SOEN 331: Introduction to Formal Methods for Software Engineering

Tutorial exercises on propositional logic

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1. For propositions p and q, provide answers to the following:

   (a) When is $p \rightarrow q$ false?

   When p is true and q is false.

   (b) When is a biconditional true?

   When the two propositions have the same truth value.

   (c) Explain whether the expression $p \wedge \neg p$ is a tautology or a contradiction.

   It is a contradiction since by definition it is always false regardless of the values of its variables.

   (d) Explain whether the expression $p \vee \neg p$ is a tautology or a contradiction.

   It is a tautology since by definition it is always true regardless of the values of its variables.

   (e) Define *consistency*.

   A set of sentences is consistent (or satisfiable) if and only if it is possible that every sentence in the set is true.

(f) For a conditional statement $p \to q$ write down its inverse, its converse and its contrapositive.

The inverse is $\neg p \to \neg q$, the converse is $q \to p$, and the contrapositive is $\neg q \to \neg p$.

(g) Determine logical equivalences between a conditional statement, its converse and its contrapositive.

The inverse and the converse of an implication statement are logically equivalent. The implication statement is logically equivalent to its contrapositive.

(h) Explain what is meant by "p is a sufficient condition for q" and translate the statement into formal logic.

The statement implies that the occurrece of p guarantees the occurrence of q. Formally we write $p \to q$.

(i) Explain what is meant by "p is a necessary condition for q" and translate the statement into formal logic.

The statement implies that if p does not occur, then q cannot occur either. Formally we write $\neg p \to \neg q \equiv q \to p$.

(j) When is p considered to be a *criterion* for q?

p is called a *criterion* for q iff $p \leftrightarrow q$.

(k) Define *validity*.

An argument is *valid* if and only if it is impossible that all its premises are true while its conclusion is false.

(l) Explain whether or not an argument can be valid but not sound.

An argument may have a validating form, but at least one of its premises is false. In this case it cannot be sound.

(m) Explain whether or not an argument can be sound but not valid.

By definition soundness implies validity, so such argument cannot exist.

(n) Explain whether or not a valid argument can have false conclusion.

When the conclusion is based on a false premise.

(o) Explain when the following statement is true and when it is false: $\forall x \exists y P(x, y)$.

It is true when for every x there is a y for which $P(x, y)$ is true. It is false when there is an x such that $P(x, y)$ is false for every y.

(p) Explain when the following statement is true and when it is false: $\exists x \forall y P(x, y)$.
It is true when there is an x for which $P(x, y)$ is true for every y. It is false when For every x there is a y for which $P(x, y)$ is false.

2. For each one of the following arguments determine whether it is valid or invalid and write down the underlying rule that applies.

(a) Argument:

If it is raining, **then** the streets are wet.
It is not raining.
Therefore, the streets are not wet.

This argument is invalid by *inverse error* ("Denying the antecedent").

(b) Argument:

If taxes are lowered, **then** we will have more money to spend.
We have more money to spend.
Therefore, taxes must have been lowered.

This argument is invalid by *converse error* ("affirming the consequent").

3. Discuss the properties of the following argument and decide whether it should be accepted or rejected.

"If we were at the beginning of a long-term global warming trend, then El Nino would be causing a northern shift in the jet stream. We have just seen such a northern shift. Therefore, we must be at the beginning of a long-term global warming trend."

Solution:

The argument is invalid by affirmation of the consequent (or "converse error") and by definition it cannot be sound. As non-sound, the argument must be rejected.

4. You are shown a set of four cards placed on a table, each of which has a **number** on one side and a **symbol** on the other side. The visible faces of the cards show the numbers **2** and **7**, and the symbols $\square$, and $\bigcirc$.

   Which card(s) must you turn over in order to test the truth of the proposition that *"If a card has an odd number on one side, then it has the symbol $\square$ on the other side"*? Explain your reasoning by deciding for <u>each</u> card whether it should be turned over and why.

   <u>Solution</u>:

   The proposition can be expressed as $odd \rightarrow \square$.

   - By *modus ponens* we must turn over **7** and expect to see $\square$.
   - By *modus tollens* (i.e. taking the contrapositive of the implication statement, $\neg\square \rightarrow \neg odd$), we must turn over the card with $\bigcirc$ and expect to see an even number.
   - We cannot make any claim based on 2, as it not the case that $\neg odd \rightarrow \neg\square$ (Denying the antecedent, or "inverse error").
   - Finally, we cannot make any claim based on $\square$, as it is not the case that $\square \rightarrow odd$ (Affirming the consequent, or "converse error").

5. You are shown a set of four cards placed on a table, each of which has a **letter** on one side and a **symbol** on the other side. The visible faces of the cards show the letters **L** and **A**, and the symbols $\square$, and $\Diamond$.

   Which card(s) must you turn over in order to test the truth of the proposition that *"If a card has an consonant on one side, then it has the symbol $\Diamond$ on the other side"*? Explain your reasoning <u>in detail</u> by deciding for <u>each</u> card whether it should be turned over and why. In your answers, apply any and all appropriate validating or non-validating patterns where applicable.

Solution:

The proposition to be tested can be expressed as $consonant \to \Diamond$.

- By *modus ponens*, we have $consonant \to \Diamond, consonant \vdash \Diamond$. We must turn over **L** and expect to see $\Diamond$.
- By *modus tollens*, we have $consonant \to \Diamond, \neg\Diamond \vdash \neg consonant$. We must turn over the card with $\Box$ and expect to see **A**.
- We cannot make any claim based on **A**, as it not the case that $\neg consonant \to \neg\Diamond$ (Denying the antecedent, or "inverse error").
- Finally, we cannot make any claim based on $\Box$, as it is not the case that $\neg\Diamond \to \neg consonant$ (Affirming the consequent, or "converse error").

6. You are working as security in a bar, and it is your job to make sure that the rule that governs the consumption of alcohol is enforced. The rule states that "If a person consumes an alcoholic beverage, then they must be over 21 years old." The cards below represent the drink of choice and the age of four customers at the bar: **Beer**, **Soda**, **23**, **16**. In order to enforce the policy on the consumption of alcohol, explain your reasoning in detail by deciding for each card whether it should be turned over or not and why.

Solution:

The proposition to be tested can be expressed as: "`alcohol` $\to$ `21+ yrs old`."

- By *modus ponens*, we have `alcohol` $\to$ `(21+ yrs old)`, `alcohol` $\vdash$ `(21+ yrs old)`.

  We must turn over **Beer** and expect to see a number greater than 21.

- By *modus tollens*, we have

  `alcohol` $\rightarrow$ `(21+ yrs old)`, $\neg$ `(21+ yrs old)` $\vdash \neg$ `alcohol`.

  We must turn over the card with `16` and expect to see `non-alcohol`.

- We cannot make any claim based on **Soda**, as it not the case that

  $\neg$ `alcohol` $\rightarrow \neg$ `(21+ yrs old)` (Denying the antecedent, or "inverse error").

- Finally, we cannot make any claim based on `23`, as it is not the case that

  `(21+ yrs old)` $\rightarrow$ `alcohol`. (Affirming the consequent, or "converse error").

7. For each of the following indicate whether the statement is true or false and provide a reasoning.

   (a) A proposition can be valid or invalid.

   False. Validity is a property of arguments, not a property of propositions.

   (b) A proposition can be sound or unsound.

   False. Soundness is a property of arguments, not a property of propositions.

   (c) An argument can be true or false.

   False. Truthfulness is a property of propositions, not a property of arguments.

   (d) If the conclusion to an argument is true, then the argument is sound.

   False.

   (e) It is impossible for a valid argument to have a false conclusion.

   False.

   (f) If an argument has all premises true and a false conclusion, then the argument is invalid.

   True.

(g) It is impossible for an argument to have all premises true, the conclusion true and be invalid.

True.

8. For each one of the following arguments, a) determine whether it is valid or invalid (in each case write down the underlying rule that applies), and b) where applicable state whether it is sound.

(a) **If** there is fire, **then** there is smoke.
There is smoke.
Therefore, there is fire.

This argument is invalid by *converse error*.

(b) **If** there is a storm, **then** school will be closed.
School is not closed.
Therefore, there is no storm.

The argument is valid by *modus tollens*.

(c) **If** I win the lotto, **then** I will take a vacation or I will buy a car.
I don't buy a car but I take a vacation.
Therefore, I have won the lotto.

This argument is invalid by *converse error*.