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Exercise set 9
Topological Data Analysis, FS 23

2023-10-11

Theory Recap

Definition 1 (Shifted homomorphism)**.** *Let* $\mathbb{U}, \mathbb{V}$ *be persistence modules over* $\mathbb{R}$*, and let* ϵ *be any real number. A* homomorphism of degree ϵ *is a collection* Φ *of linear maps*

$$\phi_a : U_a \to V_{a+\epsilon}$$

for all $a \in \mathbf{R}$*, such that the diagram*

$$\begin{tikzcd} U_a \arrow[r, "u_{a,a'}"] \arrow[d, "\phi_a"] & U_{a'} \arrow[d, "\phi_{a'}"] \\ V_{a+\epsilon} \arrow[r, "v_{a+\epsilon,a'+\epsilon}"] & V_{a'+\epsilon} \end{tikzcd}$$

commutes whenever $a \leqslant a'$*. We write*

$$\begin{aligned} \mathrm{Hom}^{\epsilon}(\mathbb{U}, \mathbb{V}) &= \{ \textit{ homomorphisms } \mathbb{U} \to \mathbb{V} \textit{ of degree } \epsilon \} \\ \mathrm{End}^{\epsilon}(\mathbb{V}) &= \{ \textit{ homomorphisms } \mathbb{V} \to \mathbb{V} \textit{ of degree } \epsilon \} \end{aligned}$$

Composition gives a map

$$\mathrm{Hom}^{\epsilon_2}(\mathbb{V}, \mathbb{W}) \times \mathrm{Hom}^{\epsilon_1}(\mathbb{U}, \mathbb{V}) \to \mathrm{Hom}^{\epsilon_1+\epsilon_2}(\mathbb{U}, \mathbb{W})$$

For $\epsilon \geqslant 0$, the *shift map*

$$1^{\epsilon}_{\mathbb{V}} \in \mathrm{End}^{\epsilon}(\mathbb{V})$$

is the degree- ϵ endomorphism given by the collection of maps $\{v_{a,a+\epsilon}\}$ from the persistence structure on $\mathbb{V}$. If Φ is a homomorphism $\mathbb{U} \to \mathbb{V}$ of any degree, then by definition

$$\Phi 1^{\epsilon}_{\mathbb{U}} = 1^{\epsilon}_{\mathbb{V}} \Phi$$

for all $\epsilon \geqslant 0$.

Definition 2 (Interleaving)**.** *Let* $\epsilon \geq 0$*. Two persistence modules* $\mathbb{U}, \mathbb{V}$ *are said to be* ϵ-interleaved *if there are maps*

$$\Phi \in \mathrm{Hom}^{\epsilon}(\mathbb{U}, \mathbb{V}), \quad \Psi \in \mathrm{Hom}^{\epsilon}(\mathbb{V}, \mathbb{U})$$

such that

$$\Psi\Phi = 1^{2\epsilon}_{\mathbb{U}}, \quad \Phi\Psi = 1^{2\epsilon}_{\mathbb{V}}.$$

The interleaving distance *between the two modules* $\mathbb{U}, \mathbb{V}$ *is defined as*

$$d_I(\mathbb{U}, \mathbb{V}) = \inf_{\epsilon} \{\epsilon \,|\, \mathbb{U}, \mathbb{V} \textit{ are } \epsilon \textit{ interleaved}\}. \tag{1}$$

Exercise 1

Proposition 1. *The interleaving distance is a pseudo-metric.*

Proof. 1. *Interleaving distance between isomorphic persistence modules is 0:* It is sufficient to prove that the two persistence modules say $\mathbb{U}, \mathbb{V}$, are 0-interleaved. The commutativity of first two "square" diagrams (we refer to Definition 3 of SN17 for notation) follows directly from the fact that $\mathbb{U}, \mathbb{V}$ are isomorphic. Since $f_a^{-1} \circ f_a = \mathrm{Id}_{U_a}$, $f_a \circ f_a^{-1} = \mathrm{Id}_{V_a}$, also the following diagrams commute and so $\mathbb{U}, \mathbb{V}$ are 0-interleaved.

$$\begin{tikzcd} U_a \arrow[rr, "\mathrm{i}_{U_a}"] \arrow[dr, "f_a"'] & & U_a \\ & V_a \arrow[ur, "f_a^{-1}"'] & \end{tikzcd} \qquad \begin{tikzcd} & U_a \arrow[dr, "f_a"] & \\ V_a \arrow[ur, "f_a^{-1}"] \arrow[rr, "\mathrm{i}_{V_a}"'] & & V_a \end{tikzcd}$$

2. *Interleaving distance is non-negative:* by contradiction, $d_I(\mathbb{U}, \mathbb{V}) < 0$ implies that there exists an $\epsilon < 0$ such that the two persistance modules are ϵ-interleaved. But this cannot happen by definition since $\epsilon \geq 0$.

3. *Interleaving distance fulfils the triangle inequality:* we have to prove that for any three persistence modules $\mathbb{U}, \mathbb{V}, \mathbb{W}$ it holds that

   $$\mathrm{d_I}(\mathbb{U}, \mathbb{W}) \leqslant \mathrm{d_I}(\mathbb{U}, \mathbb{V}) + \mathrm{d_I}(\mathbb{V}, \mathbb{W})$$

   Given a δ_1-interleaving between $\mathbb{U}, \mathbb{V}$ and a δ_2-interleaving between $\mathbb{V}, \mathbb{W}$ we can construct a $\delta = (\delta_1 + \delta_1)$-interleaving between $\mathbb{U}, \mathbb{W}$ by composing the following interleaving maps:

   $$\mathbb{U} \xrightarrow{\Phi_1} \mathbb{V} \xrightarrow{\Phi_2} \mathbb{W}$$
   $$\mathbb{U} \xleftarrow{\Psi_1} \mathbb{V} \xleftarrow{\Psi_2} \mathbb{W}$$

   We have that $\Phi = \Phi_2\Phi_1$ and $\Psi = \Psi_1\Psi_2$ are interleaving maps, indeed

   $$\Psi\Phi = \Psi_1\Psi_2\Phi_2\Phi_1 = \Psi_1 1_{\mathbb{V}}^{2\delta_2}\Phi_1 = \Psi_1\Phi_1 1_{\mathbb{U}}^{2\delta_2} = 1_{\mathbb{U}}^{2\delta_1}1_{\mathbb{U}}^{2\delta_2} = 1_{\mathbb{U}}^{2\delta}$$
   $$\Phi\Psi = \Phi_2\Phi_1\Psi_1\Psi_2 = \Phi_2 1_{\mathbb{V}}^{2\delta_1}\Psi_2 = \Phi_2\Psi_2 1_{\mathbb{W}}^{2\delta_1} = 1_{\mathbb{W}}^{2\delta_2}1_{\mathbb{W}}^{2\delta_2} = 1_{\mathbb{W}}^{2\delta}$$

   Therefore

   $$\inf\{\delta \mid \mathbb{U} \text{ and } \mathbb{W}, \delta\text{-interleaved }\} \leq \inf\{\delta \mid \mathbb{U} \text{ and } \mathbb{V}, \delta\text{-interleaved }\} + \inf\{\delta \mid \mathbb{V} \text{ and } \mathbb{W}, \delta\text{-interleaved }\}$$

□

Remark 1. *The interleaving distance is not a true metric because* $\mathrm{d_I}(\mathbb{U}, \mathbb{V}) = 0$ *does not imply* $\mathbb{U} \cong \mathbb{V}$ *(the two "square" diagram do not commute both ways in general. In fact, two q-tame persistence modules have interleaving distance 0 if and only if their undecorated persistence diagrams are the same. This is a consequence of the isometry theorem (Theorem 9 of SN17).*

Example 0-interleaved non isomorphic: Let $\mathbb{U}$ be the all-0 module (the groups are trivial at all values a, i.e., $U_a = 0$). $\mathbb{V}$ is the module that has $V_a = 0$ for all a except for some single $a' \in R$ it has $V_{a'} = \mathbb{Z}_2$. Note that these are NOT 0-interleaved. However, for any epsilon¿0 they are epsilon-interleaved, and since the interleaving distance is an infimum, their interleaving distance is 0.

Exercise 2

Lemma 1. *Let X be a triangulable topological space and let $\mathcal{F}, \mathcal{G}$ be filtrations over $\mathbb{R}$ of the two tame functions* $\mathrm{f}, \mathrm{g} : \mathrm{X} \to \mathbb{R}$. *Then it holds that $\mathcal{F}, \mathcal{G}$ are $\|f - g\|_\infty$-interleaved.*

Proof. Let $\epsilon = \|f - g\|_\infty$.
Why do the diagonal inclusions hold (consider the case $f^{-1}(-\infty, t]$ and $g^{-1}(-\infty, t+\epsilon]$, the others are "similar": let $x \in X$ be any element of X. Then, if $x \in f^{-1}(-\infty, t]$, we have that $f(x) \leq t$. Since the infinity norm of $f - g$ is ϵ, we have that $g(x) \leq t + \epsilon$. Thus $x \in g^{-1}(\infty, t + \epsilon]$.

$$\begin{tikzcd}
f^{-1}(-\infty, t] \arrow[r] \arrow[dr] & f^{-1}(-\infty, t+\epsilon] \arrow[r] \arrow[dr] & f^{-1}(-\infty, t+2\epsilon] \\
g^{-1}(-\infty, t] \arrow[r] \arrow[ur] & g^{-1}(-\infty, t+\epsilon] \arrow[r] \arrow[ur] & g^{-1}(-\infty, t+2\epsilon].
\end{tikzcd}$$

□

Since f, g are tame functions, then the persistence modules of the filtrations $\mathcal{F}, \mathcal{G}$ are q-tame. We compute so

$$\begin{aligned}
\mathrm{d_b}\left(\mathrm{Dgm_p}\left(\mathcal{F}_\mathrm{f}\right), \mathrm{Dgm_p}\left(\mathcal{F}_\mathrm{g}\right)\right) &= \mathrm{d_I}\left(\mathrm{H_p}\mathcal{F}, \mathrm{H_p}\mathcal{G}\right) \quad \text{(Theorem 9 SN17)} \\
&\leq \mathrm{d_I}(\mathcal{F}, \mathcal{G}) \quad \text{(Observation 7 SN17)} \\
&\leq \|f - g\|_\infty \quad \text{(Lemma 1)}
\end{aligned}$$

Exercise 3

By Definition 2 (see (1)), it is enough to prove that the two persistence modules $\mathbb{U}$ and $\mathbb{V}$ are ϵ-interleaved with $\epsilon = \max\{\frac{x-w}{2}, \frac{z-y}{2}\}$. The following two diagrams

$$
\begin{tikzcd}
U_a \arrow[r, "u_{a,a'}"] \arrow[d, "\phi_a"] & U_{a'} \arrow[d, "\phi_{a'}"] \\
V_{a+\epsilon} \arrow[r, "v_{a+\epsilon,a'+\epsilon}"] & V_{a'+\epsilon}
\end{tikzcd}
\qquad
\begin{tikzcd}
U_{a+\epsilon} \arrow[r, "u_{a+\epsilon,a'+\epsilon}"] & U_{a'+\epsilon} \\
V_a \arrow[u, "\psi_a"] \arrow[r, "v_{a,a'}"] & V_{a'} \arrow[u, "\psi_{a'}"]
\end{tikzcd}
$$

commutes for any a, a', ϵ if ϕ_a, $\phi_{a'}$, ψ_a, $\psi_{a'}$ are the zero maps. So let's focus the attention on the remaining two diagrams:

$$
\begin{tikzcd}
U_a \arrow[rr, "u_{a,a+2\epsilon}"] \arrow[dr, "\phi_a"] & & U_{a+2\epsilon} \\
& V_{a+\epsilon} \arrow[ur, "\psi_{a+\epsilon}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& U_{a+\epsilon} \arrow[dr, "\phi_{a+\epsilon}"] & \\
V_a \arrow[ur, "\psi_a"] \arrow[rr, "v_{a,a+2\epsilon}"] & & V_{a+2\epsilon}
\end{tikzcd}
$$

Let's assume $\epsilon = \max\{\frac{x-w}{2}, \frac{z-y}{2}\} = \frac{z-y}{2}$.

Case $a = y$ For the rightmost diagram, since $a + 2\epsilon = z$, $v_{a,a+2\epsilon} = v_{y,z}$ is the zero map. Then the diagram commutes if $\psi_a = \psi_y, \phi_{a+\epsilon} = \phi_{\frac{z+y}{2}}$ are the zero map. For the leftmost diagram, we consider the two cases $y \geq w$ and $y < w$. If $y \geq w$ then, since $z \geq y + x - w$, we get $z \geq x$ and so $u_{a,a+2\epsilon} = u_{y,z}$ is the zero map. If instead $y < w$, then again $u_{a,a+2\epsilon} = u_{y,z}$ is the zero map. Then the diagram commutes when $\phi_a = \phi_y, \psi_{a+\epsilon} = \psi_{\frac{z+y}{2}}$ are both the zero map, both if $y \geq w$ and $y < w$.

Case $a < y$ Since $a < y$, $v_{a,a+2\epsilon} = v_{y,z}$ is the zero map and so the rightmost diagrams commutes if $\psi_a, \phi_{a+\epsilon}$ are both the zero map. Then, by considering the three cases $a < y < w$, $a < w < y$ and $w \leq a < y$ separately and recalling that we are assuming $\max\{\frac{x-w}{2}, \frac{z-y}{2}\} = \frac{z-y}{2}$, it's easy to check that also the leftmost diagram commutes if $\phi_a, \psi_{a+\epsilon}$ are both the zero map.

Case $a > y$ Since $a + 2\epsilon = a + z - y > z$,$v_{a,a+2\epsilon} = v_{y,a+2\epsilon}$ is the zero map and so the rightmost diagrams commutes if $\psi_a, \phi_{a+\epsilon}$ are both the zero map. By considering the three cases $w \leq y < a$, $y < w < a$ and $y < a < w$ separately and recalling that we are assuming $\max\{\frac{x-w}{2}, \frac{z-y}{2}\} = \frac{z-y}{2}$, it's easy to check that also the leftmost diagram commutes if $\phi_a, \psi_{a+\epsilon}$ are both the zero map.

We can reason analogously if $\epsilon = \max\{\frac{x-w}{2}, \frac{z-y}{2}\} = \frac{x-w}{2}$. Thus the two persistence modules $\mathbb{U}$ and $\mathbb{V}$ are ϵ-interleaved with $\epsilon = \max\{\frac{x-w}{2}, \frac{z-y}{2}\}$.