Analysis III Zusammenfassung

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0 Contents

1 Laplace Transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

1.1 General Formulas:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	Inverse
$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$	Linearity
$\mathcal{L}\{e^{+at} f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$	s-Shifting
$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	t-Shifting
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) -$ $-s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$ $\mathcal{L}\left(\int_0^t f(\tau)\, d\tau\right) = \frac{1}{s}\mathcal{L}(f)$	Differentiation of Function Integration of Function
$\mathcal{L}(tf(t)) = -F'(s)$ $\mathcal{L}(t^n f(t)) = (-1)^n F^{(n)}(s)$ $\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\tilde{s})\, d\tilde{s}$	Differentiation of Transform Integration of Transform

Theorem: Partial fraction decomposition

- Single zero: $\frac{A}{x-x_0}$
- Double zeros: $\frac{A}{x-x_0} + \frac{B}{(x-x_0)^2}$
- Complex zeros: $\frac{A \cdot x + B}{\text{i.e.} \to x^2+1}$

Theorem: Existence

If $f(t)$ is defined on piecewise continuous on every finite interval on the semi-axis $t \geq 0$ and satisfies (1) $\forall t \geq 0$ and some constant M, k, the the Laplace transform exists $\forall s > k \to |f(t)| \leq Me^{kt}$

Theorem: Uniqueness

If two continuous functions have the same transform, they are identical.

1.2 Heavyside Function or Unit Step Function:

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \qquad \mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

1.3 Heavyside Expansions:

The subsidiary equation usually appears as a quotient $Y(s) = F(s)/G(s)$, so that partial fraction is necessary.

- **Repeated real factors** (in $G(s)$)

  $(s-a)^n \longrightarrow \sum_{i=1}^n \frac{A_i}{(s-a)^i}$

  $\xRightarrow{\mathcal{L}^{-1}} e^{at}(A_1 + A_2 t + \frac{1}{2}A_3 t^2 + \cdots + \frac{1}{n-1}A_n t^{n-1})$

- **Unrepeated complex factors**

  $(s-a)(s-\overline{a}),\ a = \alpha + i\beta \longrightarrow \frac{As+B}{(s-\alpha)^2+\beta^2}$

1.4 Dirac's Delta:

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^\infty g(t)\delta(t-a)\, dt = g(a) \qquad \textbf{sifting property}$$

$\int_0^\infty \delta(t-a)\, dt = 1 \qquad \mathcal{L}\{\delta(t-a) = e^{-as}\}$

1.5 Convolution:

Caution: $\mathcal{L}\{fg\} \neq \mathcal{L}\{f\}\mathcal{L}\{g\}$ (in general)

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau$$

$$\mathcal{L}(h) = \mathcal{L}(f)\mathcal{L}(g) \Leftrightarrow H = FG$$

$$\mathcal{L}^{-1}(F(s) \cdot G(s)) = \mathcal{L}^{-1}(F(s)) * \mathcal{L}^{-1}(G(s))$$

Example: Convolution

$y(t) - \int_0^t y(\tau)\sin(t-\tau)\, d\tau = t$

This equation can be written as: $y - y * \sin(t) = t$ (convolution theorem) $\Rightarrow Y - Y\frac{1}{s^2+1} = \frac{s^2}{s^2+1}Y = \frac{1}{s^2} \Leftrightarrow Y = \frac{s^2+1}{s^4}$

$\Rightarrow Y(s) = \frac{1}{s^2} + \frac{1}{s^4} \Rightarrow y(t) = t + \frac{t^3}{6}$

1.5.1 Properties:

$f * g = g * f$	$f(g_1 * g_2) = f * g_1 + f * g_2$
$(f * g) * v = f * (g * v)$	$f * 0 = 0 * f = 0$
$f * 1 \neq f$	$f * f \geq 0$ not necessarily

Example: Laplace Transform of a function

$f(t) = g(t) + \int_0^t f(\tau) \cdot h(t-\tau) d\tau$

$\Rightarrow f(t) = \mathcal{L}^{-1}(\frac{G(s)}{1-H(s)})$

1.5.2 Application to Non-Homogeneous ODEs:

$y(0) = y'(0) = 0 \Rightarrow Y = QR, y(t) = \int_0^t q(t-\tau)r(\tau)\, d\tau$

Example: Laplace application to an NHODE

Calculate: $\mathcal{L}^{-1} \ln\left(\frac{s^2+\omega^2}{s^2}\right)$

Differentiation:

$F'(s) = \frac{d}{ds}[\ln(s^2+\omega^2) - \ln(s^2)] = \frac{2s}{s^2+\omega^2} - \frac{2s}{s^2}$

$\Rightarrow \mathcal{L}^{-1}\{F'(s)\} = 2\cos\omega t - 2 = -tf(t)$

Differentiation & Integration:

$G(s) = F'(s) \Rightarrow g(t) = \mathcal{L}^{-1}\{G\} = 2(\cos(\omega t - 1)$

$\mathcal{L}^{-1}\left\{\ln\frac{s^2+\omega^2}{s^2}\right\} = \mathcal{L}^{-1}\left\{\int_s^\infty G(\tilde{s})\, d\tilde{s}\right\} = -\frac{g(t)}{t}$

1.6 ODE/ Initial Value Problems:

$$y'' + ay' + by = r(t),\ y(0) = K_0,\ y'(0) = K_1$$
$$r(t): \textbf{input} \quad y(t): \textbf{output}$$

1. **Setting up subsidiary equation**:
   $Y = \mathcal{L}(y),\ R = \mathcal{L}(r)$
   $[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$
   $(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s)$

2. **Solution of subsidiary equation:**
   $Q = \dfrac{1}{s^2 + as + b} = \dfrac{1}{(s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2}$ **transfer function**
   $\Rightarrow Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s)$
   $Q(s)$ doesn't depend on $r(t)$ or on the initial conditions

3. **Inversion of Y:**
   Reduce to sum of terms (inverses can be found in tables).

$$y(t) = \mathcal{L}^{-1}(Y)$$

1.7 Special Linear ODE with Variable Coeff.:

$$\mathcal{L}(ty') = -\frac{d}{ds}[sY - y(0)] = -Y - s\frac{dY}{ds}$$
$$\mathcal{L}(ty'') = -\frac{d}{ds}[s^2 - sy(0) - y'(0)] = -2sY - s^2\frac{dY}{ds} + y(0)$$

1.8 Systems of ODEs:

We consider a **first-order** linear system with constant coefficients. $\left.\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + g_1(t) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + g_2(t) \end{aligned}\right\} \Rightarrow y' = Ay + g \overset{\mathcal{L}}{\Rightarrow}$

$sY - y(0) = AY + G \Leftrightarrow \begin{aligned} (a_{11} - s)Y_1 + a_{12}Y_2 &= -y_1(0) - G_1(s) \\ a_{21}Y_1 + (a_{22} - s)Y_2 &= -y_2(0) - G_2(s) \end{aligned}$

$\Rightarrow (A - s\mathbb{I})Y = -y(0) - G$

$\rightarrow$ solve algebraically and take the inverse transform of Y

Theorem: Cramer's Rule

$Ax = b$ with $A \in \mathbb{R}^{n\times n},\ x, b \in \mathbb{R}^n,\ D := \det A$, and D_k is the determinant obtained by replacing k^{th} column by b

$\Rightarrow x_k = \dfrac{D_k}{D}$

Example: Laplace res of ODE impulse function

$y'' + 3y' + 2y = r(t),$

$r(t) = \begin{cases} 1 & \text{if } 1 < t < 2 \\ 0 & \text{otherwise} \end{cases},\ y(0) = y'(0) = 0$

$\Rightarrow Q(s) = \dfrac{1}{s^2 + 3s + 2} = \dfrac{1}{(s + 1)(s + 2)} = \dfrac{1}{s + 1} - \dfrac{1}{s + 2}$

$\Rightarrow q(t) = e^{-t} - e^{-2t}$

$$\begin{aligned} \Rightarrow y(t) &= \int q(t - \tau) \cdot 1\, d\tau = \int [e^{-(t-\tau)} - e^{-2(t-\tau)}]\, d\tau \\ &= e^{-(t-\tau)} - \frac{1}{2}e^{-2(t-\tau)} \end{aligned}$$

$r(\tau) = 1$ if $1 < t < 2$ only. So if $t < 1$, the intregal is zero. If $1 < t < 2$, we have to integrate from $\tau = 1$ to t:

$$\begin{aligned} y_2(t) &= e^{-0} - \frac{1}{2}e^{-0} - (e^{-(t-1)} - \frac{1}{2}e^{-2(t-1)}) \\ &= \frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)} \end{aligned}$$

If $t > 2$, we have to integrate from $\tau = 1$ to 2:

$y_3(t) = e^{-(t-2)} - \frac{1}{2}e^{-2(t-2)} - (e^{-(t-1)} - \frac{1}{2}e^{-2(t-1)})$

$$\Rightarrow y(t) = \begin{cases} y_1(t) = 0, & t < 1 \\ y_2(t), & 1 < t < 2 \\ y_3(t), & t > 2 \end{cases}$$

2 Fourier Analysis

2.1 Periodicity of functions:

Theorem: Fundamental Theorem of Periodicity

A function $f : \mathbb{R} \to \mathbb{R}$ is periodic of period $P > 0$ if $f(x + P) = f(x) \forall x \in \mathbb{R}$.
A fundamental period is (if it exists) the smallest positive number P for which f is periodic of period P. Ex: for $\sin(mx)$,$\cos(mx)$ $P = \frac{2\pi}{m}$. **Attention:** $\cos(x)^2$ and $\cos(x)^4$ as well as sin ... have period of π.

Theorem: Period of the sum of functions

- Let be P_f anf P_g the period of 2 periodic functions f and g. The function $f + g$ or $f \cdot g$ is periodic iff $\frac{P_f}{P_g} \in \mathbb{Q}$. The Period of $f + g$ is also the l.c.m(P_f, P_g).
- If f, g are functions periodic of period p and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is periodic of period p

Theorem: Property of boundedness

Let $f : \mathbb{R} \to \mathbb{R}$ be any function:

- If f is periodic and continuous, then it is bounded.
- If f is differentiable and periodic of period P, then also f' is periodic with the same period.
- If f is periodic and smooth, then it is bounded and all its derivative are bounded as well.

2.2 Fourier Series:

$$F(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x) \quad \textbf{Fourier series}$$

Beachte: The Fourier series $F(x)$ converges to the function $f(x)$ if the function is continuous.

(0) $a_0 = \dfrac{1}{2L}\int_{-L}^{L} f(x)\, dx$

(a) $a_n = \dfrac{1}{L}\int_{-L}^{L} f(x) \cos\dfrac{n\pi x}{L}\, dx$

(b) $b_n = \dfrac{1}{L}\int_{L}^{L} f(x) \sin\dfrac{n\pi x}{L}\, dx$

Theorem: Fourier coefficient

The Fourier coefficients of $f_1 + f_2$ are the sums of the corresponding coefficients of f_1 and f_2.
The Fourier coefficients of αf are α times the corresponding coefficients of f.

Theorem: Representation of a function by FS/ FI

Let f be the 2L-periodic function, piecewise continuous:

- $\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = f'_+(x_0),$
- $\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = f'_-(x_0)$

Then:

1. If x_0 is a point of continuity of $f \Rightarrow$ the Fourier Series (FS/FI) converges to $F(x) = f(x_0)$
2. If x_0 is a point of discontinuity of $f \Rightarrow$ the Fourier Series (FS/FI) converges to $F(x) = \frac{f(x_0^+)+f(x_0^-)}{2}$

Example: Fourier coefficients

$x(\pi - x) \overset{!}{=} \sum_{n=1}^{\infty} b_n \sin nx$

Solution:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\,dx$$

$$= \underbrace{-\frac{2x(\pi - x)}{n\pi}\cos nx\Big|_0^{\pi}}_{=0} + \frac{2}{n\pi}\int_0^{\pi}(\pi - 2x)\cos nx\,dx$$

$$= \underbrace{\frac{2(\pi - 2x)}{n^2\pi}\sin nx\Big|_0^{\pi}}_{=0} + \frac{4}{n^2\pi}\int_0^{\pi}\sin nx\,dx = -\frac{4}{n^3\pi}\cos nx\Big|_0^{\pi}$$

2.3 Better look: $\cos(nx), \sin(mx)$:

Theorem: Orthogonality of Trigonometric System

The *trigonometric system* is orthogonal on any interval of length 2π; that is: $n, m \in \mathbb{N},\ n \neq m$

(a) $\int_{-\pi}^{\pi} \cos nx \cos mx\,dx = 0$

(b) $\int_{-\pi}^{\pi} \sin nx \sin mx\,dx = 0$

(c) $\int_{-\pi}^{\pi} \sin nx \cos mx\,dx = 0$

$$\int \sin(mx)\cdot\cos(nx)dx = \begin{cases} \frac{-\cos(mx-nx)}{2\cdot(m-n)} - \frac{\cos(mx+nx)}{2\cdot(m+n)}, & m \neq n \\ \frac{-(\cos(mx))^2}{2m}, & m = n \end{cases}$$

$$\int \sin(mx)\cdot\sin(nx)dx = \begin{cases} \frac{\sin(mx-nx)}{2\cdot(m-n)} - \frac{\sin(mx+nx)}{2\cdot(m+n)}, & m \neq n \\ \frac{x}{2} - \frac{\sin 2mx}{ma}, & m = n \end{cases}$$

$$\int \cos(mx)\cdot\cos(nx)dx = \begin{cases} \frac{\sin(mx-nx)}{2\cdot(m-n)} + \frac{\sin(mx+nx)}{2\cdot(m+n)}, & m \neq n \\ \frac{x}{2} + \frac{\sin 2mx}{ma}, & m = n \end{cases}$$

2.3.1 Transform properties:

Let $n \in \mathbb{N}$:

- $\sin(n\pi) = 0$
- $\cos(n\pi) = (-1)^n$
- $\cos(n\frac{\pi}{2}) = (\frac{1+(-1)^n}{2})\cdot(-1)^{\frac{n}{2}} = \begin{cases} 0, n = 2j+1 \\ (-1)^j, n = 2j \end{cases}$
- $\sin(n\frac{\pi}{2}) = (\frac{1+(-1)^n}{2})\cdot(-1)^{\frac{n+2}{2}} = \begin{cases} 0, n = 2j \\ (-1)^j, n = 2j+1 \end{cases}$
- $\sin((n \pm 1)\frac{\pi}{2}) = \pm\cos(\frac{n\pi}{2})$
- $\cos((n \pm 1)\frac{\pi}{2}) = \pm\sin(\frac{n\pi}{2})$
- $\cos((n \pm \frac{1}{2})\pi) = 0, \quad \sin((n \pm \frac{1}{2})\pi) = \pm\cos(n\pi)$

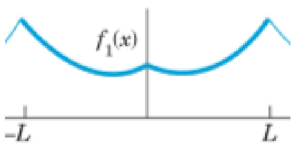
2.4 Even and Odd Functions/Expansions:

Theorem: Even and odd functions

Let $f, g : \mathbb{R} \to \mathbb{R}$ be any differentiable functions:

- if f is even, f' is odd
- if f is odd, f' is even
- $e_1(x) + e_2(x)$ is even
- $e_1(x) \cdot e_2(x)$ is even
- $o_1(x) + o_2(x)$ is odd
- $o_1(x) \cdot o_2(x)$ is even
- $o_1(x) \cdot e_1(x)$ is odd
- $\int_{-a}^{a} e_1(x)dx = 2 \cdot \int_0^a e_1(x)dx$
- $\int_{-a}^{a} o_1(x)dx = 0$

Even Expansion (gerade):

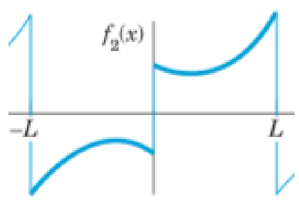


If $f(x)$ is an **even** $2L$ **periodic function** $(f(-x) = f(x))$

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x}$$

$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad n \in \mathbb{N}$

Odd Expansion (ungerade):



If $f(x)$ is an **odd** $2L$ **periodic function** $(f(-x) = -f(x))$

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x}$$

$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

2.5 Complex Fourier Series:

$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{L}x}}$ $\boxed{c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi}{L}x}\,dx}$

$e^{\pm it} = \cos t \pm i\sin t \qquad \cos t = \frac{e^{it} + e^{-it}}{2} \qquad \sin t = \frac{e^{it} - e^{-it}}{2i}$

comp $\to$ real	real $\to$ comp
$a_n = c_n + c_{-n} = 2\cdot Re(c_n)$	$c_n = \frac{1}{2}(a_n - ib_n)$
$b_n = i(c_n - c_{-n}) = -2Re(c_n)$	$c_{-n} = \frac{1}{2}(a_n + ib_n)$

- $e^{\pm i\pi n} = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$
- $c_n = \overline{c}_{-n}$

2.6 Approximation by Trigonometric Polynomials:

$x \in [-\pi, \pi]$, N fixed
Trigonometric polynomial of degree N:

$A_0 + \sum_{n=1}^{N}(A_n \cos nx + B_n \sin nx)$

$\boxed{E* = \int_{-L}^{L} f^2\,dx - L\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right]}$ **Square Error**

2.7 Fourier Integral:

- $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv$
- $B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\, dv$

$$F(x) = \int_0^{\infty} [A(w)\cos wx + B(w)\sin wx]\, dw$$

Theorem: Existance condition, Fourier integral

The Fourier integral exists if $f(x)$:

- is piecewise continuous in every finite interval
- has a left- and right-hand derivative at every point
- is absolutely integrable $\Leftrightarrow \lim_{a\to-\infty}\int_a^0 |f(x)|\,dx + \lim_{b\to\infty}\int_0^b |f(x)|\,dx$ exists

2.7.1 Fourier Cosine and Sine Integral:

Fourier cosine integral (f even):

- $f(x) = \int_0^{\infty} A(w)\cos wx\, dw$
- $A(w) = \frac{2}{\pi}\int_0^{\infty} f(v)\cos wv\, dv$

Fourier sine integral (f odd):

- $f(x) = \int_0^{\infty} B(w)\sin wx\, dw$
- $B(w) = \frac{2}{\pi}\int_0^{\infty} f(v)\sin wv\, dv$

Laplace integrals:

- $\int_0^{\infty}\frac{\cos wx}{k^2+w^2} = \frac{\pi}{2k}e^{-kx}$
- $\int_0^{\infty}\frac{w\sin wx}{k^2+w^2} = \frac{\pi}{2}e^{-kx}$

2.8 Fourier Transforms:

Fourier Transform:

- $\mathcal{F}(f) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\, dx$
- $\mathcal{F}^{-1}(\hat{f}) = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}\, dw$

Fourier cosine transform (f even):

- $\mathcal{F}_c(f) = \hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos wx\, dx$
- $\mathcal{F}_c^{-1}(\hat{f}_c) = f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \hat{f}_c(w)\cos wx\, dw$

Fourier sine transform (f odd):

- $\mathcal{F}_s(f) = \hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin wx\, dx$
- $\mathcal{F}_s^{-1}(\hat{f}_s) = f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \hat{f}_s(w)\sin wx\, dw$

Fourier Transforms are **linear operations**:

$$\mathcal{F}(af+bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

Example: Fourier Transform

(i) $\int\limits_{\mathbb{R}} e^{-ax^2}\, dx$
(ii) $\int\limits_{\mathbb{R}} x\, e^{-ax^2}\, dx$
(iii) $\int\limits_{\mathbb{R}} x^2 e^{-ax^2}\, dx$

Hint: $\mathcal{F}\left[e^{-ax^2}\right](w) = \frac{1}{\sqrt{2a}}e^{-\frac{w^2}{4a}}$

$$\mathcal{F}\left[x^k f(x)\right] = i^k\frac{d^k}{dw^k}\mathcal{F}[f(x)](w)$$

Solution:

(i) $\int\limits_{\mathbb{R}} e^{-ax^2}\, dx = \sqrt{2\pi}\mathcal{F}[e^{-ax^2}](0) = \sqrt{\frac{\pi}{a}}$

(ii) $\int\limits_{\mathbb{R}} x\, e^{-ax^2}\, dx = \sqrt{2\pi}i\frac{d}{dw}\mathcal{F}[e^{-ax^2}](0) = 0$

(iii)
$$\int\limits_{\mathbb{R}} x^2 e^{-ax^2}\, dx = \sqrt{2\pi}i^2\frac{d^2}{dw^2}\mathcal{F}[e^{-ax^2}](0)$$
$$= -\frac{d^2}{dw^2}\sqrt{\frac{\pi}{a}}e^{-\frac{w^2}{4a}}\Big|_{w=0}$$
$$= \left(\frac{d}{dw}\frac{w}{2a}\right)\sqrt{\frac{\pi}{a}}e^{-\frac{w^2}{4a}}\Big|_{w=0}$$
$$= \sqrt{\frac{\pi}{4a^3}}$$

2.8.1 Transforms of Derivatives:

- $\mathcal{F}\{f^{(n)}(x)\} = (iw)^n\mathcal{F}\{f(x)\}$
- $\mathcal{F}_c\{f'(x)\} = w\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}f(0)$
- $\mathcal{F}_c\{f''(x)\} = -w^2\mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}}f'(0)$
- $\mathcal{F}_s\{f'(x)\} = -w\mathcal{F}_c\{f(x)\}$
- $\mathcal{F}_s\{f''(x)\} = -w^2\mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}}wf(0)$

Theorem: Fourier and ramp function

Let $f:\mathbb{R}\to\mathbb{C}$ differentiable and such that:

- f is absolutely integrable
- xf is absolutely integrable.

So is:

$$\left(\hat{f}\right)' = -i(\hat{xf})$$

2.8.2 Other properties:

- $\mathcal{F}(f(\lambda x))(\omega) = \begin{cases}\frac{1}{\lambda}\mathcal{F}(f(x))(\frac{\omega}{\lambda}), \lambda>0\\ -\frac{1}{\lambda}\mathcal{F}(f(x))(\frac{\omega}{\lambda}), \lambda<0\end{cases}$
- $\mathcal{F}(f(x-a))(\omega) = e^{-iwa}\mathcal{F}(f(x))(\omega)$
- $\mathcal{F}(x^k\cdot f(x))(w) = i^k\frac{d^k}{dw^k}\mathcal{F}(f(x))(w)$

Theorem: Convolution

Let be f and g piecewise continuous, bounded and absolutely integrable, then $\mathcal{F}(f*g) = \sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(g)$

Theorem: Bounded conditions

Let $f:\mathbb{R}\to\mathbb{R}$ be any function:

- if f is periodic and continuous, then it is bounded.
- if f is differentiable and periodic of period P, then also f' is periodic with the same period

3 Partial Differential Equation (PDE)

3.1 Basic Concepts:

Classification

- A PDE is linear if it is of first degree in the unknown function u and its partial derivatives
  An equation of the form
  $a_1(x,y)u_{xx} + a_2(x,y)u_{xy} + a_3(x,y)u_{yx} + a_4(x,y)u_{yy} + a_5(x,y)u_x + a_6(x,y)u_y + a_7(x,y)u = f(x,y)$ is linear (a_n must not contain u or it's derivatives)

- A *linear* PDE is homogeneous if each of its terms contains either u or one of its partial derivatives (See linear, the term $f(x,y)$ must be $= 0$)

- The order of the highest derivative is called the order of the PDE

Theorem: Superposition

If u_1, u_2 are solutions of a *homogeneous linear* PDE in some region R, then $u = c_1u_1 + c_2u_2$ with any constant c_1, c_2 is also a solution of that PDE in R.

3.1.1 Simple Approaches:

- **Substitution**
  $u_{xy} = -u_x$ setting $u_x = p$
  $\Rightarrow p_y = -p \Leftrightarrow p_y/p = -1 \Leftrightarrow \ln|p| = -y + c(x) \Leftrightarrow p = \tilde{c}(x)e^{-y} \Leftrightarrow u(x,y) = f(x)e^{-y} + g(y)$

- **Like an ODE**
  $u_{xx} - u = 0$ (no y-derivatives occur)
  $\Rightarrow u'' - u = 0 \Rightarrow u = A(y)e^x + B(y)e^{-x}$

Example: PDE

$$\begin{cases} t^2u_x - u_t = 0 \\ u(x,0) = 3\cos x \end{cases}$$

Solution:

1. Take Fourier Transform
   $\mathcal{F}(t^2u_x) = t^2 iw\hat{u}(w)$
   $$\mathcal{F}(u_t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t e^{-iwx}\,dx = \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} ue^{-iwx}\,dx = \frac{\partial \hat{u}}{\partial t}$$

2. Separation of variables
   $$\frac{\partial \hat{u}}{\partial t} = t^2 iw\hat{u} \Leftrightarrow \frac{1}{\hat{u}}\partial\hat{u} = iwt^2\partial t$$
   $$\Leftrightarrow \ln\hat{u} = C(w)\frac{1}{3}t^2 iw \Leftrightarrow \hat{u} = C(w)e^{\frac{1}{3}t^2 iw}$$

3. Initial Condition
   $\hat{u}(w,0) = C(w) = \hat{f}(w) = \mathcal{F}(3\cos x)$

Example: PDE

$$\begin{cases} u_t - u_x x = 0 & (x,t) \in \mathbb{R}\times\mathbb{R}_+ \\ u(x,0) = x^2 & x \in \mathbb{R} \end{cases}$$

$$\Rightarrow u(x,t) = \int_{\mathbb{R}} f(y)K(x-y,t)\,dy$$

$$K(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$$

Solution: Ansatz: $z = x - y$

$$u(x,t) = \int_{\mathbb{R}} (x-z)^2 K(z,t)\,dz$$
$$= \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} (x^2 - 2xz + z^2)e^{-\frac{z^2}{4t}}\,dz$$
$$= \frac{1}{4\pi t}\left(x^2\sqrt{\frac{\pi}{a}} + 0 + \sqrt{\frac{\pi}{4a^3}}\right)\Bigg|_{a=\frac{1}{4t}}$$
$$= (x^2 + 2t)$$

3.2 First order PDE:

3.2.1 Method of Characteristics:

A PDE of the form:

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C_1(x,y)u = C_0(x,y)$$

is called a (first order) linear PDE (in two variables). It is called homogeneous if $C_0 = 0$. More generally, a PDE of the form:

$$A(x,y,u)\frac{\partial u}{\partial x} + B(x,y,u)\frac{\partial u}{\partial y} = C(x,y,u)$$

with the initial condition $u(x_0,y_0) = f(x,y) := z_0$, will be called a (first order) quasi-linear PDE (in two variables).

Remark: Every linear PDE is also quasi-linear since we may set: $C(x,y,u) = C_0(x,y) - C_1(x,y)u$

1. **Parametrize the initial curve** Γ: write:
   $$\Gamma := \begin{cases} x = x_0(a), \\ y = y_0(a), \\ z = z_0(a) \end{cases}$$

2. **Solve ODE System**: For each a, find the stream line of F that passes through $\Gamma(a)$. That is, solve the system of ODE with initial value problems:
   $$\begin{cases} \frac{dx}{ds} = A(x,y,z), x(0) = x_0(a) \\ \frac{dy}{ds} = B(x,y,z), y(0) = y_0(a) \\ \frac{dz}{ds} = C(x,y,z), z(0) = z_0(a) \end{cases}$$
   These are the *Characteristic equations of the PDE*. The Solution to the system will be in terms of the parameters a and s:
   $$x = X(a,s), y = Y(a,s), z = Z(a,s)$$
   This is a parametric expression for the graph of the solution surface $z = u(x,y)$ (in terms of the variables a, s).

3. **Find** a, s: Solve the previous equations for a, s in terms of x, y:
   $$a = \Lambda(x,y), s = S(x,y)$$

4. **Substitute the results** of Step 3 into $z = Z(a,s)$ to get the solution to the PDE: $u(x,y) = Z(\Lambda(x,y), S(x,y))$.

Example: First Order PDE

Find the solution to $x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = u^2$ that satisfies $u(x,x) = x^3$.

This is a quasi-linear PDE with:

$$A(x,y,u) = x, B(x,y,u) = -2y, C(x,y,u) = u^2,$$

so we may apply the method of characteristics. The initial curve Γ can be parametrized as:

$$x = a, y = a, z = a^3.$$

Hence the characteristic ODEs are:

$$\begin{cases} \frac{dx}{ds} = x, x(0) = a \\ \frac{dy}{ds} = -2y, y(0) = a \\ \frac{dz}{ds} = z^2, z(0) = a^3 \end{cases}$$

We find immediately that: $x(s) = ae^s$ and $y(s) = ae^{-2s}$ and $z(s) = \frac{a^3}{1-sa^3}$. We now need to solve for a and s. We have $\frac{x}{a} = e^s$ so that:

$$y = a(e^s)^{-2} = a(\frac{x}{a})^{-2} = \dots \Rightarrow a = x^{2/3}y^{1/3}$$

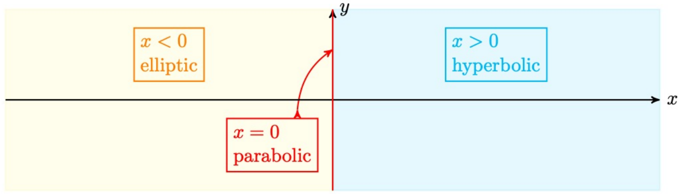
$$e^s = \frac{x}{a} = x^{1/3}y^{-1/3} = \dots \Rightarrow s = \frac{1}{3}\ln(x\frac{x}{y})$$

Substituting is z we get:

$$u(x,y) = z(s) = \frac{x^2 y}{1 - \frac{1}{3}x^2 y \ln(\frac{x}{y})}.$$

3.3 Second order PDE:

3.3.1 Method of Characteristics:



$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = F(x,y,u,u_x,u_x)$$

Type	Condition	Normal Form
Hyperbolic	$AC - B^2 < 0$	$u_{vw} = F(v,w,u,u_v,u_w)$
Parabolic	$AC - B^2 = 0$	$u_{vv} = F(v,w,u,u_v,u_w)$
Elliptic	$AC - B^2 > 0$	$u_{vv} + u_{ww} = F(\dots)$

Characteristic equation:

$A(x,y)(y')^2 - 2B(x,y)y' + C(x,y) = 0$ with $y' = \frac{dy}{dx} \overset{sol'n}{\Longrightarrow}$ $y' = \frac{2B \pm \sqrt{((2B)^2 - 4AC}}{2A}$

$\xi(x,y) = c_1 =$ const, $\zeta(x,y) = c_2 =$ const
(Solve for $y =$ and then for $c_1 =, c_2 =$)

Type	New Variables	
Hyperbolic	$v = \xi$	$w = \zeta$
Parabolic	$v = x$	$w = \xi = \zeta$
Elliptic	$v = \dfrac{\xi + \zeta}{2}$	$w = \dfrac{\xi - \zeta}{2i}$

Theorem: Kettenregel/chain rule

$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$
$E(t) = f(g(t)) \to E'(t) = f'(g(t)) \cdot g'(t) \leftrightarrow \frac{dE}{dt} = \frac{dE}{dv} \cdot \frac{dv}{dt}$

Example: Normal form of a PDE

Solve: $u_{xx} + 2u_{xy} - 3u_{yy} = e^{x+2y}$

1. $u_{xx} + 2u_{xy} - 3u_{yy} = e^{x+2y} \leftrightarrow Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x,y) \to A = 1, B = 1, C = -3$
2. $AC - B^2 < 0$ PDE is hyperbolic
3. Char eq: $A(y')2 - 2By' + C = 0 \leftrightarrow (y'-3)(y'+1) = 0$
4. Solutions: $\begin{cases} y' = 3 \leftrightarrow y - 3x = c_1 \\ y' = -1 \leftrightarrow y + x = c_2 \end{cases}$
5. $\begin{cases} v = \xi(x,y) = y - 3x \\ w = \zeta x, y = y + x \end{cases} \to \begin{cases} v_x = -3, v_y = 1 \\ w_x = 1, w_y = 1 \end{cases}$
6. Partial derivatives:
   (a) $u_x = u_v v_x + u_w w_x = -3u_v + u_w$
   (b) $u_y = u_v v_y + u_w w_y = u_v + u_w$
   (c) $u_{xx} = -3u_{vv}v_x - 3u_{vw}w_x + u_{wv}v_x + u_{ww}w_x = 9u_{vv} - 6u_{vw} + u_{ww}$
   (d) $u_{xy} = -3u_{vv}v_y - 3u_{vw}w_y + u_{wv}v_y + u_{ww}w_y = -3u_{vv} - 2u_{vw} + u_{ww}$
   (e) $u_{yy} = u_{vv}v_y + u_{vw}w_y + u_{wv}v_y + u_{ww}w_y = u_{vv} + 2u_{vw} + u_{ww}$
7. Transform left side (new coord): $u_{xx} + 2u_{ux} - 3u_{yy} = -16u_{vw}$
8. Transform right side (new coord): $x + 2y = \frac{v+7w}{4} \leftrightarrow e^{e+2y} = e^{\frac{v+7w}{4}}$
9. Togheter: $\frac{1}{16}e^{\frac{v+7w}{4}} = u_{vw}$
10. Integrate and substitute original variables: $u(x,y) = -\frac{1}{7}e^{x+2y} + \varphi(y-3x) + \psi y + x$

4 Wave Equation

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}} \text{ with } c^2 = \frac{T}{\rho}$$

- **Boundary conditions:** $u(0,t) = 0$, $u(L,t) = 0$, $\forall t \geq 0$
- **Initial conditions:** $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $x \in [0,L]$

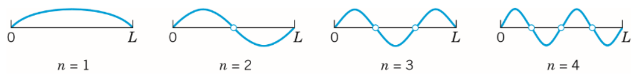
1. **Separating Variables**
Setting $u(x,t) = F(x)G(t)$
$\Rightarrow \dfrac{\partial^2 u}{\partial t^2} = F\ddot{G},\ \dfrac{\partial^2 u}{\partial x^2} = F''G \Rightarrow \dfrac{\ddot{G}}{c^2 G} = \dfrac{F''}{F} = k = const$
$\Rightarrow F'' - kF = 0,\ \ddot{G} - c^2 kG = 0$

2. **Boundary Conditions**

- $\underline{\mathbf{k = 0}:}$ $F = ax + b \Rightarrow a = b = 0 \Rightarrow F \equiv 0$
$\underline{\mathbf{k = \mu^2 > 0}: F = Ae^{\mu x} + Be^{-\mu x} \Rightarrow A = -B = 0}$
$\underline{\Rightarrow F \equiv 0}$
$\underline{\mathbf{k = -p^2 < 0}:}$ $F(x) = A\cos px + B\sin px$
$\overset{B.C.}{\Rightarrow} F(0) = A = 0,\ F(L) = B\sin pL = 0$
$\Rightarrow p = \dfrac{n\pi}{L},\ n \in \mathbb{N} \Rightarrow F_n(x) = B_n \sin\dfrac{n\pi}{L}x$

- $\ddot{G} + \lambda_n^2 G = 0,\ \lambda_n = cp = \dfrac{cn\pi}{L}$
$\Rightarrow G_n(t) = B_n \cos\lambda_n t + B_n^* \sin\lambda_n t$

$\Rightarrow u_n(x,t) = (B_n\cos\lambda_n t + B_n^*\sin\lambda_n t)\sin\dfrac{n\pi}{L}x$
These functions are called eigenfunctions and $\boxed{\lambda_n = \dfrac{cn\pi}{L}}$ are called eigenvalues. The set $\{\lambda_1, \lambda_2, \ldots\}$ is called spectrum.



3. **Fourier Series**
$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) =$
$= \boxed{\sum_{n=1}^{\infty}(B_n\cos\lambda_n t + B_n^*\sin\lambda_n t)\sin(\frac{n\pi}{L}x)}$

- **Displacement condition**
$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\dfrac{n\pi}{L}x = f(x)$ (Fourier)
$\Rightarrow \boxed{B_n = \dfrac{2}{L}\int_0^L f(x)\sin\dfrac{n\pi x}{L}\,dx}$

- **Velocity condition**
$\dfrac{\partial u}{\partial t}\Big|_{t=0} = \sum_{n=1}^{\infty} B_n^*\lambda_n \sin\dfrac{n\pi x}{L} = g(x)$
$\Rightarrow B_n^*\lambda_n = \dfrac{2}{L}\int_0^L g(x)\sin\dfrac{n\pi x}{L}\,dx$
$\Leftrightarrow \boxed{B_n* = \dfrac{2}{cn\pi}\int_0^L g(x)\sin\dfrac{n\pi x}{L}\,dx}$

If $g(x) = 0 \Rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n\cos\lambda_n t\sin\dfrac{n\pi x}{L}$

$$\Leftrightarrow u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x-ct)\right\} + \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x+ct)\right\}$$

4.1 D'Alembert Solution of the Wave Equation:

- **Solve characteristic eqn.**
$u_{tt} = c^2 u_{xx} \Leftrightarrow u_{tt} - c^2 u_{xx} = 0$
$\Rightarrow A = -c^2,\ B = 0,\ C = 1 \Rightarrow$ hyperbolic
char.eqn.: $-c^2(y')^2 + 1 = 0 \Leftrightarrow 1 \pm cy' = 0$
$\Rightarrow v = \xi = x + ct = c_1,\ w = \zeta = x - ct = c_2$

- **Bring into normal form**

$$\begin{aligned}
u_x &= u_v v_x + u_w w_x = u_v + u_w \\
u_{xx} &= (u_{vv}v_x + u_{vw}w_x) + (u_{wv}v_x + u_{ww}w_x) \\
&= u_{vv} + 2u_{vw} + u_{ww} \\
u_t &= u_v v_t + u_w w_t = cu_v - cu_w \\
u_{tt} &= c(u_{vv}v_t + u_{vw}w_t) - c(u_{wv}v_t + u_{ww}w_t) \\
&= c^2(u_{vv} - 2u_{vw} + u_{ww})
\end{aligned}$$

$c^2(u_{vv} - 2u_{vw} + u_{ww}) - c^2(u_{vv} + 2u_{vw} + u_{ww}) = -4c^2 u_{vw} = 0$
$\Rightarrow u_{vw} = 0 \Rightarrow u(v,w) = \phi(v) + \psi(w) \Leftrightarrow u(x,t) = \phi(x+ct) + \psi(x-ct)$

- **Solve for initial conditions**
I.C.: $u(x,0) = f(x) \quad u_t(x,0) = g(x),\ x \in \mathbb{R}, t \geq 0$
$f(x) = \phi(x) + \psi(x) \quad g(x) = c\phi'(x) - c\psi'(x)$

$$\phi(x) + \psi(x) = f(x)$$
$$\phi(x) - \psi(x) = \frac{1}{c}\int_0^x g(s)\,ds + \underbrace{\phi(0) - \psi(0)}_{k_0}$$

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds + \frac{1}{2}k_0$$
$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds - \frac{1}{2k_0}$$

by replacing x with $x \pm ct$ and adding the two sol'ns:
$\boxed{u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$

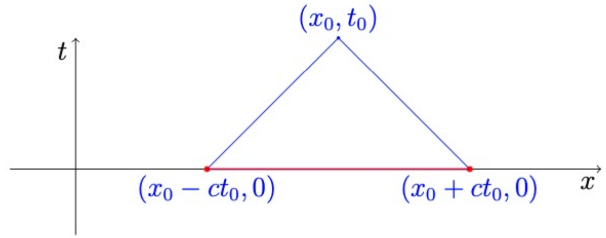
4.2 Graphic solution (Characteristic lines):

- **What kind of information has an influence on the solution u at the point (x,t)?**

Through any point (x_0, t_0) with $t_0 > 0$, there are exactly two characteristics, namely:

$$x - ct = x_0 - ct_0 \text{ and } x + ct = x_0 + ct_0.$$

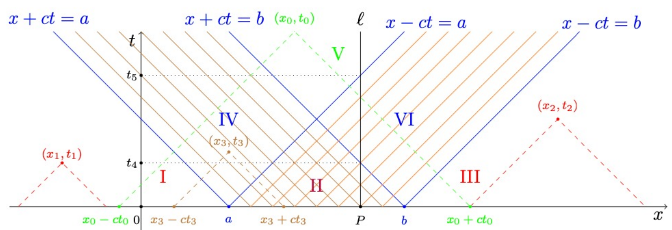
These are straight lines whose intersections with the x-axis are respectively the points $(x_0 - ct_0, 0)$ and $(x_0 + ct_0)$. The triangle with vertices $(x_0 - ct_0, 0)$, $(x_0 + ct_0)$ and (x_0, t_0) is called **characteristic triangle.**



The interval $[x_0 - ct0, x_0 + ct_0]$ is called the **domain of dependence** of u at (x_0, t_0). Changing f or g outside this domain of dependence will not affect the value $u(x_0, t_0)$.

- **What region of the (x,t) in the upper half plane is affected by the initial data on an interval $[a,b]$?**

The endpoints of the interval define four characteristics: $x \pm ct = a$ and $x \pm ct = b$, whose intersections define six regions indicated in the picture with I, II, III, IV, V and VI.

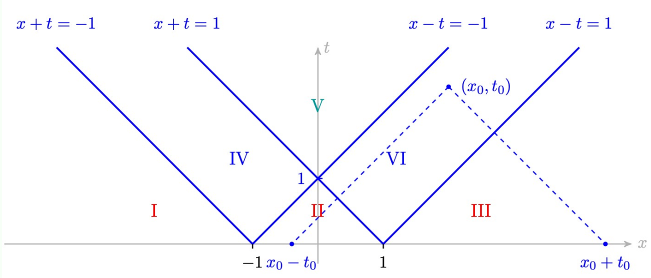


The points that are affected by the initial conditions are exactly the points (x,t) whose domain of dependence $[x - ct, x + ct]$ intersects the interval $[a,b]$ in a non-trivial way. In particular, as shown in the picture, any point in the regions I and III are such that $u(x,t)=0$. For values of $t=\infty$ the point is at rest in a position $u(x,t)=\frac{1}{2c}\int_a^b g(s)ds$.

Example: Region of influence

Let $u(x,t)$ be the solution of the problem $(x\in\mathbb{R}, t>0)$:

$$\begin{cases} u_{tt}=u_{xx} \\ u(x,0)=f(x)=\begin{cases}1, |x|\le 1\\ 0, |x|>1\end{cases} \\ u_t(x,0)=\begin{cases}1, |x|\le 1\\ 0, |x|>1\end{cases}\end{cases}.$$



- If $(x_0,t_0)\in$ I: $u(x_0,t_0)=0$
- if $(x_0,t_0)\in$ II: $u(x_0,t_0)=\frac{f(x_0+t_0)+f(x_0-t_0)}{2}+\frac{1}{2}\in_{x_0-t_0}^{x_0+t_0} f(y)dy=1+\frac{1}{2}(x_0+t_0-x_0+t_0)=1+t_0$
- If $(x_0,t_0)\in$ III: $u(x_0,t_0)=0$
- If $(x_0,t_0)\in$ IV: $u(x_0,t_0)=\frac{f(x_0+t_0)}{2}+\frac{1}{2}\int_{-1}^{x_0+t_0} g(y)dy=\frac{1}{2}+\frac{x_0+t_0+1}{2}=\frac{x_0+t_0}{2}+1$
- If $(x_0,t_0)\in$ V: $u(x_0,t_0)=\frac{1}{2}\int_{x_0-t_0}^{x_0+t_0} g(y)dy=\frac{1}{2}\int_{-1}^{1} g(y)dy=1$
- If $(x_0,t_0)\in$ VI: $u(x_0,t_0)=\frac{f(x_0-t_0)}{2}+\frac{1}{2}\int_{x_0-t_0}^{1} g(y)dy=\frac{1}{2}+\frac{-x_0+t_0+1}{2}=\frac{-x_0+t_0}{2}+1$

Observe that the Max value of $u(x,t)$ is obtained at the point $(x_0,t_0)\in$ II satisfying: $\begin{cases}x_0-t_0=-1\\x_0+t_0=1\end{cases}\Rightarrow$ $x_0=0$ and $t_0=1$. $\forall x\in\mathbb{R}$: $\lim t\to\infty u(x,t)=\frac{1}{2}\int -1^1 g(y)dy=\frac{1}{2}2=1.$

5 Heat Equation

$$\frac{\partial u}{\partial t}=c^2\nabla^2 u \text{ with } c^2=\frac{K}{\rho\sigma},$$

where $\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$
(K =conductivity,ρ = Density,σ = Spec. heat, c = Thermal Diffusivity)

5.1 One-Dimensional Heat Equation:

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$$

- **Assumptions:** laterally insulated, thin, homogeneous metal bar, heat flows in $x-$direction only
- **Boundary conditions:** $u(0,t)=0$, $u(L,t)=0$, $\forall t\ge 0$
- **Initial condition:** $u(x,0)=f(x)$

1. **Separating variables**
   Substituting $u(x,t)=F(x)G(t)$
   $\Rightarrow F\dot{G}=c^2F''G \Leftrightarrow \frac{\dot{G}}{c^2G}=\frac{F''}{F}=-p^2=\text{const}$
   $\Rightarrow F''+p^2F=0,\ \dot{G}+c^2p^2G=0$
   Note: Look at the wave equation to see why it is $-p^2$!
2. **Boundary Condition**
   - $F(x)=A\cos px+B\sin px \overset{B.C.}{\Rightarrow} A=0,\ p=\frac{n\pi}{L},\ n\in\mathbb{N}$
     Setting $B=1\Rightarrow F_n(x)=\sin\frac{n\pi x}{L}$
   - $\dot{G}+\lambda_n^2G=0,\ \lambda_n=\frac{cn\pi}{L}$
     $\Rightarrow G_n(t)=B_ne^{-\lambda_n^2t}$

   $\Rightarrow u_n(x,t)=F_n(x)G_n(t)=B_n\sin\frac{n\pi x}{L}e^{-\lambda_n^2t}$
3. **Fourier Series**
   - $u(x,0)=\sum_{n=1}^{\infty}u_n(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}=f(x)$
   - $\Rightarrow B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$
   - $u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}e^{-\lambda_n^2t}$

Example: Heat equation on a bar

Find $u(x,t)$ in a laterally insulated bar of length 80cm with initial temperature $f(x)=100\sin(\pi x/80)$°C.
$u(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{80}=f(x)=100\sin\frac{\pi x}{80}$ By inspection: $B_1=100, B_2=B_3=\cdots=0$
$\Rightarrow u(x,t)=100\sin\frac{\pi x}{80}e^{-\lambda^2t}$

Example: Isolated bar

Find $u(x,t)$, bar with insulated ends.
$u_x(0,t)=0\quad u_x(L,t)=0$
$u(x,t)=F(x)G(t)\Rightarrow u_x(0,t)=F'(0)G(t)=0,\ u_xL,t=F'(L)G(t)=0$
$F'=-Ap\sin px+Bp\cos px\Rightarrow F'(0)=Bp=0,\ F'(L)=-Ap\sin pL=0$
$\Rightarrow p_n=\frac{n\pi}{L}\Rightarrow u_n(x,t)=F_n(x)G_n(t)=A_n\cos\frac{n\pi x}{L}e^{-\lambda_n^2t}$
$\Rightarrow u(x,t)=\sum_{n=0}^{\infty}A_n\cos\frac{n\pi x}{L}e^{-\lambda_n^2t}$
with $A_0=\frac{1}{L}\int_0^L f(x)\,dx,\quad A_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$
Note: Here, $\lambda_0=0$ is an eigenvalue too.

5.2 One - Dimensional Heat Equation with inomogeneus BC:

$$\begin{cases}u_t(x,t)=c^2u_{xx}(x,t)\\u(0,t)=2\\u(\pi,t)=3\\u(x,0)=f(x)\end{cases}$$

Solution:

1. Construct a function $w(x)$ with $w(0)=2,\ w(\pi)=3,\ w''=0$
   $\Rightarrow w=\frac{x}{\pi}+2$
2. State the boundary value problem for $v(x,t):=u(x,t)-w(x)$
   $$\Rightarrow\begin{cases}v_t(x,t)=c^2v_{xx}(x,t)\\v(0,t)=0\\v(\pi,t)=0\\v(x,0)=f(x)-w(x)\end{cases}$$
3. Solve by separation of variables.

5.3 Inomogeneus One Dimensional Heat Equation:

$$\begin{cases} u_t = a^2 u_{xx} + B & 0 < x < L, t > 0 \\ u(0,t) = u(L,t) & t \geq 0 \\ u(x,0) = \sin(\frac{\pi x}{L}) & 0 \leq x \leq L \end{cases}$$

Solution:

1. Find the stationary solution:
   $v : x \to v(x)$ which fulfills the boundary conditions.

   $$\begin{cases} a^2 v_{xx} + b = 0 \\ v(0) = v(L) = 0 \end{cases} , x \in \mathbb{R}$$

   The unique solution to this PDE is $v(x) = -\frac{b}{2a^2}(x-L)x$ (consider zeros in 0 and L)

2. Construct a function $w(x)$ with $w(0) = 2,\ w(\pi) = 3$, $w'' = 0$
   $\Rightarrow w = \dfrac{x}{\pi} + 2$

3. Set $w(x,t) = u(x,t) - v(x)$

   $$\Rightarrow \begin{cases} w_t = u_t = a^2(w_{xx} + v_{xx}) + b = a^2 w_{xx} \\ w(0,t) = w(L,t) = 0 \\ w(x,0) = \sin(\frac{\pi x}{L}) - v(x) \end{cases}$$

4. $\dots \Rightarrow w(x,t) = \sum_{n \leq 1} \alpha_n \sin(\frac{n\pi x}{L}) = sin(\frac{\pi x}{L}) - v(x)$. We extend v to an odd 2L periodic function $\tilde{v}$ and determinate its Fourier coefficients:

   $$B_n := \frac{1}{L}\int_{-L}^{L} \tilde{v(x)} \sin(\frac{n\pi x}{L}) dx.$$

5. By comparing we get $\alpha_n = -B_n$ and for $n = 1 \Rightarrow \alpha_1 = 1 - B_1$.

6. $u(x,t) = w(x,t) + v(x)$

5.4 Heat Equation for Infinite Bars:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

1. **Separating variables**
   $$u(x,t) = F(x)G(t) \Rightarrow \begin{cases} F(x) = A\cos px + b \sin px \\ G(t) = e^{-c^2p^2t} \end{cases}$$
   $$\Rightarrow u(x,t;p) = FG = (A\cos px + B\sin px)e^{-c^2p^2t}$$

2. **Fourier Integrals**
   A, B are arbitrary, regard them as $A = A(p), B = B(p)$
   $$u(x,t) = \int_0^\infty u(x,t;p)\,dp = \int_0^\infty [A(p)\cos px + B(p)\sin px]e^{-c^2p^2t}\,dp$$

3. $\mathbf{A(p), B(p)}$ **from Initial Condition**
   $u(x,0) = \int_0^\infty [A(p)\cos px + B(p)\sin px]\,dp = f(x)$ $A(p) = \dfrac{1}{\pi}\int_{-\infty}^\infty f(v)\cos pv\,dv$, $B(p) = \dfrac{1}{\pi}\int_{-\infty}^\infty f(v)\sin pv\,dv$ using:
   $$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$$
   $$\cos x \cos y = \frac{1}{2}(\cos(x-y) + \cos(x+y))$$
   $$\Rightarrow u(x,0) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(v)\cos(px - pv)\,dv\right] dp$$
   $$\Rightarrow u(x,t) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\left[\int_0^\infty e^{-c^2p^2t}\cos(px-pv)\,dp\right] dv$$
   $\int_0^\infty e^{-s^2}\cos 2bs\,ds = \dfrac{\sqrt{\pi}}{2}e^{-b^2}$ with $p = \dfrac{s}{c\sqrt{t}},\ b = \dfrac{x-v}{2c\sqrt{t}}$
   $$\Rightarrow \int_0^\infty e^{-c^2p^2t}\cos(px-pv)\,dp = \frac{\sqrt{\pi}}{2c\sqrt{t}}\exp\left(-\frac{(x-v)^2}{4c^2t}\right)$$
   $$\Rightarrow \quad u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^\infty f(v)\exp\left(-\frac{(x-v)^2}{4c^2t}\right) dv$$
   with $z = \dfrac{v-x}{2c\sqrt{t}} \Rightarrow$
   $$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty f(x + 2cz\sqrt{t})e^{-z^2}\,dz$$

5.4.1 Using Fourier Transform:

1. **Take Fourier Transform**
   $\hat{u} = \mathcal{F}(u)$
   - $c^2\mathcal{F}(u_{xx}) = c^2(-w^2)\mathcal{F}(u) = -c^2w^2\hat{u}$
   - $$\mathcal{F}(u_t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty u_t e^{-iwx}\,dx = \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_{-\infty}^\infty u e^{-iwx}\,dx = \frac{\partial \hat{u}}{\partial t}$$

   $$\Rightarrow \frac{\partial \hat{u}}{\partial t} = -c^2w^2\hat{u} \Rightarrow \hat{u}(w,t) = C(w)e^{-c^2w^2t}$$

2. **Initial Condition**
   $\hat{u}(w,0) = \hat{f}(w) = C(w) \Rightarrow \hat{u}(w,t) = \hat{f}(w)e^{-c^2w^2t}$
   $\hat{f}(w) = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(v)e^{-iwv}\,dv$
   $u(x,t) = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{f}(w)e^{-c^2w^2t}e^{iwx}\,dw$
   $\Rightarrow u(x,t) = \dfrac{1}{2\pi}\int_{-\infty}^\infty f(v)\left[\int_{-\infty}^\infty e^{-c^2w^2t}e^{i(wx-wv)}\,dw\right] dv$
   $e^{i(wx-wv)} = \cos(wx-wv) + i\sin(wx-wv)$
   $\rightarrow$ the imaginary part is odd so its integral is 0

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\left[\int_{-\infty}^\infty e^{-c^2w^2t}\cos(wx-wv)\,dw\right] dv$$

5.4.2 Method of Convolution:

$u(x,t) = \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{f}(w)e^{-c^2w^2t}e^{iwx}\,dw$
$u(x,t) = (f*g)(x) = \int_{-\infty}^\infty \hat{f}(x)\hat{g}(w)e^{iwx}\,dw$
$\Rightarrow \hat{g}(w) = \dfrac{1}{\sqrt{2\pi}}e^{-c^2w^2t}$
Def. of convolution: $(f*g)(x) = \int_{-\infty}^\infty f(p)g(x-p)\,dp$
Inverse transform of $\hat{g}$: $\mathcal{F}\left(e^{-ax^2}\right) = \dfrac{1}{\sqrt{2a}}e^{-w^2/(4a)}$
with $a = 1/(ac^2t) \Rightarrow \mathcal{F}\left(e^{-x/(4ac^2t)}\right) = \sqrt{2c^2t}e^{-c^2w^2t} = \sqrt{2c^2t}\sqrt{2\pi}\hat{g}(w)$
$\Rightarrow \mathcal{F}^{-1}(\hat{g}) = \dfrac{1}{\sqrt{2c^2t}\sqrt{2\pi}}e^{-x^2/(2c^2t)}$

$$u(x,t) = (f*g)(x) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^\infty f(p)\exp\left(-\frac{(x-p)^2}{4c^2t}\right) dp$$

6 Laplace eq. (Steady 2D Heat Problem)

steady = time-independent $\Rightarrow \dfrac{\partial u}{\partial t} = 0$

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ Laplace's equation}$$

Theorem: Liouville's

Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded. Then u is constant.

Theorem: Weak Minimum Principle

Suppose $u_{xx} + u_{yy} = 0$, $B(0,R)$ and u is continuous in $\bar{B}(0,R) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le R^2\}$. Than the Maximum and the Minimum values of u are obtained on $\partial\bar{B} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = R^2\}$.

$$\max u(x,y) = \max f(\theta)$$

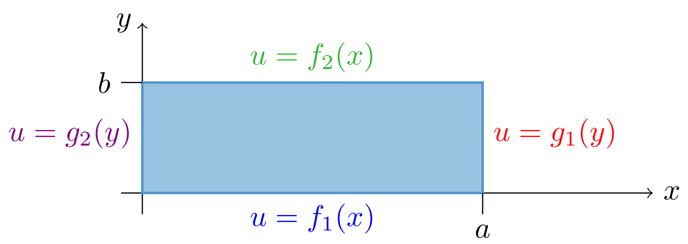
This principle holds also for a square bound, i.e Section 6.1:

$$\Rightarrow \max u(x,y) = \max f(x)$$

Theorem: Strong Minimum Principle

Let $B \subseteq \mathbb{R}^2$ be a domain (can also not be bounded) and $u : B \to \mathbb{R}$ be a harmonic function (function that solves the Laplace equation). If u ottains its maximum or minimum in B, then u is constant.

6.1 Dirichlet problem:



Since the equation is linear we can break the problem into simpler problems which do have sufficient homogeneous BC and use superposition to obtain the solution

$$u(x,y) = u_A + u_B + u_C + u_D$$

6.1.1 Solution to problem with $f_1(x)$ (u_A):

1. **Assume** $f_2(x) = g_1(y) = g_2(y) = 0$

2. **Separating variables**
$u_{xx} = -u_{yy},\ u(x,y) = F(x)G(y)$
$\Rightarrow \dfrac{1}{F}\dfrac{d^2F}{dx^2} = -\dfrac{1}{G}\dfrac{d^2G}{dy^2} = -k^2$
$\Rightarrow \dfrac{d^2F}{dx^2} + k^2F = 0,\ \dfrac{d^2G}{dy^2} - k^2G = 0$
Note: Look at the wave equation to see why it is $-k^2$

3. **Boundary conditions**

   - $F(x) = A\sin kx + B\cos kx$
   left/right B.C.: $F(0) = 0,\ F(a) = 0$
   $\Rightarrow A = 1,\ B = 0,\ k = \dfrac{n\pi}{a}$
   $\Rightarrow F(x) = F_n(x) = \sin\dfrac{n\pi}{a}x$

   - $G(y) = C\cosh(ky) + D\sinh(ky)$
   upper B.C.: $G_n(b) = C\cosh(kb) + D\sinh(kb) = 0 \Rightarrow C = -D\tanh(ka)$
   $\Rightarrow G(y) = -D\tanh(kb)\cosh(ky) + D\sinh(ky) = D\left(\frac{\sinh(ky)\cosh(kb) - \cosh(ky)\sinh(kb)}{\cosh(kb)}\right) = \frac{D}{\cosh(kb)}\sinh(k(y-b)) = \tilde{D}\sinh((k(y-b))$
   $\Rightarrow G_n(y) = \tilde{D}\sinh((k(y-b))$

$$u_n(x,y) = F_n(x)G_n(y) = A_n^*\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi}{a}(y-b)\right)$$

4. **Fourier Series**
$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$
lower BC $u_{x,0} = f_1(x) = \sum_{n=1}^{\infty}\left(-A_n^*\sinh\dfrac{n\pi b}{a}\right)\sin\dfrac{n\pi x}{a}$
$b_n = -A_n^*\sinh\dfrac{n\pi b}{a} = \dfrac{2}{a}\int_0^a f(x)\sin\dfrac{n\pi x}{a}\,dx$
$\Leftrightarrow A_n^* = -\dfrac{2}{a\sinh(n\pi b/a)}\int_0^a f_1(x)\sin\dfrac{n\pi x}{a}\,dx$

$$u_A(x,y) = \sum_{n=1}^{\infty} A_n^*\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi}{a}(y-b)\right)$$

6.1.2 Solution to problem with $f_2(x)$ (u_C):

1. **Assume** $f_1(x) = g_1(y) = g_2(y) = 0$

2. **Separating variables**
$u_{xx} = -u_{yy},\ u(x,y) = F(x)G(y)$
$\Rightarrow \dfrac{1}{F}\dfrac{d^2F}{dx^2} = -\dfrac{1}{G}\dfrac{d^2G}{dy^2} = -k^2$
$\Rightarrow \dfrac{d^2F}{dx^2} + k^2F = 0,\ \dfrac{d^2G}{dy^2} - k^2G = 0$
Note: Look at the wave equation to see why it is $-k^2$

3. **Boundary conditions**

   - $F(x) = A\sin kx + B\cos kx$
   left/right B.C.: $F(0) = 0,\ F(a) = 0$
   $\Rightarrow A = 1,\ B = 0,\ k = \dfrac{n\pi}{a}$
   $\Rightarrow F(x) = F_n(x) = \sin\dfrac{n\pi}{a}x$

   - $G_n(y) = A_n\overbrace{e^{n\pi y/a}}^{ky} + B_n\overbrace{e^{-n\pi y/a}}^{-ky}$
   lower B.C.: $G_n(0) = A_n + B_n = 0 \Leftrightarrow B_n = -A_n$
   $\Rightarrow G_n(y) = A_n(e^{n\pi y/a} - e^{-n\pi y/a}) = \underbrace{2A_n}_{A_n^*}\sinh\dfrac{n\pi y}{a}$

   $\Rightarrow u_n(x,y) = F_n(x)G_n(y) = C_n^*\sin\dfrac{n\pi x}{a}\sinh\dfrac{n\pi y}{a}$

4. **Fourier Series**
$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$
upper B.C $u_{x,b} = f_2(x) = \sum_{n=1}^{\infty}\left(C_n^*\sinh\dfrac{n\pi b}{a}\right)\sin\dfrac{n\pi x}{a}$
$b_n = C_n^*\sinh\dfrac{n\pi b}{a} = \dfrac{2}{a}\int_0^a f(x)\sin\dfrac{n\pi x}{a}\,dx$
$\Leftrightarrow C_n^* = \dfrac{2}{a\sinh(n\pi b/a)}\int_0^a f_2(x)\sin\dfrac{n\pi x}{a}\,dx$

$$u_C(x,y) = \sum_{n=1}^{\infty} C_n^*\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

6.1.3 Solution to problem with $g_1(x)$ (u_B):

1. **Assume** $f_1(x) = f_2(x) = g_2(y) = 0$ (for $g_2(y)$ analogous)

2. **Separating variables**
$u_{xx} = -u_{yy},\ u(x,y) = F(x)G(y)$

$$\Rightarrow \frac{1}{F}\frac{d^2F}{dx^2} = -\frac{1}{G}\frac{d^2G}{dy^2} = -k^2$$

$$\Rightarrow \frac{d^2F}{dx^2} - k^2F = 0,\ \frac{d^2G}{dy^2} + k^2G = 0$$

Note: Look at the wave equation to see why it is k^2

3. **Boundary conditions**
   - $F(x) = c_1\cosh(kx) + c_2\sinh(kx)$
     upper/lower B.C.: $F_n(0) = A_n + B_n = 0 \Leftrightarrow B_n = -A_n$
     $\Rightarrow F_n(x) = A_n(e^{n\pi x/b} - e^{-n\pi x/b}) = \underbrace{2A_n}_{A_n^*}\sinh\frac{n\pi x}{b}$
   - $G(y) = A\cos ky + B\sin ky$
     left B.C.: $G(0) = 0,\ G(b) = 0$
     $\Rightarrow A = 0,\ B = 1,\ k = \frac{n\pi}{b}$
     $\Rightarrow G(y) = G_n(y) = \sin\frac{n\pi}{b}y$

   $\Rightarrow u_n(x,y) = F_n(x)G_n(y) = B_n^*\sin\frac{n\pi y}{b}\sinh\frac{n\pi x}{b}$

4. **Fourier Series**
   $u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$
   right B.C $u_{y,a} = g_1(x) = \sum_{n=1}^{\infty}\left(B_n^*\sinh\frac{n\pi a}{b}\right)\sin\frac{n\pi y}{b}$
   $b_n = B_n^*\sinh\frac{n\pi a}{b} = \frac{2}{b}\int_0^b f(x)\sin\frac{n\pi y}{b}\,dy$
   $\Leftrightarrow \boxed{B_n^* = \frac{2}{b\sinh(n\pi a/b)}\int_0^b g_1(y)\sin\frac{n\pi y}{b}\,dy}$

$$\boxed{u_B(x,y) = \sum_{n=1}^{\infty} B_n^*\sin\frac{n\pi y}{b}\sinh\frac{n\pi x}{b}}$$

6.2 Laplacian in Polar Coordinates:

Polar Coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \end{cases}$$

$$\begin{cases} u_x = u_r r_x + u_\theta\theta_x \\ u_{xx} = (u_{rr}r_x + u_{r\theta}\theta_x)r_x + u_r r_{xx} + (u_{\theta r}r_x + u_{\theta\theta}\theta_x)\theta_x + u_\theta\theta_{xx} \end{cases}$$

$$\begin{cases} r_x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} r_{xx} = \frac{r - xr_x}{r^2} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{y^2}{r^3} \\ \theta_x = \frac{1}{1+(y/x)^2}\left(-\frac{x}{x^2}\right) = -\frac{y}{r^2}\theta_{xx} = -y\left(-\frac{2}{r^3}\right)r_x = \frac{2xy}{r^4} \end{cases}$$

$$\Rightarrow u_{xx} = \frac{x^2}{r^2}u_{rr} - 2\frac{xy}{r^3}u_{r\theta} + \frac{y^2}{r^4}u_{\theta\theta} + \frac{y^2}{r^3}u_r + 2\frac{xy}{r^4}u_\theta$$

Analogously: $u_{yy} = \frac{y^2}{r^2}u_{rr} + 2\frac{xy}{r^3}u_{r\theta} + \frac{x^2}{r^4}u_{\theta\theta} + \frac{x^2}{r^3} - 2\frac{xy}{r^4}u_\theta$

$$\Rightarrow \boxed{\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2}}$$

6.3 Dirichlet Problem for a Disk:

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0}$$

- $u(R,\theta) = f(\theta)$ **Boundary Condition**
- $u(r,0) = u(r,2\pi),\ u_\theta(r,0) = u_\theta(r,2\pi)$ **Continuity Conditions**

1. **Separating Variables**
   Setting $u(r,\theta) = F(r)G(\theta)$
   $\Rightarrow F''G + \frac{1}{r}F'G + \frac{1}{r^2}FG'' = 0$
   $\Leftrightarrow \frac{r^2F'' + rF'}{F} = -\frac{G''}{G} = k = \text{ const}$

   $$r^2F'' + rF' - kF = 0 \qquad G'' + kG = 0$$

2. **Continuity Conditions** $G(0) = G(2\pi)$ $G'(0) = G'(2\pi)$
   - $\mathbf{k < 0}$: $G(\theta) = Ae^{\sqrt{-k}\theta} + Be^{-\sqrt{-k}\theta}$

     $$A + B = Ae^{\sqrt{-k}2\pi} + Be^{\sqrt{-k}2\pi}$$
     $$\sqrt{-k}A - \sqrt{-k}B = \sqrt{-k}Ae^{k2\pi} - \sqrt{-k}Be^{k2\pi}$$

     $$A + B = Ae^{\sqrt{-k}2\pi} + Be^{\sqrt{-k}2\pi}$$
     $$A - B = Ae^{\sqrt{-k}2\pi} - Be^{\sqrt{-k}2\pi}$$

     $$\Rightarrow 2A = 2Ae^{\sqrt{-k}2\pi} \Rightarrow A = 0 \Rightarrow B = 0$$
   - $\mathbf{k = 0}$: $G(\theta) = A\theta + B$
     $G(0) = B = G(2\pi) = 2\pi A + B \Rightarrow A = 0$
     $\Rightarrow G(\theta) = B = \text{ const}$
   - $\mathbf{k > 0}$: $G(\theta) = A\cos(\sqrt{k}\theta) + B\sin(\sqrt{k}\theta)$

     $$A = A\cos(2\pi\sqrt{k}) + B\sin(2\pi\sqrt{k}) \qquad |\cdot B$$
     $$\sqrt{k}B = -\sqrt{k}A\sin(2\pi\sqrt{k}) + \sqrt{k}B\cos(2\pi\sqrt{k}) \qquad |\cdot A$$

     $\Rightarrow B^2\sin(2\pi\sqrt{k}) = -A^2\sin(2\pi\sqrt{k})$
     $\Rightarrow \sin(2\pi\sqrt{k}) = 0 \Rightarrow \sqrt{k} := n \in \mathbb{N}$
     $\Rightarrow G_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$

   $r^2F'' + rF' - n^2F = 0 \Rightarrow F(r) = r^\alpha,\ \alpha \in \mathbb{Q}$

   $$r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2r^\alpha = 0$$
   $$\Leftrightarrow \alpha(\alpha-1) + \alpha - n^2 = 0$$
   $$\Leftrightarrow \alpha^2 - n^2 = 0 \Rightarrow \alpha = \pm n$$

   Note: The general solution is $F_n(r) = P_nr^n + Q_nr^{-n}$, which is not bounded for $r = 0$. Therefore the solution to this particular problem is $F_n(r) = P_nr^n$

   $$\Rightarrow \boxed{u(r,\theta) = \sum_{n=0}^{\infty} r^n(A_n\cos(n\theta) + B_n\sin(n\theta))}$$

3. **Boundary Condition** $u(R,\theta) = \sum_{n=0}^{\infty} R^n(A_n\cos(n\theta) + B_n\sin(n\theta)) = f(\theta)$

   $$A_n = \frac{1}{R^n\pi}\int_0^{2\pi} f(\phi)\cos(n\phi)\,d\phi$$
   $$B_n = \frac{1}{R^n\pi}\int_0^{2\pi} f(\phi)\sin(n\phi)\,d\phi$$

4. **Poisson Integral Form**

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi \\ &+ \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\int_0^{2\pi}\underbrace{[\cos n\theta\cos\phi + \sin n\theta\sin n\phi]}_{\cos(n(\theta-\phi))}f(\phi)\,d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi + \\ &+ \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\int_0^{2\pi}\cos(n(\theta-\phi))f(\phi)\,d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\cos(n(\theta-\phi))\right]f(\phi)\,d\phi \\ &\overset{*}{=} \frac{1}{2\pi}\int_0^{2\pi}\frac{1-\left(\frac{r}{R}\right)^2}{1 - 2\frac{r}{R}\cos(\theta-\phi) + \left(\frac{r}{R}\right)^2}f(\phi)\,d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2 - 2rR\cos(\theta-\phi) + r^2}f(\phi)\,d\phi \end{aligned}$$

$* \sum_{n=1}^{\infty} t^n \cos(n\alpha) = \Re\left(\sum_{n=1\infty} t^n e^{in\alpha}\right)$
$= \Re\left(\frac{te^{i\alpha}}{1-te^{i\alpha}}\right) = \cdots = \frac{t\cos\alpha - t^2}{1-2t\cos\alpha + t^2}$

Theorem: Poisson Integral Form

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} K(r,\theta,R,\phi) f(\phi)\, d\phi,$$

where $K(r,\theta,R\phi) := \dfrac{R^2-r^2}{R^2-2rR\cos(\theta-\phi)+r^2}$

Theorem: Mean value property

If the disk has center in (x_0,y_0). Then using the Poisson Integral Form with $K(0,\theta,R,\phi) \Rightarrow u(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)d\phi = \frac{1}{2\pi}\int_0^{2\pi} u(x_0+R\cos(\tilde{\sigma}), y_0+R\sin(\tilde{\sigma}))d\tilde{\sigma}$.

Example: Poisson Kessel integral

Prove that $\frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)(\cos^3(\vartheta)\sin(\vartheta)-\sin^3(\vartheta)\cos(\vartheta))}{1-2r\cos(\vartheta-\varphi)+r^2} d\varphi = \frac{r^4}{4}\sin(4\vartheta)$

- Let $D = (r,\vartheta)|0\le r\le 1, 0\le\vartheta\vartheta 2\pi$, $u(r,\vartheta) = 1$ (constant function)
- $u(r,\vartheta) = \frac{1}{2\pi}\int_0^{2\pi} K(r,\vartheta,1,\varphi)u(1,\varphi)d\varphi \leftrightarrow 1 = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\vartheta-\varphi)+r^2}d\varphi$
- Cosider the function $\cos^3(\vartheta)\sin(\vartheta)-\sin^3(\vartheta)\cos(\vartheta)$
- $\begin{cases}\Delta u = 0, in D\\ u(1,\vartheta) = f(\vartheta), 0\le\vartheta\le 2\pi\end{cases}$
- Using the boundary condition $\cos^3(\vartheta)\sin(\vartheta) - \sin^3(\vartheta)\cos(\vartheta) = x^3y + xy^3$
- $\Delta(x^3y+xy^3) = 6xy - 6xy = 0$ is a well defined harmonic function. It is the solution of the problem
- Convert in polar coordinates: $x^3y + xy^3 = \cdots \frac{r^4}{4}\sin(4\vartheta)$

Example: Property of Laplace on Circular Membrane

D_1 = unit disk
$\begin{cases}\Delta u = 0\\ u(x,y) = xy+3, (x,y)\in\partial D_1\end{cases}$

1. Find $u(x,y)$: $\Delta(xy+3) = \partial_{xx}(xy+3)+\partial_{yy}(xy+3) = 0$. By uniqueness of the solution for the Dirichlet problem the solution must be: $u(x,y) = xy+3$
2. Find $u(0,0)$: $u(0,0) = 0\cdot 0+3 = 3$
3. Find maximum of $u(x,y)$
   - Convert in polar coordinates: $u(r,\vartheta) = r^2\cos(\vartheta)\sin(\vartheta)+3 = \frac{r^2}{2}\sin(2\vartheta)+3$
   - Maximise every components (indipendent variables)
   - Maximum are $u(1,\frac{\pi}{4}) = u(1,\frac{5}{4}\pi) = \frac{7}{2}$

Example: Circular Membrane of Lagrange Equation

Find sol of $u(r,\vartheta)$ from $\begin{cases}\Delta u = 0, 0\le r\le R, 0\le\vartheta\le 2\pi\\ u(R,\vartheta) = \sin^2(\vartheta), 0\le\vartheta\le 2\pi\end{cases}$

1. Solution: $u(r,\vartheta) = \sum_{n=0}^{\infty} r^n(A_n\cos(n\vartheta) + B_n\sin(n\vartheta))$,coefficients found with:$u(R,\vartheta) = \sum_{n=0}^{\infty} R_n(A_n\cos(n\vartheta)+B_n\sin(n\vartheta)) = \sin^2(\vartheta)$
2. Use $\sin^2(\vartheta) = \frac{1}{2}-\frac{1}{2}\cos(2\vartheta)$ to obtain the coefficient: $\begin{cases}B_n = 0, \forall n\ge 0\\ A_n = 0, \forall n\ge 0, n\ne 0,2\\ A_0 = \frac{1}{2}\\ A_2 = -\frac{1}{2R^2}\end{cases}$
3. The solution is $u(r,\vartheta) = \frac{1}{2}-\frac{1}{2R^2}\cos(2\vartheta)$

Maximum: constant doesn't play a role other function components are indipendent, maximise individually. $\begin{cases}-\cos(2\vartheta) = 1 \leftrightarrow \vartheta = \frac{\pi}{2},\frac{3}{2}\pi\\ r = R\leftrightarrow r = R\end{cases}$.
In Cartesian coordinates: $\cos(2\vartheta) = \cos^2(\vartheta) - \sin^2(\vartheta) \to r^2\cos(2\vartheta) = x^2-y^2$. The solution is then $u(x,y) = \frac{1}{2}-\frac{1}{2r^2}(x^2-y^2)$

6.4 Dirichlet Problem for a Ring:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0$$

- $u(R_1,\theta) = f_1(\theta)$ **Boundary Condition** $(R_1 < R_2)$
- $u(R_2,\theta) = f_2(\theta)$ **Boundary Condition** $(R_2 > R_1)$
- $u(r,0) = u(r,2\pi)$, $u_\theta(r,0) = u_\theta(r,2\pi)$ **Continuity Conditions**

6.4.1 Cosinus and Sinus variant:

1. **Separating variables:**
$$u(r,\varphi) = R(r)\Phi(\varphi) \Rightarrow \begin{cases}r^2R''(r)+rR'(r)+kR(r) = 0\\ \dot{\Phi}(\varphi) = k\Phi(\varphi)\\ \Phi(\varphi+2\pi) = \Phi(\varphi)\end{cases}$$
2. **Hence Φ is periodic**, k must be negativ: the general solution is so:
$$\Phi_n(\varphi) = a_n\cos(n\varphi)+b_n\sin(n\varphi)$$
For the fist we take $R(r) = r^\alpha$. The solution to this is:
$$R(r) = r^n, R(r) = r^{-n}, R(r) = \ln(r)$$
We can now use Superposition:
$$u(r,\varphi) = \sum_{n=1}^{\infty}((a_nr^n + a_{-n}r^{-n})\cos(n\varphi) + (b_nr^n+b_{-n}r^{-n})\sin(n\varphi)) + c_0 + d_0\ln(r)$$
3. **Boundary Condition**: from now suppose that $f_1(\theta) = 1-3\cos(\varphi)$, $f_2(\theta) = 6\sin(\varphi)+60\sin(\varphi)\cos(\varphi)$ and $R_1 = 1, R_2 = 2$.
We observe, by coefficient comparison for $f_1(\theta)$:
$$u(1,\varphi) = \sum_{n=1}^{\infty}((a_n+a_{-n})\cos(n\varphi) + (b_n+b_{-n})\sin(n\varphi)) + c_0 = 1-3\cos(\varphi)$$
   - $a_1 + a_{-1} = -3$
   - $a_n + a_{-n} = 0\ \forall n\ne 1$s
   - $b_n + b_{-n} = 0\ \forall n\in\mathbb{N}$
   - $c_0 = 1$

And by coefficient comparison for $f_2(\theta)$:

$$u(2,\varphi) = \sum_{n=1}^{\infty}((a_n 2^n + a_{-n}2^{-n})\cos(n\varphi) + (b_n 2^n + b_{-n}2^{-n})\sin(n\varphi)) + 1 + d_0\ln(2) = 6\sin(\varphi) + 30\sin(2\varphi)$$

- $2^n a_n + 2^{-n}a_{-n} = 0 \ \forall n \in \mathbb{N}$
- $b_n(2^n - 2^{-n}) = 0$ fuer $n \neq 1, 2$
- $b_1(2 - \frac{1}{2}) = 6$
- $b_2(4 - \frac{1}{4}) = 30$
- $d_0 = -\frac{1}{\ln(2)}$

4. **Find all the coefficient**: $u(r,\varphi) = \dots$

6.4.2 Complex variant:

1. **Separating variables:** as 6.5.1
2. **Hence Φ is periodic**, k must be negativ: the general solution is so:

$$u(r,\varphi) = \sum_{n=1}^{\infty}((A_n r^n + B_n r^{-n})e^{in\varphi}) + C_0 + D_0\ln(r)$$

Attention: The solution cal also be written as:

$$u(r,\varphi) = \sum_{n=-\infty}^{\infty}(A_n e^{in\varphi} r^{-|n|}) + C_0 + D_0\ln(r)$$

3. **Boundary Condition**: by comparison as 6.5.1. Remember to transform $\sin(x)$ and $\cos(x)$ of the Boundary function in complex exponential form!
4. **Find all the coefficient**: $u(r,\varphi) = \dots$

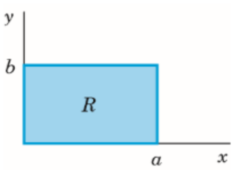
7 Two-Dimensional Wave Equation

Physical Assumptions

- Constant density and perfect flexibility of membrane
- Constant tension T
- small deflection compared to the size of the membrane

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = c^2\Delta^2 u \qquad c^2 = \frac{T}{\rho}$$

7.1 Rectangular Membrane:



- $u = 0$ on the boundary
- $u(x,y,0) = f(x,y)$ (initial displacement)
- $u_t(x,y,0) = g(x,y)$ (initial velocity)

1. **ODEs From the Wave Equation**

$u(x,y,t) = F(x,y)G(t) \Rightarrow F\ddot{G} = c^2(F_{xx}G + F_{yy}G)$

$$\Leftrightarrow \frac{\ddot{G}}{c^2 G} = \frac{1}{F}(F_{xx} + F_{yy}) = -\nu^2 = \text{const}$$

$$\ddot{G} + \lambda^2 G = 0 \quad \text{with } \lambda = c\nu, \quad F_{xx} + F_{yy} + \nu^2 F = 0$$

$$F(x,y) = H(x)Q(y) \Rightarrow \frac{d^2H}{dx^2}Q = -\left(H\frac{d^2Q}{dy^2} + \nu^2 HQ\right)$$

$$\Leftrightarrow \frac{1}{H}\frac{d^2H}{dx^2} = -\frac{1}{Q}\left(\frac{d^2Q}{dy^2} + \nu^2 Q\right) = -k^2 = \text{const}$$

$$\frac{d^2Q}{dy^2} + p^2 Q = 0 \quad \text{with } p^2 = \nu^2 - k^2, \quad \frac{d^2H}{dx^2} + k^2 H = 0$$

2. **Boundary Condition**

$H(x) = A\cos kx + B\sin kx \quad Q(y) = C\cos py + D\sin py$

$H(0) = A = 0,\ H(a) = B\sin ka = 0 \Rightarrow k = \frac{m\pi}{a}$

$Q(0) = C = 0,\ Q(b) = D\sin pb \Rightarrow p = \frac{n\pi}{b}$

$$H_m(x) = \sin\frac{m\pi x}{a} \qquad Q_n(y) = \sin\frac{n\pi y}{b}$$

$\Rightarrow F_{mn}(x,y) = H_m(x)Q_n(y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

Since $p^2 = \nu^2 - k^2$, $\lambda = c\nu \Rightarrow \lambda = \lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$

$\Rightarrow G_{mn}(t) = B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t$

$u_{mn}(x,y,t) = $

$= (B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

These functions are called eigenfunctions and λ_{mn} are called eigenvalues.

3. **Double Fourier Series**

$u(x,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$

$u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = f(x,y)$

- $K_m(y) := \sum_{n=1}^{\infty} B_{mn}\sin\frac{n\pi y}{b}$
  $\Rightarrow B_{mn} = \frac{2}{b}\int_0^b K_m(y)\sin\frac{n\pi y}{b}$
- $f(x,y) = \sum_{m=1}^{\infty} K_m(y)\sin\frac{m\pi x}{a}$
  $\Rightarrow K_m(y) = \frac{2}{a}\int_0^a f(x,y)\sin\frac{m\pi x}{a}\,dx$

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B^*_{mn}\lambda_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = g(x,y)$

Analogously:

$$B^*_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

8 Tables

8.1 Laplace Transforms:

$\mathcal{L}\{f(t)\}$	$f(t)$
$1/s$	1
$1/s^2$	t
$1/s^n$	$t^{n-1}/(n-1)!$
$1/\sqrt{s}$	$1/\sqrt{\pi t}$
$1/s^{3/2}$	$2\sqrt{t/\pi}$
$1/s^k$	$t^{k-1}/\Gamma(k)$
$\frac{1}{s-a}$	e^{at}
$\frac{1}{(s-a)^2}$	te^{at}
$\frac{1}{(s-a)^n}$	$\frac{1}{(n-1)!}t^{n-1}e^{at}$
$\frac{1}{(s-a)^k}$	$\frac{1}{\Gamma(k)}t^{k-1}e^{at}$
$\frac{\omega}{s^2+\omega^2}$	$\sin\omega t$
$\frac{a}{s^2-a^2}$	$\sinh at$
$\frac{\omega}{(s-a)^2+\omega^2}$	$e^{at}\sin\omega t$
$\frac{\omega}{(s-a)^2-\omega^2}$	$e^{at}\sinh\omega t$
$\frac{s}{s^2+\omega^2}$	$\cos\omega t$
$\frac{s}{s^2-a^2}$	$\cosh at$
$\frac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos\omega t$
$\frac{s-a}{(s-a)^2-\omega^2}$	$e^{at}\cosh\omega t$
$\frac{\omega^2}{s(s^2+\omega^2)}$	$1-\cos\omega t$
$\frac{\omega^3}{s^2(s^2+\omega^2)}$	$\omega t-\sin\omega t$

$\mathcal{L}\{f(t)\}$	$f(t)$
e^{-as}/s	$u(t-a)$
e^{-as}	$\delta(t-a)$
$\frac{1}{\sqrt{s}}e^{-\omega/s}$	$\frac{1}{\sqrt{\pi t}}\cos 2\sqrt{\omega t}$
$e^{-k\sqrt{s}}$	$\frac{k}{2\sqrt{\pi t^3}}e^{-k^2/4t}$
$\frac{1}{(s-a)(s-b)}$	$\frac{1}{a-b}(e^{at}-e^{bt})$
$\frac{s}{(s-a)(s-b)}$	$\frac{1}{a-b}(ae^{at}-be^{at})$
$\frac{2\omega^3}{(s^2+\omega^2)^2}$	$\sin\omega t-\omega t\cos\omega t$
$\frac{2\omega s}{(s^2+\omega^2)^2}$	$t\sin\omega t$
$\frac{2\omega s^2}{(s^2+\omega^2)^2}$	$\sin\omega t+\omega t\cos\omega t$
$\frac{s^2}{(s^2+a^2)(s^2+b^2)}$	$\frac{1}{b^2-a^2}(\cos at-\cos bt)$
$\frac{4\omega^3}{s^4+4\omega^4}$	$\sin\omega t\cos\omega t-\cos\omega t\sinh\omega t$
$\frac{2\omega^2 s}{s^4+4\omega^4}$	$\sin\omega t\sinh\omega t$
$\frac{2\omega^3}{s^4-\omega^4}$	$\sinh\omega t-\sin\omega t$
$\frac{2\omega^2 s}{s^4-\omega^4}$	$\cosh\omega t-\cos\omega t$
$\ln\frac{s-a}{s-b}$	$\frac{1}{t}(e^{bt}-e^{at})$
$\ln\frac{s^2+\omega^2}{s^2}$	$\frac{2}{t}(1-\cos\omega t)$
$\ln\frac{s^2-\omega^2}{s^2}$	$\frac{2}{r}(1-\cosh\omega t)$

$k>0,\ n\in\mathbb{N},\ a\neq b,\ \gamma\approx 0.5772$

8.2 Fourier Transforms:

$a>0$

$f(x)$	$\hat{f}_c(w)$
$\begin{cases}1 & 0<x<a\\ 0 & \text{otherwise}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin aw}{w}$
e^{-ax}	$\sqrt{\frac{2}{\pi}}\frac{a}{a^2+w^2}$
$e^{-x^2/2}$	$e^{-w^2/2}$
e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{-w^2/4a}$
$x^n e^{-ax}$	$\sqrt{\frac{2}{\pi}}\frac{n!}{(a^2+w^2)^{n+1}}\Re(a+iw)^{n+1}$
$\begin{cases}\cos x & 0<x<a\\ 0 & \text{otherwise}\end{cases}$	$\frac{1}{\sqrt{2\pi}}\left[\frac{\sin a(1-w)}{1-w}+\frac{\sin a(1+w)}{1+w}\right]$
$\cos(ax^2)$	$\frac{1}{\sqrt{2\pi}}\cos\left(\frac{w^2}{4a-\frac{\pi}{4}}\right)$
$\sin(ax^2)$	$\frac{1}{\sqrt{2a}}\cos\left(\frac{w^2}{4a}+\frac{\pi}{4}\right)$
$\frac{\sin ax}{x}$	$\sqrt{\frac{\pi}{2}}[1-u(w-a)]$
$\frac{e^{-x}\sin x}{x}$	$\frac{1}{\sqrt{2\pi}}\arctan\frac{2}{w^2}$

$f(x)$	$\hat{f}_s(w)$
$\begin{cases}1 & 0<x<a\\ 0 & \text{otherwise}\end{cases}$	$\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos aw}{w}\right]$
$1/\sqrt{x}$	$1/\sqrt{w}$
$1/x^{3/2}$	$2\sqrt{w}$
e^{-ax}	$\sqrt{\frac{2}{\pi}}\left(\frac{w}{a^2+w^2}\right)$
$\frac{e^{-ax}}{x}$	$\sqrt{\frac{2}{\pi}}\arctan\frac{w}{a}$
$x^n e^{-ax}$	$\sqrt{\frac{2}{\pi}}\frac{n!}{(a^2+w^2)^{n+1}}\Im(a+iw)^{n+1}$
$xe^{-x^2/2}$	$we^{w^2/2}$
xe^{-ax^2}	$\frac{2}{(2a)^{3/2}}e^{-w^2/4a}$
$\begin{cases}\sin x & 0<x<a\\ 0 & \text{otherwise}\end{cases}$	$\frac{1}{\sqrt{2\pi}}\left[\frac{\sin a(1-w)}{1-w}-\frac{\sin a(1+w)}{1+w}\right]$
$\frac{\cos ax}{x}$	$\sqrt{\frac{\pi}{2}}u(w-a)$
$\arctan\frac{2a}{x}$	$\sqrt{2\pi}\frac{\sin aw}{w}e^{-aw}$

$f(x)$	$\hat{f}(w)$
$\begin{cases}1 & -b<x<b\\ 0 & \text{otherwise}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin bw}{w}$
$\begin{cases}1 & b<x<c\\ 0 & \text{otherwise}\end{cases}$	$\frac{e^{-ibw}-e^{-icw}}{iw\sqrt{2\pi}}$
$\frac{1}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert w\rvert}}{a}$
$\begin{cases}x & 0<x<b\\ 2x-b & b<x<2b\\ 0 & \text{otherwise}\end{cases}$	$\frac{-1+2e^{ibw}-e^{-2ibw}}{\sqrt{2\pi}w^2}$
$\begin{cases}e^{-ax} & x>0\\ 0 & \text{otherwise}\end{cases}$	$\frac{1}{\sqrt{2\pi}(a+iw)}$
$\begin{cases}e^{ax} & b<x<c\\ 0 & \text{otherwise}\end{cases}$	$\frac{e^{(a-iw)c}-e^{(a-iw)b}}{\sqrt{2\pi}(a-iw)}$
$\begin{cases}e^{iax} & -b<x<b\\ 0 & \text{otherwise}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin b(w-a)}{w-a}$
$\begin{cases}e^{iax} & b<x<c\\ 0 & \text{otherwise}\end{cases}$	$\frac{i}{\sqrt{2\pi}}\frac{e^{ib(a-w)}-e^{ic(a-w)}}{a-w}$
e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{-w^2/4a}$
$\frac{\sin ax}{x}$	$\begin{cases}\sqrt{\pi/2} & \lvert w\rvert<a\\ 0 & \lvert w\rvert>a\end{cases}$

8.3 Some Integrals:

	$\int_0^{\pi/4}$	$\int_0^{\pi/2}$	$\int_0^{\pi}$	$\int_0^{2\pi}$	$\int_{-\pi/4}^{\pi/4}$	$\int_{\pi/2}^{\pi/2}$	$\int_{-\pi}^{\pi}$
sin	$\frac{\sqrt{2}-1}{\sqrt{2}}$	1	2	0	0	0	0
$\sin^2$	$\frac{\pi-2}{8}$	$\pi/4$	$\pi/2$	π	$\frac{\pi-2}{4}$	$\pi/2$	π
$\sin^3$	$\frac{8-5\sqrt{2}}{12}$	$2/3$	$4/3$	0	0	0	0
$\sin^4$	$\frac{3\pi-8}{32}$	$\frac{3\pi}{16}$	$\frac{3\pi}{8}$	$\frac{3\pi}{4}$	$\frac{3\pi-8}{16}$	$\frac{3\pi}{8}$	$\frac{3\pi}{4}$
cos	$1/\sqrt{2}$	1	0	0	$\sqrt{2}$	2	0
$\cos^2$	$\frac{2+\pi}{8}$	$\pi/4$	$\pi/2$	π	$\frac{2+\pi}{4}$	$\pi/2$	π
$\cos^3$	$\frac{5}{6\sqrt{2}}$	$2/3$	0	0	$\frac{5}{3\sqrt{2}}$	$4/3$	0
$\cos^4$	$\frac{8+3\pi}{32}$	$\frac{3\pi}{16}$	$\frac{3\pi}{8}$	$\frac{3\pi}{4}$	$\frac{8+3\pi}{16}$	$\frac{3\pi}{8}$	$\frac{3\pi}{4}$
sin cos	$1/4$	$1/2$	0	0	0	0	0
$\sin^2\cos$	$\frac{2}{6\sqrt{2}}$	$1/3$	0	0	$\frac{1}{3\sqrt{2}}$	$2/3$	0
$\sin\cos^2$	$\frac{4-\sqrt{2}}{12}$	$1/3$	$2/3$	0	0	0	0

$$\int_{r\frac{\pi}{2}}^{s\frac{\pi}{2}} \sin^n x\,dx = \frac{n-1}{n}\int_{r\frac{\pi}{2}}^{s\frac{\pi}{2}} \sin^{(n-2)} x\,dx \quad \forall n \geq 2$$

$$\int_{r\frac{\pi}{2}}^{s\frac{\pi}{2}} \cos^n x\,dx = \frac{n-1}{n}\int_{r\frac{\pi}{2}}^{s\frac{\pi}{2}} \cos^{(n-2)} x\,dx \quad r, s \in \mathbb{Z}$$

$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1}$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|$

$\int (ax^p+b)^n x^{p-1}\,dx = \frac{(ax^p+b)^{n+1}}{ap(n+1)}$

$\int (ax^p+b)^{-1} x^{p-1}\,dx = \frac{1}{ap}\ln|ax^p+b|$

$\int \frac{ax+b}{cs+d}\,dx = \frac{ax}{c} - \frac{ad-bc}{c^2}\ln|cx+d|$

$\int x(ax+b)^n\,dx = \frac{(ax+b)^{n+2}}{(n+2)a^2} - \frac{b(ax+b)^{n-1}}{(n+1)a^2}$

$\int \frac{x}{(ax+b)^n}\,dx = -\frac{1}{(n-2)a^2(ax+b)^{n-2}} + \frac{b}{(n-1)a^2(ax+b)^{n-1}}$

$\int \frac{x}{(x^2+a)^n}\,dx = -\frac{1}{2(n-1)(a^2+x^2)^{n-1}}$

$\int \frac{x}{(x^2-a^2)^n}\,dx = \frac{1}{2(n-1)(a^2+x^2)^{n-1}}$

$\int \frac{x}{x^2+a}\,dx = \frac{1}{2}\ln|x^2+a|$

$\int \frac{x}{ax^2+b}\,dx = \frac{1}{2a}\ln|ax^2+b|$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a}$

$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln|x+\sqrt{x^2+a^2}|$

$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x} + \frac{a^2}{2}\arcsin\frac{x}{|a|}$

$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}|$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln|x+\sqrt{a^2+x^2}|$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln|x+\sqrt{x^2-a^2}|$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{|a|}$

$\int \frac{x^2-y^2}{x^2+y^2}\,dx = x - 2\tan^{-1}(\frac{x}{y})\cdot y$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x$

$\int \frac{1}{1-x^2}\,dx = \operatorname{artanh} x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \ |x|<1$

$\int \frac{1}{\sqrt{x^2+1}}\,dx = \operatorname{arsinh} x = \ln|x+\sqrt{x^2+1}|$

$\int \frac{1}{\sqrt{x^2-1}}\,dx = \operatorname{arcosh} x = \ln|x+\sqrt{x^2-1}|, |x|>1$

$\int \frac{1}{\sin^2 x}\,dx = -\cot x$

$\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x\cos x)$

$\int \sin^3 x\,dx = \frac{1}{12}(\cos 3x - 9\cos x)$

$\int \sin^4\,dx = \frac{1}{32}(12x - 8\sin 2x + \sin 4x)$

$\int \frac{1}{\sin x}\,dx = \ln\left|\tan\frac{x}{2}\right|$

$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

$\int \arcsin x\,dx = x\arcsin x + \sqrt{1-x^2}$

$\int \cos^2 x\,dx = \frac{1}{2}(x + \sin x\cos x)$

$\int \cos^3 x\,dx = \frac{1}{12}(9\sin x + \sin 3x)$

$\int \cos^4 x\,dx = \frac{1}{32}(12x + 8\sin 2x + \sin 4x)$

$\int \frac{1}{\cos x}\,dx = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$

$\int \cos^n x\,dx = \frac{1}{n}\sin x\cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

$\int \arccos x\,dx = x\arccos x - \sqrt{1-x^2}$

$\int \sin 2x\,dx = -\frac{1}{2}\cos 2x$

$\int \cos 2x\,dx = \sin x\cos x$

$\int \sin x\cos x\,dx = -\frac{1}{2}\cos^2 x$

$\int \sin^2 x\cos x\,dx = \frac{1}{3}\sin^3 x$

$\int \sin x\cos^2 x\,dx = -\frac{1}{3}\cos^3 x$

$\int \sin^2 x\cos^2 x\,dx = \frac{1}{32}(4x - \sin 4x)$

$\int \sin^n ax\cos ax\,dx = \frac{\sin^{n+1} ax}{a(n+1)}$

$\int \sin ax\cos^n ax\,dx = -\frac{\cos^{n+1} ax}{a(n+1)}$

$\int \tan x\,dx = -\ln|\cos x|$

$\int \tan^2 x\,dx = \tan x - x$

$\int \tan^3 x\,dx = \frac{1}{2\cdot\cos(x)^2} + \ln|\cos x|$

$\int \tan^4 x\,dx = x + \frac{1}{3}\tan x(\frac{1}{\cos(x)^2} - 4)$

$\int \frac{1}{\tan x}\,dx = \ln|\sin x|$

$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln|1+x^2|$

$\int \coth x\,dx = \ln|\sinh x|$

$\int \frac{\cos ax}{\sin^n ax}\,dx = -\frac{1}{a(n-1)\sin^{n-1} ax}$

$\int \tanh x\,dx = \ln\cosh x$

$\int \operatorname{arsinh} x\,dx = x\operatorname{arsinh} x - \sqrt{x^2+1}$

$\int \operatorname{arcosh} x\,dx = x\operatorname{arcosh} x - \sqrt{x^2-1}$

$\int \operatorname{artanh} x\,dx = x\operatorname{artanh} x + \frac{1}{2}ln|1-x^2|$

$\int \frac{1}{e^x+a}\,dx = \frac{x - \ln|a+e^x|}{a}$

$\int \frac{1}{x^2+x}\,dx = \ln|x| - \ln|x+1|$

$\int \frac{1}{ax^2+\sqrt{bx+c}}\,dx = \frac{2\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}{\sqrt{4ac-b^2}}$

$\int xe^{ax}\,dx = \frac{ax-1}{a^2}e^{ax}$

$\int x^2e^{ax}\,dx = \frac{a^2x^2-2ax+2}{a^3}e^{ax}$

$\int \frac{1}{p+qe^{ax}}\,dx = \frac{x}{p} - \frac{1}{ap}\ln|p+qe^{ax}|$

$\int xe^{kx^2}\,dx = \frac{1}{2k}e^{kx^2}$

$\int \frac{\ln^n |x|}{x} dx = \frac{\ln^{n+1}|x|}{x+1}$

$\int \sin^2 ax\, dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$

$\int x \cdot \sin ax\, dx = \frac{\sin ax}{a^2} - \frac{x \cdot \cos ax}{a}$

$\int \cos^2 ax\, dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$

$\int x \cdot \cos ax\, dx = \frac{\cos ax}{a^2} + \frac{x \cdot \sin ax}{a}$

$\int \sin ax \cos ax\, dx = -\frac{\cos^2 ax}{2a}$

$\int e^x \sin x\, dx = \frac{e^x}{2}(\sin x - \cos x)$

$\int e^x \cos x\, dx = \frac{e^x}{2}(\sin x + \cos x)$

$\int x^2 \cdot \sin ax\, dx = \frac{1}{a^3}\left(-a^2x^2 \cos ax + 2\cos ax + 2ax \sin ax\right)$

$\int x^2 \cdot \cos ax\, dx = \frac{1}{a^3}\left(a^2x^2 \sin ax - 2\sin ax + 2ax \cos ax\right)$

$\int \tan ax\, dx = -\frac{1}{a} \ln|\cos ax|$

$\int e^{ax} \sin nx\, dx = \frac{e^{ax}(a \sin nx - n \cos nx)}{a^2+n^2}$

$\int e^{ax} \cos nx\, dx = \frac{e^{ax}(a \cos nx + n \sin nx)}{a^2+n^2}$

$\int \cos(kx)\sin(nx)^2 dx = -\frac{1}{4}\left(\frac{\sin(x(k+2n))}{k+2n} + \frac{\sin(x(k-2n))}{k-2n}\right) + \frac{1}{2k}\sin(kx)$

$\int \sin(kx)\cos(nx)^2 dx = -\frac{1}{4}\left(\frac{\cos(x(k+2n))}{k+2n} + \frac{\cos(x(k-2n))}{k-2n}\right) - \frac{1}{2k}\cos(kx)$

$\int \sin(kx)\sin(nx)^2 dx = \frac{1}{4}\left(\frac{\cos(x(k+2n))}{k+2n} + \frac{\cos(x(k-2n))}{k-2n}\right) - \frac{1}{2k}\cos(kx)$

$\int \cos(kx)\cos(nx)^2 dx = \frac{1}{4}\left(\frac{\sin(x(k+2n))}{k+2n} + \frac{\sin(x(k-2n))}{k-2n}\right) + \frac{1}{2k}\sin(kx)$

8.3.1 Basic integrals:

$f(x)$	$F(x)$	$f(x)$	$F(x)$
a	ax	x^n	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\frac{1}{x^n}$	$\frac{-1}{(n-1)x^{n-1}}$
$\sqrt{x}$	$\frac{2}{3}x\sqrt{x}$	$\frac{1}{\sqrt{x}}$	$2\sqrt{x}$
$\frac{1}{(x-a)(x-b)}$	$\frac{1}{a-b}\ln\lvert\frac{x-a}{x-b}\rvert$	$\frac{ax+b}{cx+d}$	$\frac{ax}{c} - \frac{ad-bc}{c^2}\ln\lvert cx+d\rvert$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\arctan(\frac{x}{a})$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\lvert\frac{x-a}{x+a}\rvert$
e^x	e^x	$\ln(x)$	$x(\ln(x)-1)$
a^x	$\frac{a^x}{ln(a)}$	$\log_a(x)$	$x(\log_a(x) - log_a(e))$
xe^ax	$\frac{a^x}{\ln(a)}$	$x\ln(ax)$	$\frac{x^2}{4}(2\ln(ax)-1)$
$\sin(mx)$	$-\frac{\cos(mx)}{m}$	$\arcsin(x)$	$x\arcsin(x) + \sqrt{1-x^2}$
$\cos(mx)$	$\frac{\sin(mx)}{m}$	$\arccos(x)$	$x\arccos(x) - \sqrt{1-x^2}$
$\tan(mx)$	$-\frac{\ln\lvert\cos(mx)\rvert}{m}$	$\arctan(x)$	$x\arctan(x) - \frac{1}{2}\ln(1+x^2)$
$\cot(mx)$	$\frac{\ln\lvert\sin(mx)\rvert}{m}$	$\mathrm{arccot}(x)$	$x\,\mathrm{arccot}(x) + \frac{1}{2}\ln(1+x^2)$
$\frac{1}{1+\sin(x)}$	$\frac{-\cos(x)}{1+\sin(x)}$	$\frac{1}{1-\sin(x)}$	$\frac{\cos(x)}{1-sin(x)}$
$\frac{1}{1+\cos(x)}$	$\frac{\sin(x)}{1+\cos(x)}$	$\frac{1}{1-\cos(x)}$	$\frac{-\sin(x)}{1-\cos(x)}$
$\sinh(x)$	$\cosh(x)$	$\mathrm{arsinh}(x)$	$x\,\mathrm{arsinh}(x) - \sqrt{x^2+1}$
$\cosh(x)$	$\sinh(x)$	$\mathrm{arcosh}(x)$	$x\,\mathrm{arcosh}(x) - \sqrt{x^2-1}$
$\tanh(x)$	$\ln(\cosh(x))$	$\mathrm{artanh}(x)$	$x\,\mathrm{artanh}(x) + \frac{1}{2}\ln(1-x^2)$
$\coth(x)$	$\ln\lvert\sinh(x)\rvert$	$\mathrm{arcoth}(x)$	$x\,\mathrm{arcoth}(x) + \frac{1}{2}\ln(x^2-1)$

8.3.2 Per Parts Resolution method:

$\int_a^b f(x) \cdot g'(x)dx = f(b) \cdot g(b) - f(a) \cdot g(a) - \int_a^b f'(x)g(x)dx$

8.4 Some Series:

$$\sum_{k=0}^{\infty} aq^k = \frac{a}{1-q},\ 0 < |q| < 1$$

$$\sum_{k=0}^{\infty} (k+1)q^k = \frac{1}{(1-q)^2},\ 0 < |q| < 1$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4} \qquad \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90} \qquad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln 2$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12} \qquad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} = \frac{7\pi^4}{720}$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e \qquad \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{e}$$

8.5 Logarithm:

$\log(uv) = \log u + \log v \qquad \log\frac{u}{v} = \log u - \log v$

$\log\frac{1}{v} = -\log v \qquad \log u^r = r \cdot \log u$

$\log_a x = \frac{\log_b x}{\log_b a} \qquad y = \log_a x \Leftrightarrow a^y = x$

8.6 Differential Calculus:

$f(x) = u(x) \pm v(x) \qquad f'(x) = u'(x) \pm v'(x)$

$f(x) = c \cdot u(x) \qquad f'(x) = c \cdot u'(x)$

$f(x) = u(x) \cdot v(x) \qquad f'(x) = u'(x)v(x) + u(x) \cdot v'(x)$

$f(x) = \frac{u(x)}{v(x)} \qquad f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$

$f(x) = u(v(x)) \qquad f'(x) = u'(v(x)) \cdot v'(x)$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

$(\arctan x)' = \frac{1}{1+x^2} \qquad (\tan x)' = \frac{1}{\cos^2 x} = 1 + \tan^2 x$

$(\mathrm{arsinh}\, x)' = \frac{1}{\sqrt{x^2+1}} \qquad (\mathrm{arcosh}\, x)' = \frac{1}{\sqrt{x^2-1}}$

$(\mathrm{artanh}\, x)' = \frac{1}{1-x^2} \qquad (\tanh x)' = \frac{1}{\cosh^2 x} = 1 - \tanh^2 x$

8.7 Standard ODE:

$F''(x) = kF(x)$

- $k = 0 \to F(x) = Ax + B$
- $k < 0 \to F(x) = C\cos(\sqrt{-k}x) + D\sin(\sqrt{-k}x)$
- $k > 0 \to F(x) = Ee^{\sqrt{k}x} + Ie^{-\sqrt{k}x}$
  or $F(x) = E\sinh(\sqrt{k}x) + I\cosh(\sqrt{k}x)$
  if $E = -I \to F(x) = C\sinh(\sqrt{k}x)$
  if $E = I \to F(x) = C\cosh(\sqrt{k}x)$

$F''(x) = -kF(x)$

- $k = 0 \to F(x) = Ax + B$
- $k > 0 \to F(x) = C\cos(\sqrt{+k}x) + D\sin(\sqrt{+k}x)$
- $k < 0 \to F(x) = Ee^{\sqrt{-k}x} + Ie^{-\sqrt{-k}x}$
  or $F(x) = E\sinh(\sqrt{-k}x) + I\cosh(\sqrt{-k}x)$
  if $E = -I \to F(x) = C\sinh(\sqrt{-k}x)$
  if $E = I \to F(x) = C\cosh(\sqrt{-k}x)$

$F'(x) = aF(x) \leftrightarrow F = Le^{ax}$

8.8 Some Limits:

$\lim_{x\to 0} \frac{\sin x}{x} = 1$	$\lim_{x\to 0} \frac{e^{x-1}}{x} = 1$	$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$
$\lim_{x\to 1} \frac{\ln x}{x-1} = 1$	$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$	$\lim_{x\to\infty} \frac{\ln x}{x^a} = 0$
$\lim_{x\to\infty} \frac{x^m}{e^{ax}} = 0$	$\lim_{x\to 0} (x^a \ln x) = 0$	$\lim_{x\to 0} \frac{\ln(a+x)}{x} = \frac{1}{a}$
$\lim_{x\to\infty} \sqrt[x]{x} = 1$	$\lim_{x\to\infty} \frac{e^{ax}}{b} = \infty$	$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$
$\lim_{x\to\infty} \sqrt[x]{a} = 1$	$\lim_{x\to 0} x^a \ln^b x = 0$	$\lim_{x\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$
$\lim_{x\to\infty} \frac{x^b}{e^{ax}} = 0$		$\lim_{x\to\infty} \left(\frac{x-1}{x+1}\right)^x = \frac{1}{e^2}$
$\lim_{x\to\pm\frac{\pi}{2}} \frac{\tan x}{x} = \mp\infty$	$\lim_{x\to 0} x \sin\frac{1}{x} = 1$	$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$
$\lim_{x\to 0} \frac{x}{\sin ax} = \frac{1}{a}$	$\lim_{x\to 0} \frac{\sin ax}{\sin x} = a$	$\lim_{x\to 0} \frac{1 - \cos x}{x^2} = 1$
$\lim_{x\to 0} \frac{\arctan x}{x} = 1$	$\lim_{x\to 0} \frac{\arcsin x}{x} = 1$	$\lim_{x\to\pm\infty} \arctan x = \pm\frac{\pi}{2}$

8.9 Limit:

$$\lim_{x\to a}(f(x)\pm g(x)) = \lim_{xra} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a} c\cdot f(x) = c\cdot \lim_{x\to a} f(x)$$

$$\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{xra} f(x)\cdot \lim_{x\to a} g(x)$$

$\lim_{x\to a} f(x) = u_0,\ g(u)$ stetig bei u_0

$$\Rightarrow \lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right) = g(u_0)$$

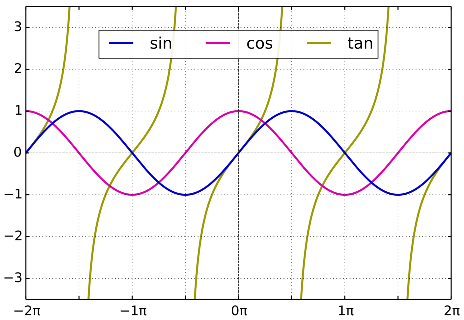
8.9.1 Bernoulli De L'Hôpital:

Sind f und g differenzierbar, $g(x)\neq 0$ auf (a,b),

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ (oder $\pm\infty$) und existiert $\lim_{x\to a}\frac{f'(x)}{g'(x)}$,

so gilt: $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

8.10 Trigonometric Functions:



- $\cos^2(\alpha)+\sin^2(\alpha) = 1$
- $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$
- $\cot(\alpha) = \frac{cos(\alpha)}{\sin(\alpha)}$
- $\cot(\alpha) = \frac{1}{\tan(\alpha)}$
- $\frac{1}{\cos^2(\alpha)} = 1+\tan^2(\alpha)$
- $\frac{1}{\sin^2(\alpha)} = 1+\cot^2(\alpha)$

8.10.1 Periodicity:

- $\cos(\alpha+2\pi) = \cos(\alpha)$
- $\sin(\alpha+2\pi) = \sin(\alpha)$
- $\tan(\alpha+2\pi) = \tan(\alpha)$

8.10.2 Relation between trigonometric functions:

$\cos(-\alpha) = \cos(\alpha)$	$\sin(-\alpha) = -\sin(\alpha)$	$\tan(-\alpha) = -\tan(\alpha)$
$\cos(\pi-\alpha) = -\cos(\alpha)$	$\sin(\pi-\alpha) = \sin(\alpha)$	$\tan(\pi-\alpha) = -\tan(\alpha)$
$\cos(\pi+\alpha) = -\cos(\alpha)$	$\sin(\pi+\alpha) = -\sin(\alpha)$	$\tan(\pi+\alpha) = \tan(\alpha)$
$\cos(\frac{\pi}{2}-\alpha) = \sin(\alpha)$	$\sin(\frac{\pi}{2}-\alpha) = \cos(\alpha)$	$\tan(\frac{\pi}{2}-\alpha) = \cot(\alpha)$
$\cos(\frac{\pi}{2}+\alpha) = -\sin(\alpha)$	$\sin(\frac{\pi}{2}+\alpha) = \cos(\alpha)$	$\tan(\frac{\pi}{2}+\alpha) = -\cot(\alpha)$

8.10.3 Exact value of trigonometric functions:

α deg	α rad	$\cos(\alpha)$	$\sin(\alpha)$	$\tan(\alpha)$	$\cot(\alpha)$
$0°$	0	1	0	0	$-$
$30°$	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
$60°$	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$
$90°$	$\frac{\pi}{2}$	0	1	$-$	0

8.10.4 Trigonometric function with $t = tan(\frac{\alpha}{2})$:

- $\cos(\alpha) = \frac{1-t^2}{1+t^2}$
- $\sin(\alpha) = \frac{2t}{1+t^2}$
- $\tan(\alpha) = \frac{2t}{1-t^2}$

8.10.5 Addition Theorems:

$$\sin x = \frac{1}{2i}(e^{ix}-e^{-ix})$$

$$\cos x = \frac{1}{2}(e^{ix}+e^{-ix})$$

$$\tan x = \frac{e^{ix}-e^{-ix}}{i(e^{ix}+e^{-ix})}$$

$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha\pm\beta) = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1-2\sin^2\alpha$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha)$$

$$\sin^4\alpha = \frac{1}{8}(\cos 4\alpha - 4\cos 2\alpha + 3)$$

$$\sin^n\alpha = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left((n-2k)(\alpha-\tfrac{\pi}{2})\right)$$

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

$$\cos^3\alpha = \frac{1}{4}(3\cos\alpha + \cos 3\alpha)$$

$$\cos^4\alpha = \frac{1}{8}(3+4\cos 2\alpha + \cos 4\alpha)$$

$$\cos^n\alpha = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos((n-k)\alpha)$$

$$\tan^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{1+\cos\alpha}$$

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

$$\binom{n}{k} = \frac{n\cdot(n-1)\cdots(n-k+1)}{n!}$$

8.10.6 Sums and Products:

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\tan(\alpha)+\tan(\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha)\cos(\beta)}$$

$$\tan(\alpha)-\tan(\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha)\cos(\beta)}$$

$$a\cos(\alpha) + b\sin(\alpha) = Acos(\alpha-\varphi), \leftrightarrow$$

$$\leftrightarrow A = \sqrt{a^2+b^2} \,\&\, \cos(\varphi) = \frac{a}{A} \,\&\, \sin(\varphi) = \frac{b}{A}$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$$

8.10.7 Inverse Functions:

$$\cos(\arcsin x) = \sqrt{1-x^2} \qquad \sin(\arccos x) = \sqrt{1-x^2}$$

$$\sin(\arctan x) = \frac{x}{\sqrt{x^2+1}} \qquad \cos(\arctan x) = \frac{1}{\sqrt{x^2+1}}$$

$$\tan(\arcsin x) = \frac{x}{(1-x)^{1/4}} \qquad \tan(\arccos x) = \frac{(1-x)^{1/4}}{x}$$

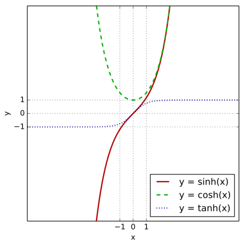
.

8.11 Hyperbolic Functions:

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \operatorname{arsinh} x = \ln(x + \sqrt{x^2+1})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \qquad \operatorname{arcosh} x = \ln(x + \sqrt{x^2-1})$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \operatorname{artanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$



$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh(x_1 \pm x_2) = \sinh x_1 \cosh x_2 \pm \cosh x_1 \sinh x_2$$

$$\cosh(x_1 \pm x_2) = \cosh x_1 \cosh x_2 \pm \sinh x_1 \sinh x_2$$

$$\tanh(x_1 \pm x_2) = \frac{\tanh x_1 \pm \tanh x_2}{1 \pm \tanh x_1 \tanh x_2}$$

$$\sinh x_1 + \sinh x_2 = 2\sinh\frac{x_1+x_2}{2}\cosh\frac{x_1-x_2}{2}$$

$$\sinh x_1 - \sinh x_2 = 2\cosh\frac{x_1+x_2}{2}\sinh\frac{x_1-x_2}{2}$$

$$\cosh x_1 + \cosh x_2 = 2\cosh\frac{x_1+x_2}{2}\cosh\frac{x_1-x_2}{2}$$

$$\cosh x_1 - \cosh x_2 = 2\sinh\frac{x_1+x_2}{2}\sinh\frac{x_1-x_2}{2}$$

$$\tanh x_1 \pm \tanh x_2 = \frac{\sinh x_1 \pm x_2}{\cosh x_1 \cosh x_1}$$

8.12 Graph of some common functions: