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Assignment 1
Topological Data Analysis, FS 23

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A variant of Helly's theorem

Recall the following definitions from [1]:

Definition 1. *The diameter of a point set Q is $\sup_{p,q\in Q}\ \mathrm{d}(p,q)$. The set Q is bounded if its diameter is finite, and is unbounded otherwise. A point set Q in a metric space is compact if it is closed and bounded.*

Definition 2. *Given a finite collection of sets $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$, we define the nerve of the set $\mathcal{U}$ to be the simplicial complex $N(\mathcal{U})$ whose vertex set is the index set A, and where a subset $\{\alpha_0, \alpha_1, \ldots, \alpha_k\} \subseteq A$ spans a k-simplex in $N(\mathcal{U})$ if and only if $U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_k} \neq \varnothing$.*

Consider now the following definition from Scribe Notes 4 (Definition 8),

Definition 3. *Let X be a metric space, and U a finite family of closed subsets of X. We call U a good cover, if every non-empty intersection of sets in U is contractible (i.e., homotopy equivalent to a point).*

Additionally consider the following proposition:

Proposition 1. *Given a finite family of compact convex sets it holds that*

1. *The sets of the family are closed,*
2. *The intersection of sets of the family is convex,*
3. *If the sets of the family are non empty, they are contractible.*

Proof 1.

1. *Follows from Definition 1*
2. *Follows from Proposition 2.2.6 in [2]*
3. *Let X be a convex set. let $x_0 \in X$ be non empty, convex. We define the following homotopy:*

$$\mathbf{H} : X \times I \to X, \quad \mathbf{H}(\mathbf{x}, t) = t \cdot x_0 + (1-t) \cdot \mathbf{x}\,.$$

This yields a homotopy between the identity map id_X (with $t = 0$) and the constant map x_0 (with $t = 1$). Since X is convex, $\mathbf{H}$ takes values in X and is a continuous function (polynomial in both $\mathbf{x}$ and t).

□

Let $\mathcal{F} = \{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ be a finite family ($n < \infty$) of compact convex k-dimensional subsets of $\mathbb{R}^{\mathrm{d}}$. By assumption we know that any subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of $k+2$ or fewer sets has a non-empty intersection, i.e., $\cap\mathcal{F}' \neq \emptyset$.
Let now $\mathcal{F}^\star$ be the subfamily of $\mathcal{F}$ whose elements (i.e the sets) are build by taking unions and intersection from the sets of $\mathcal{F}$ such that they have a non empty intersection (i.e $\mathcal{F}^\star$ contains all the sets with non empty intersection).

Remark 1. *We know that $\mathcal{F}^\star$ will contain at least all the subfamilies of $\mathcal{F}$ formed by taking maximum $k+2$ elements. Since every intersection of $k+2$ or less sets in not empty, the nerve of $\mathcal{F}^\star$, i.e $N(\mathcal{F}^\star)$ will contain all the $j-1$ simplicies spanned by any j-ple of vertices with $j \leq k+2$ (this follows from Definition 2).*

Moreover, it holds that the subfamility $F^\star$ is a cover of $\bigcup\mathcal{F}$, since each of the sets $\mathcal{F}_i$, with $i = 1, \ldots, m$ can be written as a finite union of sets of $\mathcal{F}^\star$

$$\mathcal{F}_i = \underbrace{(\mathcal{F}_i \cap \mathcal{F}_1)}_{=(*)_1} \cup \cdots \cup \underbrace{(\mathcal{F}_i \cap \mathcal{F}_m)}_{=(*)_m}\,.$$

Remark 2. *Every of the $(*)_i$ is in $\mathcal{F}^\star$ because of the first phrase in Remark 1 and the fact that $k+2 > 2$.*

From Proposition 1 it follows that $\mathcal{F}^\star$ is a good cover of $\bigcup\mathcal{F}^\star$. Recall from [1] the Nerve Theorem

Theorem 1 (Nerve Theorem). *Given a finite cover $\mathcal{U}$ (open or closed) of a metric space M, the underlying space $|N(\mathcal{U})|$ is homotopy equivalent to M if every non-empty intersection $\cap_{i=0}^{k} U_{\alpha_i}$ of cover elements is homotopy equivalent to a point, that is, contractible.*

Using Theorem 1 and Proposition 1, we have that the underlying space $|N(\mathcal{F}^\star)|$ is homotopy equivalent to $\bigcup \mathcal{F}^\star = \bigcup \mathcal{F}$.
Let's now assume by contradiction that $\bigcap \mathcal{F} = \emptyset$. Then it exists a k' that satisfies $k+2 < k' \leq m$ and a subfamily $\{\mathcal{F}_1, \ldots, \mathcal{F}_{k'}\}$ of $\mathcal{F}$ such that their intersection $\mathcal{F}_1 \cap \cdots \cap \mathcal{F}_{k'} = \emptyset$. This means that $|N(\mathcal{F}^\star)|$ does not contain the $k'-1$ dimensional simplex spanned by the correspondent vertices $\{v_1, \ldots, v_{k'}\}$. Consider now the following proposition

Proposition 2. *If $|N(\mathcal{F}^\star)|$ does not contain the $k'-1$ dimensional simplex spanned by the vertices $\{v_1, \ldots, v_{k'}\}$, it holds that $|N(\mathcal{F}^\star)|$ contains a $k'-2$ cycle $c^\star = \sum_{i=1}^{k'} \langle v_1, \ldots, \hat{v}_i, \ldots, v_{k'} \rangle$ that is not a boundary (notation of the $c^\star$ is the same used in the Scribe Notes).*

Proof 2. *We first note that if $c^\star$ was a border, it would have been the border of of k' dimensional simplex spanned by the vertices $\{v_1, \ldots, v_{k'}\}$ but since this is not in $|N(\mathcal{F}^\star)|$, this cannot be the case. Therefore it remains to prove that $c^\star$ is indeed a $k'-2$ cycle. From the Scribe Notes 6 Definition 2 we have that $c^\star$ is a $k'-2$ cycle if $\delta(c^\star) = 0$. We compute:*

$$\begin{aligned}
\delta(c^\star) &= \delta\left(\sum_{i=1}^{k'} \langle v_1, \ldots, \hat{v}_i, \ldots, v_{k'} \rangle\right) && (1)\\
&= \sum_{j=1, j\neq i}^{k'} \sum_{i=1}^{k'} \langle v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{k'} \rangle && (2)\\
&= 0 && (3)
\end{aligned}$$

where the last equality holds since every tuple of $(\hat{v}_i, \hat{v}_j)$ appears exactly two times (severy possible chains in Equation 2 appears two times) and thus applies what is written after Observation 3 in Scribe Notes 5. □

From the Proposition 2 is follows that $H_{k'-2}(|N(\mathcal{F}^\star)|) \ncong 0$. By Corollary 4 of Scribe Note 8 we have that the homology groups of $\bigcup \mathcal{F}$ are isomorphic to the homology groups of $|N(\mathcal{F}^\star)|$.

Proposition 3. *Let $\mathcal{X}$ be a k dimensional topological space. Then we have that*

$$H_p(\mathcal{X}) \cong 0, \quad \forall p > k\,.$$

Proof 3. *A proof has been given in class for the $d-$dimensional sphere. Intuitively this holds in general because in X (which is k dimensional) there cannot exists non-trivial p-dimensional holes with $p > k$.* □

Since we can assume that $\bigcup \mathcal{F}$ is k dimensional (it is the union of k dimensional sets) and $k' > k+2$, from Proposition 3 we have that $H_{k'-2}(\bigcup \mathcal{F}) \cong 0$. We arrived so at a contradiction. It must so hold that all the sets have a common intersection, i.e., $\bigcap \mathcal{F} \neq \emptyset$.

Literatur

[1] Tamal Krishna Dey and Yusu Wang. *Computational Topology for Data Analysis*. Cambridge University Press, 2022.

[2] Niels Lauritzen. *LECTURES ON CONVEX SETS*. https://users.fmf.uni-lj.si/lavric/lauritzen.pdf, 2010.