Relational Calculus

Dr. Constantinos Constantinides, P.Eng.

Department of Computer Science and
Software Engineering
Concordia University

Example 1: A declaration of a Power Set type

- Consider the set Name = {John, Myriam, Mike, Suzan}.
- Consider the power set of Name:

  P Name =

  {

  ∅,

  {John}, {Myriam}, {Mike}, {Suzan},

  {John, Myriam}, {John, Mike}, {John, Suzan},

  {Myriam, Mike}, {Myriam, Suzan}, {Mike, Suzan},

  {John, Myriam, Mike}, {John, Myriam, Suzan},

  {John, Mike, Suzan}, {Myriam, Mike, Suzan},

  {John, Myriam, Mike, Suzan}

  }
- The power set of Name is a set that contains, as its elements, all subsets of Name.

Type declarations

- How do we interpret a type declaration?

- The expression v : Type is interpreted as "The variable v can assume any value supported by Type."

- Examples:

- In $x : \mathbb{N}$, variable x can assume values 1, 2, 3, …

- In z : Boolean, variable z can assume values true, or false.

Declaring a variable to hold a set

$$
\begin{aligned}
\mathbb{P}\ Name = \{\ & \varnothing, \\
& \{John\}, \{Myriam\}, \{Mike\}, \{Suzan\}, \\
& \{John, Myriam\}, \{John, Mike\}, \{John, Suzan\}, \\
& \{Myriam, Mike\}, \{Myriam, Suzan\}, \{Mike, Suzan\}, \\
& \{John, Myriam, Mike\}, \{John, Myriam, Suzan\}, \\
& \{John, Mike, Suzan\}, \{Myriam, Mike, Suzan\}, \\
& \{John, Myriam, Mike, Suzan\}\ \}
\end{aligned}
$$

- The declaration known : P Name is interpreted as "The variable known can assume any value supported by P Name."

Since the values of P Name are sets, this implies that known is declared to be a set.

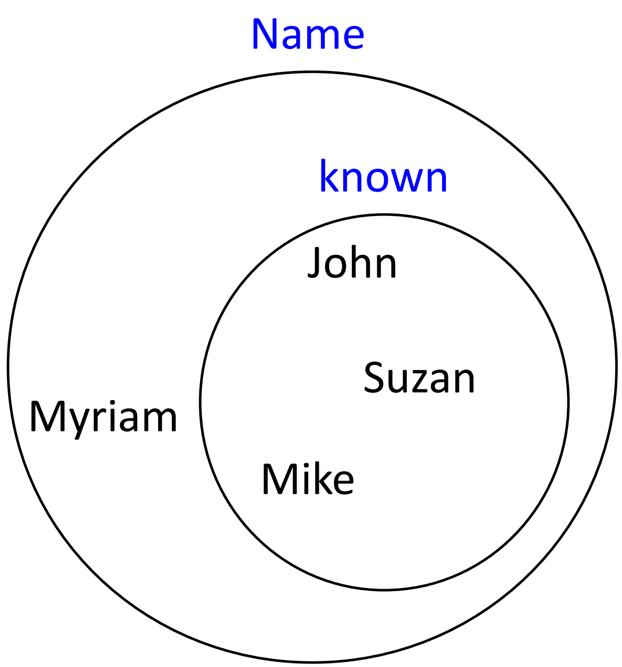
Reasoning about legitimate values for the variable

P Name = { $\varnothing$,
{John}, {Myriam}, {Mike}, {Suzan},
{John, Myriam}, {John, Mike}, {John, Suzan},
{Myriam, Mike}, {Myriam, Suzan}, {Mike, Suzan},
{John, Myriam, Mike}, {John, Myriam, Suzan},
{John, Mike, Suzan}, {Myriam, Mike, Suzan},
{John, Myriam, Mike, Suzan} }

- The declaration known : P Name is interpreted as “The variable known can assume any value supported by P Name.”

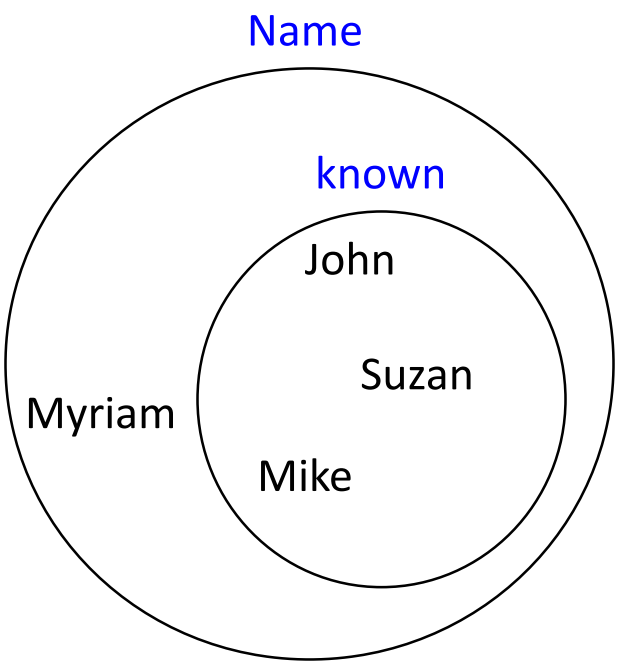
- Is {John, Mike, Suzan} a legitimate value for known? Yes.

Visualizing set relations



Is {John, Mike, Suzan} a legitimate value for known? Yes.

Variable declaration vs. set relation



- Note that known : P Name, and known $\subseteq$ Name are both correct, but they mean a different thing.

- The expression known : P Name is a variable declaration.

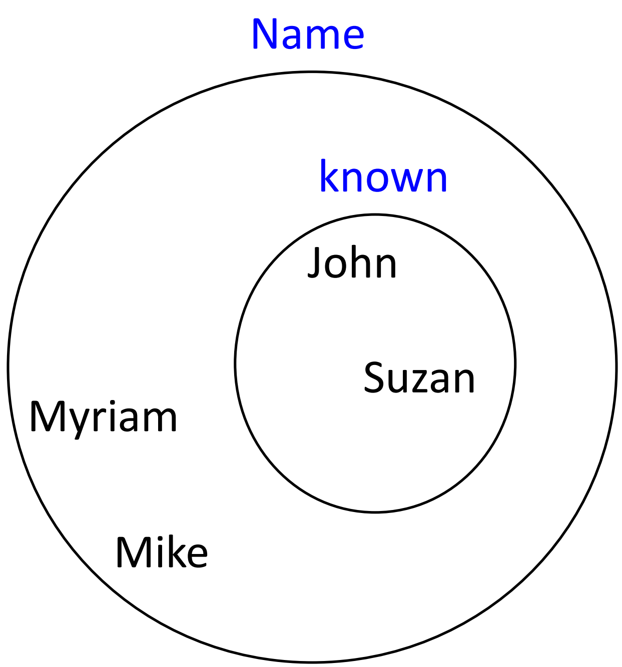
- The expression known $\subseteq$ Name describes a set relation.

Reasoning about legitimate values for the variable /cont.

P Name = { $\varnothing$,
{John}, {Myriam}, {Mike}, {Suzan},
{John, Myriam}, {John, Mike}, {John, Suzan},
{Myriam, Mike}, {Myriam, Suzan}, {Mike, Suzan},
{John, Myriam, Mike}, {John, Myriam, Suzan},
{John, Mike, Suzan}, {Myriam, Mike, Suzan},
{John, Myriam, Mike, Suzan} }

- If we now removed Mike from known, we will have {John, Suzan}.
- Is {John, Suzan} a legitimate value for known? Yes.

Visualizing set relations /cont.



Is {John, Suzan} a legitimate value for known? Yes.

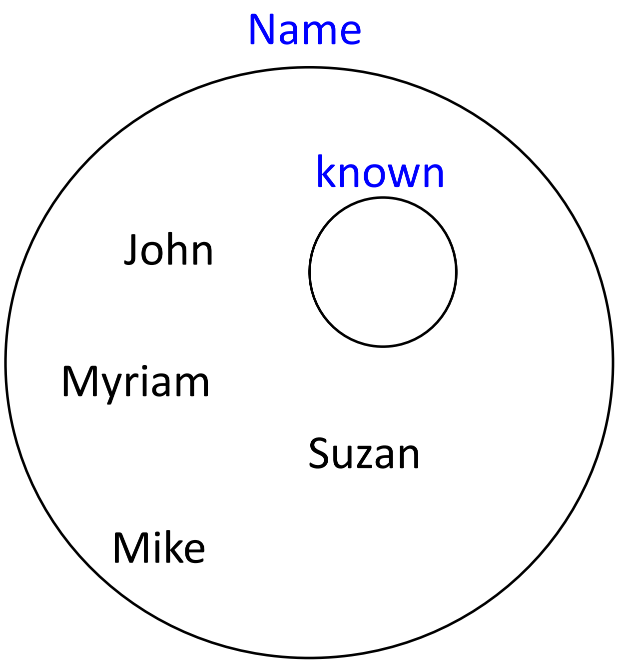
Reasoning about legitimate values for the variable /cont.

P Name = { $\varnothing$,
{John}, {Myriam}, {Mike}, {Suzan},
{John, Myriam}, {John, Mike}, {John, Suzan},
{Myriam, Mike}, {Myriam, Suzan}, {Mike, Suzan},
{John, Myriam, Mike}, {John, Myriam, Suzan},
{John, Mike, Suzan}, {Myriam, Mike, Suzan},
{John, Myriam, Mike, Suzan} }

- If we now removed all elements from known, we will have $\varnothing$.

- Is $\varnothing$ a legitimate value for known? Yes.

Visualizing set relations /cont.



Is $\varnothing$ a legitimate value for known? Yes.

Reasoning about legitimate values for the variable /cont.

P Name = { $\varnothing$,

{John}, {Myriam}, {Mike}, {Suzan},

{John, Myriam}, {John, Mike}, {John, Suzan},

{Myriam, Mike}, {Myriam, Suzan}, {Mike, Suzan},

{John, Myriam, Mike}, {John, Myriam, Suzan},

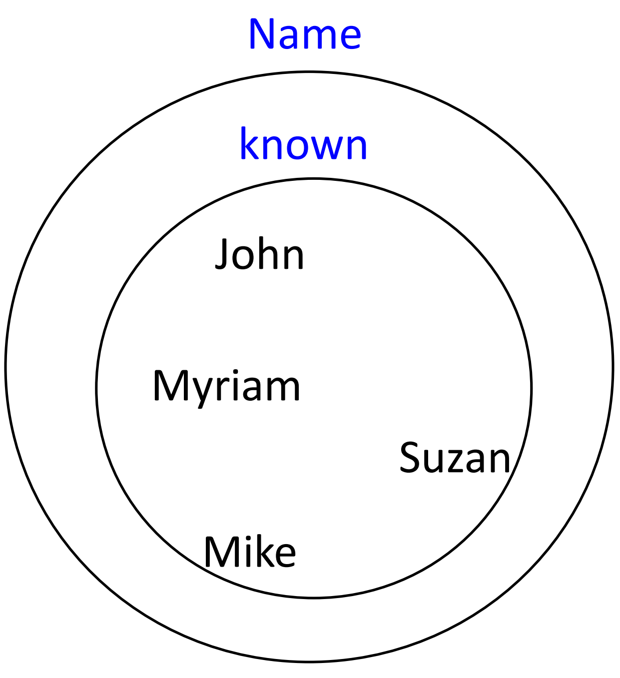
{John, Mike, Suzan}, {Myriam, Mike, Suzan},

{John, Myriam, Mike, Suzan} }

- Let us now add all names into known.

- Is {John, Myriam, Mike, Suzan} a legitimate value for known? Yes.

Visualizing set relations /cont.



Is {John, Myriam, Mike, Suzan} a legitimate value for known? Yes.

Example 2: A database table

- Let binary relation `car : Model ↔ Make`
- The relation `car` can be used to model a database table.

car =

$$
\{
\begin{array}{lcl}
\text{camry} & \mapsto & \text{toyota,} \\
\text{avalon} & \mapsto & \text{toyota,} \\
\text{highlander} & \mapsto & \text{toyota,} \\
\text{wrangler} & \mapsto & \text{jeep,} \\
\text{renegade} & \mapsto & \text{jeep,} \\
\text{compass} & \mapsto & \text{jeep,} \\
\text{altima} & \mapsto & \text{nissan,} \\
\text{pathfinder} & \mapsto & \text{nissan,} \\
\text{murano} & \mapsto & \text{nissan}
\end{array}
\}
$$

Domain and range

- Domain and range are sets of first and second elements respectively.
- `dom car` = {camry, avalon, highlander, wrangler, renegade, compass, altima, pathfinder, murano}
- `ran car` = {toyota, jeep, nissan}

car = {

camry $\mapsto$ toyota,
avalon $\mapsto$ toyota,
highlander $\mapsto$ toyota,
wrangler $\mapsto$ jeep,
renegade $\mapsto$ jeep,
compass $\mapsto$ jeep,
altima $\mapsto$ nissan,
pathfinder $\mapsto$ nissan,
murano $\mapsto$ nissan

}

Domain of car $\leftarrow$

$\rightarrow$ Range of car

Model queries (1/2)

- Restriction operators can be used to <u>model database queries</u>.

- For example: “Select the pairs based on ‘wrangler’ and ‘pathfinder’.”

$$\{wrangler, pathfinder\} \lhd car =$$
$$\{$$
$$wrangler \mapsto jeep,$$
$$pathfinder \mapsto nissan$$
$$\}$$

Domain restriction selects pairs based on the first element.

Model queries (2/2)

- Restriction operators can be used to <u>model database queries</u>.
- For example: "Select the pair(s) based on 'jeep'."

$$
\begin{aligned}
\text{car} \rhd \{\text{jeep}\} = \\
&\{ \\
&\text{wrangler} \mapsto \text{jeep}, \\
&\text{renegade} \mapsto \text{jeep}, \\
&\text{compass} \mapsto \text{jeep} \\
&\}
\end{aligned}
$$

Range restriction selects pairs based on the second element.

Model updates (1/3)

- Relational overriding can be used to model database updates.
- We have 2 types of updates: Insertions and modifications.
- For example, this update is an insertion:

car $\oplus$ {cherokee $\mapsto$ jeep} =
{
camry $\mapsto$ toyota,
avalon $\mapsto$ toyota,
highlander $\mapsto$ toyota,
wrangler $\mapsto$ jeep,
renegade $\mapsto$ jeep,
compass $\mapsto$ jeep,
altima $\mapsto$ nissan,
pathfinder $\mapsto$ nissan,
murano $\mapsto$ nissan,
cherokee $\mapsto$ jeep
}

All pairs of set on LHS are added to the set on RHS.

Model updates (2/3)

- Variable car’ holds the state of the set upon successful evaluation of the RHS expression.

$$
\boxed{\text{car'}} = \boxed{\text{car} \oplus \{\text{cherokee} \mapsto \text{jeep}\}} =
$$

$$
\begin{array}{lcl}
\{ & & \\
\text{camry} & \mapsto & \text{toyota}, \\
\text{avalon} & \mapsto & \text{toyota}, \\
\text{highlander} & \mapsto & \text{toyota}, \\
\text{wrangler} & \mapsto & \text{jeep}, \\
\text{renegade} & \mapsto & \text{jeep}, \\
\text{compass} & \mapsto & \text{jeep}, \\
\text{altima} & \mapsto & \text{nissan}, \\
\text{pathfinder} & \mapsto & \text{nissan}, \\
\text{murano} & \mapsto & \text{nissan}, \\
\text{cherokee} & \mapsto & \text{jeep} \\
\} & &
\end{array}
$$

Model updates (3/3)

- Relational overriding can be used to model database updates.
- We have 2 types of updates: Insertions and modifications.
- For example, this update is a modification:

$$car' = car \oplus \{avalon \mapsto lexus\} =$$

{
avalon ↦ lexus,
camry ↦ toyota,
highlander ↦ toyota,
wrangler ↦ jeep,
renegade ↦ jeep,
compass ↦ jeep,
altima ↦ nissan,
pathfinder ↦ nissan,
murano ↦ nissan,
cherokee ↦ jeep
}

All pairs from set on LHS except avalon ↦ toyota, are added to the set on the set of the RHS.

Model deletions (1/2)

- "Remove all pairs where model is 'camry', 'murano', or 'altima'."

$$
\begin{aligned}
car' = \{camry, murano, altima\} \lhd\ car = \\
\{ \\
& avalon \mapsto lexus, \\
& \cancel{camry \mapsto toyota,} \\
& highlander \mapsto toyota, \\
& wrangler \mapsto jeep, \\
& renegade \mapsto jeep, \\
& compass \mapsto jeep, \\
& \cancel{altima \mapsto nissan,} \\
& pathfinder \mapsto nissan, \\
& \cancel{murano \mapsto nissan,} \\
& cherokee \mapsto jeep \\
\}
\end{aligned}
$$

Domain subtraction removes all elements from the domain of the relation.

Model deletions (2/2)

- "Remove all pairs where make is 'jeep'."

$$car' = car \rhd \{jeep\} =$$

$$\{$$
$$avalon \mapsto lexus,$$
$$highlander \mapsto toyota,$$
$$\cancel{wrangler \mapsto jeep,}$$
$$\cancel{renegade \mapsto jeep,}$$
$$\cancel{compass \mapsto jeep,}$$
$$pathfinder \mapsto nissan,$$
$$\cancel{cherokee \mapsto jeep}$$
$$\}$$

Range subtraction removes all elements from the range of the relation.

Binary relations and functions revisited

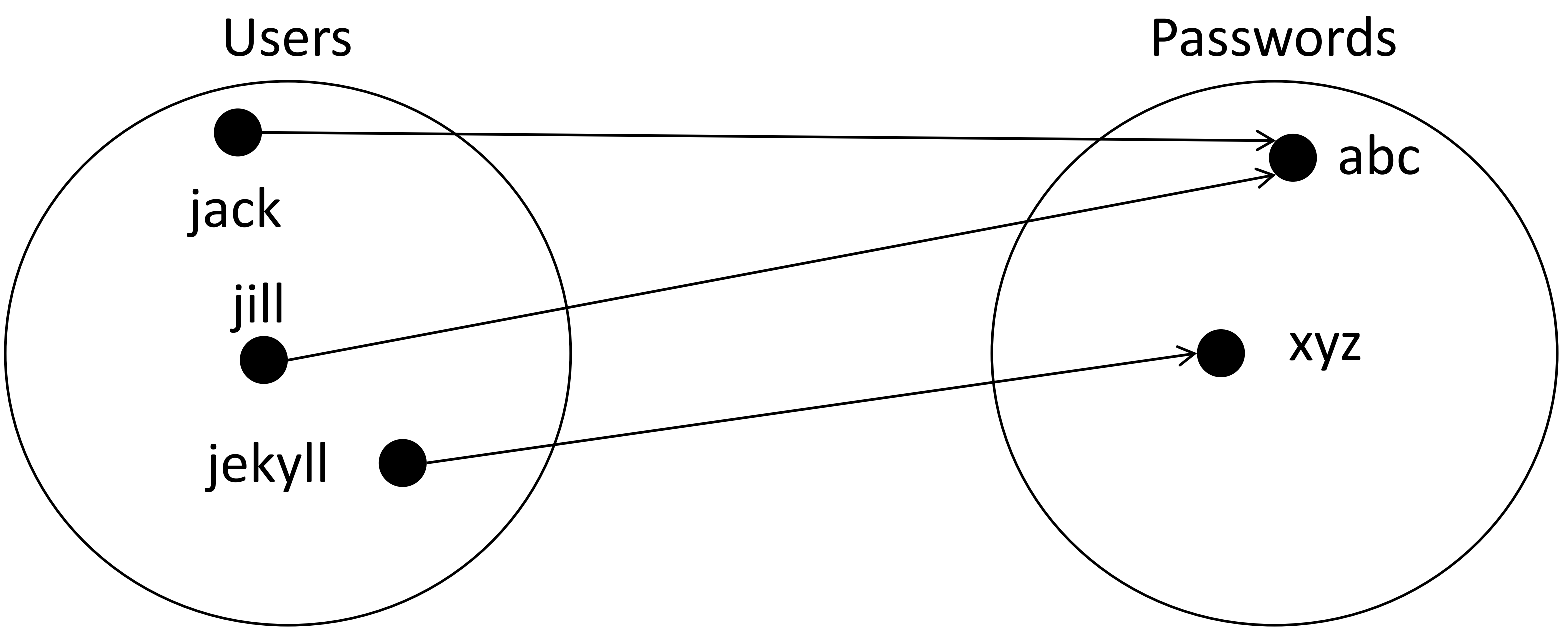
- In the example, the binary relation `car` is expressed as a set of ordered pairs.
- Formally, `car` is defined as `car: Model` $\leftrightarrow$ `Make`
- Alternatively we can formally define `car` as `car` $\subseteq$ `Model x Make`
- The relation `car` is also a function, and it can be formally defined as `car:  Model` $\rightarrow$ `Make`
- Both relations and functions define relationships between sets.
- Both relations and functions can be represented as sets.
- Not all relations are functions, but all functions are relations.

Example 3: A password file

- Requirements:
  1. Each user has a unique user-id.
  2. Each user-id is associated with only one password.
  3. Users may have a common password.

- We define types Users and Passwords.

- A possible state of the system is shown below:



Mapping and visualization

- The two types Users and Passwords are represented as sets.
- We capture the mapping from `Users` to `Passwords` by a set of ordered pairs which we will call `password`.
- The initial state of the system can thus be expressed as
  $\texttt{password} = \{\texttt{jack} \mapsto \texttt{abc},\ \texttt{jill} \mapsto \texttt{abc},\ \texttt{jekyll} \mapsto \texttt{xyz}\}$
- Note that `password` is a (binary) relation, since
  $\texttt{password} \subseteq \texttt{Users} \times \texttt{Passwords}$
- The relation `password` is also a function, i.e.
  $\texttt{password}: \texttt{Users} \rightarrow \texttt{Passwords}$

Properties of function ‘password’

- Function password is partial (as opposed to total) as it maps a subset of the domain set (Users). Formally this will be denoted as
  `password: Users` $\nrightarrow$ `Passwords`

- Is not injective (one-to-one) because it is not the case that

$$\forall user_1, user_2: Users \; (user_1 \neq user_2 \rightarrow password(user_1) \neq password(user_2))$$

- Is not surjective (onto) as it is not the case that

$$\forall passwd: Passwords \; \exists user: Users \; (password(user) = passwd)$$

- By definition it is not bijective (one-to-one correspondence).

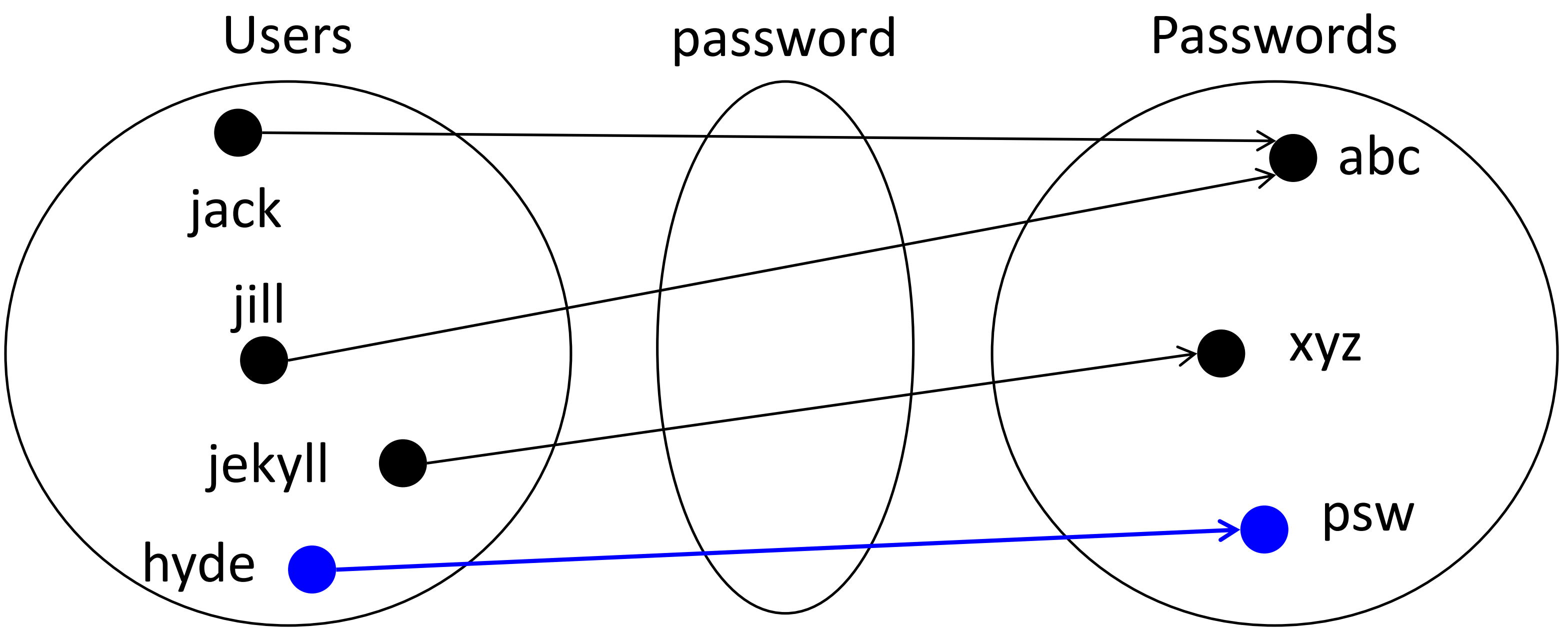
Case 1: Adding a new user (with a precondition)

password =
{
jack ↦ abc,
jill ↦ abc,
jekyll ↦ xyz,
hyde ↦ psw
}

pre: user $\notin$ dom password

Precondition successful

password' =
password $\cup$ {hyde ↦ psw}
or
password $\oplus$ {hyde ↦ psw}



Case 1: Adding a new user (with a precondition) /cont.

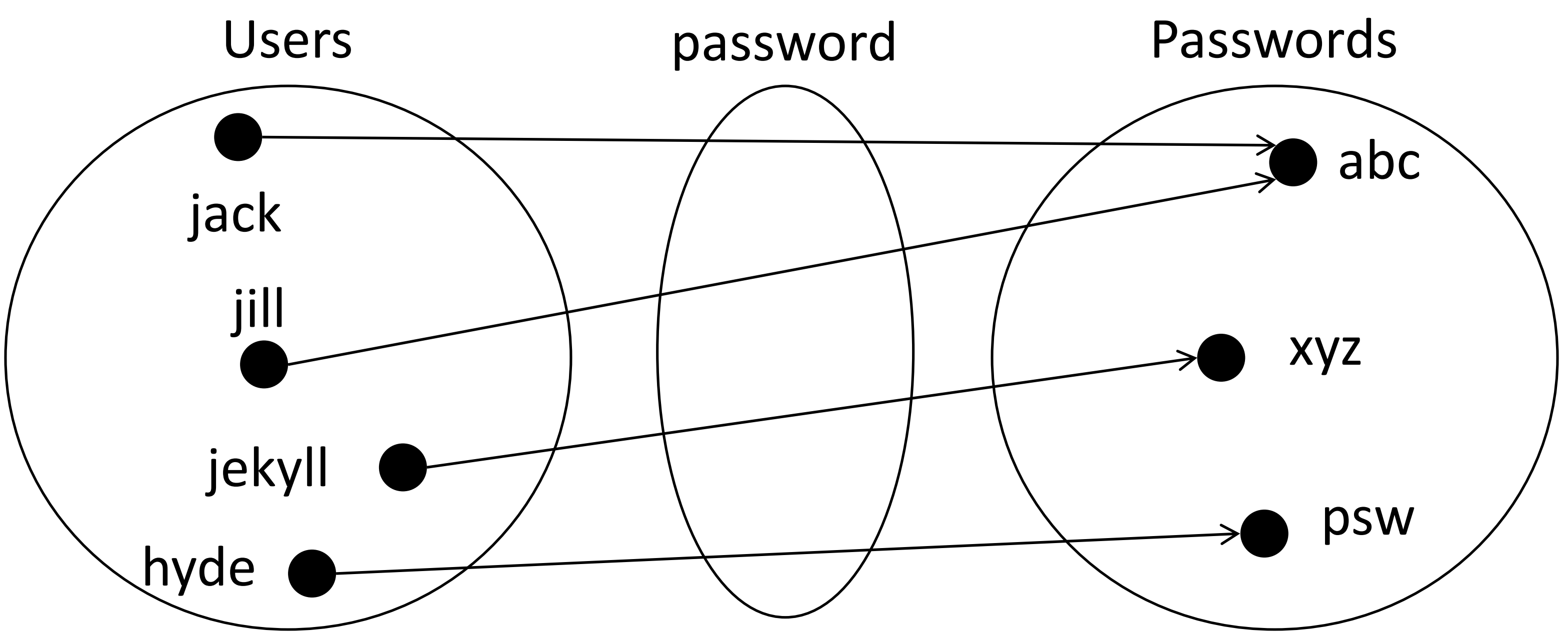
password =

$$\{ \; jack \mapsto abc, \; jill \mapsto abc, \; jekyll \mapsto xyz, \; hyde \mapsto psw \; \}$$

pre: user $\notin$ dom password

Wish to add: jekyll $\mapsto$ psw

Precondition failure

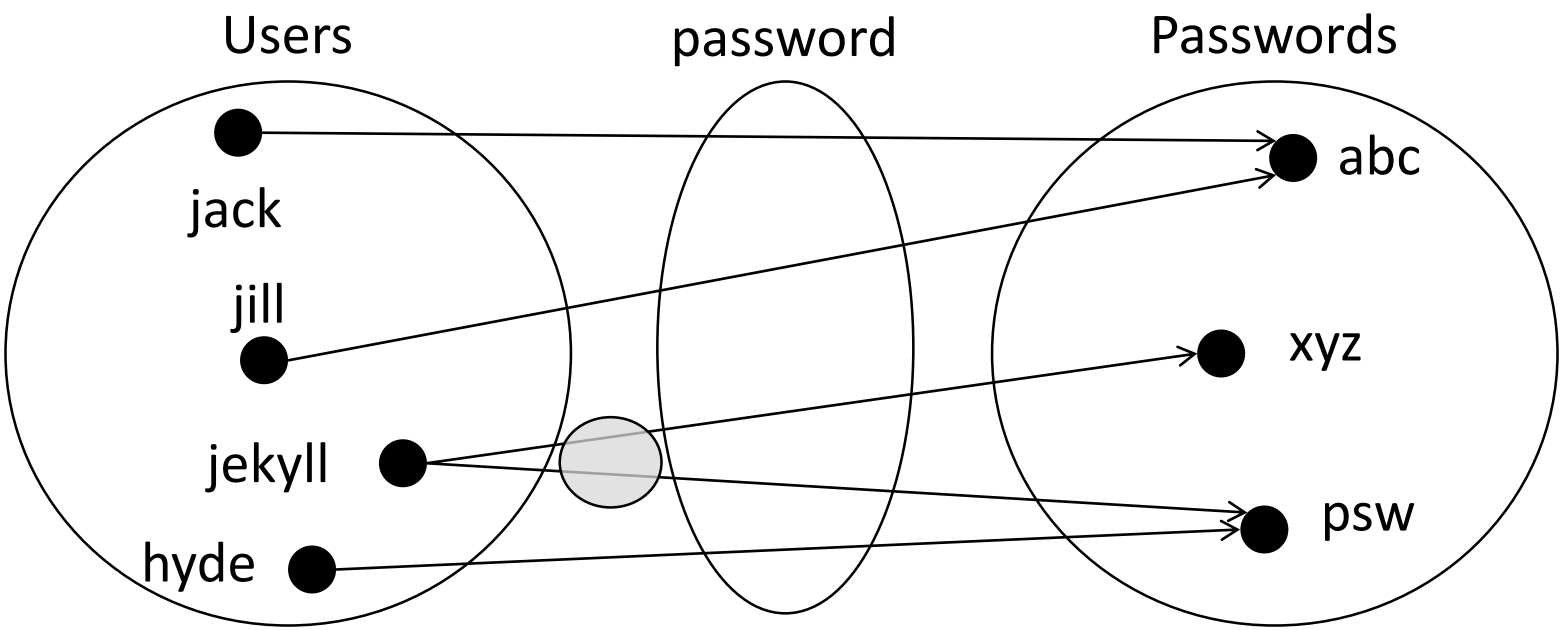


Case 2: Adding a new user with set union (without a precondition)

password =
{
jack ↦ abc,
jill ↦ abc,
jekyll ↦ xyz,
jekyll ↦ psw,
hyde ↦ psw
}

password' =
password U {jekyll ↦ psw}

Violation of requirements



Case 2: Adding a new user with set union (without a precondition) /cont.

password =

$$\{ \; jack \mapsto abc, \; jill \mapsto abc, \; jekyll \mapsto xyz, \; hyde \mapsto psw \; \}$$

password' =

$$password \; \cup \; \{jekyll \mapsto psw\}$$

- In the absence of a precondition, will there always be a violation of requirements with set union?
- If there exists no such user, then the user-password pair will be added to the file. No violation of requirements.
- If there already exists such user, then the existing user-password pair will also be added to the file. Violation of requirements (duplicate user).

Case 2: Adding a new user with relational override (without a precondition)

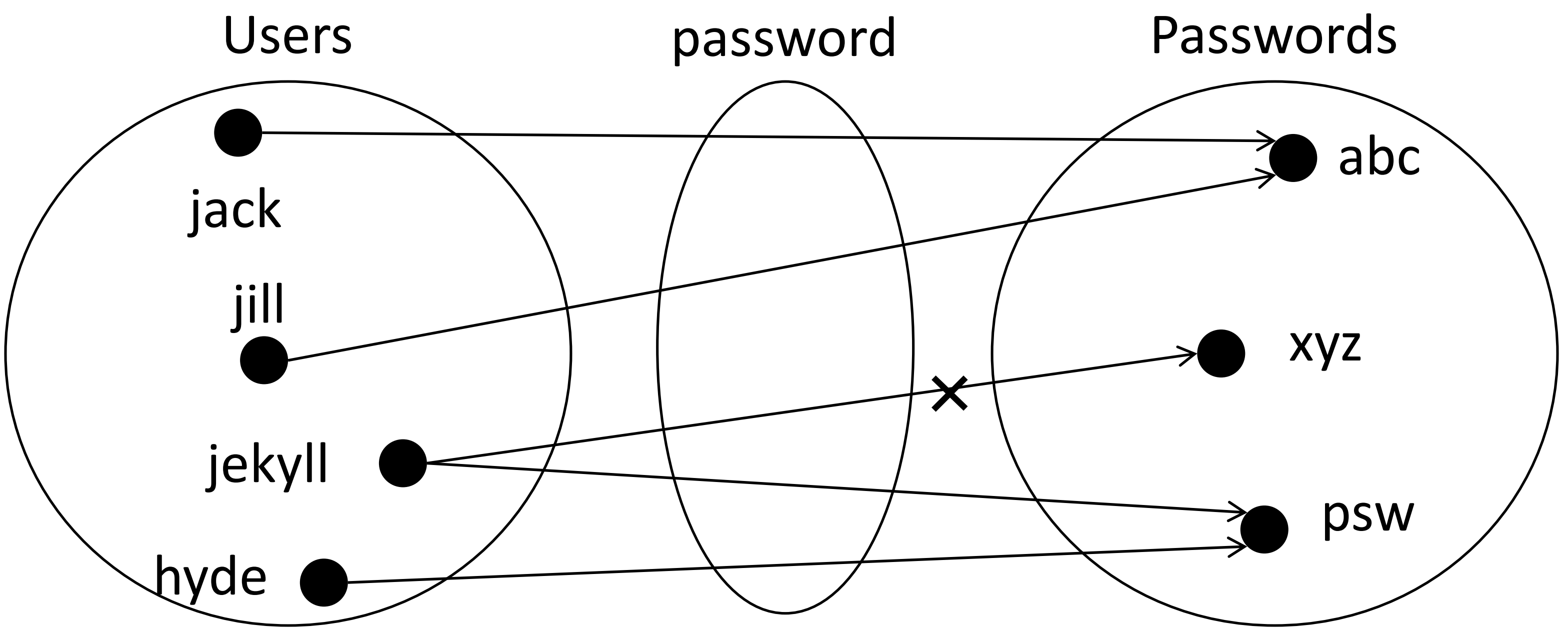
password =

$$\{ \text{jack} \mapsto \text{abc}, \; \text{jill} \mapsto \text{abc}, \; \cancel{\text{jekyll} \mapsto \text{xyz}}, \; \text{jekyll} \mapsto \text{psw}, \; \text{hyde} \mapsto \text{psw} \}$$

password' =

$$\text{password} \oplus \{\text{jekyll} \mapsto \text{psw}\}$$

Violation of requirements



Case 2: Adding a new user with relational override (without a precondition) /cont.

password =

$$\{ jack \mapsto abc, \; jill \mapsto abc, \; jekyll \mapsto xyz, \; hyde \mapsto psw \}$$

password' =

$$password \oplus \{jekyll \mapsto psw\}$$

- In the absence of a precondition, will there always be a violation of requirements with relational override?
- If there exists no such user, then the user-password pair will be added to the file. No violation of requirements.
- However, if there already exists such user, then the existing user-password pair will be replaced by a new such pair. Violation of requirements.

Case 3: Modifying the password of an existing user (with a precondition)

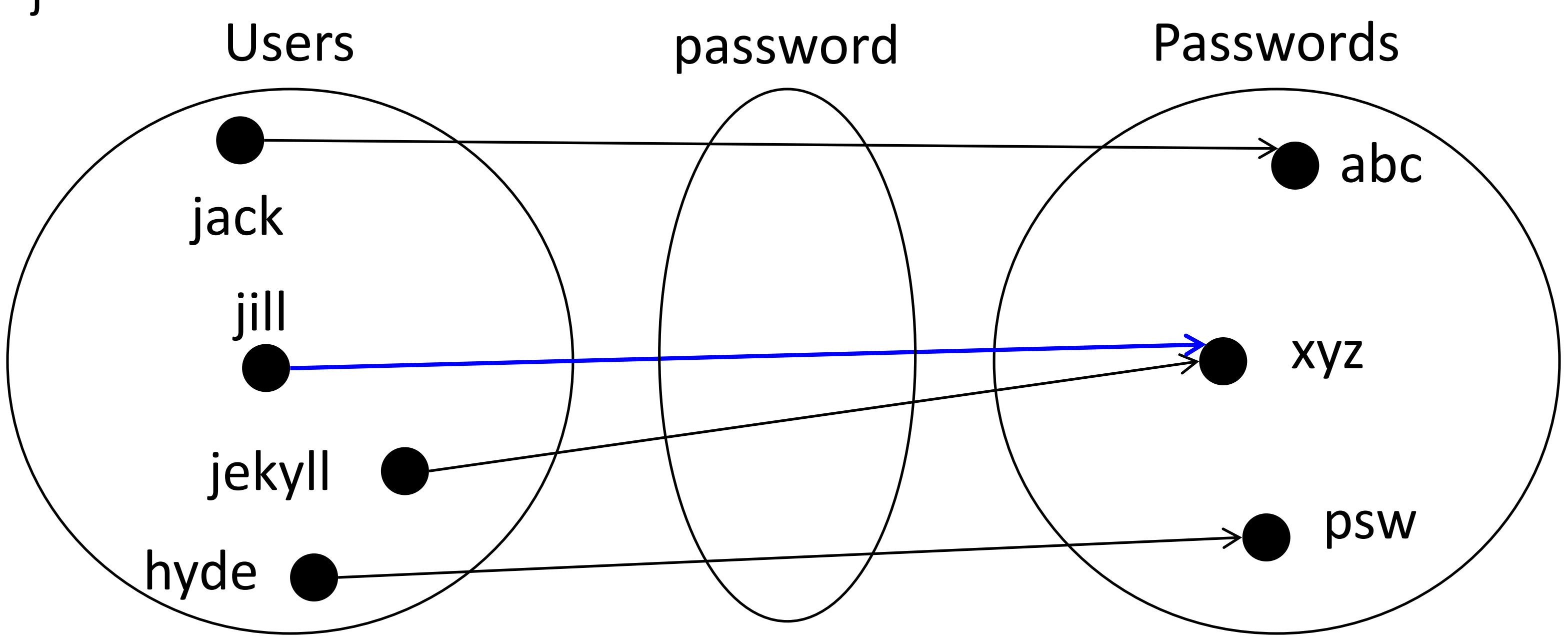
password =

$$
\{ \\
jack \mapsto abc, \\
\cancel{jill \mapsto abc,} \\
jill \mapsto xyz, \\
jekyll \mapsto xyz, \\
hyde \mapsto psw \\
\}
$$

<u>pre</u>: user $\in$ dom password

Wish to modify jill's password to xyz

password' = password $\oplus$ {jill $\mapsto$ xyz}



Case 3: Modifying the password of an existing user (with a precondition) /cont.

password =

$$\{ \text{jack} \mapsto \text{abc}, \ \text{jill} \mapsto \text{abc}, \ \text{jill} \mapsto \text{xyz}, \ \text{jekyll} \mapsto \text{xyz}, \ \text{hyde} \mapsto \text{psw} \}$$

Should be excluded !

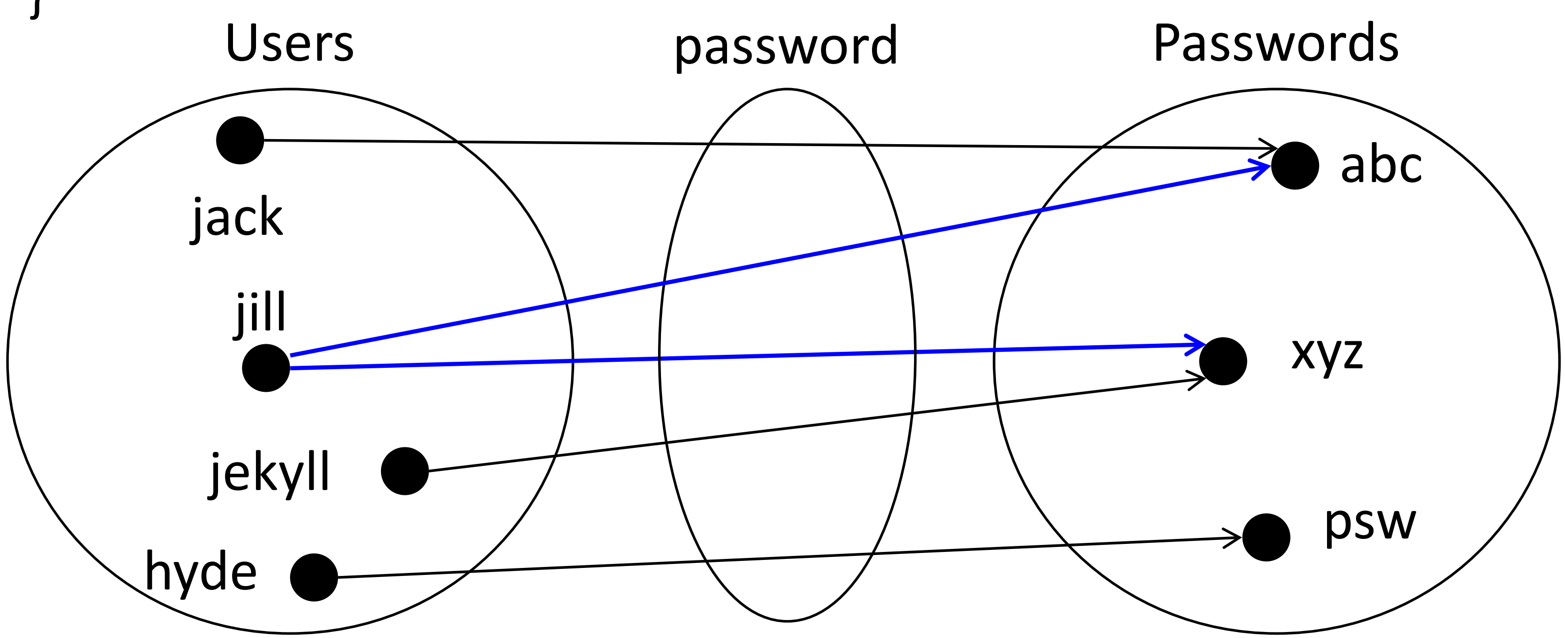
pre: user $\in$ dom password

Wish to modify jill's password to xyz

How about

password' = password U {jill $\mapsto$ xyz} ?

Violation of requirements



Example 4: Revisiting Sets, Binary Relations and Functions

- Recall that:

1. **Binary relations are sets**.
   (though not all sets are binary relations)

2. **Functions are binary relations**.
   (though not all binary relations are functions)

Relational overriding

- Consider the following relations:

R = { (Mike, 20), (Roger, 18), (Anne, 23) }

dom **R** = { Mike, Roger, Anne }, **ran** **R** = { 20, 18, 23 }

S = { (Anne, 20), (Yang, 19) }

dom **S** = { Anne, Yang }, and **ran** **S** = { 20, 19 }

- Relational override is a **binary operation** on relations.

R $\oplus$ **S** is defined as a set that contains all pairs of **S** plus those pairs from **R** whose first coordinates do not belong to the domain of **S**.

R ⊕ **S** is defined as a set that contains all pairs of **S** plus those pairs from **R** whose first coordinates do not belong to the domain of **S**.

R = { (**Mike**, **20**), (Roger, 18), (Anne, 23) }

S = { (Anne, 20), (Yang, 19) }

dom **S** = { Anne, Yang }

- Will (**Mike**, **20**) be added to the resulting set?

  Yes, because **Mike** is <u>not</u> in **dom** **S**.

  Result (so far):

  { (Anne, 20), (Yang, 19), (Mike, 20) }

R $\oplus$ S is defined as a set that contains all pairs of S plus those pairs from R whose first coordinates do not belong to the domain of S.

R = { (Mike, 20), (Roger, 18), (Anne, 23) }

S = { (Anne, 20), (Yang, 19) }

dom S = { Anne, Yang }

- Will (Roger, 18) be added to the resulting set?

  Yes, because Roger is <u>not</u> in **dom** S.

  Result (so far):

  { (Anne, 20), (Yang, 19), (Mike, 20), (Roger, 18) }

$R \oplus S$ is defined as a set that contains all pairs of S plus those pairs from R whose first coordinates do not belong to the domain of S.

R = { (Mike, 20), (Roger, 18), (**Anne**, **23**) }

S = { (Anne, 20), (Yang, 19) }

dom S = { Anne, Yang }

- Will (**Anne**, **23**) be added to the resulting set?

  No, because **Anne** is in **dom** S.

  Final result:

  { (Anne, 20), (Yang, 19), (Mike, 20), (Roger, 18) }

- In this example, the two binary relations are both functions:

R = { (Mike, 20), (Roger, 18), (Anne, 23) }, and

S = { (Anne, 20), (Yang, 19) }

- Is **R** U **S** a function? No.
- Is **R** $\oplus$ **S** or **S** $\oplus$ **R** a function? Yes.

Example 5: Set Union vs. Relational Override

- Consider the following relations:

S = { (**Ali**, **555**), (Ellie, 100), (Bruce, 430) }

dom S = { Ali, Ellie, Bruce }, **ran S** = { 555, 100, 430 }

T = { (Bruce, 400), (Bruce, 300), (Ellie, 999) }

dom T = { Bruce, Ellie }, and **ran T** = { 400, 300, 999 }

- Need to calculate **S** $\oplus$ **T**

- Will (**Ali**, **555**) be added to the resulting set?

Yes, because **Ali** is <u>not</u> in **dom T**.

Result (so far):

{ (Bruce, 400), (Bruce, 300), (Ellie, 999) , (Ali, 555) }

- Consider the following relations:

S = { (Ali, 555), (**Ellie**, **100**), (Bruce, 430) }

dom S = { Ali, Ellie, Bruce }, **ran S** = { 555, 100, 430 }

T = { (Bruce, 400), (Bruce, 300), (Ellie, 999) }

dom T = { Bruce, Ellie }, and **ran T** = { 400, 300, 999 }

- Need to calculate **S** $\oplus$ **T**

- Will (**Ellie**, **100**) be added to the resulting set?

No, because **Ellie** is in **dom T**.

Result (so far):

{ (Bruce, 400), (Bruce, 300), (Ellie, 999) , (Ali, 555) }

- Consider the following relations:

  S = { (Ali, 555), (Ellie, 100), (**Bruce**, **430**) }
  dom S = { Ali, Ellie, Bruce }, **ran S** = { 555, 100, 430 }

  T = { (Bruce, 400), (Bruce, 300), (Ellie, 999) }
  dom T = { Bruce, Ellie }, and **ran T** = { 400, 300, 999 }

- Need to calculate **S** $\oplus$ **T**

- Will (**Bruce**, **430**) be added to the resulting set?
  No, because **Bruce** is in **dom T**.

  Final result:
  { (Bruce, 400), (Bruce, 300), (Ellie, 999) , (Ali, 555) }

- In this example, it is not the case that both binary relations are functions:

S = { (Ali, 555), (Ellie, 100), (Bruce, 430) }

T = { (Bruce, 400), (Bruce, 300), (Ellie, 999) }

Example 6: A final example on Relational Override

- Consider the following relations:

S = { (Ali, 555), (Ellie, 100), (Ellie, 88), (Sanjay, 100), (Cho, 300),
(Sanjay, 999), (Bruce, 001)
}

T = { (Joseph, 530), (Bruce, 300) }

- **S** $\oplus$ **T** = {

All the pairs of **T**, ...

(Joseph, 530), (Bruce, 300),

(Ali, 555), (Ellie, 100), (Ellie, 88), (Sanjay, 100),
(Cho, 300), (Sanjay, 999)

}

...plus those pairs from **S**
whose first coordinate
is not in the domain of **T**.