Dynamics of a Single Particle:

kinematics of a single particle:

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}, \qquad \boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$$

velocity and acceleration components in **polar coordinates** (r, φ):

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\varphi}\boldsymbol{e}_\varphi, \qquad \boldsymbol{a} = (\ddot{r} - r\dot{\varphi}^2)\boldsymbol{e}_r + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\boldsymbol{e}_\varphi$$

velocity and acceleration components using the **space curve** description:

$$\boldsymbol{v} = \dot{s}\boldsymbol{e}_t, \qquad \boldsymbol{a} = \ddot{s}\boldsymbol{e}_t + \frac{v^2}{\rho}\boldsymbol{e}_n \qquad \text{with} \qquad \boldsymbol{e}_t = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \frac{\boldsymbol{v}}{v}, \quad \boldsymbol{e}_n = \frac{\dot{\boldsymbol{e}}_t}{|\dot{\boldsymbol{e}}_t|} = \frac{\rho}{v}\dot{\boldsymbol{e}}_t$$

balance of linear momentum for a particle of constant mass m:

$$\sum_i \boldsymbol{F}_i = \dot{\boldsymbol{P}} = \frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v})$$

work–energy balance for a particle of constant mass m:

$$T(t_2) - T(t_1) = W_{12}, \qquad T(t) = \frac{1}{2}m|\boldsymbol{v}(t)|^2, \qquad W_{12} = \sum_i \int_{t_1}^{t_2} \boldsymbol{F}_i \cdot \boldsymbol{v}\,\mathrm{d}t = \sum_i \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}$$

for a **conservative force**:

$$\boldsymbol{F} = -\frac{\mathrm{d}V}{\mathrm{d}\boldsymbol{r}} \qquad \Rightarrow \qquad W_{12} = V(\boldsymbol{r}_1) - V(\boldsymbol{r}_2)$$

conservation of energy for a conservative system:

$$T + V = \text{const.}$$

balance of angular momentum of a particle with respect to point B:

$$\boldsymbol{M}_\mathrm{B} = \dot{\boldsymbol{H}}_\mathrm{B} + \boldsymbol{v}_\mathrm{B} \times \boldsymbol{P} \qquad \text{with} \qquad \boldsymbol{H}_\mathrm{B} = \boldsymbol{r}_\mathrm{BP} \times \boldsymbol{P}, \quad \boldsymbol{P} = m\boldsymbol{v}_\mathrm{P}$$

special case of a **rotation in 2D** around a fixed point B at a distance R:

$$M_\mathrm{B} = I_\mathrm{B}\ddot{\varphi} \qquad \text{with} \qquad I_\mathrm{B} = mR^2$$

particle **collision** with a **frictionless rigid wall**:

$$v_n(t_+) = -e\,v_n(t_-), \qquad v_t(t_+) = v_t(t_-)$$

Dynamics of Systems of Particles:

center of mass and **total mass** of a system of n particles:

$$\boldsymbol{r}_{\mathrm{CM}}(t) = \frac{1}{M}\sum_{i=1}^{n} m_i \boldsymbol{r}_i(t), \qquad M = \sum_{i=1}^{n} m_i$$

balance of linear momentum for $M = \text{const.}$:

$$\sum_{i=1}^{n} \boldsymbol{F}_i^{\mathrm{int}} = \boldsymbol{0}, \qquad \sum_{i=1}^{n} \boldsymbol{F}_i^{\mathrm{ext}} = \dot{\boldsymbol{P}} = \frac{\mathrm{d}}{\mathrm{d}t}(M\boldsymbol{v}_{\mathrm{CM}}) \qquad \Rightarrow \qquad \sum_{i=1}^{n} \boldsymbol{F}_i^{\mathrm{ext}} = M\boldsymbol{a}_{\mathrm{CM}}$$

balance of linear momentum for a particle of varying mass $m(t)$:

$$\sum_{i} \boldsymbol{F}_i = m\boldsymbol{a} + \dot{m}(\boldsymbol{v} - \boldsymbol{v}_m)$$

conservation of linear momentum:

$$\text{if } \sum_{i=1}^{n} \boldsymbol{F}_i^{\mathrm{ext}} = \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{P} = \sum_{i=1}^{n} m_i \boldsymbol{v}_i = \text{const.}$$

work–energy balance:

$$T(t_2) - T(t_1) = W_{12} \qquad \text{with} \qquad T(t) = \frac{1}{2}\sum_{i=1}^{n} m_i |\boldsymbol{v}_i(t)|^2, \quad W_{12} = \sum_{i=1}^{n} \int_{\boldsymbol{r}_i(t_1)}^{\boldsymbol{r}_i(t_2)} \boldsymbol{F}_i \cdot \mathrm{d}\boldsymbol{r}_i$$

in the special case of **rigid** connections:

$$W_{12} = \sum_{i=1}^{n} \int_{\boldsymbol{r}_i(t_1)}^{\boldsymbol{r}_i(t_2)} \boldsymbol{F}_i^{\mathrm{ext}} \cdot \mathrm{d}\boldsymbol{r}_i$$

balance of angular momentum with respect to point B:

$$\boldsymbol{M}_{\mathrm{B}}^{\mathrm{ext}} = \dot{\boldsymbol{H}}_{\mathrm{B}} + \boldsymbol{v}_{\mathrm{B}} \times \boldsymbol{P} \qquad \text{with} \qquad \boldsymbol{P} = M\boldsymbol{v}_{\mathrm{CM}}, \qquad \boldsymbol{H}_{\mathrm{B}} = \sum_{i=1}^{n} \boldsymbol{r}_{\mathrm{BP}_i} \times \boldsymbol{P}_i.$$

coefficient of restitution for a **two-particle collision**:

$$e = -\frac{v_2^n(t_+) - v_1^n(t_+)}{v_2^n(t_-) - v_1^n(t_-)} = \frac{\hat{P}_{\mathrm{rest}}}{\hat{P}_{\mathrm{comp}}}$$

particle velocities after a **two-particle collision**:

$$v_1^{\mathrm{n}}(t_+) = \frac{m_1 v_1^{\mathrm{n}}(t_-) + m_2 v_2^{\mathrm{n}}(t_-) + e\, m_2 \left[v_2^{\mathrm{n}}(t_-) - v_1^{\mathrm{n}}(t_-)\right]}{m_1 + m_2}, \qquad v_1^{\mathrm{t}}(t_+) = v_1^{\mathrm{t}}(t_-)$$
$$v_2^{\mathrm{n}}(t_+) = \frac{m_1 v_1^{\mathrm{n}}(t_-) + m_2 v_2^{\mathrm{n}}(t_-) + e\, m_1 \left[v_1^{\mathrm{n}}(t_-) - v_2^{\mathrm{n}}(t_-)\right]}{m_1 + m_2}, \qquad v_2^{\mathrm{t}}(t_+) = v_2^{\mathrm{t}}(t_-)$$

Dynamics of Rigid Bodies:

velocity and acceleration transfer formulae:

$$\boldsymbol{v}_{\mathrm{B}} = \boldsymbol{v}_{\mathrm{A}} + \boldsymbol{\omega} \times \boldsymbol{r}_{\mathrm{AB}}$$
$$\boldsymbol{a}_{\mathrm{B}} = \boldsymbol{a}_{\mathrm{A}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{\mathrm{AB}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{\mathrm{AB}})$$

center of mass and **instantaneous center/axis of rotation** (from a point P):

$$\boldsymbol{r}_{\mathrm{CM}} = \frac{1}{M}\int_{\mathcal{B}} \boldsymbol{r}\rho\,\mathrm{d}V, \qquad \boldsymbol{r}_{\mathrm{P\Pi}} = \frac{1}{\omega}\boldsymbol{e} \times \boldsymbol{v}_{\mathrm{P}} = \frac{\boldsymbol{\omega} \times \boldsymbol{v}_{\mathrm{P}}}{\omega^2}$$

balance of linear momentum if $M = \text{const.}$:

$$\sum_i \boldsymbol{F}_i^{\mathrm{ext}} = \dot{\boldsymbol{P}} = \frac{\mathrm{d}}{\mathrm{d}t}(M\boldsymbol{v}_{\mathrm{CM}}) \qquad \Rightarrow \qquad \sum_i \boldsymbol{F}_i^{\mathrm{ext}} = M\boldsymbol{a}_{\mathrm{CM}}$$

balance of angular momentum with respect to an arbitrary point B:

$$\boldsymbol{M}_{\mathrm{B}} = \dot{\boldsymbol{H}}_{\mathrm{B}} + \boldsymbol{v}_{\mathrm{B}} \times \boldsymbol{P}$$

angular momentum with respect to a point B on body $\mathcal{B}$:

$$\boldsymbol{H}_{\mathrm{B}} = \boldsymbol{I}_{\mathrm{B}}\boldsymbol{\omega} + M(\boldsymbol{r}_{\mathrm{CM}} - \boldsymbol{r}_{\mathrm{B}}) \times \boldsymbol{v}_{\mathrm{B}}$$

moment of inertia tensor ($\mathrm{B} \in \mathcal{B}$ serves as coordinate origin):

$$[\boldsymbol{I}_{\mathrm{B}}] = \int_{\mathcal{B}} \begin{pmatrix} x_2^2 + x_3^2 & -x_1x_2 & -x_1x_3 \\ -x_1x_2 & x_1^2 + x_3^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & x_1^2 + x_2^2 \end{pmatrix} \rho\,\mathrm{d}V$$

parallel axes theorem (**Steiner's theorem**) with $\Delta\boldsymbol{x} = \boldsymbol{r}_{\mathrm{B}} - \boldsymbol{r}_{\mathrm{CM}}$:

$$[\boldsymbol{I}_{\mathrm{B}}] = [\boldsymbol{I}_{\mathrm{CM}}] + M \begin{pmatrix} (\Delta x_2)^2 + (\Delta x_3)^2 & -\Delta x_1\,\Delta x_2 & -\Delta x_1\,\Delta x_3 \\ -\Delta x_1\,\Delta x_2 & (\Delta x_1)^2 + (\Delta x_3)^2 & -\Delta x_2\,\Delta x_3 \\ -\Delta x_1\,\Delta x_3 & -\Delta x_2\,\Delta x_3 & (\Delta x_1)^2 + (\Delta x_2)^2 \end{pmatrix}$$

balance of angular momentum for $\mathrm{B} \in \mathcal{B}$ if $\mathrm{B} = \mathrm{CM}$ or $\boldsymbol{v}_{\mathrm{B}} = \boldsymbol{0}$ and if $\dot{\boldsymbol{I}}_{\mathrm{B}} = \boldsymbol{0}$:

$$\boldsymbol{M}_{\mathrm{B}} = \boldsymbol{I}_{\mathrm{B}}\dot{\boldsymbol{\omega}} \qquad \overset{\text{in 2D}}{\Rightarrow} \qquad M_{\mathrm{B}} = I_{\mathrm{B}}\dot{\omega} \quad \text{with} \quad I_{\mathrm{B}} = I_{\mathrm{CM}} + M(\Delta x)^2$$

angular momentum transfer formula for arbitrary points A and B:

$$\boldsymbol{H}_{\mathrm{B}} = \boldsymbol{H}_{\mathrm{A}} + \boldsymbol{P} \times \boldsymbol{r}_{\mathrm{AB}}$$

centroidal moments of inertia in 2D:

$$\text{slender rod}: \quad I_{\mathrm{CM}} = \frac{ML^2}{12}, \qquad \text{disk/cylinder}: \quad I_{\mathrm{CM}} = \frac{MR^2}{2}$$

kinetic energy of a rigid body for a point $\mathrm{C} \in \mathcal{B}$ with $\mathrm{C} = \mathrm{CM}$ or $\boldsymbol{v}_\mathrm{C} = \boldsymbol{0}$:

$$T = \frac{1}{2} M |\boldsymbol{v}_\mathrm{C}|^2 + \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{I}_\mathrm{C}\, \boldsymbol{\omega}$$

work–energy balance:

$$T(t_2) - T(t_1) = W_{12} \qquad W_{12} = \sum_i \int_{\boldsymbol{r}_i(t_1)}^{\boldsymbol{r}_i(t_2)} \boldsymbol{F}_i^\mathrm{ext} \cdot \mathrm{d}\boldsymbol{r}_i + \int_{\boldsymbol{\varphi}(t_1)}^{\boldsymbol{\varphi}(t_2)} \boldsymbol{M}^\mathrm{ext} \cdot \mathrm{d}\boldsymbol{\varphi}$$

time derivative relation between an **inertial frame** $\mathcal{C}$ and a **non-inertial frame** $\mathcal{M}$:

$$\dot{\boldsymbol{y}}^\mathcal{C} = \mathring{\boldsymbol{y}}^\mathcal{M} + \boldsymbol{\Omega}^\mathcal{M} \times \boldsymbol{y}$$

balance of linear momentum in a moving frame $\mathcal{M}$:

$$M \boldsymbol{a}_\mathrm{CM}^\mathcal{M} = \boldsymbol{F}_\mathrm{ext} + \boldsymbol{F}_\mathrm{Coriolis} + \boldsymbol{F}_\mathrm{Euler} + \boldsymbol{F}_\mathrm{centrifugal} - M \boldsymbol{a}_{\mathrm{O}^\mathcal{M}}$$

Coriolis, **Euler** and **centrifugal forces**:

$$\begin{aligned}
\boldsymbol{F}_\mathrm{Coriolis} &= -2M \boldsymbol{\Omega}^\mathcal{M} \times \boldsymbol{v}_\mathrm{CM}^\mathcal{M} \\
\boldsymbol{F}_\mathrm{Euler} &= -M \frac{\mathrm{d}\boldsymbol{\Omega}^\mathcal{M}}{\mathrm{d}t} \times \boldsymbol{r}_\mathrm{CM}^\mathcal{M} \\
\boldsymbol{F}_\mathrm{centrifugal} &= -M \boldsymbol{\Omega}^\mathcal{M} \times (\boldsymbol{\Omega}^\mathcal{M} \times \boldsymbol{r}_\mathrm{CM}^\mathcal{M})
\end{aligned}$$

balance of angular momentum in a moving frame $\mathcal{M}$:

$$\boldsymbol{M}_\mathrm{B} = (\boldsymbol{I}_\mathrm{B} \boldsymbol{\omega})^{\circ\mathcal{M}} + \boldsymbol{\Omega}^\mathcal{M} \times \boldsymbol{I}_\mathrm{B} \boldsymbol{\omega} \qquad \text{if} \qquad \mathrm{B} = \mathrm{CM} \quad \text{or} \quad \boldsymbol{v}_\mathrm{B} = \boldsymbol{0}$$

Euler's equations in a rotating (principal) **body frame** $\hat{\mathcal{M}}$ ($\boldsymbol{\Omega}^{\hat{\mathcal{M}}} = \boldsymbol{\omega}$):

$$\left.\begin{aligned}
\hat{I}_1 \dot{\omega}_1 + (\hat{I}_3 - \hat{I}_2)\omega_3\omega_2 &= [M_{\mathrm{B},1}]_{\hat{\mathcal{M}}} \\
\hat{I}_2 \dot{\omega}_2 + (\hat{I}_1 - \hat{I}_3)\omega_1\omega_3 &= [M_{\mathrm{B},2}]_{\hat{\mathcal{M}}} \\
\hat{I}_3 \dot{\omega}_3 + (\hat{I}_2 - \hat{I}_1)\omega_2\omega_1 &= [M_{\mathrm{B},3}]_{\hat{\mathcal{M}}}
\end{aligned}\right\} \quad \text{if} \quad \boldsymbol{\Omega}^\mathcal{M} = \boldsymbol{\omega} \quad \text{and} \quad \begin{aligned} \omega_i &= [\omega_i]_{\hat{\mathcal{M}}} \\ \dot{\omega}_i &= [\mathring{\omega}_i]_{\hat{\mathcal{M}}} = \tfrac{\mathrm{d}}{\mathrm{d}t}[\omega_i]_{\hat{\mathcal{M}}} \end{aligned}$$

angular momentum balance in a rotating (principal) frame $\mathcal{B}$ if $\mathring{\boldsymbol{I}} = \boldsymbol{0}$ ($\boldsymbol{\Omega}^{\hat{\mathcal{B}}} \neq \boldsymbol{\omega}$):

$$\left.\begin{aligned}
\hat{I}_1 \dot{\omega}_1 + \hat{I}_3 \Omega_2^\mathcal{B} \omega_3 - \hat{I}_2 \Omega_3^\mathcal{B} \omega_2 &= [M_{\mathrm{B},1}]_{\hat{\mathcal{B}}} \\
\hat{I}_2 \dot{\omega}_2 + \hat{I}_1 \Omega_3^\mathcal{B} \omega_1 - \hat{I}_3 \Omega_1^\mathcal{B} \omega_3 &= [M_{\mathrm{B},2}]_{\hat{\mathcal{B}}} \\
\hat{I}_3 \dot{\omega}_3 + \hat{I}_2 \Omega_1^\mathcal{B} \omega_2 - \hat{I}_1 \Omega_2^\mathcal{B} \omega_1 &= [M_{\mathrm{B},3}]_{\hat{\mathcal{B}}}
\end{aligned}\right\} \quad \text{where} \quad \begin{aligned} \omega_i &= [\omega_i]_{\hat{\mathcal{B}}} \\ \dot{\omega}_i &= [\mathring{\omega}_i]_{\hat{\mathcal{B}}} = \tfrac{\mathrm{d}}{\mathrm{d}t}[\omega_i]_{\hat{\mathcal{B}}} \end{aligned}$$

TSP-rule for a fast-spinning top:

$$\dot{\boldsymbol{\varphi}} \times \dot{\boldsymbol{\psi}} \approx \frac{\boldsymbol{M}_\mathrm{O}}{\hat{I}_3}$$

collision of rigid bodies: linear and angular momentum balance with contact point S:

$$\hat{\boldsymbol{P}} = m_1 \left[\boldsymbol{v}_{\mathrm{CM}_1}(t_+) - \boldsymbol{v}_{\mathrm{CM}_1}(t_-) \right], \qquad \boldsymbol{r}_{\mathrm{CM}_1\mathrm{S}} \times \hat{\boldsymbol{P}} = \boldsymbol{I}_{\mathrm{CM}_1} \left[\boldsymbol{\omega}_1(t_+) - \boldsymbol{\omega}_1(t_-) \right]$$

$$-\hat{\boldsymbol{P}} = m_2 \left[\boldsymbol{v}_{\mathrm{CM}_2}(t_+) - \boldsymbol{v}_{\mathrm{CM}_2}(t_-) \right], \qquad \boldsymbol{r}_{\mathrm{CM}_2\mathrm{S}} \times (-\hat{\boldsymbol{P}}) = \boldsymbol{I}_{\mathrm{CM}_2} \left[\boldsymbol{\omega}_2(t_+) - \boldsymbol{\omega}_2(t_-) \right]$$

collision of two smooth, frictionless bodies:

$$v^t_{\mathrm{CM}_1}(t_+) = v^t_{\mathrm{CM}_1}(t_-), \qquad v^t_{\mathrm{CM}_2}(t_+) = v^t_{\mathrm{CM}_2}(t_-)$$

coefficient of restitution for a two-body collision (contact points S_1 and S_2):

$$e = -\frac{v^n_{\mathrm{S}_2}(t_+) - v^n_{\mathrm{S}_1}(t_+)}{v^n_{\mathrm{S}_2}(t_-) - v^n_{\mathrm{S}_1}(t_-)}$$

passive rotation between the $\mathcal{M}$- and $\mathcal{C}$-frames:

$$[v_i]_{\mathcal{C}} = \sum_{j=1}^{3} [R^{\mathcal{CM}}_{ij}]^{\mathrm{T}} [v_j]_{\mathcal{M}}, \quad [v_i]_{\mathcal{M}} = \sum_{j=1}^{3} [R^{\mathcal{MC}}_{ij}]^{\mathrm{T}} [v_j]_{\mathcal{C}} \qquad \text{with} \qquad [R^{\mathcal{MC}}_{ij}] = [R^{\mathcal{CM}}_{ji}] = \boldsymbol{e}^{\mathcal{C}}_i \cdot \boldsymbol{e}^{\mathcal{M}}_j$$

active rotations:

$$\boldsymbol{e}^{\mathcal{M}}_i = \boldsymbol{R}^{\mathcal{MC}} \boldsymbol{e}^{\mathcal{C}}_i, \qquad \boldsymbol{e}^{\mathcal{C}}_i = \boldsymbol{R}^{\mathcal{CM}} \boldsymbol{e}^{\mathcal{M}}_i \qquad \text{with} \qquad \boldsymbol{R}^{\mathcal{MC}} = \left(\boldsymbol{R}^{\mathcal{CM}} \right)^{\mathrm{T}}$$

Vibrations:

Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i^{\mathrm{nc}} \qquad \text{with} \qquad \mathcal{L} = T - V, \qquad Q_i^{\mathrm{nc}} = \sum_{j=1}^{N} \boldsymbol{F}_j^{\text{non-cons.}} \cdot \frac{\partial \boldsymbol{r}_j}{\partial q_i}$$

equilibria of a **conservative, static system** with a single DOF q:

$$\text{stable equilibrium} \quad \Leftrightarrow \quad \text{energy minimum} \quad \Leftrightarrow \quad \frac{\partial^2 V}{\partial q^2} > 0$$

$$\text{unstable equilibrium} \quad \Leftrightarrow \quad \text{energy maximum} \quad \Leftrightarrow \quad \frac{\partial^2 V}{\partial q^2} < 0$$

general form of the **equation of motion** for a single DOF:

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = F/m \qquad \text{with} \qquad \omega_0 = \sqrt{\frac{k}{m}}, \quad \delta = \frac{c}{2m}, \quad D = \frac{\delta}{\omega_0}, \quad T = \frac{2\pi}{\omega_0}$$

general solution for **undamped vibrations** ($D = 0$):

$$x(t) = A_1 \cos(\omega_0 t) + A_2 \sin(\omega_0 t) = A \sin(\omega_0 t + \varphi_0)$$

general solution for **overdamped vibrations** ($D > 1$):

$$x(t) = A_1 e^{-\left(\delta + \sqrt{\delta^2 - \omega_0^2}\right)t} + A_2 e^{-\left(\delta - \sqrt{\delta^2 - \omega_0^2}\right)t}$$

general solution for **critically damped vibrations** ($D = 1$):

$$x(t) = A_1 e^{-\delta t} + A_2 t e^{-\delta t}$$

general solution for **underdamped vibrations** ($0 < D < 1$):

$$x(t) = e^{-\delta t}\left[A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)\right], \qquad \omega_d = \sqrt{\omega_0^2 - \delta^2}$$

general solution for **forced vibrations**:

$$F(t) = \hat{F}\cos(\Omega t) \qquad \Rightarrow \qquad x(t) = x_{\mathrm{hom}}(t) + \frac{\hat{F}}{k} V \cos(\Omega t - \varphi)$$

with **magnification** and **phase**

$$V = \frac{1}{\sqrt{(1-\eta^2)^2 + 4D^2\eta^2}}, \qquad \varphi = \arctan\left(\frac{2D\eta}{1-\eta^2}\right) \qquad \text{where} \qquad \eta = \frac{\Omega}{\omega_0}$$

equations of motion for multiple-DOF vibrations:

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{F}(t)$$

kinetic and **potential energy** for multiple-DOF linear(ized) vibrations:

$$T = \frac{1}{2}\dot{\boldsymbol{x}} \cdot \boldsymbol{M}\dot{\boldsymbol{x}}, \qquad V = \frac{1}{2}\boldsymbol{x} \cdot \boldsymbol{K}\boldsymbol{x}$$

linearized system matrices around a stable equilibrium $\boldsymbol{q}_0$:

$$\boldsymbol{M} = \frac{\partial^2 T}{\partial \dot{\boldsymbol{q}} \partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0, \boldsymbol{0}), \qquad \boldsymbol{C} = -\frac{\partial \boldsymbol{Q}^{\mathrm{nc}}}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}_0, \boldsymbol{0}), \qquad \boldsymbol{K} = \frac{\partial^2 V}{\partial \boldsymbol{q} \partial \boldsymbol{q}}(\boldsymbol{q}_0) - \frac{\partial \boldsymbol{Q}^{\mathrm{nc}}}{\partial \boldsymbol{q}}(\boldsymbol{q}_0, \boldsymbol{0})$$

structural damping:

$$\boldsymbol{C} = \alpha \boldsymbol{M} + \beta \boldsymbol{K}, \qquad \alpha, \beta \geq 0$$

eigenfrequencies ω_j and **eigenmodes** $\hat{\boldsymbol{x}}_j$ are obtained from

$$\det\left(-\omega_j^2 \boldsymbol{M} + \boldsymbol{K}\right) = 0 \qquad \text{and} \qquad \left(-\omega_j^2 \boldsymbol{M} + \boldsymbol{K}\right)\hat{\boldsymbol{x}}_j = \boldsymbol{0}$$

general solution of **free vibrations** for multiple-DOF systems:

$$\boldsymbol{x}(t) = \sum_{j=1}^{n} \boldsymbol{x}_j(t) = \sum_{j=1}^{l} c_j \cos(\omega_j t + \varphi_j)\hat{\boldsymbol{x}}_j + \sum_{j=l+1}^{n} (a_j + b_j t)\hat{\boldsymbol{x}}_j$$

Dynamics of Deformable Bodies:

effective stiffness of a rod in **extension**, **bending**, and **torsion**:

$$k_{\text{eff}} = \frac{F}{\Delta l} = \frac{EA}{l}, \qquad k_{\text{eff}} = \frac{F}{w} = \frac{3EI}{l^3}, \qquad k_{\text{eff}} = \frac{M}{\Delta\theta} = \frac{GJ}{l}$$

global balance of linear momentum for bodies and sub-bodies:

$$\sum_i \boldsymbol{F}_i^{\text{ext}} = M\boldsymbol{a}_{\text{CM}}$$

local balance of linear momentum:

$$\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f} = \rho\boldsymbol{a} \qquad \text{or} \qquad \sum_{j=1}^{3}\frac{\mathrm{d}\sigma_{ij}}{\mathrm{d}x_j} + f_i = \rho a_i \qquad \text{for} \quad i = 1,2,3$$

longitudinal wave equation for stretching/compression of a homogeneous slender bar:

$$\ddot{u}(x,t) = c^2 u_{,xx}(x,t) \qquad \text{with} \qquad c = \sqrt{\frac{E}{\rho}}$$

torsional wave equation for twisting of a homogeneous slender bar:

$$\ddot{\theta}(x,t) = c_{\text{T}}^2 \theta_{,xx}(x,t) \qquad \text{with} \qquad c_{\text{T}} = \sqrt{\frac{G}{\rho}}$$

general solution for **longitudinal vibrations** (and **torsion** analogously):

$$u(x,t) = \hat{u}(x)q(t) \qquad \text{with}$$
$$\hat{u}(x) = B_1\cos\left(\frac{\omega}{c}x\right) + B_2\sin\left(\frac{\omega}{c}x\right) \qquad \text{and} \qquad q(t) = A_1\cos(\omega t) + A_2\sin(\omega t)$$

flexural wave equation for bending of a homogeneous slender bar:

$$w_{,xxxx}(x,t) + \frac{\rho A}{EI_y}\ddot{w}(x,t) = 0$$

general solution for **flexural vibrations**:

$$w(x,t) = \hat{w}(x)q(t) \qquad \text{with} \qquad q(t) = A_1\cos(\omega t) + A_2\sin(\omega t)$$
$$\text{and} \qquad \hat{w}(x) = B_1\cos(kx) + B_2\sin(kx) + B_3\cosh(kx) + B_4\sinh(kx), \quad \omega^2 = k^4\frac{EI_y}{\rho A}$$

complete solution for **longitudinal** and **flexural vibrations**:

$$u(x,t) = \sum_{n=1}^{\infty}\hat{u}_n(x)q_n(t), \qquad w(x,t) = \sum_{n=1}^{\infty}\hat{w}_n(x)q_n(t)$$