# Statistical Learning Theory summary

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## Information Theory

**Surprise:** $S(\mathbb{P}(A)) : [0,1] \to [0,\infty)$. $S(1)=0$, $S(u)>S(v) \to u<v$, continuous, $S(uv)=S(u)+S(v)$ if indip, $S(u) = -c\log u$, $c$ const **Entropy:** $H(P(x)) = H(X) = \mathbb{E}_{x\sim P}[S] = -\sum_x P(x)\log P(x)$ Cont: H can $\le 0$ unif $(0,1/2)$, Discr: $0 \le H \le \log N$, $H=0$ iff $\exists x: p(x)=1$, $N = |\mathcal{X}|$, $H_N(p(x_1)..p(x_N)) = H_{N-1}(p(x_1)+p(x_2),..)$ $+(p(x_1)+p(x_2))H_2(\frac{p(x_1)}{p(x_1)+p(x_2)}, \frac{p(x_2)}{p(x_1)+p(x_2)})$

**BinH:** $H_{bin}(\delta) = -\delta\log\delta - (1-\delta)\log(1-\delta)$

**ConH** $H(Y|X) = \int P(X=x)H(Y|X=x)dx = \int p(x)\int p_{Y|X}(y,x)\log p_{Y|X}(y,x)dydx$ $H(X,Y)=H(X)+H(Y|X)$ (CR)

**MI** $I(X;Y)=I(Y;X)=H(X)-H(X|Y) = \mathbb{E}_{X,Y}[\log\frac{p(X,Y)}{p(X)p(Y)}]$, $I(X;X)=H(X)$

**ConI** $I(X;Y|Z) = H(X|Z)-H(X|Y,Z)$ $I(X,Y;Z) = H(X,Y)-H(X,Y|Z) = H(X)+H(Y|X)-H(X|Z)-H(Y|X,Z) = I(X;Z)+I(Y;Z|X)$ Given $X \to Y \to Z$: $I(X;Z)\le I(X;Y)$ **Chain:** $I(X;Y,Z) = I(X;Y)+I(X;Z|Y) = I(X;Z)+I(X;Y|Z)$. $I(X_1,...X_n;Z) = \sum_{i=1}^n I(X_i;Z|X_1,...X_{i-1})$

**DPi:** $I(X;Y|Z)\ge 0$ and $I(X;Z|Y)=0$ $X_1\to..\to X_n$, $I(X_1;X_2..X_n) = I(X_1;X_2)$

**KL** $D_{KL}(p\|q) = \sum_x p(x)\log(\frac{p(x)}{q(x)})$

**Msc** $I(X;Y) = KL(p_{X,Y}(x,y)\|p_X(x)p_Y(y))$, $H(X|Y)\le H(X)$, $H(X,Y|Z)\ge H(X|Z)$, $H(5X)=H(X)$ if $X$ discrete, $>H(X)$ cont. $H(X,Y,Z)-H(X,Y)\le H(X,Z)-H(X)$, $H(X|Y)\ge H(X|Y,Z)$, $H(X,Y)\le H(X)+H(Y)$, $H(g(X))\le H(X)$, $X$ discr. $I(X,Y;Z) \ge I(X;Z), I(X;Z|Y) = -I(Z;Y)+I(Z;Y|X)+I(X;Z), I(g(X),Y) \le I(X;Y)$ **Codes:** source code $C$ for RV $X$ is map $\mathcal{X}\to D^\star$, domain $X$, set of finite length strings from D-ary alphabet.

**Exp length:** $L(C)=\sum_{x\in\mathcal{X}}P(x)l(x)$

**Prefix code:** no codeword is prefix of other cw. **Kraft ineq:** prefix code over |alphabet| $= D$: $\sum_{i=1}^m D^{-l_i}\le 1$ ($l_i$ = length code word)

**Opt. codes:** minimize $L=\sum_{i=1}^m p_i l_i$ s.t. $\sum_{i=1}^m D^{-l_i}\leqslant 1 \Rightarrow L^\star = -\sum_1^m p_i\log p_i$

**Inform. Bottle:** efficient code $X\mapsto C$ Preserve relevant info on context variable $Y$ $\mathcal{R}^{IB} = I(X;C)-\lambda I(C;Y), \lambda>0$

**Scheme:** Obj space → Data groups (by minze mutual info) → Minze cost → Get $c^{opt}$ → Constr on context MI → Feat space

**Rate dist theory:** $d:\mathcal{X}\times\hat{\mathcal{X}}\to\mathbb{R}^+$, w\ $\mathbb{E}_{x,\hat x}[d(x,\hat x)] = \sum_{x\in\mathcal{X}}\sum_{\hat x\in\hat{\mathcal X}} p(\hat x|x)p(x)d(x,\hat x)$

**Shannon/Kolmogorov:** opt repr satisfies $R(D) = \min_{\{p(\hat x|x):\mathbb{E}_{x,\hat x}[d(x,\hat x)]<D\}} I(x,\hat x)$ $\mathrm{L}(p(\hat x|x)) = I(x,\hat x)+\beta(\mathbb{E}_{x,\hat x}[d(x,\hat x)]-D)$ $\frac{\delta L}{\delta p(\hat x|x)} = 0 \Rightarrow p(\hat x|x) = \frac{p(\hat x)}{Z(x,\beta)}\exp(-\beta d(x,\hat x))$

## Max Entropy: $p(c|X)\propto\exp(-\beta R(c,X))$

**Requirements for** $p(\cdot|X)$: $R(c_1,X)\le R(c_2,X) \to p(c_1|X)\ge p(c_2|X)$, Jaynes's (see below) $p_\beta(c) = \exp[-\beta\cdot(R(c)-F(\beta))]$,

**Free energy** $F(\beta) = \mathbb{E}[R(c,X)] - T\cdot H = -\frac{1}{\beta}\log Z(\beta)$. **Gibbs free energy:** $G(p) = \sum_{c\in C}p(c)R(c)+\frac{1}{\beta}\sum_c p(c)\log p(c)$, $G(p) = \frac1\beta KL(p\|p_\beta)+F(\beta)$

Measurements: $r_j(x), 1\le j\le m$, yield constraints $\mathbb{E}\{r_j(X)\} = \mu_j$. **Jaynes's principle** $\sup_{p(X)}\{-\int_{\mathcal X}p(x)\log p(x)dx\}$, $\int_{\mathcal X}p(x)dx=1$ $p(x)\ge0$, $\int_{\mathcal X}p(x)r_j(x)dx = \mu_j, 1\le j\le m$ $J(p+\delta p) = \int_x(p+\delta p)\cdot (-\log(p(1+\delta p/p)) + \lambda_0 - \sum_{j=1}^m\lambda_j r_j(x))dx$. *Variation equal zero:* $p(x) = \frac1Z\exp(-\sum_1^m\lambda_j r_j(x))$, $Z=\int_x\ldots$ **TTL '84:** $0 = \int_x\frac{\partial p(x)}{\partial\mu_i}dx$, $1 = \int_x\frac{\partial p(x)}{\partial\mu_i}r_i(x)dx$ $1\le\sqrt{\mathbb{E}[(\frac{\partial\log p(x)}{\partial\mu_i})^2]}\sqrt{\mathrm{Var}[r_i(x)]}$, estimated error $\mu_i\sim\sqrt{\mathrm{Var}\{r_i\}}\Rightarrow\delta\mu_i = \beta_i\sqrt{\mathrm{Var}\{r_i\}}$ $\beta_i\le\sqrt{\mathbb{E}[(\frac{\delta_{\mu_i}p(x)}{p})^2]}$, eq. if $\frac{\partial\ln p(x)}{\partial\mu_i}\sim r_i(x)-\mu_i$

Distrib with min sensitivity: $\frac{\partial\ln p(x)}{\partial\mu_i} = \alpha_i(r_i(x)-\mu_i)$, $\alpha_i$ constants. Cov matrix $C_{i,j} = \mathbb{E}(r_i(x)-\mu_i)(r_j(x)-\mu_j)$ is diag, $\alpha_i(\mu_1,\ldots,\mu_m)$ dep only on $\mu_i$ → Gibbs distr → min sensitiv to change moments $\mu_i$.

**Kmeans:** $R^{\mathrm{km}}(c,\mathbf{Y},\mathbf{X}) = \sum_{i\le n}\|x_i-y_{c(i)}\|^2$ $\frac{\partial}{\partial y_\alpha}R^{\mathrm{km}}(\ldots) = -2\sum_{i:c(i)=\alpha}(x_i-y_\alpha) = 0$ $y_\alpha = \frac{1}{n_\alpha}\sum_{i:c(i)=\alpha}x_i$ w\ $n_\alpha = \#\{i:c(i)=\alpha\}$

**Centroid equation:** posterior $p(c|X,\theta)$ depends on $\theta$. ME for $\theta$, $S$ = entropy: $\frac{\partial}{\partial\theta}S = -\sum_c(\frac{\partial}{\partial\theta}p(c\mid x,\theta))\log p(c\mid x,\theta) + \sum_c p(c|x,\theta)\frac{\frac{\partial}{\partial\theta}p(c|x,\theta)}{p(c|x,\theta)} = \beta\sum_c R(\frac{\partial}{\partial\theta}p(c\mid x,\theta))$ $\frac{\partial}{\partial\theta}\beta\sum_c Rp(c\mid x,\theta) = \frac{\partial}{\partial\theta}\mu = 0$, $\mu$ is guaranteed by Lagr variabl $\beta$ (not dep on $\theta$) $\frac{\partial}{\partial\theta}\sum_c Rp(c|\ldots) = 0 \to \sum_c p(c|\ldots)(\frac{\partial}{\partial\theta}R) = 0$

**DA:** stop splitting via CV, MDL, PA

## Markov Chains $\sum_{c'}P(c,c')=1$

$\mathrm{P}(X_{t+1}=c'|X_t=c) = P(c,c')$, **Irreducible:** can go from any stato to any state (finite steps), **Periodic:** if state $i$ is visited after number of steps multiple of integer $d>1$ ($d=1$ aperiodic), **Stationarity:** $\sum_{c\in\mathcal C}\pi(c)P(c,c') = \pi(c')$. If all above satisfied, $\lim_{t\to\infty}\mathbb{P}[X_t=c] = \pi(c)$, $\lim_{t\to\infty}\frac1t\sum_1^t f(X_s) = \sum_c\pi(c)f(c)$, **Detailed balnc** $\pi(c)P(c,c') = \pi(c')P(c',c)$

**Mixing time** of MC $\propto\frac{1}{\lambda_1-\lambda_2}$ where $\lambda_1=1$ is first eig and $\lambda_2\le\lambda_1$ is second eig of $P$.

## Least Angle Clustering $e_i := X_i/|X_i|$

$S(x_i,x_j) = w_{ij}\cos\phi_{ij} = w_{ij}e_ie_j$, $w_{ij} = v_i\cdot v_j$ $v_i = 1/\sqrt{p_k n}, p_k = \frac1n\sum_i M_{ik}, v_iv_j = \frac{1}{p_\alpha n} = \bullet$ $R(M,X) = -\frac12\sum_\nu^k\sum_i^n\sum_j^n M_{i\nu}M_{j\nu}v_iv_j\cos\phi_{ij}$ $+\frac12\sum_\nu^k\sum_i^n M_{i\nu}^2v_i^2 = k/2$

$Z = \sum_{\{M\}}\prod_\nu^k\exp(\frac{\beta p_\nu n}{2}\|\frac{1}{p_\nu n}\sum_{i=1}^n M_{i\nu}e_i\|^2)$ $y_\alpha = \frac{1}{p_\alpha n}\sum_{i=1}^n e_ip_{i\alpha}$, $p_\alpha = \frac1n\sum_{i=1}^n p_{i\alpha}$

**Laplace:** $f(x^*)=0, I = \int d\mathbf{x}\,h(\mathbf{x})e^{-nf(\mathbf{x})}$ $f(\mathbf{x})\approx f(x^*)+\frac12(x-x^*)^\top H_f(x^*)(x-x^*)$ $I\approx e^{-nf(x^\star)}\int h(x^\star)e^{-\frac n2(x-x^*)^\top H_f(x^*)(x-x^*)}$ $H = Q^\top\Lambda Q, H^{1/2} = Q\Lambda^{1/2}, y = \sqrt n H^{1/2}(x-x^\star), x = y\frac{H^{-1/2}}{\sqrt n}+x^\star$ Jacobian, $n^{-d/2}\det(\Lambda)^{1/2}$ $I\approx e^{-nf(x^*)}h(x^*)\int\mathrm{d}y\,e^{-\frac{\|y\|^2}{2}}\cdot|\det|J_y|| = \frac{e^{-nf(x^*)}h(x^*)\cdot(2\pi)^{d/2}}{n^{d/2}\sqrt{\det(H_f(x^*))}}$ **Gibbs distribution** $p(M|X) = \frac1z\exp(\frac{\beta}{2n}\sum_{\nu=1}^k\frac{1}{p_\nu}(\sum_{i=1}^n M_{i\nu}e_i)^2)$

**BW:** $e^{b^2/2a^2} = \int_{\mathbb R}\frac{a}{\sqrt{2\pi}}\exp(-\frac{a^2x^2}{2}+bx)dx$

Nominator: $\sqrt{\beta n/\pi}^{dk}\prod_\nu^k p_\nu^{\frac d2}\int_{\mathbb{R}^d}dy_1\cdots dy_k$ $\exp(-\beta n(\frac12\sum_\nu p_\nu y_\nu^2 - \frac1n\sum_\nu y_\nu\sum_i M_{i\nu}e_i))$ $p_\alpha^*y_\alpha^* = \frac1n\sum_{i=1}^n e_ip_{i\alpha}^*$, $p_{i\alpha}^* = \frac{\exp(\beta e_iy_\alpha^*)}{\sum_{\nu=1}^k\exp(\beta e_iy_\nu^*)}$ $Z = \int dy_1..\int dy_k\exp(-\beta nf_X(y)), p(M|X)\approx \frac{\exp(-\beta n(\frac12\sum_i p_\nu^*y_\nu^{*2} - \frac1n\sum_\nu y_\nu^*\sum_i M_{i\nu}e_i))}{\exp(-\beta n(\frac12\sum_\nu^* p_\nu^* y_\nu^* - \frac{1}{\beta n}\sum_{i=1}^n\log\sum_{\nu=1}^n\exp(\beta e_iy_\nu^*)))} = \exp(\beta\sum_\nu y_\nu^*\sum_i M_{i\nu}e_i - \sum_i^n\log\sum_\nu^n\exp(\beta e_iy_\nu^*))$

**MAP:** $\frac1n\sum_i M_{i\alpha}e_i = \frac1n\sum_{i=1}^n e_ip_{i\alpha}^*$

## Histo Cluster $\Delta_{i,j} = 1(0)$ if $i=j (i\ne j)$

$n$ obj, $m$ feat, $l$ dyads $\{(x_{i(r)},y_{j(r)})\}_{r=1}^l$

**Generative model:** 1. select $x_i\in\mathcal X$ with unif $1/n$; 2. choose clust using clust membership $c(i)$ of $x_i$; select $y_j\in\mathcal Y$ from $q(y_j|c(i))$. $\hat p(x_i,y_j) = \frac1l\sum_{r=1}^l\Delta_{x_i,x_{i(r)}}\Delta_{y_j,y_{j(r)}}$, $\hat p(y_j|x_i) = \frac{\hat p(x_i,y_j)}{\hat p(x_i)} = \frac{\hat p(x_i,y_j)}{\frac1l\sum_{r=1}^l\sum_{j=1}^m\Delta_{x_i,x_{i(r)}}\Delta_{y_j,y_{j(r)}}}$

$\mathcal L = \prod_{i=1}^n\prod_{j=1}^m\mathbf{P}(x_i,y_j|c(i),q)^{l\hat p(x_i,y_j)}$ $\mathbf P(x_i,y_j|c,q) = q(y_j|c(i))p(c(i))$, $p(\alpha) = 1/k$ $R^{\mathrm{hc}}(c;\{q(\cdot|\alpha)\}) = -\log(\mathcal L)$, $x_i\to i, y_j\to j$ $R^{\mathrm{hc}}\to R^{\mathrm{hc}}+l\sum_i\sum_j \hat p(j|i)\hat p(i)\log\hat p(j|i) = l\sum_i\hat p(i)\sum_m\hat p(j|i)(\log\hat p(j|i)-\log q(j|c(i))) - l\sum_i\sum_j\hat p(j|i)\hat p(i)\log(p(c(i))) = l\sum_i\hat p(i)\sum_j \hat p(j|i)\log(\frac{\hat p(j|i)}{q(j|c(i))}) - l\sum_i\hat p(i)\log p(c(i))$

$\hat p(i)\approx 1/n$ since obj dawn from unif distrib $R^{hc}(c,q,\hat p)\propto -l\sum_i^n\sum_j^m\hat p(i,j)\log q(j|c(i))$

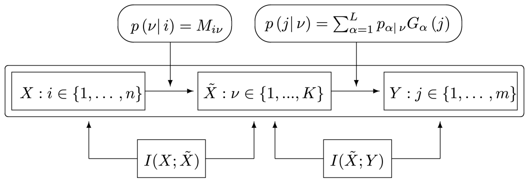
**Centr. cond:** $\mathbb{E}_{c|\hat p}\frac{\partial}{\partial q(j|\alpha)}R^{hc}(c,q,\hat p) = 0 = -\sum_i\sum_j l\hat p(i,j)\mathbb{E}_{c|\hat p}\mathbb{I}_{\{c(i)=\alpha\}}\frac{\partial}{\partial q_{j\alpha}}\log q_{j\alpha} + \frac{\partial}{\partial q_{ja}}\sum_\nu\lambda_\nu\sum_j(q_{j\nu}-1)$, $p_i(\alpha|Q,\mathcal X)$

**Limitations:** lack topology (permutations bin index do not change KL, neglect info due to noise-induced errors during the histogramming process ca), Histograms repr cat info while most feat spaces equipped w\ natural topo (feature similarity), cant impose structure (idk nothing about pixels by themselves).

**PDC:** replace empirical histogram centroids by prototypical distribution parametrized by Gaussian mixture (more robust centroids), assume given set of obj $\mathbf o_i, i\in\{1..n\}$, $M_{i\nu}=1$, if obj $\mathbf o_i$ assigned to cluster $\nu = 1..k$, $\sum_{\nu<k}M_{i\nu}=1$. Each obj $\mathbf o_i$ equipped with set of $n_i$ obs $\mathcal X_i = \{x_{i1},\ldots,x_{in_i}\}, x_{ij}\in\mathbb R^d$, $p(x|\nu) = \sum_{\alpha=1}^l p_{\alpha|\nu}g(x|\mu_\alpha,\Sigma_\alpha)$, prob of various groups $p_\nu$, feat val restr to specific domains → rectified Gauss. Domain $I = \cup_{j=1}^m I_j$, $I_j\cap I_s = \emptyset$ for $j\ne s$, region weight $G_a(j) = \int_{I_j}g_a(y)dy$,

$\Theta = \{p_\nu, p_{\alpha|\nu},\mu_\alpha|\alpha=1,\ldots,l;\nu=1,\ldots k\}$ $\mathcal L = \prod_{i\le n}\sum_{\nu\le k}M_{i\nu}p_\nu p(\mathcal X_i|\nu,\Theta) = \prod_{i\le n}\prod_{\nu\le k}[p_\nu p(\mathcal X_i|\nu,\Theta)]^{M_{i\nu}}$ $n_{ij} = \#$ observation at site $i$ is in $I_j$, $\mathcal L =$

$\prod_{i\le n}\prod_{\nu\le k}[p_\nu\prod_{j\le m}(\sum_{\alpha\le l}p_{\alpha|\nu}G_\alpha(j))^{n_{ij}}]^{M_{i\nu}}$ $\mathcal{R}^{phc}(c,\{p(a|\nu)\},\theta) = -\log\mathcal{L}$, two-part coding scheme: expected codelength when encoding the cluster memberships and encoding the individual feat values. $\mathrm{E}[\mathcal{R}] = -\sum_i\sum_\nu q_{i\nu}[\log p_\nu + \sum_j n_{ij}\log(\sum_\alpha p_{\alpha|\nu}G_\alpha(j))]$
**EM:** *E-step:* $h_{i\nu} = -\log p_\nu - \sum_j n_{ij}\log(\sum_\alpha p_{\alpha|\nu}G_\alpha(j))$, $q_{i\nu} = \mathbb{E}[M_{i\nu}] = p(M_{i\nu} = 1) \propto \exp(-\frac{h_{i\nu}}{T})$, *M-step:* $p_\nu = \frac{1}{n}\sum_i q_{i\nu}$
$X$ prob of choose obj $i$, assume $p_i = 1/n$




$H(\tilde{X}|X) = -\sum_i p_i\sum_\nu M_{i\nu}\log M_{i\nu} = 0$
$I(X;\tilde{X}) = H(\tilde{X}) = -\sum_{\nu<k}p_\nu\log p_\nu$ with $p_\nu = \sum_{i=1}^n p(C=\nu|X=i)p(X=i)$, $M_{i\nu}$
$H(Y|\tilde{X}) = -\sum_{\tilde{X},Y}p(Y,\tilde{X})\log p(Y|\tilde{X}) = -\sum_{\tilde{X},Y}\sum_X p(Y,\tilde{X},X)\log p(Y|\tilde{X})$,
$\mathcal{R}^{IB} = -\sum_i^n\sum_\nu^k M_{i\nu}[\log p_\nu + \frac{\lambda}{n_i}\sum_{j=1}^m n_{ij}\log(\sum_{\alpha=1}^L p_{\alpha|\nu}G_\alpha(j))]$ , $\lambda H(Y)$ const in $\Theta$

## Graph Clustering $S_{ij} = \exp(-D_{ij}/\Delta)$

OR: $S_{ij} = \max_{ij}D_{ij} - D_{ij}$, $W$ weight matrix of $(\mathcal{V},\mathcal{E})(A,B\subset\mathcal{V}, A\cap B=\emptyset)$
$\mathrm{cut}(A,B) = \sum_{i\in A,j\in B}W_{i,j}$, $\mathrm{assoc}(A,\mathcal{V}) = \sum_{i\in A,j\in\mathcal{V}}W_{i,j}$ (measures total connection strength from nodes A to all nodes)
**SCC:** modify CC by choosing continuous similarities ($S_{ij}\in[-1,+1]$) and sum them up relative to threshold $u$, $\frac{1}{2}(|X|\pm X) = \max\{0,\pm X\}$, $\mathcal{R}^{cc}(c;\mathcal{D}) = -\frac{1}{2}\sum_{\nu\le k}\sum_{(i,j)\in\mathcal{E}_{\nu\nu}}(|S_{ij}-u| + S_{ij}-u) - \frac{1}{2}\sum_{\nu\le k}\sum_{\substack{\mu\le k\\ \mu\ne\nu}}\sum_{(i,j)\in\mathcal{E}_{\nu\mu}}(|S_{ij}+u| - S_{ij}-u)$.
**PC & DA:** given $S = (S_{ij})_{i,j\le N}\in\mathbb{R}^{N\times N}$ specifies how much $i$ likes $j$. Want to cluster in $K$ groups based on personality, estimate hm person like another based on personality type. Cluster assignment matrix $M = (M_{ik})_{i<N.k<K}$ , $r = \{r_{kk'}\}_{k,k'<N}$ (hm person type $k$ likes type $k'$, $\sum_{k\le K}M_{ik} = 1$
$R(M,r) = \sum_{ij\le N,kk'\le K}M_{ik}M_{jk'}(S_{ij}\text{-}r_{kk'})^2$
With $r$ fix, Lagrangian $\mathcal{L}(\mathbf{P}(\cdot\mid r),\lambda_0,\lambda_1) = H[\mathbf{P}(\cdot|r)] + \lambda_0(\mathbb{E}[R]\text{-cst}) + \lambda_1(\sum_M\mathbf{P}(M|r)\text{-}1)$

$\mathbf{P}^*(S,M|r)\propto\exp(-R(M,r)/T)$, compute now values for $r$ that maximize $H[\mathbf{P}^*(\cdot|r)]$
$\arg\max_r H[\mathbf{P}^*(\cdot|r)] = \arg\max_r F[R(\cdot,r)]$
w/ $F[R(\cdot,r)] = \log(\sum_M\exp(-\frac{R(M,r)}{T}))$
EM to compute ($\star$) $= \arg\max\log\tilde{\mathbf{P}}(S|r)$
w/ $\tilde{\mathbf{P}}(S\mid r)$ is the marginal pdf of ($\star$)
$\sum_M\int_S\tilde{\mathbf{P}}(S,M|r)dS = \sum_M\prod_{i,j}\int_{S_{ij}}\prod_{k,k'}\exp(-M_{ik}M_{jk'}(S_{ij} - r_{kk'})^2/T)dS_{ij}$ **EM:** *E-step* compute $\tilde{\mathbf{P}}(\cdot\mid S,r_0) = \mathbf{P}^*(\cdot|r)$, *M-step* compute $r^* = \arg\max_r Q(r,r_0)$, w/ $Q(r,r_0) = \mathbb{E}_{M\sim\mathbf{P}^*(\cdot|r_0)}[\log\tilde{\mathbf{P}}(S,M\mid r')] = -\frac{1}{T}\sum_{ij}\sum_{kk'}\mathbb{E}_{M\sim\mathbf{P}^*(\cdot|r_0)}[M_{ik}M_{jk'}](S_{ij}\text{-}r_{kk'})^2$
$r_{kk'}^{(\ell+1)} = \frac{\sum_{i,j\le N}\mathbf{P}^*(M_{ik}M_{jk'}=1|r^{(\ell)})S_{ij}}{\sum_{i,j\le N}\mathbf{P}^*(M_{ik}M_{jk'}=1|r^{(\ell)})}$ cant use, norm const of ($\star$) is $\Theta(2^{N\times K})$ **MFA:** ($\star$) $\approx\mathbf{q}(\cdot|r) = \arg\min_q D_{KL}(q\|\mathbf{P}^*(\cdot\mid r))$ with $\mathbf{q}(M|r) = \prod_{i\le N,k\le K}\mathbf{q}_{ik}(M_{ik}\mid r)$, $\mathbf{q}_{ik}(M_{ik}|r)$ Ber, $\mathbf{P}^*(M_{ik}M_{jk'}=1|r) = \mathbf{q}(M_{ik}=1|r)\mathbf{q}(M_{jk'}=1|r)$

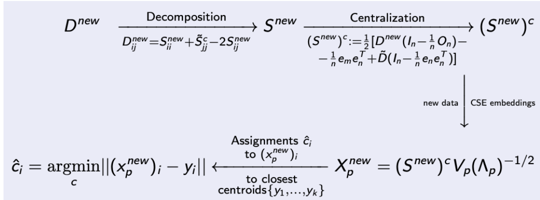
**Algorithm 2** DA

**procedure** SOLVE($\epsilon\in\mathbb{R}$, $S\in\mathbb{R}^{N\times N}$)
  $T\leftarrow\infty$
  **while** $T>\epsilon$ **do**
    For $i,k$, initialize $\mathbf{q}_{ik}(1\mid r)$ with a random value in $[0,1]$. ▷ MFA starts
    **repeat**
      $h_{ik,b}\leftarrow\mathbb{E}_{ik\mapsto b}[L(M,r)]$, for $i\le N, k\le K, b\in\{0,1\}$.
      $\mathbf{q}_{ik}(1\mid r)\leftarrow\frac{\exp(-h_{ik,1}/T)}{\sum_{b\in\{0,1\}}\exp(-h_{ik,b}/T)}$
    **until** Convergence of all $\mathbf{q}_{ik}(1\mid r)$ ▷ MFA ends
    Initialize $r^{\text{old}}$ with random values. ▷ EM starts
    **repeat**
      $r_{kk'} = \frac{\sum_{i,j\le N}\mathbf{q}_{ik}(1|r)\mathbf{q}_{jk'}(1|r^{\text{old}})S_{ij}}{\sum_{i,j\le N}\mathbf{q}_{ik}(1|r)\mathbf{q}_{jk'}(1|r^{\text{old}})}$,   for $k,k'\le K$.
      $r^{\text{old}} = r$.
    **until** Convergence of $r$. ▷ EM ends
    Reduce $T$.
  **end while**
  **Compute** $M$ **from** $\mathbf{q}(\cdot\mid r)$.
  **return** $M,r$.
**end procedure**

**Graph partitioning**, $D_{ij}\in\mathbb{R}$: $\mathcal{R}^{\text{gp}}(c;\mathcal{D}) = \sum_{\nu\le k}\sum_{(i,j)\in\mathcal{E}_{\nu\nu}}D_{ij} = \sum_{(i,j)\in\mathcal{E}}D_{ij} - \sum_{\nu,\mu:\nu<\mu}\sum_{(i,j)\in\mathcal{E}_{\nu\mu}}D_{ij} = \text{const} - \sum_{\nu\le k}\mathrm{cut}(\mathcal{G}_\nu(\mathcal{D}),\mathcal{V}\backslash\mathcal{G}_\nu(\mathcal{D})) = \text{const}_2 + \sum_{\nu\le k}\mathrm{cut}(\mathcal{G}_\nu(\mathcal{S}),\mathcal{V}\backslash\mathcal{G}_\nu(\mathcal{S}))$. *Problem:* strong bias for unbalances (very small or large) clusters due to lack normalization.
**Alternative costs:** Norm cut $\mathcal{R}^{\text{nc}}(c;\mathcal{S}) = \sum_{\nu\le k}(\frac{\mathrm{cut}(\mathcal{G}_\nu,\mathcal{V}\backslash\mathcal{G}_\nu)}{\mathrm{assoc}(\mathcal{G}_\nu,\mathcal{V})}) = k\text{-}\sum_{\nu\le k}(\frac{\mathrm{assoc}(\mathcal{G}_\nu,\mathcal{G}_\nu)}{\mathrm{assoc}(\mathcal{G}_\nu,\mathcal{V})})$, Average cut $\mathcal{R}^{\text{ac}}(c;\mathcal{S}) = \sum_{\nu\le k}(\frac{\mathrm{cut}(\mathcal{G}_\nu,\mathcal{V}\backslash\mathcal{G}_\nu)}{|\mathcal{G}_\nu|})$, Min-Max $\mathcal{R}^{\text{mmc}}(c;\mathcal{S}) = \sum_{\nu\le k}(\frac{\mathrm{cut}(\mathcal{G}_\nu,\mathcal{V}\backslash\mathcal{G}_\nu)}{\mathrm{assoc}(\mathcal{G}_\nu,\mathcal{G}_\nu)})$ (biased over equipatitions), ARC: $\mathcal{R}^{\text{rc}}(c;\mathcal{S}) = \sum\sum_{\mu=\nu+1}^k\mathrm{cut}(\mathcal{G}_\nu,\mathcal{G}_\mu)(|\mathcal{G}_\nu|^{-\frac{1}{p-1}} + |\mathcal{G}_\mu|^{-\frac{1}{p-1}})^{p-1}$ ($p=2$ standard Ratio Cut, $p=1$ multiway Cheeger Cut), Chg. Cut $\mathcal{R}^{\text{Cheeger}}(c;\mathcal{S}) = \sum_{\nu=1}^k\sum_{\mu=\nu+1}^k(\frac{\mathrm{cut}(\mathcal{G}_\nu,\mathcal{G}_\mu)}{\min\{|\mathcal{G}_\nu|,|\mathcal{G}_\mu|\}})$. **Validation** of graph based methods poor understood and no general accepted principle available.
**Data space**: pow(2, $\binom{n}{2}$), **Sol spc**: $2^n$.
Data space is exponentially larger than solution space. Estimating probability distribution over solution space feasible when data distributions cannot be estimated.
**PC & Norm Cut:** $k=2$, $G=(V,E)$, $Ncut(A,B) = \frac{\mathrm{cut}(A,B)}{\mathrm{assoc}(A,V)} + \frac{\mathrm{cut}(A,B)}{\mathrm{assoc}(B,V)} = \frac{\mathrm{assoc}(A,V)-\mathrm{assoc}(A,A)}{\mathrm{assoc}(A,V)} + \frac{\mathrm{assoc}(B,V)-\mathrm{assoc}(B,B)}{\mathrm{assoc}(B,V)} = 2 - (\frac{\mathrm{assoc}(A,A)}{\mathrm{assoc}(A,V)} + \frac{\mathrm{assoc}(B,B)}{\mathrm{assoc}(B,V)})$

## Constant Shift Embedding

$\tilde{D}$ decomposition same $D$ but tilde, $\tilde{S}^c = -\frac{1}{2}\tilde{D}^c$, $X_t = V_t(\Lambda_t)^{1/2}$, *Prob:* cluster new obj given $M\times N$ dissimilarities $D_{ij}^{\text{new}}$ betw new obj and all $n$ original
**Note:** $\tilde{S}^cV_p = V_p\Lambda_p\to X_p = \tilde{S}^cV_p(\Lambda_p)^{-1/2}$



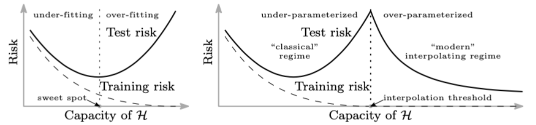

## MFA: $c(i), c(j), i\ne j$ not dependent

$\frac{\partial^2\mathcal{B}}{\partial q_u(\alpha)\partial q_v(\gamma)} = \frac{\partial h_u(\alpha)}{\partial q_v(\gamma)} = \sum_c\prod_{\substack{i=1\\ i\ne u,v}}^n q_i(c(i))\,\mathbb{I}_{\{c(u)=\alpha,\,c(v)=\gamma\}}R(c,X)$
**MDS:** given $\mathbf{D}$, $\mathcal{Z} = \{\mathbf{z}_1,\dots,\mathbf{z}_n\}\subset\mathbb{R}^d$ find $\mathbf{X} = \{\mathbf{x}_1,\dots,\mathbf{x}_n\}\subset\mathbb{R}^m$, $m=1,2,3$
$J^{\text{Sam}SStr}(\mathbf{X},\mathbf{D}) = \sum_{i\le n}\sum_{k\le n}w_{ik}(\|\mathbf{x}_i - \mathbf{x}_k\|^2 - D_{ik}^2)^2 = \sum_{i,k\le n}w_{ik}(\|\mathbf{x}_i\|^2 + \|\mathbf{x}_k\|^2 - (2\mathbf{x}_i^\top\mathbf{x}_k + D_{ik}^2))^2 = \sum_{i,k\le n}w_{ik}(2\|\mathbf{x}_i\|^4 + 2\|\mathbf{x}_i\|^2\|\mathbf{x}_k\|^2 - 8\|\mathbf{x}_i\|^2\mathbf{x}_i^\top\mathbf{x}_k - 4\|\mathbf{x}_i\|^2D_{ik}^2 + 4\,\mathrm{Tr}(\mathbf{x}_i\mathbf{x}_i^\top)(\mathbf{x}_k\mathbf{x}_k^\top) + 4\mathbf{x}_i^\top\mathbf{x}_kD_{ik}^2 + D_{ik}^4)$,
SSTRESS (quartic cost func) → analysed w\ linear algebra, SAMMONs (irrational cost func) mapping smoother embeddings than SSTRESS. **Metric MDS:** $D(\cdot)$ yields numerical values meaningful/ interpretable.
**Non-Metric MDS:** $D(\cdot)$ respects rank dissimilarities, but not necessarily their numerical values, $\delta_{ij}<\delta_{kl}\to D(\delta_{ij})<D(\delta_{kl})$.
$0 = \frac{\partial\mathcal{D}^{\text{KL}}(\mathbf{P}^0\|\mathbf{P}^G)}{\beta\partial\theta_{ip}} = \alpha_i^0\frac{\partial\mathbb{E}\{\|\mathbf{x}_i\|^4\}}{\partial\theta_{ip}} + \hat{\mathbf{h}}_i^T\frac{\partial\mathbb{E}\{\|\mathbf{x}_i\|^2\mathbf{x}_i\}}{\partial\theta_{ip}} + \frac{\partial}{\partial\theta_{ip}}\mathrm{Tr}[\mathbf{H}_i\mathbb{E}\{\mathbf{x}_i\mathbf{x}_i^T\}] + \mathbf{h}_i^T\frac{\partial\mathbb{E}\{\mathbf{x}_i\}^T}{\partial\theta_{ip}} + T\frac{\partial}{\partial\theta_{ip}}\mathbb{E}\{\log q_i\}$, $1\le i\le n$
$\hat{\mathbf{h}}_i = -8\sum_{k=1}^N w_{ik}\mathbb{E}\{\mathbf{x}_k\}$, $\alpha_i^0 = 2\sum_{k=1}^N w_{ik}$
$\mathbf{h}_i = 8\sum_{k=1}^N w_{ik}(D_{ik}^2\mathbb{E}\{\mathbf{x}_k\} - \mathbb{E}\{\|\mathbf{x}_k\|^2\mathbf{x}_k\})$,
$\mathbf{H}_i = \sum_{k=1}^N w_{ik}[8\mathbb{E}\{\mathbf{x}_k\mathbf{x}_k^T\} + 4(\mathbb{E}\{\|\mathbf{x}_k\|^2\} - D_{ik}^2)\mathbf{I}]$, $\Theta_i = (\alpha_i^0,\mathbf{h}_i,\mathbf{H}_i,\hat{\mathbf{h}}_i)$

initialize the parameters $\Theta$ of $\mathbf{P}^0(\mathbf{X}|\Theta)$ randomly
$T\leftarrow T_{max}$
**while** $T>T_{min}$ **do**
  **while** change of KL divergence $\mathcal{D}^{\text{KL}}>\varepsilon$ **do**
    $\Theta^0\leftarrow\Theta$
    **for all** $1\le i\le N$ in random order **do**
      Compute the statistics $\Phi_i$ from the expectations $\{\mathbb{E}\{\mathbf{x}_k\},\mathbb{E}\{\mathbf{x}_k\mathbf{x}_k^T\},\mathbb{E}\{\|\mathbf{x}_k\|^2\mathbf{x}_k\} : 1\le k\le n,\ k\ne i\}$, taken w.r.t. $\mathbf{P}^0(\mathbf{X}|\Theta)$
      Minimize $\mathcal{D}^{\text{KL}}(\mathbf{P}^0(\mathbf{X}|\Theta)\|\mathbf{P}^{G(\mathbf{X})})$ w.r.t. $\Theta_i$
    **end for**
  **end while**
  $T\leftarrow\eta T$, $0<\eta<1$
**end while**

## Model Selection for Clustering

**Validation Methods:** 1) procedures and concepts for quantitative and objective assessment of clustering solutions. 2) evaluate specific quality measure. 3) external (compare with ground-truth/teacher information) or internal. 4) used for model selection, i.e. use validity measures to select a model.




Second descent in the interpolation regime when the training error vanishes and the generalization error decays with a capacity increase. **Complexity MS:** assume model-based clustering, problem → log-like diverges for vanishing variance. Strategy → add regularization to neg log-like. **Occam's razor:** choose model with shortest description of the data → MDL, BIC
**MDL:** minimize description length $-\log p(\mathbf{X}|\Theta_k) - \log p(\Theta_k)$ (-data-model).
Asymptot approx (-neg log-like + penalty)
$\hat{k}\in\underset{1\le k\le K_{\max}}{\operatorname{argmin}}(-\log(\mathrm{p}(\mathbf{X}\mid\hat{\Theta}_k)) + \frac{1}{2}k'\log n)$
$k'\to\#$ param in model enc. by MLE $\hat{\Theta}_k$
**BIC:** $\dim\theta = p$, $f'(x)_{x=x_0} = 0$, flat prior
$p(M|X) = \frac{p(M)}{p(X)}\int\exp(\log p(X|M,\theta))p(\theta|M)d\theta$
$\int_{\mathbb{R}}\exp(cf(x))dx\propto\sqrt{2\pi}C|f''(x_0)|\exp(cf(x_0))$
$\bar{l}(\theta) = \frac{1}{n}\log p(X|\theta,M) = \frac{1}{n}\sum_i\log p(x_i|\theta,M)$
$\bar{l}(\theta) = \bar{l}(\hat{\theta}) - \frac{1}{2}(\hat{\theta}-\theta)^\top(-\frac{\partial^2\bar{l}(\theta)}{\partial\theta\partial\theta^\top}{}_{\theta=\hat{\theta}})(\theta-\hat{\theta})$
$p(X|M)\propto C\cdot\exp(l(\hat{\theta}))(\frac{2\pi}{n})^{\frac{k}{2}}|\mathcal{I}(\hat{\theta})|^{-\frac{1}{2}}$

$p(X|M) = \exp(\log l(\hat\theta) - \frac{k}{2}\log(n) + O(1)) = \exp(-\frac{\mathrm{BIC}}{2} + O(1))$, $p(M|X) \propto p(X|M)p(M) \approx \exp(-\frac{\mathrm{BIC}}{2})p(M)$. MDL and BIC are consistent as a model selection criterion for $n \to \infty$.
⊕ Well-motivated model selection schemes. ⊖ MDL/BIC methods to model selection rely on likelihood optimization. (not gen applicable) ⊕ Many variants, ⊖$\lambda$ tuning.
**Stability-based Validation** if no prior knowledge avaible → solutions on two data sets from same source should be similar. Stable solut. transfered to second data (same distribution) at minimal model misfit *Two sample scenario:* 1) Draw two data sets from same source 2) Cluster both data sets 3) Compute disagreement. *In Practice:* only one dataset avaible, 1) Estimate expected agreement by resampling 2) Cluster entire dataset with optimal $k$. *How to measure disagreement of two clustering $c$ and $c'$ on disjoint data?* Extend solution from $X'$ to $X$ with classifiers. 1) Train a predictor $\phi_{\mathbf{Z}'}(\cdot)$ on data $\mathbf{Z}' := (\mathbf{X}', \hat c(\mathbf{X}'))$ 2) Predict labels on $\mathbf{X}$ using $\phi_{\mathbf{Z}'}$ 3) Compare clustering solutions $(\phi_{\mathbf{Z}'}(X_i))_{i\in[n]}$ and $\hat c(\mathbf{X})$ on $\mathbf{X}$. *Symmetry Breaking:* problem → labeling unique only up to $\pi \in \mathfrak{S}_k$. Solution: Stability index $S :=$ expected minimal disagreement over all $\pi \in \mathfrak{S}_k$: Hungarian method $O(k^3)(O(n)$ preprocessing). *Final Measure for Stability:*

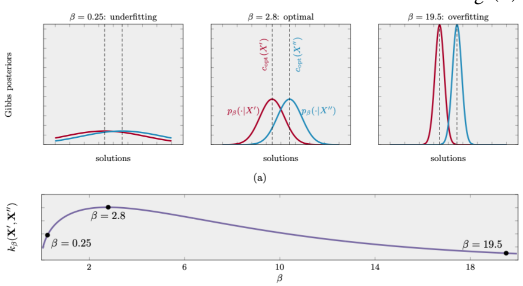
$$S_k(\hat c) = \mathbb{E}_{\mathbf{X},\mathbf{X}'}(\underbrace{\min_{\pi\in\mathfrak{S}_k}\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_{\{\hat c_k(\mathbf{X})_i \neq \pi\circ\phi_{\mathbf{z}'}(X_i)\}}}_{d_{\tilde E_k}(\hat c_k(\mathbf{X}),\phi'_{\mathbf{Z}}(\mathbf{X}))})$$

*In Practice:* Estimate $\mathbb{E}_{\mathbf{X},\mathbf{X}'}$ by resampling. $t(\mathbf{X})$ any target labelling. Then $S_k(\hat c) \le \mathbb{E}_{\mathbf{X}} d_{\mathfrak{S}_k}(\hat c_k(\mathbf{X}), t(\mathbf{X})) + \mathbb{E}_{\mathbf{X},\mathbf{X}'} d_{\mathfrak{S}_k}(t(\mathbf{X}), \phi'_{\mathbf{Z}}(\mathbf{X}))$ Large $S_k$ → not close to any target labelling on average! ⊕ Performance on experimental datasets. Stability method overlooks solution complexity. Tradeoff betw informative and stability needs quantitative eval, requiring algo validation theory.

## Nature of DSA: train $X'$, validation $X''$

Substitute data-hypothesis relation with posterior distribution. *How select scoring for posterior:* conciseness (minimize MDL), generalization (high score validation)

**Design constraints hyp $\Theta$**: a priori, all hypotheses $\theta \in \Theta$ necessary for inference (If can rule out hypotheses before measured data → exclude such hypothesis a priori) **Generalization DL:** for Gibbs distrib $\mathbb{E}_{\theta|X'}$-log $\mathbf{P}(\theta|X'') = \beta(\mathbb{E}_{\theta|X'}R(\theta,X'')\text{-}\mathcal{F}(X''))$ $X'_{\mathcal{E}}, X''_{\mathcal{E}}$ generated by same experiment $\mathcal{E}$. **Object:** MDL post given test data when hyp. sampled from post given train data **M1** Train/val data cond. independent $\mathbf{P}^{\mathcal{A}}(X'_{\mathcal{E}}, X''_{\mathcal{E}}|\varphi_{\mathcal{E}}) = \mathbf{P}^{\mathcal{A}}(X'_{\mathcal{E}}|\varphi_{\mathcal{E}})\mathbf{P}^{\mathcal{A}}(X''_{\mathcal{E}}|\varphi_{\mathcal{E}})$, **M2** valdata perturbation of train data (iid) $\mathbf{P}^{\mathcal{A}}(X'_{\mathcal{E}}, X''_{\mathcal{E}}|\varphi_{\mathcal{E}}) = \mathbf{P}^{\mathcal{A}}(X''_{\mathcal{E}}|X'_{\mathcal{E}})\mathbf{P}^{\mathcal{A}}(X'_{\mathcal{E}}|\varphi_{\mathcal{E}})$ M2 corresponds to noisy channel transmission where random codebook vector $X'_{\mathcal{E}}$ corrupted by channel noise $\mathbf{P}^{A}(X''_{\mathcal{E}}|X'_{\mathcal{E}})$. **Scale oo sample DL** M1, M2 $k_{\mathcal{A}}(X'_{\mathcal{E}}, X''_{\mathcal{E}})$: $-\log\mathbf{P}^{\mathcal{A}}_{\mathcal{E}}(\theta|\varphi_{\mathcal{E}}) = -\log\mathbb{E}_{X''_{\mathcal{E}}|\varphi_{\mathcal{E}}}\mathbf{P}^{\mathcal{A}}(\theta|X''_{\mathcal{E}},\varphi_{\mathcal{E}})$ $= -\log\mathbb{E}_{X'_{\mathcal{E}}|\varphi_{\mathcal{E}}}\mathbb{E}_{X''_{\mathcal{E}}|X'_{\mathcal{E}}}\mathbf{P}^{\mathcal{A}}(\theta|X''_{\mathcal{E}},\varphi_{\mathcal{E}})$,
**Score** $\min_{\mathcal{A}}\mathbb{E}_{X'_{\mathcal{E}},X''_{\mathcal{E}}}\mathbb{E}_{\theta|X'_{\mathcal{E}}}(-\log\frac{\mathbf{P}^{\mathcal{A}}(\theta|X''_{\mathcal{E}})}{\mathbf{P}^{\mathcal{A}}_{\mathcal{E}}(\theta)}) \ge \min_{\mathcal{A}}\mathbb{E}_{X'_{\mathcal{E}},X''_{\mathcal{E}}}(-\log\mathbb{E}_{\theta|X'_{\mathcal{E}}}\frac{\mathbf{P}^{\mathcal{A}}(\theta|X''_{\mathcal{E}})}{\mathbf{P}^{\mathcal{A}}_{\mathcal{E}}(\theta)}) \ge 0$, with $\mathbb{E}_{X'_{\mathcal{E}},X''_{\mathcal{E}}}\mathbb{E}_{\theta|X'_{\mathcal{E}}}(-\log\frac{\mathbf{P}^{\mathcal{A}}(\theta|X''_{\mathcal{E}})}{\mathbf{P}^{\mathcal{A}}_{\mathcal{E}}(\theta)}) = \mathbb{E}_{X'_{\varepsilon},X''_{\varepsilon}}$ $\mathcal{D}(\mathbf{P}^{\mathcal{A}}(\theta|X'_{\mathcal{E}})\|\mathbf{P}^{\mathcal{A}}(\theta|X''_{\mathcal{E}})) - \mathcal{I}(\theta;X_{\mathcal{E}}) \ge 0$ **Prob Richness:** sampling hyp uniform from all experiments yield uniform prior $\pi(\theta) := \mathbb{E}_{\mathcal{E}}\mathbf{P}^{\mathcal{A}}_{\mathcal{E}}(\theta) \approx |\Theta|^{-1}$. $\min_{\mathcal{A}}\mathcal{D}(\pi(\theta)\||\Theta|^{-1}) = \log|\Theta| - \max_{\mathcal{A}}H_\pi(\theta)$ **Post. selection:** $\min_{\mathcal{A}}(\mathbb{E}_{X'_{\mathcal{E}},X''_{\mathcal{E}}}\mathbb{E}_{\theta|X'_{\mathcal{E}}}(-\log\frac{\mathbf{P}^{\mathcal{A}}(\theta|X''_{\mathcal{E}})}{\mathbf{P}^{\mathcal{A}}_{\mathcal{E}}(\theta)}) + \lambda\mathcal{D}(\pi(\theta)\||\Theta|^{-1})) \ge -\max_{\mathcal{A}}(\lambda H_\pi(\theta) - \lambda\log|\Theta| + \mathbb{E}_{X',X''}\log k_{\mathcal{A}}(X'_{\mathcal{E}},X''_{\mathcal{E}})) \ge$- $\max_{\mathcal{A}}\mathbb{E}_{X',X''}\log(\frac{\exp(\lambda H_\pi(\theta))}{|\Theta|^\lambda}k_{\mathcal{A}}(X'_{\mathcal{E}},X''_{\mathcal{E}})) \le 1$ $\ge \max\mathbb{E}_{X'_{\mathcal{E}},X''_{\mathcal{E}}}\log(k_{\mathcal{A}}(X'_{\mathcal{E}},X''_{\mathcal{E}}))$, $H_\pi(\theta) = \log|\Theta|$, Kernel topo concept, part ordering normalized desc length $\mathbb{E}_{\theta|X'_{\mathcal{E}}}(-\log\frac{\mathbf{P}^{\mathcal{A}}(\theta|X''_{\mathcal{E}})}{\mathbf{P}^{\mathcal{A}}_{\mathcal{E}}(\theta)})$




**Algorithm 2** Posterior Selection Algorithm
1: Derive empirical lower bound on PA kernel score $\mathbb{E}_{X',X''}\log k_{\mathcal{A}}(X',X'') \ge \frac{1}{L}\sum_{l\le L}\log k_{\mathcal{A}}(X'_l,X''_l) -$ penalty
2: Estimate the optimal empirical posterior distribution:
3: $\mathbf{P}^{\mathcal{A}}_{\mathrm{opt}}(.\,|\,.) \in \arg\max_{\mathcal{A}}(\frac{1}{L}\sum_{l\le L}\log k_{\mathcal{A}}(X'_l,X''_l) -$ penalty)
4: Sample hypotheses $\theta \sim \mathbf{P}^{\mathcal{A}}_{\mathrm{opt}}(\theta\,|\,X''')$ from optimized posterior $\mathbf{P}^{\mathcal{A}}_{\mathrm{opt}}$ given future data $X'''$ or from "posterior agreement" $\theta \sim \mathbf{P}^{\mathcal{A}}_{\mathrm{opt}}(\theta\,|\,X')\mathbf{P}^{\mathcal{A}}_{\mathrm{opt}}(\theta\,|\,X'')$

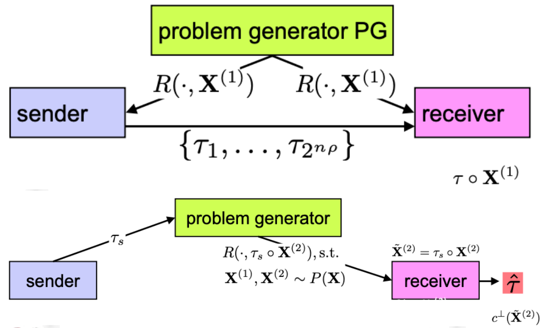
$\mathbf{P}(X'_{\mathcal{E}},X''_{\mathcal{E}}) = \sum_\theta\mathbf{P}(X'_{\mathcal{E}},X''_{\mathcal{E}}|\theta)\mathbf{P}(\theta) = \sum_\theta\frac{\mathbf{P}(X'_{\mathcal{E}},X''_{\mathcal{E}}|\theta)}{\mathbf{P}(X'_{\mathcal{E}}|\theta)\mathbf{P}(X''_{\mathcal{E}}|\theta)}\mathbf{P}(X'_{\mathcal{E}}|\theta)\mathbf{P}(X''_{\mathcal{E}}|\theta)\mathbf{P}(\theta) = \sum_\theta\frac{\mathbf{P}(X'_{\mathcal{E}},X''_{\mathcal{E}}|\theta)}{\mathbf{P}(X'_{\mathcal{E}}|\theta)\mathbf{P}(X''_{\mathcal{E}}|\theta)}\frac{\mathbf{P}(\theta|X'_{\mathcal{E}})}{\mathbf{P}(\theta)}\mathbf{P}(X'_{\mathcal{E}})\frac{\mathbf{P}(\theta|X''_{\mathcal{E}})}{\mathbf{P}(\theta)}\cdot\mathbf{P}(X''_{\mathcal{E}})\mathbf{P}(\theta)$, $\mathcal{W}(X'_{\mathcal{E}},X''_{\mathcal{E}},\theta) \le \mathbf{P}(X'_{\mathcal{E}})\mathbf{P}(X''_{\mathcal{E}})\cdot\sum_{\theta\in\Theta}\frac{\mathbf{P}(\theta|X'_{\mathcal{E}})\mathbf{P}(\theta|X''_{\mathcal{E}})}{\mathbf{P}(\theta)}\max_\theta\{\mathcal{W}(X'_{\mathcal{E}},X''_{\mathcal{E}},\theta)\}$ Insert in Mutual information $\mathcal{I}(X'_{\mathcal{E}},X''_{\mathcal{E}}) = \mathbb{E}_{X'_{\mathcal{E}},X''_{\mathcal{E}}}[\log\frac{\mathbf{P}(X'_{\mathcal{E}},X''_{\mathcal{E}})}{\mathbf{P}(X'_{\mathcal{E}})\mathbf{P}(X''_{\mathcal{E}})}] = \mathbb{E}_{X'_{\mathcal{E}},X''_{\mathcal{E}}}[\log(\sum_{\theta\in\Theta}\frac{\mathbf{P}(\theta|X'_{\mathcal{E}})\mathbf{P}(\theta|X''_{\mathcal{E}})}{\mathbf{P}(\theta)}\mathcal{W}(X'_{\mathcal{E}},X''_{\mathcal{E}},\theta))] \le \mathbb{E}_{X'_{\mathcal{E}},X''_{\mathcal{E}}}[\log\sum_{\theta\in\Theta}\frac{\mathbf{P}(\theta|X'_{\mathcal{E}})\mathbf{P}(\theta|X''_{\mathcal{E}})}{\mathbf{P}(\theta)}] + \mathbb{E}_{X'_{\mathcal{E}},X''_{\mathcal{E}}}[\log\max_\theta\{\mathcal{W}(X'_{\mathcal{E}},X''_{\mathcal{E}},\theta)\}]$ MI → weighted version of PA bounded above by PA and constant (vanishes if solutions $\theta$ are suff. stat of $X'_{\mathcal{E}},X''_{\mathcal{E}}$. **Binary sym channel:** minimize Hamming dist $\in[0,n]$, $x,c\in\{-1,1\}^n$ $d(c,x) = \sum_{i=1}^n\frac{1}{2}(1-x_ic_i) = \frac{n}{2} - \frac{1}{2}\sum_{i=1}^n x_ic_i$, $R^{\mathrm{Ham}}(c,x) = -\sum_{i=1}^n x_ic_i$ $p(c|X) = \prod_1^n\frac{\exp(\beta x_ic_i)}{\sum_c\exp(\beta\sum_1^n x_ic_i)} = \prod_1^n\frac{\exp(\beta x_ic_i)}{2\cosh(\beta x_i)}$, $Z^2 = 2^{2n}(\cosh\beta)^{2n}$ Expected alignment $\mathbb{E}_{c|x}\frac{1}{n}\sum_{i=1}^n x_ic_i = \tanh\beta$, Channel noise $\delta := \frac{1}{n}\sum_{i\le n}\mathbb{I}_{\{x'_i\neq x''_i\}}$ $\frac{1}{2}|\mathbf{x}'-\mathbf{x}''| = n\delta$, $k(x',x'') = \sum_c\frac{e^{-\beta x'c}}{Z'}\frac{e^{-\beta x''c}}{Z''} = \frac{1}{Z^2}\sum_c\prod e^{-\beta(x'_i+x''_i)c_i} = \frac{1}{Z^2}\prod e^{-\beta(x'_i+x''_i)} + e^{\beta(x'_i+x''_i)c_i} = \frac{1}{Z^2}2^n(\cosh2\beta)^{n(1-\delta)}$ $I = \mathbb{E}_{x',x''}\log|c|k(x',x'') \propto n(1-\delta)\cdot\log\cosh2\beta - 2n\log\cosh\beta \to \frac{\partial I}{\partial\beta} = 0$ $\cosh\beta = \sqrt{\frac{1}{2}(\cosh2\beta+1)} = (2\delta)^{-1/2}$ $\frac{1}{n}I(\beta^{opt}) = (1-\delta)\log\cosh2\beta - 2\log\cosh\beta = \log2 - H_{bin}(\delta)$, for generalized ofSDL: $2\delta\tanh\beta - (1-2\delta)\frac{\beta}{(\cosh\beta)^2} = 0$

## Info Theoretical Algorithm Validation

Data noise reduces solution resolution; simple solution space leads to underfitting. Data analysis focuses on concentrated data distributions with signals and noise. Worst case analysis neglects the signal. *Objective:* rank algorithms based on signal sensitivity and noise robustness.

**Algorithm 3** Structure Validation IT
1: Sample a hypothesis $\tilde c \sim p^{\mathcal{A}}(c\,|\,\mathbf{X})$
2: **for** $j = 1,\dots,M$ **do**
3: Select random transformation $\tau_j \in \mathbb{T}$
4: Define code vector $\tilde c_i \sim p^{\mathcal{A}}(c\,|\,\tau_i\circ\mathbf{X})$
5: **end for**
6: **return** Codebook $\mathcal{T} := \{\tilde c_1,\dots,\tilde c_M\}$

**Approximation sets (AS):** data in high dim space, measure sensitivity sol sets to noise. **Coding** with sets of approximate rankings: define set code problems. **Communication** by AS: estimate coding error




**Algorithm 4** Communication by approximation sets
1: Sender sends transformation $\tau_s$ to problem generator
2: Problem generator sends a new problem with transformed indices to receiver without revealing $\tau_s$.
3: Receiver identifies transformation $\hat\tau$ by comparing approximation sets.

**Approximate Sorting:** given $N$ objects $o_1..o_i..o_N$, rank them based on pairwise noisy comparisons represented by dataset $X$, with $X_{ij} = 1$ if $o_i < o_j$, 0 else, $X_{ii} = 0$ and $X_{ij} = 1 - X_{ji}\ \forall i, j \neq i$, feasible solutions = all permutations $c$ over items, where $c_i = \ell$ = position (rank) of obj $o_i$ $\mathcal{R}^{sort}(c|X) := |\{(i,j)|X_{ij} = 1 \text{ but } c_i > c_j\}|$ Let $M$ assignment matrix with $M_{il} = 1$ if $o_i$ gets rank $l$, 0 else, $\sum_{\ell i}M_{i\ell} = 1$, item/rank get unique rank/item, $M(i) = \ell \Leftrightarrow M_{i\ell} = 1$ $\mathcal{R}^{\mathrm{sort}}(M|X) = \sum_i\sum_j\sum_\ell\sum_{\bar\ell>\ell}X_{ij}M_{i\bar\ell}M_{j\ell}$ **MFA:** $q_{i\ell} := \frac{e^{-\beta\mathcal{E}_{i\ell}}}{\sum_h e^{-\beta\mathcal{E}_{ih}}}$, $Q(M|\mathcal{E}) = q_{1M(1)}\cdot q_{2M(2)}\cdots q_{NM(N)}$, $\mathcal{E}_{i\ell} = \mathbb{E}_{M\sim Q_{i\to\ell}}[\mathcal{R}^{\mathrm{sort}}]$, in before equation notation means $o_i$ to rank $l$, Q satisfies this constraint, not both C1) $X_{ij} = 1 \Rightarrow i\to\ell$ and $j\to\underline{\ell}$ with $\underline{\ell}<\ell$

C2) $X_{ji} = 1 \Rightarrow i \to \ell$ and $j \to \underline{\ell}$ with $\overline{l} > \ell$
C3) $X_{ab} = 1 \Rightarrow a \to \overline{\ell}$ and $b \to \underline{\ell}$ w/ $\underline{\ell} < \overline{\ell}$
$\mathcal{E}_{i\ell} = \sum_j \sum_{\underline{\ell}<\ell} X_{ij} q_{j\underline{\ell}} + \sum_j \sum_{\overline{l}<\ell} X_{ji} q_{j\overline{\ell}}$ $+ \sum_{a \neq i} \sum_{b \neq i} X_{ab} \sum_{\underline{\ell}} \sum_{\overline{l} > \underline{\ell}} q_{a\overline{\ell}} q_{b\underline{\ell}}$, const
$\mathcal{R}^{MF}_{fix}(M|\mathcal{E}) = \sum_{i,l} M_{il}\mathcal{E}_{il} + \sum_\ell \lambda_\ell (\sum_j M_{j\ell} - 1)$ $= \sum_{il} M_{il}(\mathcal{E}_{il} + \lambda_l) + \text{cst}$, $\lambda_l$ from $\sum_i q_{i\ell} = 1$
**sMBP:** given $G = (V, E, W)$, select subset of nodes $U$, find min bisection for $U$. If $|U| 0\Theta(n^{2/7}) \to$ NP-hard. **REM:** given $N = \Theta(\exp(n))$ states $J \in \mathcal{H}$, i.i.d. score $(R(c, \mathbf{X}))$ values score$(J) \sim \mathcal{N}(0, \sqrt{n})$. Find $\widehat{J}_{\max} = \arg\max_{J \in \mathcal{H}} \text{score}(J)$, model max disordered and without structure for searching $n^s = \Theta(n^{2/7})$. By construction, different solutions $J', J''$ do not share common parameters (statistical independent) *sMBP is lower/upper bounded by REM:*
$\lim_{n\to\infty} \frac{\mathbb{E}[\log Z(\beta,\mathbf{X})] + \hat{\beta}\mu\sqrt{N \log m}}{\log m}$
$= 1 + \frac{\hat{\beta}^2\sigma^2}{2}$ if $\hat{\beta}\sigma < \sqrt{2}$ or $= \hat{\beta}\sigma\sqrt{2}$ (else)
**Generalization capacity** of sMBP for noise perturbed random graph: $\mathbf{X}' = \mathbf{X} + \delta\mathbf{X}', \mathbf{X}'' = \mathbf{X} + \delta\mathbf{X}'', \gamma = \sigma/\tilde{\sigma}$, 2 transitions
$\lim_{n\to\infty} \frac{\mathbb{E}_{X,\delta X',\delta X''} \log(|\mathcal{C}|\hat{k}_\beta(X',X''))}{\log m} = \eta(\hat{\beta})$

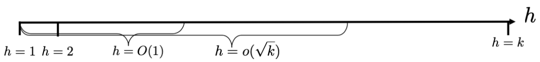
$$\eta(\hat{\beta}) = \begin{cases} (\hat{\beta}\tilde{\sigma})^2, & \hat{\beta}\tilde{\sigma} < \frac{\sqrt{2}}{\sqrt{4+2\gamma^2}} \\ \hat{\beta}\tilde{\sigma}\sqrt{2}\sqrt{4+2\gamma^2} - (\hat{\beta}\tilde{\sigma})^2(1+\gamma^2) - 1, & \frac{\sqrt{2}}{\sqrt{4+2\gamma^2}} \le \hat{\beta}\tilde{\sigma} < \frac{\sqrt{2}}{\sqrt{1+\gamma^2}} \\ \hat{\beta}\tilde{\sigma}\sqrt{2}\left(\sqrt{4+2\gamma^2} - 2\sqrt{1+\gamma^2}\right) + 1, & \frac{\sqrt{2}}{\sqrt{1+\gamma^2}} \le \hat{\beta}\tilde{\sigma} \end{cases}$$

REM has no information for searching $\to$ evaluate $M = 2^n$ cost values. Random sMBP, CDP $\approx$ REM (free E gen. func.)

## Learning and algorithmic complexity

**Strategy for combinatorial search:** 1) Define concept- and structurally simple comb. problem P1 2) Define reference prob. P2 where exhaustive search is optimal 3) Study relation betw. P1 and P2.
**PREM (P2):** given $N = O(\exp(n))$ states $J \in \mathcal{H}$ with scores $X_J \sim \mathcal{N}(X \mid 0, \sigma^2)$, define planted state $I^\star \sim \mathcal{N}(X|\mu, 1)$, find state $\widehat{J}_{\max} = \arg\max_{J \in \mathcal{H} \cup I^\star} \text{score}(J)$. *Signal:* Bias $\mu$, unbiased scores act as observation noise. Minimal bias $\mu_0$ for recovery in probability? (Unlimited positive bias can't improve recovery beyond exhaustive search due to lack of shared information with neighboring states) **Recover prob:**
$\mathbf{P}_{\mathcal{X}}(\max_{J \in \mathcal{H}\setminus\{I^\star\}} X_J < X_{I^\star}) \xrightarrow{n\uparrow\infty} 1$

**Success prob.** to recover Planted REM:
$\mathbf{P}(\widehat{J} = I^\star) \geq \exp(-\exp(\ln|\mathcal{H}| - \frac{\mu^2}{4}))$
0/1 behaviour, threshold $\mu_0 = 2\sqrt{\ln|\mathcal{H}|}$:
$\lim_{n\uparrow\infty} \mathbf{P}(\widehat{J} = I^\star) = 1$ if $\mu > \mu_0$, $= 0$ else
$\mathbf{P}(\hat{J} = I^\star) = \mathbb{E}_{X_{I^\star}} \mathbf{P}(\hat{J} = I^\star | X_{I^\star}) = \mathbb{E}_{X_{I^\star}}$
$\mathbf{P}(\bigwedge_{J \neq I^\star} X_J < X_{I^\star} | X_{I^\star}) = \mathbb{E}_{X_{I^\star}} \prod_{J \neq I^\star}$
$\mathbf{P}(X_J < X_{I^\star}|X_{I^\star}) = \mathbb{E}_{X_{I^\star}} (\int_{-\infty}^{X_{I^\star}} d\mathcal{N}(z)$
$)^{|\mathcal{H}|-1} = \mathbb{E}_{X_{I^\star}}(1 - \int_{X_{I^\star}}^\infty d\mathcal{N}(z))^{|\mathcal{H}|-1} \geq (1 -$
$\mathbb{E}_{X_{I^\star}} \int_{X_{I^\star}}^\infty d\mathcal{N}(z))^{|\mathcal{H}|} \geq (1 - \mathbb{E}_{X_{I^\star}} \frac{\mathcal{N}(X_{I^\star})}{X_{I^\star}})^{|\mathcal{H}|}$
$\approx \exp(-|\mathcal{H}| \exp(-\mu^2/4))$
**Note:** $\frac{1}{u}e^{-u^2/2}(1 + u^{-2}) \leq \int_u^\infty e^{-z^2/2} dz \leq \frac{1}{u}e^{-u^2/2}$, $d\mathcal{N}(z) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2})dz$
**Planted sub-hypergraph (P1):** Hypothesis $J$ is set of $k$ indices $J = \{j_1, \ldots, j_k\} \subset \{1, \ldots, n\} =: [n]$, $\mathcal{H} = \{J \subset [n] : |J| = k\}$
$|\mathcal{H}| = \binom{n}{k} < (\frac{ne}{k})^k$, sol $I^\star = \{i_1^\star, \ldots, i_k^\star\}$
Subsets $H$ of $h$ items assigned score $X_H, H = \{j_1, \ldots, j_h\}$ Scores $X_H$ are data of planted sub-hypergraph recovery. score$(J) = \sum_{H \subset J:|H|=h} X_H$. All proper subsets $H$ of planted sol $I^\star$ are biased.
$X_H = \begin{cases} \mu + \xi_H & H \subset I^\star \\ \xi_H & H \not\subset I^\star \end{cases}$, $\xi_H \sim \mathcal{N}(0, \sigma^2)$
**Note:** $J$ defined by Bin$(k, h)$ parameters $X_H$. When $h$ increases, localize planted solution $I^\star$ by using Bin$(n, h)$ RV as parameters (localization of solutions).
**Interaction degree** $h$: $h = 1$ sort the $X_i$ values, select $k$ largest score$(J) = \sum_{i \in J} X_i$, $h = 2$ find $k$-clique with only biased edges, score$(J) = \sum_{(i,j) \in J \times J} X_{i,j}$, $h \geq 3$ sub-hypergraph recovery, $h = k$ identify set $J = \arg\max_{J \subset [n]} X_J$ employing exhaustive search of Bin$(n, k)$ states, score$(J) = X_J$.




Increasing $h \to k$ localizes influence planted solution $I^\star$ up to single state at expense of increasing amount of (random) data.
**REM recovery:** $k$ sub-hypergraph recovery with $h = k$ interaction suppresses parameter sharing and eliminates statistical dependence between solutions. This independence implies REM behavior since all of the Bin$(n, k)$ subsets of size $k$ of $n$ items are characterized by individual score value.

**Algorithm 5** Hypergraph recovery problem
1: Define spectrum of hypergraph recovery prob $(1 \leq h \leq k)$ with increased localization of solutions.
2: Characterize the statistical dependencies of solutions.
3: Compare the community detection case $h = \mathcal{O}(n^0)$ with REM as limit case $h = k$ for $k = n^\alpha, 0 < \alpha < \frac{1}{2}$.
4: Estimate number of effectively independent solutions to measure the computational complexity of the search problem.
5: Prove absence of effective statistical tests to amplify asymptotically vanishing search information.

*How many states share param with planted solution $I^\star$?* Assume $|J \cap I^\star| = l \geq h \Rightarrow$ Bin$(k, l) \cdot$ Bin$(n - k, n - l)$. $\mathcal{D}(I^\star)$ set of solu $J$ statistically dependent on $I^\star$ (share at least 1 param). Let $|\mathcal{D}(I^\star)| = \sum_{l=h}^k \text{Bin}(k, l)\,\text{Bin}(n - k, n - l)$, $k = n^\alpha, 0 < \alpha < \frac{1}{2}$, Bin$(k, h) \cdot$ Bin$(n - k, n - h)$, then $(*) \leq |\mathcal{D}(I^\star)| \leq (*) \cdot (k - h + 1)$,
*Proof:* upper bound holds since $(*)$ monotonically decreases in $l$ for $k = o(\sqrt{n})$.
**Effective independence:** if sol $J$ shares $l > h$ items with $I^\star$, then Bin$(l, h)$ parameters of $J$ are biased and Bin$(k, h) -$ Bin$(l, h)$ are unbiased. If total bias much small than standard dev, (Bin$(l, h)\mu \ll \sqrt{\text{Bin}(k, h)\sigma}$), then cant distinguish betw sol with $l$ shared items and independent solution due to fluctuations. Assume that overlap $h \leq l \leq l_\perp$ ($l_\perp$ cutoff parameter) too weak to significant dependence, # effectively dependent states: $|\mathcal{D}^{\text{eft}}(I^\star)| = \sum_{l=l_\perp}^k \text{Bin}(k, l) \cdot \text{Bin}(n - k, k - l) \approx \text{Bin}(k, l_\perp)\,\text{Bin}(n - k, k - l_\perp)(k - l_\perp + 1)$
**Sub hyp det rate:** $\mathfrak{R} = \ln \frac{|\mathcal{H}|}{|\mathcal{D}^{\text{eft}}(I^\star)||\mathcal{X}|}$
$\geq \ln \text{Bin}(n, k) - \ln[\text{Bin}(k, l_\perp) \cdot \text{Bin}(n - k, k - l_\perp)(k - l_\perp + 1)] - \ln \text{Bin}(n, h)$
If $\mathfrak{R} = \mathcal{O}(n^\beta)$, exponentially many independent states that define REM and this REM is exponentially larger than the number of parameters! Since a REM requires exhaustive search, conclude that searching for planted sub-hypergraph is exponentially more expensive than testing. **Stirling:**
$n! = \sqrt{2\pi n}(\frac{n}{e})^n \cdot (1 + \frac{1}{12n} + \mathcal{O}(n^{-2}))$
$\ln \text{Bin}(a, b) \cong b \ln \frac{a}{b} + b(1 - r) \leq b(\ln \frac{a}{b} + 1)$
with $r = \sum_{\nu=1}^\infty \frac{1}{\nu(\nu+1)} \left(\frac{b}{a}\right)^\nu +$
$\frac{1}{2b} \ln 2\pi b \left(1 - \frac{b}{a}\right) = \mathcal{O}\left(\max\left\{\frac{b}{a}, \frac{1}{b}\right\}\right)$
**Recovery Theo:** if $k = n^\alpha, l_\perp = n^\beta, h = \mathcal{O}(n^0), 2\alpha < 1$, then $\mathfrak{R} \geq l_\perp \ln \frac{l_\perp n}{k^2} - l_\perp - h(\ln \frac{n}{h} + 1) - \ln(k - l_\perp + 1) = \mathcal{O}(n^\beta \ln n)$
For case $\mathcal{D}(I^\star) = \mathcal{D}^{\text{eft}}(I^\star)$ with $l_\perp = h$, rate scale polynomial in $n$ with $\mathfrak{R} = \mathcal{O}(h \ln n)$, $\oplus$ refined if $h = \mathcal{O}(n^\gamma)$. *Proof:* $\geq k(\ln \frac{n}{k} + 1) - kr(n, k) - l_\perp(\ln \frac{k}{l_\perp} + 1) - (k - l_\perp)(\ln \frac{n-k}{k-l_\perp} + 1) - \ln(k - l_\perp + 1) - h(\ln \frac{n}{h} + 1)$

The terms linear in $l_\perp$ define the leading contributions in $n$ (since the terms linear in $k$ vanish asymptotically), i.e.,

$$-l_\perp(\ln \frac{k}{l_\perp} + 1) + l_\perp(\ln \frac{n-k}{k-l_\perp} + 1) = l_\perp \ln \frac{(n-k)l_\perp}{(k-l_\perp)k} \geq l_\perp \ln \frac{nl_\perp}{k^2} = (1 + \beta - 2\alpha)\, n^\beta \ln n = \mathcal{O}(n^\beta \ln n)$$

The sum of terms proportional to $k$ is negative; therefore, we bound its absolute value from above, i.e.,

$$-k(\ln \frac{n}{k} + 1) + k(\ln \frac{n-k}{k-l_\perp} + 1) + kr(n,k) = k \ln \frac{k(n-k)}{n(k-l_\perp)} + kr(n,k) = k \ln \frac{1-k/n}{1-l_\perp/k} + kr(n,k)$$
$$= k \sum_{\nu=1}^\infty \frac{1}{\nu}\left(\frac{l_\perp^\nu}{k^\nu} - \frac{k^\nu}{n^\nu}\right) + kr(n,k) = l_\perp - \frac{k^2}{n} + \sum_{\nu=2}^\infty \frac{k}{\nu}\left(\frac{l_\perp^\nu}{k^\nu} - \frac{k^\nu}{n^\nu}\right) + kr(n,k) \leq l_\perp + kr(n,k) = \mathcal{O}(n^\beta),$$

since $kr(n, k) \sim \frac{k^2}{n} = O(n^{2\alpha-1}) \to 0$ for $2\alpha < 1$. Collecting all the terms yields the final rate

$$\mathfrak{R} \geq l_\perp \ln \frac{nl_\perp}{k^2} - l_\perp - h(\ln \frac{n}{h} + 1) - \ln(k - l_\perp + 1) = \mathcal{O}(n^\beta \ln n)$$

since the positive term $l_\perp \ln \frac{nl_\perp}{k^2} = \mathcal{O}(n^\beta \ln n)$ dominates the negative terms that scale as $l_\perp = \mathcal{O}(n^\beta)$ and it also dominates $h(\ln \frac{n}{h} + 1) + \ln(k - l_\perp + 1) = \mathcal{O}(\ln n)$ in this scaling limit $h = \mathcal{O}(1)$.

**Exponential scaling:** $h = O(n^\gamma), 0 < \gamma \leq \beta < \alpha, |\mathcal{X}| = \text{Bin}(n, h) = O(n^{(n^\gamma)})$, $l_\perp = \omega h$ with $\omega = O(1)$, $\mathfrak{R} \geq l_\perp \ln \frac{nl_\perp}{k^2} - l_\perp - h(\ln \frac{n}{h} + 1) - \ln(k - l_\perp + 1) \cong \omega h \ln \frac{n\omega h}{k^2} - \omega h - h(\ln \frac{n}{h} + 1)$, $\frac{\mathfrak{R}}{h \ln n} \geq \omega(1 + \gamma - 2\alpha) - 1 + \gamma + \frac{\omega(\ln \omega - 1) - 1}{\ln n} > 0$, $\alpha - \gamma = \frac{\psi}{\ln n}, \omega - 1 = \frac{\tau}{\ln n}$
Rate is positive in asymptotic limit ($\mathcal{R} \geq n^\gamma$) for $\tau > 2\frac{\psi+1}{1-\alpha} \Rightarrow \frac{\mathfrak{R}}{h \ln n} \geq 0$, so $\omega > 1 + \frac{2}{1-\alpha}(\alpha - \gamma) + \frac{2}{(1-\alpha)\ln n}$
**Small overlap:** sol $J'$ weakly stat dependent on $I^\star$ ($|J' \cap I^\star| = l > h$); sol$J''$ stat independent on $I^\star$,($|J'' \cap I^\star| < h$); $J'$ and $J''$ are stat independent ($|J' \cap J''| < h$)

$$\text{score}(J') = \sum_{H \subset J' \cap I^\star} X_H + \sum_{\tilde{H} \subset J' \wedge \tilde{H} \not\subset I^\star} X_{\tilde{H}} \sim \mathcal{N}\left(\binom{l}{h}\mu, \binom{k}{h}\sigma^2\right)$$

$$\text{score}(J'') = \sum_{H \subset J''} X_H \sim \mathcal{N}\left(0, \binom{k}{h}\sigma^2\right)$$

score$(J', J'') \in \text{Bin}(l, h)\mu \pm \sqrt{\text{Bin}(k, h)\sigma}$
select maximal overlap $l_\perp$ as Bin$(l_\perp, h) = \epsilon\sqrt{\text{Bin}(k, h)}\frac{\sigma}{\mu} \to l_\perp = \sqrt{k}(\epsilon\frac{\sigma}{\mu})^{\frac{1}{h}}$, Induce subsampled hyp class $\mathcal{H}^\Delta$, with $|\mathcal{H}^\Delta| = \frac{|\mathcal{H}|}{|\mathcal{D}^{\text{eft}}(I^\star)|} = \mathcal{O}(n \exp(n^\beta))$, REM recovery condition: $\frac{\mu}{\sigma} > 2\sqrt{\ln|\mathcal{H}^\Delta|}$, $\beta, \alpha = \frac{1}{7}, \frac{1}{3}$
$l_\perp = n^\beta < \sqrt{k}(\frac{\epsilon}{2\sqrt{\ln|\mathcal{H}^\Delta|}})^{\frac{1}{h}} = \mathcal{O}(n^{\frac{\alpha}{2}} n^{-\frac{\beta}{2h}})$
**Ising Model,** $N = \#$ **pixel,** $\lambda \,\&\, J_{ij} \geq 0$
$E(\boldsymbol{\sigma}|\mathbf{h}) = -\lambda \sum_{i=1}^N h_i \sigma_i - \sum_{\langle i,j \rangle} J_{ij} \sigma_i \sigma_j$,
$\mathbf{h} = \{h_i\}$ noisy img, $\boldsymbol{\sigma} = \{\sigma_i, h_i\}(\sigma_i \in \{-1, 1\})$ denoised img (sol), like, prior
**MCMC:** $p_i(\boldsymbol{\sigma}) = \exp(-\beta \max[0, \Delta E_i(\boldsymbol{\sigma})])$
$P(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = \frac{1}{N} p_i(\boldsymbol{\sigma})$ if $\boldsymbol{\sigma}' = \boldsymbol{\sigma}^{(i)}$ for some $i$, $= 1 - \frac{1}{N} \sum_{i=1}^N p_i(\boldsymbol{\sigma})$ if $\boldsymbol{\sigma}' = \boldsymbol{\sigma}$ , else 0.

Irreducible and stationary, detailed balance hold, periodic (if $E(\boldsymbol{\sigma}) = E$ for all state, period 2, only case → no loop = E const)

*Periodic p-prior*: $-\sum_{i=1}^{N-p} \sigma_i \sigma_{i+p}$ (E is sum of $p$ 1-periodic Ising energies)

**Algorithm 1** Metropolis-Hastings

Define $\{q(\cdot \mid c)\}_{c \in C}$ s.t $\mathcal{G}_q$ is connected, $q(c \mid c) > 0$
$c_0 \leftarrow \$$
**for** $t = 1, 2, \ldots$ **do**
  $\tilde{c} \xleftarrow{\$} q(\cdot \mid c_{t-1})$
  $b \xleftarrow{\$} \mathrm{Ber}(\min\{1, \frac{q(c_{t-1}|\tilde{c})p(\tilde{c})}{q(\tilde{c}|c_{t-1})p(c_{t-1})}\})$
  **if** $b = 1$ **then**
    Set $c_t = \tilde{c}$ (Accept the proposal)
  **else**
    Set $c_t = c_{t-1}$ (Reject the proposal)
  **end if**
  $t \leftarrow t + 1$
**end for**

**Algorithm 2** Posterior Selection Algorithm

Derive empirical lower bound on PA kernel score $\mathbb{E}_{X',X''} \log k_{\mathcal{A}}(X', X'') \geq \frac{1}{L}\sum_{l \leq L} \log k_{\mathcal{A}}(X'_l, X''_l) -$ penalty
Estimate the optimal empirical posterior distribution:
$\mathbf{P}^{\mathcal{A}}_{\text{opt}}(. \mid .) \in \arg\max_{\mathcal{A}} \left(\frac{1}{L}\sum_{l \leq L} \log k_{\mathcal{A}}(X'_l, X''_l) - \text{ penalty}\right)$
Sample hypotheses $\theta \sim \mathbf{P}^{\mathcal{A}}_{\text{opt}}(\theta \mid X''')$ from optimized posterior $\mathbf{P}^{\mathcal{A}}_{\text{opt}}$ given future data $X'''$ or from "posterior agreement" $\theta \sim \mathbf{P}^{\mathcal{A}}_{\text{opt}}(\theta \mid X')\, \mathbf{P}^{\mathcal{A}}_{\text{opt}}(\theta \mid X'')$

**Algorithm 3** Structure Validation IT

Sample a hypothesis $\tilde{c} \sim p^{\mathcal{A}}(c \mid \mathbf{X})$
**for** $j = 1, \ldots, M$ **do**
  Select random transformation $\tau_j \in \mathbb{T}$
  Define code vector $\tilde{c}_i \sim p^{\mathcal{A}}(c \mid \tau_i \circ \mathbf{X})$
**end for**
**return** Codebook $\mathcal{T} := \{\tilde{c}_1, \ldots, \tilde{c}_M\}$

**Algorithm 4** Communication by approximation sets

Sender sends transformation $\tau_s$ to problem generator
Problem generator sends a new problem with transformed indices to receiver without revealing $\tau_s$.
Receiver identifies transformation $\hat{\tau}$ by comparing approximation sets.

$$\eta(\hat{\beta}) = \begin{cases} (\hat{\beta}\tilde{\sigma})^2, & \hat{\beta}\tilde{\sigma} < \frac{\sqrt{2}}{\sqrt{4+2\gamma^2}} \\ \hat{\beta}\tilde{\sigma}\sqrt{2}\sqrt{4+2\gamma^2} - (\hat{\beta}\tilde{\sigma})^2\left(1+\gamma^2\right) - 1, & \frac{\sqrt{2}}{\sqrt{4+2\gamma^2}} \leq \hat{\beta}\tilde{\sigma} < \frac{\sqrt{2}}{\sqrt{1+\gamma^2}} \\ \hat{\beta}\tilde{\sigma}\sqrt{2}\left(\sqrt{4+2\gamma^2} - 2\sqrt{1+\gamma^2}\right) + 1, & \frac{\sqrt{2}}{\sqrt{1+\gamma^2}} \leq \hat{\beta}\tilde{\sigma} \end{cases}$$

**Algorithm 5** Hypergraph recovery problem

1: Define spectrum of hypergraph recovery prob
$(1 \leq h \leq k)$ with increased localization of solutions.
2: Characterize the statistical dependencies of solutions.
3: Compare the community detection case $h = \mathcal{O}\left(n^0\right)$ with
REM as limit case $h = k$ for $k = n^\alpha, 0 < \alpha < \frac{1}{2}$.
4: Estimate number of effectively independent solutions to
measure the computational complexity of the search problem.
5: Prove absense of effective statistical tests to
amplify asymptotically vanishing search information.

$$\text{score}\left(J'\right) = \sum_{H \subset J' \cap I^\star} X_H + \sum_{\tilde{H} \subset J' \wedge \tilde{H} \not\subset I^\star} X_{\tilde{H}} \sim \mathcal{N}\left(\left(\begin{array}{c} l \\ h \end{array}\right)\mu, \left(\begin{array}{c} k \\ h \end{array}\right)\sigma^2\right)$$

$$\text{score}\left(J''\right) = \sum_{H \subset J''} X_H \sim \mathcal{N}\left(0, \left(\begin{array}{c} k \\ h \end{array}\right)\sigma^2\right)$$