

SPCE5025

Lecture 2

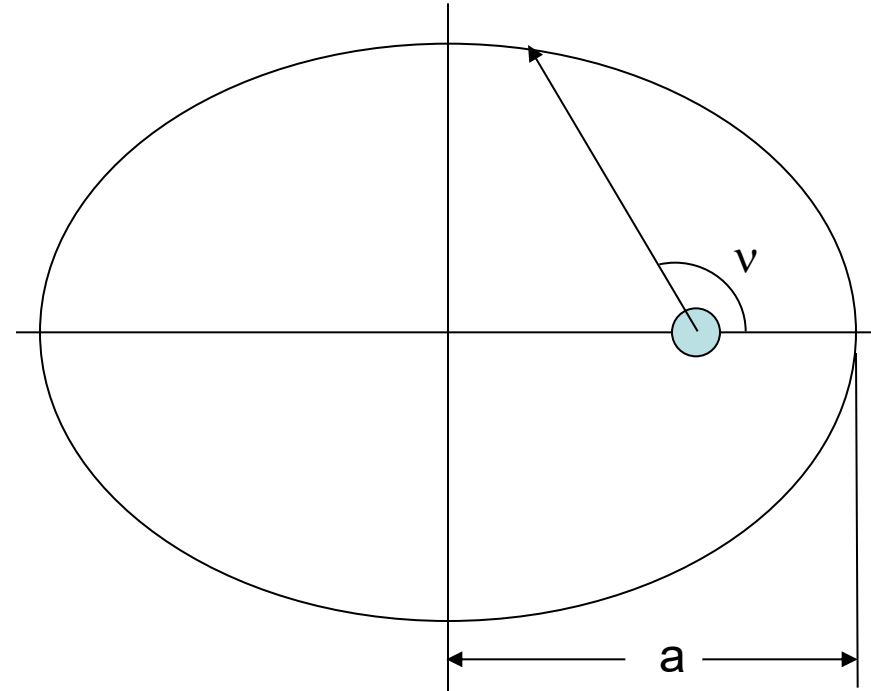
26 January 2026

Overview

- Kepler's Equation
- Kepler's Problem
- Orbit State Representations

Where we've been

- Last week covered the derivations of the two-body problem
- In particular,
 - a – semi-major axis
 - e – Eccentricity
 - ν - True Anomaly
 - Instantaneous location on the ellipse
 - ... But the satellite is actually moving

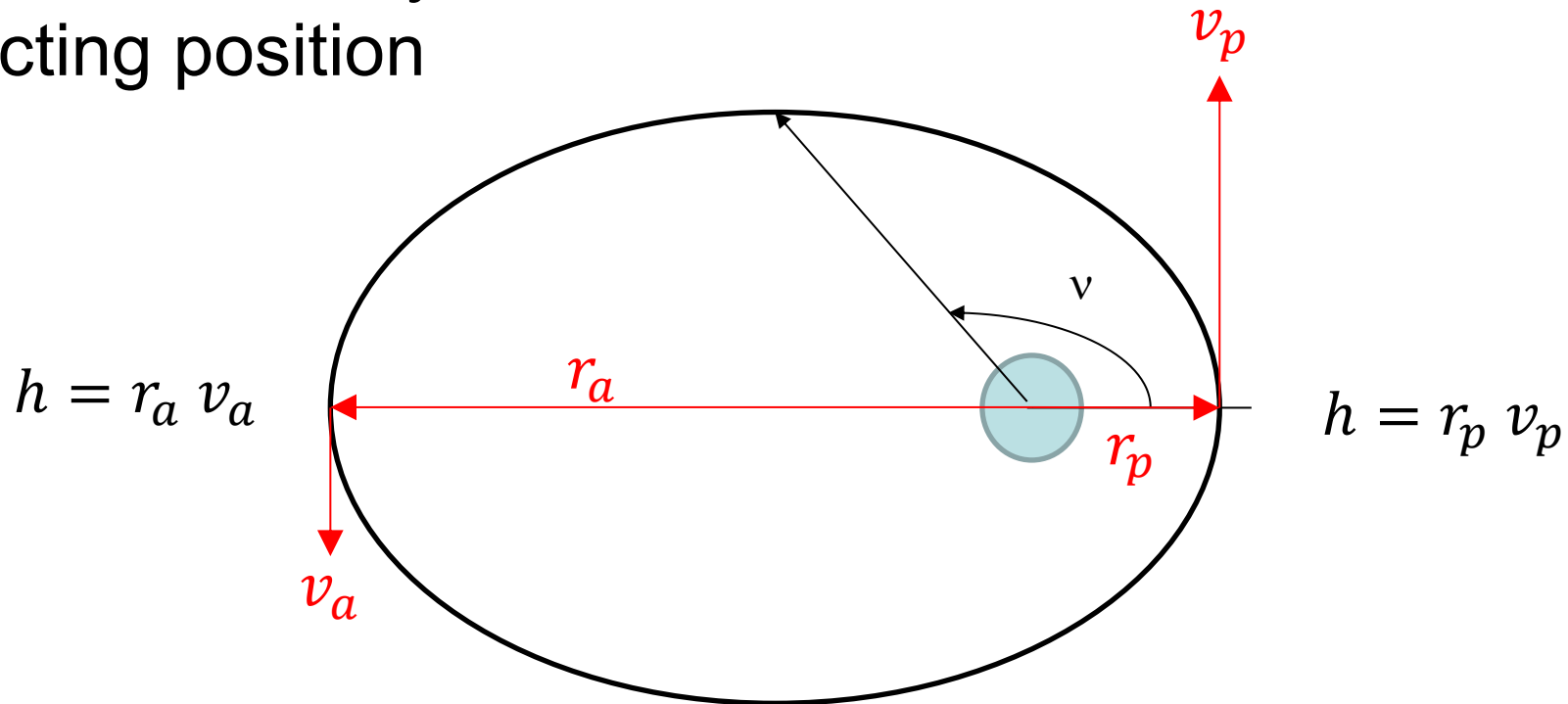


Motivating the Problem

- We know that the satellite moves on an ellipse with the central body at one focus
- Suppose we see the satellite now, and want to know where it will be some time later.....
 - True anomaly ν is not a static value – it changes as the satellite moves
- How do we figure out...
 - Where it will be after some Δt ?
 - How long it will take to get from here to there?
 - Different, though closely-related problems

Motivating the Problem (2)

- Angular momentum is constant
 - But velocity is not constant
 - Therefore, angular rate $\dot{\nu}$ is not constant
- Need to find a way to account for the actual motion when predicting position



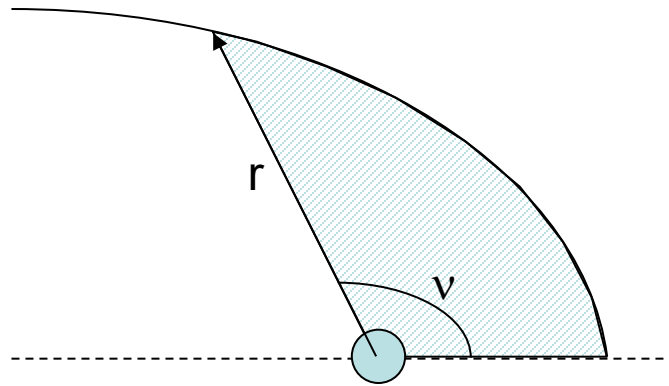
Next Step – Tying It to Time

- How long does it take to get from v_o to v_1 ?
- It's a tricky problem – analytic solution is not available
- Start out by looking at “Equal Areas in Equal Time” law
- Another way of looking at it:

$$\frac{\Delta t}{A} = \text{Constant} \longrightarrow \frac{\Delta t}{A} = \frac{TP}{\pi ab} \left(= \frac{\text{Orbit Period}}{\text{Area of Ellipse}} \right)$$

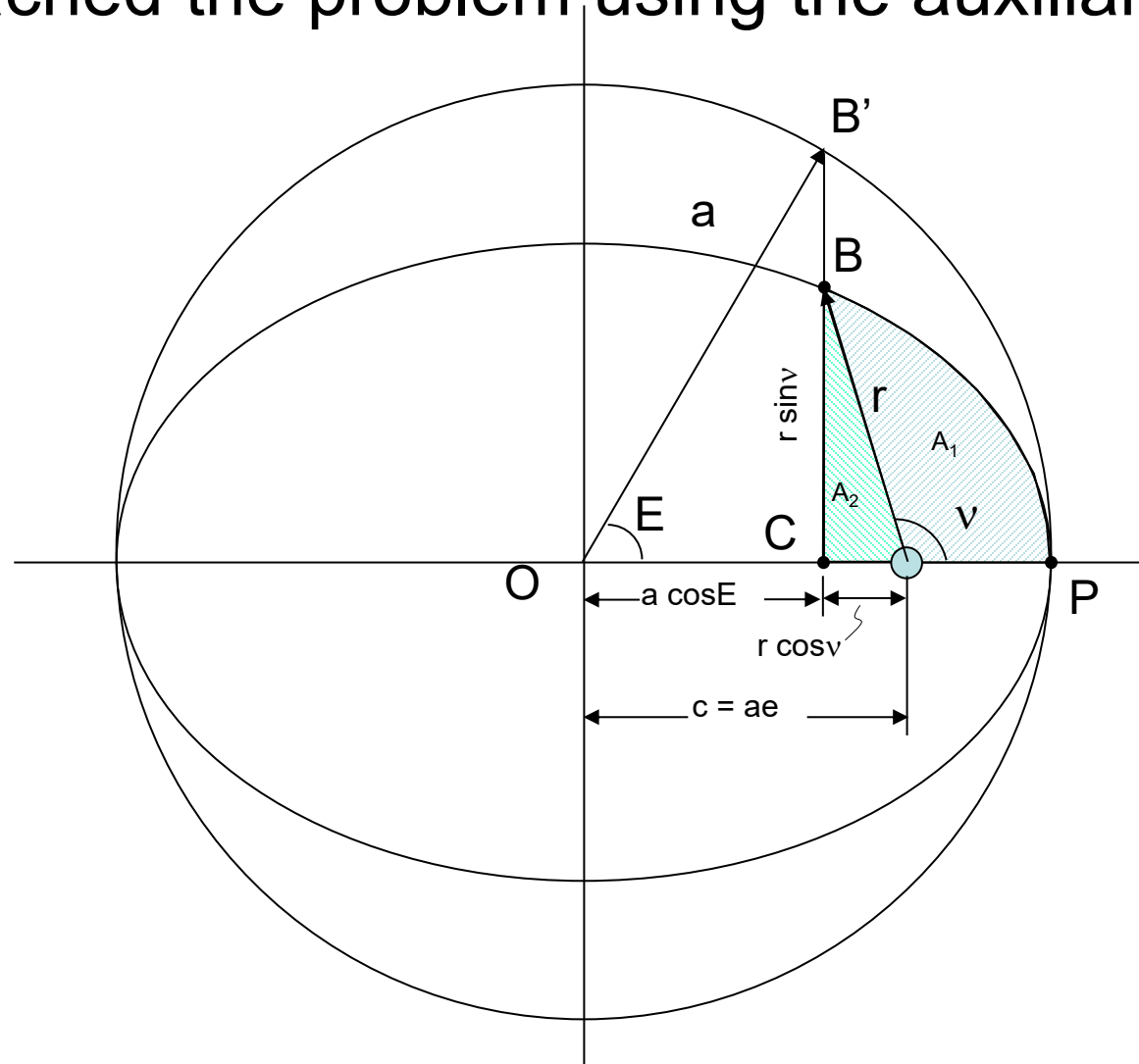
Question is: How to find A

- Finding area of an elliptical section as function of v is difficult to do directly



Auxiliary Circle

- Kepler approached the problem using the auxiliary circle (Vallado fig. 2-2)



Ellipse and Circle Relationships

- Circle

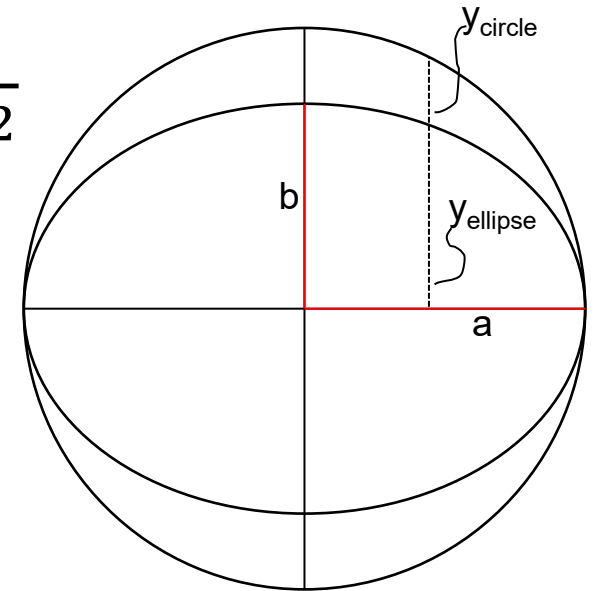
$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \longrightarrow y_{\text{circle}} = \sqrt{a^2 - x^2}$$

- Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow y_{\text{ellipse}} = \frac{b}{a} \sqrt{a^2 - x^2}$$

- Thus

$$y_{\text{ellipse}} = \frac{b}{a} y_{\text{circle}}$$



Area Relationships

$$\overline{CB} = \frac{b}{a} \overline{CB'}$$

$$\overline{CB'} = a \sin E$$

$$\overline{CB} = \frac{b}{a} a \sin E = b \sin E$$

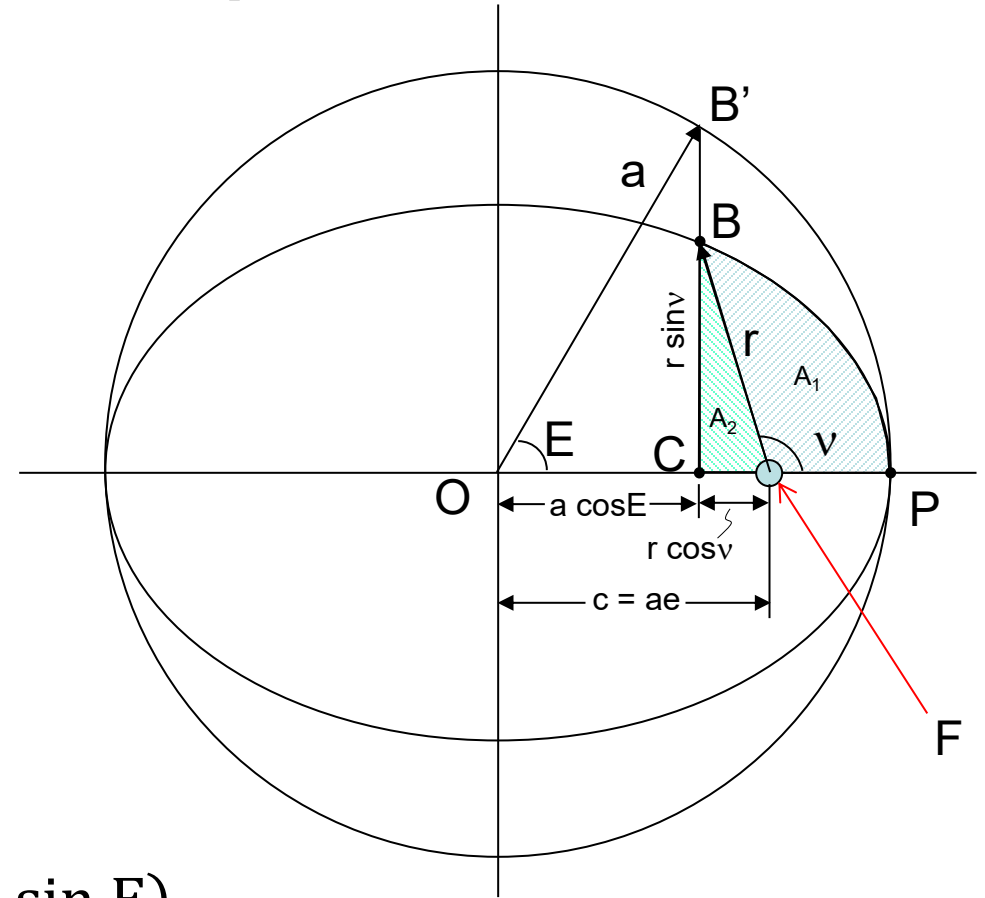
$$\overline{CF} = c - \overline{OC}$$

$$\overline{CF} = ae - a \cos E$$

$$A_1 = A_{PCB} - A_2$$

$$A_2 = \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2} (ae - a \cos E)(b \sin E)$$

$$A_2 = \frac{ab}{2} (e \sin E - \sin E \cos E)$$



Area of Circular Sector

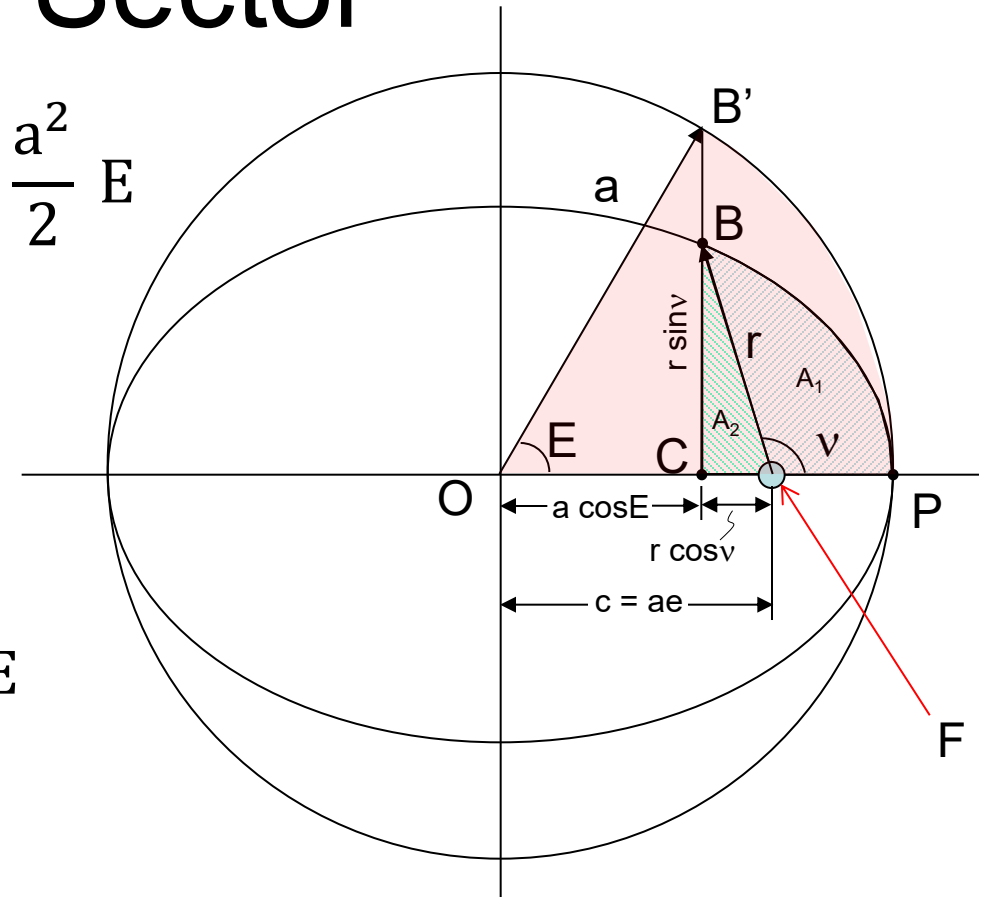
$$A_{\text{sector}} = \int_0^E \int_0^a r \, dr \, dE = \frac{a^2}{2} E$$

- Area of PCB' :

$$A_{PCB'} = A_{\text{sector}} - A_{OCB'}$$

$$A_{OCB'} = \frac{1}{2} (a \sin E) (a \cos E) = \frac{a^2}{2} \sin E \cos E$$

$$A_{PCB'} = \left(\frac{a^2 E}{2} - \frac{a^2}{2} \sin E \cos E \right)$$



Note: For this to work, all angles are expressed in radians!

- Need to find corresponding area in ellipse

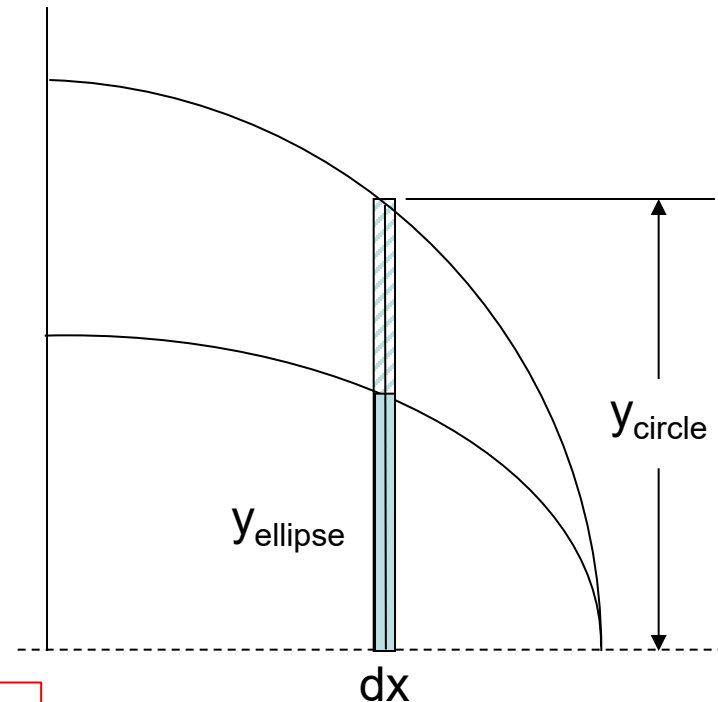
Relate area of Circle and Ellipse

$$dA_{\text{circle}} = y_{\text{circle}} dx$$

$$dA_{\text{ellipse}} = y_{\text{ellipse}} dx$$

$$y_{\text{ellipse}} = \frac{b}{a} y_{\text{circle}}$$

$$dA_{\text{ellipse}} = \frac{b}{a} dA_{\text{circle}}$$



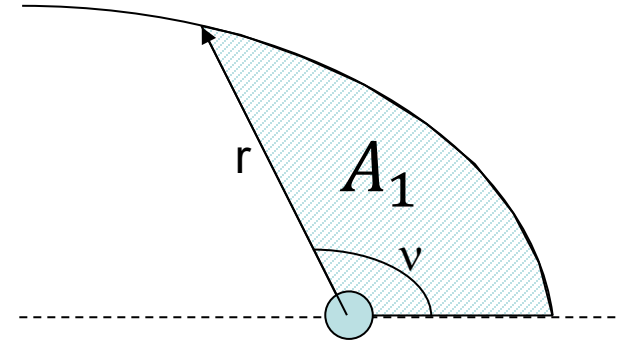
$$A_{\text{ellipse}} = \frac{b}{a} A_{\text{circle}}$$

Scale Area by b/a

$$A_1 = A_{PCB} - A_2$$

$$A_{PCB} = \frac{b}{a} A_{PCB'} = \frac{b}{a} \left(\frac{a^2 E}{2} - \frac{a^2}{2} \sin E \cos E \right)$$

$$A_2 = \frac{ab}{2} (e \sin E - \sin E \cos E)$$



$$A_{PCB} - A_2 = \frac{b}{a} \left(\frac{a^2 E}{2} - \frac{a^2}{2} \sin E \cos E \right) - \frac{ab}{2} (e \sin E - \sin E \cos E)$$

$$= \frac{ab E}{2} - \cancel{\frac{ab}{2} \sin E \cos E} - \frac{ab}{2} e \sin E + \cancel{\frac{ab}{2} \sin E \cos E}$$

$$A_1 = \frac{ab}{2} (E - e \sin E)$$

Relating Area to Time

$$\frac{\Delta t}{A_1} = \frac{TP}{\pi ab} \quad \left(= \frac{\text{Orbit Period}}{\text{Area of Ellipse}} \right)$$

$$\frac{\Delta t}{\frac{ab}{2} (E - e \sin E)} = \frac{TP}{\pi ab} \implies \frac{2\pi \Delta t}{\cancel{ab} (E - e \sin E)} = \frac{TP}{\cancel{ab}}$$

$$\frac{2\pi \Delta t}{(E - e \sin E)} = TP \implies TP = 2\pi \sqrt{\frac{a^3}{\mu}}$$

- $\Delta t = (t_1 - T)$, time from perigee to E

$$\Delta t = \sqrt{\frac{a^3}{\mu}} (E - e \sin E)$$

Standard Expression

- Time from perigee to Eccentric Anomaly E

$$(t_1 - T) = \sqrt{\frac{a^3}{\mu}} (E - e \sin E)$$

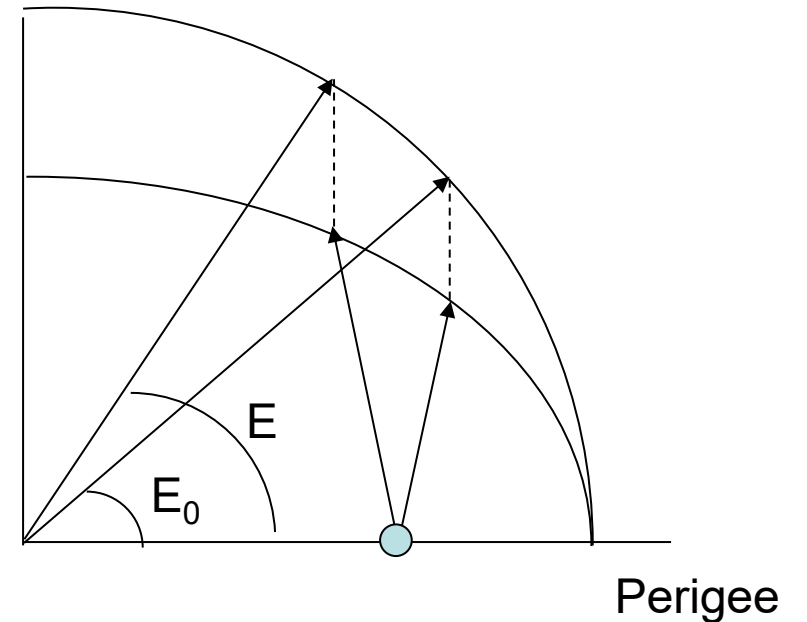
$$n = \sqrt{\frac{\mu}{a^3}} \quad (\text{mean motion})$$

$$M = (E - e \sin E) \quad (\text{mean anomaly})$$

$$n (t_1 - T) = M$$

Going from E_o to E

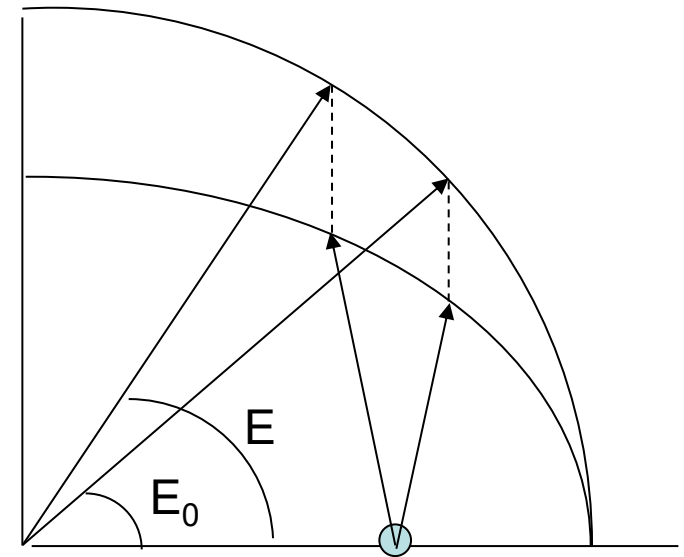
- Look at generic case, not starting at perigee
 - Going from E_o to E = Perigee to E – Perigee to E_o
- Let
 - T = Time of last perigee passage
 - t_o = Time from perigee to E_o
 - t = Time from perigee to E
 - k = perigee passages between E_o and E
 - $k \cdot TP$ = number of full orbits from epoch



General Time of Flight Equation

- Given
 - T = time of perigee just prior to stopping point
 - k perigee crossings between t_o and t

$$t - t_o = k TP + (t - T) - (t_o - T)$$



$$t - t_o = k TP + \frac{1}{n}(E - e \sin E) - \frac{1}{n}(E_0 - e \sin E_0)$$

Relating E to v

$$a \cos E = ae + r \cos v$$

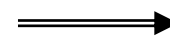
$$\cos E = \frac{ae + r \cos v}{a}$$

$$r = \frac{a(1 - e^2)}{1 + e \cos v} \quad (\text{ellipse eqn})$$

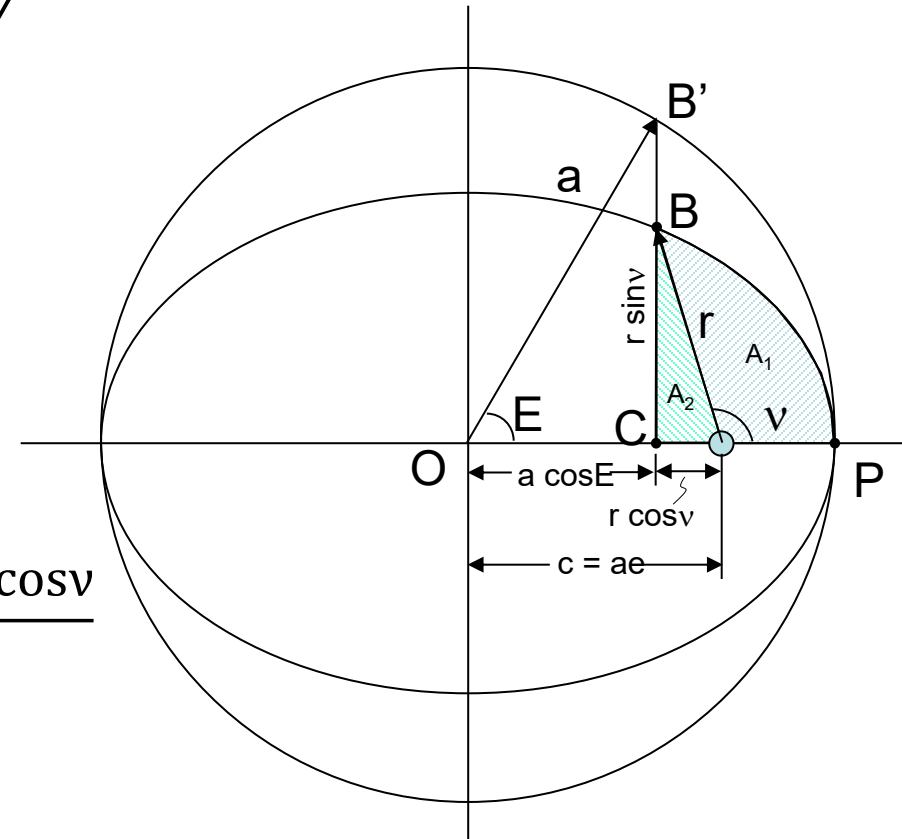
$$\cos E = \frac{\cancel{ae} + \frac{\cancel{a}(1 - e^2)}{\cancel{a}} \cos v}{\cancel{a}}$$

$$\cos E = \frac{e(1 + e \cos v) + (1 - e^2) \cos v}{1 + e \cos v}$$

$$\cos E = \frac{e + \cancel{e^2 \cos v} + \cos v - \cancel{e^2 \cos v}}{1 + e \cos v}$$



$$\cos E = \frac{e + \cos v}{1 + e \cos v}$$



Solve for v

$$\cos E = \frac{e + \cos v}{1 + e \cos v}$$

\implies

$$\cos v = \frac{e - \cos E}{e \cos E - 1}$$

From the auxiliary circle

$$b \sin E = r \sin v \quad b^2 = a^2 - c^2 \quad c = ae$$

$$b^2 = a^2 - a^2 e^2 = a^2(1 - e^2) \implies b = a\sqrt{1 - e^2}$$

$$a\sqrt{1 - e^2} \sin E = r \sin v$$

$$\sin E = \frac{\sin v \sqrt{1 - e^2}}{1 + e \cos v}$$

\implies

$$\sin v = \frac{\sin E \sqrt{1 - e^2}}{1 - e \cos E}$$

$$E = \text{ATAN2}(\sin E, \cos E)$$

$$v = \text{ATAN2}(\sin v, \cos v)$$

We could use these to express Time of Flight in terms of v , but it's easier to convert known values of v to E , and use the Time of Flight equation already derived

Eccentric Anomaly from Vectors

- Eccentric anomaly can be computed directly from position/velocity

$$N_E = \frac{\vec{r} \cdot \vec{v}}{\sqrt{\mu a}}$$

$$D_E = 1 - \frac{r}{a}$$

$$E = \text{atan2}(N_E, D_E)$$

Expressions taken from Space Shuttle Mission Control math specs. Not derived in this class.

Finding E from Time of Flight

- Starting at Perigee:
 - $n(t - T) = E - e \sin E$ (from perigee)
 - How do we find E ?
- No analytic solution, can find by iteration
- Commonly use Newton-Raphson iteration:

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

Iteration for E (Starting at Perigee)

$$f(E_k) = n(t - T) - (E_k - e \sin E_k)$$

$$f'(E_k) = -(1 - e \cos E_k)$$

$$n(t - T) = M \quad (\text{fixed})$$

$$E_{k+1} = E_k + \frac{M - (E_k - e \sin E_k)}{1 - e \cos E_k} \quad \text{Continue until } |E_{k+1} - E_k| < \text{TOL}$$

Specify iteration starting value:


$$E_0 = M \quad \text{should usually work}$$

Alternate Solution (Biondini)

- Let:
$$M = E - e \sin E$$
$$E = M + e \sin E$$

- Take sine of both sides:

$$\sin E = \sin(M + e \sin E)$$


$$\begin{aligned}x &= \sin E \\x_{i+1} &= \sin(M + ex_i) \\E_{i+1} &= \sin^{-1}(M + ex_{i+1})\end{aligned}$$

- Repeat until

$$\Delta x = |x_{i+1} - x_i| \rightarrow 0$$

General Form for Time of Flight

- Not starting at perigee, setup for time of flight from t_0 to t
- General Time of Flight Equation

$$t - t_0 = k TP + \frac{1}{n}(E - e \sin E) - \frac{1}{n}(E_0 - e \sin E_0)$$

– k = number of perigee crossings between E_0 and E

- General Iteration function

$$f(E_k) = n(t - T) - (E_k - e \sin E_k) + (E_0 - e \sin E_0)$$

- Note: the ($k TP$) term is not included: this version allows $E_k > 2\pi$

Newton-Raphson Iteration for E

- As before

$$n(t - T) = n\Delta t = \text{fixed}$$

$$M_0 = (E_0 - e \sin E_0) = \text{fixed}$$

$$f'(E_k) = -(1 - e \cos E_k)$$

$$E_{k+1} = E_k + \frac{n\Delta t + M_0 - (E_k - e \sin E_k)}{1 - e \cos E_k}$$

Specify iteration starting value:

$$E_0 = M_0 \quad \text{should usually work}$$

Note on the Iteration

- Perigee passages are not included in the iteration,

$$E_{k+1} = E_k + \frac{n\Delta t + M_0 - (E_k - e \sin E_k)}{1 - e \cos E_k}$$

- If $k > 1$, result will be a value of $E > 2\pi$
 - Take $\text{mod}(E, 2\pi)$ to get E in $[0, 2\pi]$
 - Can compute k by

$$k = \text{floor}((E - E_o) / 2\pi)$$

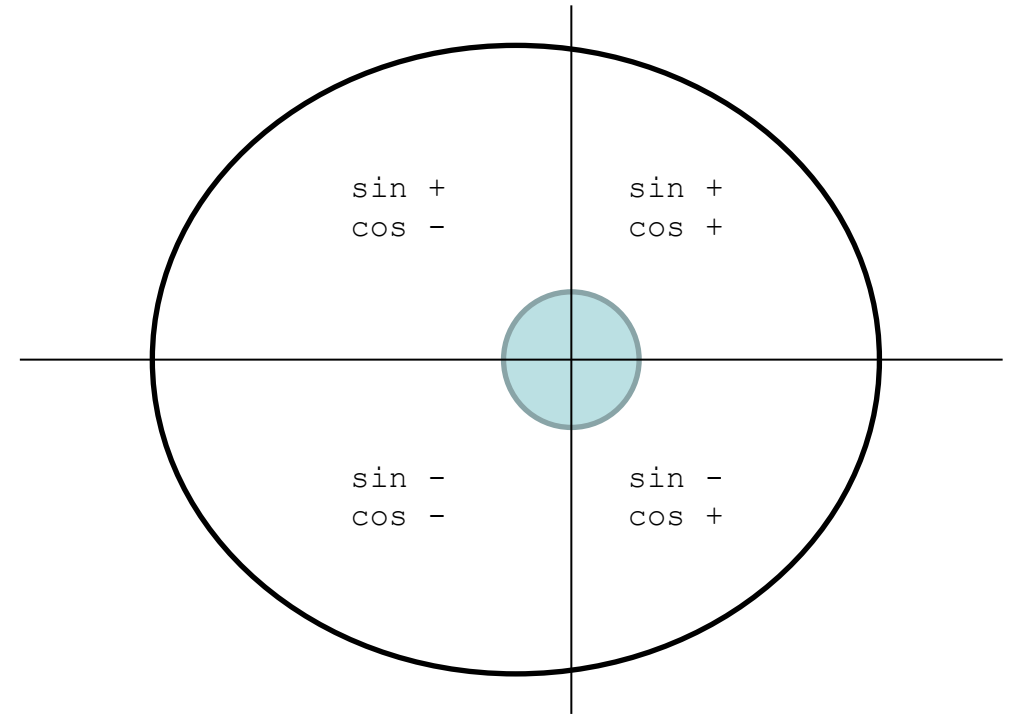
Getting Right Quadrant for v

- Use solved-for value of E to get v

$$\cos v = \frac{e - \cos E}{e \cos E - 1}$$

$$\sin v = \frac{\sin E \sqrt{1 - e^2}}{1 - e \cos E}$$

$$v = \text{atan2}(\sin v, \cos v)$$



- atan2 returns angle in range $[-\pi, \pi]$
 - If angle < 0 , add 2π to get range $[0, 2\pi]$

An Aside: Planar Motion

- Things we know:

$$\vec{h} = \vec{r} \times \vec{v}$$

$$\vec{h} \perp \vec{r} \text{ and } \vec{v}$$

$$\vec{h} = \text{constant}$$

- Therefore:
 - Satellite motion is always in the plane containing \vec{r} and \vec{v}
 - For 2-body problem, can treat prediction problem as a problem in planar motion
 - Don't worry about orientation of orbit in space (yet)

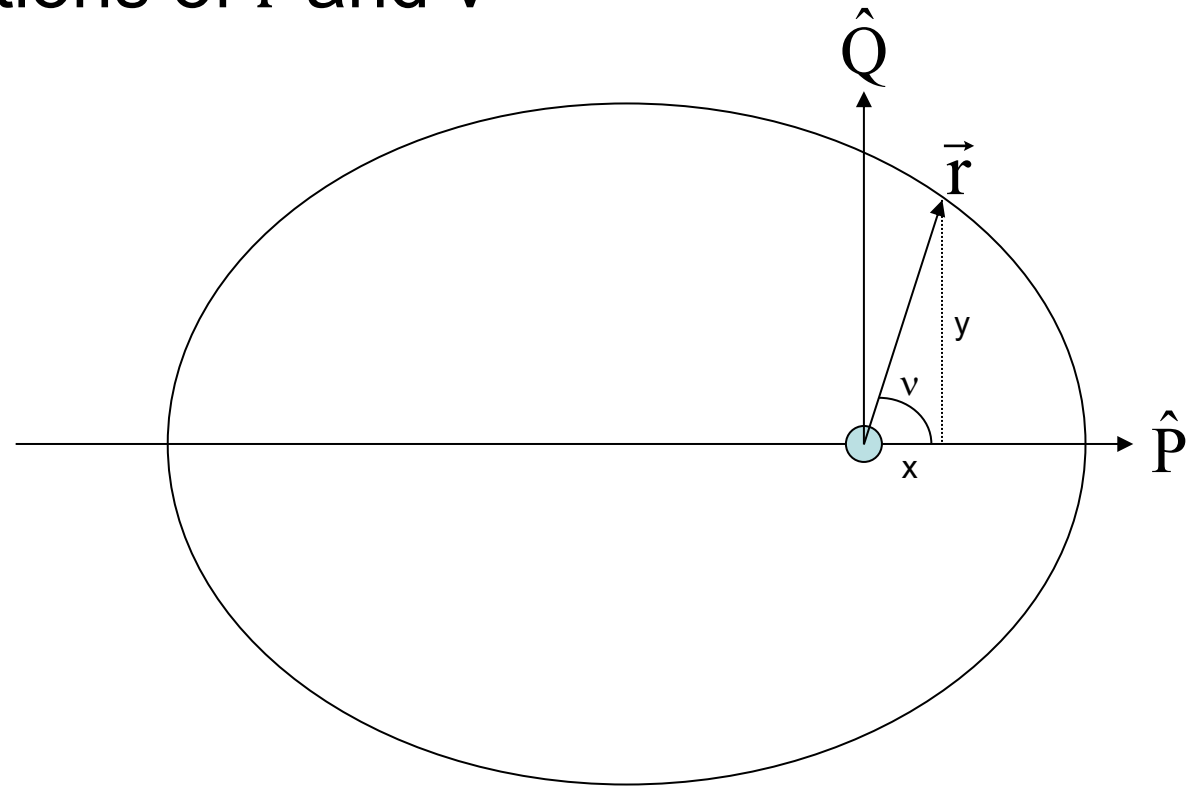
The Kepler Problem

- Orbit elements are constant functions of \vec{r} and \vec{v}
- Look at orbit in perifocal frame:
 - Motion in orbit plane

$$\vec{r} = x \hat{P} + y \hat{Q} + 0 \hat{W}$$

$$\dot{\vec{r}} = \dot{x} \hat{P} + \dot{y} \hat{Q} + 0 \hat{W}$$

(\hat{P} , \hat{Q} , \hat{W} are inertial, so $d/dt = 0$)



Prediction

- Because the orbit elements are constant, it is reasonable to suppose that future states can be found from current states:

$$\vec{r} = f \vec{r}_0 + g \dot{\vec{r}}_0$$

$$\dot{\vec{r}} = \dot{f} \vec{r}_0 + \dot{g} \dot{\vec{r}}_0$$

$$(\vec{r}_0, \dot{\vec{r}}_0 \text{ are fixed, so } d/dt = 0)$$

- Need to find expressions for f , g , etc.

Find f

- Post-cross multiply both sides by $\dot{\vec{r}}_0$

$$\vec{r} \times \dot{\vec{r}}_0 = f \vec{r}_0 \times \dot{\vec{r}}_0 + g \cancel{\dot{\vec{r}}_0 \times \dot{\vec{r}}_0} \quad \text{0}$$

$$\vec{r} \times \dot{\vec{r}}_0 = f \vec{r}_0 \times \dot{\vec{r}}_0 = f \vec{h}$$

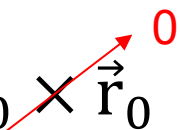
$$\vec{r} \times \dot{\vec{r}}_0 = \begin{vmatrix} \hat{P} & \hat{Q} & \hat{W} \\ x & y & 0 \\ \dot{x}_0 & \dot{y}_0 & 0 \end{vmatrix} = (x\dot{y}_0 - \dot{x}_0 y) \hat{W} \quad \text{Note : } \vec{h} = h \hat{W}$$

$$(x\dot{y}_0 - \dot{x}_0 y) \cancel{\hat{W}} = f h \cancel{\hat{W}} \implies (x\dot{y}_0 - \dot{x}_0 y) = f h$$

$$f = \frac{(x\dot{y}_0 - \dot{x}_0 y)}{h}$$

Find g

- Pre-cross multiply both sides by \vec{r}_0

$$\vec{r}_0 \times \vec{r} = f \vec{r}_0 \times \vec{r}_0 + g \vec{r}_0 \times \dot{\vec{r}}_0$$


$$\vec{r}_0 \times \vec{r} = g \vec{r}_0 \times \dot{\vec{r}}_0 = g \vec{h}$$

$$\vec{r}_0 \times \vec{r} = \begin{vmatrix} \hat{P} & \hat{Q} & \hat{W} \\ x_0 & y_0 & 0 \\ x & y & 0 \end{vmatrix} = (x_0 y - x y_0) \hat{W}$$

Similar to above:

$$g = \frac{(x_0 y - x y_0)}{h}$$

Time derivatives of f and g

$$\dot{f} = \frac{d}{dt} \left(\frac{(x\dot{y}_0 - \dot{x}_0 y)}{h} \right) = \frac{(\dot{x}\dot{y}_0 - \dot{x}_0 \dot{y})}{h}$$

(x_0 and y_0 are fixed)

$$\dot{g} = \frac{d}{dt} \left(\frac{(x_0 y - x y_0)}{h} \right) = \frac{(x_0 \dot{y} - \dot{x} y_0)}{h}$$

Express in terms of v

$$x = r \cos v \quad y = r \sin v$$

$$\dot{x} = \dot{r} \cos v - r \sin v \dot{v}$$

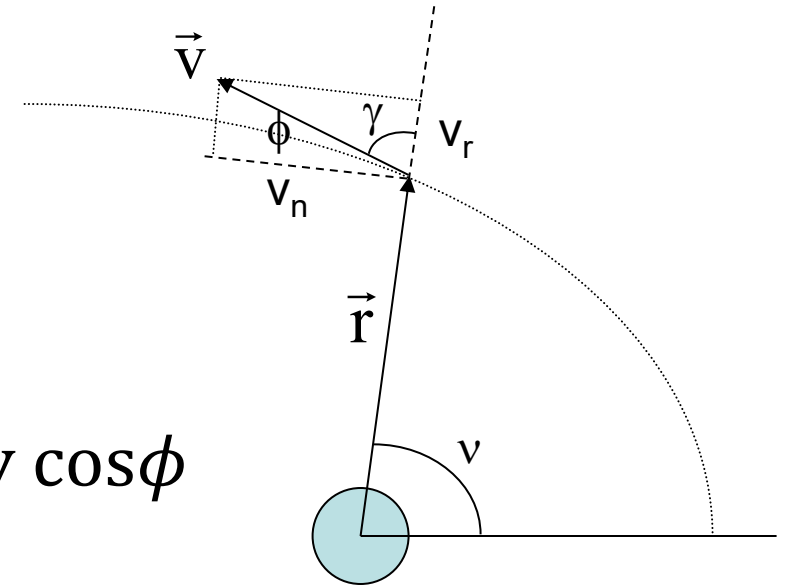
$$\dot{y} = \dot{r} \sin v + r \cos v \dot{v}$$

$$v_n = |\vec{v}| \cos \phi = r \dot{v}$$

$$|\vec{h}| = |\vec{r} \times \vec{v}| = rv \sin \gamma = rv \cos \phi$$

$$h = r (r \dot{v}) = r^2 \dot{v}$$

$$r \dot{v} = \frac{h}{r}$$



$$r = \frac{a(1-e^2)}{1+e\cos v} = \frac{p}{1+e\cos v} \quad \dot{r} = -\frac{p}{(1+e\cos v)^2}(-e\sin v)\dot{v}$$

$$\dot{r} = \frac{\cancel{p}}{(1+\cancel{e}\cos v)} \frac{(e\sin v)\dot{v}}{(1+e\cos v)} = \frac{re\sin v\dot{v}}{(1+e\cos v)}$$

$$\dot{r}\cos v = \frac{re\cos v\sin v\dot{v}}{(1+e\cos v)}$$

$$\dot{x} = \frac{re\cos v\sin v\dot{v}}{(1+e\cos v)} - r\sin v\dot{v} = \frac{re\cos v\sin v\dot{v}}{(1+e\cos v)} - \frac{r\sin v(1+e\cos v)\dot{v}}{(1+e\cos v)}$$

$$\dot{x} = \frac{\cancel{re\cos v\sin v\dot{v}} - r\sin v\dot{v} - \cancel{re\cos v\sin v\dot{v}}}{(1+e\cos v)} \implies \boxed{\dot{x} = \frac{-r\sin v\dot{v}}{(1+e\cos v)}}$$

Solving...

$$\dot{x} = \frac{-r \sin v \dot{v}}{(1 + e \cos v)}$$

$$r \dot{v} = \frac{h}{r} \implies \dot{x} = \frac{-h \sin v}{r (1 + e \cos v)}$$

$$p = r (1 + e \cos v) = \frac{h^2}{\mu} \implies h = \sqrt{\mu p}$$

$$\dot{x} = \frac{-\sqrt{\mu p} \sin v}{p} \implies \dot{x} = -\sqrt{\frac{\mu}{p}} \sin v$$

Similarly for \dot{y}

$$\dot{y} = \dot{r} \sin v + r \cos v \dot{v}$$

$$r = p (1 + e \cos v)^{-1}$$

$$\dot{r} = -p (1 + e \cos v)^{-2} (-e \sin v) \dot{v}$$

$$\dot{r} = \frac{p e \sin v}{(1 + e \cos v)^2} \dot{v}$$

$$\dot{r} = \frac{r^2 e \sin v}{p} \dot{v}$$

$$h = r^2 \dot{v}$$

$$\dot{r} = \frac{h}{p} e \sin v$$

$$h = r(r \dot{v}) = \sqrt{\mu p}$$

$$r \dot{v} = \frac{\sqrt{\mu p}}{r}$$

$$r \dot{v} = \frac{\sqrt{\mu p} (1 + e \cos v)}{p}$$

$$r \dot{v} = \sqrt{\frac{\mu}{p}} (1 + e \cos v)$$

$$\dot{r} = \sqrt{\frac{\mu}{p}} e \sin v$$

Combine

$$\dot{y} = \dot{r} \sin v + r \cos v \dot{v}$$

$$\dot{y} = \sqrt{\frac{\mu}{p}} e \sin^2 v + \sqrt{\frac{\mu}{p}} (1 + e \cos v) \cos v$$

$$\dot{y} = \sqrt{\frac{\mu}{p}} (e [\sin^2 v + \cos^2 v] + \cos v)$$

$$\dot{y} = \sqrt{\frac{\mu}{p}} (e \sin^2 v + \cos v + e \cos^2 v) \implies \boxed{\dot{y} = \sqrt{\frac{\mu}{p}} (e + \cos v)}$$

Summary so far

$$\begin{aligned}x &= r \cos v & y &= r \sin v \\ \dot{x} &= -\sqrt{\frac{\mu}{p}} \sin v & \dot{y} &= \sqrt{\frac{\mu}{p}} (e + \cos v)\end{aligned}$$

Perifocal Position and
Velocity Components

$$f = \frac{(x\dot{y}_0 - \dot{x}_0 y)}{h}$$

$$g = \frac{(x_0 y - x y_0)}{h}$$

$$\dot{f} = \frac{(\dot{x}\dot{y}_0 - \dot{x}_0 \dot{y})}{h}$$

$$\dot{g} = \frac{(x_0 \dot{y} - \dot{x} y_0)}{h}$$

$$f\dot{g} - \dot{f}g = 1$$

Combine to find expressions in v

Example: find f

$$f = \frac{(x\dot{y}_0 - \dot{x}_0 y)}{h}$$

$$f = \frac{\left(r \cos v \sqrt{\frac{\mu}{p}} (e + \cos v_0) + \sqrt{\frac{\mu}{p}} \sin v_0 r \sin v \right)}{\sqrt{\mu p}}$$

$$f = \frac{r(\cos v (e + \cos v_0) + \sin v_0 \sin v)}{p}$$

$$f = \frac{r(e \cos v + \cos v \cos v_0 + \sin v_0 \sin v)}{p}$$

Recall the trig identity...

$$\cos A \cos B \pm \sin A \sin B = \cos(A \mp B)$$

$$f = \frac{r(e \cos v + \cos(v-v_0))}{p} = \frac{r(e \cos v + \cos \Delta v)}{p}$$

Add and subtract...

$$\frac{r}{p} - \frac{r}{p} + \frac{r e \cos v}{p} = -\frac{r}{p} + \frac{r}{p}(1 + e \cos v)$$

$$p = r(1 + e \cos v)$$

$$-\frac{r}{p} + \frac{r}{p}(1 + e \cos v) = -\frac{r}{p} + 1$$

$$\frac{r e \cos v}{p} = 1 - \frac{r}{p}$$

$$f = 1 - \frac{r}{p} + \frac{r}{p} \cos \Delta v$$

$$f = 1 - \frac{r}{p}(1 - \cos \Delta v)$$

Similarly

$$g = \frac{r r_0}{\sqrt{\mu p}} \sin \Delta v$$

$$\dot{g} = 1 - \frac{r_0}{p} (1 - \cos \Delta v)$$

Using $f\ddot{g} - \dot{f}\dot{g} = 1$:

$$\dot{f} = \sqrt{\frac{\mu}{p}} \tan \frac{\Delta v}{2} \left(\frac{1 - \cos \Delta v}{p} - \frac{1}{r} - \frac{1}{r_0} \right)$$

The f&g Functions: So What?

- The f&g functions: undeniably cool, but rather esoteric-seeming
- Does anybody actually use them?
- Answer: yes!
 - Cheap lower-fidelity orbit propagation
 - Can extend to non-two-body formulations
 - Commonly used for initial orbit determination (orbits from observations)
 - Basis of “Lambert Targeting” methods for rendezvous/proximity ops
 - Provides analytic solution for orbit determination State Transition Matrix
 - Universal variable formulation generalizes to all conic sections
 - And more!

Homework 2

Due Saturday, 31 Jan 2026

Homework 2 (1)

Given:

Component	Value (meters)
Position X	326151.080726
Position Y	6077471.251787
Position Z	2944583.918767
Velocity X	-7455.178720
Velocity Y	-482.482572
Velocity Z	1910.883434

- Find E_o (slide 19)
- Find v_o (Keplerian comps)
- Find M_o
- Find time of flight from perigee to v_o

- Find time of flight from v_o to $v = 65^\circ$
- Verify: Starting at v_o , compute the true anomaly after the time of flight computed above.
- Starting at v_o , what is the true anomaly after 2700 seconds?
- Starting at v_o , what is the true anomaly after exactly two orbit periods?
- What is the true anomaly after 15000 seconds?

Homework 2 (2)

Given:

Component	Value (meters)
Position X	572461.711228
Position Y	-1015437.194396
Position Z	7707337.871302
Velocity X	-6195.262945
Velocity Y	-3575.889650
Velocity Z	-5.423283

- Compute Keplerian elements
- Find perifocal position and velocity, \vec{r}_o and \vec{v}_o
 - See equations on Slide 39
- Find f, g, \dot{f}, \dot{g} for $\Delta\nu = 33^\circ$
- Find \vec{r} and \vec{v}

Homework 2 (3)

- Derivation 1: Using energy equation and ellipse equation, find an expression for the orbit speed as a function of true anomaly
- Derivation 2: Using the above, show that

$$v_{\text{perigee}} = \sqrt{\frac{\mu}{a} \left(\frac{1+e}{1-e} \right)}$$

$$v_{\text{apogee}} = \sqrt{\frac{\mu}{a} \left(\frac{1-e}{1+e} \right)}$$