Exercise 1

See exercise 3 of set 4.

Exercise 2

For the sake of understanding, we consider first the case in which $p = 1$. The filtration shown in figure 1 satisfies:

- $H_1(X_k) \neq 0$ for all k, since there is a one-dimensional hole in each member of the filtration.
- $H_1^{i,j} = 0$ for all $i < j$, since, for a given i, none of the holes in the object X_i survive in the following objects $X_j, j > i$.

For $p > 1$, we follow the same reasoning. Let $\partial\Delta^{p+1}$ be the simplex given by the triangulation of the p-sphere. We define the filtration by

$$\begin{aligned} X_1 &= \partial\Delta^{p+1} \\ X_2 &= \Delta^{p+1} \vee \partial\Delta^{p+1} \\ &\vdots \\ X_n &= \Delta^{p+1} \vee \ldots \vee \Delta^{p+1} \vee \partial\Delta^{p+1} = \vee_{i=1}^{n-1} \Delta^{p+1} \vee \partial\Delta^{p+1} \end{aligned}$$

where $\vee$ denotes the wedge operator.

- To have $H_p(X_i) \neq 0$, we include, in each member of the filtration, the space $\partial\Delta^{p+1}$. Indeed, $H_p(\partial\Delta^{p+1}) \neq 0$.
- As was the case in the 1-dimensional example, each "border simplex" $\partial\Delta^{p+1}$ gets mapped to Δ^{p+1}, and $H_p(\Delta^{p+1}) = 0$, hence $H_p^{i,j} = 0$ for all $i < j$.

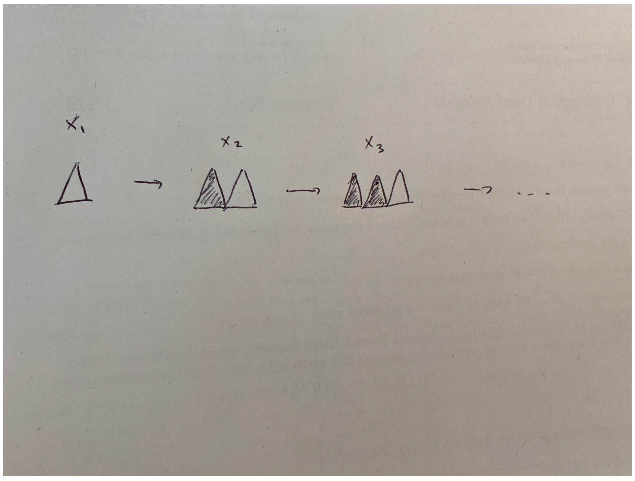

Figure 1: Filtration for $p = 1$

Exercise 3

Let the filtration defined by simplicial complexes $K_1, \ldots, K_N$ be defined by the following simplex order

$$\sigma_1, \cdots, \sigma_{k-1}, \sigma_{k+1}, \sigma_k, \sigma_{k+2}, \cdots, \sigma_N \tag{1}$$

i.e.

$$K_1 = \sigma_1, K_2 = K_1 \cup \sigma_2, \ldots, K_k = K_{k-1} \cup \sigma_{k+1}, K_{k+1} = K_k \cup \sigma_k, \ldots, K_N = K_{N-1} \cup \sigma_N.$$

This filtration is a simplex-wise filtration if $K_j \setminus K_{j-1}$ is empty or a single simplex for $j = 1, \cdots, N$. Now we have that

$$\begin{cases} K_{k+1} \setminus K_k = \sigma_k \\ K_k \setminus K_{k-1} = \sigma_{k+1} \\ K_j \setminus K_{j-1} = \sigma_j & \text{for } j \neq k, k+1 \end{cases}$$

Then, if σ_{k+1} is defined on the same set of simplices as K_{k-1} (i.e. the border $\partial\sigma_{k+1} \in K_{k-1}$) or if σ_{k+1} is just a vertex, the filtration is still simplex-wise. Notice that the second case is only a special case of the first, since for any vertex v, $\partial v = 0 \in K$ for any simplicial complex K.
To study the persistence diagrams, we will now distinguish between the following two cases:

1. One of the simplices $\{\sigma_k, \sigma_{k+1}\}$ is a destroyer, the other is a creator
2. both simplices are either destroyers or creators

Case 1:
Firstly, notice that these two simplices cannot be the destroyer/creator for the same cycle. For the sake of simplicity, assume σ_k is a creator, σ_{k+1} a destroyer. Otherwise, the simplex-wise filtration assumptions on the initial order and on the new order would not be satisfied. Then, the cycle created by σ_k would swap its birth time (previously k) with the previous death time ($k+1$) of the cycle destroyed by σ_{k+1}, and the new death time of this other cycle becomes the previous birth time (k) of the cycle created by σ_k.
In the general case, we notice that one cycle's death time is swapped with the other's birth time. The point representing the dying cycle on the persistence diagram shifts down/up by one, and the point representing the created cycle shifts right/left by one.
Case 2
In this latter case, swapping the order of σ_k and σ_{k+1} may also swap the birth/death times of two cycles. If they're creators, the birth times of two cycles will swap, though this is not guaranteed. In the example of a filtration where two vertices are then joined by an edge, swapping the order in which the vertices appear in the filtration does not change the persistence diagram, since the edge gets paired always with the youngest vertex. If they are both destructors, the death times of two cycles will swap, meaning that a point on the diagram will shift up/down by one and the other one will shift down/up by one.

Exercise 4

We define the following two filtrations:

$$\begin{aligned} X_1 &= S^2 \vee S^1 \vee S^1 \\ X_2 &= X_1 \vee S^2 \vee S^1 \vee S^1 \\ &\vdots \\ X_n &= X_{n-1} \vee S^2 \vee S^1 \vee S^1 \end{aligned}$$

and

$$\begin{aligned} Y_1 &= T^2 = S^1 \times S^1 \\ Y_2 &= T^2 \vee T^2 \\ &\vdots \\ Y_n &= \vee_{i=1}^n T^2 \end{aligned}$$

where T^2 denotes the 2-torus. For simplicity of understanding, refer to figure 2.
We consider X_1 and Y_1, show that their homologies are equivalent, and explain why they are not homotopy equivalent. This will suffice to show that these filtrations have the same persistence diagrams, despite no members being homotopy equivalent. Indeed, the birth times of any p-dimensional holes will be equal between

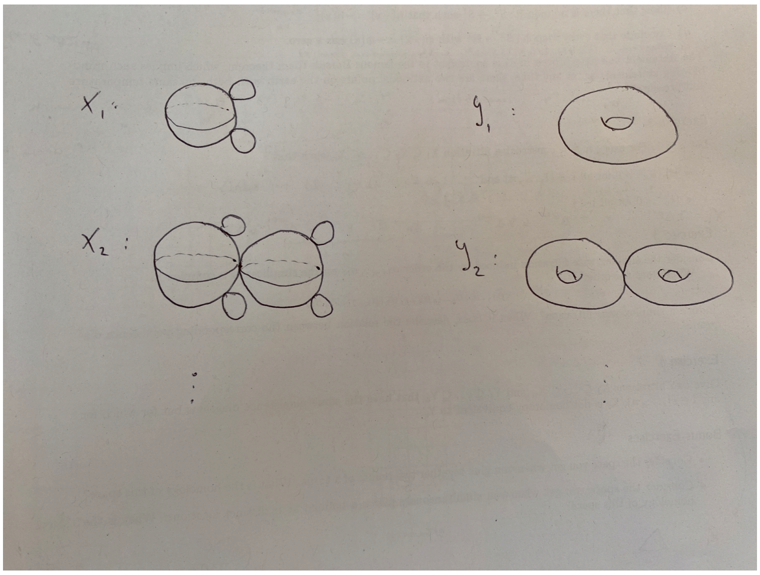

Figure 2: Filtrations with same homologies

the two filtrations, and, by how we defined them, it is clear to see that no holes will ever die, hence all points in the persistence diagram will have coordinates $(birth, \infty)$. Since the birth times are equal, this allows us to conclude.
Now we show that the homologies of X_1 and Y_1 are the same: the homology of a Torus Y_1 is given by

$$H_p(Y_1) = \begin{cases} \mathbb{Z}_2, & \text{if } p = 0 \text{ or } p = 2. \\ \mathbb{Z}_2^2, & \text{if } p = 1. \\ 0, & \text{otherwise.} \end{cases}$$

(c.f. example 2.3 AT). The homology of X_1, which is composed by $S^2 \vee S^1 \vee S^1 \cong \partial\Delta^3 \vee \partial\Delta^2 \vee \partial\Delta^2$, is given by

$$H_p(X_1) = \begin{cases} \mathbb{Z}_2, & \text{if } p = 0 \text{ or } p = 2. \\ \mathbb{Z}_2^2, & \text{if } p = 1. \\ 0, & \text{otherwise.} \end{cases}$$

since $\partial\Delta^3$ introduces a 2-dimensional hole, $\partial\Delta^2 \vee \partial\Delta^2$ introduce 2 1-dimensional holes, and by construction there is only one connected component. Hence $H_p(Y_1) = H_p(X_1)$ for all $p \geq 0$
It remains to show that X_1 and Y_1 are not homotopy equivalent, and, to prove this, we should compute the *homotopy groups* of X_1 and Y_1.