On readout and initialisation fidelity by finite demolition single shot readout

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Ideal projective quantum measurement makes the system state to collapse in one of the eigenstates $|\phi_\lambda\rangle$ of the observable $\mathbb{O}$ corresponding to the measured eigenvalue λ, making it a powerful tool for preparing system in the desired state. Albeit demonstration of single shot readout technique, experimental realisations of projective measurement are not ideal. During the time needed to overcome the classical noise of the apparatus, system state is often (slightly) perturbed. In this paper we propose an analytical model to analyse the initialisation fidelity of the system performed by the single shot readout and benchmark it experimentally on the example of NV center in diamond. Our work is important for accurate description of initialisation fidelity of the quantum bit when single shot readout is used for initialisation via post-selection or feedback control.

I. INTRODUCTION AND BACKGROUND

Quantum projective measurement of a single system is intrinsically probabilistic and probability to measure eigenvalue λ reads $p_\lambda = \mathrm{Tr}(P_\lambda \rho)$. However, once eigenvalue λ was measured, system collapses (*reduces*) to the state $\rho_\lambda = P_\lambda \rho P_\lambda^\dagger / p_\lambda$. Therefore successive projective measurements of observable $\mathbb{O}$ have probability $p'_\lambda = \mathrm{Tr}(P_\lambda P_\lambda \rho P_\lambda^\dagger / p_\lambda) = 1$ to give the same output λ and hence same post-measurement state. Due to the preservation of the once measured (*collapsed*) state its experimental realisation is denoted as quantum non-demolition (QND) measurement which was demonstrated in many systems e.g. trapped ion, dopants in diamond, superconducting qubits [1–3]. Measurements of the quantum system is associated with obtaining a macroscopically readable signal, which usually is encoded in number of photons or electrons. The counting process is stochastic and classical, meaning that it does not relate to the *quantumness* of the system being measured. It is this process which gives the measurement noise. In most cases this process could be described as a Poissonian distribution with mean number of counts $\bar{\lambda}$. Various states of the system generate distributions of meter outputs with various average values $\bar{\lambda}_i$. Due to the finite width of the distributions, there is always the overlap between them. It is this overlap, which causes the infidelity of the quantum state estimation [**?**]. With a cycling non demolition measurement the signal could be acquired as long as needed to achieve desired fidelity. In particular, when the signal-to-noise ratio acquired within the measurement time of the system is above unity it is denoted as *single shot readout* corresponding to fidelity exceeding 79 % [4]. In practice most experiments subject the system to additional decay channels which limits the available measurement time and hence the fidelity of it. Moreover it disturbs the state of the system during the measurement. And hence it could scrutinise the method of state preparation, which rely on the measurement, such as post-selection or active feedback drive for desired state preparation. **Therefore accurate estimation of the initialisation fidelity is important for benchmarking and optimising the quantum hardware. It became crucial for estimating feasibility and performance of envisioned quantum algorithms such as error correction algorithms [?].**

When the system spontaneously changes its state during the measurement, the macroscopic outputs probability distributions ($PDF_{0,1}$) are not single poissonians. Instead they present combination of them with continuously distributed parameter λ, corresponding to the occupation time distributions in each state. Additionally the system changes it states during the readout, and the moment t at which system was in $|0\rangle$ or $|1\rangle$ state shall be specified to calculate the $PDF^t_{0,1}$ accurately. One approach is to fix the time in the beginning of the measurement $t = 0$ and derive the $PDF^{t=0}_{0,1}$. Having a meter reading λ_i using maximum likelihood method one could estimate in which state system was at the moment $t = 0$ [5, 6]. Using techniques of statistical signal analysis [**?**] the fidelity of readout could be optimised with choosing the optimum detection threshold λ_{th}, as well as using filtering techniques [7–9]. However when estimating the state of the system at end of the measurement $t = t_r$ one has to make additional assumptions about state decay during the measurement, instead of directly using the measurement statistics. **In this work we follow a different approach**. By setting the aforementioned time when defining the PDFs to the end of the measurement $t = t_r$, we calculate the $PDF^{t=t_r}_{0,1}$ and directly apply the maximum likelihood method for the readout of the state at $t = t_r$. This approach allows for more accurate fidelity estimation for state preparation using finite demolition readout in the presence of finite classical noise.

As in the work of [7] different filters, including exponential, optimum linear and nonlinear could be applied to improve the fidelity. Intuitively a filter with more weight towards the end of the measurement would emphasise the final state in the distribution. We analyse performance of various filters and find that XXX. We apply this model to the NV center charge state system and validate our theory by experiment. The paper is organised as follows

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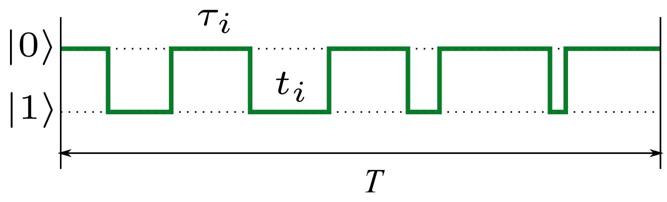

FIG. 1. The example of the switching dynamics of the two level system

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II. THEORY

A. Switching dynamics and point poisson process

We consider a case of two level system (TLS) with states $|0\rangle$ and $|1\rangle$. Under the asymmetric stationary decay with rates γ_1, γ_2 between the two states 0 and 1 respectively system performs the sequence of transitions (*jumps*) events which forms a point poisson process. The time intervals $\{\tau_i\}$ and $\{t_i\}$ spend in state 0 and 1 between the switches are then random variables which have exponential distributions: $\tau \sim \text{Exp}(\gamma_1)$, $t \sim \text{Exp}(\gamma_2)$. We recall following known properties related with the exponential distributions.

a. Lemma 1 : The sum of n exponentially distributed random variables x_i with rate γ: $X_n = \sum_i^n x_i$ is a random variable. It has Erlang distribution with probability density function

$$p(X_n = x) = \gamma \frac{e^{-\gamma x}(\gamma x)^{n-1}}{(n-1)!} \tag{1}$$

Additionally, we introduce a random variable N_x [10] which is defined as $N_x = \min(n | \sum_i^n x_i \geq x)$ and represents minimum number of elements from a given set of random variable sample, which sum exceeds x.

b. Lemma 2 : Variable $N_x - 1$ has Poisson distribution, and N_x has a probability density function as follows

$$p(N_x = n) = \frac{e^{-\gamma x}(\gamma x)^{n-1}}{(n-1)!}. \tag{2}$$

Similar to work [5] we consider cases of odd and even number of switching events separately. We introduce n as a number of intervals spent in state $|0\rangle$. We can now estimate the probability that system spend time τ in state $|0\rangle$ during the measurement time T. When having odd number of switching, and starting from state $|0\rangle$ the intervals between switches are sets of random variables $\{\tau_i\} \sim \text{Exp}(\gamma_1)$ and $\{t_i\} \sim \text{Exp}(\gamma_2)$ with rates $\gamma_{1,2}$. Probability that system spends total time τ in state $|0\rangle$ is a sum of products of the Erlang-n distribution that sum of n variables equals τ with probability that n intervals occur, which is the probability that the residual time $t = T - x$ is exceeded in n increments of process $\{t_i\}$, hence:

$$\begin{aligned} p(\tau|odd, |0\rangle) &= \sum_{n=1}^{\infty} P(\tau_n = \tau) P(N_{t=T-\tau} = n) \\ &= \gamma_1 e^{(\gamma_2-\gamma_1)\tau - \gamma_2 T} \sum_{n=1}^{\infty} \frac{(\gamma_1 \tau \gamma_2 (T-\tau))^{n-1}}{(n-1)!^2} \\ &= \gamma_1 e^{(\gamma_2-\gamma_1)\tau - \gamma_2 T} I_0\left(2\sqrt{\gamma_1\gamma_2\tau(T-\tau)}\right), \end{aligned} \tag{3}$$

where $I_0(z) = \sum_{i=0}^{\infty} \frac{(z/2)^{2n}}{n!^2}$ is modified Bessel function of the first kind of $0-th$ order. For the case of even number of switches we have to take the opposite consideration. Now the total interval $t = T - \tau$ has a fixed length, and the length τ has to be exceeded, because system could stay in final state after time T. Hence the probability is sum over n of probability that process $\{\tau_i\}$ exceeds τ in n increments (steps), times the probability that $n-1$ intervals $\{t_i\}$ sum to $T - x$, hence:

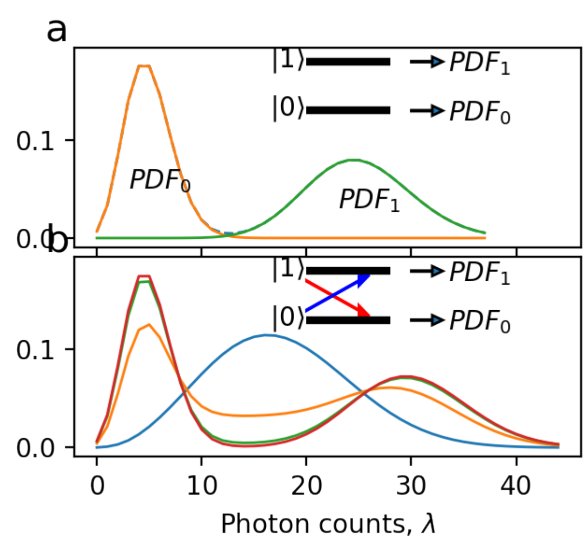

FIG. 2. Two level system being measured with an apparatus. a) Distributions of the outputs of the non demolition measurement. b) Distribution of the outputs with various rates of demolishing. $\gamma = 10, 1, 0.1, 0.01 \times T_r^{-1}$

$$\begin{aligned} p(\tau|even, |0\rangle) &= \sum_{n=2}^{\infty} P(t_{n-1} = T - \tau) P(N_\tau = n) \\ &= \gamma_2 e^{(\gamma_2-\gamma_1)\tau - \gamma_2 T} \sum_{n=2}^{\infty} \frac{(\gamma_1\tau)^{n-1}(\gamma_2(T-\tau))^{n-2}}{(n-1)!(n-2)!} \\ &= e^{(\gamma_2-\gamma_1)\tau - \gamma_2 T} \cdot \sqrt{\frac{\gamma_1\gamma_2\tau}{T-\tau}} I_1\left(2\sqrt{\gamma_1\gamma_2\tau(T-\tau)}\right), \end{aligned} \tag{4}$$

where $I_1(z) = \sum_{i=0}^{\infty} \frac{(z/2)^{2n+1}}{n!n+1!}$ is modified Bessel function of the first kind of $1-st$ order. Additionally we consider case where no switches happens which simply reads:

$$p(\tau = T|switchless, |0\rangle) = e^{-\gamma_1 T} \quad (5)$$

The probability conditioned on initial state $|1\rangle$ could be obtained by substituting $\gamma_1 \leftrightarrow \gamma_2$ and $\tau \leftrightarrow T - \tau$. This concludes the derivation of the distribution of the time spent by the system in the state $|0\rangle$ conditioned on initial state:

$$p(\tau||0\rangle) = e^{(\gamma_2-\gamma_1)\tau-\gamma_2 T}\left(\gamma_1 I_0\left(2\sqrt{\gamma_1\gamma_2\tau(T-\tau)}\right) + \sqrt{\frac{\gamma_1\gamma_2\tau}{T-\tau}} I_1\left(2\sqrt{\gamma_1\gamma_2\tau\,(T-\tau)}\right)\right) + e^{-\gamma_1 T}\delta(\tau - T) \quad (6)$$

$$p(\tau||1\rangle) = e^{(\gamma_1-\gamma_2)(T-\tau)-\gamma_1 T}\left(\gamma_2 I_0\left(2\sqrt{\gamma_1\gamma_2\tau\,(T-\tau)}\right) + \sqrt{\frac{\gamma_1\gamma_2\,(T-\tau)}{\tau}} I_1\left(2\sqrt{\gamma_1\gamma_2\tau\,(T-\tau)}\right)\right) + e^{-\gamma_2 T}\delta\,(\tau) \quad (7)$$

, where T is a total counting time and τ is total time spent in state $|0\rangle$. Using the expressions for probability density for τ eq. 6, eq.7, combining with eq.8 and integrating τ over the interval $\tau \in [0, T]$ we obtain expression for the photon counting statistics similar to [5].

B. Photon counting statistics conditioned on initial state

Assuming that system emits photon and they arrive on the photodetector at random times with constant rate λ_1 and λ_2 conditioned on the system state. The number of photon counts is a random variable

$$\lambda \sim \text{Poisson}\,(\lambda_1\tau + \lambda_2(\text{T} - \tau)) \quad (8)$$

$$p\,(\lambda||1\rangle) = \int_0^T \left\{ e^{(\gamma_1-\gamma_2)(T-\tau)-\gamma_1 T}\left(\gamma_2 I_0\left(2\sqrt{\gamma_1\gamma_2\tau\,(T-\tau)}\right) + \sqrt{\frac{\gamma_1\gamma_2\,(T-\tau)}{\tau}} I_1\left(2\sqrt{\gamma_1\gamma_2\tau\,(T-\tau)}\right)\right) \times \right.$$
$$\left. \text{Poisson}\,(\lambda, \lambda_1\tau + \lambda_2(T-\tau))\,d\tau \right\} + e^{-\gamma_2 T}\text{Poisson}\,(\lambda, \lambda_2 T) \quad (9)$$

$$p\,(\lambda||0\rangle) = \int_0^T \left\{ e^{(\gamma_2-\gamma_1)\tau-\gamma_2 T}\left(\gamma_1 I_0\left(2\sqrt{\gamma_1\gamma_2\tau\,(T-\tau)}\right) + \sqrt{\frac{\gamma_1\gamma_2\tau}{T-\tau}} I_1\left(2\sqrt{\gamma_1\gamma_2\tau\,(T-\tau)}\right)\right) \times \right.$$
$$\left. \text{Poisson}\,(\lambda, \lambda_1\tau + \lambda_2(T-\tau))\,d\tau \right\} + e^{-\gamma_1 T}\text{Poisson}\,(\lambda, \lambda_1 T) \quad (10)$$

The photon counting statistics is numerically simulated and presented in Fig. 2. The distributions deviates from Poissonian and the overlap between the distributions increases with increasing switching rate. Note in the case where $\gamma = 10T^{-1}$ the distributions almost fully overlap, hence the measurement has no information to resolve the state and the fidelity of the measurement $\mathcal{F} = 0$, while in the case where $\gamma \leq T^{-1}$ despite a significant overlap between the distributions one could discriminate the initial states.

C. Photon counting statistics conditioned on the final state

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D. calculation of the fidelity with box car filter

The estimation uncertainty could be minimised [**?** **?**] by choosing the threshold value λ_t at which normalised distributions of outputs $PDE_0(\lambda_t) = PDE_1(\lambda_t)$ The error in the state discrimination between 0 and 1 is thus the surface under the overlapping of PDE0 and PDE1

[?]. For the general case of uniform in time decay rate (Exponential decay) of TLS with rates γ_1, γ_0.

E. Optimum linear and nonlinear filtering

III. EXPERIMENT

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