Finding E from Time of Flight

- Starting at Perigee:
  - $n\,(t - T) = E - e\,sinE$ (from perigee)
  - How do we find E?

- No analytic solution, can find by iteration

- Commonly use Newton-Raphson iteration:

$$\mathrm{x_{n+1}} = \mathrm{x_n} - \frac{\mathrm{f(x)}}{\mathrm{f'(x)}}$$

Iteration for E (Starting at Perigee)

$$f(E_k) = n(t - T) - (E_k - e \sin E_k)$$

$$f'(E_k) = -(1 - e \cos E_k)$$

$$n\,(t - T) = M \qquad \text{(fixed)}$$

$$E_{k+1} = E_k + \frac{M - (E_k - e \sin E_k)}{1 - e \cos E_k}$$

Continue until $|E_{k+1} - Ek| <$TOL

Specify iteration starting value:

$$E_o = M \qquad \text{should usually work}$$

Alternate Solution (Biondini)

- Let: $M = E - e \sin E$

  $E = M + e \sin E$

- Take sine of both sides:

$$\sin E = \sin(M + e \sin E)$$

$$x = \sin E$$

$$x_{i+1} = \sin(M + e x_i)$$

$$E_{i+1} = \sin^{-1}(M + e x_{i+1})$$

- Repeat until

$$\Delta x = |x_{i+1} - x_i| \to 0$$

General Form for Time of Flight

- Not starting at perigee, setup for time of flight from t_0 to t
- General Time of Flight Equation

$$t - t_o = k\,TP + \frac{1}{n}\underbrace{(E - e\sin E)}_{M} - \frac{1}{n}\underbrace{(E_0 - e\sin E_0)}_{Mo}$$

  – k = number of perigee crossings between E_0 and E

- General Iteration function

$$f(E_k) = n(t - T) - (E_k - e\sin E_k) + (E_0 - e\sin E_0)$$

  - Note: the $(k\,TP)$ term is not included: this version allows $E_k > 2\pi$

Newton-Raphson Iteration for E

- As before

$$n\,(t - T) \;=\; n\Delta t \;= \text{fixed}$$

$$M_0 \;= (E_0 - e\,\sin E_0) \;=\; \text{fixed}$$

$$f'(E_k) \;=\; -(1 - e\,\cos E_k)$$

$$\boxed{E_{k+1} \;=\; E_k \;+\; \frac{n\Delta t \;+\; M_0 - (E_k - e\,\sin E_k)}{1 - e\,\cos E_k}}$$

Specify iteration starting value:

$$E_o \;=\; M_o \qquad \text{should usually work}$$

Note on the Iteration

- Perigee passages are not included in the iteration,

$$\mathrm{E}_{\mathrm{k+1}} = \mathrm{E}_{\mathrm{k}} + \frac{\mathrm{n}\Delta\mathrm{t} + \mathrm{M}_0 - (\mathrm{E}_{\mathrm{k}} - \mathrm{e}\,\sin\mathrm{E}_{\mathrm{k}})}{1 - \mathrm{e}\,\cos\mathrm{E}_{\mathrm{k}}}$$

- If k > 1, result will be a value of E > 2π
  - Take $mod(E, 2\pi)$ to get E in $[0, 2\pi]$
  - Can compute k by

$$\mathrm{k} = \mathrm{floor}(\,(\mathrm{E} - \mathrm{E}_o)\,/\,2\pi\,)$$

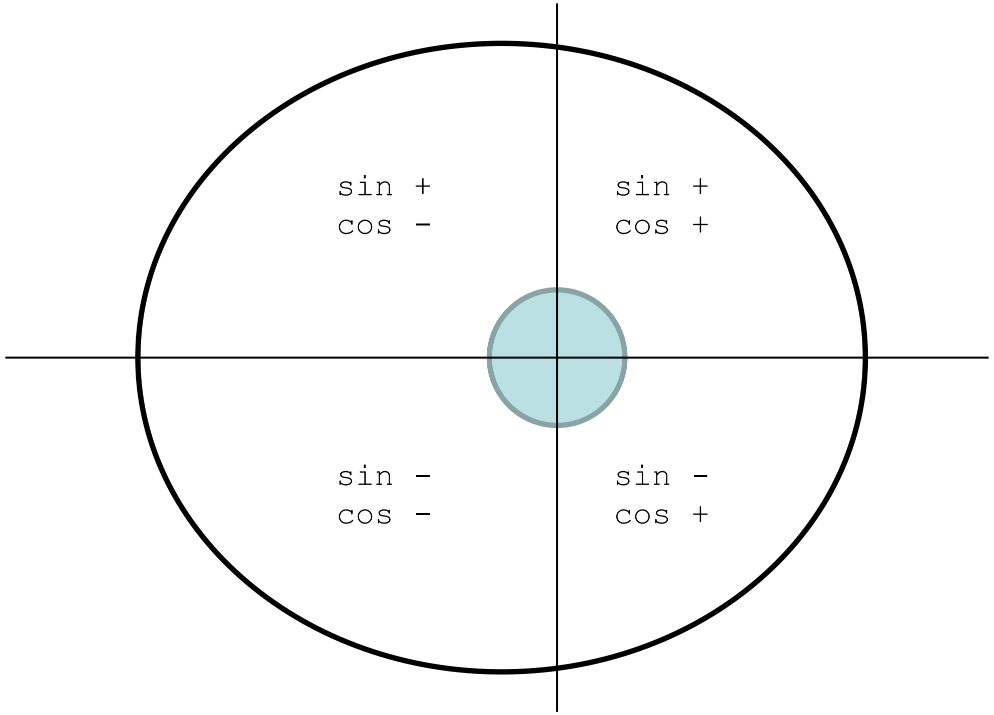
Getting Right Quadrant for ν

- Use solved-for value of E to get ν

$$\cos\nu = \frac{e - \cos E}{e \cos E - 1}$$

$$\sin\nu = \frac{\sin E\sqrt{1-e^2}}{1 - e\cos E}$$

$$\nu = \operatorname{atan2}(\sin\nu, \cos\nu)$$



- atan2 returns angle in range [-π, π]
  - If angle < 0, add 2π to get range [0, 2π]

An Aside: Planar Motion

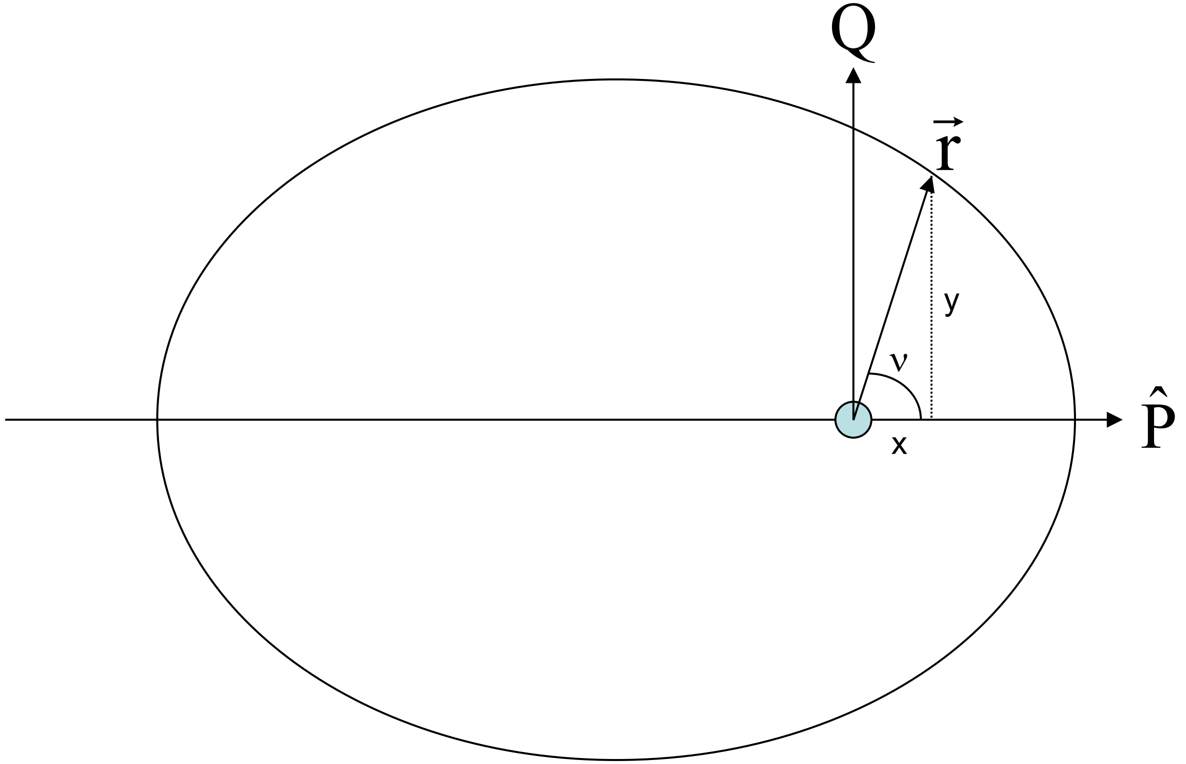
- Things we know:

$$\vec{h} = \vec{r} \times \vec{v}$$
$$\vec{h} \perp \vec{r} \text{ and } \vec{v}$$
$$\vec{h} = \text{constant}$$

- Therefore:
  - Satellite motion is always in the plane containing $\vec{r}$ and $\vec{v}$
  - For 2-body problem, can treat prediction problem as a problem in planar motion
    - Don’t worry about orientation of orbit in space (yet)

The Kepler Problem

- Orbit elements are constant functions of $\vec{r}$ and $\vec{v}$
- Look at orbit in perifocal frame:
  - Motion in orbit plane



$$\vec{r} = x\,\hat{P} + y\,\hat{Q} + 0\,\hat{W}$$

$$\dot{\vec{r}} = \dot{x}\,\hat{P} + \dot{y}\,\hat{Q} + 0\,\hat{W}$$

$(\hat{P}, \hat{Q}, \hat{W}$ are inertial, so $d/dt = 0)$

Prediction

- Because the orbit elements are constant, it is reasonable to suppose that future states can be found from current states:

$$\vec{r} = f\,\vec{r}_0 + g\,\dot{\vec{r}}_0$$

$$\dot{\vec{r}} = \dot{f}\,\vec{r}_0 + \dot{g}\,\dot{\vec{r}}_0$$

$$(\vec{r}_0,\ \dot{\vec{r}}_0 \text{ are fixed, so } d/dt = 0)$$

- Need to find expressions for f, g, etc.

Find f

- Post-cross multiply both sides by $\dot{\vec{r}}_0$

$$\vec{r} \times \dot{\vec{r}}_0 = f\,\vec{r}_0 \times \dot{\vec{r}}_0 + \cancel{g\,\dot{\vec{r}}_0 \times \dot{\vec{r}}_0} \to 0$$

$$\vec{r} \times \dot{\vec{r}}_0 = f\,\vec{r}_0 \times \dot{\vec{r}}_0 = f\,\vec{h}$$

$$\vec{r} \times \dot{\vec{r}}_0 = \begin{vmatrix} \hat{P} & \hat{Q} & \hat{W} \\ x & y & 0 \\ \dot{x}_0 & \dot{y}_0 & 0 \end{vmatrix} = (x\dot{y}_0 - \dot{x}_0 y)\,\hat{W}$$

$\boxed{\text{Note: } \vec{h} = h\,\hat{W}}$

$$(x\dot{y}_0 - \dot{x}_0 y)\cancel{\hat{W}} = f\,h\,\cancel{\hat{W}} \longrightarrow (x\dot{y}_0 - \dot{x}_0 y) = f\,h$$

$$\boxed{f = \frac{(x\dot{y}_0 - \dot{x}_0 y)}{h}}$$

Find g

- Pre-cross multiply both sides by $\vec{r}_o$

$$\vec{r}_0 \times \vec{r} = f\,\underbrace{\cancel{\vec{r}_0 \times \vec{r}_0}}_{0} + g\,\vec{r}_0 \times \dot{\vec{r}}_0$$

$$\vec{r}_0 \times \vec{r} = g\,\vec{r}_0 \times \dot{\vec{r}}_0 = g\,\vec{h}$$

$$\vec{r}_0 \times \vec{r} = \begin{vmatrix} \hat{P} & \hat{Q} & \hat{W} \\ x_0 & y_0 & 0 \\ x & y & 0 \end{vmatrix} = (x_0 y - x y_0)\,\hat{W}$$

Similar to above:

$$\boxed{g = \frac{(x_0 y - x y_0)}{h}}$$

Time derivatives of f and g

$$\dot{f} = \frac{d}{dt}\left(\frac{(x\dot{y}_0 - \dot{x}_0 y)}{h}\right) = \frac{(\dot{x}\dot{y}_0 - \dot{x}_0\dot{y})}{h}$$

(x_0 and y_0 are fixed)

$$\dot{g} = \frac{d}{dt}\left(\frac{(x_0 y - x y_0)}{h}\right) = \frac{(x_0\dot{y} - \dot{x} y_0)}{h}$$

Express in terms of ν

$$x = r\cos\nu \qquad y = r\sin\nu$$

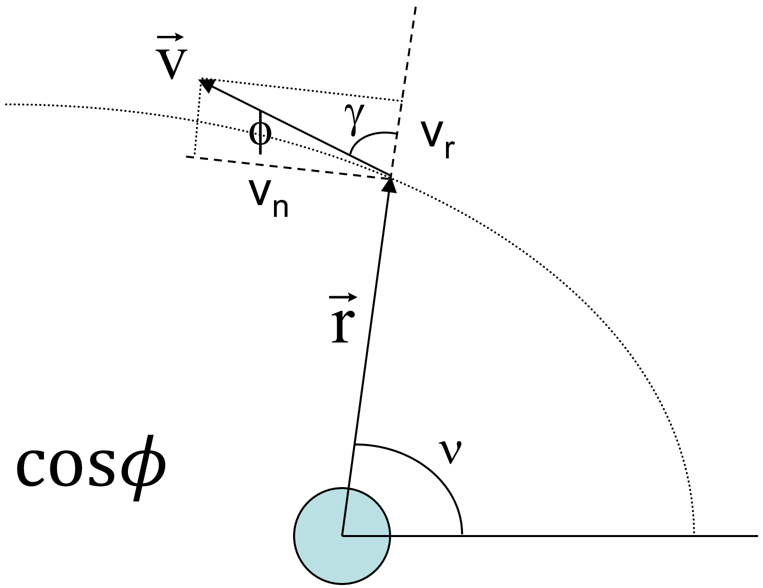
$$\dot{x} = \dot{r}\cos\nu - r\sin\nu\,\dot{\nu}$$
$$\dot{y} = \dot{r}\sin\nu + r\cos\nu\,\dot{\nu}$$

$$v_n = |\vec{v}|\cos\phi = r\,\dot{\nu}$$

$$|\vec{h}| = |\vec{r}\times\vec{v}| = rv\sin\gamma = rv\cos\phi$$

$$h = r\,(r\,\dot{\nu}) = r^2\dot{\nu}$$

$$\boxed{r\,\dot{\nu} = \frac{h}{r}}$$



$$r = \frac{a\,(1-e^2)}{1 + e\cos\nu} = \frac{p}{1 + e\cos\nu} \qquad \dot{r} = -\frac{p}{(1 + e\cos\nu)^2}(-e\sin\nu)\,\dot{\nu}$$

$$\dot{r} = \frac{\cancelto{r}{p}}{(1 + e\cos\nu)}\frac{(e\sin\nu)\,\dot{\nu}}{(1 + e\cos\nu)} = \frac{re\sin\nu\,\dot{\nu}}{(1 + e\cos\nu)}$$

$$\dot{r}\cos\nu = \frac{re\cos\nu\sin\nu\,\dot{\nu}}{(1 + e\cos\nu)}$$

$$\dot{x} = \frac{re\cos\nu\sin\nu\,\dot{\nu}}{(1 + e\cos\nu)} - r\sin\nu\,\dot{\nu} = \frac{re\cos\nu\sin\nu\,\dot{\nu}}{(1 + e\cos\nu)} - \frac{r\sin\nu(1 + e\cos\nu)\,\dot{\nu}}{(1 + e\cos\nu)}$$

$$\dot{x} = \frac{\cancel{re\cos\nu\sin\nu\,\dot{\nu}} - r\sin\nu\,\dot{\nu} - \cancel{re\cos\nu\sin\nu\,\dot{\nu}}}{(1 + e\cos\nu)} \Longrightarrow$$

$$\boxed{\dot{x} = \frac{-r\sin\nu\,\dot{\nu}}{(1 + e\cos\nu)}}$$

Solving…

$$\dot{x} = \frac{-\,r \sin\nu\, \dot{\nu}}{(1 + e\cos\nu)}$$

$$r\,\dot{\nu} = \frac{h}{r} \Longrightarrow \dot{x} = \frac{-\,h \sin\nu}{r\,(1 + e\cos\nu)}$$

$$p = r\,(1 + e\cos\nu) = \frac{h^2}{\mu} \Longrightarrow h = \sqrt{\mu p}$$

$$\dot{x} = \frac{-\sqrt{\mu p}\,\sin\nu}{p} \Longrightarrow \boxed{\dot{x} = -\sqrt{\frac{\mu}{p}}\sin\nu}$$

Similarly for $\dot{y}$

$$\dot{y} = \dot{r}\sin\nu + r\cos\nu\,\dot{\nu}$$

$$r = p\,(1 + e\cos\nu)^{-1}$$

$$\dot{r} = -p\,(1 + e\cos\nu)^{-2}(-e\sin\nu)\,\dot{\nu}$$

$$\dot{r} = \frac{p\,e\sin\nu}{(1 + e\cos\nu)^2}\dot{\nu}$$

$$\dot{r} = \frac{r^2\,e\sin\nu}{p}\dot{\nu}$$

$$h = r^2\dot{\nu}$$

$$\dot{r} = \frac{h}{p}e\sin\nu$$

$$h = r(r\,\dot{\nu}) = \sqrt{\mu p}$$

$$r\,\dot{\nu} = \frac{\sqrt{\mu p}}{r}$$

$$r\,\dot{\nu} = \frac{\sqrt{\mu p}\,(1 + e\cos\nu)}{p}$$

$$r\,\dot{\nu} = \sqrt{\frac{\mu}{p}}(1 + e\cos\nu)$$

$$\boxed{\dot{r} = \sqrt{\frac{\mu}{p}}e\sin\nu}$$

Combine

$$\dot{y} = \dot{r}\sin\nu + r\cos\nu\,\dot{\nu}$$

$$\dot{y} = \sqrt{\frac{\mu}{p}}\, e\sin^2\nu + \sqrt{\frac{\mu}{p}}(1 + e\cos\nu)\cos\nu$$

$$\dot{y} = \sqrt{\frac{\mu}{p}}(e\,[\sin^2\nu + \cos^2\nu] + \cos\nu)$$

$$\dot{y} = \sqrt{\frac{\mu}{p}}(e\sin^2\nu + \cos\nu + e\cos^2\nu) \Longrightarrow \boxed{\dot{y} = \sqrt{\frac{\mu}{p}}(e + \cos\nu)}$$

Summary so far

$$x = r\cos\nu \qquad y = r\sin\nu$$

$$\dot{x} = -\sqrt{\frac{\mu}{p}}\sin\nu \qquad \dot{y} = \sqrt{\frac{\mu}{p}}(e + \cos\nu)$$

Perifocal Position and Velocity Components

$$f = \frac{(x\dot{y}_0 - \dot{x}_0 y)}{h} \qquad g = \frac{(x_0 y - x y_0)}{h}$$

$$\dot{f} = \frac{(\dot{x}\dot{y}_0 - \dot{x}_0\dot{y})}{h} \qquad \dot{g} = \frac{(x_0\dot{y} - \dot{x}y_0)}{h}$$

$$f\dot{g} - \dot{f}g = 1$$

Combine to find expressions in ν

Example: find f

$$f = \frac{(x\dot{y}_0 - \dot{x}_0 y)}{h}$$

$$f = \frac{\left(r\cos\nu \sqrt{\frac{\mu}{p}}(e + \cos\nu_0) + \sqrt{\frac{\mu}{p}}\sin\nu_0\, r\sin\nu \right)}{\sqrt{\mu p}}$$

$$f = \frac{r(\cos\nu\,(e + \cos\nu_0) + \sin\nu_0 \sin\nu\,)}{p}$$

$$f = \frac{r(e\cos\nu + \cos\nu\cos\nu_0 + \sin\nu_0\sin\nu\,)}{p}$$

Recall the trig identity…

$$\cos A\ \cos B \pm \sin A \sin B = \cos(A \mp B)$$

$$f = \frac{r(e\cos\nu + \cos(\nu - \nu_0)\,)}{p} = \frac{r(e\cos\nu + \cos\Delta\nu\,)}{p}$$

Add and subtract…

$$\frac{r}{p} - \frac{r}{p} + \frac{r\,e\cos\nu}{p} = -\frac{r}{p} + \frac{r}{p}(1 + e\cos\nu)$$

$$p = r\,(1 + e\cos\nu)$$

$$-\frac{r}{p} + \frac{r}{p}(1 + e\cos\nu) = -\frac{r}{p} + 1$$

$$\frac{r\,e\cos\nu}{p} = 1 - \frac{r}{p}$$

$$f = 1 - \frac{r}{p} + \frac{r}{p}\cos\Delta\nu$$

$$\boxed{f = 1 - \frac{r}{p}(1 - \cos\Delta\nu)}$$

Similarly

$$\boxed{g = \frac{r\, r_0}{\sqrt{\mu p}} \sin\Delta\nu}$$

$$\boxed{\dot{g} = 1 - \frac{r_0}{p}(1 - \cos\Delta\nu)}$$

Using $f\dot{g} - \dot{f}g = 1$:

$$\boxed{\dot{f} = \sqrt{\frac{\mu}{p}}\ \tan\frac{\Delta\nu}{2}\left(\frac{1 - \cos\Delta\nu}{p} - \frac{1}{r} - \frac{1}{r_o}\right)}$$

The f&g Functions: So What?

- The f&g functions: undeniably cool, but rather esoteric-seeming
- Does anybody actually use them?

- Answer: yes!
  – Cheap lower-fidelity orbit propagation
    - Can extend to non-two-body formulations
  – Commonly used for initial orbit determination (orbits from observations)
  – Basis of “Lambert Targeting” methods for rendezvous/proximity ops
  – Provides analytic solution for orbit determination State Transition Matrix
  – Universal variable formulation generalizes to all conic sections
  – And more!

Homework 2

Due Saturday, 31 Jan 2026

Homework 2 (1)

Given:

Component	Value (meters)
Position X	326151.080726
Position Y	6077471.251787
Position Z	2944583.918767
Velocity X	-7455.178720
Velocity Y	-482.482572
Velocity Z	1910.883434

- Find E_o (slide 19)
- Find ν_o (Keplerian comps)
- Find M_o
- Find time of flight from perigee to ν_o

- Find time of flight from ν_o to ν = 65°
- Verify: Starting at ν_o, compute the true anomaly after the time of flight computed above.
- Starting at ν_o, what is the true anomaly after 2700 seconds?
- Starting at ν_o, what is the true anomaly after exactly two orbit periods?
- What is the true anomaly after 15000 seconds?

Homework 2 (2)

Given:

Component	Value (meters)
Position X	572461.711228
Position Y	-1015437.194396
Position Z	7707337.871302
Velocity X	-6195.262945
Velocity Y	-3575.889650
Velocity Z	-5.423283

- Compute Keplerian elements
- Find perifocal position and velocity, $\vec{r}_o$ and $\vec{v}_o$
  - See equations on Slide 39
- Find $f, g, \dot{f}, \dot{g}$ for $\Delta\nu = 33°$
- Find $\vec{r}$ and $\vec{v}$

Homework 2 (3)

- Derivation 1: Using energy equation and ellipse equation, find an expression for the orbit speed as a function of true anomaly
- Derivation 2: Using the above, show that

$$\mathrm{v}_{\mathrm{perigee}} = \sqrt{\frac{\mu}{\mathrm{a}}\left(\frac{1+\mathrm{e}}{1-\mathrm{e}}\right)}$$

$$\mathrm{v}_{\mathrm{apogee}} = \sqrt{\frac{\mu}{\mathrm{a}}\left(\frac{1-\mathrm{e}}{1+\mathrm{e}}\right)}$$