Exercise 1

The number of holes is homotopy invariant (see Cor. 4 scribe notes 8). So, we can divide the alphabet letters in three classes: $\{B\}, \{D, O, P, Q, A, R\}$,
$\{C, G, L, M, N, S, U, V, W, Z, E, F, J, T, Y, H, I, K, X\}$. While the "B" is homotopy equivalent to the figure-of-eight graph, all the letters in the third class are contractible (all finite trees are contractible). The letters in the second class are homotopy equivalent to the circle.

Exercise 2

Definition 1. *Let* g, h *be maps* $\mathrm{X} \to \mathrm{Y}$. *A homotopy connecting* g *and* h *is a map* $\mathrm{H} : \mathrm{X} \times [0, 1] \to \mathrm{Y}$ *such that* $\mathrm{H}(\cdot, 0) = \mathrm{g}$ *and* $\mathrm{H}(\cdot, 1) = \mathrm{h}$. *In this case* g *and* h *are called homotopic.*

Definition 2. *Two spaces* X, Y *are homotopy equivalent if there exist maps* $\mathrm{g} : \mathrm{X} \to \mathrm{Y}$ *and* $\mathrm{h} : \mathrm{Y} \to \mathrm{X}$ *such that:*

- $\mathrm{h} \circ \mathrm{g}$ *is homotopic to* id_X *(the identity map* $\mathrm{x} \mapsto \mathrm{x}$ *), and*
- $\mathrm{g} \circ \mathrm{h}$ *is homotopic to* id_Y.

Definition 3. *Let* $A \subseteq X$. *A deformation retract of* X *onto* A *is a map* $\mathrm{R} : \mathrm{X} \times [0, 1] \to X$*, such that*

- $\mathrm{R}(\cdot, 0) = \mathrm{id}_X$
- $\mathrm{R}(x, 1) \in A, \forall x \in X$
- $\mathrm{R}(a, t) = a, \forall a \in A, t \in [0, 1]$

If such a deformation retract of X *onto* A *exists, we also say that* A *is a deformation retract of X.*

Proposition 1. *If* A *is a deformation retract of* X *(there exists a deformation retract of* X *onto* A*), then* A *and* X *are homotopy equivalent.*

2.1 Cylinder is homotopy equivalent to a circle

To prove that the cylinder is homotopy equivalent to the circle, it is sufficient to prove that the circle is a deformation retract of the cylinder (cfr Proposition 1).

We define

$$R : (S^1 \times I) \times I \to S^1 \times I, \text{ with } ((x, y), t) \mapsto (x, (1 - t) \cdot y) \tag{1}$$

Clearly R is continuous and we have that $R((x, y), 0) = \mathrm{id}_{S^1 \times I}(x, y)$, $R((x, y), 1) = (x, 0) \in S^1$ and $R((x, 0), t) = (x, 0), \forall t \in [0, 1]$. We have so constructed a deformation retract of $(S^1 \times I)$ into S^1. Using Proposition 1 the proof concludes. $\square$

2.1 Moebius strip is homotopy equivalent to a circle

Definition 4. *The Möbius strip* M *can be defined as* $I \times I$ *by identifying* $(0, x)$ *with* $(1, 1 - x)$ *for* $x \in I$. *Equivalently* $M = I^2 \backslash \sim$

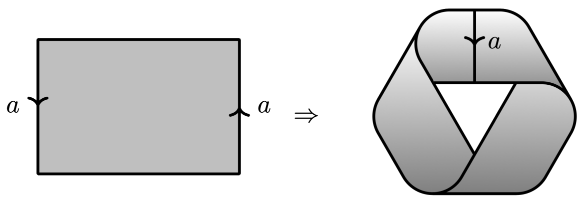

Figure 1: Moebius strip

Similarly to the cylinder, to prove that the Moebius strip is homotopy equivalent to the circle, it is sufficient to prove that the circle is a deformation retract of the Moebius strip (cfr Proposition 1).
The Möbius strip has a circle at its centre, namely the image of $I \times \{1/2\}$ (since $(0, 1/2) \sim (1, 1/2)$). The Möbius strip deformation retracts to this circle by taking the deformation retract on the rectangle

$$R : (I \times I) \times I \to I \times I, \quad ((x, y), t) \mapsto (x, (1 - t)(y - 1/2) + 1/2). \tag{2}$$

Since $1 - [(1-t)(y-1/2)+1/2] = (1-t)(1-y-1/2)+1/2$ if we pass to the quotient M we get a continuous map $\tilde{R} : M \times I \to M$ (i.e the homotopy carries over to a homotopy in the quotient) given by:

$$\tilde{R}([(x, y)], t) = [R((x, y), t)] \tag{3}$$

□

Exercise 3

Proposition 2. *X, Y are homotopy equivalent if and only if there exists a space* Z *such that X and Y are deformation retracts of Z.*

Consider the space X obtained from S^2 by attaching the two ends of an arc A to two distinct points on the sphere, say the north and south poles. Let B be an arc in S^2 joining the two points where A attaches. Then $X' = X/A$ and $Y = X/B$ are both deformations retracts of X (informally shown by the picture). It follows from Preposition 2 that X' and Y are homotopy equivalent.

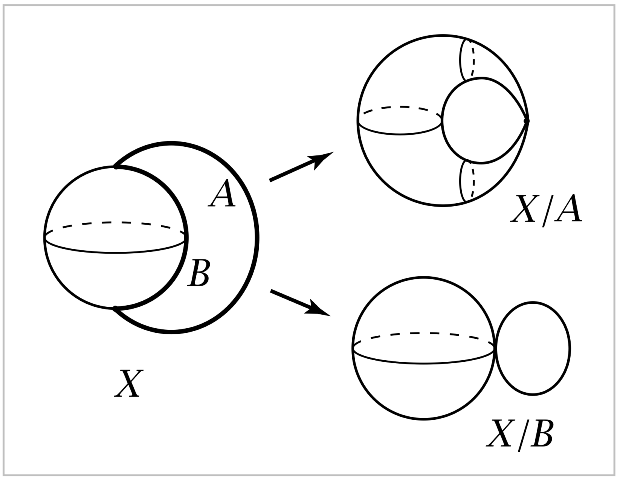

Figure 2: The space X/A is the quotient S^2/S^0, the sphere with two points identified, and X/B is $S^1 \vee S^2$. Image taken from [1].

The two spaces are not homeomorphic. Consider the 1-dimensional loop in X/b. Such a 1-dimensional part doesn't exist anywhere in X/A. (homeomorphism cannot map it anywhere)

Exercise 4

Definition 5. *A group* $(\mathrm{G}, +)$ *is a set* G *together with a binary operation "+" such that*

1. $\forall a, b \in G : a + b \in G$
2. $\forall a, b, c \in G : (a + b) + c = a + (b + c)$ *(Associativity)*
3. $\exists 0 \in \mathrm{G} : \mathrm{a} + 0 = 0 + a = a \forall a \in G$
4. $\forall a \in G \exists - a \in G : a + (-a) = 0$

$(G, +)$ *is abelian if we also have*

5. $\forall a, b \in G : a + b = b + a$ *(Commutativity)*

Definition 6. *A subset $A \subseteq G$ is a generator if every element of* G *can be written as a finite sum of elements of A and their inverses.*

Definition 7. *A cyclic group is a group* G *that contains an element* $\mathrm{g} \in \mathrm{G}$ *such that* $\{\mathrm{g}\}$ *is a generator of* G.

To prove that every cyclic group is abelian we can procede as follows: consider two elements $x, y \in G$. Since G is cyclic and g is a generator we can write every element of G as an alternating sum of g and $-g$. However note that every mixed sequence of g and $-g$ can be reduced to a sum of exclusively g or $-g$ terms since $g + (-g) = 0$. Therefore we write x, y as

$$x = n \cdot g, \quad y = m \cdot g \tag{4}$$

for some integers n, m. The product of these two elements is given by

$$\begin{aligned} x + y &= (n \cdot g) + (m \cdot g) &(5)\\ &= (n + m) \cdot g &(6)\\ &= (m + n) \cdot g &(7)\\ &= y + x\,. &(8) \end{aligned}$$

Where in the second to last line we used commutativity of integer numbers. □

Exercise 5

If by keeping applying X we would solve every cube, then there must be a generator X that solves every cube. By the previous exercise then the Rubic's cube would be commutative. We therefore we have a contradiction.

Exercise 6

If H is abelian, then, by taking $f(x) = 0$ as neutral element and $-f(x)$ as inverse element it is trivial to prove that $(\mathrm{Hom}(\mathrm{G}, \mathrm{H}), \oplus)$ is a group.
To show that $\mathrm{Hom}(\mathbb{Z}_2^2, \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2^2$, we need to find a bijective group homomorphism from $\mathrm{Hom}(\mathbb{Z}_2^2, \mathbb{Z}_2)$ to $\mathbb{Z}_2^2$.
Let f be a function in $\mathrm{Hom}(\mathbb{Z}_2^2, \mathbb{Z}_2)$, and let $a = f((1, 0))$ and $b = f((0, 1))$ be the values of f on the standard basis vectors of $\mathbb{Z}_2^2$. Since every element of $\mathbb{Z}_2^2$ is a linear combination of $(1, 0)$ and $(0, 1)$ with coefficients in $\mathbb{Z}_2$, the function f is completely determined by its values on $(1, 0)$ and $(0, 1)$. Therefore, we can define a function ϕ from $\mathrm{Hom}(\mathbb{Z}_2^2, \mathbb{Z}_2)$ to $\mathbb{Z}_2^2$ as follows:

$$\phi(f) = (a, b)$$

We claim that ϕ is a bijective group homomorphism.
First, we need to show that ϕ preserves the group operation. Let f, g be functions in $\mathrm{Hom}(\mathbb{Z}_2^2, \mathbb{Z}_2)$, and let $a = f((1, 0))$, $b = f((0, 1))$, $c = g((1, 0))$, and $d = g((0, 1))$ be their respective values on the standard basis vectors of $\mathbb{Z}_2^2$. Then,

$$\phi(f + g) = (f((1, 0)) + g((1, 0)), f((0, 1)) + g((0, 1))) = (a + c, b + d)$$

and

$$\phi(f) + \phi(g) = (a, b) + (c, d) = (a + c, b + d)$$

Therefore, ϕ preserves the group operation.
Next, we need to show that ϕ is injective. Suppose $\phi(f) = \phi(g)$ for two functions f, g in $\mathrm{Hom}(\mathbb{Z}_2^2, \mathbb{Z}_2)$. Then, $(a, b) = \phi(f) = \phi(g) = (c, d)$. This implies that $a = c$ and $b = d$, which means that f and g agree on the standard basis vectors of $\mathbb{Z}_2^2$. Since every element of $\mathbb{Z}_2^2$ is a linear combination of $(1, 0)$ and $(0, 1)$ with coefficients in $\mathbb{Z}_2$, f and g must be equal. Therefore, ϕ is injective.
Finally, we need to show that ϕ is surjective. Let (a, b) be an element of $\mathbb{Z}_2^2$. We can define a function f in Hom$(\mathbb{Z}_2^2, \mathbb{Z}_2)$ as follows:

$$f(x, y) = ax + by \pmod 2$$

for any (x, y) in $\mathbb{Z}_2^2$. It is easy to verify that f is a homomorphism from $\mathbb{Z}_2^2$ to $\mathbb{Z}_2$, and that $\phi(f) = (a, b)$. Therefore, ϕ is surjective.
Since ϕ is a bijective group homomorphism, Hom$(\mathbb{Z}_2^2, \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2^2$.

References

[1] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.