Exercise 1

Recall from the lecture the following definitions

Definition 1. *A p-chain* c *(in* K *) is a formal sum of* p*-simplices added with some coefficients from some ring* R.

$$c = \sum_{i=1}^{m_p} \alpha_i \sigma_i$$

where $\alpha_i \in \mathrm{R}$ *and* $\sigma_i \in \mathrm{K}$ *are* p*-simplices.*

Definition 2. *Let* $\sigma = \{v_0, \ldots, v_p\}$ *be a* p*-simplex. Then,* $\delta_{\mathrm{p}}(\sigma)$ *is defined by*

$$\{v_1, \ldots, v_p\} + \{v_0, v_2, \ldots, v_p\} + \ldots + \{v_0, \ldots, v_{\mathrm{p}-1}\} = \sum_{i=0}^{p} \{v_0, \ldots, \hat{v}_i, \ldots, v_{\mathbf{p}}\}$$

In the above notation, $\hat{v}_i$ *denotes that the element* v_i *is omitted from the set. Note that* $\delta_{\mathrm{p}}(\sigma)$ *is a* $(p-1)$*-chain.*

Definition 3. *A p-chain* c *is a* p*-cycle if* $\delta(\mathrm{c}) = 0$. Z_p *is the* p*-th cycle group, consisting of all* p*-cycles.*

Definition 4. *A p-chain c is a p-boundary if* $\exists \mathrm{c}' \in \mathrm{C}_{\mathrm{p}+1}$ *such that* $\delta(\mathrm{c}') = \mathrm{c}$. B_{p} *is the* p*-th boundary group, consisting of all* p*-boundaries.*

Definition 5. *The* p*-th homology group* $\mathrm{H}_{\mathrm{p}}(\mathrm{K}; \mathbb{Z}_2)$ *is the quotient group* $\mathrm{Z}_{\mathrm{p}}(\mathrm{K})/\mathrm{B}_{\mathrm{p}}(\mathrm{K})$.

The simplicial complex K can be drawn as follows

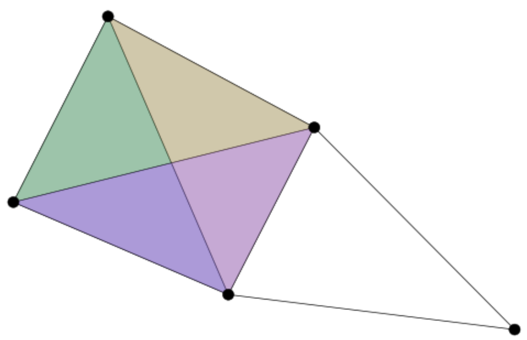


Figure 1: Simplicial complex K

Boundary

- $B_0 = \mathrm{Im}_{\delta_1} = \{$all 0-chains c such that $\exists \mathrm{c}' \in \mathrm{C}_1$ with $\delta_1(\mathrm{c}') = \mathrm{c}\}$ hence $B_0 = \langle\{a+d, b+c, c+d, c+e\}\rangle \cong \mathbb{Z}_2^4$.
- $B_1 = \delta_2(C_2(K)) = \langle \delta_2(\text{basis of } C_2(K))\rangle = \langle\{ab+bc+ac, ab+bd+ad, ac+ad+cd\}\rangle \cong \mathbb{Z}_2^3$ (the cycle $bc+bd+cd$ can be obtained by summing the 3 previous basis elements) and $\{abc, abd, acd, bcd\}$ is a basis of C_2 since these 2-simplices cannot be obtained as formal sums of other 2-simplices.
- $B_2 = \mathrm{Im}_{\delta_3} = \delta_3(C_3) = \delta_3(\{\}) = \{\}$.

Cycle

- $Z_0 = \langle\{\text{all 0-chains c such that } \delta_0(\mathrm{c}) = 0\}\rangle = \langle\{a, b, c, d, e\}\rangle \cong Z_2^5$
- $Z_1 = \ker(\delta_1)$. For each 1-chain $c \in C_1(K)$, $\delta_1(c) = 0 \iff$ each 0-simplex in c is repeated twice. A basis of the group of all 1-chains in which each 0-simplex is repeated twice is given by $\{ab + bd + ad, ac + bc + ab, ac + cd + ad, ce + de + cd\}$, so $Z_1(K) = \langle\{ab + bd + ad, ac + bc + ab, ac + cd + ad, ce + de + cd\}\rangle \cong \mathbb{Z}_2^4$
- $Z_2 = \{\text{all 2-chains c such that } \delta_2(\mathrm{c}) = 0\}$. We have that

  $$\delta_2(abc + abd + acd + bcd) = ab + bc + ac + ab + bd + ad + ac + cd + ad + bc + cd + bd = 0$$

  Since cde is not a 2 dimensional face, the only non zero 2 dimensional cycle is given by $\{abc + abd + acd + bcd\}$, so $Z_2 \cong \mathbb{Z}_2$.

Note: The algorithm in general is quite simple, for example the p–th boundary group: Take the p-chains corresponding to the boundaries of each $p + 1$ simplex. This is a generating set of B_p. Now take the matrix of this set and reduce it to RREF form. This is the stardard lin alg algo to find a vector space basis.)

Since we always take our coefficients to be $\mathbb{Z}_2$, the resulting chain, cycle, boundary & homology groups are also vector spaces, and by fundamental algebra these will be multiple of the coefficient vector space.

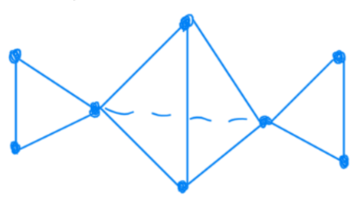
The spanning tree shortly mentioned in class (not exam relevant) is one way to think about finding a cycle group basis, but only for dimension 1! By calculating the min. spanning tree of the 1-skeleton, the number of non-included edges in the tree corresponds exactly to the cardinality of the basis of $\mathbb{Z}_1$. The reason being, every non-included edge closes exactly one loop of the tree.

Homology group

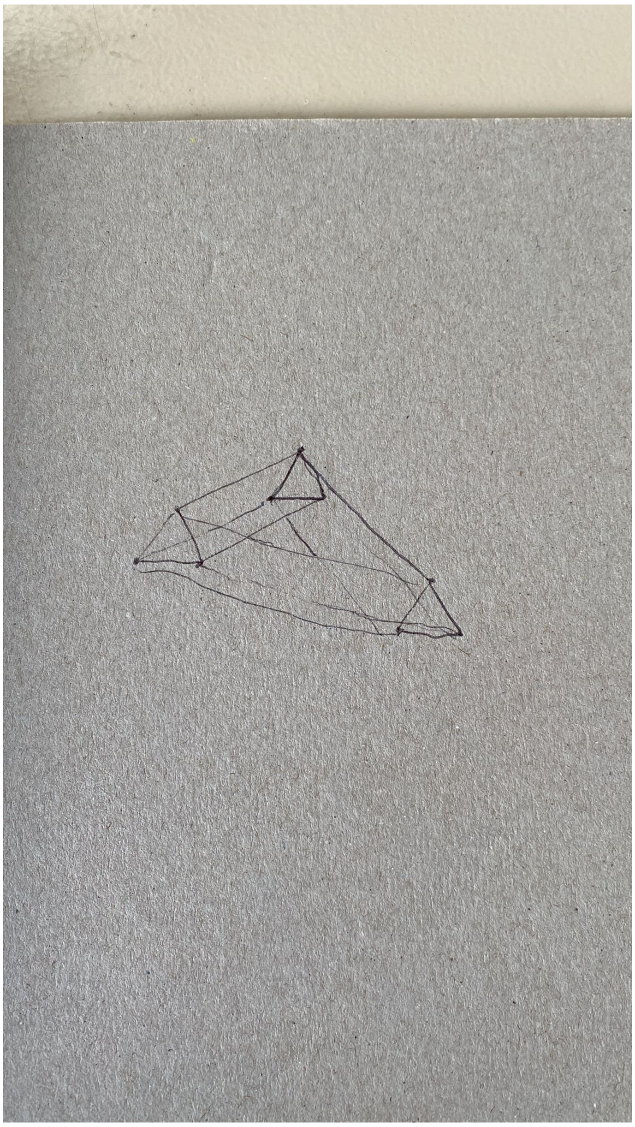
- $H_1(K) = Z_1(K)/B_1(K) \cong \mathbb{Z}_2^4/\mathbb{Z}_2^3 \cong \mathbb{Z}_2$. Note that the last isomorphism holds since $\mathbb{Z}_2^4/\mathbb{Z}_2^3$ is a finite abelian group with cardinality given by $|\mathbb{Z}_2^4|/|\mathbb{Z}_2^3| = 2$ (Lagrange Theorem) and so $\mathbb{Z}_2^4/\mathbb{Z}_2^3 \cong \mathbb{Z}_2$ (Fundamental structure Theorem for finite abelian groups).
- For the 2^{nd} homology group we have that $H_2(K) = Z_2 \backslash B_2 = Z_2 \cong \mathbb{Z}_2$

Exercise 2

Take three circles with each having 3 points on it. If we connect two of these circles with a "tube" we get a cylinder. Doing this with all 3 pairs of circles, we get a torus. Now every tube segment of the torus consists of 3 rectangles, given by the two corresponding points of each circle bordering the tube. These rectabgles can be split into two triangles by diagonally connnecting the two non-corresponding points of the circles. We just defined a triangulation of the torus on the torus itself, with 3 (tubes) $\times 3$ (rectangles) $2 = 18$ triangles. This is the smallest triangulation of the 2 -torus.
Below wedge space have the same homology of the torus but is not homeomorphic to it.



The k^{th} homology group $H_k\left(T^n\right)$ is a free abelian group $\mathbb{Z}^{\operatorname{Bin}(n,k)}$.



Exercise 3

Definition 6. $\beta_p := \dim H_p = \dim Z_p - \dim B_p$ *is the* p*-th Betti number.*

Definition 7. *The Euler characteristic of a simplicial complex* K *is defined as*

$$\chi = \mathrm{k}_0 - \mathrm{k}_1 + \mathrm{k}_2 - \dots$$

with k_i *denoting the number of i-dimensional simplices in* K.

Proposition 1. *The Euler characteristic can then be defined as the alternating sum*

$$\chi = \beta_0 - \beta_1 + \beta_2 - \beta_3 + \dots = \sum_{i=0}^{\infty} (-1)^i \beta_i \tag{1}$$

Proof 1. *We have that* $\delta_p : C_p(K) \to C_{p-1}(K)$ *is a linear map. By the Rank-Nullity theorem*

$$\dim(Im_{\delta_p}) + \dim(\ker_{\delta_p}) = \dim(C_p) = k_p \iff \dim(\ker_{\delta_p}) = k_p - \dim(Im_{\delta_p})$$

By Definition 6 we also have that

$$\beta_p = \dim(\ker_{\delta_p}) - \dim(Im_{\delta_{p+1}})$$

so

$$\beta_p = k_p - \dim(Im_{\delta_p}) - \dim(Im_{\delta_{p+1}}).$$

By plugging this in Equation 1 we get

$$\chi = \sum_{i=0}^{\infty}(-1)^i\beta_i = k_0 - \underbrace{\dim(Im_{\delta_0})}_{=0} - \dim(Im_{\delta_1}) - (k_1 - \dim(Im_{\delta_1}) - \dim(Im_{\delta_2})) + \dots$$
$$= \sum_{i=0}^{\infty}(-1)^i k_i$$

□

Proposition 1 tells us that the Euler characteristic can also be computed as an alternating sum of betti numbers, which only depends on the homology groups of K. The homology groups of different triangulations of the same topological space are all isomorphic (cfr Theorem 10 Scribe Notes 6). In particular they all have the same dimensions and so the alternating sum that computes the Euler characteristic χ does not change with the triangulation.

Exercise 4

Cone CK

Intuitively, the cone CK of a simplicial complex K can be thought of as the cone over K, where the new vertex z represents the apex of the cone, and each simplex in K forms a face of the cone.

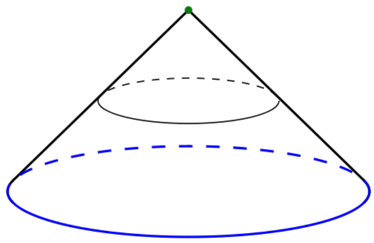


Figure 2: Cone of a circle. The original space X is in blue, and the collapsed end point v is in green.

The cone vertex's z is a deformation retract of the cone CK. Indeed the map

$$R : CK \times I \to CK \text{ given by } R(\sigma, t) = (1-t)\sigma + tz$$

is such that $R(\cdot, 0) = Id_{CK}$, $R(CK, 1) = z$, and $R(z,t) = z, \forall t \in [0,1]$. Note that $R(\cdot, 0) = Id_{CK}$ by definition of CK: CK is built by simply adding the vertex z to any simplex in K, so $Id_K = Id_{CK}$.
The cone CK is so homotopy equivalent to a point (is contractible) and so by Corollary 4 of Scribe Notes 8 its homology groups are given by $H_p(CK) = 0, \forall p > 0$.

Suspension SK

One can view SK as two cones on K, glued together at their base.

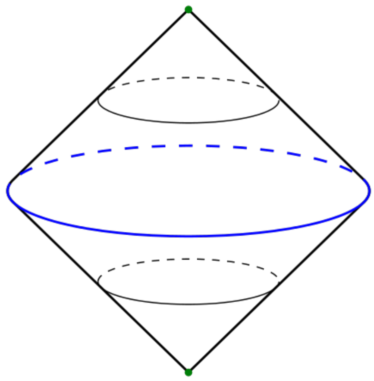


Figure 3: Suspension of a circle. The original space is in blue, and the collapsed end points are in green.

In this case (see wiki, where Maier-Vietoris is used) we have that

$$H_p(SK) \cong \begin{cases} \mathbb{Z}_2 & \text{if } p = 0 \\ H_{p-1}(K) & \text{if } p \geq 2. \end{cases}$$

Moreover $H_0(K) \cong H_1(SK) \oplus \mathbb{Z}_2$

Exercise 5

The subspace of $\mathbb{R}^{d+1}$ given by the wedge sum of a_1 copies of S^1, ..., a_d copies of S^d and $a_0 - 1$ copies of a point $a \in \mathbb{R}$,

$$\underbrace{S^1 \vee \cdots \vee S^1}_{a_1 \text{ times}} \vee \cdots \vee \underbrace{S^d \vee \cdots \vee S^d}_{a_d \text{ times}} \vee \underbrace{a \vee \cdots \vee a}_{a_0 - 1 \text{ times}},$$

is a simplicial complex with $(a_0, a_1, \cdots, a_d)$ as its Betti numbers. This holds since (see example in ethz):

$$dim\left(H_p(\underbrace{S^n \vee \cdots \vee S^n}_{h \text{ times}})\right) = \begin{cases} h & \text{if } p = n \\ 0 & \text{otherwise} \end{cases}$$

and $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}_2$ for any topological space X.
Remember the topological meaning of a wedge sum of two spaces. It's the disjoint union of the two spaces, modulo one random point from each. That just means that you glue the two spaces together at one singular point. Now observe the wedge of a sphere (tetrahedron) and circles (triangles). A fundamental result from general homology (singular homology for general topological spaces, not covered in the course) is that the reduced homology group of a wedge space is the sum of the reduced homology groups of it's parts.
This becomes clear in the simplicial Z_2 case. Every cycle must contain the wedge point an even amount of times, therefore any cycle consists only of certain full triangles around the wedge point. It follows that the basis of generators of Z must be exactly the parts around the wedge point. So the wedge and the disjoint sum of its parts have the same homology.