SPCE5025

Lecture 2

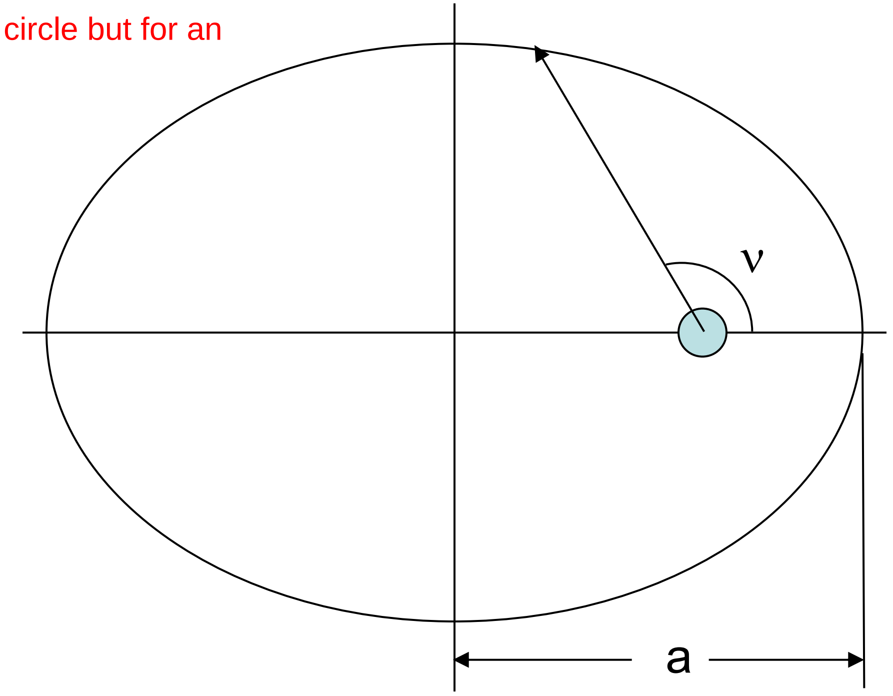
26 January 2026

Overview

- Kepler's Equation
- Kepler's Problem
- Orbit State Representations

Where we've been

- Last week covered the derivations of the two-body problem
- In particular,
  - a – semi-major axis a is similar to the radius of a circle but for an elliptical orbit
  - e – Eccentricity
  - ν - True Anomaly
    - Instantaneous location on the ellipse
    - … But the satellite is actually moving

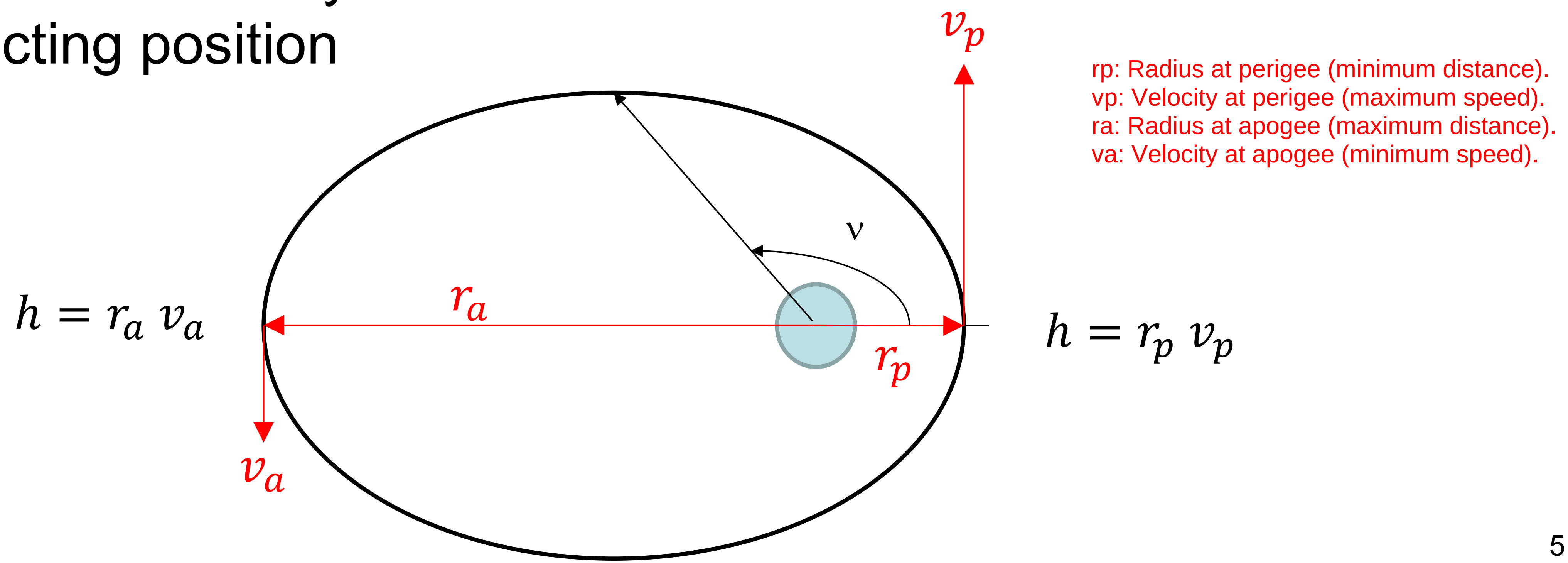


Motivating the Problem

- We know that the satellite moves on an ellipse with the central body at one focus
- Suppose we see the satellite now, and want to know where it will be some time later…..
  - True anomaly ν is not a static value – it changes as the satellite moves
- How do we figure out…
  - Where it will be after some Δt?
  - How long it will take to get from here to there?
  - Different, though closely-related problems

Motivating the Problem (2)

- Angular momentum is constant
  - But velocity is not constant
  - Therefore, angular rate $\dot{\nu}$ is not constant
- Need to find a way to account for the actual motion when predicting position



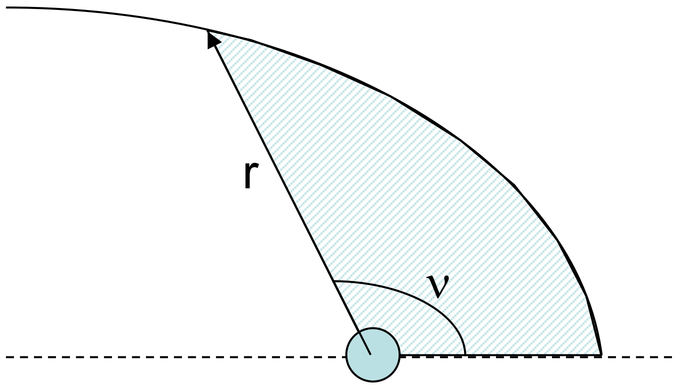
Next Step – Tying It to Time

- How long does it take to get from ν_o to ν_1?
- It's a tricky problem – analytic solution is not available
- Start out by looking at "Equal Areas in Equal Time" law
- Another way of looking at it:

$$\frac{\Delta t}{A} = \text{Constant} \longrightarrow \frac{\Delta t}{A} = \frac{TP}{\pi\ ab} \quad \left(= \frac{\text{Orbit Period}}{\text{Area of Ellipse}} \right)$$

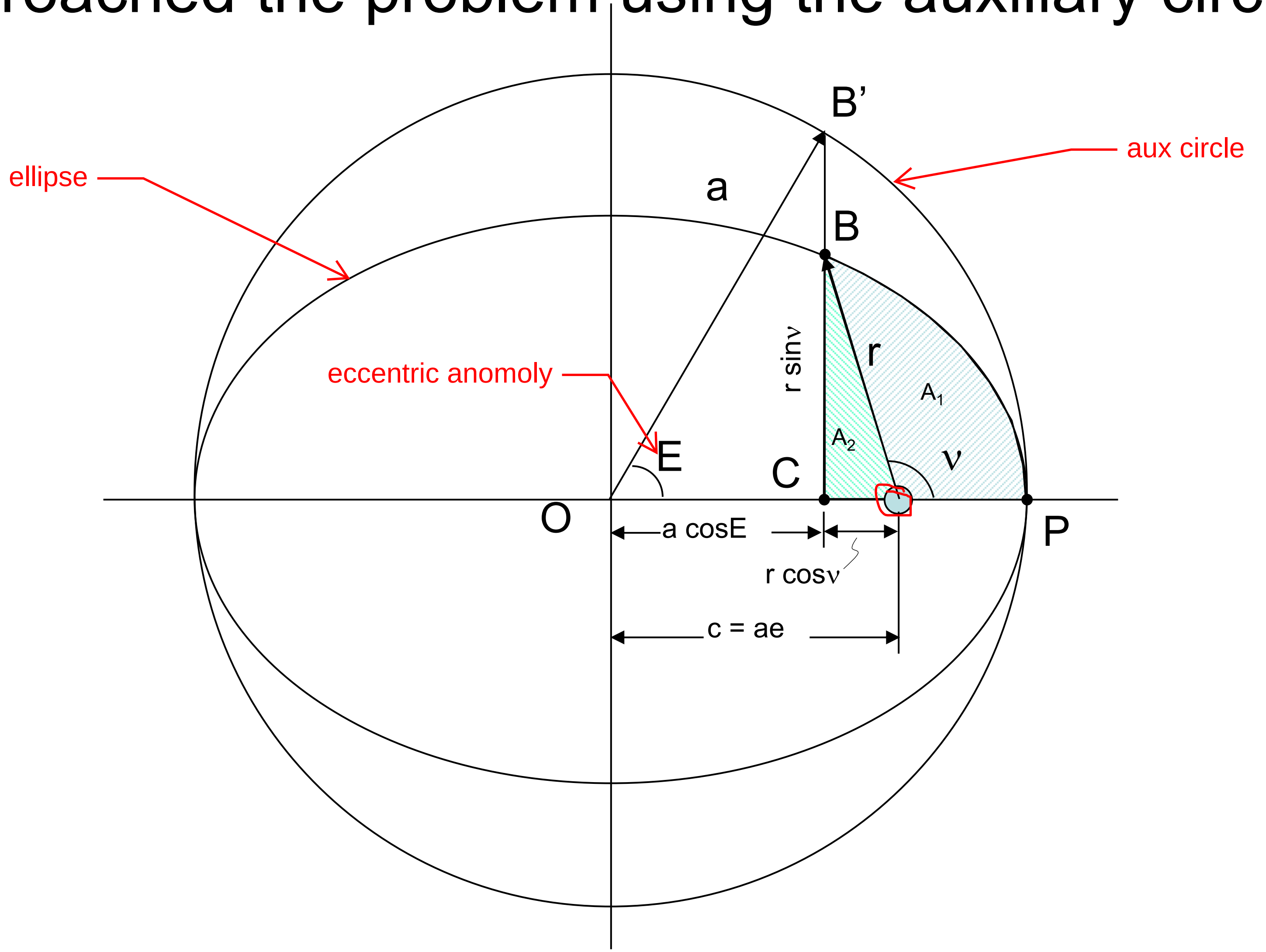
Question is: How to find A

- Finding area of an elliptical section as function of ν is difficult to do directly



Auxiliary Circle

- Kepler approached the problem using the auxiliary circle (Vallado fig. 2-2)



Ellipse and Circle Relationships

- Circle

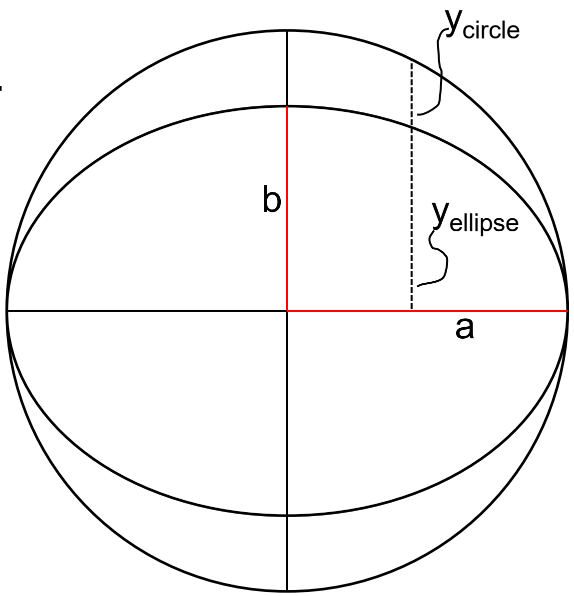
$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \longrightarrow y_{circle} = \sqrt{a^2 - x^2}$$

- Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow y_{ellipse} = \frac{b}{a}\sqrt{a^2 - x^2}$$

- Thus

$$\boxed{y_{ellipse} = \frac{b}{a} y_{circle}}$$



Area Relationships

$$\overline{CB} = \frac{b}{a}\overline{CB'}$$

$$\overline{CB'} = a\sin E$$

$$\overline{CB} = \frac{b}{a} a\sin E = b\sin E$$

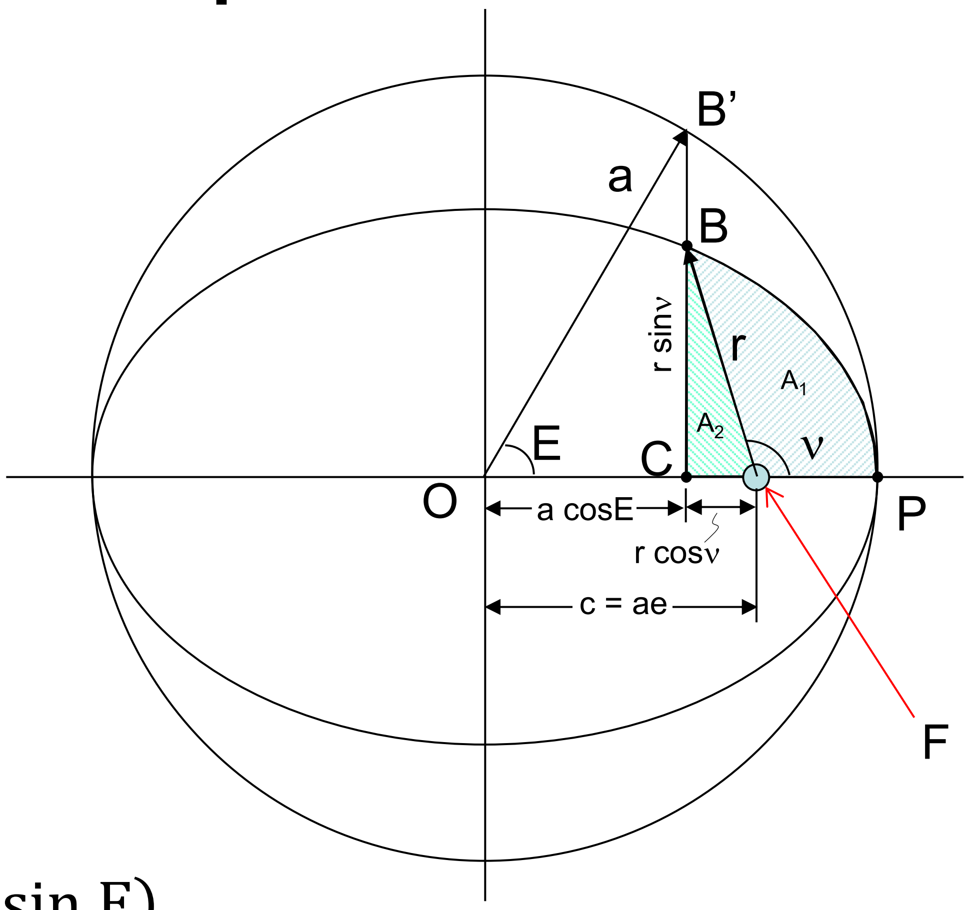
$$\overline{CF} = c - \overline{OC}$$

$$\overline{CF} = ae - a\cos E$$

$$A_1 = A_{PCB} - A_2$$

$$A_2 = \frac{1}{2}\text{ base} \times \text{height} = \frac{1}{2}(ae - a\cos E)(b\sin E)$$

$$\boxed{A_2 = \frac{ab}{2}\left(e\sin E - \sin E\cos E\right)}$$



Area of Circular Sector

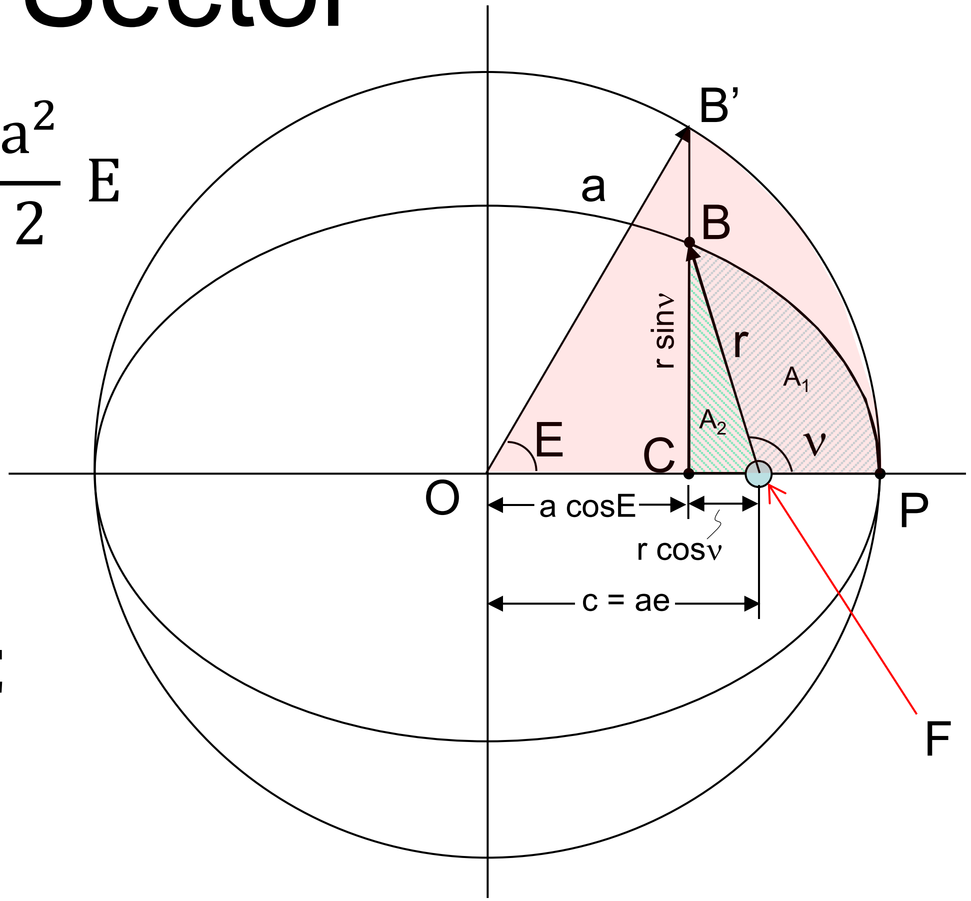
$$A_{sector} = \int_0^E \int_0^a r\,dr\,dE = \frac{a^2}{2} E$$

- Area of PCB':

$$A_{PCB'} = A_{sector} - A_{OCB'}$$

$$A_{OCB'} = \frac{1}{2}(a \sin E)\,(a \cos E) = \frac{a^2}{2} \sin E \cos E$$

$$\boxed{A_{PCB'} = \left(\frac{a^2 E}{2} - \frac{a^2}{2} \sin E \cos E\right)}$$



Note: For this to work, all angles are expressed in radians!

- Need to find corresponding area in ellipse

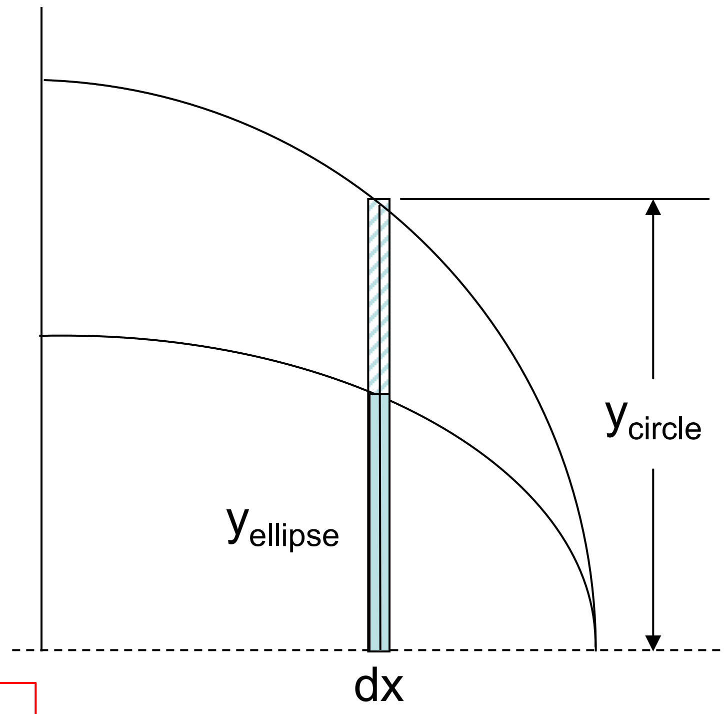
Relate area of Circle and Ellipse

$dA_{circle} = y_{circle}\, dx$

$dA_{ellipse} = y_{ellipse}\, dx$

$$y_{ellipse} = \frac{b}{a} y_{circle}$$

$$dA_{ellipse} = \frac{b}{a} dA_{circle}$$

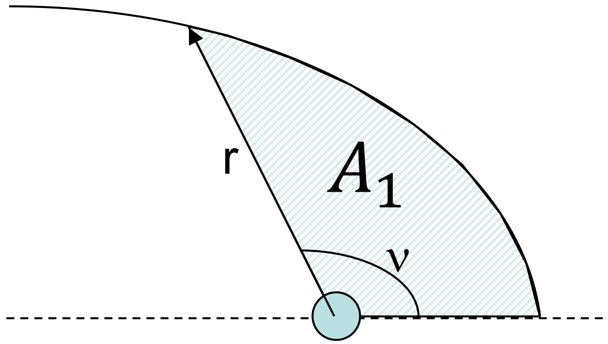


$$\boxed{A_{ellipse} = \frac{b}{a} A_{circle}}$$

Scale Area by b/a

$$A_1 = A_{PCB} - A_2$$

$$\boxed{A_{PCB} = \frac{b}{a} A_{PCB'} = \frac{b}{a}\left(\frac{a^2 E}{2} - \frac{a^2}{2}\sin E \cos E\right)}$$



$$A_2 = \frac{ab}{2}(e \sin E - \sin E \cos E)$$

$$A_{PCB} - A_2 = \frac{b}{a}\left(\frac{a^2 E}{2} - \frac{a^2}{2}\sin E \cos E\right) - \frac{ab}{2}(e \sin E - \sin E \cos E)$$

$$= \frac{ab\,E}{2} - \cancel{\frac{ab}{2}\sin E \cos E} - \frac{ab}{2} e \sin E + \cancel{\frac{ab}{2}\sin E \cos E}$$

$$\boxed{A_1 = \frac{ab}{2}(E - e \sin E)}$$

Relating Area to Time

$$\frac{\Delta t}{A_1} = \frac{TP}{\pi\, ab} \quad \left(= \frac{\text{Orbit Period}}{\text{Area of Ellipse}} \right)$$

$$\frac{\Delta t}{\frac{ab}{2}\,(E - e\sin E)} = \frac{TP}{\pi\, ab} \Longrightarrow \frac{2\pi\,\Delta t}{\cancel{ab}\,(E - e\sin E)} = \frac{TP}{\cancel{ab}}$$

$$\frac{2\pi\,\Delta t}{(E - e\sin E)} = TP \longrightarrow TP = 2\pi\sqrt{\frac{a^3}{\mu}}$$

- $\Delta t = (t_1 - T)$, time from perigee to E

$$\boxed{\Delta t = \sqrt{\frac{a^3}{\mu}}\,(E - e\sin E)}$$

Standard Expression

- Time from perigee to Eccentric Anomaly E

$$(t_1 - T) = \sqrt{\frac{a^3}{\mu}}(E - e \sin E)$$

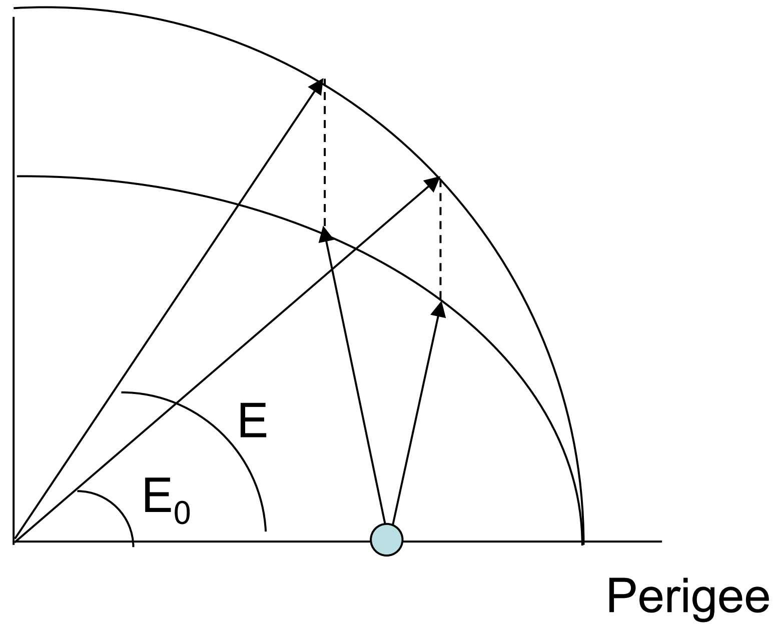
$$n = \sqrt{\frac{\mu}{a^3}} \quad \text{(mean motion)}$$

$$M = (E - e \sin E) \quad \text{(mean anomaly)}$$

$$\boxed{n\,(t_1 - T) = M}$$

Going from E_o to E

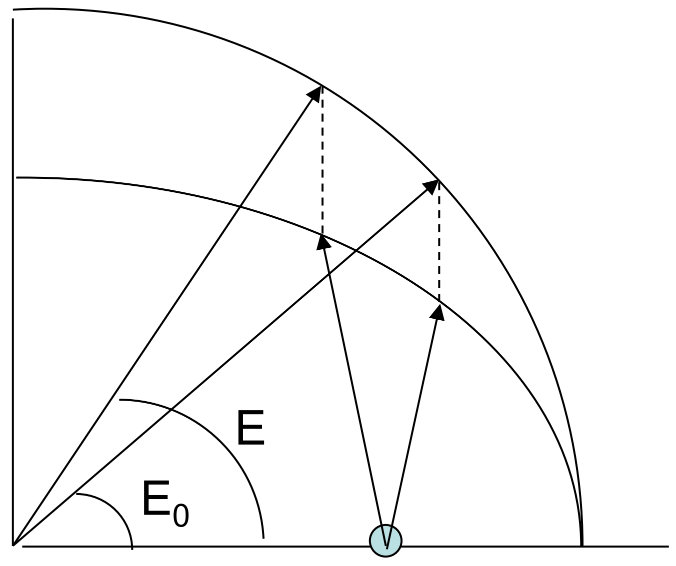
- Look at generic case, not starting at perigee
  - Going from E_o to E = Perigee to $E -$ Perigee to E_o
- Let
  - T = Time of last perigee passage
  - t_o = Time from perigee to E_o
  - t = Time from perigee to E
  - k = perigee passages between E_o and E
  - $k \cdot TP$ = number of full orbits from epoch



General Time of Flight Equation

- Given
  - T = time of perigee just prior to stopping point
  - k perigee crossings between t_o and t

$$t - t_o = k\,TP + (t - T) - (t_o - T)$$



$$\boxed{t - t_o = k\,TP + \frac{1}{n}(E - e\sin E) - \frac{1}{n}(E_0 - e\sin E_0)}$$

Relating E to ν

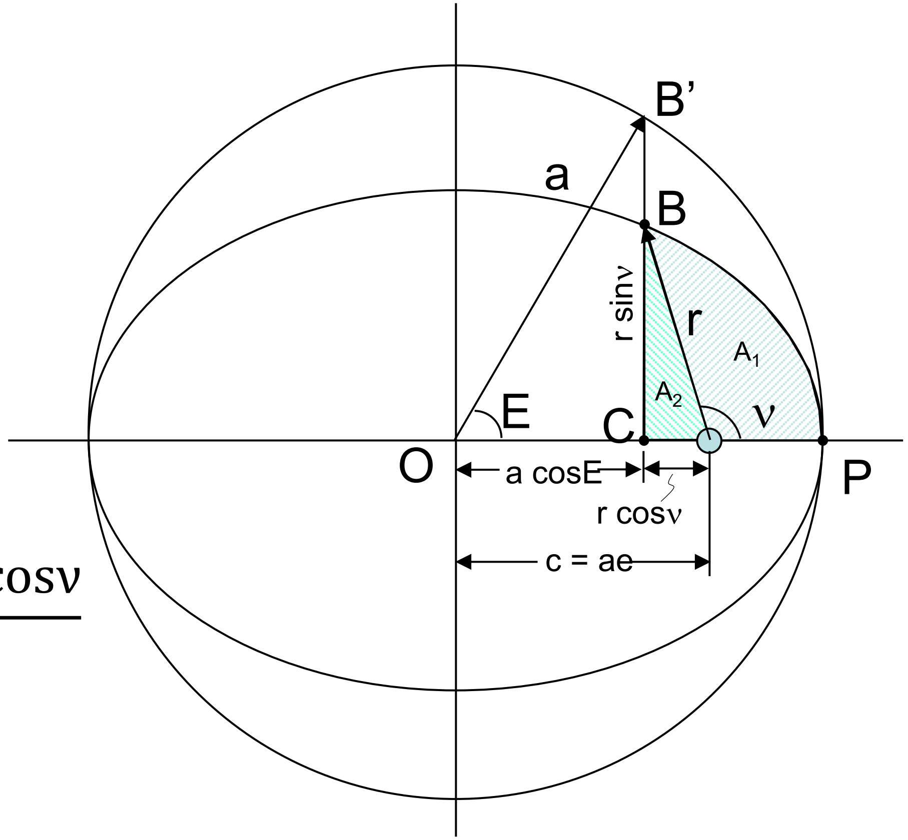
$$a\cos E = ae + r\cos\nu$$

$$\cos E = \frac{ae + r\cos\nu}{a}$$

$$r = \frac{a(1-e^2)}{1+e\cos\nu} \quad \text{(ellipse eqn)}$$

$$\cos E = \frac{\cancel{a}e + \frac{\cancel{a}(1-e^2)}{1+e\cos\nu}\cos\nu}{\cancel{a}}$$

$$\cos E = \frac{e(1+e\cos\nu) + (1-e^2)\cos\nu}{1+e\cos\nu}$$



$$\cos E = \frac{e + \cancel{e^2\cos\nu} + \cos\nu - \cancel{e^2\cos\nu}}{1+e\cos\nu} \Longrightarrow \boxed{\cos E = \frac{e+\cos\nu}{1+e\cos\nu}}$$

Solve for ν

$$\boxed{\cos E = \frac{e + \cos\nu}{1 + e\cos\nu}} \Longrightarrow \boxed{\cos\nu = \frac{e - \cos E}{e\cos E - 1}}$$

From the auxiliary circle

$$b\sin E = r\sin\nu \qquad b^2 = a^2 - c^2 \qquad c = ae$$

$$b^2 = a^2 - a^2e^2 = a^2(1 - e^2) \Longrightarrow b = a\sqrt{1 - e^2}$$

$$a\sqrt{1 - e^2}\,\sin E = r\sin\nu$$

$$\boxed{\sin E = \frac{\sin\nu\sqrt{1 - e^2}}{1 + e\cos\nu}} \Longrightarrow \boxed{\sin\nu = \frac{\sin E\sqrt{1 - e^2}}{1 - e\cos E}}$$

$$\boxed{E = ATAN2(\sin E,\ \cos E)} \qquad \boxed{\nu = ATAN2(\sin\nu,\ \cos\nu)}$$

We could use these to express Time of Flight in terms of ν, but it's easier to convert known values of ν to E, and use the Time of Flight equation already derived

Eccentric Anomaly from Vectors

- Eccentric anomaly can be computed directly from position/velocity

$$N_E = \frac{\vec{r} \cdot \vec{v}}{\sqrt{\mu a}}$$

$$D_E = 1 - \frac{r}{a}$$

$$E = \mathrm{atan2}(N_E, D_E)$$

Expressions taken from Space Shuttle Mission Control math specs. Not derived in this class.