

Copula Entropy

Theory and Applications

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Copula Theory

Definition (Copula)

^a Given N random variables $\mathbf{X} = (X_1, \dots, X_N) \in \mathcal{R}^N$. Let $\{u_i = F_i(x_i), i = 1, \dots, N\}$ be the marginal distribution functions of \mathbf{X} . A N -dimensional copula $C : \mathcal{I}^N \rightarrow \mathcal{I} (\mathcal{I} = [0, 1])$ of \mathbf{X} is a function with following properties:

- ① C is grounded and N -increasing;
- ② $C(1, \dots, 1, u_i, 1, \dots, 1) = u_i$.

^aRoger B Nelsen. *An introduction to copulas*. Springer, 2007.

- the theory on **representation** of statistical dependence in probability
- copula function contains all the dependence information between random variables
- a probability function on unit cubic

Copula Theory

Corollary

The probabilistic density function (PDF) $p(\mathbf{x})$ of \mathbf{X} can be represented as

$$p(\mathbf{x}) = c(\mathbf{u}) \prod_{i=1}^N p_i(x_i), \quad (2)$$

where $\{p_i, i = 1, \dots, N\}$ are marginal density functions of \mathbf{X} , and c is copula density.

- separating dependence representation with properties of individual variables

Copula Entropy: Theory

Theorem

Mutual Information of \mathbf{X} is equivalent to its negative copula entropy.

$$I(\mathbf{x}) = -H_c(\mathbf{x}). \quad (4)$$

Corollary

$$H(\mathbf{x}) = \sum_i H_i(x_i) + H_c(\mathbf{x}). \quad (5)$$

- the bridge between copula theory and information theory¹

¹ Jian Ma and Zengqi Sun. "Mutual information is copula entropy". In: *Tsinghua Science & Technology* 16.1 (2011). See also arXiv preprint arXiv:0808.0845 (2008), pp. 51–54.

Copula Entropy: Estimation

- **Non-Parametric Estimation Method²**
 - ① estimating empirical copula density with rank statistics
 - ② estimating copula entropy with kNN entropy estimation method
- Advantages
 - distribution-free, non-parametric
 - tuning-free, insensitive to parameters
 - good convergence
 - easy to implement
 - low computation burden

² Jian Ma and Zengqi Sun. "Mutual information is copula entropy". In: *Tsinghua Science & Technology* 16.1 (2011). See also arXiv preprint arXiv:0808.0845 (2008), pp. 51–54.

Copula Entropy: Application I

Association Discovery³

³ Jian Ma. "Discovering Association with Copula Entropy". In: *arXiv preprint arXiv:1907.12268* (2019).

Copula Entropy: Association Discovery

- Problem
 - To discover association relationship between random variables from data
- History
 - An old and fundamental problem since statistics birth
- Related Methods
 - Pearson Correlation Coefficient
 - Regression

Copula Entropy: Association Discovery

- Traditional association measures
 - Pearson Correlation Coefficient

$$r_{XY} = \text{corr}(X, Y) = \frac{\text{cov}(X, Y)}{\delta_X \delta_Y} \quad (6)$$

- Spearman's ρ and Kendall's τ

$$\rho_{XY} = 12 \int_u \int_v C(u, v) dudv - 3 \quad (7)$$

$$\tau_{XY} = 4 \int_u \int_v C(u, v) dC(u, v) - 1 \quad (8)$$

- Why Copula Entropy?

Table: Theoretical comparison between CE and CC.

	CC	CE
linearity	linear	nonlinear
Order	2	≥ 2
Assumption	Gaussian	None
variate	bivariate	multivariate

Copula Entropy: Association Discovery

Experiments on the NHANES data

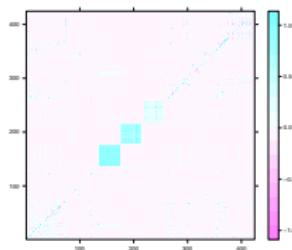
- Objectives of NHANES
 - to monitor trends and emerging issues of population health
 - to investigate its relationship with risk factors, nutritions and environmental exposures, etc.
- NHANES (2013-2014)
 - 14,332 persons from 30 different survey locations were selected;
 - Of those selected, 10,175 interviewed and 9,813 examined;
 - 5 groups of data: demographics, dietary, examination, laboratory, and questionnaire.
- Experimental data

The laboratory data, which includes 423 variables from blood, urine, oral rinse and vaginal/Penile swabs.
- Missing values

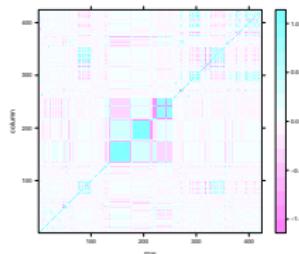
The missing values were filled with the mean of their corresponding variables.

Copula Entropy: Association Discovery

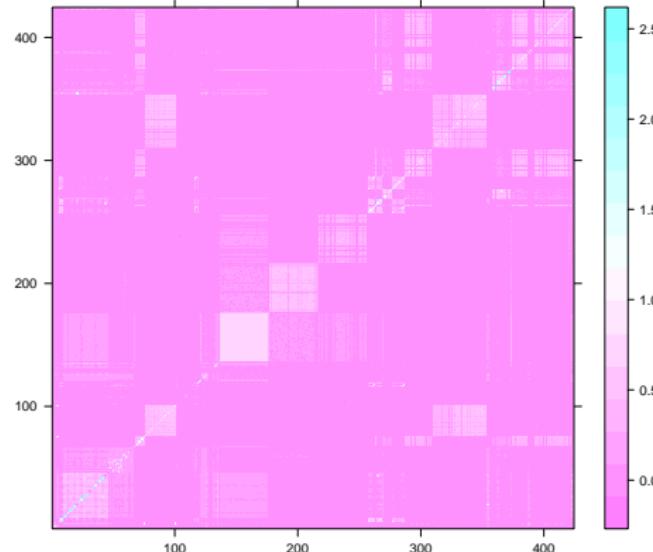
- Results - Correlation matrices



Pearson's r



Spearman's ρ



Copula Entropy

Copula Entropy: Association Discovery

- Results - Variable groups with meanings

Table: Variable groups with biomedical meanings discovered with CE.

Group	Index	Variables
1	288-302	Polycyclic Aromatic Hydrocarbones (PAH) - Urine
	68-75	Copper, Selenium & Zinc - Serum
	395-420	Urine Metals
2	358-373	Blood Lead, cadmium, total Mercury, Selenium, and Manganese
	269-276	Blood mercury: inorganic, ethyl and methyl
3	277-287	Oral Glucose Tolerance Test
	258-262	Insulin
	7-9	Cholesterol-LDL, Triglyceride&Apolipoprotein(ApoB), WTSAF2YR-Fasting Subsample 2 Year MEC Weight, LBXAPB-Apolipoprotein (B) (mg/dL), LBDAPBSI-Apolipoprotein (B) (g/L)
4	10-46	Standard Biochemistry Profile
	137-176	Human Papillomavirus (HPV) - Oral Rinse
5	76-101	Personal Care and Consumer Product chemicals and Metabolites
	327-353	Phthalates and Plasticizers Metabolites - Urines

Copula Entropy: Application II

Structure Learning⁴

⁴ Jian Ma and Zengqi Sun. "Dependence structure estimation via copula". In: *arXiv preprint arXiv:0804.4451* (2008).

Copula Entropy: Structure Learning

- Problem
 - To learn statistical structure among random variables from data
- Graph Representation
 - A probability density is represented with a directed or undirected graph, of which each node represents a random variable, and each edge represents a (conditional) dependence relation between two random variables
- Related Methods
 - Chow-Liu Algorithm

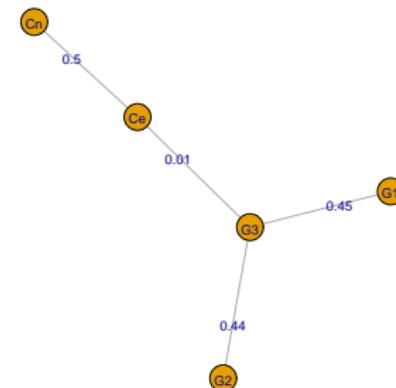
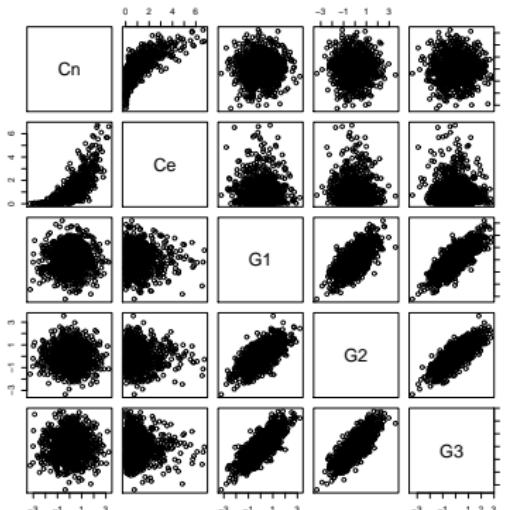
Copula Entropy: Structure Learning

- Our Algorithm
 - ① computing dependence matrix \mathbf{W}_x of data x with CE estimation
 - ② constructing dependence structure T from \mathbf{W}_x with MST algorithm
- Advantages
 - distribution-free, non-parametric
 - tuning-free, insensitive to parameters
 - easy to implement
 - low computation burden

Copula Entropy: Structure Learning

- Simulated Experiment

5 random variables: the first three are Gaussian and the others two are governed by Gaussian copula with margins as normal distribution and exponential distribution respectively

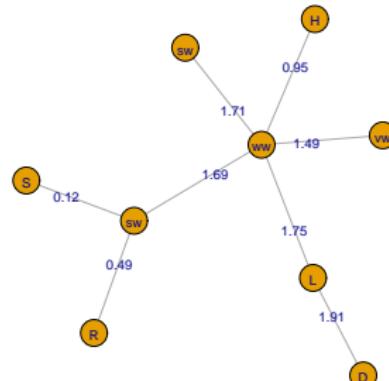


Copula Entropy: Structure Learning

Experiment on real data

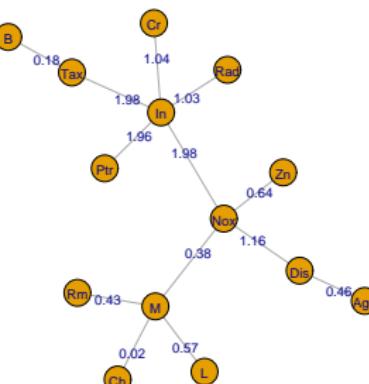
- Abalone data

Predicting the age of abalone from physical measurements



- Boston housing data

Concerns housing values in suburbs of Boston



Copula Entropy: Application III

Variable Selection⁵

⁵ Jian Ma. "Variable Selection with Copula Entropy". In: *Chinese Journal of Applied Probability and Statistics* 37.4 (2021), pp. 405–420.

Copula Entropy: Variable Selection

- Problem
 - To select a 'right' subset of variables from the whole group for building classification or regression models with good predictability and interpretability
- History
 - An old and basic problem in statistics and machine learning
- Related Problems
 - Feature Selection
 - Model Selection

Copula Entropy: Variable Selection

Existing methods - Likelihood with penalty

- Information Criteria
with penalty on the number of parameters in the models

$$\text{AIC} = -2L + 2p \quad (9)$$

$$\text{BIC} = -2L + p \log N \quad (10)$$

- Penalized GLMs
with penalty on the nonzero coefficients in the GLMs

- LASSO
- Ridge Regression
- Elastic Net

$$\min_{\beta} \{L(\beta; y, \mathbf{X}) + \lambda_1 \|\beta\|_1 + \lambda_2 \|\beta\|_2^2\} \quad (11)$$

- Adaptive LASSO

$$\min_{\beta} \{L(\beta; y, \mathbf{X}) + \lambda \sum_{j=1}^p w_j |\beta_j|\} \quad (12)$$

Copula Entropy: Variable Selection

Existing methods - Statistical independence measures

- Distance Correlation

$$\text{dCor}(X, Y) = \frac{\nu^2(X, Y)}{\sqrt{\nu^2(X)\nu^2(Y)}}, \quad (13)$$

where $\nu^2(X, Y)$ be distance covariance.

- Hilbert-Schmidt Independence Criterion (HSIC)

$$\text{dHSIC}(P(\mathbf{X})) = ||\Pi(P(X_1) \otimes \dots \otimes P(X_d)) - \Pi(P(\mathbf{X}))||, \quad (14)$$

where Π be the mean embedding function associated with kernel functions.

Copula Entropy: Variable Selection

- CE based method

To select variables based on ranks of their negative CE values with target

- Advantages

- model-free, non-parametric
- tuning-free, insensitive to parameters
- interpretable with physical meanings
- supported by rigorous math
- science instead of art, compared with existing methods
- easy to implement, low computation burden

Copula Entropy: Variable Selection

Experiments on the UCI heart disease data⁶

- Overview of the data

The data set contains 4 databases (899 samples) concerning heart disease diagnosis. All attributes are numeric-valued. The data was collected from the four following locations:

- Cleveland clinic foundation;
- Hungarian Institute of Cardiology, Budapest;
- V.A. medical center, long beach, CA;
- University hospital, Zurich, Switzerland.

- Attributes

The data has 76 attributes (#58 'num' for diagnosis). Of them, 13 attributes are recommended by professionals as clinical relevant.

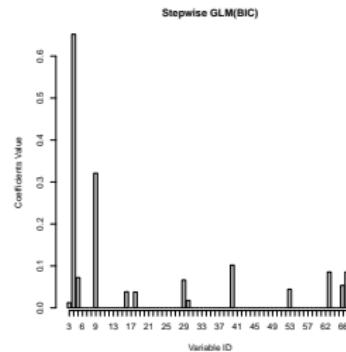
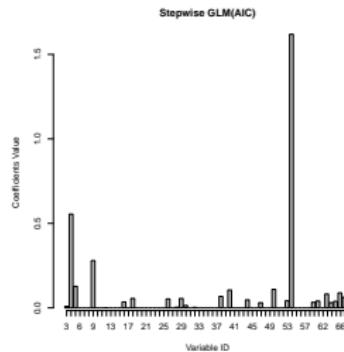
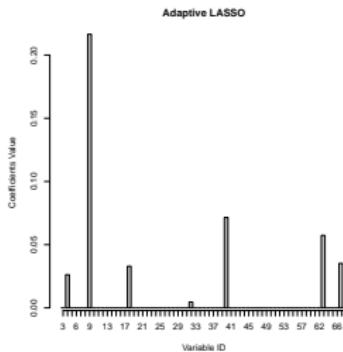
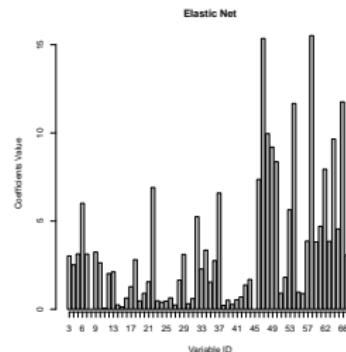
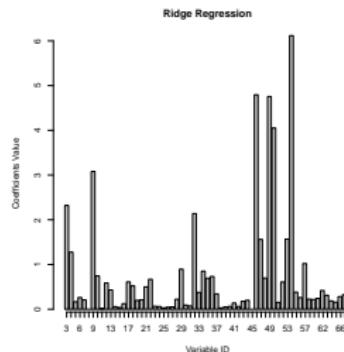
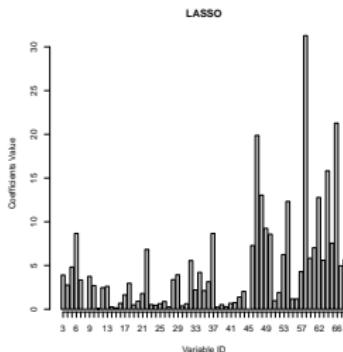
Table: Recommended attributes.

ID	3	4	9	10	12	16	19
Name	age	sex	cp	trestbps	chol	fbs	restecg
ID	32	38	40	41	44	51	58
Name	thalach	exang	oldpeak	slope	ca	thal	num

⁶Arthur Asuncion and David Newman. *UCI machine learning repository*. 2007.

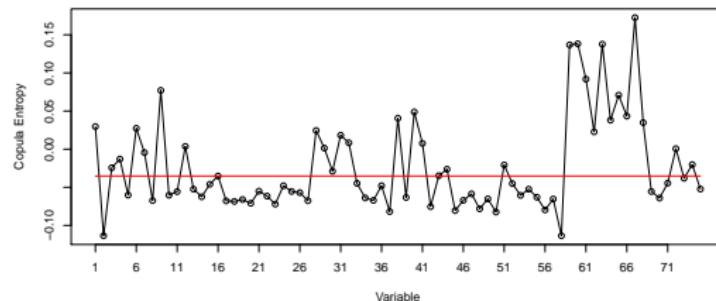
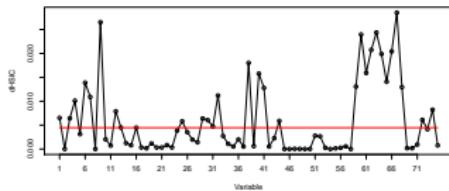
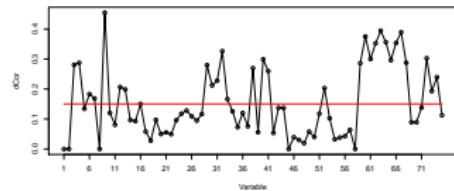
Copula Entropy: Variable Selection

- Results - Coefficients of penalized likelihood based models



Copula Entropy: Variable Selection

- Results - with statistical dependence measures (dCor, dHSIC, CE)



Copula Entropy: Variable Selection

- Results - Prediction accuracy

the selected variables present the best prediction accuracy.

Model	Accuracy(%)
SVM(Recommended variables)	84.20
SVM(CE)	84.76
SVM(dCor)	82.76
SVM(dHSIC)	84.54
Stepwise GLM(AIC)	51.8
Stepwise GLM(BIC)	49.1
LASSO	79.2
Ridge Regression	63.0
Elastic Net	75.9
Adaptive LASSO	35.7

Copula Entropy: Variable Selection

- Results - Selected variables

Copula Entropy selects more 'right' variables than the other methods do.

Method	Selected Variables' ID	✓
Recommended variables	3,4,9,10,12,16,19,32,38,40,41,44,51	13
CE	3,4,6,7,9,12,16,28-32,38,40,41,44,51,59-68	11
dHSIC	3,4,6,7,9,12,13,16,25,29-32,38,40,41,44,59-68	10
dCor	3,4,6,7,9,12,13,16,28-33,38,40,41,52,59-68	9
Stepwise GLM(AIC)	3,4,5,9,12,16,18,20,26,29,30,32,40,44,47,50,53,54,60,61,63,65-67	8
Stepwise GLM(BIC)	3,4,5,9,16,18,29,30,40,53,63,66,67	5
Adaptive LASSO	4,6,9,18,32,40,63,67	4
LASSO		
Ridge Regression	all except 8,45	-
Elastic Net		

Copula Entropy: Application IV

Causal Discovery⁷

⁷ Jian Ma. "Estimating Transfer Entropy via Copula Entropy". In: *arXiv preprint arXiv:1910.04375* (2019).

Copula Entropy: Causal Discovery

- Problem
 - To infer causality from time series data by *estimating Transfer Entropy*
- History & Significance
 - Causality is one of the oldest topics in philosophy.
 - Causal discovery is a central problem of all sciences.
- Correlation vs Causality
 - Correlation does not mean causation.
 - Correlation is only helpful for prediction while causality means intervention and control.

Copula Entropy: Causal Discovery

- Causality measures

- Wiener's Principle

Cause should improve the prediction of effect.

- Granger Causality

improvement measured by the variance of prediction error

$$\delta^2(Y_{t+1}|Y_t, X_t) < \delta^2(Y_{t+1}|Y_t) \quad (15)$$

- Transfer Entropy

improvement on the uncertainty of prediction measured by Shannon entropy

$$TE = \sum p(Y_{t+1}, Y^t, X_t) \log \frac{p(Y_{t+1}|Y^t, X_t)}{p(Y_{t+1}|Y^t)} \quad (16)$$

$$= H(Y_{t+1}|Y^t) - H(Y_{t+1}|Y^t, X_t) \quad (17)$$

$$= I(Y_{t+1}, X_t | Y^t) \quad (18)$$

- Issue on TE

difficult to estimate, some think impossible without model assumptions

Copula Entropy: Causal Discovery

- TE via CE

Proposition

Transfer Entropy can be represented with only Copula Entropy.

$$T_{x \rightarrow y} = -H_c(Y_{t+1}, Y^t, X_t) + H_c(Y_{t+1}, Y^t) + H_c(Y^t, X_t) - H_c(Y^t) \quad (19)$$

- Non-parametric Estimator of TE
 - ① estimating three or four CE terms in (19);
 - ② calculating TE for these estimated CEs.
- inheriting all the merits of non-parametric CE estimation

Copula Entropy: Causal Discovery

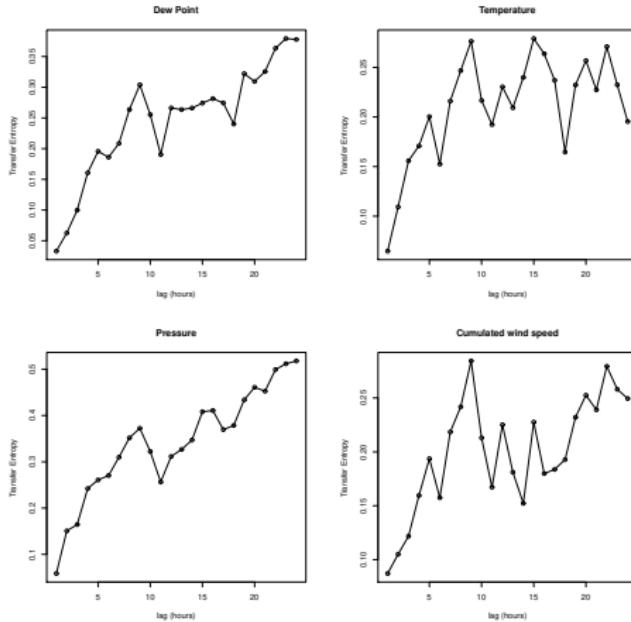
Experiments on the UCI Beijing PM2.5 data⁸

- Overview of the data
 - Time
 - hourly data from 2010-01-01 to 2014-12-31, which results in 43824 samples with missing values.
 - Observations
 - PM2.5 data of US Embassy in Beijing
 - Meteorological data from Beijing Capital International Airport
 - Meteorological factors
 - dew point, temperature, pressure, cumulated wind speed, combined wind direction, cumulated hours of snow, cumulated hours of rain.
- Experimental data
 - the first four factors used in the experiments;
 - 1000 samples without missing values (2010-04-02~2010-05-14).

⁸Arthur Asuncion and David Newman. *UCI machine learning repository*. 2007.

Copula Entropy: Causal Discovery

Results: Effects of meteorological factors on PM2.5



- Two phases

- Sharp increase phase: the first 9 hours time lag, and peak at about 9 hours lag;
- Flat increase phase: TE of Dew point and pressure increase with relatively flat rate while TE of temp. and cumulated wind speed does increase any more.

- Interpretation

- The effects do not show immediately and are cumulating processes.

Copula Entropy: Application V

Time Lag Estimation⁹

⁹ Jian Ma. "Identifying Time Lag in Dynamical Systems with Copula Entropy based Transfer Entropy". In: *arXiv preprint arXiv:2301.06037* (2023).

Copula Entropy: Time Lag Estimation

- Problem
 - To identify time lag in dynamical systems with copula entropy based transfer entropy
- Significance
 - Time lag is ubiquitous in physical, social, and biological systems.
 - Identifying time lag is of fundamental importance in applications of dynamical systems.
- Related Methods
 - Auto-correlation
 - Time-delayed mutual information

Copula Entropy: Time Lag Estimation

- Our method
 - ① estimating transfer entropies on time lag horizon from data with the CE-based estimator
 - ② identifying the time lag associated with the maximum TE value

Copula Entropy: Time Lag Estimation

- Simulations

- ① generate trajectories from four simulated dynamical system with respect to different state or output lags
- ② identify the time lag with our method

- Simulated systems

- a system driven by random walk with output lag
- a system driven by sine function with output lag
- Wiener process with output lag
- a first-order linear system with state lag

Copula Entropy: Time Lag Estimation

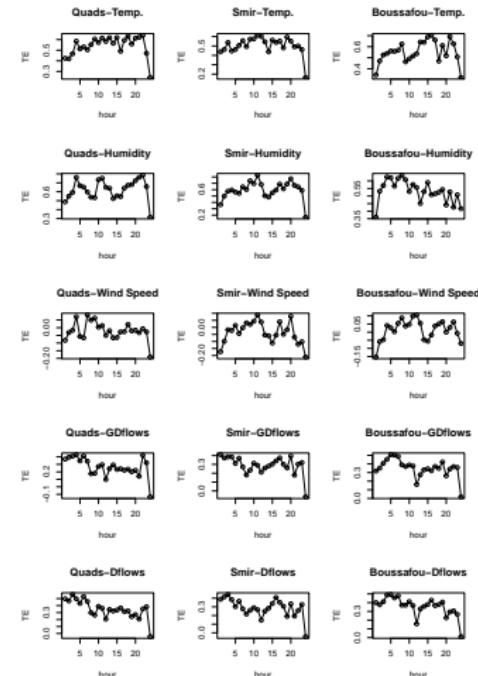
Power consumption of the Tetouan city¹⁰

• Data

- power consumption of 3 networks in 2017
- weather factors, including temperature, humidity, wind speed, general diffuse flows, diffuse flows

• Power consumption forecast

- To identify time lags from weather to power consumption



¹⁰ Arthur Asuncion and David Newman. UCI machine learning repository. 2007.

Copula Entropy: Application VI

System Identification¹¹

¹¹ Jian Ma. "System Identification with Copula Entropy". In: *arXiv preprint arXiv:2304.12922* (2023).

Copula Entropy: System Identification

- Problem
 - To discover differential equation from time series data
- Significance
 - differential equations are the main mathematical tools for modelling dynamical systems.
 - discovering differential equations of dynamical systems has wide applications in many scientific fields.
- Related Methods
 - SINDy
 - Gaussian processes

Copula Entropy: System Identification

- Idea

considering system identification as a variable selection problem

$$\frac{dx_i}{dt} = f(\mathbf{x}, t). \quad (20)$$

- Our method

- calculating the derivative of system variables with differential operator;
- estimating the CEs between the calculated derivatives and the covariates of the system;
- selecting the covariates with high CE value for each derivatives.

Copula Entropy: System Identification

- Simulations

- simulating time series data from the 3D Lorenz system
- identifying the system equation from data with our method

- Results

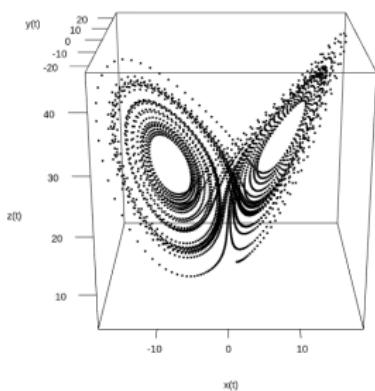


Figure: 3D plot of the simulated data.

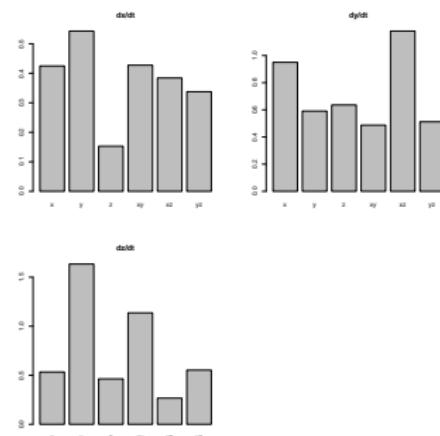


Figure: Identification results.

Copula Entropy: Application VII

Multivariate Normality Test¹²

¹² [Jian Ma](#). "Multivariate Normality Test with Copula Entropy". In: *arXiv preprint arXiv:2206.05956* (2022).

Copula Entropy: Multivariate Normality Test

- Problem
 - To test the hypothesis that the distribution of data is normal distribution
- Significance
 - Normal distribution is the most important distribution in probability theory;
 - Normality is a common assumption of many statistical tools;
 - Testing normality is widely needed in real applications.
- Related Methods
 - characteristics function based
 - moments based
 - skewness and kurtosis
 - energy distance based
 - entropy based
 - Wasserstein distance based

Copula Entropy: Multivariate Normality Test

- The proposed statistic

$$T_{ce} = H_c(\mathbf{x}) - H_c(\mathbf{x}_n), \quad (21)$$

where \mathbf{x}_n is the Gaussian random vector with the same covariances as \mathbf{x} .

- defined as the difference of copula entropies
- $T_{ce} = 0$ if normal distributions

- The estimator

- the first term in (21) can be estimated with the non-parametric CE estimator;
- the second term in (21) can be estimated easily by first estimating the covariances V_x of \mathbf{x} and then calculating the result according to (22).

$$H_c(\mathbf{x}_n) = \frac{1}{2} \log |V_x|. \quad (22)$$

Copula Entropy: Multivariate Normality Test

Simulation Experiments

- Data
 - bivariate normal copula with normal and exponential marginals
 - bivariate Gumbel copula with normal marginals
- Compared methods
 - Mardia's
 - Royston's
 - Henze and Zirkler's
 - Doornik and Hansen's, and
 - the energy distance based test by Rizzo and Székely

Copula Entropy: Application VIII

Two-Sample Test¹³

¹³ [Jian Ma.](#) "Two-Sample Test with Copula Entropy". In: *arXiv preprint arXiv:2307.07247* (2023).

Copula Entropy: Two-Sample Test

- Problem
 - To test the hypothesis that two samples are from a same distribution
- Significance
 - a basic hypothesis testing problem;
 - Symmetry test and change point detection can be formulated as two-sample test problem;
 - has many real applications in many areas, such as politics, medicine, etc.
- Related Methods
 - T-test or F-test
 - Kernel-based two-sample test
 - Kolmogorov-Smirnov test
 - Mutual information based test

Copula Entropy: Two-Sample Test

- The proposed statistic

$$T_{ce} = H_c(\mathbf{X}, Y_0) - H_c(\mathbf{X}, Y_1), \quad (23)$$

where $\mathbf{X} = (X_1, X_2)$ is for two samples $X_1 = \{X_{11}, \dots, X_{1m}\}$ and $X_2 = \{X_{21}, \dots, X_{2n}\}$, and $Y_1 = (0_1, \dots, 0_m, 1_1, \dots, 1_n)$ and $Y_0 = (1_1, \dots, 1_{m+n})$ are the labels for the null and the alternative hypothesis.

- non-parametric multivariate two-sample test
- defined as the difference between the copula entropies of the null and the alternative hypothesis;
- T_{ce} is small if H_0 is true.

- The estimator

- estimating the two terms in (23);
- calculating the estimated statistic.

Copula Entropy: Two-Sample Test

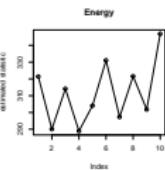
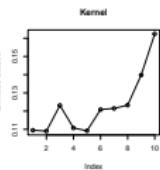
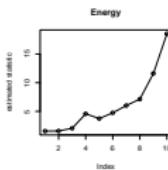
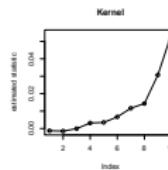
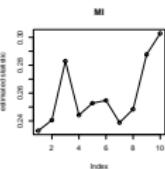
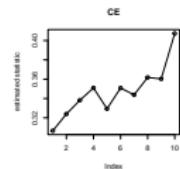
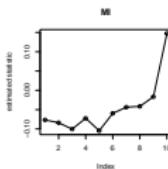
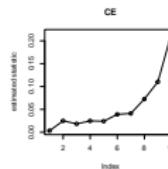
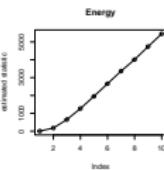
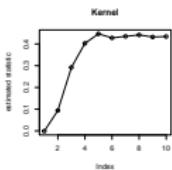
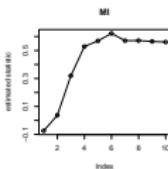
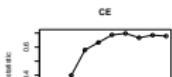
Simulation Experiments

- Data
 - bivariate normal distribution with different means
 - bivariate normal distribution with different variances
 - bivariate Gaussian copula with normal and exponential marginals
- Compared methods
 - Kernel-based test
 - Energy distance-based test
 - Mutual information-based test

Copula Entropy: Two-Sample Test

Simulation Results

- Bivariate normal distribution with different means
- Bivariate normal distribution with different variances
- Bivariate normal copula with different variances



Copula Entropy: Application IX

Change Point Detection¹⁴

¹⁴ Jian Ma. "Change Point Detection with Copula Entropy based Two-Sample Test". In: *arXiv preprint arXiv:2403.07892* (2024).

Copula Entropy: Change Point Detection

- Problem
 - To detect single or multiple disrupt change in time series data
- Significance
 - a basic problem in time series analysis;
 - can be solved with two-sample test problem;
 - has aboard real applications in many areas, such as geoscience, biology, manufacturing, etc.
- Existing Methods
 - CUSUM
 - Kernel-based method
 - two-sample test based
 - multiple detection with binary segmentation

Copula Entropy: Change Point Detection

- The proposed method
 - Single change point detection
solved with copula entropy based two-sample test
 - Multiple change point detection
Single change point detection with binary segmentation
- Merits
 - nonparametrical method based on CE estimator
 - both univariate and multivariate
 - distribution-free, universally applicable
 - almost tuning-free, universal threshold for test statistic
- Implementation
 - performance boosted with parallel computing

Copula Entropy: Change Point Detection

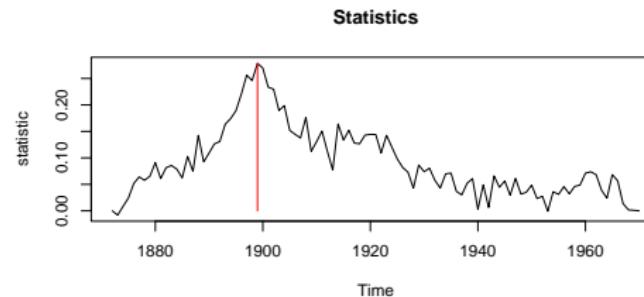
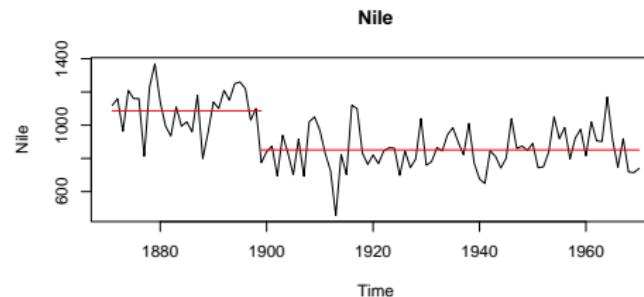
Simulation Experiments

- Univariate and multivariate multiple change points
 - changes in mean
 - changes in mean and variance
 - changes in variance
- Compared methods
 - The methods in the R package **changepoint**
 - Kernel based method in the R package **ecp**

Copula Entropy: Change Point Detection

Real data experiment

- The Nile data



Summary

- The theory of Copula Entropy was developed from copula theory, and a non-parametric method for estimating CE was proposed.
- CE was proposed to test statistical independence and conditional independence (transfer entropy).
- CE was applied to solve 9 fundamental statistical problems, including association discovery, structure learning, variable selection, causal discovery, time lag estimation, system identification, multivariate normality test, two-sample test, and change point detection.

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- ② Jian Ma. "Discovering Association with Copula Entropy". In: *arXiv preprint arXiv:1907.12268* (2019)
- ③ Jian Ma and Zengqi Sun. "Dependence structure estimation via copula". In: *arXiv preprint arXiv:0804.4451* (2008)
- ④ Jian Ma. "Variable Selection with Copula Entropy". In: *Chinese Journal of Applied Probability and Statistics* 37.4 (2021), pp. 405–420
- ⑤ Jian Ma. "Estimating Transfer Entropy via Copula Entropy". In: *arXiv preprint arXiv:1910.04375* (2019)
- ⑥ Jian Ma. "Multivariate Normality Test with Copula Entropy". In: *arXiv preprint arXiv:2206.05956* (2022)
- ⑦ Jian Ma. "Identifying Time Lag in Dynamical Systems with Copula Entropy based Transfer Entropy". In: *arXiv preprint arXiv:2301.06037* (2023)
- ⑧ Jian Ma. "System Identification with Copula Entropy". In: *arXiv preprint arXiv:2304.12922* (2023)
- ⑨ Jian Ma. "Two-Sample Test with Copula Entropy". In: *arXiv preprint arXiv:2307.07247* (2023)
- ⑩ Jian Ma. "Change Point Detection with Copula Entropy based Two-Sample Test". In: *arXiv preprint arXiv:2403.07892* (2024)
- ⑪ Jian Ma. "copent: Estimating Copula Entropy and Transfer Entropy in R". In: *arXiv preprint arXiv:2005.14025* (2020)



http://arxiv.org/a/ma_j_3

Softwares

- Official

The **copent**¹⁵ package in R and Python for estimating copula entropy, transfer entropy, and the statistic for multivariate normality test are available on CRAN and PyPI respectively. The source codes are provided on GitHub.



<https://cran.r-project.org/package=copent>



<https://pypi.org/project/copent/>



<https://github.com/majianthu>

- Third-Party

The third-party implementations of the CE estimator include the **cylcop** package in R, the **MLFinLab**, **ArbitrageLab** and **Polars-ds** package in Python, the **CopEnt.jl** package and the **CausalityTools.jl** package in Julia, and the **gcmi** package in Matlab and Python.

¹⁵ Jian Ma. "copent: Estimating Copula Entropy and Transfer Entropy in R". In: *arXiv preprint arXiv:2005.14025* (2020).

My Golf



Enjoy the Power of Copula Entropy!