

Relations

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Binary relations over sets

- ▶ A *binary relation* R between sets A and B (sometimes written as $R : A \leftrightarrow B$) is defined as $R \subseteq A \times B$.
- ▶ Given $A = \{1, 2\}$ and $B = \{2, 3\}$, and $R = \text{"is less than"}$, then
 $A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$, and
 $R = \{(1, 2), (1, 3), (2, 3)\}$.
- ▶ We can also observe that $R \in \mathcal{P}(A \times B)$.
- ▶ The statement $(a, b) \in R$ is written as aRb , or $R(a, b)$.

Binary relations over sets /cont.

- ▶ We view a binary relation R as associating every ordered pair (a, b) the value true, if $(a, b) \in R$, or the value false if $(a, b) \notin R$.
- ▶ It is, thus, evident that $<$ (less than) is a binary relation, whereas $+$ (addition) or $-$ (subtraction) are not binary relations.

Binary relations over sets /cont.

- ▶ Using sets of ordered pairs is just one example we can express relations.
- ▶ Other methods include words, directed graphs or matrices.
- ▶ For example (Noodles likes Deborah), or likes(Noodles, Deborah), is an element of the binary relation *likes* over the set of all people.

Properties of relations

- ▶ A relation R on a set A is:
 - 1 Reflexive when $\forall a \in A : aRa$,
e.g. “divides”, “is equal to.”
 - 2 Irreflexive (or anti-reflexive) when no element of A is related to itself, i.e. $\forall a \in A : \neg(aRa)$,
e.g. “is greater than”, “is not equal to.”
- ▶ Note that “not reflexive” does not mean “irreflexive.”
- ▶ For example, the relation “likes” (between people) is not reflexive since not everybody likes themselves, but it is not irreflexive either since not everybody dislikes themselves either.

Properties of relations /cont.

- 3 Symmetric, when $\forall a, b \in A : aRb \rightarrow bRa$,
e.g. “is a blood relative of”, “is married to.”
- 4 Asymmetric, when $\forall a, b \in A : aRb \rightarrow \neg(bRa)$,
e.g. “is father of”, “is less than.”
 - ▶ A relation cannot be both symmetric and asymmetric.
 - ▶ Note that “asymmetric” does not mean “not symmetric.”
 - ▶ A relation can be neither symmetric nor asymmetric, e.g. “likes.”

Properties of relations /cont.

5 Antisymmetric when

$$\forall a, b \in A : (aRb \wedge bRa) \rightarrow a = b.$$

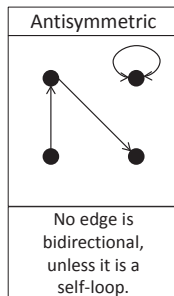
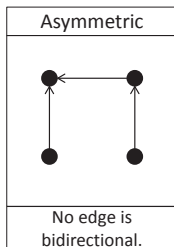
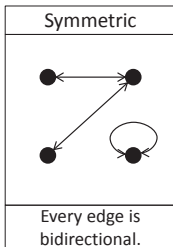
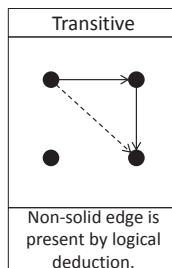
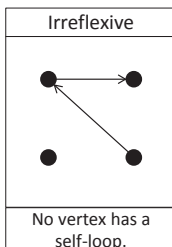
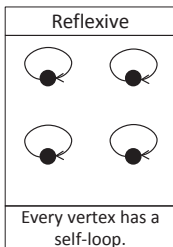
Alternatively, R is antisymmetric if whenever $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$

e.g. \leq is antisymmetric, but “likes” is not.

Properties of relations /cont.

- 6 Transitive when $\forall a, b, c \in A : (aRb \wedge bRc) \rightarrow aRc$,
e.g. “is ancestor of.”
- ▶ A relation is intransitive iff it is not transitive.
 - ▶ For example, the relation “likes” (between people) is intransitive.

Visualization of properties of binary relations



Example 1: Relation “likes”

- ▶ We have seen above that the proposition *Noodles likes Deborah* defines the binary relation *likes* over the set of all people. The relation *likes* is
 - (a) not reflexive (not everybody likes themselves),
 - (b) not symmetric (not everybody's feelings are reciprocated), and
 - (c) not transitive.

Example 2: Relation “is son of”

- ▶ The relation *is son of* is
 - (a) irreflexive, as no person can be his own son,
 - (b) not symmetric, and
 - (c) not transitive.

Domain and range in binary relations

- ▶ A binary relation R can be modeled as a set of ordered pairs.
- ▶ The *domain* of R , $\text{dom } R$, is the set of all first elements of ordered pairs.
- ▶ The *range* of R , $\text{ran } R$, is the set of all second elements of ordered pairs.
- ▶ For example, for $R = \{(0, 1), (0, 2), (1, 1), (3, 5)\}$, then $\text{dom } R = \{0, 1, 3\}$, and $\text{ran } R = \{1, 2, 5\}$

Inverse of binary relations

- ▶ Any binary relation R has an *inverse* relation (denoted by R^{-1} , or R^\sim , also called *converse* relation) which is obtained by changing the order of the elements in the relation.
- ▶ For a relation R on a set A

$$R^{-1} = \{(b, a) \in A \times A \mid (a, b) \in R\}$$

- ▶ A binary relation over a set is equal to its inverse if and only if it is symmetric.

Example: Inverse relation

- ▶ For $R = \{(0, 1), (0, 2), (1, 1), (3, 5)\}$, then
 $R^{-1} = \{(1, 0), (2, 0), (1, 1), (5, 3)\}$,
 $\text{dom } R^{-1} = \{1, 2, 5\}$, and
 $\text{ran } R^{-1} = \{0, 1, 3\}$
- ▶ We observe that
 $\text{dom } R = \text{ran } R^{-1}$, and
 $\text{ran } R = \text{dom } R^{-1}$.

Equivalence relations

- ▶ **Definition:** Any relation that is reflexive, symmetric and transitive is called an *equivalence relation*,
e.g. the relation “is equal to” on the set of numbers.

Example 3: Relation “has the same birthday as”

- ▶ For any $a, b, c \in A$, the relation R : “has the same birthday as” is
 - 1 Reflexive: $\forall a \in A : aRa$,
 - 2 Symmetric: $\forall a, b \in A : aRb \rightarrow bRa$, and
 - 3 Transitive: $\forall a, b, c \in A : (aRb \wedge bRc) \rightarrow aRc$.

Partition

- ▶ A *partition* of a set A is a collection, P , of non-empty subsets that satisfy the following:

- 1 $A = A_1 \cup A_2 \cup \cdots \cup A_n$, and

- 2 $A_i \cap A_j = \emptyset$, for $i \neq j$.

where the sets (i.e. the elements of P) are called the *blocks* (also: *parts*, or *cells*) of the partition.

Partial order

- ▶ **Definition:** Any relation that is reflexive, antisymmetric and transitive is called a *partial order*,
e.g. the relation “is less than or equal to” on the set of real numbers.
- ▶ A set for which a partial order is defined is called a *poset*.
- ▶ A *total order* on a set A is a partial ordering in which every pair of elements is related.

Predecessors and successors in posets

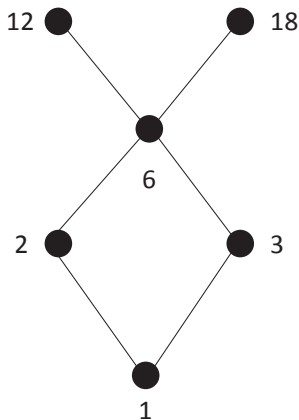
- ▶ If R is a partial order on a set A and xRy , $x \neq y$, we call x a *predecessor* of y (and subsequently we call y a *successor* of x).
- ▶ An element may possess many predecessors.
- ▶ If x is a predecessor of y and there is no z for which xRz and zRy , then we call x the *immediate predecessor* of y , denoted by $x < y$.

Hasse diagrams

- ▶ A *Hasse diagram* is a graph where vertices represent the elements of a poset A , and whenever $x < y$, vertex x is placed below vertex y and the vertices are joined by an edge.

Example: Hasse diagram

- ▶ Given the relation “is divisor of” on the set $A = \{1, 2, 3, 6, 12, 18\}$, the corresponding Hasse diagram is shown:



Minimal and maximal elements in Hasse diagrams

- ▶ In any poset there are *minimal elements* (ones without predecessors) and *maximal elements* (ones without successors).
- ▶ In the example, the poset has one minimal element (1) and two maximal elements (12, 18).

Composition of binary relations

- ▶ Given two binary relations, we can form a new one by a process called a *composition*.
- ▶ For example, given “is brother of (x, y) ” and “is parent of (y, z) ”, we can combine the two to form “is uncle of (x, z) .”
- ▶ Formally, if $R \subseteq X \times Y$, and $S \subseteq Y \times Z$, then their *composition*, denoted by $R \circ S$, is the relation
$$R \circ S = \{(x, z) \in X \times Z \mid (\exists y \in Y : (x, y) \in R \wedge (y, z) \in S)\}$$
- ▶ In other words, the tuple (x, z) is an element of the new composition relation, if there exists a $y \in Y$ such that $(x, y) \in R \wedge (y, z) \in S$.

Relational override

- ▶ The *override* of R with S , denoted by $R \oplus S$ is obtained by adding to S all those ordered pairs from R whose first coordinates are not in the domain of S .
- ▶ Given the relations
 $R = \{(0, 1), (0, 2), (2, 3)\}$, and $S = \{(0, 1), (1, 3), (3, 0)\}$.
Then
 $R \oplus S = \{(0, 1), (1, 3), (3, 0), (2, 3)\}$,
i.e. the ordered pair $(2, 3)$ has been added to S since $2 \notin \text{dom } S$.