Exercise 1

(Proof based on [1]). Given two merge trees T_f and T_g, equipped with functions $f : T_f \to \mathbb{R}$ and $g : T_g \to \mathbb{R}$, we want to prove that $d_I(T_f, T_g) = d_{FD}(T_f, T_g)$. We break down the proof in two Lemmas:

Lemma 1. $d_I(T_f, T_g) \leq d_{FD}(T_f, T_g)$

Proof. In what follows, $\mathrm{j}^a : T_g \to T_g$ and $\mathrm{i}^a : T_f \to T_f$ are the two a-shift maps for T_f and T_g respectively.
Let $\delta = d_{FD}(T_f, T_g)$ denote the functional distortion-distance between two merge trees T_f and T_g, and let $\phi^* : T_f \to T_g$ and $\psi^* : T_g \to T_f$ be the optimal continuous maps achieving δ. We will now construct a pair of δ-compatible maps for T_f and T_g using ϕ^* and ψ^*. This then implies that $d_I(T_f, T_g) \leq d_{FD}(T_f, T_g)$ as claimed.
First, we construct the map $\alpha^\delta : T_f \to T_g$ as follows: for every point $x \in T_f$, let $y = \phi^*(x)$. Now set $\rho = f(x) + \delta - g \circ \phi^*(x)$: by the definition of d_{FD}, ρ is a non-negative real value in the range $[0, 2\delta]$. We now set $\alpha^\delta(x) = \mathrm{j}^\rho(y) = \mathrm{j}^\rho \circ \phi^*(x)$. Easy to see that by the choice of $\rho, g\left(\alpha^\delta(x)\right) = f(x) + \delta$. Since ϕ^* is continuous, the function $\rho : T_f \to \mathbb{R}$ is continuous, and the map α^δ is thus also a continuous map. Similarly, we construct $\beta^\delta : T_g \to T_f$. By their construction, the first two requirements on the exercise sheet are satisfied. We now show that the other two requirements also hold for α^δ and β^δ.
Indeed, consider a point $x \in T_f$, and let $y = \phi^*(x)$ and $y' = \alpha^\delta(x)$. By the definition of $\alpha^\delta, g(y') = f(x) + \delta \geq g(y)$ and there is a monotone path π connecting y to y' (in particular, y' is along the path from y to the root of the merge tree T_g). Now map π back to T_f via the map β^δ, which is necessarily a monotone path π' connecting $\tilde{x} := \beta^\delta(y)$ and $x' := \beta^\delta(y') = \beta^\delta \circ \alpha^\delta(x)$. In other words, x' is along the path from $\tilde{x}$ to the root of the merge tree T_f. By the definition of α^δ and $\beta^\delta, f(x') = f(x) + 2\delta$. We could also show that x' is along the path from x to the root of the merge tree T_f: this implies that $x' = \mathrm{i}^{2\delta}$, namely, $\beta^\delta \circ \alpha^\delta = \mathrm{i}^{2\delta}$. A symmetric argument shows that $\alpha^\delta \circ \beta^\delta = \mathrm{j}^{2\delta}$. Putting everything together, we have that α^δ and β^δ form a δ-compatible pair of maps between T_f and T_g. As such, $d_I(T_f, T_g) \leq \delta = d_{FD}(T_f, T_g)$. □

Lemma 2. $\mathrm{d}_I(T_f, T_g) \geq d_{FD}(T_f, T_g)$

Proof. Let $\varepsilon = d_I(T_f, T_g)$. Suppose that ε is obtained by a pair of ε-compatible maps $\alpha^\varepsilon : T_f \to T_g$ and $\beta^\varepsilon : T_g \to T_f$. We will show that the correspondances generated by these two maps α^ε and β^ε induce a distance distortion at most ε. This implies that $d_{FD}(T_f, T_g) \leq \varepsilon$. Specifically, let $C(\alpha^\varepsilon, \beta^\varepsilon)$ and $D(\alpha^\varepsilon, \beta^\varepsilon)$ be defined as in Definition 4.25 of the Lecture Notes. We now bound $D(\alpha^\varepsilon, \beta^\varepsilon)$.
Consider two pairs $(x_1, y_1), (x_2, y_2) \in C(\alpha^\varepsilon, \beta^\varepsilon)$. We first aim to bound $|d_f(x_1, x_2) - d_g(y_1, y_2)|$ from above. Consider $y_1 = \alpha^\varepsilon(x_1)$ and $y_2 = \alpha^\varepsilon(x_2)$. Let π_1 be the optimal path connecting x_1 to x_2 that achieves $d_f(x_1, x_2)$, which is necessarily the unique simple path connecting x_1 to x_2 in the tree T_f. Its image $\pi_1' = \alpha^\varepsilon(\pi_1)$ is a path connecting y_1 and y_2. By the first and second properties of ε-compatible maps written on the exercise sheet, we have that α^ε shift every point up by ε in the corresponding function value. Hence the range of π_1 is shifted up by ε to the range of π_1' while their heights are the same. Hence we have $d_g(y_1, y_2) \leq d_f(x_1, x_2)$. We have 2 different cases:

1. Consider the optimal path π_2 connecting y_1 to y_2 to achieve $d_g(y_1, y_2)$ in T_g. Let $x_1' = \beta^\varepsilon(y_1)$, $x_2' = \beta^\varepsilon(y_2)$. The image $\pi_2' = \beta^\varepsilon(\pi_2)$ of π_2 under the map β^ε is a path connecting x_1' to x_2' in T_f. Similarly, we have that height $(\pi_2) =$ height (π_2') and the range of π_2 is translated up by ε to π_2'. On the other hand, by the third and fourth properties of ε-compatible maps written on the exercise sheet , we have $x_1' = \mathrm{i}^{2\varepsilon}(x_1)$, and $x_2' = \mathrm{i}^{2\varepsilon}(x_2)$. By the definition of the shift map, there is a monotone path from x_1 to x_1' (along the path from x_1 to the root of the merge tree T_f) in T_f; and similarly for x_2 and x_2'. Concatenating these two montone paths with π_2' we obtain a path π_3 connecting x_1 to x_2. Since the two new paths are monotone, of height 2ε each, and both going up, we have that $|d_f(x_1, x_2) - d_g(y_1, y_2)| \leq 2\varepsilon$. If the two pairs are obtained via $x_1 = \beta^\varepsilon(y_1)$ and $x_2 = \beta^\varepsilon(y_2)$, a symmetric argument will show $| d_f(x_1, x_2) - d_g(y_1, y_2) | \leq 2\varepsilon$ as well.

2. We now consider the remaining case where $y_1 = \alpha^\varepsilon(x_1)$ but $x_2 = \beta^\varepsilon(y_2)$. Let π be the optimal path connecting x_1 to x_2 in T_f to achieve $d_f(x_1, x_2)$. Let $\pi' = \beta^\varepsilon(\pi)$ be its image in T_g : note π' connects y_1 to $y_2' = \beta^\varepsilon(x_2)$. By first and second properties on exercise sheet we have that π' is of the same height of π (and its range is that of π shifted upward by ε). By third and fourth properties on the exercise sheet we have that $y_2' = \mathrm{j}^{2\varepsilon}(y_2)$ and thus there is a monotone path π_4 of height 2ε connecting y_2 to y_2'.

Hence the concatenation $\pi_5 = \pi' \circ \pi_4$ is a path connecting y_1 to y_2. Thus height $(\pi_5) \leq$ height $(\pi') + 2\varepsilon =$ height $(\pi) + 2\varepsilon$, implying that $d_g(y_1, y_2) \leq d_f(x_1, x_2) + 2\varepsilon$.

A symmetric argument shows that $d_f(x_1, x_2) \leq d_g(y_1, y_2) + 2\varepsilon$. Hence $|d_f(x_1, x_2) - d_g(y_1, y_2)| \leq 2\varepsilon$.

It then follows that $D(\alpha^\varepsilon, \beta^\varepsilon) \leq \varepsilon$. On the other hand, by the first and second properties of ε-compatible maps written on the exercise sheet, $\|f - g \circ \alpha^\varepsilon\|_\infty = \varepsilon$ and $\|f \circ \beta_\epsilon - g\|_\infty = \varepsilon$. By definition 4.25 of the Lecture Notes, it then follows that $d_{FD}(T_f, T_g) \leq \varepsilon$.

Intuition: we basically just create our bounds by taking the maps guaranteed by one distance to build the maps for the other distance. □

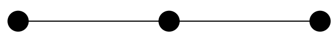
Exercise 2

(a) By definition, $\mathcal{M}(\mathcal{U}, f)$ is the nerve of the pullback cover $f^*(\mathcal{U})$. By assumption, $f^*(\mathcal{U})$ is a good cover of X. Thus, by the Nerve Theorem, $\mathcal{M}(\mathcal{U}, f)$ is homotopy equivalent to $\bigcup f^*(\mathcal{U}) = \bigcup V_\beta$, where each V_β is a path connected component of some $f^{-1}(U_\alpha)$. Since $f^*(\mathcal{U})$ is a (good) cover of X, we get

$$\bigcup f^*(\mathcal{U}) = X$$

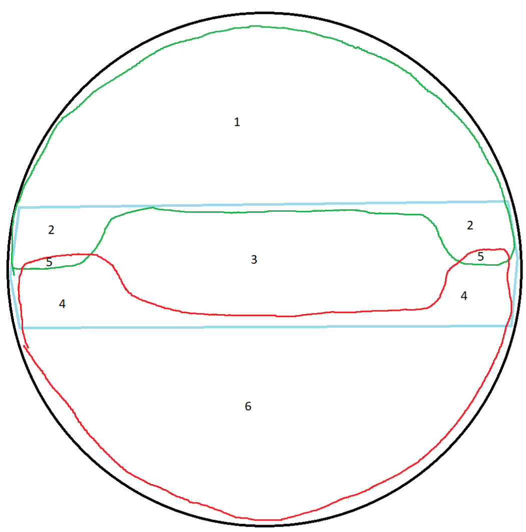
and this proves (a).

(b) Let $X = S^2$, the 2-sphere of radius 1 centered in the origin. We take $Z = [-1, 1]$, as filter function f we consider the projection onto the vertical axis $Z = [-1, 1]$ and as good cover $\mathcal{U}$ of Z we take $\mathcal{U} = \{[-1, -0.2), (-0.3, 0.3), (0.2, 1]\}$.
Let $\mathcal{V} = f^{-1}(\mathcal{U})$ be the cover of S^2 given by the lower cap $\mathcal{V}_1 = f^{-1}([-1, -0.2))$, the upper cap $\mathcal{V}_2 = f^{-1}((0.2, 1])$ and the belt $\mathcal{V}_3 = f^{-1}((-0.3, 0.3))$. Each $\mathcal{V}_i$, $i = 1, \cdots, 3$, is path-connected and so the Mapper $\mathcal{M}(\mathcal{U}, f)$ is the nerve of $\mathcal{V}$, that is the following simplicial complex:



This simplicial complex is not homotopy equivalent to S^2.

(c) Consider the split of the X = 2D-disk into the three sets as follows, $Z = \{1, 2, 3, 4, 5, 6\}$:



The numbers are the value of the filtration f on these areas. Now, let the open cover be $(4 - \epsilon, 6 + \epsilon)$ for the red part, $(1 - \epsilon, 2 + \epsilon) \cup (5 - \epsilon, 5 + \epsilon)$ for the green part, and $(2 - \epsilon, 5 + \epsilon)$ for the blue part. Obviously, the pullback cover is exactly these three areas as I drew them above. This is clearly not a good cover, since red and green intersect in two connected components, thus this intersection is not contractible. However, the nerve of the three areas is just a filled in triangle, which is homotopy equivalent to the disk.

References

[1] Ulrich Bauer, Xiaoyin Ge, and Yusu Wang. Measuring distance between reeb graphs, 2016.