Statistical analysis of the inhomogeneous telegrapher's process

Stefano Maria Iacus
Department of Economics, University of Milan
Via Conservatorio 7, I-20123 Milan - Italy
email: *stefano.iacus@unimi.it*

Abstract

We consider a problem of estimation for the telegrapher's process on the line, say $X(t)$, driven by a Poisson process with non constant rate. It turns out that the finite-dimensional law of the process $X(t)$ is a solution to the telegraph equation with non constant coefficients. We give the explicit law $(\mathbf{P}_\theta)$ of the process $X(t)$ for a parametric class of intensity functions for the Poisson process. We propose an estimator for the parameter θ of $\mathbf{P}_\theta$ and we discuss its properties as a first attempt to apply statistics to these models.

Keywords : *telegraph equation, inhomogeneous Poisson process, minimax estimation, random motions.*

MSC: *primary 60K99*, *secondary 62M99*;

1 Introduction

We consider a class of random processes governed by hyperbolic equations. These kind of processes have been proposed in the literature to describe motions of particles with finite velocities as opposed to diffusion-type models. The first contribution in this area goes back to Goldstein (1951). He considered the simplest random evolution on the real line where a particle, placed in the origin at time 0, moves with two finite velocities $\pm c$ changing its current velocity according to a Poisson process of constant rate λ. He found that the distribution of the position of the particle x at time t, is a solution to the telegraph equation, namely

$$u_{tt}(x,t) + 2\lambda u_t(x,t) = c^2 u_{xx}(x,t) \qquad (*)$$

where $u_{tt}(x,t) = \frac{\partial^2}{\partial t^2}u(x,t)$ and so forth. This model has received a great attention in the last decades. Many generalizations have been studied and the probabilistic properties of these models have been presented. In particular, the finite dimensional and the first passage time laws have been presented in an explicit form in a series of paper (see Orsingher 1985, 1990 and 1995, Foong 1992, Foong and Kanno, 1994). Different generalizations in the presence of a finite set of velocities c_i and Poisson rates λ_i (see e.g. Beghin *et al* 1999) as well as motions in two or more dimensions (Orsingher 1986 and 2000, Orsingher and Kolesnik 1996, Kolesnik and Turbin 1991, Iacus 1995) have also been considered. In general, for these models it is quite hard to find explicitly the distribution laws of interest.

In this paper we present a generalization of the model when the underlying Poisson process is not homogeneous that are interesting from a statistical point of view. Up to our knowledge, this is the first attempt to apply statistics in this field.

The paper is divided into two parts: the first concerns the probabilistic analysis of the model and the latter is devoted to statistical estimation. In Section 2 we introduce the model, we study the so called velocity process and we derive the following partial differential equation analogous to (*) :

$$u_{tt}(x,t) + 2\lambda(t)u_t(x,t) = c^2 u_{xx}(x,t) \tag{**}$$

where $\lambda(\cdot)$ is the intensity function of the Poisson process (see e.g. Kutoyants 1998). Then (in §2.2) we give conditions on $\lambda(\cdot)$ under which an explicit solution to (**) can be found. It turns out that it is a parametric family of intensity measures $\lambda(\cdot) = \lambda_\theta(\cdot)$. In the second part of the paper (Section 3) we consider the problem of estimation of the parameter θ for this particular class of solutions and we present asymptotically efficient –in the minimax sense– estimators for θ.

2 The model and the velocity process

Consider a particle placed in $x = 0$ at time $t = 0$. It can move leftward or rightward with finite velocity c. Changes of direction occur at Poisson times. This means that the particle will move at constant speed (either $+c$ or $-c$) among two successive Poisson events. We introduce the velocity process $V(t)$:

$$V(t) = V(0)(-1)^{N(t)}, \quad t > 0\,, \tag{1}$$

where $V(0)$ is a random variable taking values $\pm c$ with probability $\frac{1}{2}$ and independent from the Poisson process $N(t)$ with intensity measure $\Lambda(\cdot)$:

$$\Lambda(t) = \int_0^t \lambda(s)\mathrm{d}s\,, \qquad t > 0\,,$$

where $\lambda(\cdot)$ – the intensity function – belongs to $\mathcal{C}^1(\mathbb{R})$. The increments $N(t) - N(s)$, $0 < s < t$, are then distributed according to the Poisson law with parameter $\Lambda(t) - \Lambda(s)$.

Our main goal is to determine the law governing the process $X(t)$ that represents the position of the particle at time t :

$$X(t) = V(0)\int_0^t (-1)^{N(s)}\mathrm{d}s\,. \tag{2}$$

We start with the analysis of the velocity process $V(t)$, $t > 0$. At first, we note that the two probabilities

$$\begin{aligned} p^{(c)}(t) &= \mathbf{P}(V(t) = c) \\ p^{(-c)}(t) &= \mathbf{P}(V(t) = -c) \end{aligned}$$

are solutions of the following system of differential equations

$$\begin{cases} p_t^{(c)}(t) = \Lambda_t(t)(p^{(-c)}(t) - p^{(c)}(t)) \\ p_t^{(-c)}(t) = \Lambda_t(t)(p^{(c)}(t) - p^{(-c)}(t)) = -p_t^{(c)}(t) \end{cases} \tag{3}$$

This can be proved by Taylor expansion of the two functions $p_t^{(c)}(t)$ and $p_t^{(-c)}(t)$. The following conditional laws:

$$\mathbf{P}(V(t) = c|V(0) = c) \qquad \text{and} \qquad \mathbf{P}(V(t) = -c|V(0) = c)\,,$$

characterize the velocity process. Their explicit form are as follows.

Proposition 2.1.

$$\mathbf{P}(V(t) = c|V(0) = c) = 1 - \int_0^t \lambda(s)e^{-2\Lambda(s)}\mathrm{d}s = \frac{1 + e^{-2\Lambda(t)}}{2}\,, \tag{4}$$

$$\mathbf{P}(V(t) = -c|V(0) = c) = \int_0^t \lambda(s)e^{-2\Lambda(s)}\mathrm{d}s = \frac{1 - e^{-2\Lambda(t)}}{2} \tag{5}$$

Proof. We give only the derivation of (4) because the other follows by symmetry. Conditioning on $V(0) = c$ implies that: $p^{(c)}(0) = 1$, $p^{(c)}_t(0) = -\Lambda_t(0) = -\lambda(0)$, $p^{(-c)}(0) = 0$ and $p^{(-c)}_t(0) = \lambda(0)$. From (3) we obtain that

$$\frac{p^{(c)}_{tt}(t)}{p^{(c)}_t(t)} = \frac{\mathrm{d}}{\mathrm{d}t} \log p^{(c)}_t(t) = -\frac{2\Lambda_t(t)^2 - \Lambda_{tt}(t)}{\Lambda(t)} ,$$

and by simple integration by parts it emerges that:

$$p^{(c)}_t(t) = \exp\left\{-2\Lambda(t) + \log\left(\frac{\lambda(t)}{\lambda(0)}\right) + \log(-\lambda(0))\right\} ,$$

from which follows (4). $\square$

Remark that it is also possible to derive (4) by simply noting that

$$\begin{aligned}
\mathbf{P}(V(t) = c|V(0) = c) &= \mathbf{P}\left(\bigcup_{k=0}^{\infty} N(t) = 2k\right) \\
&= \sum_{k=0}^{\infty} \frac{\Lambda(t)^{2k}}{(2k)!} e^{-\Lambda(t)} \\
&= \frac{1 + e^{-2\Lambda(t)}}{2}
\end{aligned}$$

and then $\mathbf{P}(V(t) = -c|V(0) = c) = 1 - \mathbf{P}(V(t) = c|V(0) = c)$.

We conclude the analysis of the velocity process by giving also the covariance function of the couple $(V(t), V(s))$, $s, t > 0$. In fact, it can be easily proven that the characteristic function of the couple $(V(t), V(s))$ is, for all $(\alpha, \beta) \in \mathbb{R}^2$,

$$\mathbf{E}\left(e^{i\alpha V(s) + i\beta V(t)}\right) = \cos(\alpha c)\cos(\beta c) - e^{-2(\Lambda(t)-\Lambda(s))}\sin(\alpha c)\sin(\beta c),$$

and the covariance function is then

$$\mathbf{E}\left(V(s)V(t)\right) = -\frac{\partial^2}{\partial\alpha\partial\beta}\mathbf{E}\left(e^{i\alpha V(s) + i\alpha V(t)}\right)\Big|_{\alpha=\beta=0} = c^2 e^{-2|\Lambda(t)-\Lambda(s)|} .$$

2.1 Derivation of the telegraph equation

In order to analyze the distribution of the position of the particle

$$P(x, t) = \mathbf{P}(X(t) < x) \tag{6}$$

we introduce the two distribution functions $F(x,t) = \mathbf{P}\left(X(t) < x, V(t) = c\right)$ and $B(x,t) = \mathbf{P}\left(X(t) < x, V(t) = -c\right)$, so that $P(x,t) = F(x,t) + B(x,t)$ and $W(x,t) = F(x,t) - B(x,t)$. The function $W(\cdot,\cdot)$ is usually called the "flow function". Next result gives the analogous to (*) in the case of nonhomogeneous Poisson process.

Proposition 2.2. *Suppose that $F(\cdot,\cdot)$ and $B(\cdot,\cdot)$ are two times differentiable in x and t, then*

$$\begin{cases} F_t(x,t) = -cF_x(x,t) - \lambda(t)(F(x,t) - B(x,t)) \\ B_t(x,t) = cB_x(x,t) + \lambda(t)(F(x,t) - B(x,t)) \end{cases} \tag{7}$$

moreover $P(x,t)$ is a solution to the following telegraph equation with non constant coefficients

$$\frac{\partial^2}{\partial t^2}u(x,t) + 2\lambda(t)\frac{\partial}{\partial t}u(x,t) = c^2\frac{\partial^2}{\partial x^2}u(x,t)\,. \tag{8}$$

Proof. By Taylor expansion, one gets that $F(\cdot,\cdot)$ and $B(\cdot,\cdot)$ are solutions to (7) and rewriting system (7) in terms of the functions $W(\cdot,\cdot)$ and $P(\cdot,\cdot)$ it emerges that

$$\begin{cases} P_t(x,t) = -cW_x(x,t) \\ W_t(x,t) = cP_x(x,t) - 2\lambda(t)W(x,t) \end{cases} \tag{9}$$

The conclusion arises by direct substitutions. In fact, by the first of system (9) we have

$$P_{tt}(x,t) = \frac{\partial}{\partial t}P_t(x,t) = -c\frac{\partial}{\partial t}W_x(x,t)\,.$$

Furthermore

$$\begin{aligned} W_{tx}(x,t) &= \frac{\partial}{\partial x}(cP_x(x,t) - 2\lambda(t)W(x,t)) \\ &= cP_{xx}(x,t) - 2\lambda(t)W_x(x,t) \\ &= cP_{xx}(x,t) + \frac{2}{c}\lambda(t)P_t(x,t) \end{aligned}$$

by using respectively the second and the first equation of system (9). □

2.2 The explicit law of the telegraph process

In this section we give the explicit form of distribution function (6) for a particular class of intensity functions. The idea is to reduce equation (8) to a partial differential equation for which the general solution is available. This is done by imposing conditions on the family of intensity functions $\lambda(\cdot)$. Here we give only one type of solution that is interesting from the statistical point of view. The result presented in the next theorem is interesting in itself.

Theorem 2.3. *Suppose that the intensity function of the Poisson process $N(t)$ in (2) is*

$$\lambda(t) = \lambda_\theta(t) = \theta \tanh(\theta t), \quad \theta \in \mathbb{R}\,. \tag{10}$$

Then, the absolutely continuous component $p_\theta(\cdot)$ of distribution (6), is given by

$$p_\theta(x,t,c) = \begin{cases} \frac{\theta\, t}{\cosh(\theta\, t)} \frac{I_1\left(\frac{\theta}{c}\sqrt{c^2\, t^2 - x^2}\right)}{2\sqrt{c^2\, t^2 - x^2}}, & |x| < ct \\ 0, & \textit{otherwise} \end{cases} \tag{11}$$

Proof. We start without assuming (10) and by noting that $p(x,t)$ is a solution to (8). Moreover, the process $X(t) \in (-ct, ct)$ only if there occur at least one Poisson event up to time t, thus

$$\int_{-ct}^{ct} p(x,t)\mathrm{d}x = 1 - \mathbf{P}(N(t) = 0) = 1 - e^{-\Lambda(t)} \tag{12}$$

that gives one of the conditions to solve (8). We now search for solutions of the following type:

$$v(x,t) = e^{\Lambda(t)} u(x,t)\,.$$

Thus the function $v(\cdot,\cdot)$ satisfies the following partial differential equation

$$v_{tt}(x,t) - v(x,t)\left(\lambda'(t) + \lambda^2(t)\right) = c^2 v_{xx}(x,t)\,. \tag{13}$$

For a generic function $\lambda(\cdot)$ a solution to (13) not available. If the intensity function $\lambda(\cdot)$ satisfies the following ordinary differential equation

$$\lambda'(t) + \lambda^2(t) = \theta^2, \quad t > 0, \quad \theta \in \mathbb{R}\,, \tag{14}$$

then equation (13) becomes

$$v_{tt}(x,t) - v(x,t)\theta^2 = c^2 v_{xx}(x,t) \tag{15}$$

and the solution to it can be written explicitly[1]. With the initial condition $\lambda(0) = 0$ the solution to equation (14) is condition (10) and then $\Lambda_\theta(t) =$

[1]Remark that, if one solves the problem $\lambda'(t) + \lambda^2(t) = 0$ equation (13) reduces to the standard one-dimensional wave equation $v_{xx} = c^2 v_{xx}$ and its solution can also be determined but it is of no statistical interest.

$\ln(\cosh(\theta t))$. Following Orsingher (1985), by the change of variable $s = \sqrt{c^2t^2 - x^2}$ we transform equation (15) into the following standard Bessel's equation

$$v_{ss} + \frac{1}{s} v_s - \left(\frac{\theta}{c}\right)^2 v = 0 \tag{16}$$

whose general integral is

$$v(x,t) = A I_0\left(\frac{\theta}{c}\sqrt{c^2t^2 - x^2}\right) + B K_0\left(\frac{\theta}{c}\sqrt{c^2t^2 - x^2}\right), \quad |x| < ct\,.$$

The function $I_k(x)$ is the modified Bessel function of first kind and order k while $K_0(x)$ is the second type Bessel function of order 0 with the unpleasant property that $\lim_{x\to 0^+} K_0(x) = \infty$, so we put $B = 0$. In terms of $p_\theta(\cdot,\cdot)$ the solution is of the following form

$$p_\theta(x,t) = K e^{-\Lambda_\theta(t)} I_0\left(\frac{\theta}{c}\sqrt{c^2t^2 - x^2}\right), \quad |x| < ct\,. \tag{17}$$

From (12) it follows that

$$\int_{-ct}^{ct} p_\theta(x,t)\mathrm{d}x = 1 - e^{-\Lambda_\theta(t)} = 1 - \frac{1}{\cosh(\theta t)} = 1 - \frac{2}{e^{-\theta t} + e^{\theta t}}$$

and so, there is no constant value K that satisfies (17). To turn around this problem, we note that if $I_0\left(\frac{\theta}{c}\sqrt{c^2t^2 - x^2}\right)$ is a solution to (16) so is its partial derivative with respect to t. Hence we search for solutions of the form

$$p_\theta(x,t) = e^{-\Lambda_\theta(t)}\left(A I_0\left(\frac{\theta}{c}\sqrt{c^2t^2 - x^2}\right) + B\frac{\partial}{\partial t} I_0\left(\frac{\theta}{c}\sqrt{c^2t^2 - x^2}\right)\right).$$

Form the following two equalities (see Orsingher, 1995)

$$\int_{-ct}^{ct} I_0\left(\frac{\theta}{c}\sqrt{c^2t^2 - x^2}\right)\mathrm{d}x = \frac{\theta}{c}\left(e^{\theta t} - e^{-\theta t}\right)$$

and

$$\int_{-ct}^{ct} \frac{\partial}{\partial t} I_0\left(\frac{\theta}{c}\sqrt{c^2t^2 - x^2}\right)\mathrm{d}x = c\left(e^{\theta t} + e^{-\theta t}\right) - 2c$$

condition (12) implies that

$$1 - e^{-\Lambda_\theta(t)} = e^{-\Lambda_\theta(t)}\left(A(1 - \lambda_\theta(t))\frac{c}{\theta}\left(e^{\theta t} - e^{-\theta t}\right) + Bc\left(e^{\theta t} + e^{-\theta t} - 2\right)\right).$$

Observe that for $\lambda_\theta(t)$ as in (10) we have $1-\lambda_\theta(t) = 1-\theta\,\frac{e^{\theta t}-e^{-\theta t}}{e^{-\theta t}+e^{\theta t}}$ thus the equality in the above equation is attained by taking $A=0$ and $B=\frac{1}{2c}$. The solution is finally

$$p_\theta(x,t) = \frac{\theta\, t\, I_1\left(\frac{\theta}{c}\sqrt{c^2\,t^2-x^2}\right)}{(e^{-\theta\, t}+e^{\theta\, t})\sqrt{c^2\,t^2-x^2}}, \quad |x|<ct$$

that is non negative and satisfies the conditions required. It also verifies equation (8) and this is a boring calculus' exercise. $\square$

Note that the non-null component of (6) can be written as

$$\frac{1}{2\,\cosh(\theta\, t)}\left((\delta(x-ct)+\delta(x+ct)) + \frac{\theta\, t I_1\left(\frac{\theta}{c}\sqrt{c^2\,t^2-x^2}\right)}{\sqrt{c^2\,t^2-x^2}}\right),\ x\in[-ct,ct]$$

where $\delta(x)$ is the Dirac delta. By the result of last theorem we can now state for the seek of completeness the following result.

Proposition 2.4. *Suppose that the intensity function of the Poisson process $N(t)$ in* (2) *is*

$$\lambda(t) = \lambda_\theta(t) = \theta\tanh(\theta t), \quad \theta\in\mathbb{R}\,.$$

Then, the distribution function (6) *is*

$$P_\theta(x,t) = \begin{cases} 0, & x<-ct \\ \frac{1}{2\,\cosh(\theta\, t)} + \int\limits_{-ct}^{x} p_\theta(u,t)\mathrm{d}u, & -ct\le x<ct \\ 1, & x\ge ct \end{cases}$$

3 Parameter estimation

We consider the problem of estimation of the parameter θ for the model discussed in §2.2. We suppose that the velocity c is known, if it is not, one can easily determine without error its value, by observing two successive switches of velocity (or just the first switch) and the time between the two occurrences. Two possibile schemes of observations can be considered: *i*) one trajectory is observed up to some time T, letting $T\to\infty$, and *ii*) n independent and identically distributed observation $X_i(t)$ of the trajectories are observed on a fixed time interval $[0,T]$, letting $n\to\infty$. Instead of working on the law of $X(t)$ we do inference on θ via the Poisson process.

We use the method of moments to estimate θ. For scheme $i)$, recall that $\mathbf{E}N(t) = \Lambda_\theta(t)$ thus, if π_T is the number of observed switches up to time T, it suffices to find the solution to

$$\Lambda_\theta(T) = \ln(\cosh(\theta T)) = \pi_T$$

that is

$$\tilde{\theta}_T = \theta(\pi_T) = \frac{1}{T} \operatorname{arcosh}\left(e^{\pi_T}\right) .$$

Good asymptotic properties of this estimator cannot be established without assuming more conditions on the Poisson process. We give now the optimal solution in terms of the second scheme.

Suppose now to be able to observe n independent copies $X_i(t)$ of the process $X(t)$ up to a fixed horizon T, as to say, we observe the trajectories of n particles that do not interact. Denote by π_i the number of switches in each replication. Then

$$\hat{\pi}_n = \frac{1}{n}\sum_{i=1}^{n} \pi_i$$

is a consistent estimator of $\pi = \Lambda_\theta(T)$ as $n \to \infty$, moreover $\sqrt{n}(\hat{\pi}_n - \pi)$ is asymptotically Gaussian $\mathcal{N}(0, \pi)$. The estimator $\hat{\pi}_n$ – called empirical measure – is also asymptotically efficient in the minimax sense (for these and many other results see Kutoyants, 1998). In this particular case, an asymptotically efficient estimator for θ exists. In fact,

$$\pi = \ln(\cosh(\theta T)) \quad \text{and} \quad \theta = \theta(\pi) = \frac{1}{T} \operatorname{arcosh}\left(e^{\pi}\right)$$

so the estimator of moments is $\hat{\theta}_n = \theta(\hat{\pi}_n)$. By the so-called δ-method we desume the asymptotic behavior of $\theta(\hat{\pi}_n)$. In fact

$$\sqrt{n}(\theta(\hat{\pi}_n) - \theta(\pi)) = \sqrt{n}(\hat{\pi}_n - \pi)\theta'(\pi) + o(|\hat{\pi}_n - \pi|)$$

where the last term converges to 0 in probability by consistency. Thus, $\sqrt{n}(\theta(\hat{\pi}_n) - \theta(\pi))$ has, asymptotically, a centered Gaussian law with variance given by

$$\pi/\theta'(\pi)^2 = \frac{2\,T^2\pi}{1 + \coth(\pi)}$$

or better

$$\sqrt{n}(\hat{\theta}_n - \theta) \Longrightarrow \mathcal{N}\left(0, T^2 \frac{\ln(\cosh(\theta T))}{\coth(\theta T)^2}\right)$$

This estimator is also the maximum likelihood estimator for θ that is asymptotically efficient in this regular problem. To conclude, the estimator $\hat{\theta}_n$ is consistent for θ and asymptotically Gaussian, moreover it is asymptotically efficient for θ with asymptotic variance given by the inverse of the Fisher information in this problem.

References

- Beghin, L., Nieddu, L., Orsingher, E. (1999), Probabilistic analysis of the telegrapher's process with drift by means of relativistic transformations, to appear in *Jour. Appl. Math. and Stoch. Anal.*

- Foong, S.K. (1992), First passage time, maximum displacement and Kac's solution of the telegraphers equation, *Physical Review A*, **46**, 707-710.

- Foong, S.K., Kanno S. (1994), Properties of the telegrapher's random process with or without a trap, *Stoch. Proc. and Their Appl.*, 147-173.

- Goldstein, S. (1951), On diffusion by discontinuous movements and the telegraph equation, *Quart. J. Mech. Appl. Math.*, **4**, 129-156.

- Iacus, S.M. (1995), Random motions governed by hyperbolic equations, in *Acts of European Young Statisticians Meeting EYSM95*, (Eramus University, Rotterdam), 44-49.

- Kolesnik, A.D., Turbin, A.F. (1991), Infinitesimal hyperbolic operator of the Markovian random evolutions in R^n, *Doklad. Akadem. Nauk. Ukrain*, 11-14 (in russian).

- Kutoyants, Yu.A. (1998), *Statistical inference for spatial Poisson processes*, Lecture Notes in Statistics (Springer-Verlag).

- Orsingher, E. (1985), Hyperbolic equations arising in random models, *Stoch. Proc. and Their Appl.*, 49-66.

- Orsingher, E. (1986), A planar random motion governed by the two-dimensional telegraph equation, *Jour. Appl. Prob.*, **22**, 385-397.

- Orsingher, E. (1990), Probability law, flow function, maximum distribution of wave-governed random motions and their connections with Kirchoff's laws, *Stoch. Proc. and Their Appl.*, 49-66.

- Orsingher, E. (1995), Motions with reflecting and absorbing barriers driven by the telegraph equation, *Random Operators and Stochastic Equations*, **3:1**, 9-21.

- Orsingher, E. (2000), Exact joint distribution in a model of planar random motion, *Stoch. and Stoch. Reports*, **1-2**, 1-10.

- Orsingher, E., Kolesnik, A. (1996), The exact probability law of a planar random motion governed by a fourth-order hyperbolic equation, *Theory of Probability and its Appl.*, **41**, 379-386.