Exercise 1

Observation: Place $2k$ points around a circle, where always two points are exactly antipodal. Then take a distance for the Vietoris-Rips complex which is just slightly less than the radius of the circle. Thus, $\mathbb{VR}(P)$ is the complex on the points $\{a, a', b, b', \dots\}$ which contains all faces that do not both contain both points of any pair $\{x, x'\}$. The chain consisting of all $(k-1)$-dimensional faces (which are thus maximal) can obviously not be a boundary, and it must be a cycle since every $k-2$ simplex is contained in exactly two $k-1$ faces.
It follows that for this construction:

$$H_{k-1}(\mathbb{VR}(P)) \ncong 0$$

1. Use above Observation for $k = 3$
2. False, follows from above observation

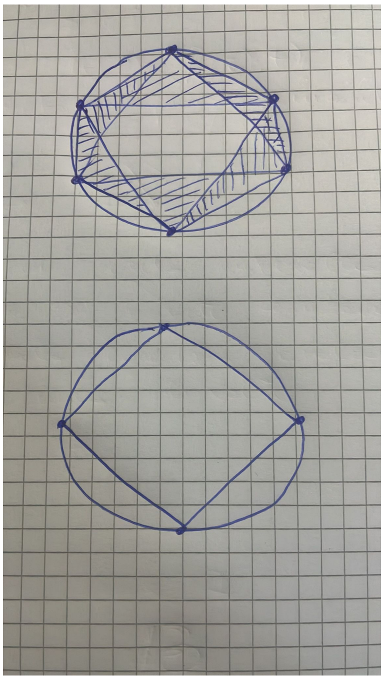

Figure 1: Above $k = 3$, below $k = 2$

Exercise 2

1. *The Nerve of the Voronoi cells $\mathcal{N}$ is contained in the extended Delaunay complex $\mathcal{D}$:*
   For any simplex $\sigma \in \mathcal{N}$, pick a point certifying $\mathcal{N}$, i.e., a point p in the intersection belonging to the Voronoi cells corresponding to the vertices of σ. This point p is equidistant to all the points corresponding to σ, and also these points are its nearest neighbor. Thus, if you draw a circle centered at p going through these points, you have an empty circumcircle for all subsets of 3 points of σ, showing that σ is also in $\mathcal{D}$.

   A simplex in $\mathcal{D}$ is also in $\mathcal{N}$:
   Consider only simplices of dimension 3 (i.e., triangles). Here, it clearly holds, since the center of the circumcircle of the triangle has the three points as nearest neighbors, and thus certifies a triple intersection between their Voronoi cells. Now, to see that larger simplices of $\mathcal{D}$ are also in $\mathcal{N}$, we can use that the Voronoi cells are convex, and thus apply Helly's theorem: If for a set of points Q any three cells intersect, then also all cells must have a common point, and the simplex is in $\mathcal{N}$.

2. For points in general position, see Theorem 2.11 in Berkeley

Uniqueness of Delaunay triangulation alternative informal proof
If the points are in general position, the Voronoi cells only have triple intersections, and two Voronoi cells intersect in some line segment, which can only have triple intersections with two more Voronoi cells, one on each end. Thus, each edge between two points is part of at most two Delaunay triangles, at most one on each side of the edge. If we had multiple Delaunay triangulations, there would need to be a triangle $T = \{u, v, w\}$ which is in the first, but not the second. But, the second triangulation would also need to cover the area of T. It must do that with an edge $\{u, v\}$ and another point w', which is on the same side of $\{u, v\}$ as w. This is a contraddiction since we showed before that for every edge there is at most one Delaunay triangle to each side.

Exercise 3

The Alpha Complex $Del^r(P)$ is the nerve of the sets $R_p(r)$, where $R_p(r) := B(p, r) \cap V_p$.
Since $R_p(r) \subseteq B(p, r)$ for all $p \in P$, the nerve of the sets $R_p(r)$, for $p \in P$, is contained in the nerve of the sets $B(p, r)$, for $p \in P$. Thus

$$Del^r(P) \subseteq \mathbb{C}^r(P). \tag{1}$$

Now we also have that $R_p(r) \subseteq V_p$ for all $p \in P$ and so, the nerve of the sets $R_p(r)$, for $p \in P$, is contained in the nerve of the sets V_p, for $p \in P$. Moreover, if the points in P are in general position, the extended Delaunay complex, that is the nerve of the the Voronoi cells V_p, coincides with the unique Delaunay triangulation of the point set in general position P. So we also have that

$$Del^r(P) \subseteq Del(P), \tag{2}$$

and, by (1) and (2), we deduce that

$$Del^r(P) \subseteq \mathbb{C}^r(P) \cap Del(P).$$

The reverse inclusion does not hold and so the statement is false. Consider the following example in which we shift the lower point towards the other two points:

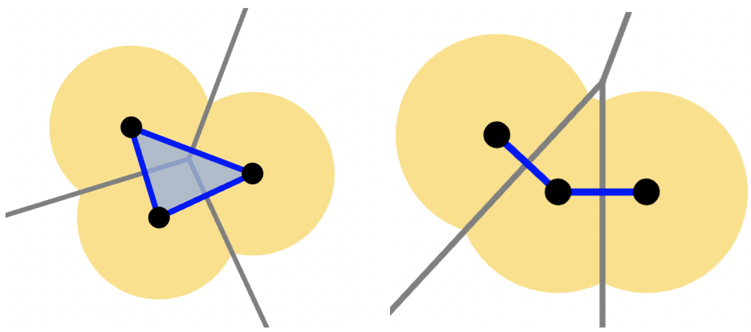

Figure 2: Alpha complex: the triangle between them is contained in the Delaunay triangulation, and also filled in the Cech complex, but since the triangle is very flat, its circumscribing ball is large and it is thus not in the Alpha complex.