Exercise 1

Recall from the lecture the following definitions

Definition 1. *A topological space* (X, T) *is a set of points* X*, with a system* T *of subsets of* X *(called the topology on* X *), such that*

1. $\emptyset \in \mathrm{T}, X \in \mathrm{T}$.
2. *For every* $\mathrm{S} \subseteq \mathrm{T}, \cup\mathrm{S} \in \mathrm{T}$.
3. *For every finite* $S \subseteq \mathrm{T}, \cap\mathrm{S} \in \mathrm{T}$.

The sets in T *are called the open sets of* X.

Definition 2. *A set* $\mathrm{Q} \subseteq \mathrm{X}$ *is called closed, if its complement* $\mathrm{X} \backslash \mathrm{Q}$ *is open. The closure* clQ *is the smallest closed set containing* Q*. The interior int* Q *is the union of all open subsets of* Q*. The boundary bnd* Q *is the set minus its interior: bnd* $\mathrm{Q} = Q \backslash \operatorname{int} Q$

To prove that the finite union of closed sets is closed is sufficient to prove that its complement (hence the finite intersection of open sets) is open. However, this follows directly from axiom 3 of Definition 1.

Exercise 2

Definition 3. *A topological space* (X, T) *is disconnected, if there are two disjoint nonempty open sets* $\mathrm{U}, \mathrm{V} \in \mathrm{T}$*, such that* $\mathrm{X} = \mathrm{U} \cup \mathrm{V}$*. A topological space is connected, if it is not disconnected.*

Definition 4. *A topological space* $\mathbb{T}$ *is called path connected if any two points* $x, y \in \mathbb{T}$ *can be joined by a path, i.e. there exists a continuous map* $f : [0, 1] \to \mathbb{T}$ *of the segment* $[0, 1] \subset \mathbb{R}$ *onto* $\mathbb{T}$ *so that* $f(0) = x$ *and* $f(1) = y$*.*

In the following we will prove that a path connected space is connected.

Proof 1. *Suppose for the sake of contradiction that* $(X, \mathcal{T})$ *is path-connected but not connected. Then there are two disjoint nonempty open sets* $\mathrm{U}, \mathrm{V} \in \mathrm{T}$*, such that* $\mathrm{X} = \mathrm{U} \cup \mathrm{V}$*. Let* $u \in U$ *and* $v \in V$*. Let* $f : [0, 1] \to X$ *be a path from* $f(0) = u$ *to* $f(1) = v$*. Then*

$$[0, 1] = f^{-1}(X) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)\,. \tag{1}$$

The last term in the above equality is the union of two open, non empty, since $0 \in f^{-1}(U)$ *and* $1 \in f^{-1}(V)$*, and disjoint sets. We therefore have a contradiction since [0, 1] is connected.* □

Exercise 3

Definition 5. *A function* $\mathrm{f} : \mathrm{X} \to \mathrm{Y}$ *is continuous if for every open set* $\mathrm{U} \subseteq \mathrm{Y}$*, its pre-image* $\mathrm{f}^{-1}(\mathrm{U}) \subseteq X$ *(the set of all elements* $x \in X$ *such that* $\mathrm{f}(\mathrm{X}) \in \mathrm{U}$ *) is open. Continuous functions are also called maps. If* f *is injective, it is called an embedding.*

Definition 6. *An open map is a function between two topological spaces that maps open sets to open sets. That is, a function* $f : X \to Y$ *is open if for any open set* U *in* X*, the image* $f(U)$ *is open in* Y*. Likewise, a closed map is a function that maps closed sets to closed sets.*

Definition 7. *A homeomorphism is a bijective map* $\mathrm{f} : \mathrm{X} \to \mathrm{Y}$ *whose inverse is also continuous. Two topological spaces* X, Y *are homeomorphic, if there is a homeomorphism between them. We also write* $\mathrm{X} \simeq \mathrm{Y}$ *to say that* X, Y *are homeomorphic.*

Proposition 1. *A bijective continuous map is a homeomorphism if and only if it is open, or equivalently, if and only if it is closed.*

Let $f : \mathrm{X} \to \mathrm{Y}$ be the identity function, where $X = Y$. It is obvious a bijection. We need f to be continuous mapping but not a homeomorphism. Therefore we choose the topology on Y, we say T_Y, to be strictly weaker that the topology T_X on X. Then we have that f is continuous, however it is not open and so it is not a homeomorphism.

Proof 2.

- *f is continuous: let $U \in T_Y \subseteq T_X$, then $f^{-1}(U) = U \in T_X$, which proves f is continuous.*
- *f is not open: there exist $U \in T_X \backslash T_Y$, then $f(U) = U \notin T_Y$, which shows f is not open.*

□

Exercise 4

(a)

We first verify the axioms in Definition 1:

1. $\varnothing$ is open by definition (set U can be the union of zero sets), and $\mathbb{Z}$ is the sequence $S(1, 0)$, and so is open as well.
2. By the second definition of open sets, the finite or infinite union of sets $S(a, b)$ is open. So, using the first definition of open sets, we get that the finite or infinite union of open sets is open.
3. Let U_1, U_2 be open, and $x \in U_1 \cap U_2$. Then exist a_1 and a_2 such that $S(a_1, x) \subset U_1$ and $S(a_2, x) \subset U_2$. In particular $S(a_1, x) \cap S(a_2, x) \subset U_1 \cap U_2$ and, since $S(a_1, x) \cap S(a_2, x) = S(lcm(a_1, a_2), x)$, we have that $U_1 \cap U_2$ is open. This can be generalized to finite intersections of more than 2 open sets.

We therefore have a topology on $\mathbb{Z}$.

(b)

If $\mathbb{Z} \setminus A$ is closed, then its complement A is open. But then for each $x \in A$ exists $b \in \mathbb{Z}$ such that $S(b, x) \subset A$ and since $S(b, x)$ is infinite, we have a contradiction.

(c)

The sets $S(a, b)$ are open by definition. To prove that they are also closed it is sufficient to write S as the complement of an open set:

$$S(a, b) = \mathbb{Z} \backslash \bigcup_{j=1}^{a-1} S(a, b + j) \tag{2}$$

(d)

The only integers that are not integer multiples of prime numbers are -1 and $+1$.

(e)

By sub-question (b) $\mathbb{Z} \backslash \{-1, 1\}$ cannot be closed. On the other hand, by sub-question (c), the sets $S(p, 0)$ are closed. So, if there were only finitely many prime numbers, then $\bigcup_{p \text{ prime}} S(p, 0)$ would be a finite union of closed sets which is closed (see exercise 1 for prove). This would be a contradiction, so there must be infinitely many prime numbers.

Bonus Exercise

See: Topological Arts in Simple Galleries