# Exercise 1

The following pictures contains the Reeb graphs for the object for the respective filter functions:

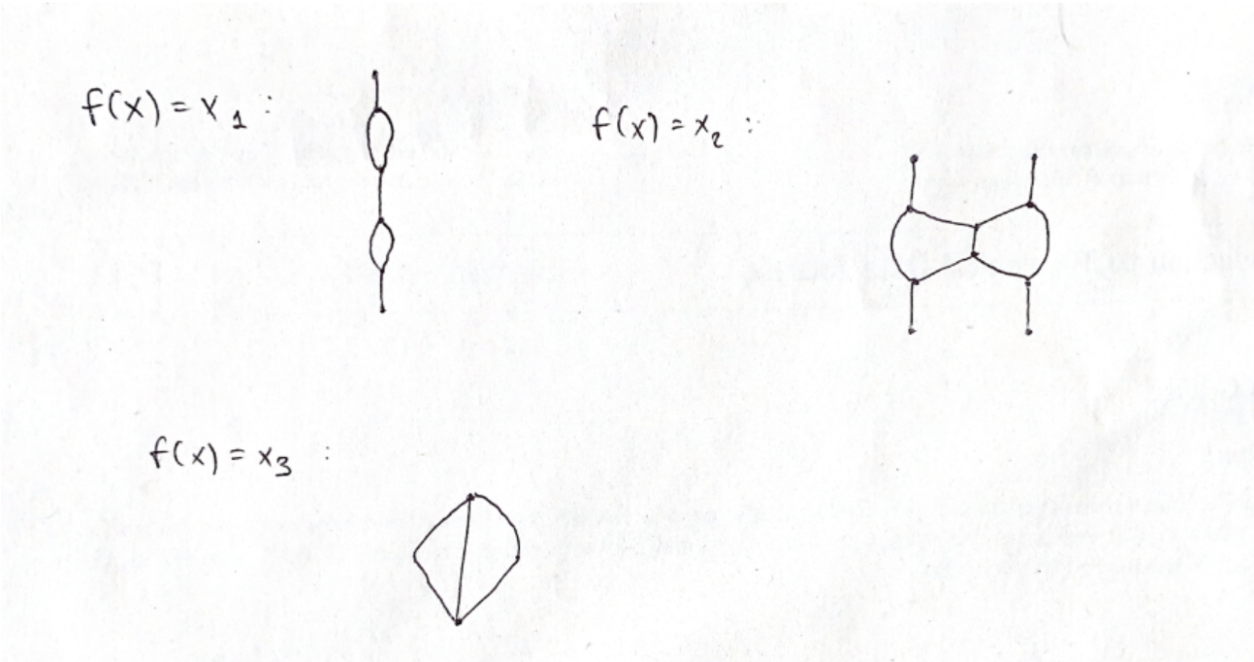


Figure 1: Reeb graphs for the double torus with their respective filter functions.

As a sanity check, we can verify that these can be correct, as for a 2-manifold $X$ that is orientable (which is the case for the double torus), the number of 1-dimensional holes in the Reeb graph should be half the number of 1-dimensional holes in the object $X$. The double Torus has 4 one-dimensional holes, hence we expect two holes to be represented in each Reeb graph.

# Exercise 2

## a)

To show this result, we show that the horizontal homology of a 2-dimensional simplicial complex $K$ embedded in $\mathbb{R}^2$ under the filter function $f(x) = x_1$ has empty horizontal homology, i.e. $\bar{H}_1(K) = 0$. Then, it follows that the vertical homology group $\check{H}_1(K) = H_1(K)/\bar{H}_1(K) = H_1(K)$, and by fact 11 of scribe notes 19 (there exists an isomorphism $H_1(R_f) \cong \check{H}_1(K)$), we can conclude.
To show that $\bar{H}_1(K) = 0$, consider $h \in \bar{H}_1(K)$. By the definition of horizontal homology, there must exist a finite family $A = \{a_1, \ldots, a_k\}$ such that $h$ has a pre-image under

$$H_1(\cup_{a\in A} K_a) \to H_1(K)$$

where $K_a = f^{-1}(a)$. Now, $f^{-1}(a) = \{x \in K : f(x) = a\} = \{x \in K : x_1 = a\}$, thus

$$\begin{aligned}\cup_{a\in A} K_a &= \cup_{a\in A}\{x \in K : x_1 = a\} \\ &= \cup_{a\in A}\{\text{intervals}\}\end{aligned}$$

thus $H_1(\cup_{a\in A} K_a) = 0$ for any collection $A$, hence $\bar{H}_1(K) = 0$ and we can conclude. This is just a (finite, since K is finite) subset of the parallel lines through the points in A, which is always going to be a uniion of intervals.

**b)**

Imagine $K$ as an empty triangle embedded in $\mathbb{R}^2$. Then, taking the constant map $f : K \to c$ for a $c \in \mathbb{R}$ works, as $\beta_1(K) = 1 > 0 = \beta_1(R_f)$.

# Exercise 3

- If we add a 0-face (i.e. a vertex $v$) to $K$, we extend $f$ accordingly (e.g. $f(v) = c$), there is only one possible scenario:

  the new vertex $v$ is not path connected to the rest of the simplices in $K$ (if it would be path connected then we would also need to add an edge but we insert maximum one simplex, hence a new connected component is created and this is reflected by a single point in the Reeb graph at the level set $c$. This point will remain isolated.

- If we add a $p$-face, with $p > 2$, there is no change in the Reeb graph. This is because for a piecewise-linear function, the Reeb graph depends only on the $2$-skeleton of the simplicial complex (c.f. top of page 65 in the lecture notes).

- If we add a $1$-face (i.e. an edge), there are two scenarios. Suppose that we can orient $K$ and $f$ such that, as seen in many examples during the lecture, $f$ is increasing as we go ”up” vertically on the plane. With this orientation, we can distinguish:

  - If we add a horizontal edge, it creates a connection in the Reeb graph. Indeed the horizontal edge will connect two different disconnected components in the pre-image of that level set.
  - If we add a non-horizontal edge $uv$, if the edge $uv$ goes between levels (function values) $c < c'$, it adds another connected component between $c$ and $c'$, which forks out at $c$ from the connected component containing $u$, and merges back at $c'$ with the connected component containing $v$. Thus will will have an up-fork in $u$ and a down-fork in $v$.

- If we add a $2$-face, intuitively we are ”filling up a triangle” that exists within $K$.

  - If $f$ takes the same value on all the vertices of the triangle, in this case we speak about a *horizontal 2-face*, there is no change in the Reeb graph.
  - Otherwise, i.e. for a *non-horizontal 2-face* $\sigma$, you can see that the intersection of $\sigma$ with every levelset c is an edge. The endpoints of this edge are the points which are the intersections of the boundary of $\sigma$ with the levelset. If these endpoints were previously in distinct connected components, adding $\sigma$ joins them in the Reeb graph. Otherwise, there is no change at c.