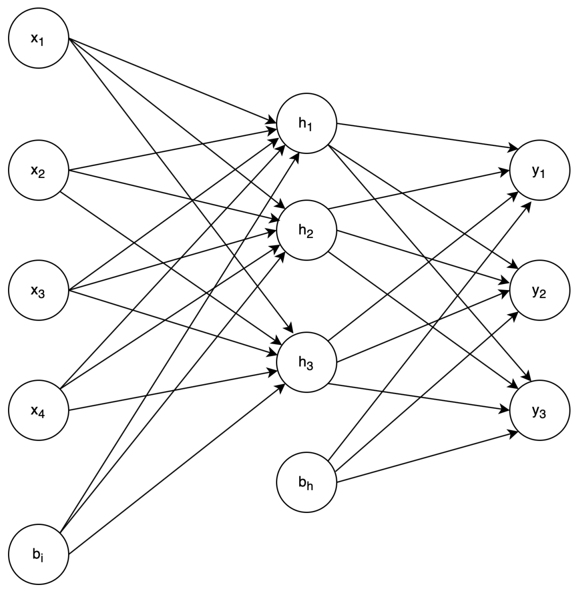
COMP 472: Artificial Intelligence
Artificial Neural Networks
Solutions

1 Question

Assume the following neural network with 4 input nodes, 1 hidden layer and 3 output layers.



Now suppose we have a binary classification task on a small dataset where each sample has five features. Does the above network suitable for this task? Explain your answer.

Solution

The answer is No! Since each input has five features, the network with four input nodes cannot be used. Note that in the above network, there are five nodes where the first four nodes are the input nodes and the last one is the input bias.

On the other hand, for a binary classification tasks we have two outputs; class one and class two. Hence two output nodes would be enough.

2 Question

Suppose the same neural network as the question above, with the input, bias, weights and outputs indicated below. Find the output values for an input x.

Assume that all the bias values are +1, and have the initial weight of 1. All other network weights are initialized to 0.5 and sigmoid is used as the activation function. The input values, initial values of hidden nodes, and the outputs are as follows:

$$x = [1, 0, 0.3, 0.7] \qquad w_{ih} = \begin{bmatrix} 0.5, 0.5, 0.5 \\ 0.5, 0.5, 0.5 \\ 0.5, 0.5, 0.5 \\ 0.5, 0.5, 0.5 \end{bmatrix} \qquad w_{ho} = \begin{bmatrix} 0.5, 0.5, 0.5 \\ 0.5, 0.5, 0.5 \\ 0.5, 0.5, 0.5 \end{bmatrix} \qquad y = [?, ?, ?]$$

Solution

To find out the output values, we have to multiply each values of x with the corresponding values of $w_i h$ which is the weights from input nodes to the hidden layer. After that, the biases b_{ih} will be added and the sigmoid activation function will be applied on them.

weighted avg + bias

$$h_{in} = xw_{ih} + b_{ih} = [1., 1., 1.] + [1, 1, 1] = [2., 2., 2.]$$

$$h = \sigma(h_{in}) = [0.8808, 0.8808, 0.8808]$$

Then to compute the output values, the values of the previous layer h_{in} should be multiplied by its weights w_{ho} and summed with the bias b_{ho}

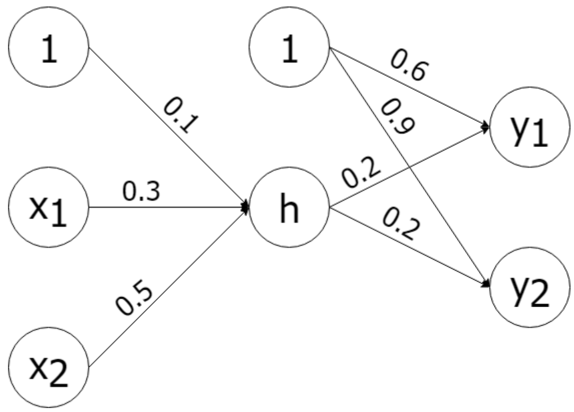
product matrix

$$y_{in} = hw_{ho} + b_{ho} = [1.3212, 1.3212, 1.3212] + [1, 1, 1] = [2.3212, 2.3212, 2.3212]$$

$$y = \sigma(y_{in}) = [0.9106, 0.9106, 0.9106]$$

3 Question

Consider the neural network shown below. It consists of 2 input nodes, 1 hidden node, and 2 output nodes, with addition to a bias at the input layer and a bias at the hidden layer. All nodes in the hidden and output layers use the sigmoid activation function (σ).



(a) Calculate the output of y1 and y2 if the network is fed $x = (1, 0)$ as input.

Solution

$$h_{in} = b_h + w_{x_1\text{-}h}x_1 + w_{x_2\text{-}h}x_2 = (0.1) + (0.3 \times 1) + (0.5 \times 0) = 0.4$$
$$h = \sigma(h_{in}) = \sigma(0.4) = \frac{1}{1 + e^{-0.4}} = 0.599$$

$$y_{1,in} = b_{y_1} + w_{h\text{-}y_1}h = 0.6 + (0.2 \times 0.599) = 0.72$$
$$y_1 = \sigma(0.72) = \frac{1}{1 + e^{-0.72}} = 0.673$$

$$y_{2,in} = b_{y_2} + w_{h\text{-}y_2}h = 0.9 + (0.2 \times 0.599) = 1.02$$
$$y_2 = \sigma(1.22) = \frac{1}{1 + e^{-1.02}} = 0.735$$

As a result, the output is calculated as $y = (y1, y2) = (0.673, 0.735)$.

(b) Assume that the expected output for the input $x = (1, 0)$ is supposed to be $t = (0, 1)$, calculate the updated weights after the backpropagation of the error for this sample. Assume that the learning rate $\eta = 0.1$.

Error for hidden layer + output

Solution

$$\delta_{y_1} = y_1(1 - y_1)(y_1 - t_1) = 0.673(1 - 0.673)(0.673 - 0) = 0.148$$

$$\delta_{y_2} = y_2(1 - y_2)(y_2 - t_2) = 0.735(1 - 0.735)(0.735 - 1) = -0.052$$

$$\delta_h = h(1 - h) \sum_{i=1,2} w_{h\text{-}y_i}\delta_{y_i} = 0.599(1 - 0.599)[0.2 \times 0.148 + 0.2 \times (-0.052)] = 0.005$$

$$\Delta w_{x_1\text{-}h} = -\eta\delta_h x_1 = -0.1 \times 0.005 \times 1 = -0.0005$$

$$\Delta w_{x_2\text{-}h} = -\eta\delta_h x_2 = -0.1 \times 0.005 \times 0 = 0$$

$$\Delta b_h = -\eta\delta_h = -0.1 \times 0.005 = -0.0005$$

$$\Delta w_{h\text{-}y_1} = -\eta\delta_{y_1} h = -0.1 \times 0.148 \times 0.599 = -0.0088652$$

$$\Delta b_{y_1} = -\eta\delta_{y_1} = -0.1 \times 0.148 = -0.0148$$

$$\Delta w_{h\text{-}y_2} = -\eta\delta_{y_2} h = -0.1 \times (-0.052) \times 0.599 = 0.0031148$$

$$\Delta b_{y_2} = -\eta\delta_{y_2} = -0.1 \times (-0.052) = 0.0052$$

Update weights

$$w_{x_1\text{-}h,new} = w_{x_1\text{-}h} + \Delta w_{x_1\text{-}h} = 0.3 + (-0.0005) = 0.2995$$

$$w_{x_2\text{-}h,new} = w_{x_2\text{-}h} + \Delta w_{x_2\text{-}h} = 0.5 + 0 = 0.5$$

$$b_{h,new} = b_h + \Delta b_h = 0.1 + (-0.0005) = 0.0995$$

$$w_{h\text{-}y_1,new} = w_{h\text{-}y_1} + \Delta w_{h\text{-}y_1} = 0.2 + (-0.0088652) = 0.1911348$$

$$b_{y_1,new} = b_{y_1} + \Delta b_{y_1} = 0.6 + (-0.0148) = 0.5852$$

$$w_{h\text{-}y_2,new} = w_{h\text{-}y_2} + \Delta w_{h\text{-}y_2} = 0.2 + 0.0031148 = 0.2031148$$

$$b_{y_2,new} = b_{y_2} + \Delta b_{y_2} = 0.9 + 0.0052 = 0.9052$$

Calculate new weights

4 Question

Recalculate your answers to the previous question, using matrix notation.

Solution

First, we need to visualize our parameters in matrix format.

$$x = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad t = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$w_{x\text{-}h} = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix} \qquad b_h = \begin{bmatrix} 0.1 \end{bmatrix} \qquad w_{h\text{-}y} = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix} \qquad b_y = \begin{bmatrix} 0.6 & 0.9 \end{bmatrix}$$

(a) In the formulas below, $\sigma(X)$ stands for element-wise sigmoid of matrix X.

$$h_{in} = xw_{x\text{-}h} + b_h = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix} + \begin{bmatrix} 0.1 \end{bmatrix} = \begin{bmatrix} 0.4 \end{bmatrix}$$

$$h = \sigma(h_{in}) = \sigma\left(\begin{bmatrix} 0.4 \end{bmatrix}\right) = \begin{bmatrix} 0.599 \end{bmatrix}$$

$$y_{in} = hw_{h\text{-}y} + b_y = \begin{bmatrix} 0.599 \end{bmatrix} \begin{bmatrix} 0.2 & 0.2 \end{bmatrix} + \begin{bmatrix} 0.6 & 0.9 \end{bmatrix} = \begin{bmatrix} 0.72 & 1.02 \end{bmatrix}$$

$$y = \sigma\left(\begin{bmatrix} 0.72 & 1.02 \end{bmatrix}\right) = \begin{bmatrix} 0.673 & 0.735 \end{bmatrix}$$

(b) In the formulas below, $X \circ Y$ stands for element-wise multiplication of matrices X and Y, and $1 - X$ is equivalent to subtracting the matrix X from a matrix with the same size of X, but with all elements equal to 1.

$$\delta_y = y \circ (1 - y) \circ (y - t) =$$

$$\begin{bmatrix} 0.673 & 0.735 \end{bmatrix} \circ \left(1 - \begin{bmatrix} 0.673 & 0.735 \end{bmatrix}\right) \circ \left(\begin{bmatrix} 0.673 & 0.735 \end{bmatrix} - \begin{bmatrix} 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0.148 & -0.052 \end{bmatrix}$$

$$\boxed{\delta_h} = h \circ (1 - h) \circ (\delta_y w_{h\text{-}y}^T) =$$

$$\begin{bmatrix} 0.599 \end{bmatrix} \circ \left(1 - \begin{bmatrix} 0.599 \end{bmatrix}\right) \circ \left(\begin{bmatrix} 0.148 & -0.052 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}\right) = \boxed{\begin{bmatrix} 0.005 \end{bmatrix}}$$

$$\Delta w_{x\text{-}h} = -\eta(x^T \boxed{\delta_h}) = -0.1 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0.005 \end{bmatrix}\right) = \begin{bmatrix} -0.0005 \\ 0 \end{bmatrix}$$

$$\Delta b_h = \boxed{-\eta\delta_h} = -0.1 \begin{bmatrix} 0.005 \end{bmatrix} = \begin{bmatrix} -0.0005 \end{bmatrix}$$

$$\Delta w_{h\text{-}y} = -\eta(h^T \delta_y) = -0.1 \left(\begin{bmatrix} 0.599 \end{bmatrix} \begin{bmatrix} 0.148 & -0.052 \end{bmatrix}\right) = \begin{bmatrix} -0.00886 & 0.00311 \end{bmatrix}$$

$$\Delta b_y = -\eta\delta_y = -0.1 \begin{bmatrix} 0.148 & -0.052 \end{bmatrix} = \begin{bmatrix} -0.0148 & 0.0052 \end{bmatrix}$$

$$w_{x\text{-}h,new} = w_{x\text{-}h} + \Delta w_{x\text{-}h} = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix} + \begin{bmatrix} -0.0005 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2995 \\ 0.5 \end{bmatrix}$$

$$b_{h,new} = b_h + \Delta b_h = \begin{bmatrix} 0.1 \end{bmatrix} + \begin{bmatrix} -0.0005 \end{bmatrix} = \begin{bmatrix} 0.0995 \end{bmatrix}$$

$$w_{h\text{-}y,new} = w_{h\text{-}y} + \Delta w_{h\text{-}y} = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix} + \begin{bmatrix} -0.0088652 & 0.0031148 \end{bmatrix} = \begin{bmatrix} 0.1911 & 0.2031 \end{bmatrix}$$

$$b_{y,new} = b_y + \Delta b_y = \begin{bmatrix} 0.6 & 0.9 \end{bmatrix} + \begin{bmatrix} -0.0148 & 0.0052 \end{bmatrix} = \begin{bmatrix} 0.5852 & 0.9052 \end{bmatrix}$$