Copula Entropy

Theory and Applications

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Theory

Theory

Copula Theory

Definition (Copula)

[a] Given N random variables $\mathbf{X} = (X_1, \ldots, X_N) \in \mathcal{R}^N$. Let $\{u_i = F_i(x_i), i = 1, \ldots, N\}$ be the marginal distribution functions of $\mathbf{X}$. A N-dimensional copula $C : \mathcal{I}^N \to \mathcal{I}(\mathcal{I} = [0, 1])$ of $\mathbf{X}$ is a function with following properties:

1. C is grounded and N-increasing;
2. $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$.

[a] Roger B Nelsen. *An introduction to copulas.* Springer, 2007.

- the theory on **representation** of statistical dependence in probability
- copula function contains all the dependence information between random variables
- a probability function on unit cubic

Copula Theory

Theorem (Sklar's Theorem)

[a] *Given a random vector* $\mathbf{X} = (X_1, \ldots, X_N)$, *its CDF* $\mathbf{F}(\mathbf{x})$ *can be represented as*

$$\mathbf{F}(\mathbf{x}) = C(u_1, \ldots, u_N), \tag{1}$$

where C *is a copula function,* $\{u_i\}$ *are marginal distribution functions of* $\mathbf{X}$. *If* $\{F_i\}$ *are continuous, then* C *is unique.*

[a] M. Sklar. "Fonctions de repartition an dimensions et leurs marges". In: *Publ. Inst. Statist. Univ. Paris* 8 (1959), pp. 229–231.

- the core of copula theory
- there exists a copula function for each multivariate probability function

Copula Theory

Corollary

The probabilistic density function (PDF) $p(x)$ of $\mathbf{X}$ can be represented as

$$p(\mathbf{x}) = c(\mathbf{u}) \prod_{i=1}^{N} p_i(x_i), \tag{2}$$

where $\{p_i, i = 1, \ldots, N\}$ are marginal density functions of $\mathbf{X}$, and c is copula density.

- seperating dependence representation with properties of individual variables

Information Theory: Definitions

Definition (Shannon Entropy)

[a] Given random variables $X \in R^n$ and their pdf $p(\mathbf{x})$, Shannon entropy is defined as

$$H(\mathbf{x}) = -\int_{\mathbf{x}} p(\mathbf{x}) \log p(\mathbf{x}) d\mathbf{x}. \tag{3}$$

[a] Thomas M Cover and Joy A. Thomas. *Elements of Information Theory*. John Wiley & Sons, 2005.

Definition (Mutual Information)

[a] Given a pair of random variables (X, Y) and their pdf $p(x, y)$ and margins $p(x), p(y)$, mutual information is defined as

$$I(x; y) = \int_x \int_y p(x, y) \log \frac{p(x, y)}{p(x)p(y)} dxdy. \tag{4}$$

[a] Thomas M Cover and Joy A. Thomas. *Elements of Information Theory*. John Wiley & Sons, 2005.

Information Theory: Definitions and Theorem

Definition (Conditional Mutual Information)

[a] Given random variables (X, Y, Z), conditional mutual information for (X, Y) given Z is defined as

$$I(x; y|z) = \int_x \int_y \int_z p(x, y, z) \log \frac{p(x, y|z)}{p(x|z)p(y|z)} dxdydz. \quad (5)$$

[a] Thomas M Cover and Joy A. Thomas. *Elements of Information Theory*. John Wiley & Sons, 2005.

Theorem

[a] *Mutual information equals the difference between marginal entropies and joint entropy.*

$$I(x; y) = H(x) + H(y) - H(x, y). \quad (6)$$

[a] Thomas M Cover and Joy A. Thomas. *Elements of Information Theory*. John Wiley & Sons, 2005.

Copula Entropy: Theory

Definition (Copula Entropy)

[a] Let $\mathbf{X}$ be random variables with marginals $\mathbf{u}$ and copula density $c(\mathbf{u})$. Copula Entropy of $\mathbf{X}$ is defined as

$$H_c(\mathbf{x}) = -\int_{\mathbf{u}} c(\mathbf{u}) \log c(\mathbf{u}) d\mathbf{u}. \quad (7)$$

[a] Jian Ma and Zengqi Sun. "Mutual information is copula entropy". In: *Tsinghua Science & Technology* 16.1 (2011). See also arXiv preprint arXiv:0808.0845 (2008), pp. 51–54.

- a special type of Shannon entropy
- an ideal measure of statistical independence
- distribution-free

Copula Entropy: Theory

Theorem

[a] *Mutual Information of* $\mathbf{X}$ *is equavalent to its negative copula entropy.*

$$I(\mathbf{x}) = -H_c(\mathbf{x}). \tag{8}$$

[a] Jian Ma and Zengqi Sun. "Mutual information is copula entropy". In: *Tsinghua Science & Technology* 16.1 (2011). See also arXiv preprint arXiv:0808.0845 (2008), pp. 51–54.

Corollary

[a]

$$H(\mathbf{x}) = \sum_i H_i(x_i) + H_c(\mathbf{x}). \tag{9}$$

[a] Jian Ma and Zengqi Sun. "Mutual information is copula entropy". In: *Tsinghua Science & Technology* 16.1 (2011). See also arXiv preprint arXiv:0808.0845 (2008), pp. 51–54.

- the bridge between copula theory and information theory[1]

[1] Jian Ma and Zengqi Sun. "Mutual information is copula entropy". In: *Tsinghua Science & Technology* 16.1 (2011). See also arXiv preprint arXiv:0808.0845 (2008), pp. 51–54.

Copula Entropy: Theory

Theorem

[a] *Given random variables* (X, Y, Z)*, their conditional mutual information can be represented as follows:*

$$I(x; y|z) = H_c(x, z) + H_c(y, z) - H_c(x, y, z). \tag{10}$$

[a] Jian Ma. "Estimating Transfer Entropy via Copula Entropy". In: *arXiv preprint arXiv:1910.04375* (2019).

- build a framework of the concepts of information theory based on copula entropy

Copula Entropy: Theory

- Axiomatic Properties of Copula Entropy
  - multivariate
  - symmetric
  - non-negative, 0 iff independence
  - invariant to monotonic transformation
  - equivalent to correlation coefficient in Gaussian cases
- An ideal measure compared with others

Table: Comparison with other independence measures.

	Copula Entropy	Distance Correlation	HSIC
Definition	copula based	generalised corr	corr in RKHS
Multivariate	Yes	distance multivariance	dHSIC
Invariance	monotonic trans	No	No
Gaussanity	equivalent to cc	unclear	unclear
Computation	low	high	high

Copula Entropy: Estimation

- **Parametric** Estimation by Definition

$$H_c(\mathbf{x}) = -E(\log c(\mathbf{u})). \tag{11}$$

Copula Entropy: Estimation

- **Non-Parametric** Estimation Method[2]
  1. estimating empirical copula density with rank statistics
  2. estimating copula entropy with kNN entropy estimation method
- Advantages
  - distribution-free, non-parametric
  - tuning-free, insensitive to parameters
  - good convergence
  - easy to implement
  - low computation burden

[2] Jian Ma and Zengqi Sun. "Mutual information is copula entropy". In: *Tsinghua Science & Technology* 16.1 (2011). See also arXiv preprint arXiv:0808.0845 (2008), pp. 51–54.

Copula Entropy: Estimation

- **Conditional Mutual Information** Estimation

can be done with copula entropy estimators according to (10).

Applications

Applications

Copula Entropy: Application I

Association Discovery[3]

[3] Jian Ma. "Discovering Association with Copula Entropy". In: *arXiv preprint arXiv:1907.12268* (2019).

Copula Entropy: Association Discovery

- Problem
  - To discover association relationship between random variables from data
- History
  - An old and fundamental problem since statistics birth
- Related Methods
  - Pearson Correlation Coefficient
  - Regression

Copula Entropy: Association Discovery

- Traditional association measures
  - Pearson Correlation Coefficient

$$r_{XY} = corr(X,Y) = \frac{cov(X,Y)}{\delta_X \delta_Y} \quad (12)$$

  - Spearman's ρ and Kendall's τ

$$\rho_{XY} = 12 \int_u \int_v C(u,v)dudv - 3 \quad (13)$$

$$\tau_{XY} = 4 \int_u \int_v C(u,v)dC(u,v) - 1 \quad (14)$$

- Why Copula Entropy?

Table: Theoretical comparison between CE and CC.

	CC	CE
linearity	linear	nonlinear
Order	2	≥ 2
Assumption	Gaussian	None
variate	bivariate	multivariate

Copula Entropy: Association Discovery

Experiments on the NHANES data

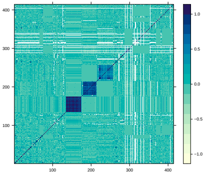
- Objectives of NHANES
  - to monitor trends and emerging issues of population health
  - to investigate its relationship with risk factors, nutritions and environmental exposures, etc.
- NHANES (2013-2014)
  - 14,332 persons from 30 different survey locations were selected;
  - Of those selected, 10,175 interviewed and 9,813 examined;
  - 5 groups of data: demographics, dietary, examination, laboratory, and questionnaire.
- Experimental data

  The laboratory data, which includes 423 variables from blood, urine, oral rinse and vaginal/Penile swabs.
- Missing values

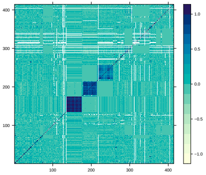
  The missing values were filled with the mean of their corresponding variables.

Copula Entropy: Association Discovery

- Results - Correlation matrices

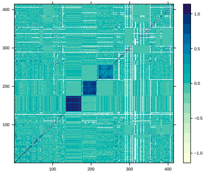


Pearson's r

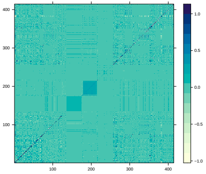


Kendall's τ

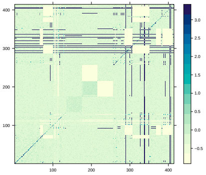
Schweizer & Wolff's σ



Spearman's ρ



Gini's γ



CE

Copula Entropy: Application II

Structure Learning[4]

[4] Jian Ma and Zengqi Sun. "Dependence structure estimation via copula". In: *arXiv preprint arXiv:0804.4451* (2008), Jian Ma. "Learning in Copula Structure". PhD thesis. Tsinghua University, 2009.

Copula Entropy: Structure Learning

- Problem
  - To learn statistical structure among random variables from data
- Graph Representation
  - A probability density is represented with a directed or indirected graph, of which each node represents a random variable, and each edge represents a (conditional) dependence relation between two random variables
- Related Methods
  - Chow-Liu Algorithm
  - Vine Copula

Copula Entropy: Structure Learning

- Copula Likelihood[5]

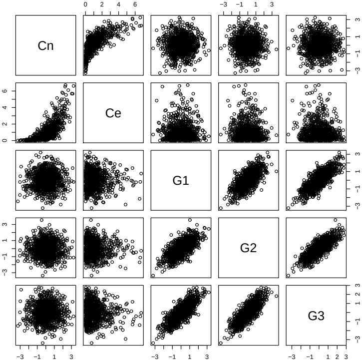
$$\mathcal{L}_c(\theta_c; \mathbf{X}) = \sum_{t=1}^{T} \log c(\mathbf{u}_t; \theta_c). \tag{15}$$

  - equivalent to copula entropy
  - guide principle of copula based structure learning
- Our Algorithm
  1. computing dependence matrix $\mathbf{W}_x$ of data x with CE estimation
  2. constructing dependence structure T from $\mathbf{W}_x$ with MST algorithm
- Advantages
  - distribution-free, non-parametric
  - tuning-free, insensitive to parameters
  - easy to implement
  - low computation burden

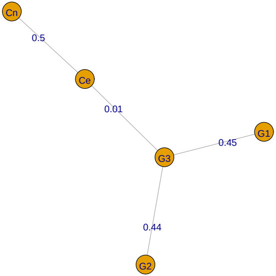
[5] Jian Ma. "Learning in Copula Structure". PhD thesis. Tsinghua University, 2009.

Copula Entropy: Structure Learning

- Simulated Experiment
  5 random variables: the first three are Gaussian and the others two are governed by Gaussian copula with margins as normal distribution and exponential distribution respectively



Simulated data

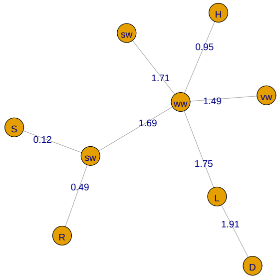


Learned graph

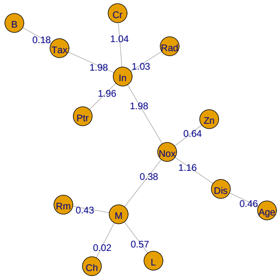
Copula Entropy: Structure Learning

Experiment on real data

- Abalone data
  Predicting the age of abalone from physical measurements



- Boston housing data
  Concerns housing values in suburbs of Boston



Copula Entropy: Application III

Variable Selection[6]

[6] Jian Ma. "Variable Selection with Copula Entropy". In: *Chinese Journal of Applied Probability and Statistics* 37.4 (2021), pp. 405–420.

Copula Entropy: Variable Selection

- Problem
  - To select a 'right' subset of variables from the whole group for building classification or regression models with good predictability and interpretability
- History
  - An old and basic problem in statistics and machine learning
- Related Problems
  - Feature Selection
  - Model Selection

Copula Entropy: Variable Selection

Existing methods - Likelihood with penalty

- Information Criteria
  with penalty on the number of parameters in the models

$$\mathbf{AIC} = -2L + 2p \tag{16}$$

$$\mathbf{BIC} = -2L + p \log N \tag{17}$$

- Penalized GLMs
  with penalty on the nonzero coefficients in the GLMs
  - LASSO
  - Ridge Regression
  - Elastic Net

$$\min_{\beta}\{L(\beta; y, \mathbf{X}) + \lambda_1 ||\beta||_1 + \lambda_2 ||\beta||_2^2\} \tag{18}$$

  - Adaptive LASSO

$$\min_{\beta}\{L(\beta; y, \mathbf{X}) + \lambda \sum_{j=1}^{p} w_j |\beta_j|\} \tag{19}$$

Copula Entropy: Variable Selection

Existing methods - Statistical independence measures

- Distance Correlation

$$\mathbf{dCor}(X, Y) = \frac{\nu^2(X, Y)}{\sqrt{\nu^2(X)\nu^2(Y)}}, \tag{20}$$

where $\nu^2(X, Y)$ be distance covariance.

- Hilbert-Schmidt Independence Criterion (HSIC)

$$\mathbf{dHSIC}(P(\mathbf{X})) = ||\Pi(P(X_1)\otimes, \ldots, \otimes P(X_d)) - \Pi(P(\mathbf{X}))||, \tag{21}$$

where Π be the mean embedding function associated with kernel functions.

Copula Entropy: Variable Selection

- CE based method
  To select variables based on ranks of their negative CE values with target
- Advantages
  - model-free, non-parametric
  - tunning-free, insensitive to parameters
  - interpretable with physical meanings
  - supported by rigorous math
  - science instead of art, compared with existing methods
  - easy to implement, low computation burden

Copula Entropy: Variable Selection

Experiments on the UCI heart disease data[7]

- Overview of the data
  The data set contains 4 databases (899 samples) concerning heart disease diagnosis. All attributes are numeric-valued. The data was collected from the four following locations:
  - Cleveland clinic foundation;
  - Hungarian Institute of Cardiology, Budapest;
  - V.A. medical center, long beach, CA;
  - University hospital, Zurich, Switzerland.
- Attributes
  The data has 76 attributes (#58 'num' for diagnosis). Of them, 13 attributes are recommended by professionals as clinical relevant.

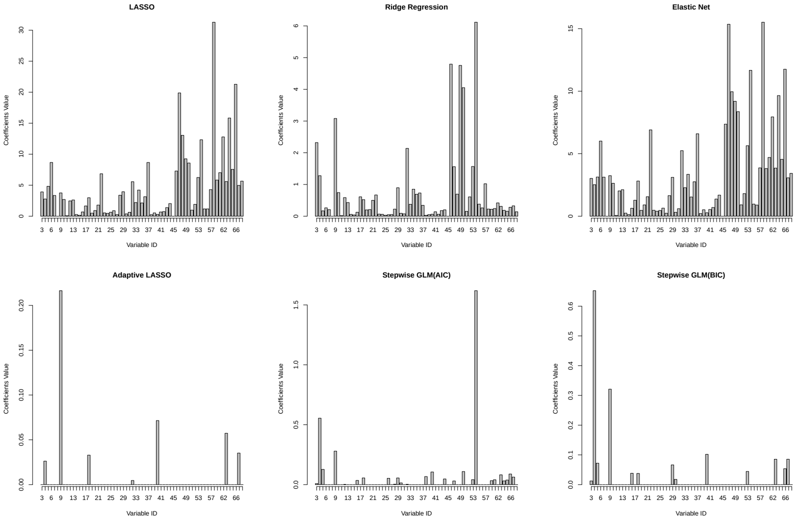
Table: Recommended attributes.

ID	3	4	9	10	12	16	19
Name	age	sex	cp	trestbps	chol	fbs	restecg
ID	32	38	40	41	44	51	58
Name	thalach	exang	oldpeak	slope	ca	thal	num

[7] Arthur Asuncion and David Newman. *UCI machine learning repository*. 2007.

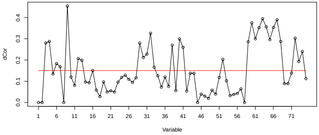
Copula Entropy: Variable Selection

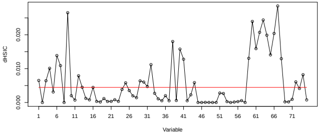
- Results - Coefficients of penalized likelihood based models

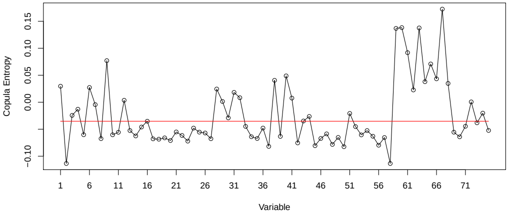


Copula Entropy: Variable Selection

- Results - with statistical dependence measures (dCor, dHSIC, CE)







Copula Entropy: Variable Selection

- Results - Prediction accuracy
  the selected variables present the best prediction accuracy.

Model	Accuracy(%)
SVM(Recommended variables)	84.20
SVM(CE)	**84.76**
SVM(dCor)	82.76
SVM(dHSIC)	84.54
Stepwise GLM(AIC)	51.8
Stepwise GLM(BIC)	49.1
LASSO	79.2
Ridge Regression	63.0
Elastic Net	75.9
Adaptive LASSO	35.7

Copula Entropy: Variable Selection

- Results - Selected variables
  Copula Entropy selects more 'right' variables than the other methods do.

Method	Selected Variables' ID	✓
Recommended variables	3,4,9,10,12,16,19,32,38,40,41,44,51	13
CE	3,4,6,7,9,12,16,28-32,38,40,41,44,51,59-68	**11**
dHSIC	3,4,6,7,9,12,13,16,25,29-32,38,40,41,44,59-68	10
dCor	3,4,6,7,9,12,13,16,28-33,38,40,41,52,59-68	9
Stepwise GLM(AIC)	3,4,5,9,12,16,18,20,26,29,30,32,40,44,47,50,53,54,60,61,63,65-67	8
Stepwise GLM(BIC)	3,4,5,9,16,18,29,30,40,53,63,66,67	5
Adaptive LASSO	4,6,9,18,32,40,63,67	4
LASSO Ridge Regression Elastic Net	all except 8,45	-

Copula Entropy: Application IV

Causal Discovery[8]

[8] Jian Ma. "Estimating Transfer Entropy via Copula Entropy". In: *arXiv preprint arXiv:1910.04375* (2019).

Copula Entropy: Causal Discovery

- Problem
  - To infer causality from time series data by *estimating Transfer Entropy*
- History & Significance
  - Causality is one of the oldest topics in philosophy.
  - Causal discovery is a central problem of all sciences.
- Correlation vs Causality
  - Correlation does not mean causation.
  - Correlation is only helpful for prediction while causality means intervention and control.

Copula Entropy: Causal Discovery

- Causality measures
  - Wiener's Principle
    Cause should improve the prediction of effect.
  - Granger Causality
    improvement measured by the variance of prediction error

$$\delta^2(Y_{t+1}|Y_t, X_t) < \delta^2(Y_{t+1}|Y_t) \tag{22}$$

  - Transfer Entropy
    improvement on the uncertainty of prediction measured by Shannon entropy
    essentially conditional mutual information

$$\begin{aligned} TE &= \sum p(Y_{t+1}, Y^t, X_t) \log \frac{p(Y_{t+1}|Y^t, X_t)}{p(Y_{t+1}|Y^t)} && (23) \\ &= H(Y_{t+1}|Y^t) - H(Y_{t+1}|Y^t, X_t) && (24) \\ &= I(Y_{t+1}, X_t|Y^t) && (25) \end{aligned}$$

- Issue on TE
  difficult to estimate, some think impossible without model assumptions

Copula Entropy: Causal Discovery

- TE via CE

Proposition

[a] *Transfer Entropy can be represented with only Copula Entropy.*

$$T_{x \to y} = -H_c(Y_{t+1}, Y^t, X_t) + H_c(Y_{t+1}, Y^t) + H_c(Y^t, X_t) - H_c(Y^t) \quad (26)$$

[a] Jian Ma. "Estimating Transfer Entropy via Copula Entropy". In: *arXiv preprint arXiv:1910.04375* (2019).

- Non-parametric Estimator of TE
  1. estimating three or four CE terms in (26);
  2. calculating TE for these estimated CEs.
- inheriting all the merits of non-parametric CE estimation

Copula Entropy: Causal Discovery

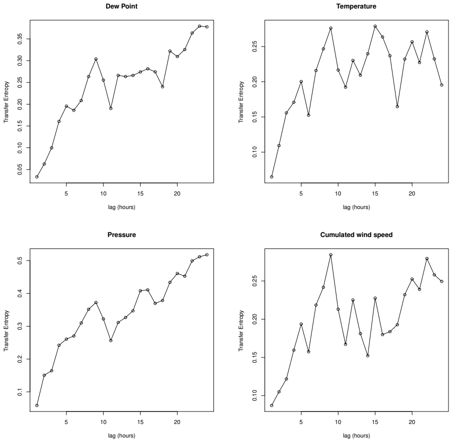
Experiments on the UCI Beijing PM2.5 data[9]

- Overview of the data
  - Time
    hourly data from 2010-01-01 to 2014-12-31, which results in 43824 samples with missing values.
  - Observations
    - PM2.5 data of US Embassy in Beijing
    - Meteorological data from Beijing Capital International Airport
  - Meteorological factors
    dew point, temperature, pressure, cumulated wind speed, combined wind direction, cumulated hours of snow, cumulated hours of rain.
- Experimental data
  - the first four factors used in the experiments;
  - 1000 samples without missing values (2010-04-02∼2010-05-14).

[9] Arthur Asuncion and David Newman. *UCI machine learning repository*. 2007.

Copula Entropy: Causal Discovery

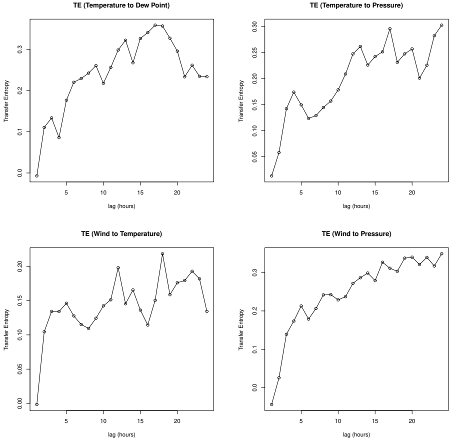
Results: Effects of meteorological factors on PM2.5



- Two phrases
  - Sharp increase phrase: the first 9 hours time lag, and peak at about 9 hours lag;
  - Flat increase phrase: TE of Dew point and pressure increase with relatively flat rate while TE of temp. and cumulated wind speed does increase any more.
- Interpretation
  - The effects do not show immediately and are cumulating processes.

Copula Entropy: Causal Discovery

Results - Effects between meteorological factors



- Temp. to Dew Point & Pressure
- Wind to Temp. & Pressure
  - Wind changes temperature in 3 hours later and
  - Wind changes pressure in 5 hours later.

Copula Entropy: Application V

Time Lag Estimation[10]

[10] Jian Ma. "Identifying Time Lag in Dynamical Systems with Copula Entropy based Transfer Entropy". In: *arXiv preprint arXiv:2301.06037* (2023).

Copula Entropy: Time Lag Estimation

- Problem
  - To identify time lag in dynamical systems with copula entropy based transfer entropy
- Significance
  - Time lag is ubiquitous in physical, social, and biological systems.
  - Identifying time lag is of fundamental importance in applications of dynamical systems.
- Related Methods
  - Auto-correlation
  - Time-delayed mutual information

Copula Entropy: Time Lag Estimation

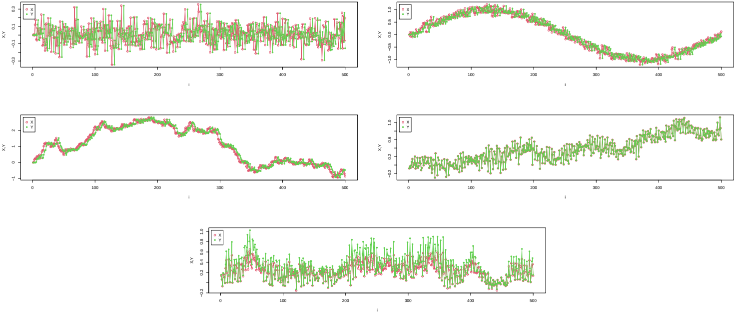
- Our method
  1. estimating transfer entropies on time lag horizon from data with the CE-based estimator
  2. identifying the time lag associated with the maximum TE value

Copula Entropy: Time Lag Estimation

- Simulations
  1. generate trajectories from four simulated dynamical system with respect to different state or ouput lags
  2. identify the time lag with our method
- Simulated systems
  - a system driven by random walk with output lag
  - a system driven by sine function with output lag
  - Wiener process with output lag
  - a first-order linear system with state lag
  - a first-order nonlinear system with state lag

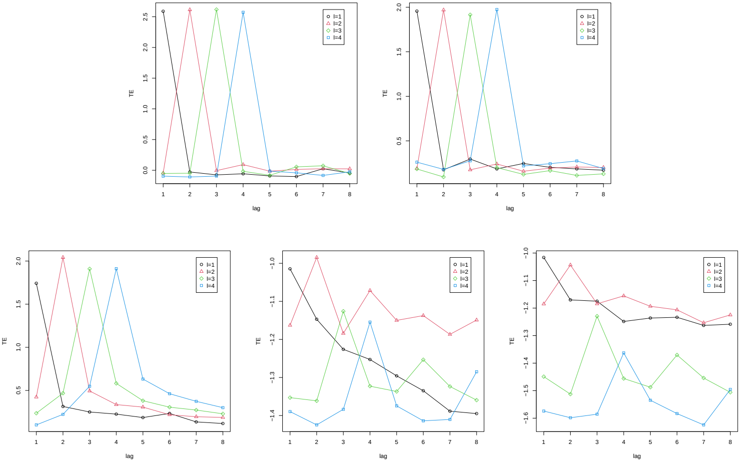
Copula Entropy: Time Lag Estimation

- Simulated trajectories



Copula Entropy: Time Lag Estimation

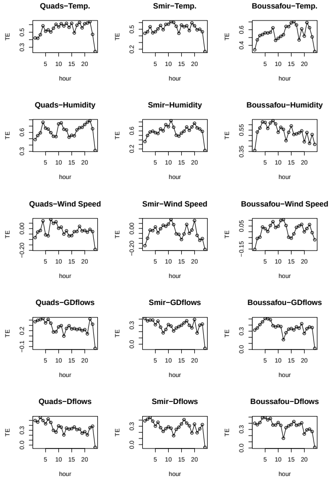
- Simulation: Results



Copula Entropy: Time Lag Estimation

Power consumption of the Tetouan city[11]

- Data
  - power consumption of 3 networks in 2017
  - weather factors, including temperature, humidity, wind speed, general diffuse flows, diffuse flows
- Power consumption forecast
  - To identify time lags from weather to power consumption



[11] Arthur Asuncion and David Newman. *UCI machine learning repository*. 2007.

Copula Entropy: Application VI

System Identification[12]

[12] Jian Ma. "System Identification with Copula Entropy". In: *arXiv preprint arXiv:2304.12922* (2023).

Copula Entropy: System Identification

- Problem
  - To discover differential equation from time series data
- Significance
  - differential equations are the main mathematical tools for modelling dynamical systems.
  - discovering differential equations of dynamical systems has wide applications in many scientific fields.
- Related Methods
  - SINDy
  - Gaussian processes

Copula Entropy: System Identification

- Idea
  considering system identification as a variable selection problem

$$\frac{dx_i}{dt} = f(\mathbf{x}, t). \tag{27}$$

- Our method
  1. calculating the derivative of system variables with differential operator;
  2. estimating the CEs between the calculated derivatives and the covariates of the system;
  3. selecting the covariates with high CE value for each derivatives.

Copula Entropy: System Identification

- Simulations
  simulating time series data from Lorenz system and Rössler system
- Results

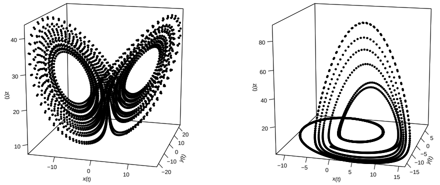


Figure: 3D plot of the simulated data.

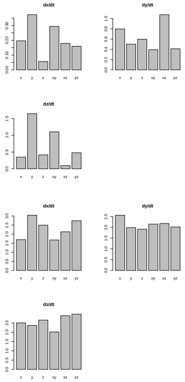


Figure: Identification results.

Copula Entropy: Application VII

Multivariate Normality Test[13]

[13] Jian Ma. "Multivariate Normality Test with Copula Entropy". In: *arXiv preprint arXiv:2206.05956* (2022).

Copula Entropy: Multivariate Normality Test

- Problem
  - To test the hypothesis that the distribution of data is normal distribution
- Significance
  - Normal distribution is the most important distribution in probability theory;
  - Normality is a common assumption of many statistical tools;
  - Testing normality is widely needed in real applications.
- Related Methods
  - characteristics function based
  - moments based
  - skewness and kurtosis
  - energy distance based
  - entropy based
  - Wasserstein distance based

Copula Entropy: Multivariate Normality Test

- The proposed statistic

$$T_{mvnt} = H_c(\mathbf{x}) - H_c(\mathbf{x}_n), \tag{28}$$

where $\mathbf{x}_n$ is the Gaussian random vector with the same covariances as $\mathbf{x}$.

  - defined as the difference of copula entropies
  - $T_{ce} = 0$ if normal distributions

- The estimator
  - the first term in (28) can be estimated with the non-parametric CE estimator;
  - the second term in (28) can be estimated easily by first estimating the covariances V_x of $\mathbf{x}$ and then calculating the result according to (29).

$$H_c(\mathbf{x}_n) = \frac{1}{2}\log|V_x|. \tag{29}$$

Copula Entropy: Multivariate Normality Test

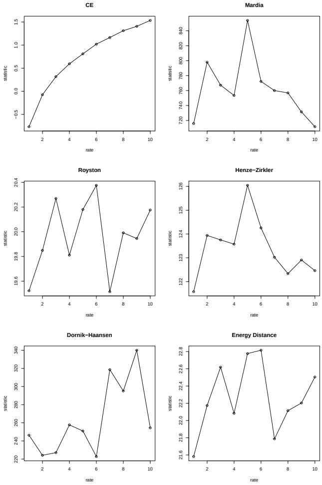
Simulation Experiments

- Data
  - bivariate normal copula with normal and exponential marginals
  - bivariate Gumbel copula with normal marginals
- Compared methods
  - Mardia's
  - Royston's
  - Henze and Zirkler's
  - Doornik and Hansen's, and
  - the energy distance based test by Rizzo and Székely

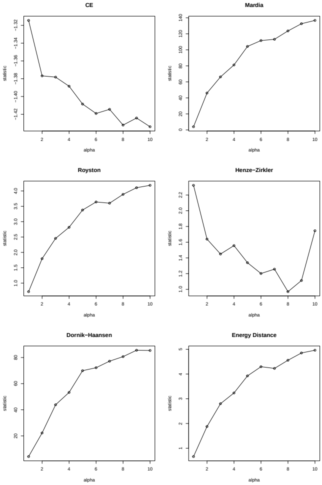
Copula Entropy: Multivariate Normality Test

Simulation Results

- Bivariate normal copula



- Bivariate Gumbel copula



Copula Entropy: Application VIII

Copula Hypothesis Testing[14]

[14] Jian Ma. "Testing Copula Hypothesis with Copula Entropy". In: *arXiv preprint arXiv:2510.22722* (2025).

Copula Entropy: Copula Hypothesis Testing

- Problem
  - To test which copula hypothesis is accepted for sample data
- Significance
  - Tesing copula hypothesis is a fundamental problem in applications of copula theory;
  - Many copula hypothesis test exist for specific types of copula function;
  - A general method for any copula function is needed.
- Related Methods
  - Gaussian copula hypothesis testing
  - Archimedeanity test
  - Copula Information Criteria
  - Goodness-of-Fit tests of copulas

Copula Entropy: Copula Hypothesis Testing

- The proposed statistic

$$T_c(\mathbf{X}_T|c) = H_c(\mathbf{X}_T|c) - H_c(\mathbf{X}_T|c_{\mathbf{x}}) \qquad (30)$$

  - defined as the difference of true copula entropy and the copula entropy of hypothesis c
  - $T_c(\mathbf{X}_T|c) = 0$ if the hypothesis is true

- The estimator

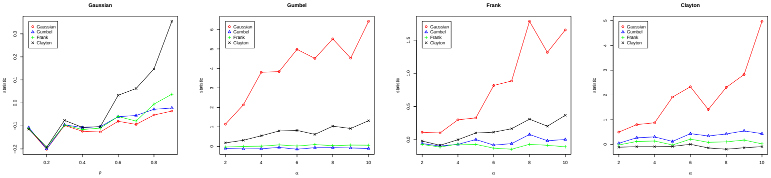
  The statistic in (30) can be estimated as two part. The second term is true CE and therefore can be estimated directly from data with the nonparametric estimator of CE. The first term is the CE of copula hypothesis which can be estimated in the following 3 steps:

  1. estimate empirical copula density $\hat{\mathbf{u}}$ from $\mathbf{X}_T$;
  2. estimate the parameters α of copula c with $\hat{\mathbf{u}}$;
  3. calculate the CE of the copula hypothesis with the following equation:

$$H_c(\mathbf{X}_T|c) = -E(\log c(\hat{\mathbf{u}}; \alpha)). \qquad (31)$$

Copula Entropy: Copula Hypothesis Testing

Simulation



(a) Gaussian (b) Gumbel (c) Frank (d) Clayton

Figure: Simulation results on Gaussian Copula and Archimedean Copulas.

Copula Entropy: Application IX

Two-Sample Test[15]

[15] Jian Ma. "Two-Sample Test with Copula Entropy". In: *arXiv preprint arXiv:2307.07247* (2023).

Copula Entropy: Two-Sample Test

- Problem
  - To test the hypothesis that two samples are from a same distribution
- Significance
  - a basic hypothesis testing problem;
  - Symmetry test and change point detection can be formulated as two-sample test problem;
  - has many real applications in many areas, such as politics, medicine, etc.
- Related Methods
  - T-test or F-test
  - Kernel-based two-sample test
  - Kolmogorov-Smirnov test
  - Mutual information based test

Copula Entropy: Two-Sample Test

- The proposed statistic

$$T_{tst} = H_c(\mathbf{X}, Y_0) - H_c(\mathbf{X}, Y_1), \tag{32}$$

where $\mathbf{X} = (X_1, X_2)$ is for two samples $X_1 = \{X_{11}, \cdots, X_{1m}\}$ and $X_2 = \{X_{21}, \cdots, X_{2n}\}$, and $Y_1 = (0_1, \cdots, 0_m, 1_1, \cdots, 1_n)$ and $Y_0 = (1_1, \cdots, 1_{m+n})$ are the labels for the null and the alternative hypothesis.

  - non-parametric multivariate two-sample test
  - defined as the difference between the copula entropies of the null and the alternative hypothesis;
  - T_{ce} is small if H_0 is true.

- The estimator
  - estimating the two terms in (32);
  - calculating the estimated statistic.

Copula Entropy: Two-Sample Test

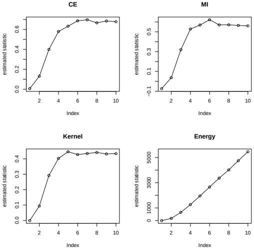
Simulation Experiments

- Data
  - bivariate normal distribution with different means
  - bivariate normal distribution with different variances
  - bivariate Gaussian copula with normal and exponential marginals
- Compared methods
  - Kernel-based test
  - Energy distance-based test
  - Mutual information-based test

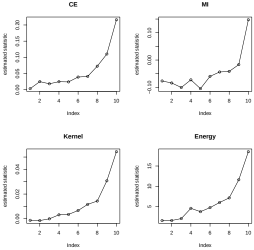
Copula Entropy: Two-Sample Test

Simulation Results

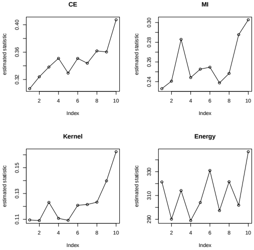
- Bivariate normal distribution with different means



- Bivariate normal distribution with different variances



- Bivariate normal copula with different variances



Copula Entropy: Application X

Change Point Detection[16]

[16] Jian Ma. "Change Point Detection with Copula Entropy based Two-Sample Test". In: *arXiv preprint arXiv:2403.07892* (2024).

Copula Entropy: Change Point Detection

- Problem
  - To detect single or multiple disrupt change in time series data
- Significance
  - a basic problem in time series analysis;
  - can be solved with two-sample test problem;
  - has aboard real applications in many areas, such as geoscience, biology, manufacturing, etc.
- Existing Methods
  - CUSUM
  - Kernel-based method
  - two-sample test based
  - multiple detection with binary segmentation

Copula Entropy: Change Point Detection

- The proposed method
  - Single change point detection
    solved with copula entropy based two-sample test
  - Multiple change point detection
    Single change point detection with binary segmentation
- Merits
  - nonparametrical method based on CE estimator
  - both univariate and multivariate
  - distribution-free, universally applicable
  - almost tuning-free, universal threshold for test statistic
- Implementation
  - performance boosted with parallel computing

Copula Entropy: Change Point Detection

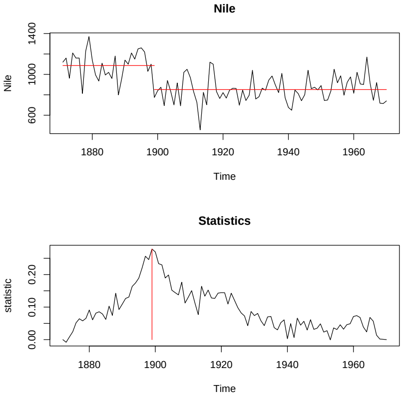
Simulation Experiments

- Univariate and multivariate multiple change points
    - changes in mean
    - changes in mean and variance
    - changes in variance
- Compared methods
    - The methods in the R package **changepoint**
    - Kernel based method in the R package **ecp**

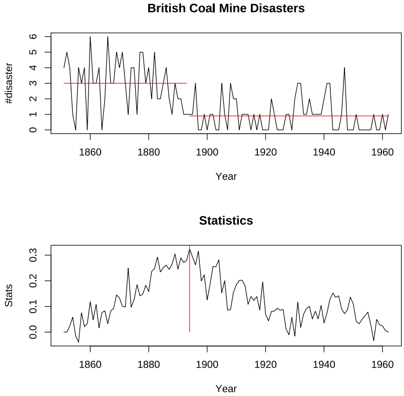
Copula Entropy: Change Point Detection

Real data experiment

- The Nile data



- British coal mine disasters data



Copula Entropy: Application XI

Symmetry Test[17]

[17] Jian Ma. "Testing Symmetry with Copula Entropy based Two-Sample Test". In: *ChinaXiv preprint ChinaXiv:202505.00167* (2025).

Copula Entropy: Symmetry Test

- Problem
    - To test the symmetry of distributions of samples
- Significance
    - Symmetry is a basic property of physics
    - is the assumptions of many statistical models and methods
- Related work
    - Symmetry tests with copulas
    - Testing symmetry of copulas
    - Entropy based symmetry tests

Copula Entropy: Symmetry Test

- The proposed statistic

$$T_{sym}(X) = T_{tst}(\tilde{X}, -\tilde{X}). \tag{33}$$

  - defined as the statistic of two-sample test on sample and its symmetric transformation
  - $T_{sym}(X) = 0$ if the distribution is symmetric

- The estimator

  Then the proposed method composed of two simple steps:

  1. deriving $\tilde{X}$ from X by $\tilde{X} = X - \tilde{u}$;
  2. estimating T_{sym} by performing two-sample test on $\tilde{X}$ according to (33).

Copula Entropy: Symmetry Test

Simulation Experiments
See Section 6

Evaluation

Evaluation

Independence Measures : Implementations

Table: Independence Measures and their implementations.

Package	Measure	Language
`copent`	CE	R
`stats`	Ktau	R
`energy`	dCor	R
`dHSIC`	dHSIC	R
`HHG`	HHG.chisq, HHG.lr	R
`independence`	Hoeff, BDtau	R
`Ball`	Ball	R
`qad`	QAD	R
`BET`	BET	R
`MixedIndTests`	Mixed	R
`subcopem2D`	subcopula	R
`EDMeasure`	MDM	R
`FOCI`	CODEC	R
`NNS`	NNS	R

Independence Measures : Results I

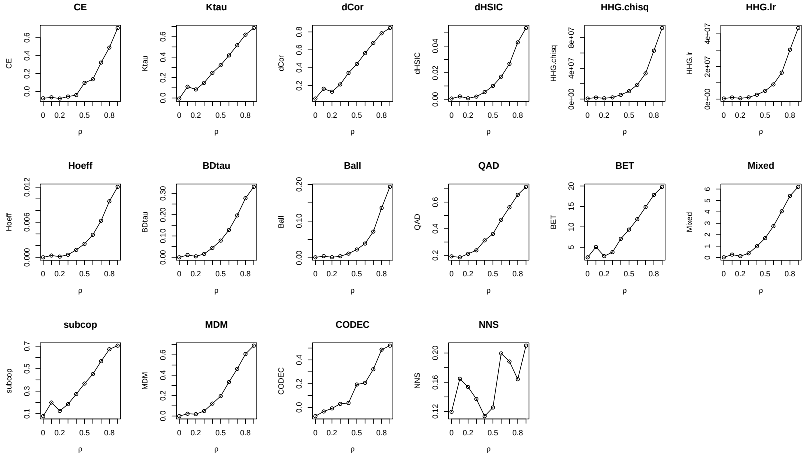


Figure: Experiments with bivariate normal distributions.

Independence Measures : Results II

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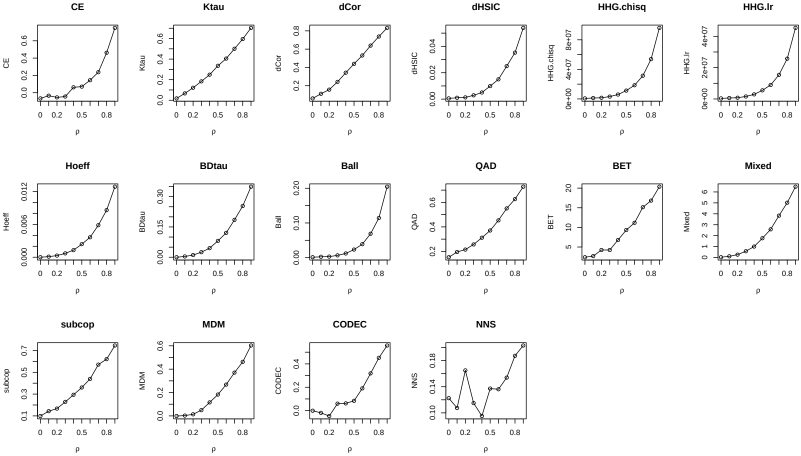
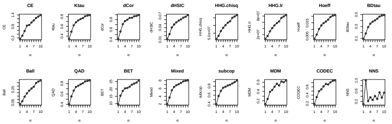
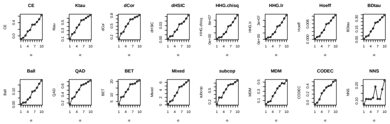


Figure: Experiments with bivariate copula functions.

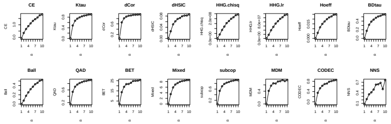
Independence Measures : Results III



(a) Clayton copula



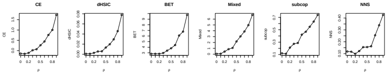
(b) Frank copula



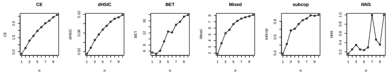
(c) Gumbel copula

Figure: Experiments with bivariate Archimedean copula functions.

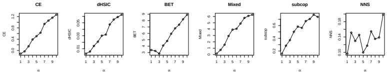
Independence Measures : Results IV



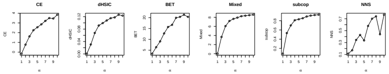
(a) Trivariate normal distribution



(b) Trivariate Clayton copula function



(c) Trivariate Frank copula function



(d) Trivariate Gumbel copula function

Figure: Experiments on multivariate measures.

Independence Measures : Results V

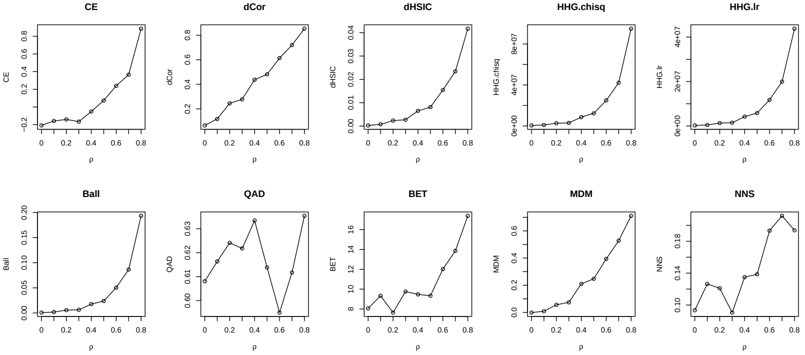


Figure: Experiments on independence between random vectors with quadvariate normal distributions.

Conditional Independence Measures : Implementations

Table: Conditional independence measures and their implementations.

Package	**Measure**	**Language**
`copent`	CE	R
`EDMeasure`	CMDM	R
`FOCI`	CODEC	R
`RCIT`	RCoT	R
`cdcsis`	CDC	R
`GeneralisedCovarianceMeasure`	GCM&R	
`weightedGCM`	wGCM	R
`comets`	PCM	R
`KPC`	KPC	R
`ppcor`	pcor	R
`parCopCITest`	pcop	R
`causallearn`	KCI	Python
`pycit`	CMI1	Python
`knncmi`	CMI2	Python
`fcit`	FCIT	Python
`CCIT`	CCIT	Python
`pcit`	PCIT	Python

Conditional Independence Measures : Results I

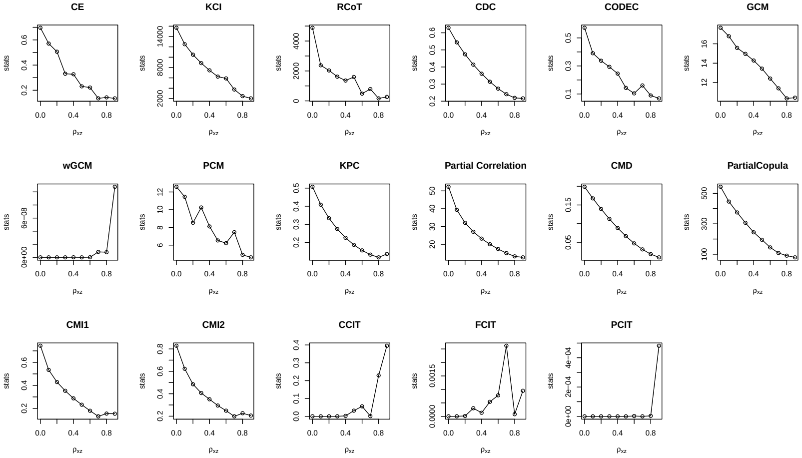


Figure: Experiments with trivariate normal distributions.

Conditional Independence Measures : Results II

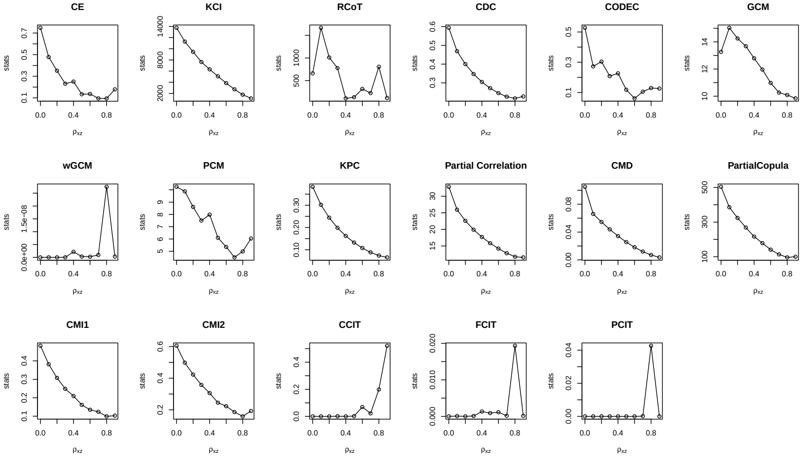


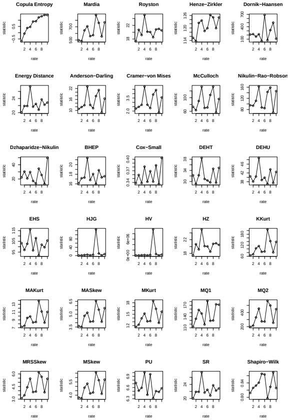
Figure: Experiments with trivariate normal copula functions.

Multivariate Normality Tests : Implementations

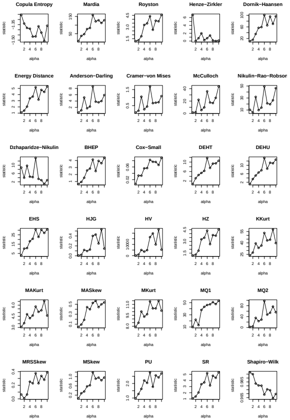
Table: Multivariate normality tests and their implementations in R.

Package	Test
`copent`	CE
`MVN`	Mardia, Royston, Henze-Zirkler Dornik-Haansen, energy distance
`mvnTest`	Anderson-Darling, Cramer-von Mises McCulloch, Nikulin-Rao-Robson Dzhaparidze-Nikulin
`mnt`	BHEP, Cox-Small, DEHT, DEHU, EHS, HJG HV, HZ, KKurt, MAKurt, MASkew, MKurt MQ1, MQ2, MRSSkew, MSkew, PU, SR
`mvnormtest`	Shapiro-Wilk

Multivariate Normality Tests : Results



(a) Varing marginals



(b) Varing copulas

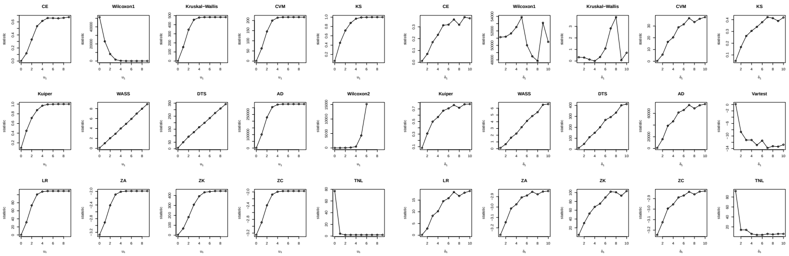
Figure: Experiments with multivariate normal distributions.

Univariate Two-Sample Tests : Implementations

Table: Univariate two-sample tests and their implementations in R.

Package	Test
`copent`	CE
`stat`	Wilcoxon1 Kruskal-Wallis
`twosamples`	CVM, KS, Kuiper WASS, DTS, AD
`robusTest`	Wilcoxon2, Vartest
`R2sample`	LR, ZA,ZK,ZC
`tnl.Test`	TNL

Univariate Two-Sample Tests : Results



(a) Mean (b) Variance

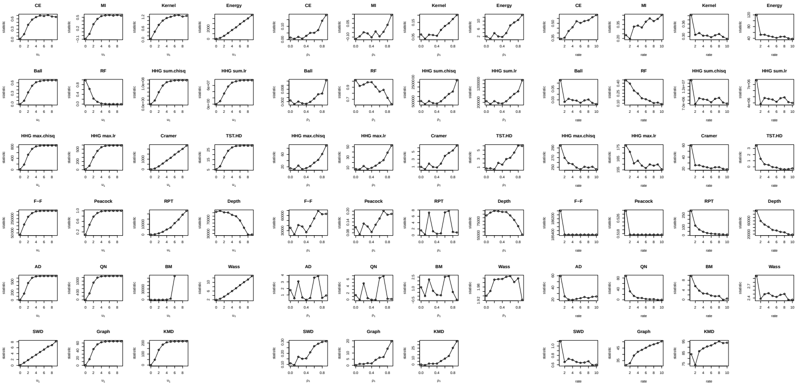
Figure: Experiments on univariate two-sample tests.

Multivariate Two-Sample Tests : Implementations

Table: Multivariate two-sample tests and their implementations in R.

Package	Test
`copent`	CE MI
`kernlab`	Kernel
`energy`	Energy statistics
`Ball`	Ball divergence
`hypoRF`	Random Forest
`HHG`	HHG {sum.chisq,sum.lr,max.chisq,max.lr}
`cramer`	Cramer
`TwoSampleTest.HD`	TST.HD
`fasano.franceschini.test`	F-F
`Peacock.test`	Peacock
`RandomProjectionTest`	RPT
`DepthProc`	Depth
`kSamples`	AD,QN
`lawstat`	BM
`T4transport`	WASS,SWD
`rgTest`	Graph
`KMD`	KMD

Multivariate Two-Sample Tests : Results



(a) Mean (b) Covariance (c) Normal copula

Figure: Experiments on multivariate two-sample tests.

Change Point Detection : Implementations

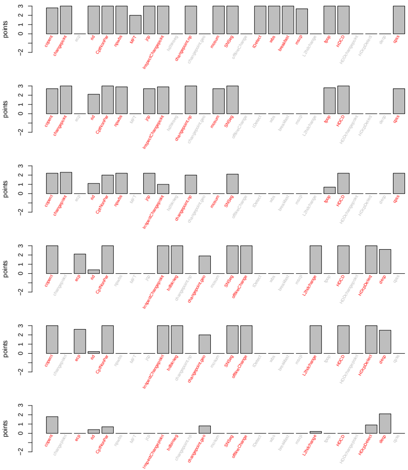
Table: Methods for change-point detection and their implementations in R.

Package	Univariate			Multivariate		
	Mean	Mean-Var	Var	Mean	Mean-Var	Var
`copent`	✓	✓	✓	✓	✓	✓
`changepoint`	✓	✓	✓			
`ecp`				✓	✓	✓
`rid`	✓	✓	✓	✓	✓	✓
`CptNonPar`	✓	✓	✓	✓	✓	✓
`npwbs`	✓	✓	✓			
`MFT`	✓					
`jcp`	✓	✓				
`InspectChangepoint`	✓	✓	✓	✓	✓	✓
`hdbinseg`				✓	✓	✓
`changepoint.np`	✓	✓	✓			
`changepoint.geo`				✓	✓	✓
`mosum`	✓	✓	✓			
`SNSeg`	✓	✓	✓	✓	✓	✓
`offlineChange`				✓	✓	✓
`IDetect`	✓					
`wbs`	✓					
`breakfast`	✓					
`mscp`	✓					
`L2hdchange`				✓	✓	✓
`fpop`	✓	✓	✓			
`HDCD`	✓	✓	✓	✓	✓	✓
`HDDchangepoint`				✓	✓	✓
`HDcpDetect`				✓	✓	✓
`decp`				✓	✓	✓

Change Point Detection: Results

Simulated change points:

- Univariate cases
  - Mean
  - Variance
  - Mean-Variance
- Multivariate cases
  - Mean
  - Variance
  - Mean-Variance



Symmetry Test: Simulation Experiments

Simulation Experiments

Simulated distributions

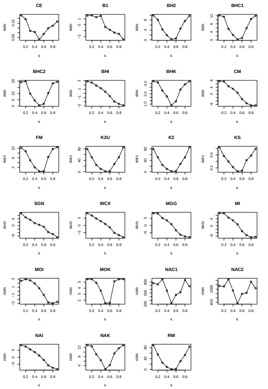
- Asymmetric Laplace distributions
- Beta distributions
- bimodal normal distributions

Compared methods:

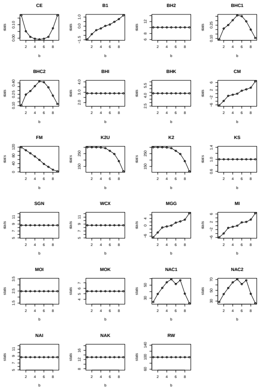
- MI : The Mira test statistic;
- CM : The Cabilio–Masaro test statistic ;
- MGG : The Miao, Gel and Gastwirth test statistic;
- B1 : The $\sqrt{b_1}$ test statistic;
- KS : The Kolmogorov–Smirnov test statistic;
- SGN : The Sign test statistic;
- WCX : The Wilcoxon test statistic;
- FM : The characterization based test;
- RW : The Rothman-Woodroofe test statistic;
- BHI : The Litvinova test statistic;
- BHK : The Baringhaus and Henze supremum-type test statistic;
- BH2 : The Baringhaus-Henze test statistic;
- MOI and MOK : The Milošević and Obradović test statistics;
- NAI and NAK : The Nikitin and Ahsanullah test statistics;
- K2 and K2U : The Božin, Milošević, Nikitin and Obradović Kolmogorov type statistics based on V- and U- statistics respectively;
- NAC1, NAC2, BHC1 and BHC2 : The Allison and Pretorius test statistics.

Symmetry Tests: Results

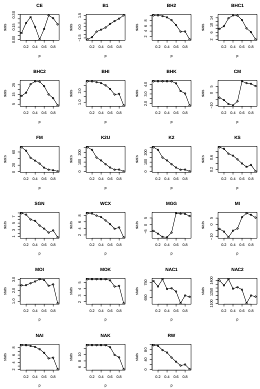
Simulation results



(a) Asymmetric Laplace distributions



(b) Beta distributions



(c) Bimodal normal distributions

Figure: Simulation results on three types of distributions.

Summary

- The theory of Copula Entropy was developed from copula theory, and parametric and non-parametric method for estimating CE was proposed.
- CE was proposed to test statistical independence and conditional independence (transfer entropy).
- CE was applied to solve 11 fundamental statistical problems, including association discovery, structure learning, variable selection, causal discovery, time lag estimation, system identification, multivariate normality test, copula hypothesis testing, two-sample test, change point detection, and symmetry test.
- CE was evaluated with its counterparts on independence / conditional independence measures, multivariate normality tests, two-sample tests, change-point detection, and symmetry tests.

References

1. Jian Ma. "Learning in Copula Structure". PhD thesis. Tsinghua University, 2009
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4. Jian Ma and Zengqi Sun. "Dependence structure estimation via copula". In: *arXiv preprint arXiv:0804.4451* (2008)
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`http://arxiv.org/a/ma_j_3`

Softwares

- Official

  The **copent**[18] package in R and Python for estimating copula entropy, transfer entropy, the statistic for multivaritate normality test, and change point detection are available on CRAN and PyPI respectively. The source codes are provided on GitHub.

  https://cran.r-project.org/package=copent

  https://pypi.org/project/copent/

  https://github.com/majianthu

- Third-Party

  The third-party implementations of the CE estimator include the **cylcop** package in R, the **MLFinLab**, **ArbitrageLab** and **Polars-ds** package in Python, the **CopEnt.jl** package, the **CausalityTools.jl** package and the **Copulas.jl** package in Julia, and the **gcmi** package in Matlab and Python.

[18] Jian Ma. "copent: Estimating Copula Entropy and Transfer Entropy in R". In: *arXiv preprint arXiv:2005.14025* (2020).

My Golf

Enjoy the Power of Copula Entropy!