SPCE5025

Lecture 2

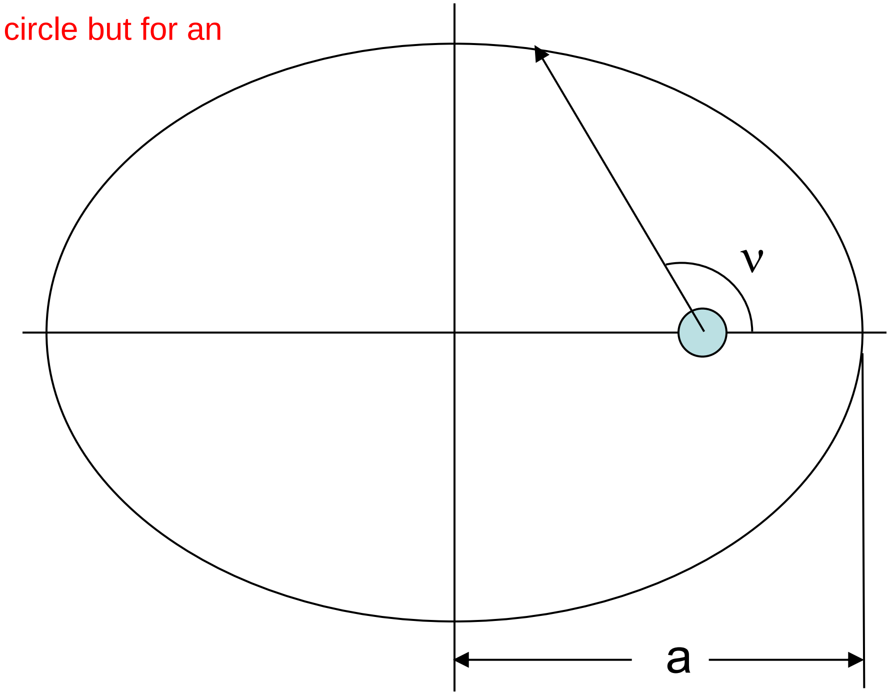
26 January 2026

Overview

- Kepler's Equation
- Kepler's Problem
- Orbit State Representations

Where we've been

- Last week covered the derivations of the two-body problem
- In particular,
  - a – semi-major axis a is similar to the radius of a circle but for an elliptical orbit
  - e – Eccentricity
  - ν - True Anomaly
    - Instantaneous location on the ellipse
    - … But the satellite is actually moving

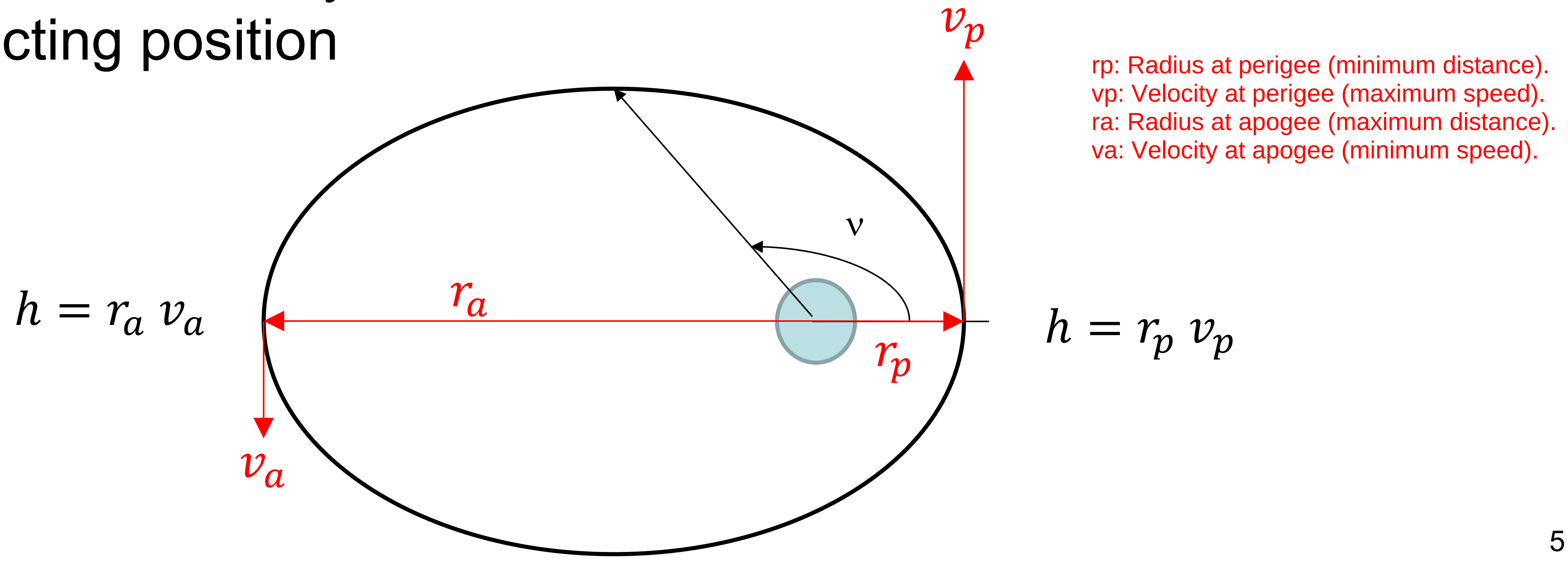


Motivating the Problem

- We know that the satellite moves on an ellipse with the central body at one focus
- Suppose we see the satellite now, and want to know where it will be some time later…..
  - True anomaly ν is not a static value – it changes as the satellite moves
- How do we figure out…
  - Where it will be after some Δt?
  - How long it will take to get from here to there?
  - Different, though closely-related problems

Motivating the Problem (2)

- Angular momentum is constant
  - But velocity is not constant
  - Therefore, angular rate $\dot{\nu}$ is not constant
- Need to find a way to account for the actual motion when predicting position



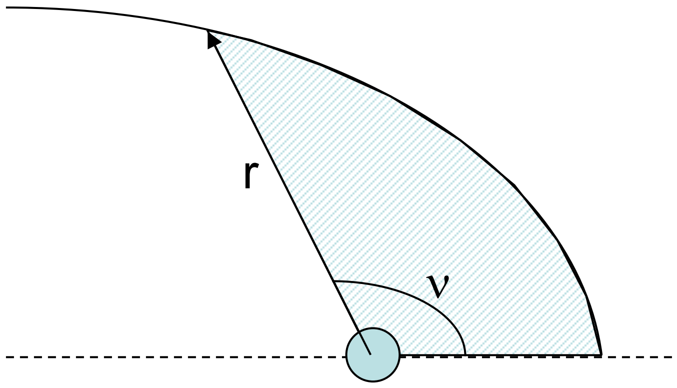
Next Step – Tying It to Time

- How long does it take to get from ν_o to ν_1?
- It's a tricky problem – analytic solution is not available
- Start out by looking at "Equal Areas in Equal Time" law
- Another way of looking at it:

$$\frac{\Delta t}{A} = \text{Constant} \quad \longrightarrow \quad \frac{\Delta t}{A} = \frac{TP}{\pi\,ab} \quad \left(= \frac{\text{Orbit Period}}{\text{Area of Ellipse}} \right)$$

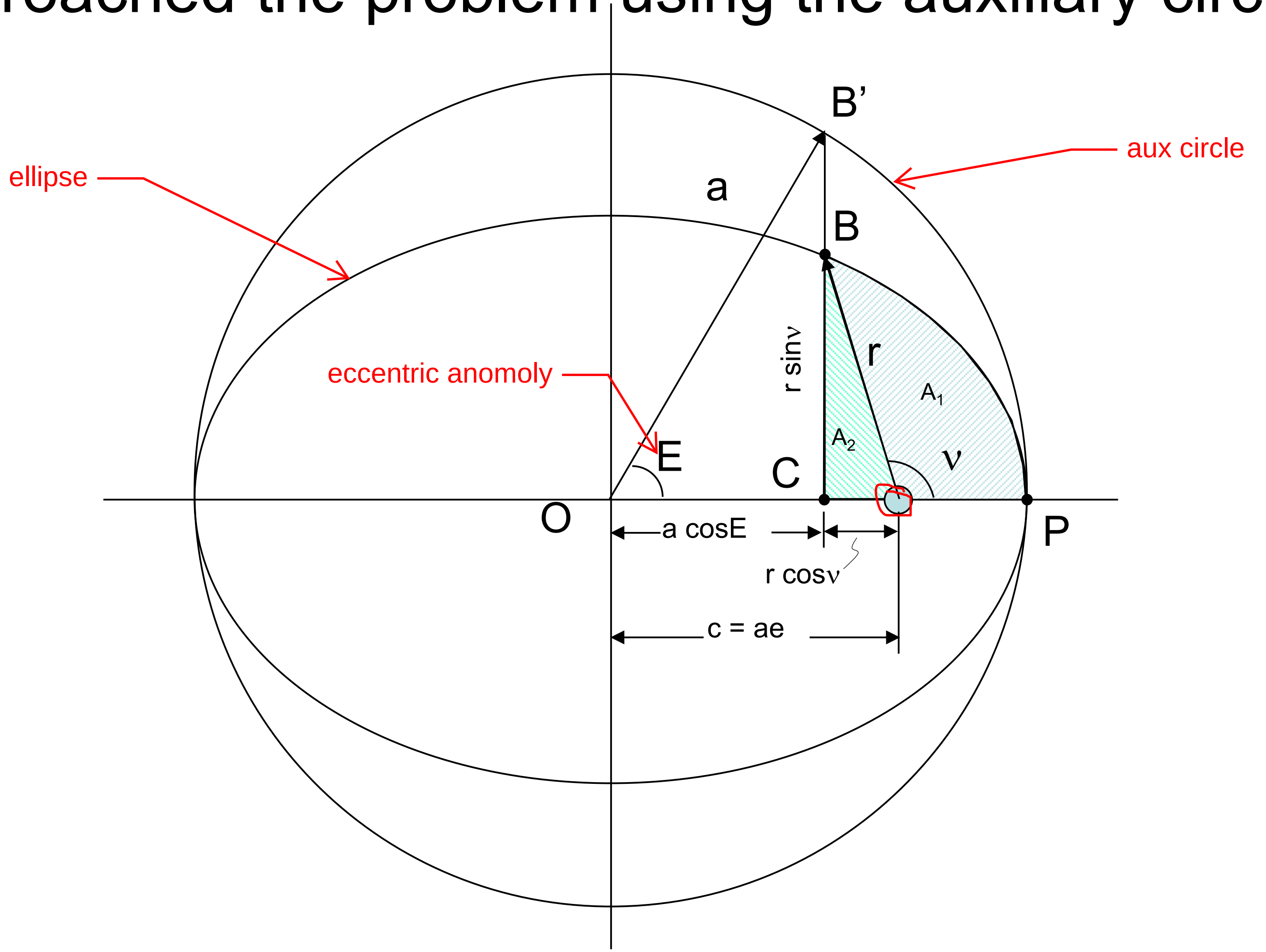
Question is: How to find A

- Finding area of an elliptical section as function of ν is difficult to do directly



Auxiliary Circle

- Kepler approached the problem using the auxiliary circle (Vallado fig. 2-2)



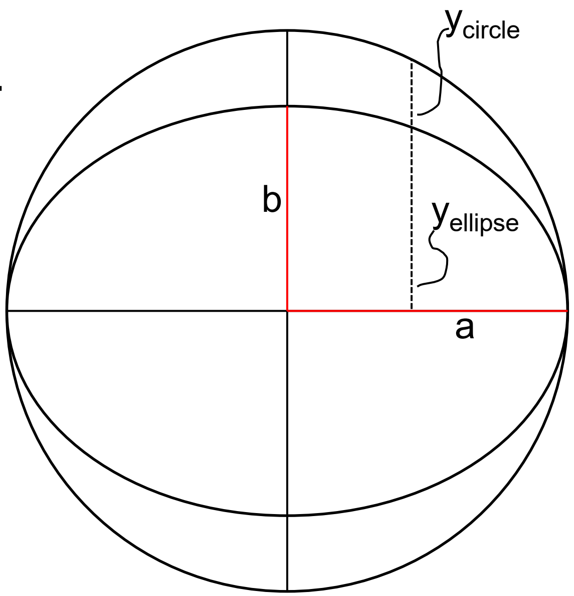
Ellipse and Circle Relationships

- Circle

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \longrightarrow y_{circle} = \sqrt{a^2 - x^2}$$

- Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow y_{ellipse} = \frac{b}{a}\sqrt{a^2 - x^2}$$



- Thus

$$\boxed{y_{ellipse} = \frac{b}{a} y_{circle}}$$

Area Relationships

$$\overline{CB} = \frac{b}{a}\overline{CB'}$$

$$\overline{CB'} = a\sin E$$

$$\overline{CB} = \frac{b}{a} a\sin E = b\sin E$$

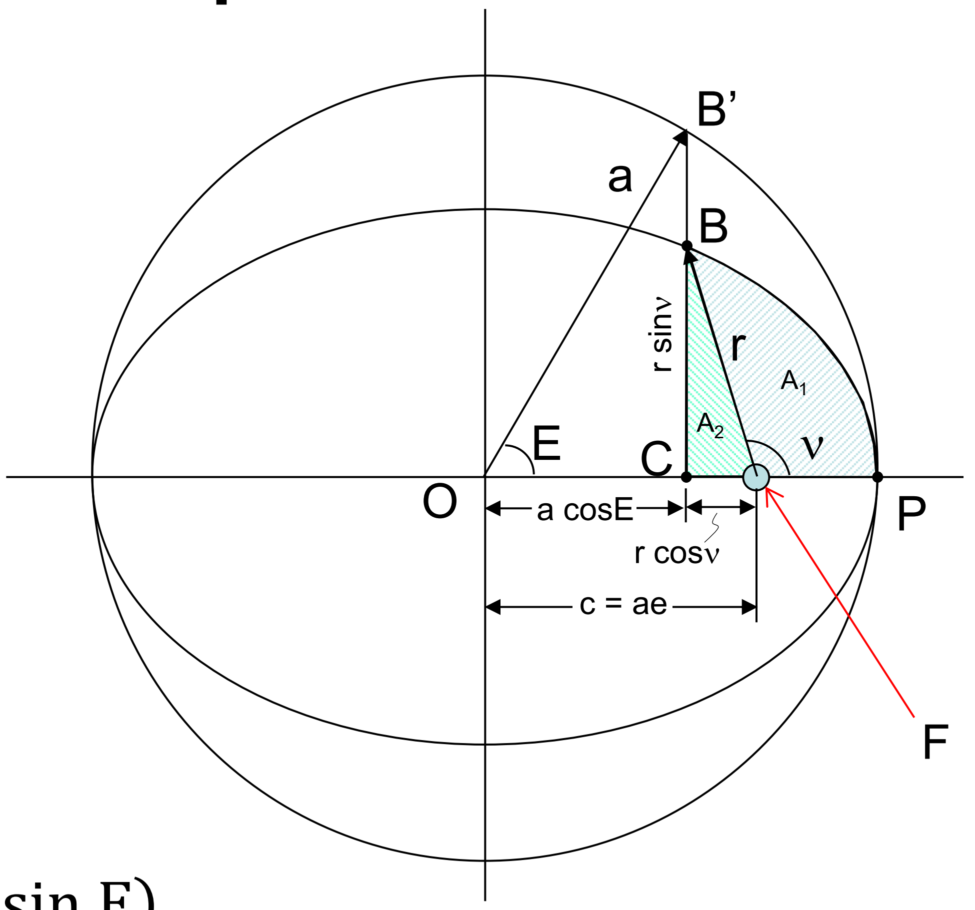
$$\overline{CF} = c - \overline{OC}$$

$$\overline{CF} = ae - a\cos E$$

$$A_1 = A_{PCB} - A_2$$

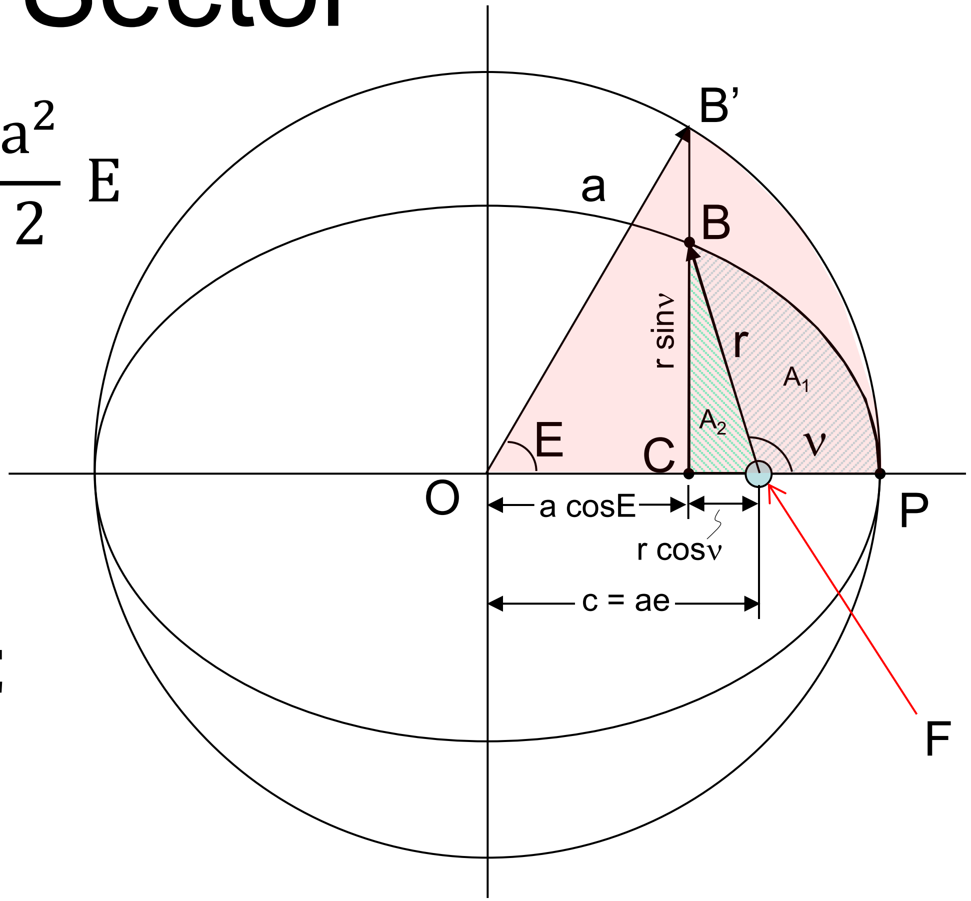
$$A_2 = \frac{1}{2}\,\text{base} \times \text{height} = \frac{1}{2}(ae - a\cos E)(b\sin E)$$

$$\boxed{A_2 = \frac{ab}{2}\left(e\sin E - \sin E\cos E\right)}$$



Area of Circular Sector

$$A_{sector} = \int_0^E \int_0^a r\,dr\,dE = \frac{a^2}{2} E$$



- Area of PCB':

$$A_{PCB'} = A_{sector} - A_{OCB'}$$

$$A_{OCB'} = \frac{1}{2}(a \sin E)\,(a \cos E) = \frac{a^2}{2} \sin E \cos E$$

$$\boxed{A_{PCB'} = \left(\frac{a^2 E}{2} - \frac{a^2}{2} \sin E \cos E\right)}$$

Note: For this to work, all angles are expressed in radians!

- Need to find corresponding area in ellipse

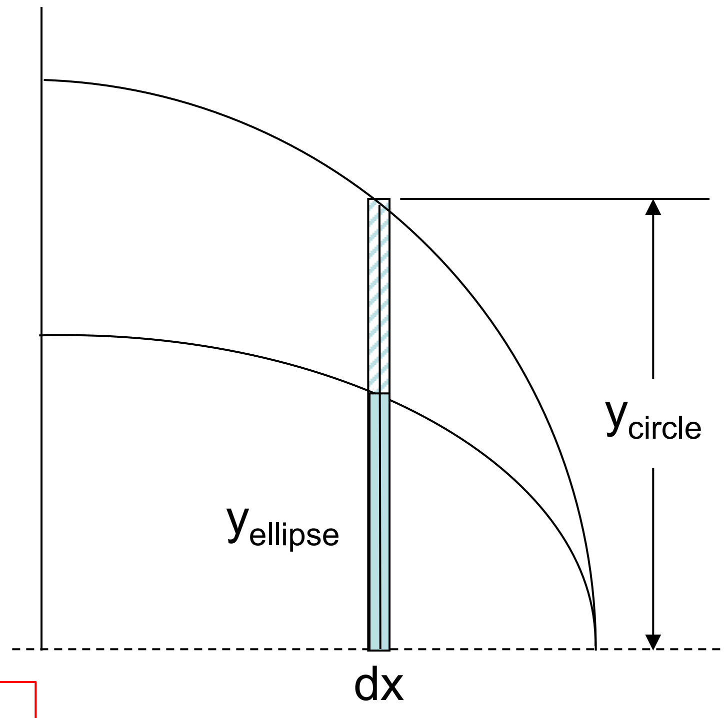
Relate area of Circle and Ellipse

$dA_{circle} = y_{circle}\ dx$

$dA_{ellipse} = y_{ellipse}\ dx$

$$y_{ellipse} = \frac{b}{a} y_{circle}$$

$$dA_{ellipse} = \frac{b}{a} dA_{circle}$$

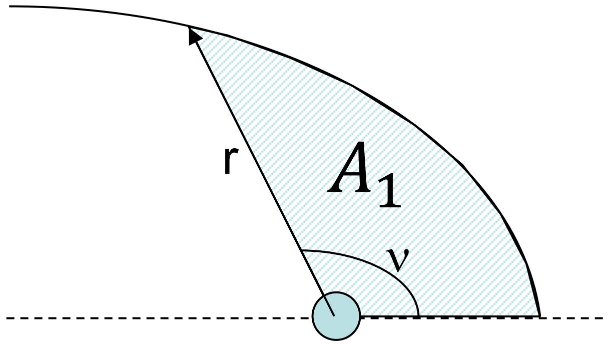


$$\boxed{A_{ellipse} = \frac{b}{a} A_{circle}}$$

Scale Area by b/a

$$A_1 = A_{PCB} - A_2$$

$$\boxed{A_{PCB} = \frac{b}{a}A_{PCB'} = \frac{b}{a}\left(\frac{a^2E}{2} - \frac{a^2}{2}\sin E\cos E\right)}$$



$$A_2 = \frac{ab}{2}(e\sin E - \sin E\cos E)$$

$$A_{PCB} - A_2 = \frac{b}{a}\left(\frac{a^2E}{2} - \frac{a^2}{2}\sin E\cos E\right) - \frac{ab}{2}(e\sin E - \sin E\cos E)$$

$$= \frac{ab\,E}{2} - \cancel{\frac{ab}{2}\sin E\cos E} - \frac{ab}{2}e\sin E + \cancel{\frac{ab}{2}\sin E\cos E}$$

$$\boxed{A_1 = \frac{ab}{2}(E - e\sin E)}$$

Relating Area to Time

$$\frac{\Delta t}{A_1} = \frac{TP}{\pi\, ab} \quad \left(= \frac{\text{Orbit Period}}{\text{Area of Ellipse}} \right)$$

$$\frac{\Delta t}{\frac{ab}{2}\,(E - e\sin E)} = \frac{TP}{\pi\, ab} \Longrightarrow \frac{2\pi\,\Delta t}{\cancel{ab}\,(E - e\sin E)} = \frac{TP}{\cancel{ab}}$$

$$\frac{2\pi\,\Delta t}{(E - e\sin E)} = TP \Longrightarrow TP = 2\pi\sqrt{\frac{a^3}{\mu}}$$

- $\Delta t = (t_1 - T)$, time from perigee to E

$$\boxed{\Delta t = \sqrt{\frac{a^3}{\mu}}\,(E - e\sin E)}$$

Standard Expression

- Time from perigee to Eccentric Anomaly E

$$(t_1 - T) = \sqrt{\frac{a^3}{\mu}}(E - e\sin E)$$

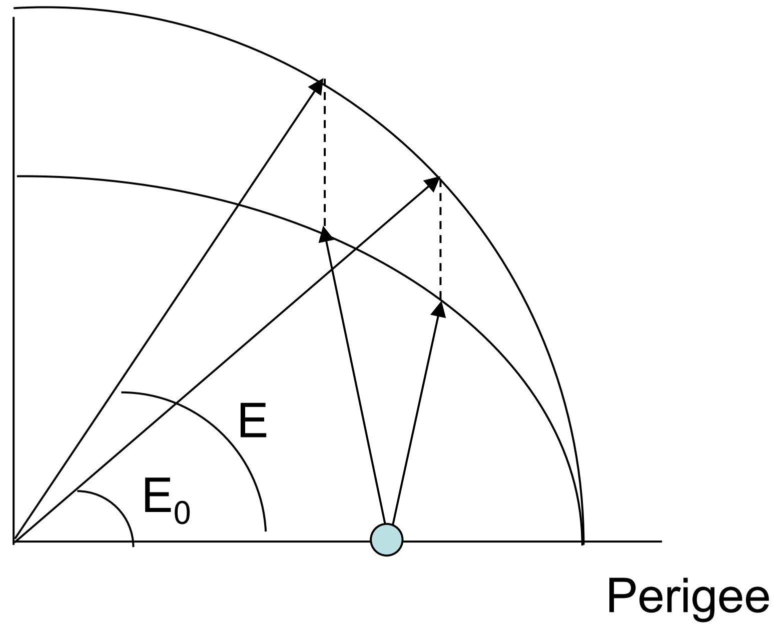
$$n = \sqrt{\frac{\mu}{a^3}} \quad \text{(mean motion)}$$

$$M = (E - e\sin E) \quad \text{(mean anomaly)}$$

$$\boxed{n\,(t_1 - T) = M}$$

Going from E_o to E

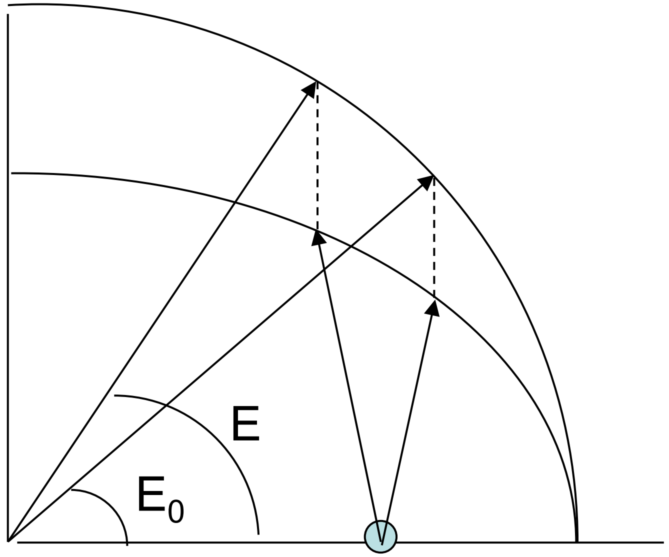
- Look at generic case, not starting at perigee
  - Going from E_o to E = Perigee to E − Perigee to E_o
- Let
  - T = Time of last perigee passage
  - t_o = Time from perigee to E_o
  - t = Time from perigee to E
  - k = perigee passages between E_o and E
  - $k \cdot TP$ = number of full orbits from epoch



General Time of Flight Equation

- Given
  - T = time of perigee just prior to stopping point
  - k perigee crossings between t_o and t

$$t - t_o = k\,TP + (t - T) - (t_o - T)$$



$$\boxed{t - t_o = k\,TP + \frac{1}{n}(E - e\sin E) - \frac{1}{n}(E_0 - e\sin E_0)}$$

Relating E to ν

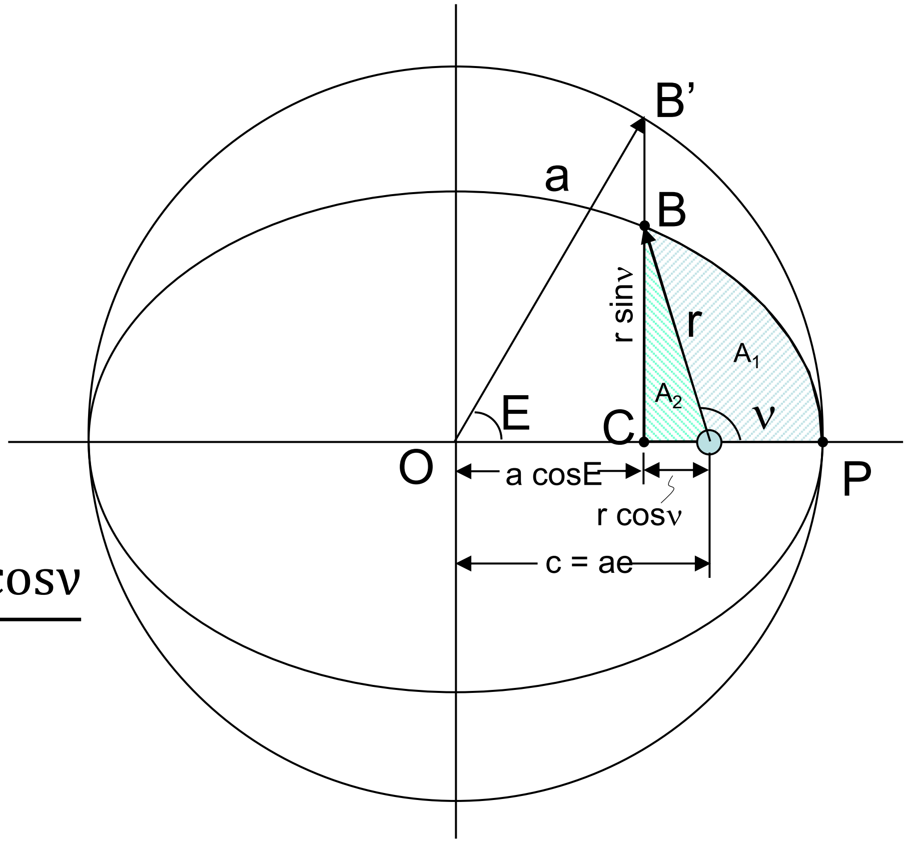
$$a\cos E = ae + r\cos\nu$$

$$\cos E = \frac{ae + r\cos\nu}{a}$$

$$r = \frac{a(1-e^2)}{1+e\cos\nu} \quad \text{(ellipse eqn)}$$

$$\cos E = \frac{\cancel{a}e + \frac{\cancel{a}(1-e^2)}{1+e\cos\nu}\cos\nu}{\cancel{a}}$$

$$\cos E = \frac{e(1+e\cos\nu) + (1-e^2)\cos\nu}{1+e\cos\nu}$$



$$\cos E = \frac{e + \cancel{e^2\cos\nu} + \cos\nu - \cancel{e^2\cos\nu}}{1+e\cos\nu} \Longrightarrow \boxed{\cos E = \frac{e+\cos\nu}{1+e\cos\nu}}$$

Solve for ν

$$\boxed{\cos E = \frac{e + \cos\nu}{1 + e\cos\nu}} \Longrightarrow \boxed{\cos\nu = \frac{e - \cos E}{e\cos E - 1}}$$

From the auxiliary circle

$$b\sin E = r\sin\nu \qquad b^2 = a^2 - c^2 \qquad c = ae$$

$$b^2 = a^2 - a^2e^2 = a^2(1 - e^2) \Longrightarrow b = a\sqrt{1 - e^2}$$

$$a\sqrt{1 - e^2}\sin E = r\sin\nu$$

$$\boxed{\sin E = \frac{\sin\nu\sqrt{1 - e^2}}{1 + e\cos\nu}} \Longrightarrow \boxed{\sin\nu = \frac{\sin E\sqrt{1 - e^2}}{1 - e\cos E}}$$

$$\boxed{E = ATAN2(\sin E,\ \cos E)} \qquad \boxed{\nu = ATAN2(\sin\nu,\ \cos\nu)}$$

We could use these to express Time of Flight in terms of ν, but it's easier to convert known values of ν to E, and use the Time of Flight equation already derived

Eccentric Anomaly from Vectors

- Eccentric anomaly can be computed directly from position/velocity

$$N_E = \frac{\vec{r} \cdot \vec{v}}{\sqrt{\mu a}}$$

$$D_E = 1 - \frac{r}{a}$$

$$E = \mathrm{atan2}(N_E, D_E)$$

Expressions taken from Space Shuttle Mission Control math specs. Not derived in this class.

Finding E from Time of Flight

- Starting at Perigee:
  - $n\,(t - T) = E - e\,sinE$ (from perigee)
  - How do we find E?

- No analytic solution, can find by iteration

- Commonly use Newton-Raphson iteration:

$$\mathrm{x_{n+1}} = \mathrm{x_n} - \frac{\mathrm{f(x)}}{\mathrm{f'(x)}}$$

Iteration for E (Starting at Perigee)

$$f(E_k) = n(t - T) - (E_k - e \sin E_k)$$

$$f'(E_k) = -(1 - e \cos E_k)$$

$$n\,(t - T) = M \qquad \text{(fixed)}$$

$$E_{k+1} = E_k + \frac{M - (E_k - e \sin E_k)}{1 - e \cos E_k}$$

Continue until $|E_{k+1} - Ek| <$TOL

Specify iteration starting value:

$$E_o = M \qquad \text{should usually work}$$

Alternate Solution (Biondini)

- Let:

$$M = E - e\sin E$$
$$E = M + e\sin E$$

- Take sine of both sides:

$$\sin E = \sin(M + e\sin E)$$

$$x = \sin E$$
$$x_{i+1} = \sin(M + e x_i)$$
$$E_{i+1} = \sin^{-1}(M + e x_{i+1})$$

- Repeat until

$$\Delta x = |x_{i+1} - x_i| \rightarrow 0$$

General Form for Time of Flight

- Not starting at perigee, setup for time of flight from t_0 to t
- General Time of Flight Equation

$$\mathrm{t} - \mathrm{t_o} = \mathrm{k\,TP} + \frac{1}{\mathrm{n}}\underbrace{(\mathrm{E} - \mathrm{e}\sin\mathrm{E})}_{\text{M}} - \frac{1}{\mathrm{n}}\underbrace{(\mathrm{E_0} - \mathrm{e}\sin\mathrm{E_0})}_{\text{Mo}}$$

  - k = number of perigee crossings between E_0 and E

- General Iteration function

$$\mathrm{f(E_k)} = \mathrm{n(t - T)} - (\mathrm{E_k} - \mathrm{e}\sin\mathrm{E_k}) + (\mathrm{E_0} - \mathrm{e}\sin\mathrm{E_0})$$

  - Note: the $(k\,TP)$ term is not included: this version allows $E_k > 2\pi$

Newton-Raphson Iteration for E

- As before

$$n\,(t - T) = n\Delta t = \text{fixed}$$

$$M_0 = (E_0 - e\,\sin E_0) = \text{fixed}$$

$$f'(E_k) = -(1 - e\,\cos E_k)$$

$$\boxed{E_{k+1} = E_k + \frac{n\Delta t + M_0 - (E_k - e\,\sin E_k)}{1 - e\,\cos E_k}}$$

Specify iteration starting value:

$$E_o = M_o \qquad \text{should usually work}$$

Note on the Iteration

- Perigee passages are not included in the iteration,

$$\mathrm{E}_{\mathrm{k}+1} = \mathrm{E}_{\mathrm{k}} + \frac{\mathrm{n}\Delta\mathrm{t} + \mathrm{M}_0 - (\mathrm{E}_{\mathrm{k}} - \mathrm{e}\sin\mathrm{E}_{\mathrm{k}})}{1 - \mathrm{e}\cos\mathrm{E}_{\mathrm{k}}}$$

- If k > 1, result will be a value of E > 2π
  - Take $mod(E, 2\pi)$ to get E in $[0, 2\pi]$
  - Can compute k by

$$\mathrm{k} = \mathrm{floor}(\,(\mathrm{E} - \mathrm{E}_o)\ /\ 2\pi\,)$$

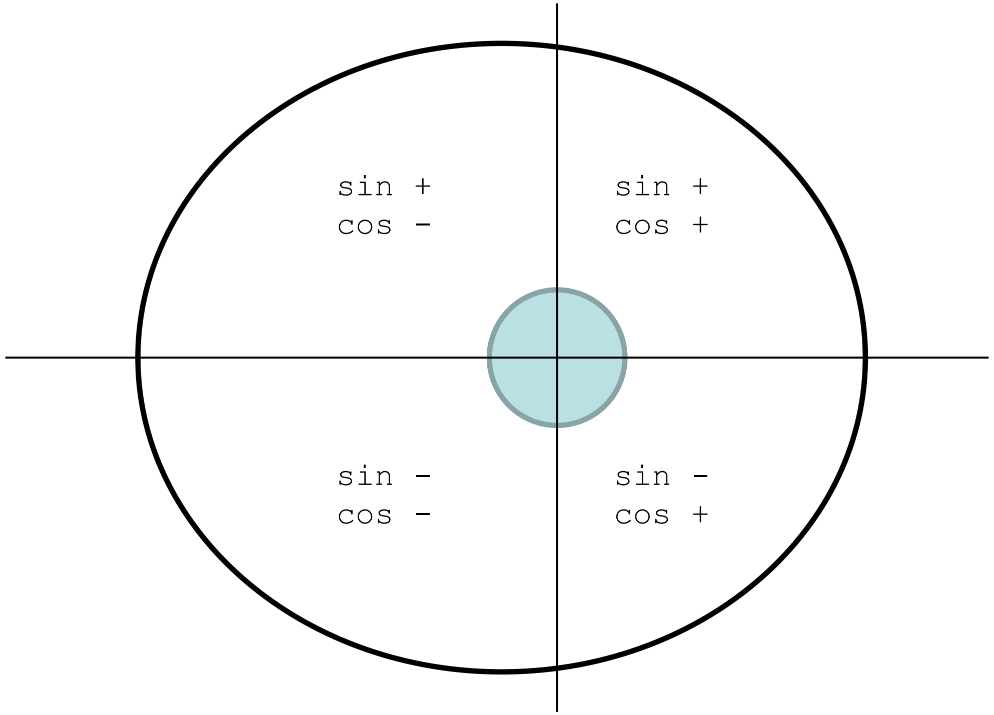
Getting Right Quadrant for ν

- Use solved-for value of E to get ν

$$\cos\nu = \frac{e - \cos E}{e\cos E - 1}$$

$$\sin\nu = \frac{\sin E\sqrt{1-e^2}}{1 - e\cos E}$$

$$\nu = \operatorname{atan2}(\sin\nu, \cos\nu)$$



- atan2 returns angle in range [$-\pi$, π]
  - If angle < 0, add 2π to get range [0, 2π]

An Aside: Planar Motion

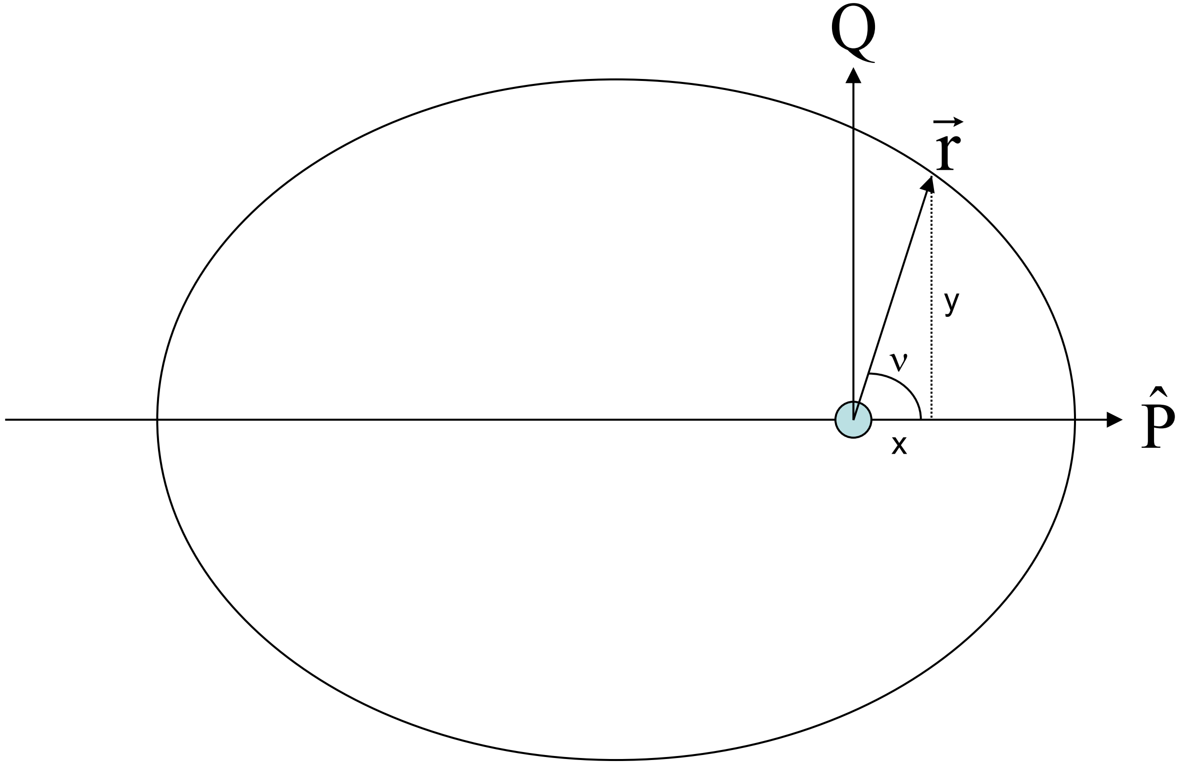
- Things we know:

$$\vec{h} = \vec{r} \times \vec{v}$$
$$\vec{h} \perp \vec{r} \text{ and } \vec{v}$$
$$\vec{h} = \text{constant}$$

- Therefore:
  - Satellite motion is always in the plane containing $\vec{r}$ and $\vec{v}$
  - For 2-body problem, can treat prediction problem as a problem in planar motion
    - Don't worry about orientation of orbit in space (yet)

The Kepler Problem

- Orbit elements are constant functions of $\vec{r}$ and $\vec{v}$
- Look at orbit in perifocal frame:
  - Motion in orbit plane



$$\vec{r} = x\,\hat{P} + y\,\hat{Q} + 0\,\hat{W}$$

$$\dot{\vec{r}} = \dot{x}\,\hat{P} + \dot{y}\,\hat{Q} + 0\,\hat{W}$$

$$(\hat{P}, \hat{Q}, \hat{W} \text{ are inertial, so } d/dt = 0)$$

Prediction

- Because the orbit elements are constant, it is reasonable to suppose that future states can be found from current states:

$$\vec{r} = f\,\vec{r}_0 + g\,\dot{\vec{r}}_0$$

$$\dot{\vec{r}} = \dot{f}\,\vec{r}_0 + \dot{g}\,\dot{\vec{r}}_0$$

$$(\vec{r}_0,\ \dot{\vec{r}}_0 \text{ are fixed, so } d/dt = 0)$$

- Need to find expressions for f, g, etc.

Find f

- Post-cross multiply both sides by $\dot{\vec{r}}_0$

$$\vec{r} \times \dot{\vec{r}}_0 = f\,\vec{r}_0 \times \dot{\vec{r}}_0 + \cancel{g\,\dot{\vec{r}}_0 \times \dot{\vec{r}}_0} \rightarrow 0$$

$$\vec{r} \times \dot{\vec{r}}_0 = f\,\vec{r}_0 \times \dot{\vec{r}}_0 = f\,\vec{h}$$

$$\vec{r} \times \dot{\vec{r}}_0 = \begin{vmatrix} \hat{P} & \hat{Q} & \hat{W} \\ x & y & 0 \\ \dot{x}_0 & \dot{y}_0 & 0 \end{vmatrix} = (x\dot{y}_0 - \dot{x}_0 y)\,\hat{W}$$

$\boxed{\text{Note: } \vec{h} = h\,\hat{W}}$

$$\left(x\dot{y}_0 - \dot{x}_0 y\right)\cancel{\hat{W}} = f\,h\,\cancel{\hat{W}} \longrightarrow \left(x\dot{y}_0 - \dot{x}_0 y\right) = f\,h$$

$$\boxed{f = \frac{(x\dot{y}_0 - \dot{x}_0 y)}{h}}$$

Find g

- Pre-cross multiply both sides by $\vec{r}_o$

$$\vec{r}_0 \times \vec{r} = f\,\underbrace{\cancel{\vec{r}_0 \times \vec{r}_0}}_{0} + g\,\vec{r}_0 \times \dot{\vec{r}}_0$$

$$\vec{r}_0 \times \vec{r} = g\,\vec{r}_0 \times \dot{\vec{r}}_0 = g\,\vec{h}$$

$$\vec{r}_0 \times \vec{r} = \begin{vmatrix} \hat{P} & \hat{Q} & \hat{W} \\ x_0 & y_0 & 0 \\ x & y & 0 \end{vmatrix} = (x_0 y - x y_0)\,\hat{W}$$

Similar to above:

$$\boxed{g = \frac{(x_0 y - x y_0)}{h}}$$

Time derivatives of f and g

$$\dot{f} = \frac{d}{dt}\left(\frac{(x\dot{y}_0 - \dot{x}_0 y)}{h}\right) = \frac{(\dot{x}\dot{y}_0 - \dot{x}_0\dot{y})}{h}$$

(x_0 and y_0 are fixed)

$$\dot{g} = \frac{d}{dt}\left(\frac{(x_0 y - x y_0)}{h}\right) = \frac{(x_0\dot{y} - \dot{x}y_0)}{h}$$

Express in terms of ν

$$x = r\cos\nu \qquad y = r\sin\nu$$

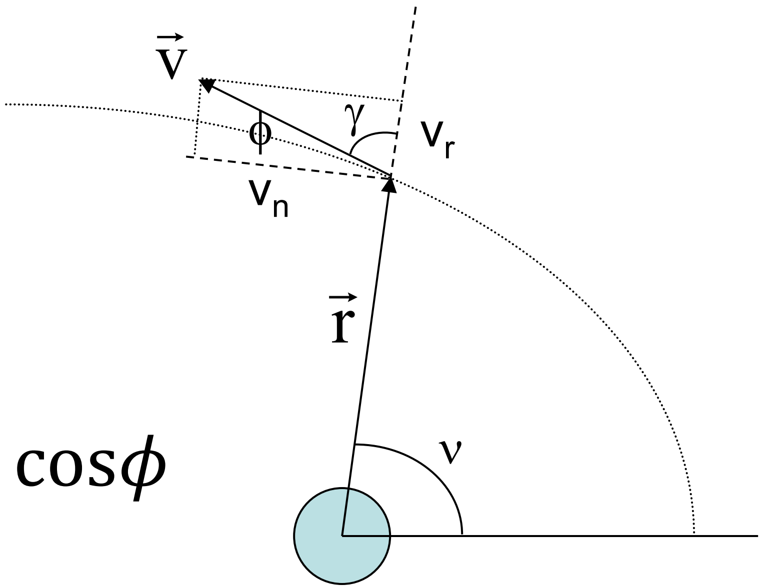
$$\dot{x} = \dot{r}\cos\nu - r\sin\nu\,\dot{\nu}$$
$$\dot{y} = \dot{r}\sin\nu + r\cos\nu\,\dot{\nu}$$

$$v_n = |\vec{v}|\cos\phi = r\,\dot{\nu}$$

$$|\vec{h}| = |\vec{r}\times\vec{v}| = rv\sin\gamma = rv\cos\phi$$

$$h = r\,(r\,\dot{\nu}) = r^2\dot{\nu}$$

$$\boxed{r\,\dot{\nu} = \frac{h}{r}}$$



$$r = \frac{a\,(1-e^2)}{1 + e\cos\nu} = \frac{p}{1 + e\cos\nu} \qquad \dot{r} = -\frac{p}{(1 + e\cos\nu)^2}(-e\sin\nu)\,\dot{\nu}$$

$$\dot{r} = \frac{\cancel{p}^{\,r}}{\cancel{(1 + e\cos\nu)}}\frac{(e\sin\nu)\,\dot{\nu}}{(1 + e\cos\nu)} = \frac{re\sin\nu\,\dot{\nu}}{(1 + e\cos\nu)}$$

$$\dot{r}\cos\nu = \frac{re\cos\nu\sin\nu\,\dot{\nu}}{(1 + e\cos\nu)}$$

$$\dot{x} = \frac{re\cos\nu\sin\nu\,\dot{\nu}}{(1 + e\cos\nu)} - r\sin\nu\,\dot{\nu} = \frac{re\cos\nu\sin\nu\,\dot{\nu}}{(1 + e\cos\nu)} - \frac{r\sin\nu(1 + e\cos\nu)\,\dot{\nu}}{(1 + e\cos\nu)}$$

$$\dot{x} = \frac{\cancel{re\cos\nu\sin\nu\,\dot{\nu}} - r\sin\nu\,\dot{\nu} - \cancel{re\cos\nu\sin\nu\,\dot{\nu}}}{(1 + e\cos\nu)} \Longrightarrow \boxed{\dot{x} = \frac{-r\sin\nu\,\dot{\nu}}{(1 + e\cos\nu)}}$$

Solving…

$$\dot{x} = \frac{-r\sin\nu\,\dot{\nu}}{(1 + e\cos\nu)}$$

$$r\,\dot{\nu} = \frac{h}{r} \Longrightarrow \dot{x} = \frac{-h\sin\nu}{r\,(1 + e\cos\nu)}$$

$$p = r\,(1 + e\cos\nu) = \frac{h^2}{\mu} \Longrightarrow h = \sqrt{\mu p}$$

$$\dot{x} = \frac{-\sqrt{\mu p}\,\sin\nu}{p} \Longrightarrow \boxed{\dot{x} = -\sqrt{\frac{\mu}{p}}\sin\nu}$$

Similarly for $\dot{y}$

$$\dot{y} = \dot{r}\sin\nu + r\cos\nu\,\dot{\nu}$$

$$r = p\,(1 + e\cos\nu)^{-1}$$

$$\dot{r} = -p\,(1 + e\cos\nu)^{-2}(-e\sin\nu)\,\dot{\nu}$$

$$\dot{r} = \frac{p\,e\sin\nu}{(1 + e\cos\nu)^2}\dot{\nu}$$

$$\dot{r} = \frac{r^2\,e\sin\nu}{p}\dot{\nu}$$

$$h = r^2\dot{\nu}$$

$$\dot{r} = \frac{h}{p}e\sin\nu$$

$$h = r(r\,\dot{\nu}) = \sqrt{\mu p}$$

$$r\,\dot{\nu} = \frac{\sqrt{\mu p}}{r}$$

$$r\,\dot{\nu} = \frac{\sqrt{\mu p}\,(1 + e\cos\nu)}{p}$$

$$r\,\dot{\nu} = \sqrt{\frac{\mu}{p}}(1 + e\cos\nu)$$

$$\boxed{\dot{r} = \sqrt{\frac{\mu}{p}}e\sin\nu}$$

Combine

$$\dot{y} = \dot{r}\sin\nu + r\cos\nu\,\dot{\nu}$$

$$\dot{y} = \sqrt{\frac{\mu}{p}}\,e\sin^2\nu + \sqrt{\frac{\mu}{p}}(1 + e\cos\nu)\cos\nu$$

$$\dot{y} = \sqrt{\frac{\mu}{p}}(e\,[\sin^2\nu + \cos^2\nu] + \cos\nu)$$

$$\dot{y} = \sqrt{\frac{\mu}{p}}(e\sin^2\nu + \cos\nu + e\cos^2\nu) \Longrightarrow \boxed{\dot{y} = \sqrt{\frac{\mu}{p}}(e + \cos\nu)}$$

Summary so far

$$x = r\cos\nu \qquad y = r\sin\nu$$

$$\dot{x} = -\sqrt{\frac{\mu}{p}}\sin\nu \qquad \dot{y} = \sqrt{\frac{\mu}{p}}(e + \cos\nu)$$

Perifocal Position and Velocity Components

$$f = \frac{(x\dot{y}_0 - \dot{x}_0 y)}{h} \qquad g = \frac{(x_0 y - x y_0)}{h}$$

$$\dot{f} = \frac{(\dot{x}\dot{y}_0 - \dot{x}_0\dot{y})}{h} \qquad \dot{g} = \frac{(x_0\dot{y} - \dot{x}y_0)}{h}$$

$$f\dot{g} - \dot{f}g = 1$$

Combine to find expressions in ν

Example: find f

$$f = \frac{(x\dot{y}_0 - \dot{x}_0 y)}{h}$$

$$f = \frac{\left(r\cos\nu \sqrt{\frac{\mu}{p}}(e + \cos\nu_0) + \sqrt{\frac{\mu}{p}}\sin\nu_0 \, r\sin\nu \right)}{\sqrt{\mu p}}$$

$$f = \frac{r(\cos\nu\,(e + \cos\nu_0) + \sin\nu_0 \sin\nu)}{p}$$

$$f = \frac{r(e\cos\nu + \cos\nu\cos\nu_0 + \sin\nu_0 \sin\nu)}{p}$$

Recall the trig identity…

$$\cos A \cos B \pm \sin A \sin B = \cos(A \mp B)$$

$$f = \frac{r(e\cos\nu + \cos(\nu - \nu_0))}{p} = \frac{r(e\cos\nu + \cos\Delta\nu)}{p}$$

Add and subtract…

$$\frac{r}{p} - \frac{r}{p} + \frac{r\,e\cos\nu}{p} = -\frac{r}{p} + \frac{r}{p}(1 + e\cos\nu)$$

$$p = r(1 + e\cos\nu)$$

$$-\frac{r}{p} + \frac{r}{p}(1 + e\cos\nu) = -\frac{r}{p} + 1$$

$$\frac{r\,e\cos\nu}{p} = 1 - \frac{r}{p}$$

$$f = 1 - \frac{r}{p} + \frac{r}{p}\cos\Delta\nu$$

$$\boxed{f = 1 - \frac{r}{p}(1 - \cos\Delta\nu)}$$

Similarly

$$\boxed{g = \frac{r\, r_0}{\sqrt{\mu p}} \sin\Delta\nu}$$

$$\boxed{\dot{g} = 1 - \frac{r_0}{p}(1 - \cos\Delta\nu)}$$

Using $f\dot{g} - \dot{f}g = 1$:

$$\boxed{\dot{f} = \sqrt{\frac{\mu}{p}}\, \tan\frac{\Delta\nu}{2}\left(\frac{1 - \cos\Delta\nu}{p} - \frac{1}{r} - \frac{1}{r_o}\right)}$$

The f&g Functions: So What?

- The f&g functions: undeniably cool, but rather esoteric-seeming
- Does anybody actually use them?

- Answer: yes!
  - Cheap lower-fidelity orbit propagation
    - Can extend to non-two-body formulations
  - Commonly used for initial orbit determination (orbits from observations)
  - Basis of “Lambert Targeting” methods for rendezvous/proximity ops
  - Provides analytic solution for orbit determination State Transition Matrix
  - Universal variable formulation generalizes to all conic sections
  - And more!

Homework 2

Due Saturday, 31 Jan 2026

Homework 2 (1)

Given:

Component	Value (meters)
Position X	326151.080726
Position Y	6077471.251787
Position Z	2944583.918767
Velocity X	-7455.178720
Velocity Y	-482.482572
Velocity Z	1910.883434

- Find E_o (slide 19)
- Find ν_o (Keplerian comps)
- Find M_o
- Find time of flight from perigee to ν_o

- Find time of flight from ν_o to $\nu = 65°$
- Verify: Starting at ν_o, compute the true anomaly after the time of flight computed above.
- Starting at ν_o, what is the true anomaly after 2700 seconds?
- Starting at ν_o, what is the true anomaly after exactly two orbit periods?
- What is the true anomaly after 15000 seconds?

Homework 2 (2)

Given:

Component	Value (meters)
Position X	572461.711228
Position Y	-1015437.194396
Position Z	7707337.871302
Velocity X	-6195.262945
Velocity Y	-3575.889650
Velocity Z	-5.423283

- Compute Keplerian elements
- Find perifocal position and velocity, $\vec{r}_o$ and $\vec{v}_o$
  - See equations on Slide 39
- Find $f, g, \dot{f}, \dot{g}$ for $\Delta\nu = 33°$
- Find $\vec{r}$ and $\vec{v}$

Homework 2 (3)

- Derivation 1: Using energy equation and ellipse equation, find an expression for the orbit speed as a function of true anomaly
- Derivation 2: Using the above, show that

$$v_{perigee} = \sqrt{\frac{\mu}{a}\left(\frac{1+e}{1-e}\right)}$$

$$v_{apogee} = \sqrt{\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)}$$