Exercise 1

Recall the definition of the induced chain map:

$$f_\sharp : c = \sum_i \alpha_i \sigma_i \in C_p(K_1) \mapsto \sum_i \alpha_i \tau_i \in C_p(K_2)$$

where $\tau_i = \begin{cases} f(\sigma_i), & \text{if } f(\sigma_i) \text{ p-simplex in } K_2. \\ 0, & \text{otherwise.} \end{cases}$

- $p = 0$:

  Since all 0-simplices of K_1 and are mapped to 0-simplices of K_2 by f, $f_\sharp(c) = f(c)$ for all $c \in C_0(K_1)$

  Notice that in K_1 there is one connected component, therefore $H_0(K_1) \cong \mathbb{Z}_2$. Take any 0-simplex c as class representative:

  $$f_*([c]) = f_\sharp(c) + B_0(K_2) = [f_\sharp(c)] = [f(c)]$$

  We conclude that f_* maps a class in $H_0(K_1)$ to a class in $H_0(K_2)$ according to the simplicial map f, and since both simplicial complexes have one connected component, this map is equivalent to the identity map $f_* : \mathbb{Z}_2 \to \mathbb{Z}_2$.

- $p = 1$:

  We calculate the homology group of K_1.

  $$B_1(K_1) = \langle \{ab + bc + ca\} \rangle = \langle \{\partial abc\} \rangle \cong \mathbb{Z}_2$$

  $$Z_1(K_1) = \langle \{ab + bc + ca, bc + cd + db, de + ec + cd\} \rangle \cong \mathbb{Z}_2^3$$

  therefore

  $$H_1(K_1) = \langle \{bc + cd + db, de + ec + cd\} \rangle \cong \mathbb{Z}_2^2$$

  Notice furthermore that $B_1(K_2) = \emptyset$. Therefore $f_*(c) = f_\sharp(c)$ for all $c \in H_1(K_1)$.

  We calculate

  $$f_*(bc + cd + db) = f_\sharp(bc + cd + db) = xy + yz + xz$$

  $$f_*(de + ec + cd) = f_\sharp(de + ec + cd) = yz + yz + 0 = 0$$

- $p = 2$:

  $H_2(K_1) = \emptyset$ since there are no 2-cycles nor 2-boundaries in K_1. Therefore, $f_* = 0$ is the induced homomorphism.

Exercise 2

1. f injective does not imply f_* injective. In the corresponding counter example, $H_1(K_1) \cong \mathbb{Z}_2$ where as $H_1(K_2) \cong 0$, thus f_* cannot be injective. However, f is injective since different simplices in K_1 are mapped to distinct simplices in K_2.

2. f surjective does not imply f_* injective. The same example as the above case suffices, since f is also a surjective simplicial map if we add the filled triangle to its image (remark that f does not need to be injective so this can be done). Thus f is surjective and f_* is not injective.

3. f injective does not imply f_* surjective. In the corresponding example, H_0 gains an element in K_2, since there is one more connected component. Here f is injective, but f_* is not surjective since it doesn't map any element to the new connected component.

4. f surjective does not imply f_* surjective. In the corresponding example, f is surjective, but the hole in $H_1(K_2)$ is not the image of any class in $H_1(K_1)$.

The corresponding counter examples are:

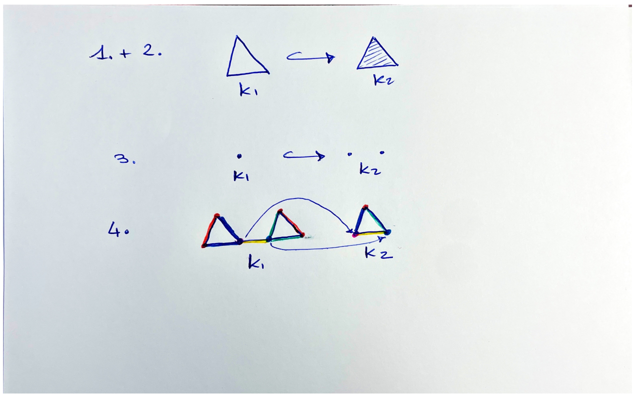

Figure 1: Counter examples

Exercise 3

a)

Case $f(-x) = f(x)$.
Since we are in the setting of $S^1 \to S^1$, the only cycle of the input just consists of the whole S^1. Because of the condition on f, this thus produces every point in S^1 an even number of times, thus $f_\#(S^1)$ is the empty set in $\mathbb{Z}_2$. Therefore, f_* is trivial.

Case $g(-x) = -g(x)$.
To show that this is an isomorphism, we show that $g_\#(S^1) = S^1$. Consider a fixed x. Then you see that $g(x)$ and $g(-x)$ must be antipodal. Thus, when travelling along one half of S^1 from x to -x, the image of g must travel from $g(x)$ to $g(-x)$. Since g is continuous, this image must sweep through at least one half of S^1. Since for every point y that is an image of some x', we also have that $-y$ is an image of $-x'$, we have that $g_\#(S^1) = S^1$, and the claim follows. ($g_\#$ maps only cicle to itself)

b)

Suppose by contradiction that f and g are homotopic. By Fact 2.44 of the lecture notes, we would then get that $f_* = g_*$, which is not the case by part a).

c)

Suppose by contradiction that there exists a map $h : S^2 \to S^1$ such that $h(-x) = -h(x)$. We get the following maps, where i denotes the inclusion map:

$$\begin{tikzcd} S^2 \arrow[r, "h"] & S^1 \\ S^1 \arrow[u, "i"] \arrow[ur, "g"'] & \end{tikzcd}$$

where we define $g := h \circ i$. These maps induce the following homomorphisms:

$$\begin{tikzcd} H_1(S^2) \arrow[r, "h_*"] & H_1(S^1) \\ H_1(S^1) \arrow[u, "i_*"] \arrow[ur, "g_*"'] & \end{tikzcd}$$

and, since $g : S^1 \to S^1$ is an odd map, by point **a)**, g_* is an isomorphism. Moreover, by the *functorial property* (cf. lecture notes page 24), we have that $g_* = h_* \circ i_*$.
As $H_1(S^2) \cong 0$, this diagram gives a contradiction: the image of g_* is $H_1(S^1)$, but the image $h_*(i_*(H_1(S^1))) \subseteq h_*(H_1(S^2)) = h_*(0) = 0$ since h_* is a homomorphism. Since i is a well-defined map, we conclude that h must not exist.

d)

Suppose that there exists a map $\Phi : S^2 \to \mathbb{R}^2$ with $\Phi(-x) = -\Phi(x)$, such that it does not have a zero. Then $h(x) := \frac{\Phi(x)}{|\Phi(x)|}$ is a well defined, continuous and odd map from S^2 to S^1, and this is a contradiction because of point **c)**.