrix to a form (like row echelon form) to identify pivot columns, which correspond to basis vectors of the column space.

Summary Table

Concept	Definition	Key Property	Example
Basis	Minimal set of vectors that span a vector space and are linearly independent	Span entire space; no vector is redundant	$\{(1,0,0), (0,1,0), (0,0,1)\}$ for $\mathbb{R}^3$
Rank	Dimension of the column space of a matrix (number of linearly independent columns)	Number of pivot columns after row reduction	Matrix A above has rank 2

Understanding basis helps you grasp the structure of vector spaces, while rank tells you about the linear independence and dimension of the space spanned by matrix columns, crucial for solving linear systems and understanding transformations.