Energy Concepts (cont'd)

- To be completely general, could write

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\mathrm{C} - \frac{\mu}{\mathrm{r}}\right) = \frac{\mu}{\mathrm{r}^2}\dot{\mathrm{r}} \qquad \text{C=constant}$$

- C can take any value. By convention, we set $C = 0$ so that

$$\mathrm{PE} \to 0 \quad \text{as} \quad \mathrm{r} \to \infty$$

  – Consequence:
    - Energy for an elliptical orbit has a negative sign
    - It's <u>*not*</u> "negative energy,": it's just a consequence of where we define zero

Trajectory Equation

- Start with

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$$

- Post-cross multiply by $\vec{h}$

$$\ddot{\vec{r}} \times \vec{h} = -\frac{\mu}{r^3}\vec{r} \times \vec{h}$$

$$\vec{r} \times \vec{h} = -\vec{h} \times \vec{r} = -\left(\vec{r} \times \dot{\vec{r}}\right) \times \vec{r}$$

$$\ddot{\vec{r}} \times \left(\vec{r} \times \dot{\vec{r}}\right) = -\frac{\mu}{r^3}\vec{r} \times \left(\vec{r} \times \dot{\vec{r}}\right) = \frac{\mu}{r^3}\left(\vec{r} \times \dot{\vec{r}}\right) \times \vec{r}$$

- Identities

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \cdot \vec{b}\right)\vec{c} - (\vec{a} \cdot \vec{c})\vec{b}$$

$$\left(\vec{a} \times \vec{b}\right) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - \left(\vec{b} \cdot \vec{c}\right)\vec{a}$$

Look at Each Side

- LHS

$$\ddot{\vec{r}} \times \vec{h} \Rightarrow \frac{d}{dt}\left(\dot{\vec{r}} \times \vec{h}\right) = \ddot{\vec{r}} \times \vec{h} + \xrightarrow{0}\cancel{\dot{\vec{r}} \times \frac{d}{dt}\left(\vec{h}\right)}$$

- RHS

$$\left(\vec{r} \times \dot{\vec{r}}\right) \times \vec{r} \;=\; (\vec{r} \cdot \vec{r})\dot{\vec{r}} - \left(\dot{\vec{r}} \cdot \vec{r}\right)\vec{r} \longleftarrow \left(\vec{a} \times \vec{b}\right) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - \left(\vec{b} \cdot \vec{c}\right)\vec{a}$$

$$\left(\vec{r} \times \dot{\vec{r}}\right) \times \vec{r} \;=\; r^2\dot{\vec{r}} - r\dot{r}\,\vec{r}$$

$$\frac{\mu}{r^3}\left(\vec{r} \times \dot{\vec{r}}\right) \times \vec{r} \;=\; \frac{\mu}{r^3}\left(r^2\dot{\vec{r}} - r\dot{r}\,\vec{r}\right)$$

- Combine

$$\boxed{\frac{d}{dt}\left(\dot{\vec{r}} \times \vec{h}\right) = \frac{\mu}{r^3}\left(r^2\dot{\vec{r}} - r\dot{r}\,\vec{r}\right) = \mu\left(\frac{\dot{\vec{r}}}{r} - \frac{\dot{r}}{r^2}\vec{r}\right)}$$

More Term-By-Term Analysis

Recall: $\frac{d}{dt}(\dot{\vec{r}} \times \vec{h}) = \frac{\mu}{r^3}(r^2\dot{\vec{r}} - r\dot{r}\,\vec{r}) = \mu\left(\frac{\dot{\vec{r}}}{r} - \frac{\dot{r}}{r^2}\vec{r}\right)$

- Note

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{\dot{\vec{r}}}{r} - \frac{\dot{r}}{r^2}\vec{r}$$

- So RHS becomes

$$\mu\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

- So

$$\frac{d}{dt}(\dot{\vec{r}} \times \vec{h}) = \mu\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

$$\boxed{\dot{\vec{r}} \times \vec{h} = \mu\frac{\vec{r}}{r} + \vec{B}}$$

$\vec{B} =$ constant of integration

Still more…

- Dot multiply

$$\vec{r}\cdot(\dot{\vec{r}}\times\vec{h}) = \mu\frac{\vec{r}\cdot\vec{r}}{r} + \vec{r}\cdot\vec{B} = \mu\cancelto{r}{\frac{r^2}{r}} + \vec{r}\cdot\vec{B}$$

- Identity:

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$$

- So

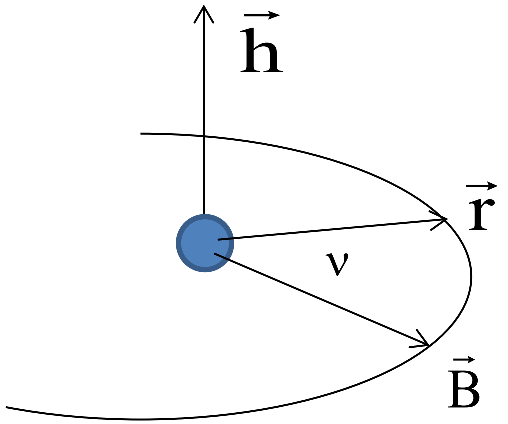
$$(\vec{r}\times\dot{\vec{r}})\cdot\vec{h} = \mu r + \vec{r}\cdot\vec{B}$$

$$\vec{h}\cdot\vec{h} = h^2 = \mu r + \vec{r}\cdot\vec{B}$$

- By definition:

$$\vec{r}\cdot\vec{B} = r\,B\cos\nu$$

$$\boxed{h^2 = \mu r + r\,B\cos\nu}$$



Note:
$\vec{h}\perp\vec{r}$ and $\dot{\vec{r}}$,
$\vec{B}$ is coplanar with $\vec{r}$ and $\dot{\vec{r}}$

And yet more….

- Divide through by μ and collect terms

$$\frac{h^2}{\mu} = r + r\,\frac{B}{\mu}\,\cos\nu$$

$$\frac{h^2}{\mu} = r\left(1 + \frac{B}{\mu}\,\cos\nu\right)$$

$$\boxed{r = \frac{h^2/\mu}{\left(1 + \frac{B}{\mu}\,\cos\nu\right)}}$$

Ellipse Equations

- Note

$$r = \frac{h^2/\mu}{\left(1 + \frac{B}{\mu}\cos\nu\right)}$$

  - This has the mathematical form of the polar equation for an ellipse relative to a focus:

$$r = \frac{a(1 - e^2)}{1 + e\cos\nu}$$

- We have shown that the trajectory follows an elliptical path with the central body at a focus
  - Actually, a “conic section”: equation also applies to parabolas and hyperbolas

Match terms

$$h^2/\mu = p = a(1 - e^2)$$

$$r = \frac{h^2/\mu}{\left(1 + \frac{B}{\mu}\cos\nu\right)} = \frac{a(1 - e^2)}{1 + e\cos\nu}$$

$$e = \frac{B}{\mu}$$

- This gives us the magnitude of e
- Define co-linear "eccentricity vector," $\vec{e}$

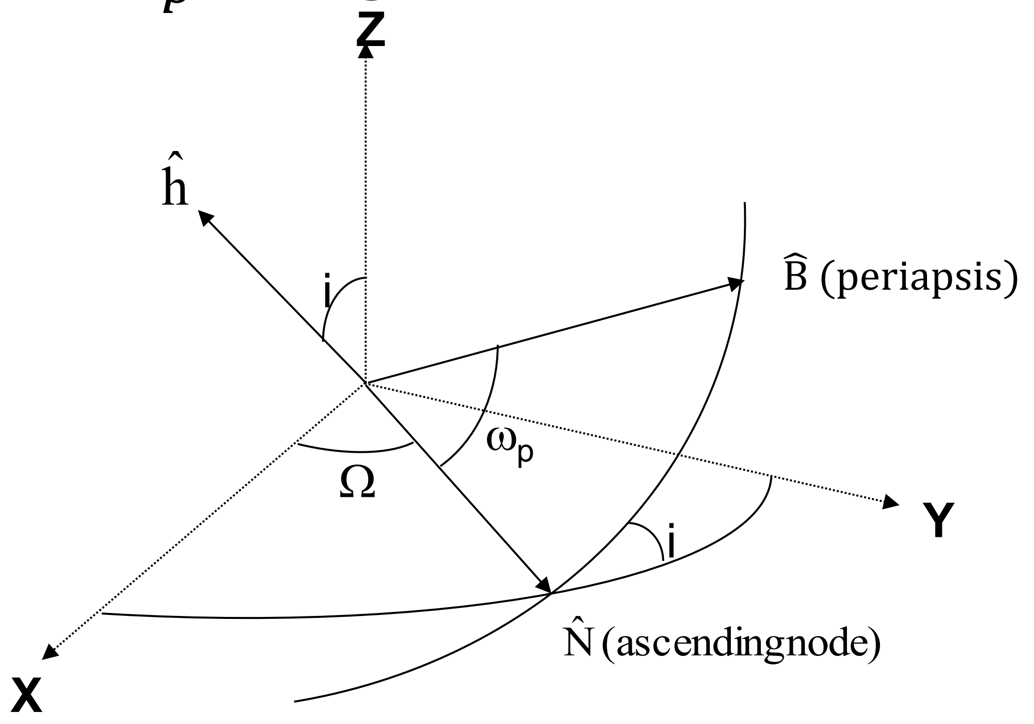
$$\vec{e} = \frac{\vec{B}}{\mu}$$

Argument of Periapsis

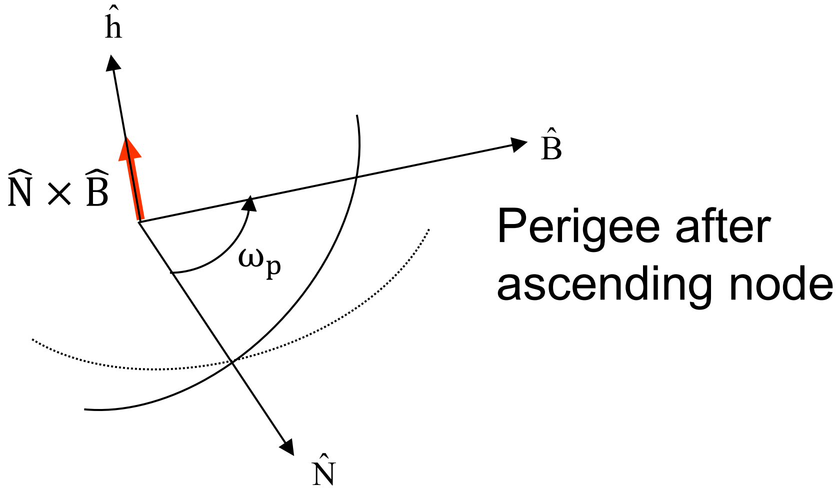
- $\vec{\mathrm{B}}$ points to direction of minimum radius, since r is minimum for ν=0, and ν is the angle between $\vec{\mathrm{B}}$ and $\vec{\mathrm{r}}$

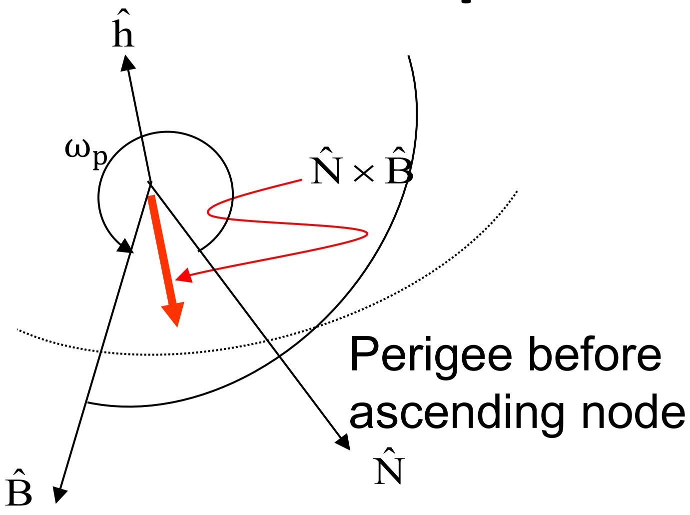
$$\mathrm{r} = \frac{\mathrm{a}(1-\mathrm{e}^2)}{1+\mathrm{e}\cos\nu}$$

- Argument of periapsis ω_p is angle between line of nodes and periapsis



Computing Argument of Periapsis





$$\widehat{\mathrm{N}} \cdot \widehat{\mathrm{B}} = \cos\omega_p$$

Definition of dot product between two unit vectors. Projection of Perigee vector on Node Vector.

$$\widehat{\mathrm{N}} \times \widehat{\mathrm{B}} \rightarrow \ || \ \hat{\mathrm{h}}$$

Angle θ between $\widehat{\mathrm{N}} \times \widehat{\mathrm{B}}$ and $\hat{\mathrm{h}}$ is either $0°$ or $180°$ so $\cos\theta = \pm 1$

$$\left|\widehat{\mathrm{N}} \times \widehat{\mathrm{B}}\right| = \sin\omega_p$$

$\left|\vec{a} \times \vec{b}\right| = |\vec{a}|\left|\vec{b}\right| \sin\theta$ where θ is the angle between the two vectors. For unit vectors, $|\hat{a}| = \left|\hat{b}\right| = 1.$

$$\hat{\mathrm{h}} \cdot \left(\widehat{\mathrm{N}} \times \widehat{\mathrm{B}}\right) = \sin\omega_p$$

Sign of $\sin\omega_p$ is either + or –, depending on directions of $\widehat{\mathrm{N}} \times \widehat{\mathrm{B}}$ and $\hat{\mathrm{h}}$. Dot product gives projection of cross product onto angular momentum, preserving sign for ATAN2.

$$\boxed{\omega_p = \mathrm{atan2}\left(\hat{\mathrm{h}} \cdot \left(\widehat{\mathrm{N}} \times \widehat{\mathrm{B}}\right), \widehat{\mathrm{N}} \cdot \widehat{\mathrm{B}}\right)}$$

Continuing…

- Combining terms $$\dot{\vec{r}} \times \vec{h} = \mu \frac{\vec{r}}{r} + \vec{B} = \mu \frac{\vec{r}}{r} + \mu\vec{e}$$

$$\mu\vec{e} = \dot{\vec{r}} \times \vec{h} - \mu \frac{\vec{r}}{r}$$

$$\mu\vec{e} = \dot{\vec{r}} \times \left(\vec{r} \times \dot{\vec{r}}\right) - \mu \frac{\vec{r}}{r}$$

- Apply vector identity: $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \cdot \vec{b}\right)\vec{c} - (\vec{a} \cdot \vec{c})\vec{b}$

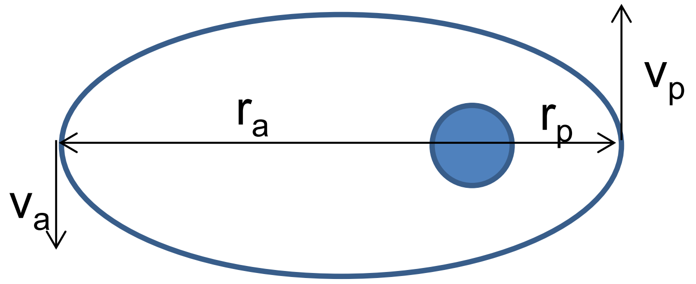
$$\mu\vec{e} = \left(\dot{\vec{r}} \cdot \dot{\vec{r}}\right)\vec{r} - \left(\dot{\vec{r}} \cdot \vec{r}\right)\dot{\vec{r}} - \mu \frac{\vec{r}}{r}$$

$$\mu\vec{e} = \left(v^2 - \frac{\mu}{r}\right)\vec{r} - \left(\dot{\vec{r}} \cdot \vec{r}\right)\dot{\vec{r}}$$

  - Use to find eccentricity from position and velocity
  - Direction of perigee from position and velocity

Look at Angular Momentum

- At periapsis and apoapsis…



$$\vec{h} = \vec{r} \times \vec{v}$$

$$h = r\,v\,\sin\theta \;=\; \text{constant}$$

$$h \;=\; r_p\,v_p = \; r_a\,v_a \longrightarrow$$ $\sin\theta = 1$ at these points

$$r_p = \frac{a(1-e^2)}{1 + e\cos 0} = \frac{a(1-e^2)}{1+e} = a(1-e)$$

$$r_a = \frac{a(1-e^2)}{1 + e\cos\pi} = \frac{a(1-e^2)}{1-e} = a(1+e)$$

Look at Energy

$$\xi = \frac{v^2}{2} - \frac{\mu}{r} = \text{constant}$$

$$\xi = \frac{v_p^2}{2} - \frac{\mu}{r_p} \quad \text{(at perigee)}$$

$$h = r_p\, v_p$$

$$v_p^2 = \left(\frac{h}{r_p}\right)^2$$

$$\xi = \frac{h^2}{2r_p^2} - \frac{\mu}{r_p}$$

$$a(1-e^2) = {h^2}/{\mu}$$

$$h^2 = \mu\, a(1-e^2)$$

$$r_p^2 = a^2(1-e)^2$$

$$r_p = a(1-e)$$

$$\boxed{\xi = -\frac{\mu}{2a}}$$

$$\xi = \frac{\mu\, a(1-e^2)}{2a^2(1-e)^2} - \frac{\mu}{a(1-e)}$$

$$\xi = \frac{\mu\, a(1-e)(1+e)}{2a^2(1-e)(1-e)} - \frac{\mu}{a(1-e)}$$

$$\xi = \frac{\mu}{2a}\left(\frac{1+e}{1-e}\right) - \frac{\mu}{a(1-e)}$$

$$\xi = \frac{\mu}{2a}\left(\frac{1+e}{1-e} - \frac{2}{1-e}\right) = \frac{\mu}{2a}\left(\frac{e-1}{1-e}\right)$$

Consider Eccentricity

$$\xi = -\frac{\mu}{2a}$$

$$a(1-e^2) = {}^{h^2}/_{\mu}$$

$$a = \frac{-\mu}{2\xi}$$

$$\frac{h^2}{\mu} = -\frac{\mu}{2\xi}(1-e^2)$$

$$1-e^2 = -\frac{2h^2\xi}{\mu^2}$$

$$e^2 = 1 + \frac{2h^2\xi}{\mu^2}$$

$$\boxed{e = \sqrt{1 + \frac{2h^2\xi}{\mu^2}} \quad (0 \leq e < 1 \text{ for ellipses})}$$

Properties of the Ellipse

$$r_1 + r_2 = constant$$

$$r_1 + r_2 = 2a$$

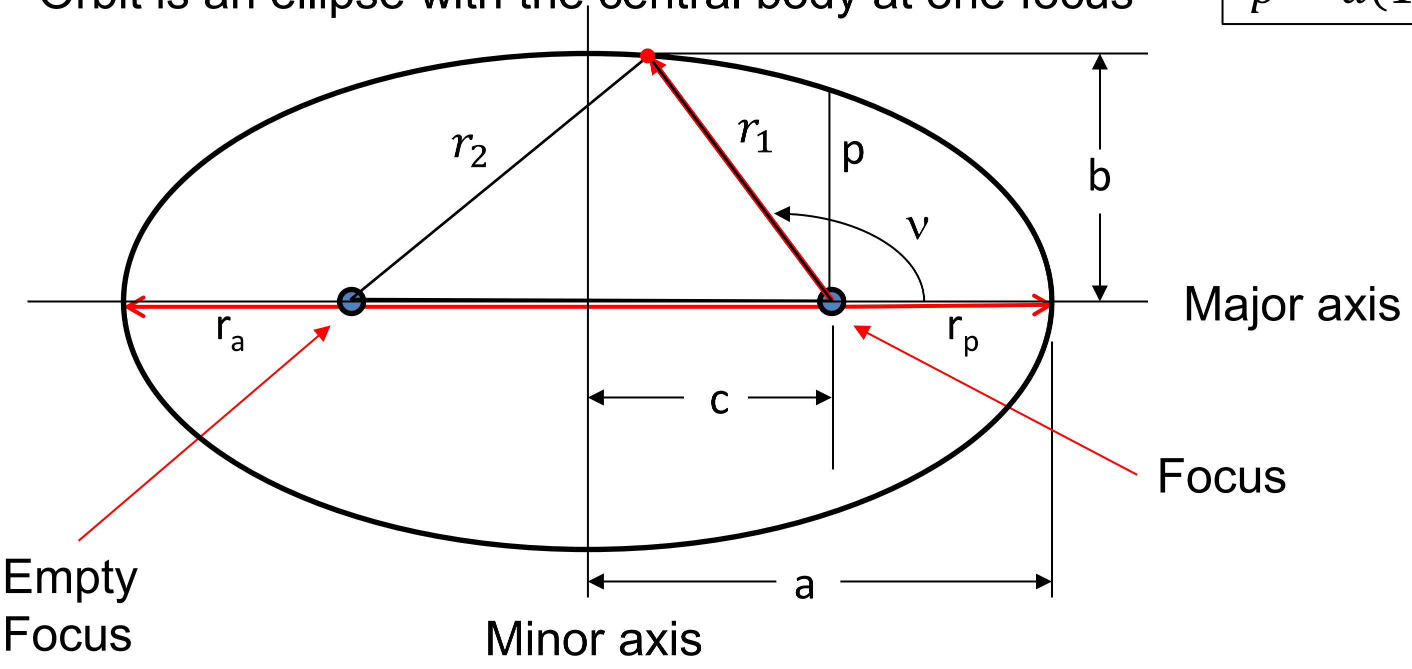
$$a = \frac{r_a + r_p}{2}$$

$$|r_2 - r_1| = 2c = constant$$

$$e = \frac{c}{a} = \frac{r_a - r_p}{r_a + r_p}$$

$$p = a(1 - e^2)$$

"Orbit is an ellipse with the central body at one focus"

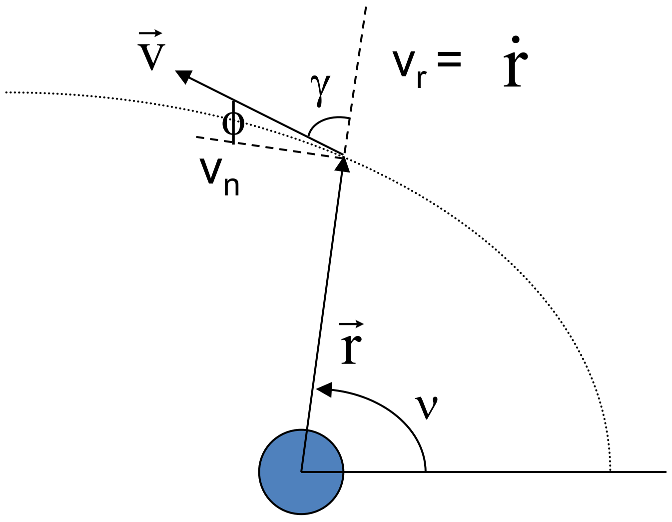


$$r = \frac{a(1 - e^2)}{1 + e\cos\nu}$$

Orbit Period

- Kepler: satellites (planets) sweep out equal areas in equal times
- Look at angular momentum

$$\vec{h} = \vec{r} \times \vec{v}$$
$$|\vec{h}| = h = r\,v\,\sin\gamma$$



- Let γ and ϕ be complementary angles, so that

$$\sin\gamma = \cos\phi$$
$$h = r\,v\,\cos\phi$$

  - ϕ is often called the flight path angle

Tangential Velocity

- Consider

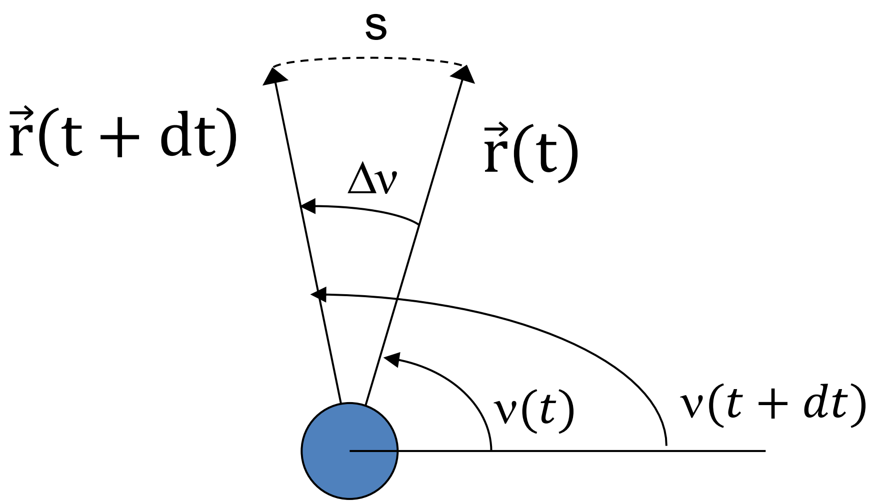
$$s = r\,\Delta\nu$$

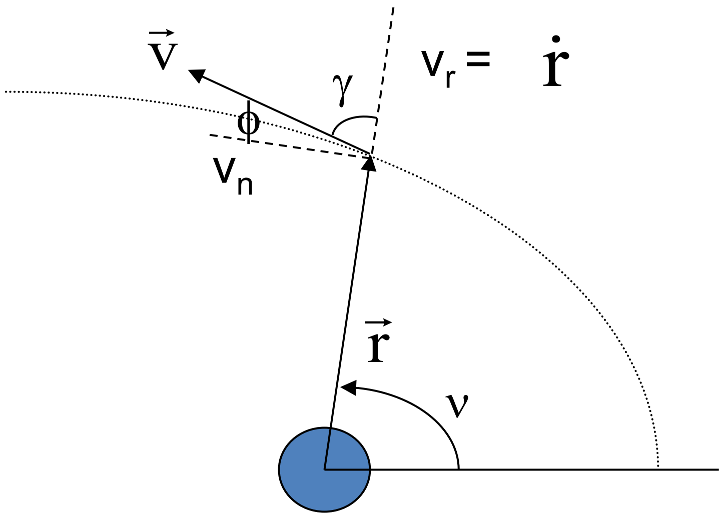
$$v_n = \frac{s}{\Delta t} = r\frac{\Delta\nu}{\Delta t} \rightarrow r\,\dot{\nu}$$

$$v_n = v\cos\phi$$

$$v\cos\phi = r\,\dot{\nu}$$

$$h = rv\cos\phi = r^2\,\dot{\nu}$$

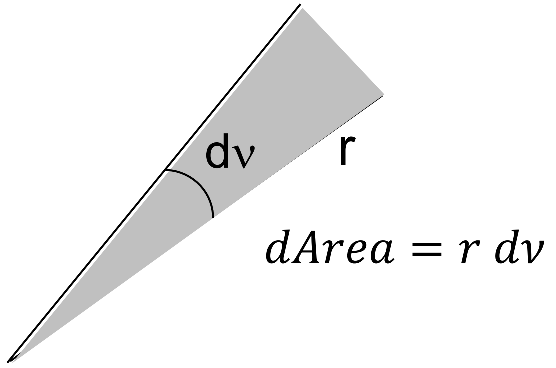




Area Swept Out Over Time dt

- Relate dν and dt

$$h = r^2 \dot{\nu} = r^2 \frac{d\nu}{dt}$$



$$dt = \frac{r^2}{h} d\nu$$

- Recall integration in polar coordinates

$$dA = \int_{r=0}^{r} r\, dr\, d\nu = \frac{r^2}{2} d\nu$$

Integrate in r only

$$d\nu = \frac{2\, dA}{r^2}$$

- So

$$\boxed{dt = \frac{2}{h} dA}$$

Orbital Period

$$\mathrm{dt} = \frac{2}{\mathrm{h}}\,\mathrm{dA}$$

- Define Orbit Period
  - Time required for satellite to sweep out the area of the entire ellipse
  - Notation: "TP" following Bate, Mueller, White

$$TP = \int \mathrm{dt} = \int \frac{2}{\mathrm{h}}\,\mathrm{dA} = \frac{2}{\mathrm{h}}\,\mathrm{A}_{\mathrm{ellipse}}$$

- Area of Ellipse = πab

$$\mathrm{TP} = \frac{2\pi\,\mathrm{ab}}{\mathrm{h}}$$

More Ellipse Properties

$$TP = \frac{2\pi\, ab}{h}$$

$$e = \frac{c}{a} \Rightarrow c = ae$$

$$b = \sqrt{a^2 - c^2} = \sqrt{a^2 - a^2e^2} = \sqrt{a^2(1 - e^2)} = \sqrt{ap}$$

$$p = \frac{h^2}{\mu}$$

$$b = \sqrt{\frac{ah^2}{\mu}}$$

$$TP = \frac{2\pi\, a}{h}\sqrt{\frac{ah^2}{\mu}}$$

$$TP = 2\pi\, a\sqrt{\frac{a}{\mu}}$$

$$TP = 2\pi\sqrt{\frac{a^3}{\mu}}$$

Orbit Period: $\boxed{TP = 2\pi\sqrt{\frac{a^3}{\mu}}}$

Summary of Important Results

$$TP = 2\pi\sqrt{\frac{a^3}{\mu}}$$

$$\vec{h} = \vec{r} \times \dot{\vec{r}} = \text{Constant}$$

$$p = h^2/\mu = a(1-e^2)$$

$$\xi = \frac{v^2}{2} - \frac{\mu}{r} = \text{constant}$$

$$\xi = -\frac{\mu}{2a}$$

$$i = \cos^{-1}\frac{\vec{h}\cdot\hat{Z}}{|\vec{h}|}$$

$$\hat{N} = \frac{\hat{Z}\times\vec{h}}{|\hat{Z}\times\vec{h}|} = h\ \sin\Omega$$

$$\Omega = ATAN2(N_y, N_x)$$

$$\vec{B} = \dot{\vec{r}} \times \vec{h} - \mu\frac{\vec{r}}{r}$$

$$\omega_p = \text{atan2}\left(\hat{h}\cdot(\hat{N}\times\hat{B}), \hat{N}\cdot\hat{B}\right)$$

$$a = -\frac{\mu}{2\xi}$$

$$e = \frac{B}{\mu} = \sqrt{1 + \frac{2h^2\xi}{\mu^2}}$$

$$\vec{r}\cdot\vec{B} = r\,B\cos\nu$$

$$\cos\nu = \frac{\vec{r}\cdot\vec{B}}{r\,B}$$ (be careful of quadrant)

“Be Careful of Quadrant”

- Above, it said “be careful of quadrant” when solving for ν
- The only equation provided was for $\cos\nu$
- Using only that equation will result in quadrant errors
  - Inverse cosine functions give angles in range $[0, \pi]$
  - I.e., you’ll have a quadrant problem if true anomaly is in 3rd or 4th quadrant

Finding True Anomaly in $[0,2\pi]$

- Previous equation for $\cos\nu$:

$$\cos\nu = \frac{\vec{r} \cdot \vec{B}}{r\,B} \qquad \text{(be careful of quadrant)}$$

- How do we find correct quadrant from cosine alone?
- From Vallado Equation 2-86,
  - If $\vec{r} \cdot \vec{v} < 0 \;\rightarrow\; \nu = 360° - \nu \quad \_or\_ \quad \nu = 2\pi - \nu$
  - depending on whether you're using degrees or radians

Homework 1

Due Saturday, 24 Jan 2026

Homework 1: Orbit Data

Vector 1

Component	Value (meters)
Position X	-464836.978606
Position Y	-6191644.716805
Position Z	-2961635.481039
Velocity X	7322.77235464
Velocity Y	406.01896116
Velocity Z	-1910.89281450

Vector 2

Component	Value (meters)
Position X	572461.711228
Position Y	-1015437.194396
Position Z	7707337.871302
Velocity X	-6195.262945
Velocity Y	-3575.889650
Velocity Z	-5.423283

Vector 3

Component	Value (meters)
Position X	-5142754.617115
Position Y	16130814.767566
Position Z	20434322.229790
Velocity X	-2924.287128
Velocity Y	-2303.326264
Velocity Z	1084.798834

Vector 4

Component	Value (meters)
Position X	-21100299.894024
Position Y	36462486.120500
Position Z	69117.555126
Velocity X	-2664.268125
Velocity Y	-1539.996659
Velocity Z	1.834442

Homework 1

- For each of the provided orbits compute:
  - $a, e, i, \Omega, \omega_p, \nu$
  - Orbit period TP
  - Apogee and perigee radii, r_a and r_p

- Let $\mu = 3.986004418 \times 10^{14} m^3/\text{sec}^2$