Animation for Computer Games
COMP 477/6311

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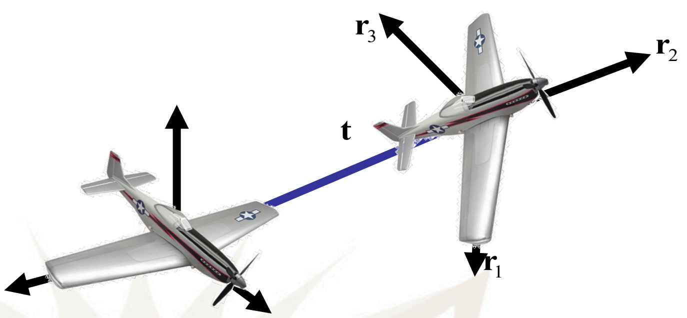
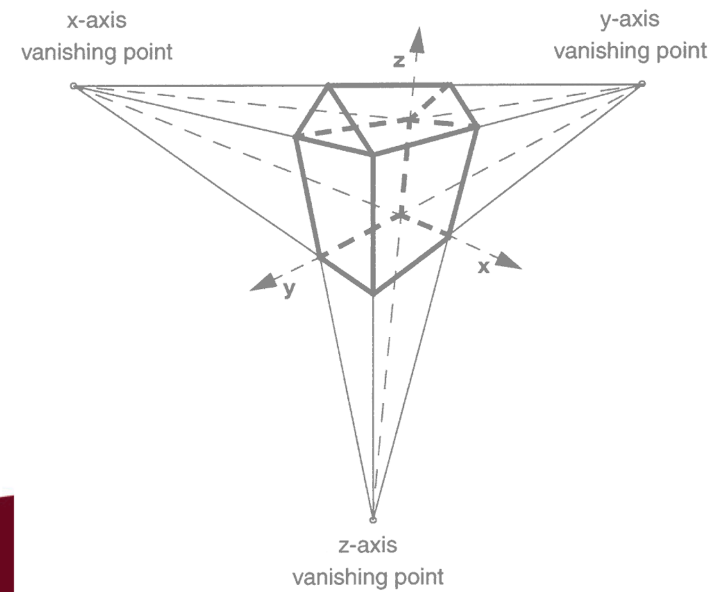
Transformations

Transformations

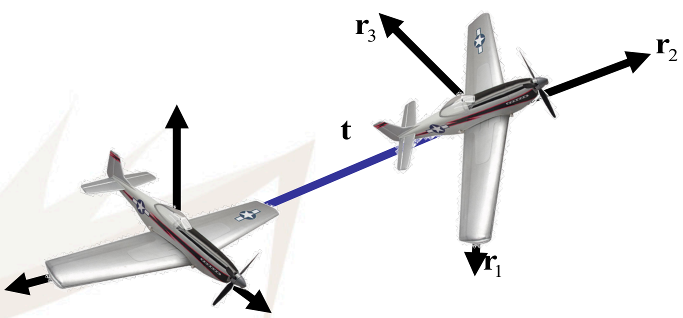
- At the core of Computer Graphics and Computer Animation
- Functions that map points from one locations to another
- We need mathematical tools to represent/compute both the **points** and the **transformations**

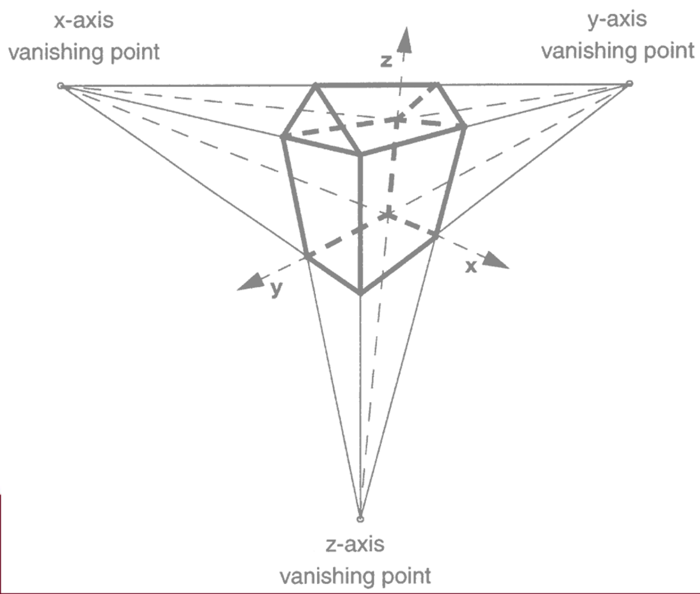




Transformations

- Common transformations
- Rigid transformations:
  - Translations
  - Rotations
- Non-rigid transformations:
  - Uniform scale
  - Non-uniform scale
  - Projective
  - Affine
  - Etc.





Transformations

- Euclidian spaces / Euclidian geometry
- Algebraically vector space in: R^3
  - Convention always column-wise!!!
- Transformations are functions:

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

- Represent them as 3 by 3 matrices

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Transformations

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- Is this a good representation?

Transformations

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- Is this a good representation?
- What does it mean to have a good representation?

Transformations

- What does it mean to have a good representation?
  1. **Complete**: to represent all transformations that are needed (i.e. represent translations, rotations, scale, etc.)
  2. **Efficient** application of the transformation (i.e. we need to transform potentially hundreds of thousands of points, need to do this fast!!!
  3. **Composition**: the ability to combine multiple transformation into one (i.e. remember the skeleton: the transformation of a bone is the composition of all transformations of its parent – we'd like to be able to compose all those transformation into one
  4. **Interpolation**: the ability to compute in-between transformations (i.e. when we have to compute the in-between frames in keyframe animation

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Transformations

- Is the 3 by 3 matrix representation a good representation?
  1. **Complete**: to represent all transformations that are needed (i.e. represent translations, rotations, scale, etc.)

Is it complete?

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Transformations

- Is the 3 by 3 matrix representation a good representation?
  1. **Complete**: to represent all transformations that are needed (i.e. represent translations, rotations, scale, etc.)

Is it complete?

NO – translations cannot be expressed as a 3 by 3 matrix; can you prove it?

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Transformations

- What does it mean to have a good representation?

  2. Efficient application of the transformation (i.e. we need to transform potentially hundreds of thousands of points, need to do this fast!!!

  Is it efficient?

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Transformations

- What does it mean to have a good representation?

  2. Efficient application of the transformation (i.e. we need to transform potentially hundreds of thousands of points, need to do this fast!!!

  Is it efficient?

  Oh yeah!!! (as efficient as it comes)

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Transformations

- What does it mean to have a good representation?

  3. Composition: the ability to combine multiple transformation into one (i.e. remember the skeleton: the transformation of a bone is the composition of all transformations of its parent – we'd like to be able to compose all those transformation into one

Can we compose transformations using this representation?

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Transformations

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  3. Composition: the ability to combine multiple transformation into one (i.e. remember the skeleton: the transformation of a bone is the composition of all transformations of its parent – we'd like to be able to compose all those transformation into one

Can we compose transformations using this representation?

Oh yeah!!! (using matrix multiplication)

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Transformations

- What does it mean to have a good representation?

  4. Interpolation: the ability to compute in-between transformations (i.e. when we have to compute the in-between frames in keyframe animation

  Can we interpolate transformations in this representation?

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Transformations

- What does it mean to have a good representation?

  4. Interpolation: the ability to compute in-between transformations (i.e. when we have to compute the in-between frames in keyframe animation

  Can we interpolate transformations in this representation?

  Think more….

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Transformations

- What does it mean to have a good representation?

  4. Interpolation: the ability to compute in-between transformations (i.e. when we have to compute the in-between frames in keyframe animation

  Can we interpolate transformations in this representation?

  Think more….

  Not really – we will leave it at that for now

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Transformations

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- Is this a good representation?

Transformations

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- Is this a good representation?
- Not really -
- Mixed bag:
  - Efficient, allows efficient composition
  - **Not complete!!!**
  - **Does not interpolate well!!!**

Homogeneous Coordinates

- Elements: $\begin{pmatrix} x & y & z & w \end{pmatrix}'$

$$f\left(\begin{matrix} x \\ y \\ z \\ w \end{matrix}\right) = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

- How do I get a 3D point/vector:
  - Points $\begin{pmatrix} x/w & y/w & z/w & 1 \end{pmatrix}'$
  - Vectors $\begin{pmatrix} x & y & z & 0 \end{pmatrix}'$

Homogeneous Coordinates

$$f\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$\begin{pmatrix} x & y & z & 1 \end{pmatrix}'$$

- Basically adds translation component when w = 1

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

Transformations

$$f\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

- Is this a good representation?
- Let's see

Transformations

- Is the 3 by 3 matrix representation a good representation?
  1. **Complete**: to represent all transformations that are needed (i.e. represent translations, rotations, scale, etc.)

Is it complete?

$$f\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Transformations

- Is the 3 by 3 matrix representation a good representation?
  1. **Complete**: to represent all transformations that are needed (i.e. represent translations, rotations, scale, etc.)

  Is it complete? - Yes –
- Basic transforms
  - Rotations (yes)
  - Translations (yes)
  - Scale (yes)
  - Projective (yes)
  - Etc.

$$f\left(\begin{matrix} x \\ y \\ z \\ w \end{matrix}\right) = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Transformations

- What does it mean to have a good representation?

  2. Efficient application of the transformation (i.e. we need to transform potentially hundreds of thousands of points, need to do this fast!!!

  Is it efficient?

$$f\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Transformations

- What does it mean to have a good representation?

  2. Efficient application of the transformation (i.e. we need to transform potentially hundreds of thousands of points, need to do this fast!!!

  Is it efficient?

  Oh yeah!!! (tiny bit more work than before)

$$f\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

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Can we compose transformations using this representation?

Oh yeah!!! (using matrix multiplication)

$$f\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

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  Can we interpolate transformations in this representation?

$$f\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Transformations

- What does it mean to have a good representation?

  4. Interpolation: the ability to compute in-between transformations (i.e. when we have to compute the in-between frames in keyframe animation

  Can we interpolate transformations in this representation?

  Think more….

$$f\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Transformations

- What does it mean to have a good representation?

  4. Interpolation: the ability to compute in-between transformations (i.e. when we have to compute the in-between frames in keyframe animation

  Can we interpolate transformations in this representation?

  Think more….

  Still not really… ☹ … sorry!

$$f\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

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Transformations

$$f\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

- Is this a good representation?

Transformations

$$f\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

- Is this a good representation?
- Yes – can't have everything

Translation and Scaling

- Representation by 4x4 matrix

$$\boldsymbol{T}(t_x, t_y, t_z) = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Scaling

$$\boldsymbol{S}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation

- Rotation matrices for x-,y-, and z-axis

$$\boldsymbol{R}_x(\theta)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{R}_y(\theta)=\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{R}_z(\theta)=\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about Arbitrary Axis

- Normalized axis $\mathbf{u}$, angle θ

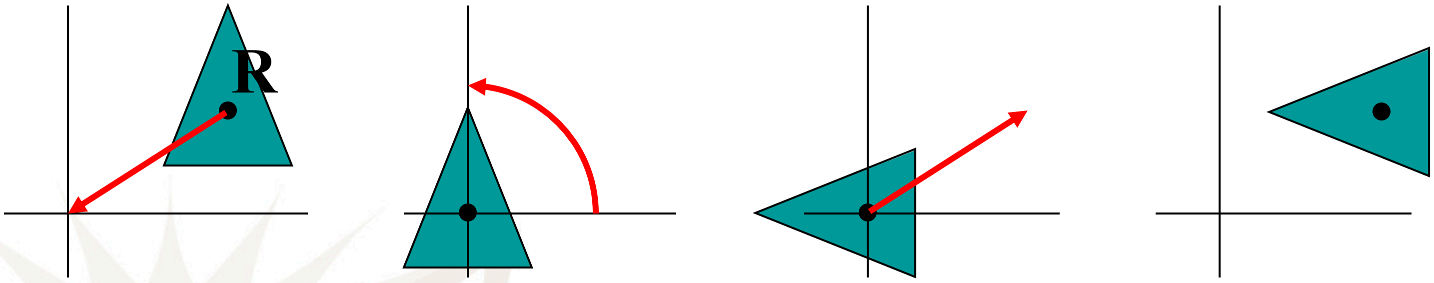
$$\mathbf{R}(\mathbf{u},\theta) =$$

$$\begin{bmatrix} u_x^2 + \cos\theta(1-u_x^2) & u_x u_y(1-\cos\theta) - u_z \sin\theta & u_x u_z(1-\cos\theta) + u_y \sin\theta & 0 \\ u_x u_y(1-\cos\theta) + u_z \sin\theta & u_y^2 + \cos\theta(1-u_y^2) & u_y u_z(1-\cos\theta) - u_x \sin\theta & 0 \\ u_x u_z(1-\cos\theta) - u_y \sin\theta & u_y u_z(1-\cos\theta) + u_x \sin\theta & u_z^2 + \cos\theta(1-u_z^2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation around a point 1

Rotate over angle θ around point **R**:

1) Translate such that **R** coincides with origin;
2) Rotate over angle θ around origin;
3) Translate back.



1) 2) 3)

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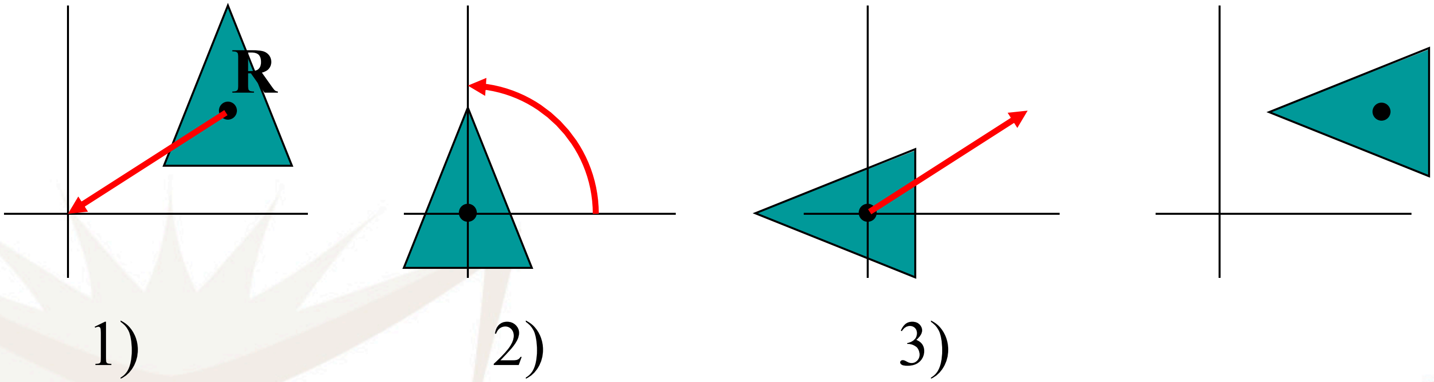
Rotation around a point 2

Rotate over angle θ around point $\mathbf{R}$:

1) $\mathbf{P}' = \mathbf{T}(-R_x, -R_y)\mathbf{P}$

2) $\mathbf{P}'' = \mathbf{R}(\theta)\mathbf{P}'$

3) $\mathbf{P}''' = \mathbf{T}(R_x, R_y)\mathbf{P}''$



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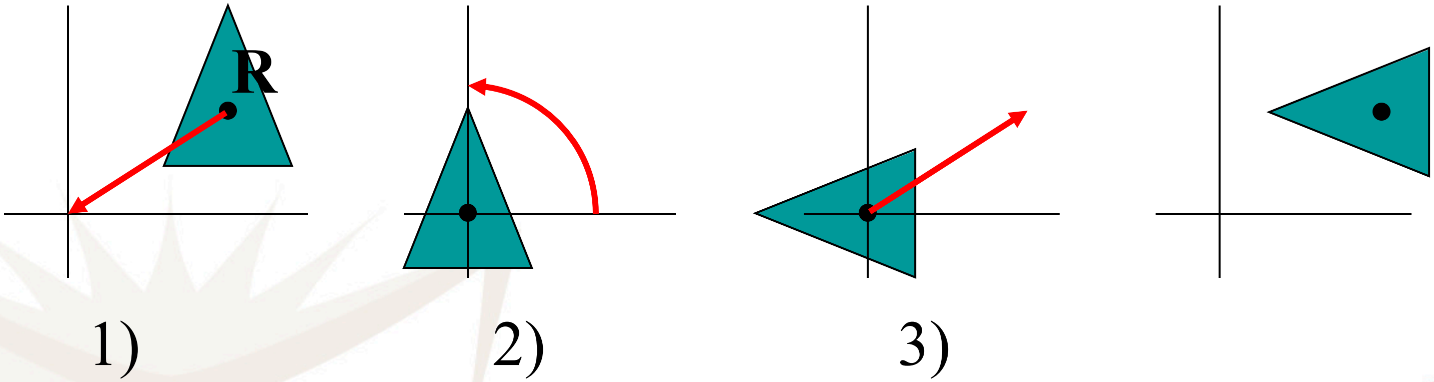
Rotation around point 3

$$1)\ \mathbf{P}' = \mathbf{T}(-R_x, -R_y)\mathbf{P}$$

$$2)\ \mathbf{P}'' = \mathbf{R}(\theta)\mathbf{P}' = \mathbf{R}(\theta)\mathbf{T}(-R_x, -R_y)\mathbf{P}$$

$$\begin{aligned} 3)\ \mathbf{P}''' &= \mathbf{T}(R_x, R_y)\mathbf{P}'' \\ &= \mathbf{T}(R_x, R_y)\mathbf{R}(\theta)\mathbf{P}' \\ &= \mathbf{T}(R_x, R_y)\mathbf{R}(\theta)\mathbf{T}(-R_x, -R_y)\mathbf{P} \end{aligned}$$



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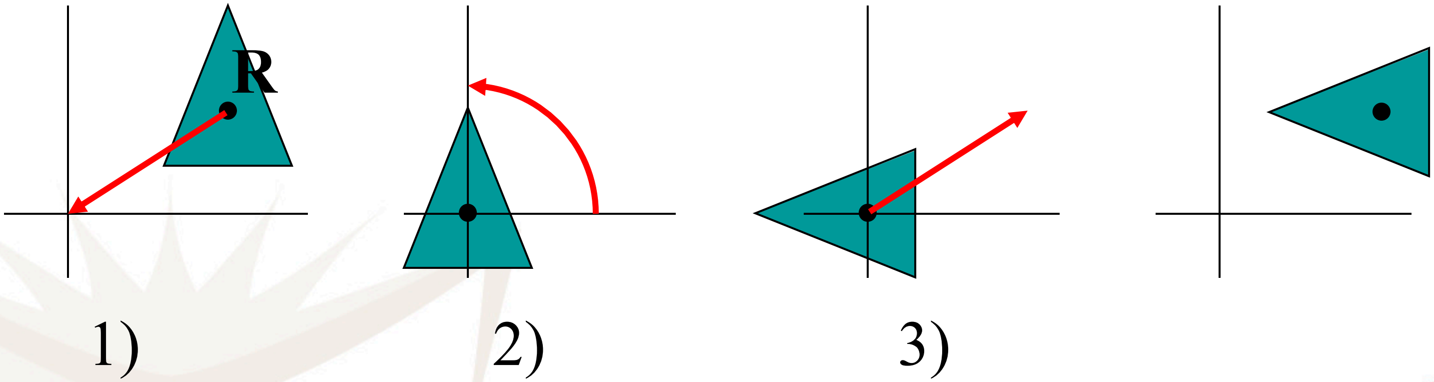
Rotation around point 4

$$1\text{-}3)\ \mathbf{P}''' = \mathbf{T}(R_x, R_y)\mathbf{R}(\theta)\mathbf{T}(-R_x, -R_y)\mathbf{P}$$

or

$$\mathbf{P}''' = \begin{pmatrix} \cos\theta & -\sin\theta & R_x(1-\cos\theta) + R_y \sin\theta \\ \sin\theta & \cos\theta & R_y(1-\cos\theta) - R_x \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \mathbf{P}$$



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Complex 3D Rotation

Compute the rotation around the axis $(1, 1, 0)$ that passes through the point (-1, 0, 0) by 90 degrees?

First translation where we put the pivot point to the origin

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Complex 3D Rotation

Compute the rotation around the axis (1, 1, 0) that passes through the point (-1, 0, 0) by 90 degrees?

WARNING: axis not unit length – normalize: $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$

$$\cos(90) = 0, \sin(90) = 1$$

$$R = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}(\mathbf{u}, \theta) = \begin{bmatrix} u_x^2 + \cos\theta(1 - u_x^2) & u_x u_y (1 - \cos\theta) - u_z \sin\theta & u_x u_z (1 - \cos\theta) + u_y \sin\theta & 0 \\ u_x u_y (1 - \cos\theta) + u_z \sin\theta & u_y^2 + \cos\theta(1 - u_y^2) & u_y u_z (1 - \cos\theta) - u_x \sin\theta & 0 \\ u_x u_z (1 - \cos\theta) - u_y \sin\theta & u_y u_z (1 - \cos\theta) + u_x \sin\theta & u_z^2 + \cos\theta(1 - u_z^2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Complex 3D Rotation

Compute the rotation around the axis (1, 1, 0) that passes through the point (-1, 0, 0) by 90 degrees?

Second translation where translate the scene back

$$T_2 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Complex 3D Rotation

Compute the rotation around the axis (1, 1, 0) that passes through the point (-1, 0, 0) by 90 degrees?

What is the final transformation?

Complex 3D Rotation

Compute the rotation around the axis $(1, 1, 0)$ that passes through the point (-1, 0, 0) by 90 degrees?

What is the final transformation?

$$\mathrm{T} = \mathrm{T}_2\mathrm{R}\mathrm{T}_1 =$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Complex 3D Rotation

Compute the rotation around the axis $(1, 1, 0)$ that passes through the point (-1, 0, 0) by 90 degrees?

What is the final transformation?

$$T = T_2 R T_1 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Complex 3D Rotation

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Complex 3D Rotation

Verify the computation using the point $P = (0, 0, 0)$

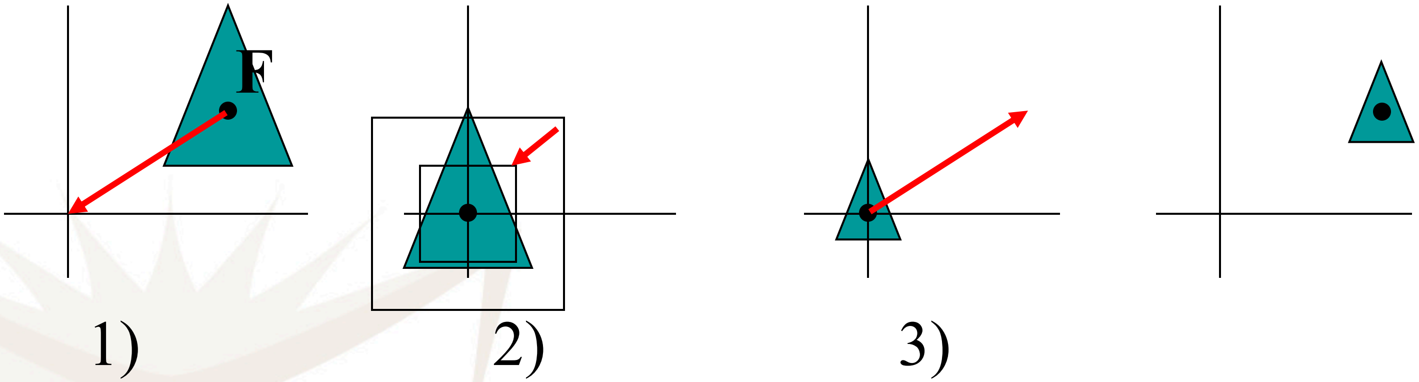
$$P' = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ 1 \end{pmatrix}$$

Is it correct? – will discuss this example in more detail in the labs

Scaling w.r.t. point 1

Scale with factors s_x and s_x w.r.t. point $\mathbf{F}$:

1) Translate such that **F** coincides with origin;
2) Schale w.r.t. origin;
3) Translate back again.



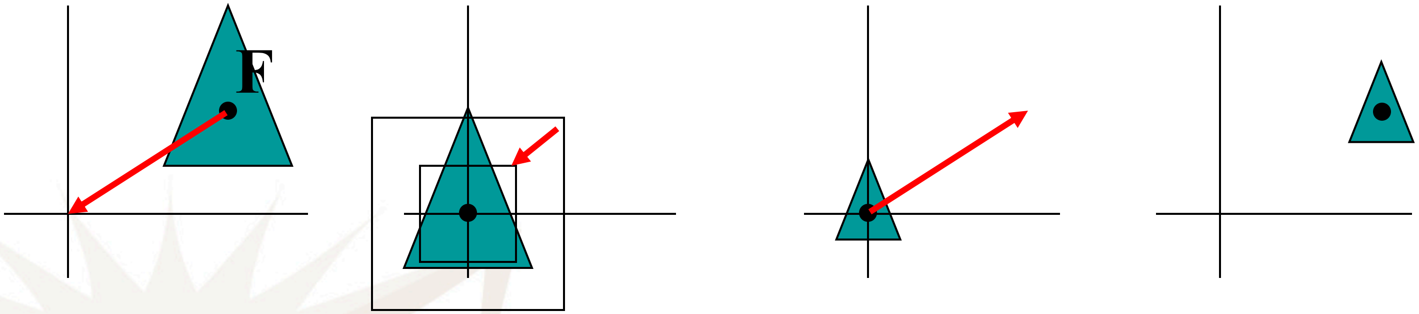
Scaling w.r.t.point 2

Schale w.r.t. point **F** :

1) $\mathbf{P}' = \mathbf{T}(-F_x, -F_y)\mathbf{P}$

2) $\mathbf{P}'' = \mathbf{S}(s_x, s_y)\mathbf{P}'$

3) $\mathbf{P}''' = \mathbf{T}(F_x, F_y)\mathbf{P}''$



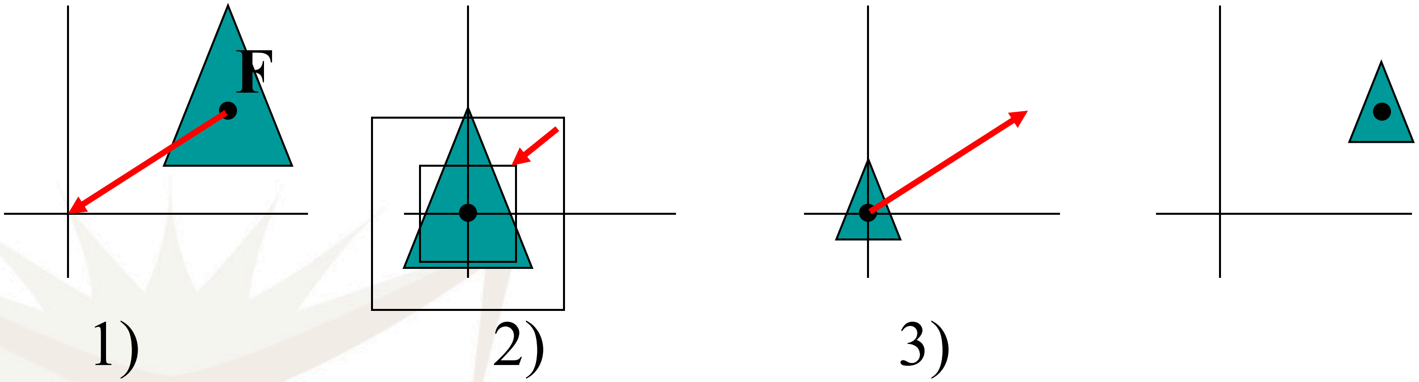
1) 2) 3)

Scaling w.r.t.point 3

$$1\text{-}3)\ \mathbf{P}''' = \mathbf{T}(F_x, F_y)\mathbf{S}(s_x, s_y)\mathbf{T}(-F_x, -F_y)\mathbf{P}$$

or

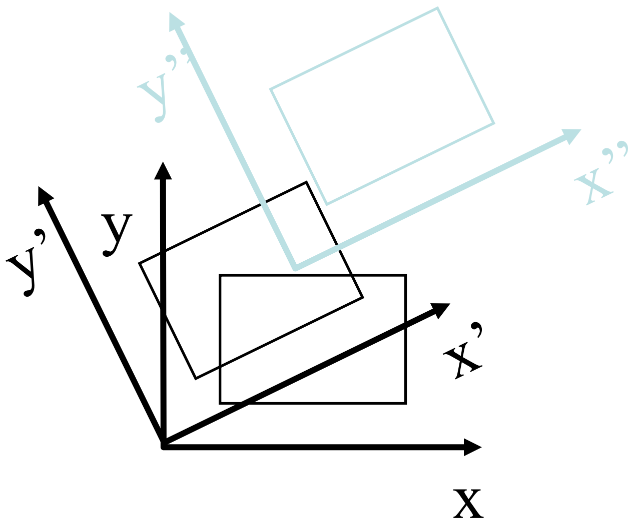
$$\mathbf{P}''' = \begin{pmatrix} s_x & 0 & F_x(1-s_x) \\ 0 & s_y & F_y(1-s_y) \\ 0 & 0 & 1 \end{pmatrix}\mathbf{P}$$

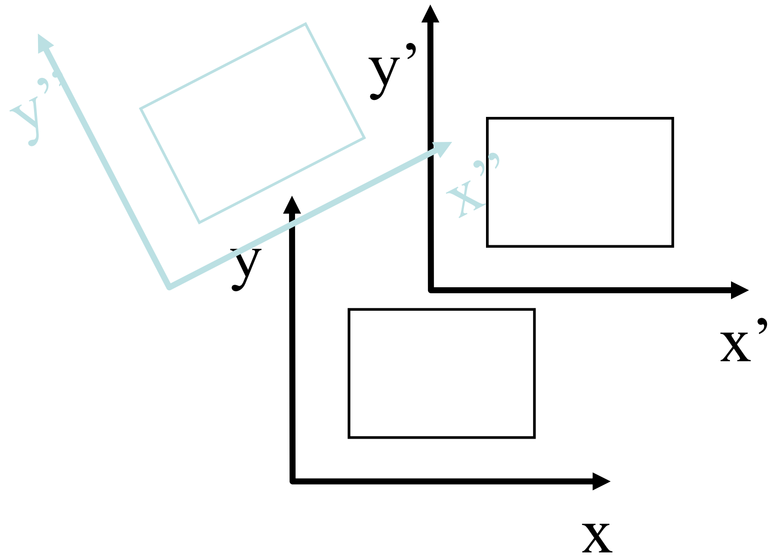


Order of transformations 1

Matrix multiplication does **not** commute.

The order of transformations makes a difference!





Order of transformations 1

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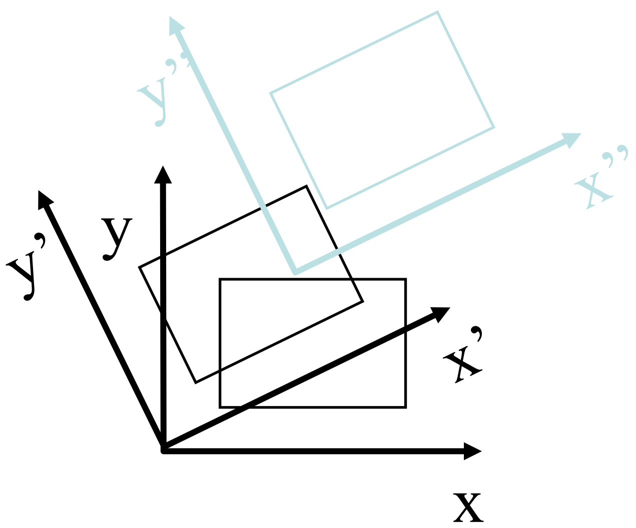
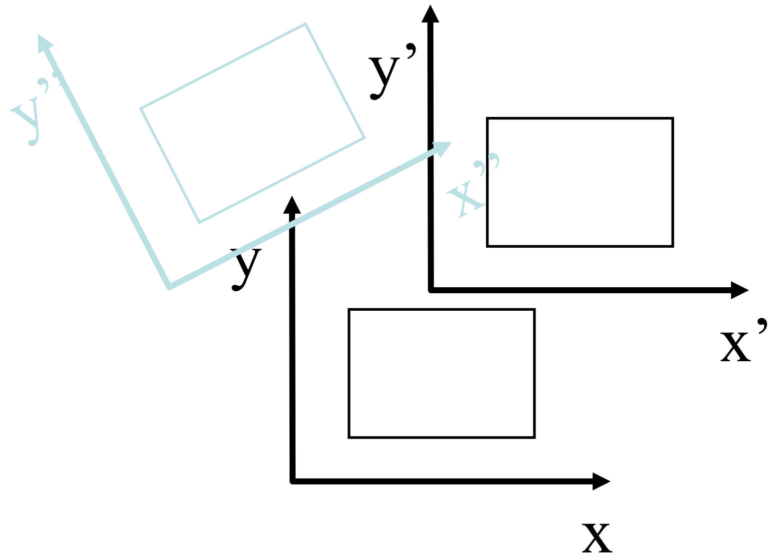
Rotation, translation…



$$\mathbf{x}'' = T(2,3)R(30)\mathbf{x}$$

Translation, rotation…



$$\mathbf{x}'' = R(30)T(2,3)\mathbf{x}$$