

# Wavelet transform & Multi-resolution processing

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Materials adapted from Dr. T. D. Bui

# Reading

The content of the lecture corresponds to Chapter 7 of the textbook for the following sections

Page 481-483, 504-526

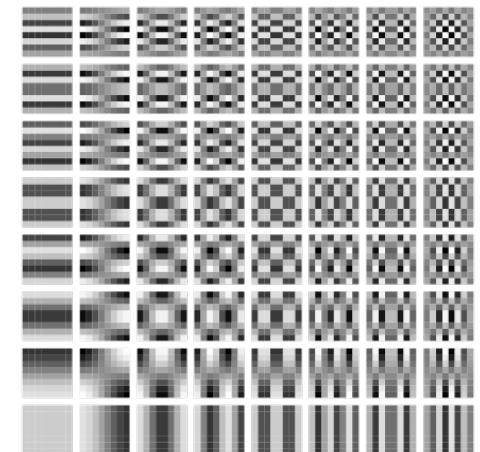
For related math operations and concepts, please refer to Page 464-470

# What is a transform?

- A **transform** can be applied to a raw signal to convert it from one domain to another (e.g Fourier).
- Enables the identification of features that may not be as easily observed in the original domain.
- The original image is usually in the time/spatial domain

# Transforms

- FT is probably the most popular transform.
- Many other transforms are used quite often by engineers and mathematicians
  - *Hilbert transform*
  - *Discrete Cosine Transform (DCT) -> JPEG format*
  - *Short-time Fourier transform (STFT)*
  - *Wavelet transform (we will see today)*
- Basis functions & coefficients



*Basis functions for DCT*

# Fourier Transform (FT)

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\pi ft} dt$$

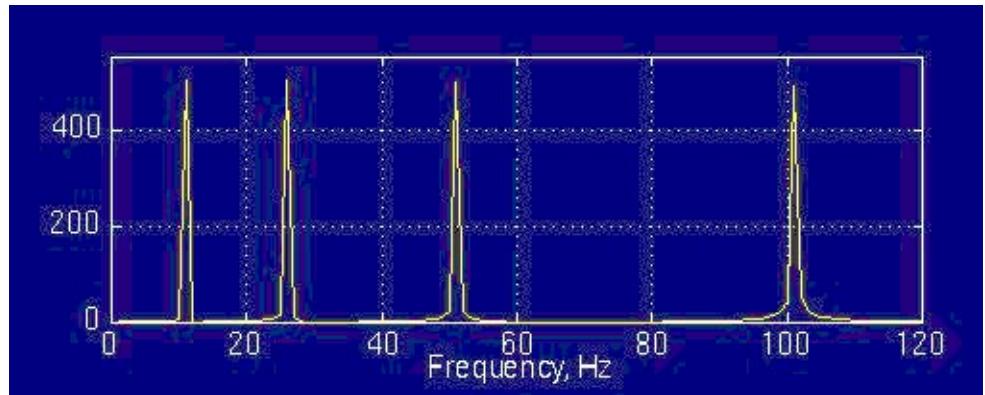
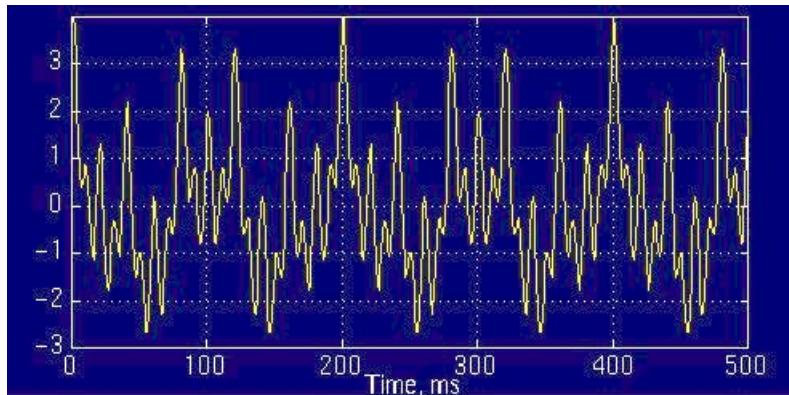
$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{j\pi ft} df$$

- No frequency information is available in the time-domain signal.
  - No time information is available in the Fourier transformed signal.
- ✓ Is it necessary to have both the time and the frequency information at the same time?

# Stationary Signals

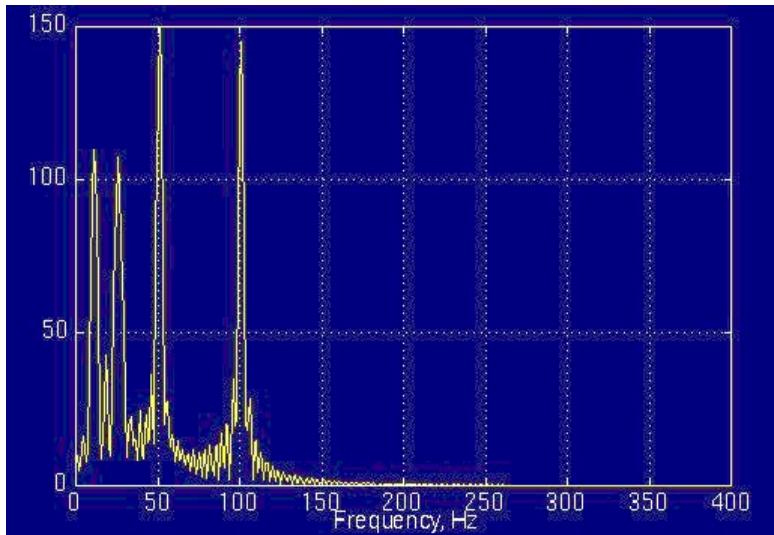
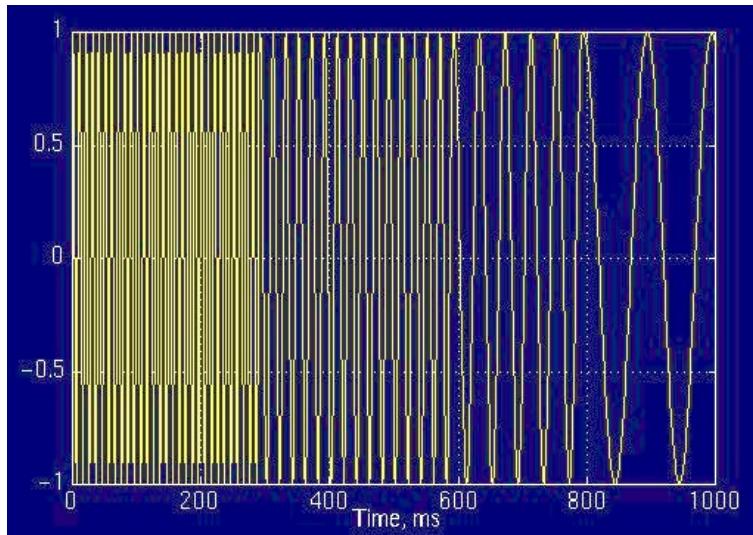
- Signals whose frequency content do not change in time:

$$x(t) = \cos(2\pi \cdot 10 \cdot t) + \cos(2\pi \cdot 25 \cdot t) + \cos(2\pi \cdot 50 \cdot t) + \cos(2\pi \cdot 100 \cdot t)$$

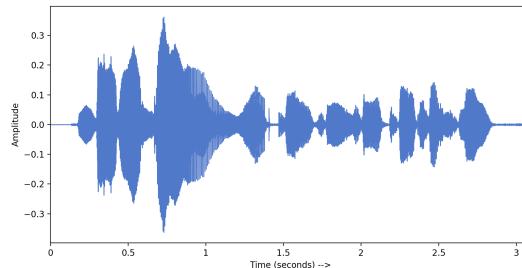


# Multiple Freq Presented at Different Time

– non-stationary



The interval 0 to 300 ms has a **100 Hz** sinusoid, the interval 300 to 600 ms has a **50 Hz** sinusoid, the interval 600 to 800 ms has a **25 Hz** sinusoid, and finally the interval 800 to 1000 ms has a **10 Hz** sinusoid



# Time-Frequency Relationship

- FT is not a suitable technique for **non-stationary** signals.
- FT gives what frequency components (spectral components) exist in the signal - Nothing more, nothing less.
- When the **time localization** of the spectral components are needed, a transform giving the *TIME-FREQUENCY REPRESENTATION* of the signal is needed.

# Short-Time Fourier Transform

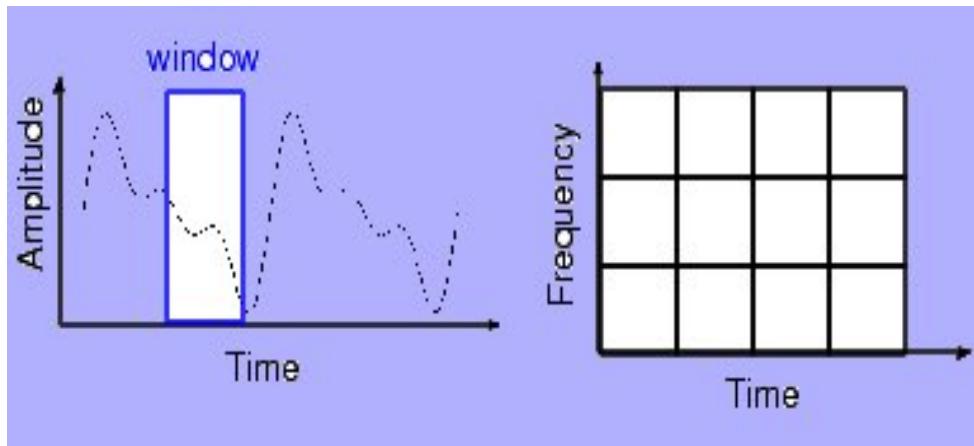
- In STFT, the signal is divided into small enough segments, where these segments of the signal can be assumed to be stationary.
- A window function "**w**" is chosen. The width of this window must be equal to the segment of the signal where its stationarity is valid.

# STFT

$$\text{STFT}_x^{(\omega)}(t', f) = \int_t [x(t)\omega(t-t')]e^{-j2\pi ft}dt$$

$\omega(t)$ : the window function

A function of time and frequency:



# Drawbacks of STFT

- Unchanged window.
- Dilemma of resolution:

Narrow window => poor frequency resolution

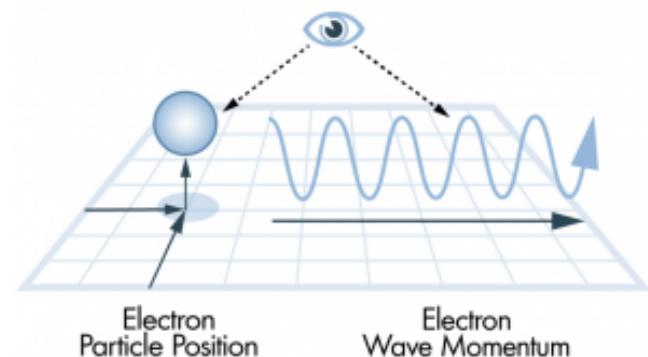
Wide window => poor time resolution

- Heisenberg Uncertainty Principle

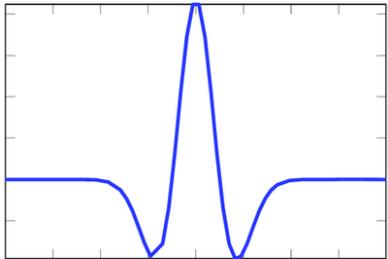
Cannot precisely know what frequency exists at what time intervals!

$$\Delta x \Delta p \geq \frac{h}{4\pi}$$

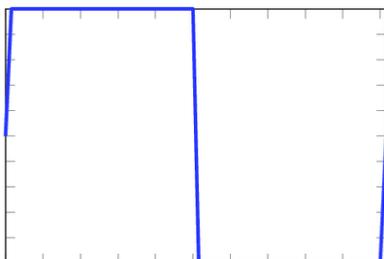
Uncertainty in position      Uncertainty in momentum      A really small number



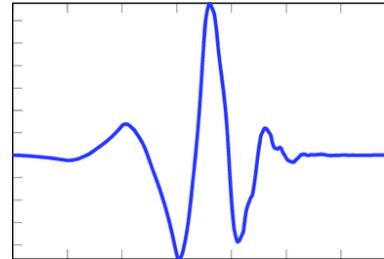
# Wavelets



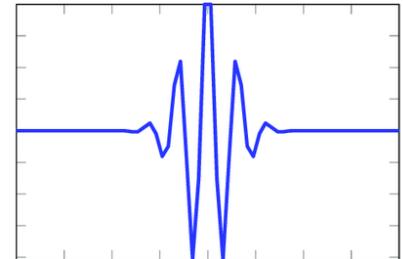
(e) Mexican Hat Wavelet



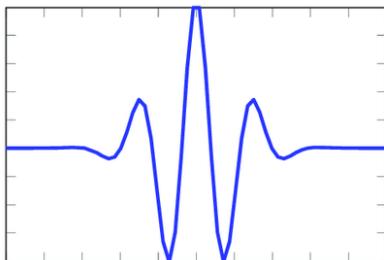
(f) Haar Wavelet



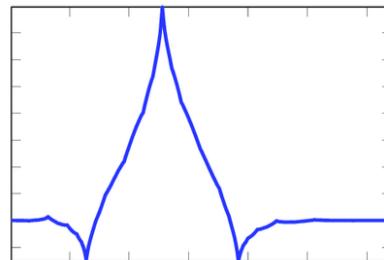
(a) db Wavelet



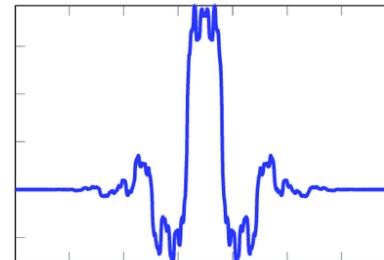
(b) Morlet Wavelet



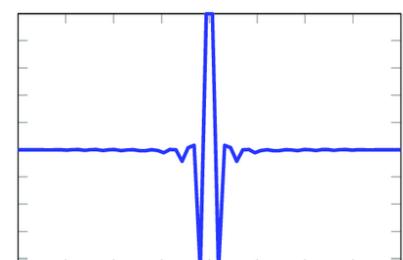
(g) Gaussian Wavelet



(h) Coiflet Wavelet



(c) Biorthogonal Wavelet



(d) Spline Wavelet

- A waveform with a *finite duration*, *average value of 0*, and *non-zero norm*
- Different types of wavelets exist
- Wavelet transform (WT)

# Uncertainty Principle

- We cannot know exactly **what frequency exists at what time instance**, but we can only know **what frequency bands exist at what time intervals.**
- This is a problem of resolution, and it is the main reason why WT was invented:
  - ✓ *STFT gives a fixed resolution at all times*
  - ✓ *Whereas WT gives a variable resolution.*

# The Continuous Wavelet Transform (CWT)

- The kernel functions used in Wavelet transform are all obtained from one prototype function, by scaling and translating the prototype function.
- This prototype is called the **Mother wavelet**

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)$$

Mother wavelet

Scale parameter

Translation parameter

$$CWT_x^{(\psi)}(a,b) = W(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \cdot \psi^*\left(\frac{t-b}{a}\right) dt$$

CWT of  $f(t)$  at scale  $a$  and translation  $b$

Normalization factor

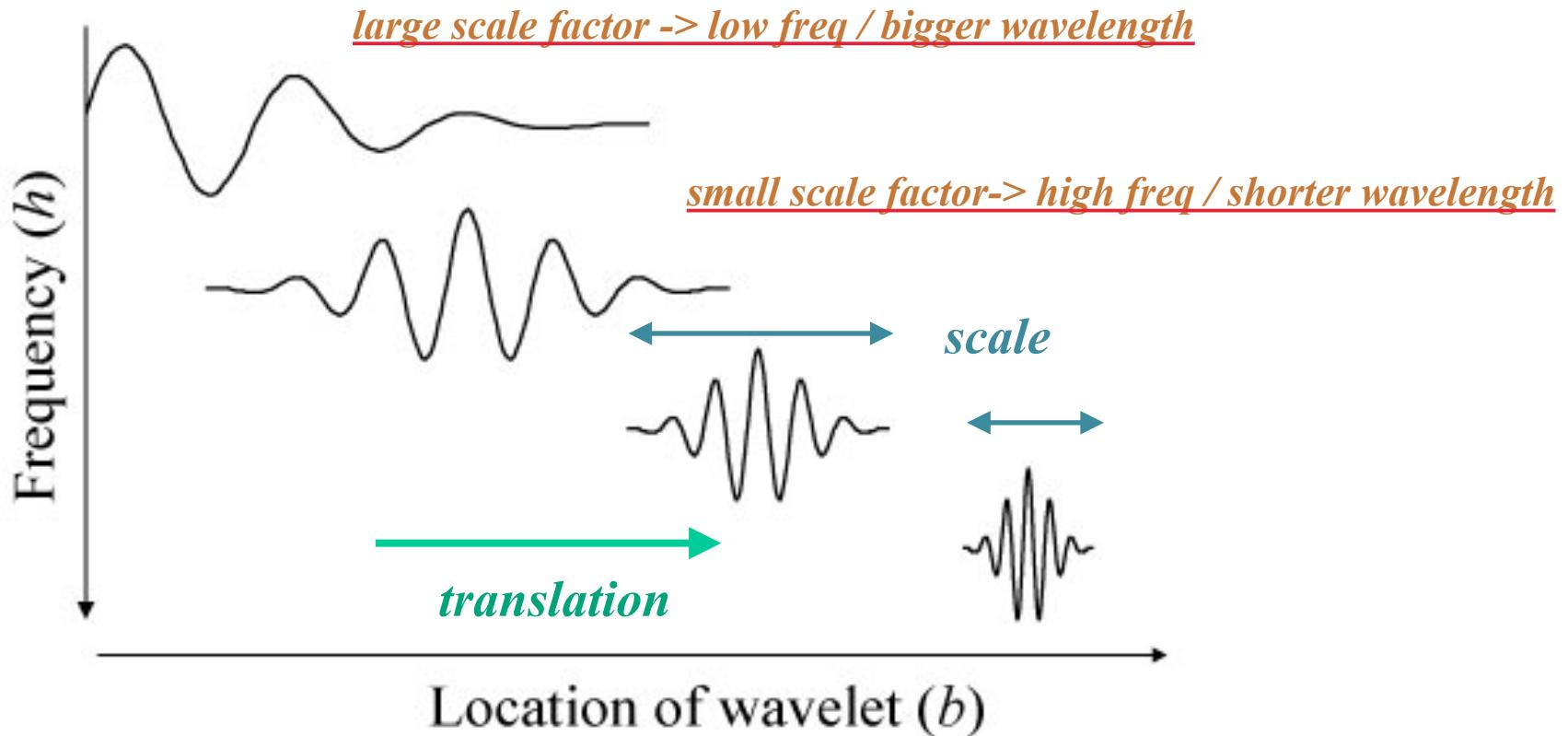
translation

Scaling

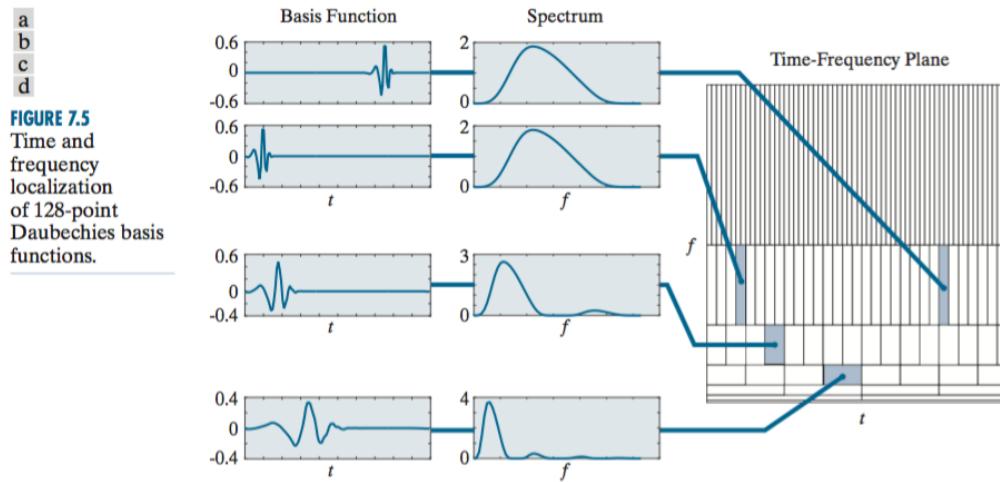
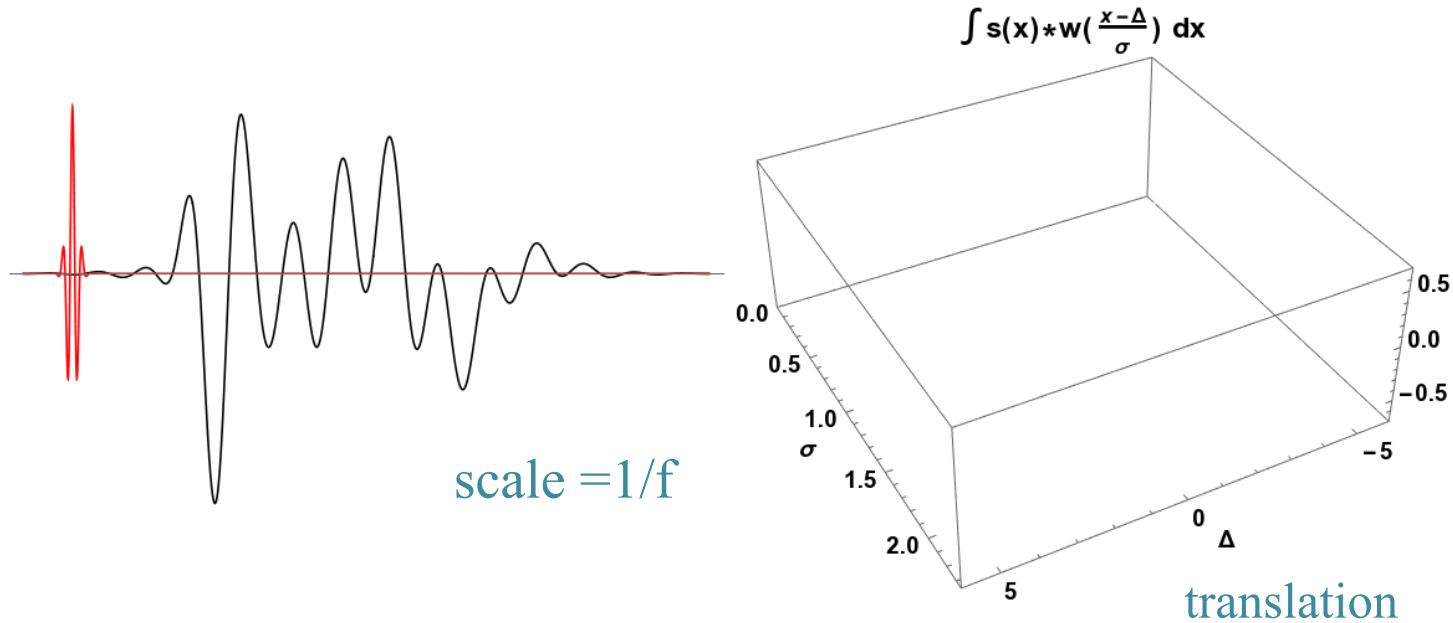
low scale → high frequency

*a and b are continuous parameters*

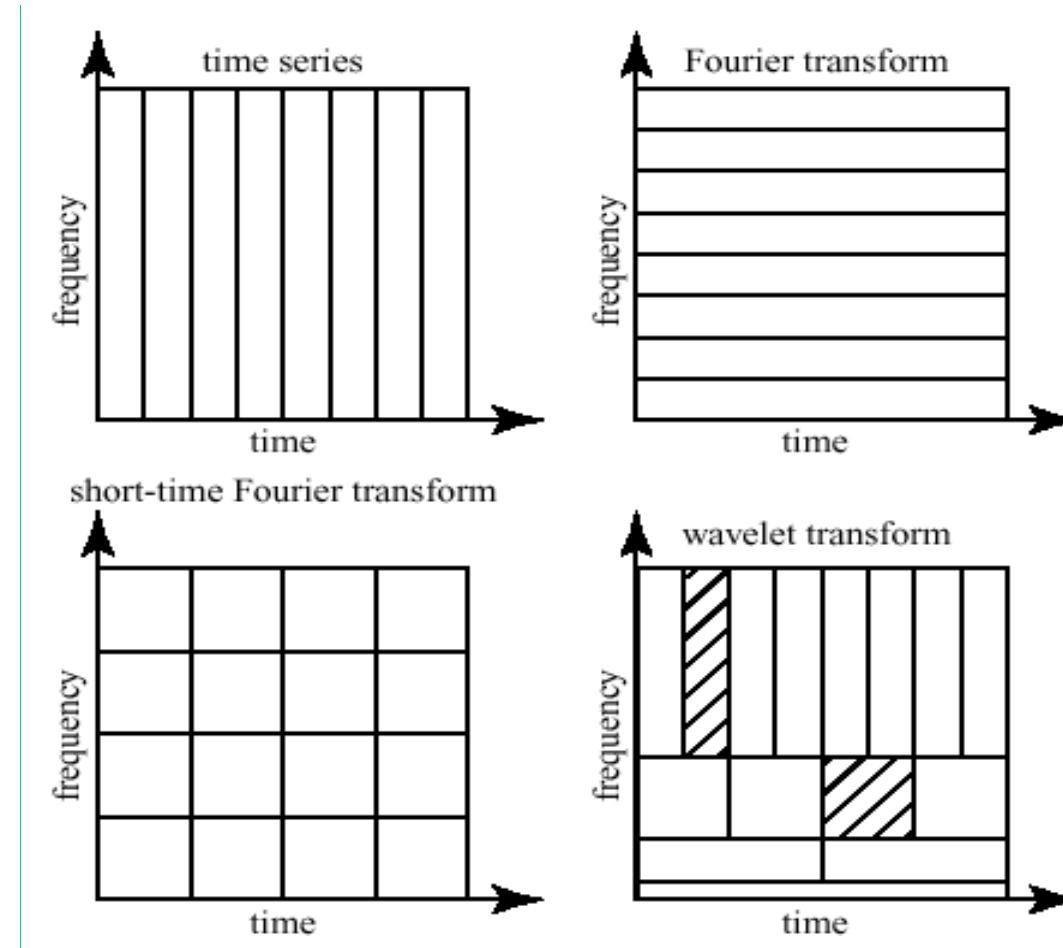
# Wavelets



# Wavelets



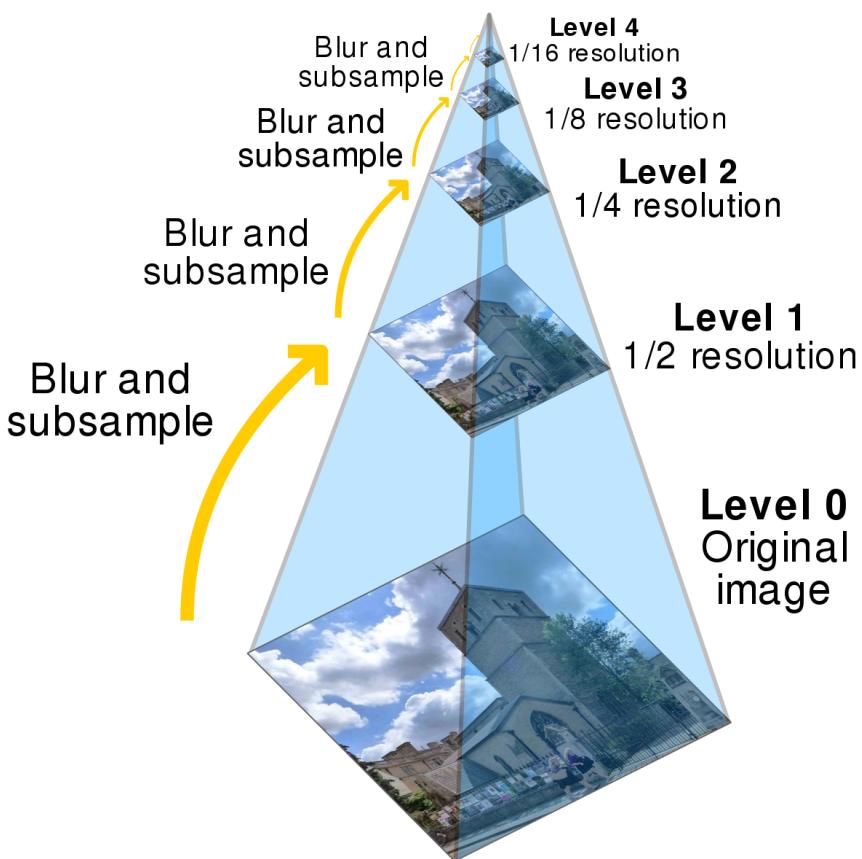
# Comparison of Transformations



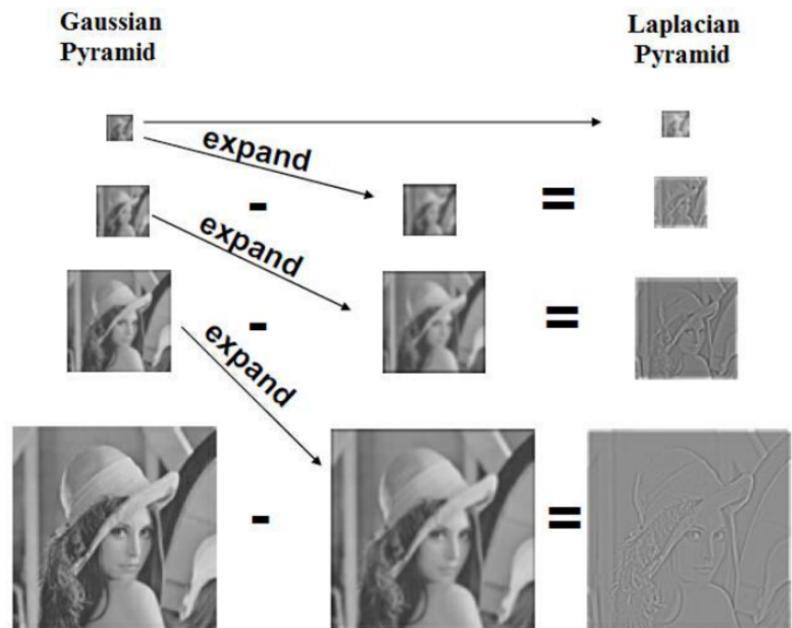
From [http://www.cerm.unifi.it/EUcourse2001/Gunther\\_lecturenotes.pdf](http://www.cerm.unifi.it/EUcourse2001/Gunther_lecturenotes.pdf), p.10

# Image Pyramid

## Gaussian pyramid



## Laplacian pyramid

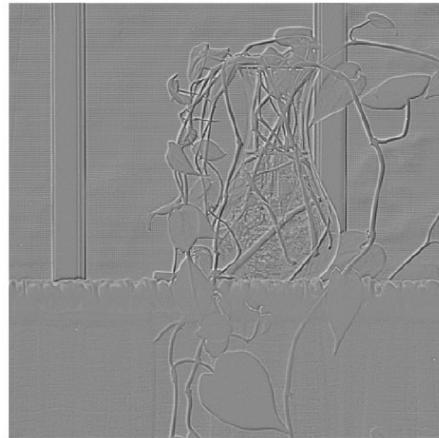
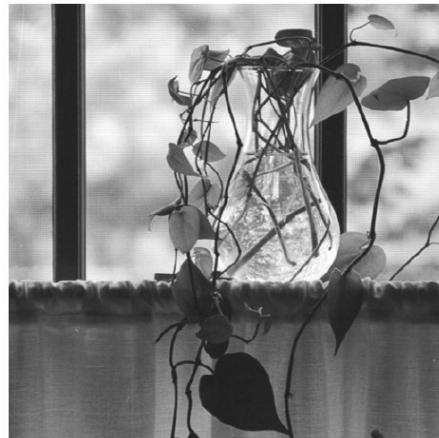
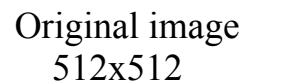


Using Level defined as above then  
Level is the *inverse to Scale*

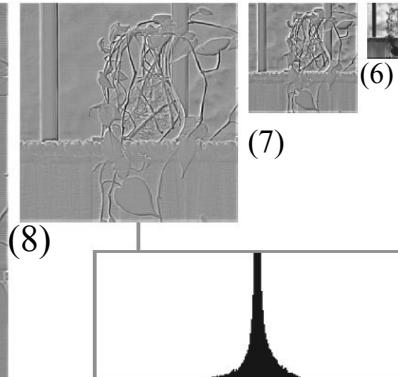
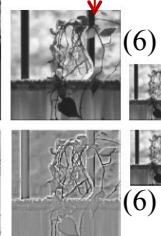
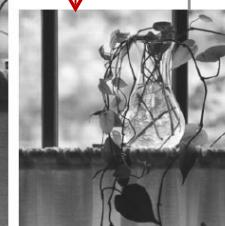
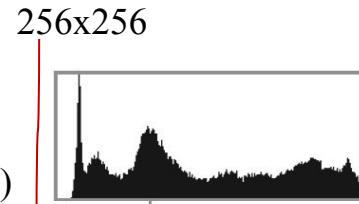
# Image Pyramid

# Gaussian Pyramid

# Laplacian Pyramid using bilinear interpolation



# Three low resolution approximations



a  
b

**FIGURE 7.3**  
Two image pyramids and their histograms:  
(a) an approximation pyramid;  
(b) a prediction residual pyramid.

# Image Pyramid

Scale space representation is very useful in image processing

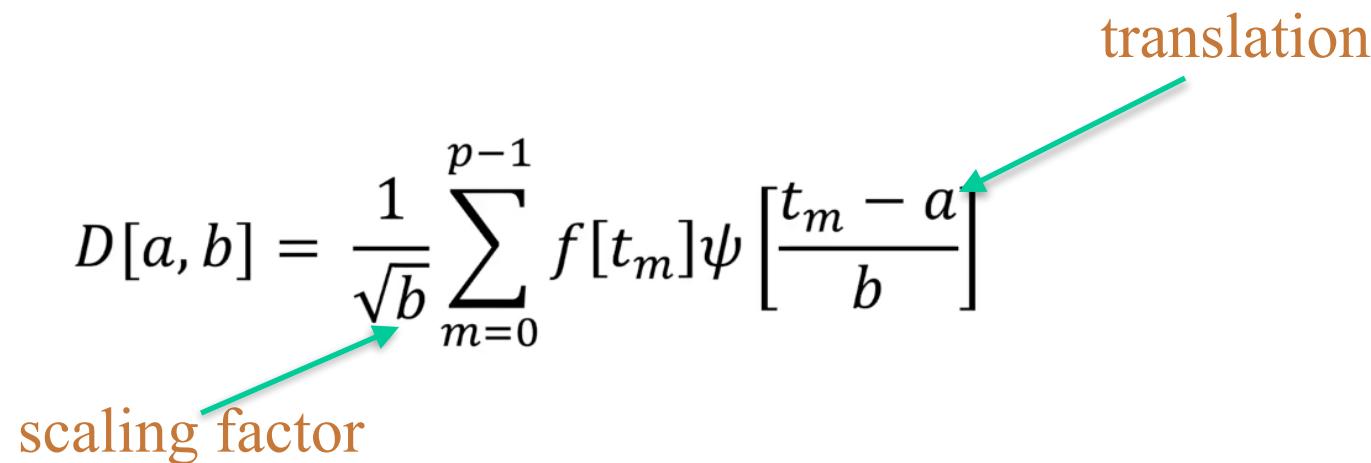
- Extract image features such as edges at multiple scales
- Redundancy reduction and image modelling for
  - ✓ efficient coding
  - ✓ image enhancement/restoration
  - ✓ image analysis/synthesis

# Discrete Wavelet transform (DWT)

$$D[a, b] = \frac{1}{\sqrt{b}} \sum_{m=0}^{p-1} f[t_m] \psi \left[ \frac{t_m - a}{b} \right]$$

translation

scaling factor

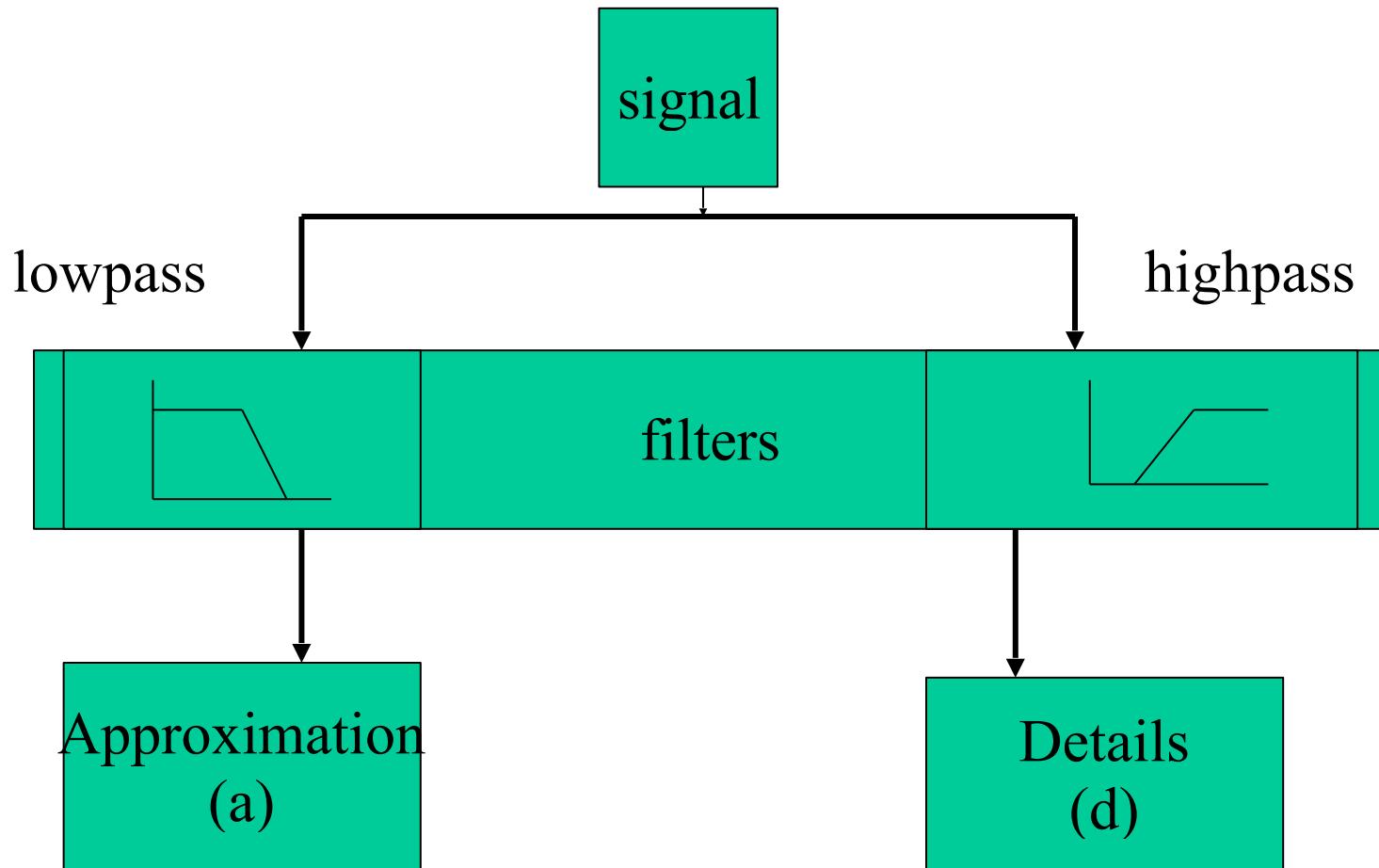


- Use a summation instead of integral
- Integer and fixed scaling (often in the multiple of 2)
- Integer translation: e.g., in the step of 1

*DWT uses the wavelets, together with a single scaling function, to represent a function or image as a linear combination of the wavelets and scaling function.*

We will see the scaling function later

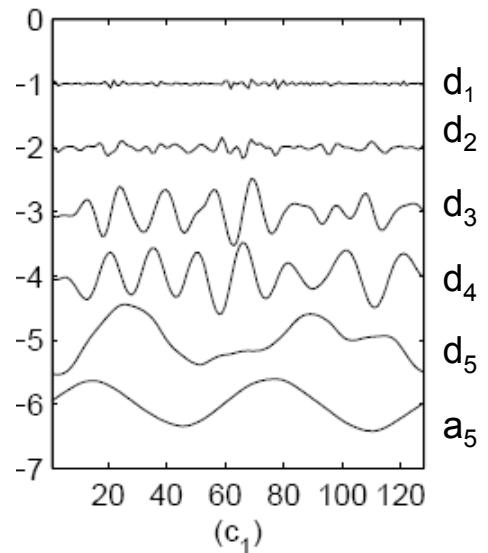
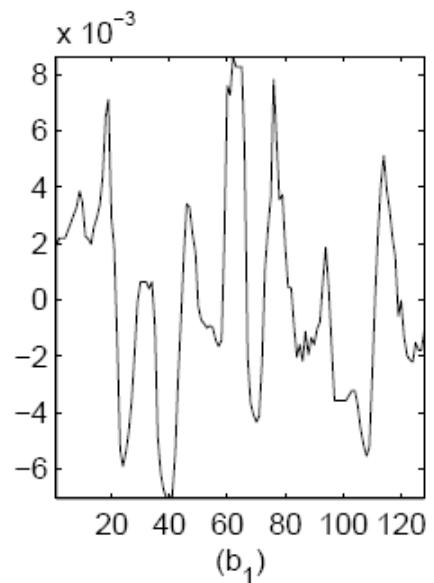
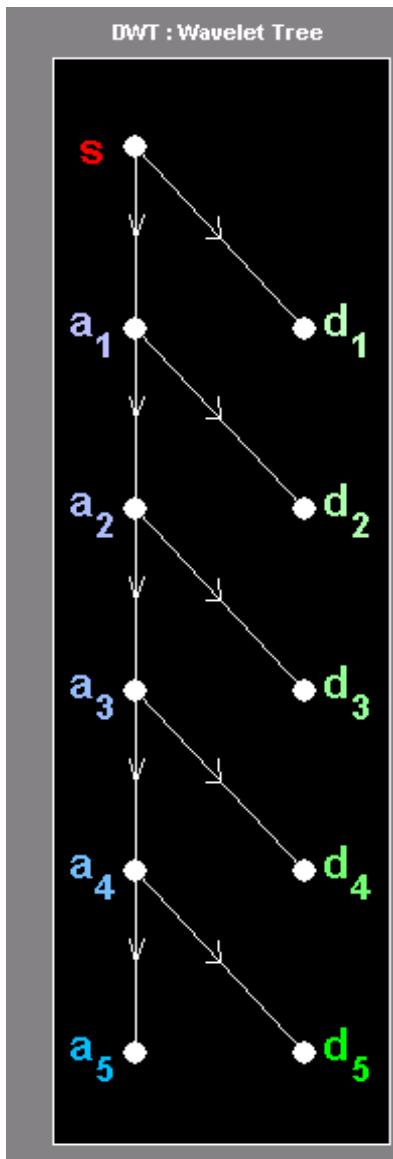
# Discrete Wavelet transform (DWT)



*(we will see more about this later)*

# Levels of decomposition

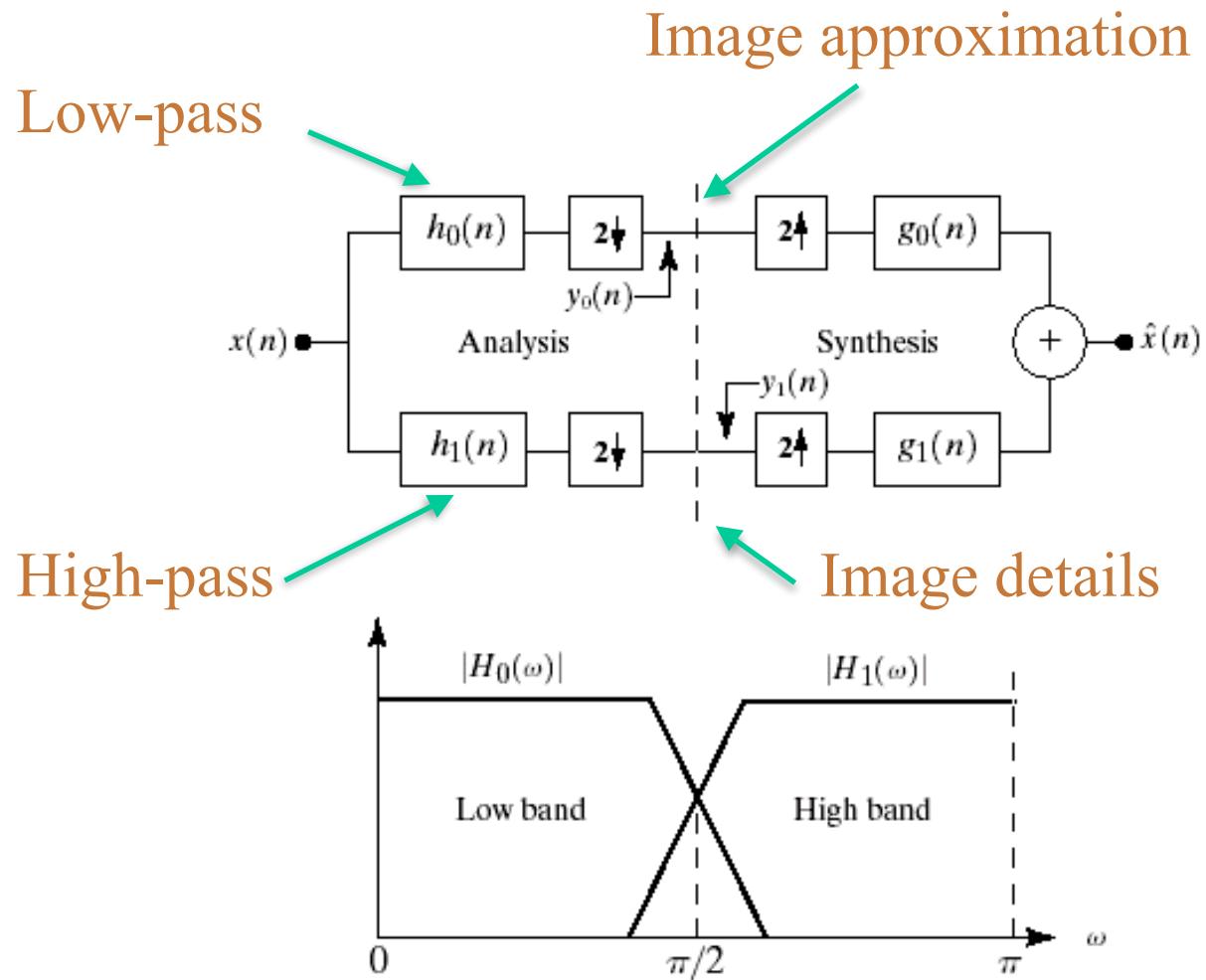
- Successively decompose the approximation
- Level 5 decomposition =  
 $a_5 + d_5 + d_4 + d_3 + d_2 + d_1$
- No limit to the number of decompositions performed



# Sub-band Coding

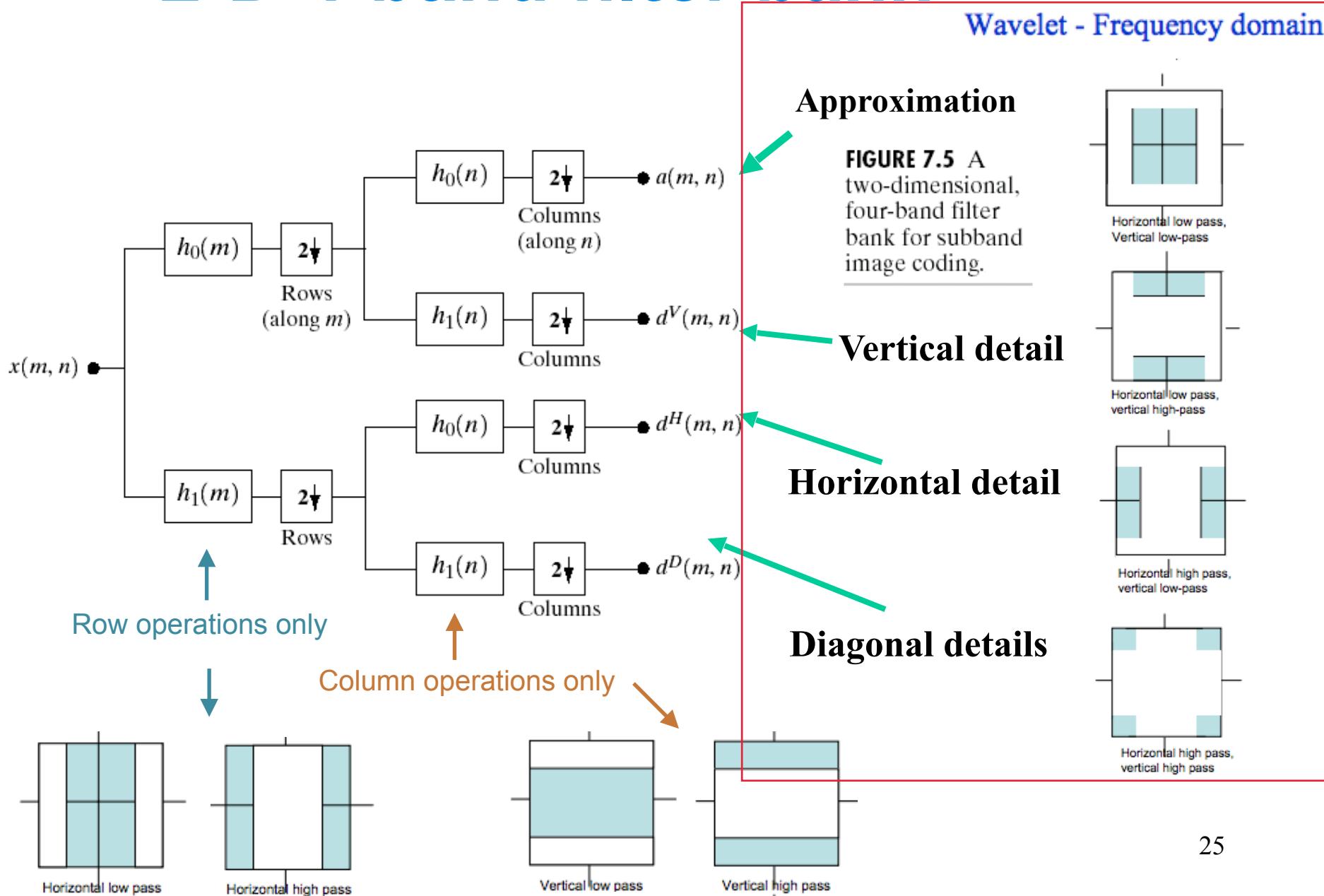
a  
b

**FIGURE 7.4** (a) A two-band filter bank for one-dimensional subband coding and decoding, and (b) its spectrum splitting properties.

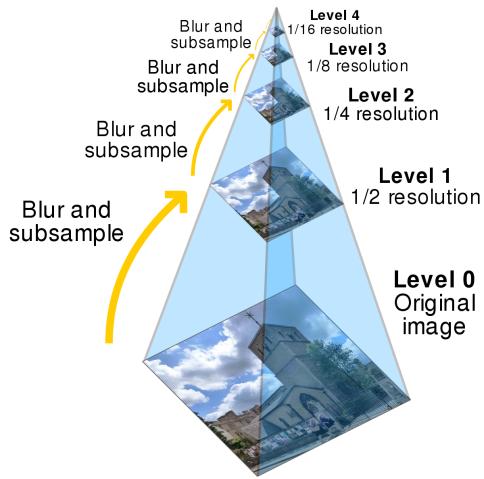


Need to pick the right filters for perfect reconstruction

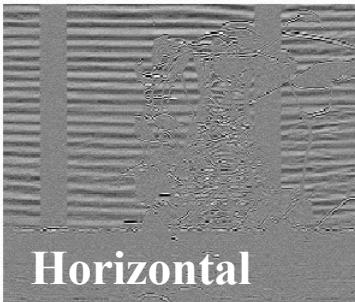
# 2-D 4-band filter bank



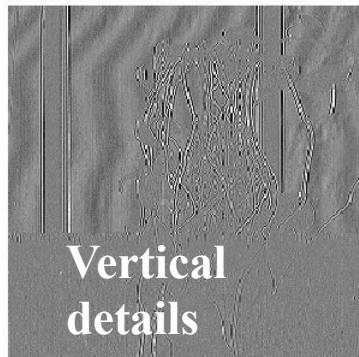
# Sub-band filter bank



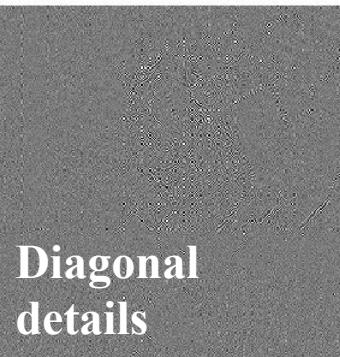
Approximation



Horizontal details

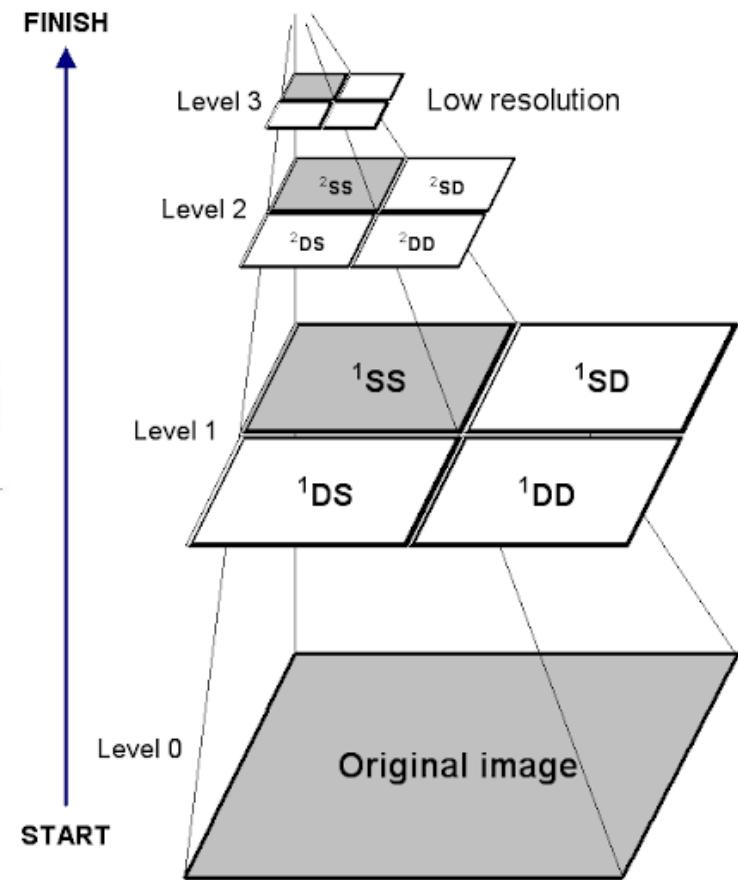


Vertical details

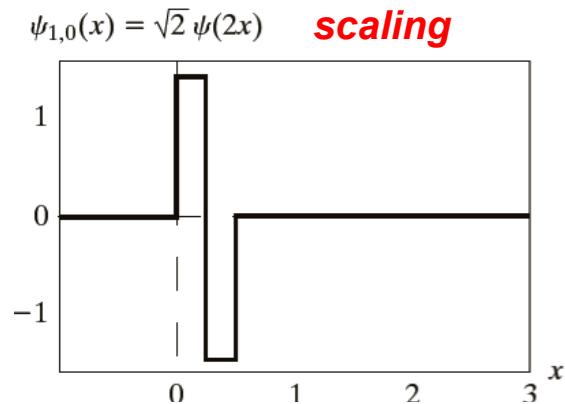
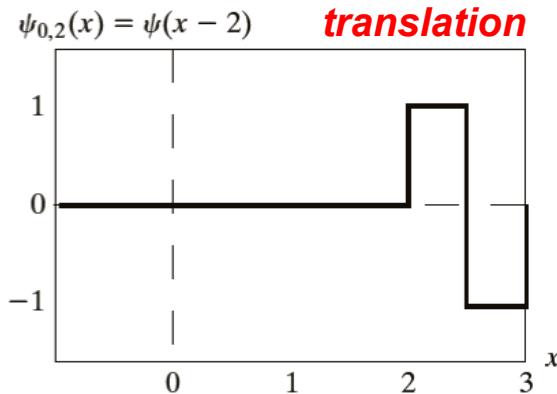
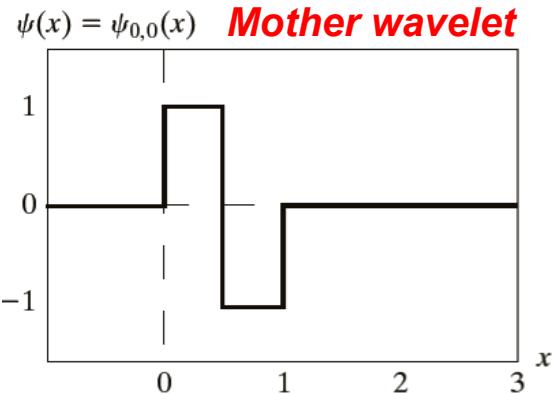


Diagonal details

**FIGURE 7.7** A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.5.



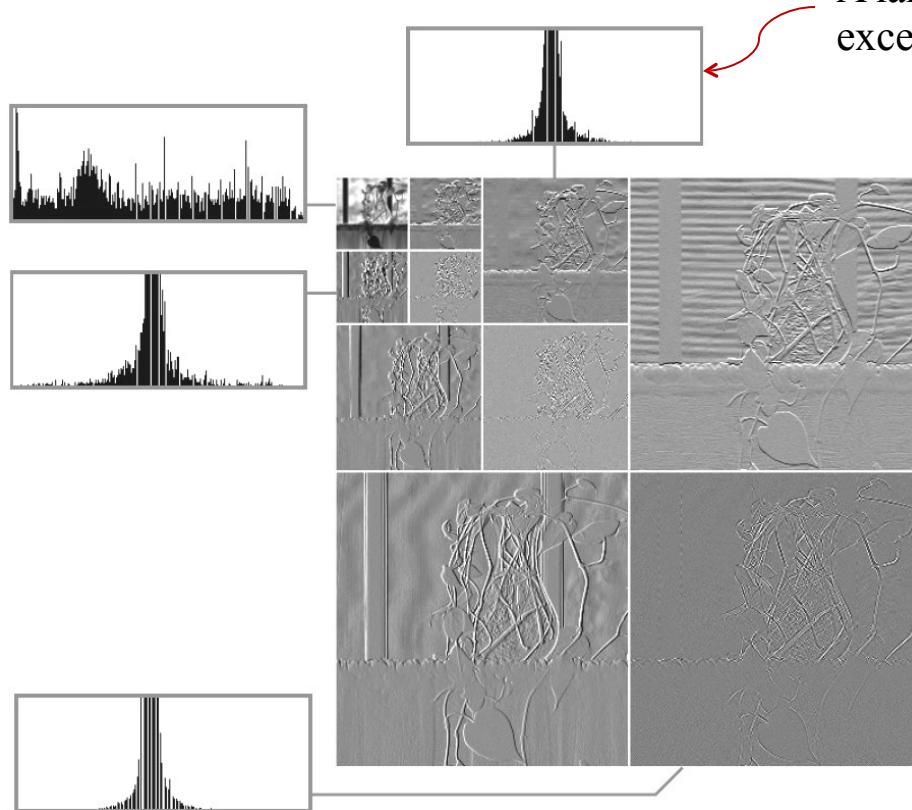
# Haar Transform



a	b
c	d
e	f

**FIGURE 7.14**  
Haar wavelet  
functions in  $W_0$   
and  $W_1$ .

# Haar Transform



a  
b c d

**FIGURE 7.10**  
(a) A discrete wavelet transform using Haar  $\mathbf{H}_2$  basis functions. Its local histogram variations are also shown. (b)–(d) Several different approximations ( $64 \times 64$ ,  $128 \times 128$ , and  $256 \times 256$ ) that can be obtained from (a).

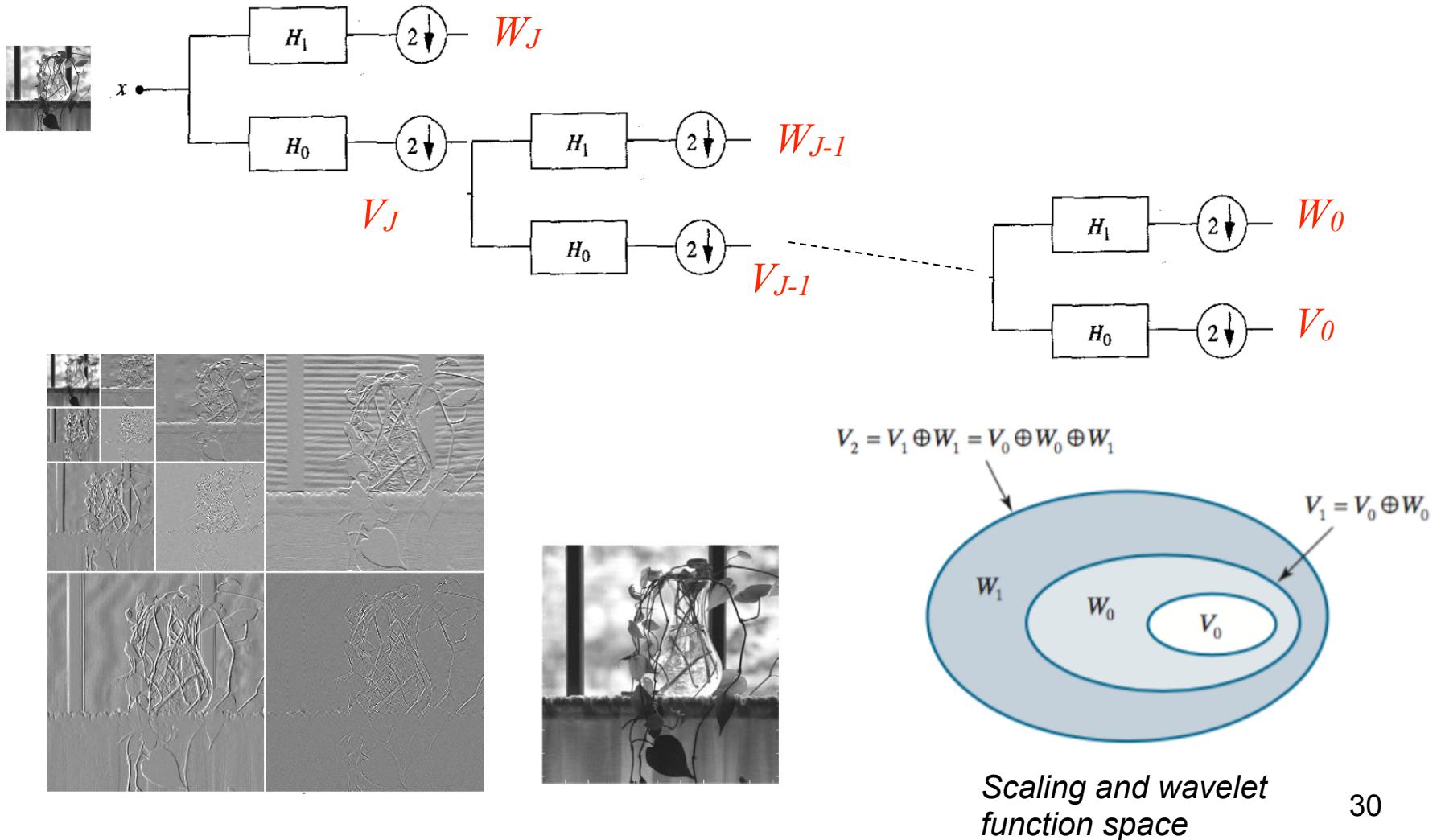




Take a break!

# Multi Resolution Analysis

MRA provides a framework for examining functions at different scales



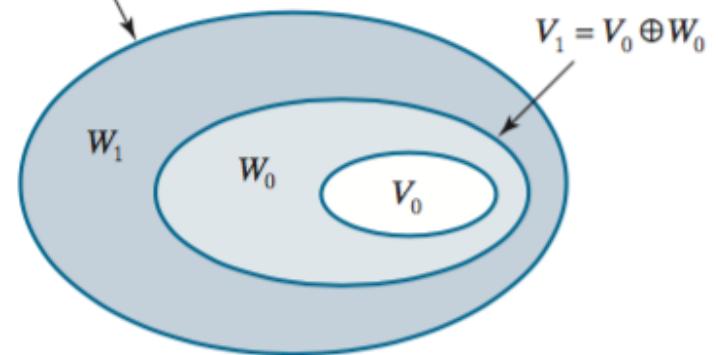
# Multi Resolution Analysis

*Allows inspection of details  
at different levels*

Definition:

A Multi-resolution analysis is a nested sequence of closed subspaces

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1$$



*Scaling and wavelet  
function space*

$$V_j \subset L^2(R) \quad \text{for} \quad j \in \mathbb{Z}$$

$$\dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \dots$$

# Multi Resolution Analysis

(1) The subspaces have a zero intersection

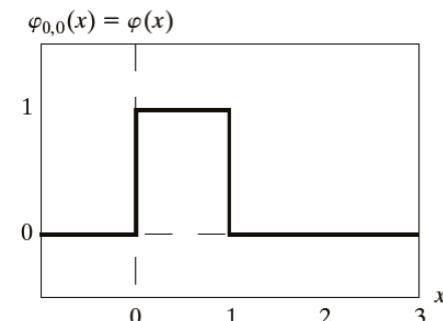
(2) The union is  $L^2(R)$

(3) Two scale relations exist

$$f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1} \quad \forall j \in \mathbb{Z}$$

$$f(x) \in V_0 \Leftrightarrow f(x - k) \in V_0 \quad \forall k \in \mathbb{Z}$$

(4) Finally there exists a **scaling function**  $\phi(x) \in V_0$  whose integer translations  $\{\phi_{0,k} \mid k \in \mathbb{Z}\}$  constitute an orthonormal basis for  $V_0$ .



# MRA: Scaling Function

1. The scaling function  $\phi_{j,k}(x)$ :

$$\int \phi(x) dx = 1$$

$$\phi_{j,k}(x) = 2^{j/2} \phi(2^j x - k) \text{ for all } j, k \in \mathbb{Z}$$

$k$  determines the position of  $\phi_{j,k}(x)$  along the  $x$ -axis,  
 $j$  determines the scale (width) of  $\phi_{j,k}(x)$ , and  $2^{j/2}$  the height  
or amplitude. The shape of  $\phi_{j,k}(x)$  change with  $j$ .

2. The refinement/dilation equation for the scaling function:

$$\varphi(x) = \sum_{k \in \mathbb{Z}} h_\varphi(k) \sqrt{2} \varphi(2x - k) \quad h_\varphi(k) = \langle \varphi(x), \sqrt{2} \varphi(2x - k) \rangle$$

i.e. the expansion functions of any subspace can be built  
from the next higher resolution space

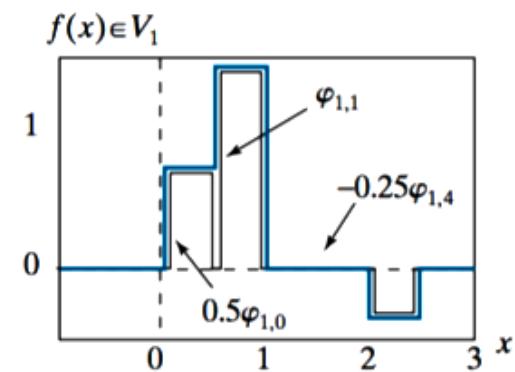
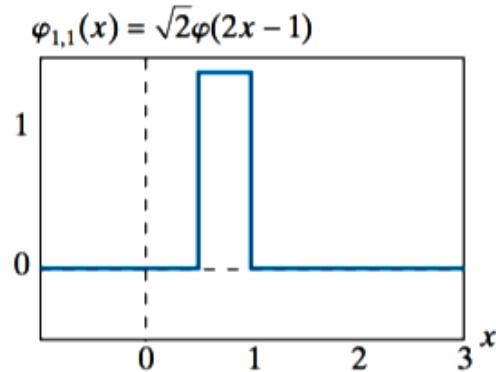
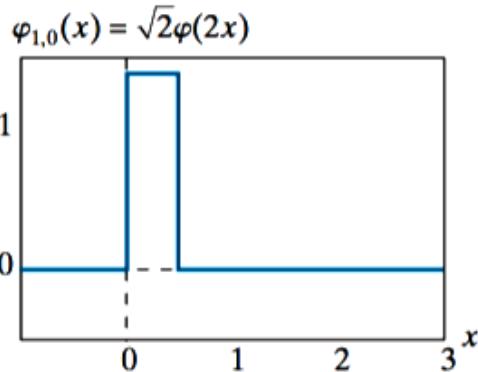
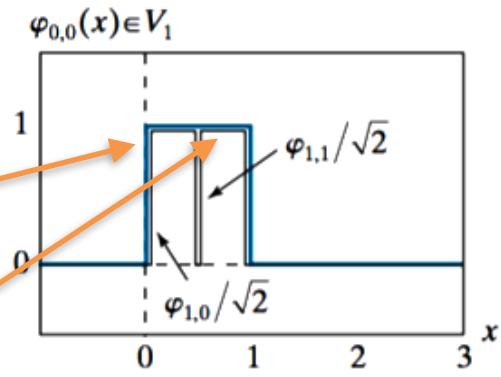
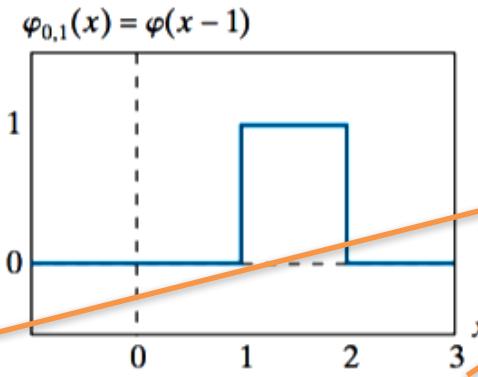
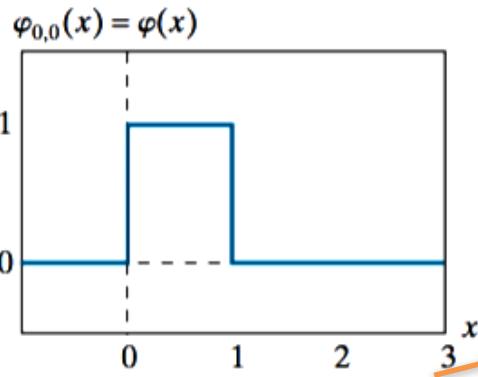
# Fundamental Requirements of MRA

- The scaling function is orthogonal to its integer translate
- The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales
- The only function that is common to all  $V_j$  is  $f(x) = 0$
- Any function can be represented with arbitrary precision

# Haar Scaling Function

Dilation

$$\begin{aligned}\varphi(x) &= \frac{1}{\sqrt{2}} [\sqrt{2}\varphi(2x)] + \frac{1}{\sqrt{2}} [\sqrt{2}\varphi(2x-1)] \\ &= \varphi(2x) + \varphi(2x-1)\end{aligned}$$



a b c  
d e f

FIGURE 7.19 The Haar scaling function.

# Haar Scaling Function

Dilation

$$\begin{aligned}\varphi(x) &= \frac{1}{\sqrt{2}} [\sqrt{2}\varphi(2x)] + \frac{1}{\sqrt{2}} [\sqrt{2}\varphi(2x-1)] \\ &= \varphi(2x) + \varphi(2x-1)\end{aligned}$$

Scaling function  
coefficients:

$$\varphi(x) = \sum_{k \in \mathbb{Z}} h_\varphi(k) \sqrt{2} \varphi(2x - k)$$
$$\{h_\varphi(n) | n = 0, 1\} = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}$$

*n* is the integer position shift

# MRA: Wavelet Functions

*Similar to the Scaling Function*

1. The wavelet functions satisfy:

$$f(x) \in W_j \Leftrightarrow f(2x) \in W_{j+1}$$

Thus the wavelet function  $\psi$  can be expressed as follows:

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k), \quad j = 0, 1, \dots \text{ and } k = 0, 1, \dots, 2^j - 1.$$

2. The dilation equation for the wavelet function:

$$\psi(x) = \sum h_\psi(k) \sqrt{2} \varphi(2x - k)$$

$$\varphi(x) = \sum_{k \in \mathbf{Z}} h_\varphi(k) \sqrt{2} \varphi(2x - k)$$

Complementary scaling and wavelet functions are orthogonal

$$\langle \varphi_{j_0,k}(x), \psi_{j_0,l}(x) \rangle = 0 \quad \text{for } k \neq l$$

Wavelet function coefficients:

$$h_\psi(k) = (-1)^k h_\varphi(1 - k)$$

# Haar Wavelet Functions

$$h_\psi(k) = (-1)^k h_\varphi(1-k)$$

$$\{h_\varphi(n) | n = 0, 1\} = \{1/\sqrt{2}, 1/\sqrt{2}\}$$

$$\psi(x) = \sum_k h_\psi(k) \sqrt{2} \varphi(2x - k)$$

It follows that

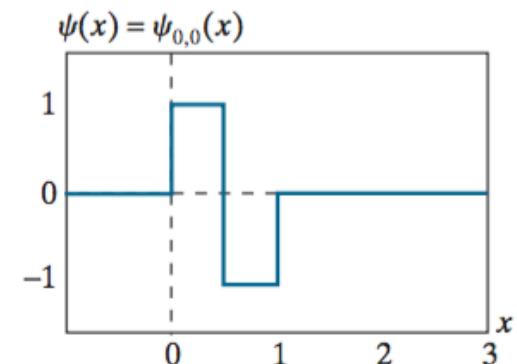
Haar wavelet  
coefficients:

$$h_\psi(k) = \begin{cases} 2^{-1/2} & \text{for } k = 0 \\ -2^{-1/2} & \text{for } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

Haar wavelet:

$$\psi(x) = \phi(2x) - \phi(2x-1) \Rightarrow \psi(x) = \begin{cases} 1 & \text{for } 0 \leq x < \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$



# Haar Wavelet Functions

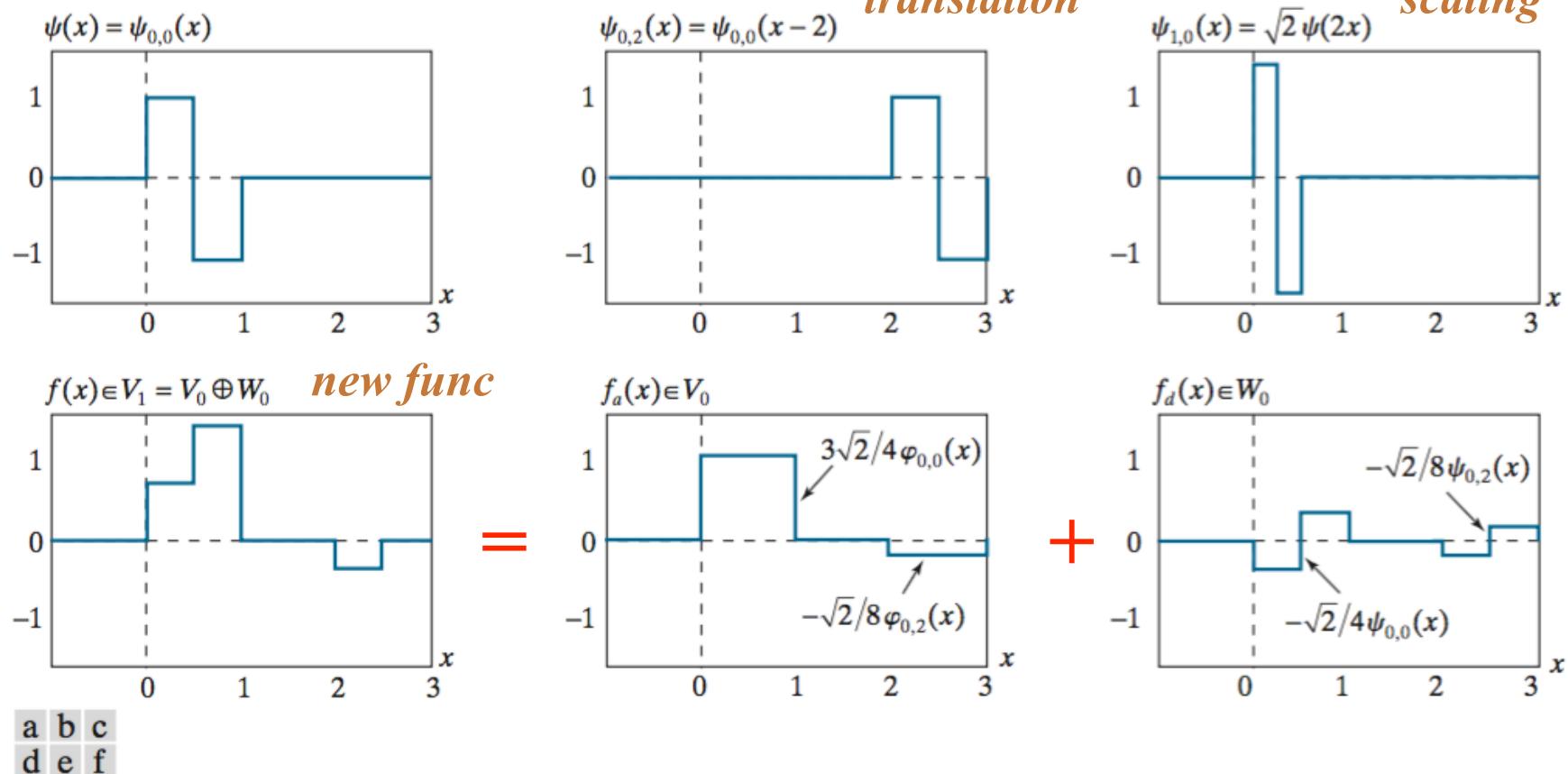


FIGURE 7.21 Haar wavelet functions.

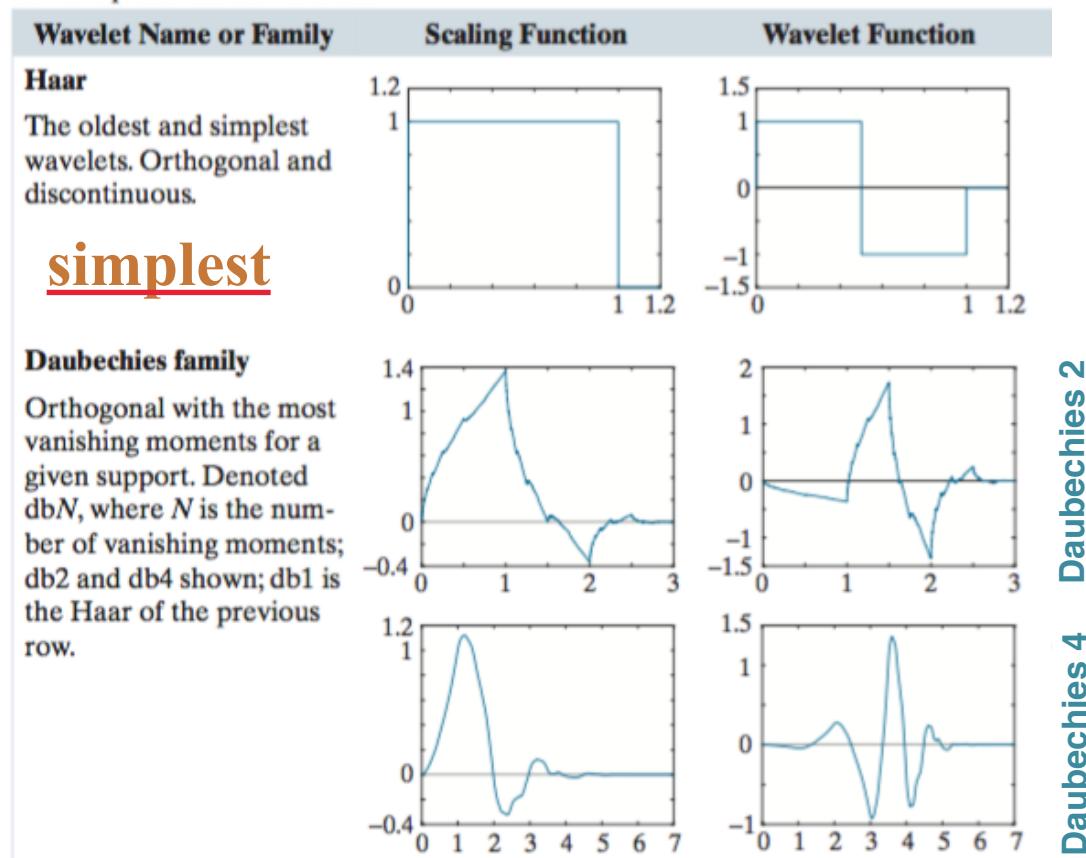
$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k), \quad j, k \in \mathbb{Z}$$

# MRA: Series Expansion

By solving the dilation equation of the scaling function with increasing number of expansions, we are able to obtain other families of wavelets + scaling function

**TABLE 7.1**

Some representative wavelets.



## Scaling function coefficients of Daubechies 4

$$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}$$

$$h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}$$

$$h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}$$

$$h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$$

More vanishing moments,  
More detailed representation

# Wavelet Series Expansion

Express a function with scaling and wavelet functions

$$f(x) = \sum_k c_{j_0}(k) \varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_k d_j(k) \psi_{j,k}(x)$$

*Approximation  
Coefficient*

$$c_{j_0} = \langle f(x), \varphi_{j_0,k}(x) \rangle$$

*Detail  
Coefficient*

$$d_j = \langle f(x), \psi_{j,k}(x) \rangle$$

Example 7.18 in the textbook:  
expand the function  $y$  with Haar wavelets

$$y = \begin{cases} x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$y = \underbrace{\frac{1}{3} \varphi_{0,0}(x)}_{V_0} + \underbrace{\left[ -\frac{1}{4} \psi_{0,0}(x) \right]}_{W_0} + \underbrace{\left[ -\frac{\sqrt{2}}{32} \psi_{1,0}(x) - \frac{3\sqrt{2}}{32} \psi_{1,1}(x) \right]}_{W_1} + \dots$$

$V_1 = V_0 \oplus W_0$

$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1$

# Discrete wavelet transform

In the discrete cases, where we have N-point discrete function and N is a power of 2 ( $N=2^J$ )

$$f(x) = \frac{1}{\sqrt{N}} \left[ T_\varphi(0,0)\varphi(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} T_\psi(j,k)\psi_{j,k}(x) \right]$$

where

Approximation Coefficient

$$\boxed{T_\varphi(0,0)} = \langle f(x), \varphi_{0,0}(x) \rangle = \langle f(x), \varphi(x) \rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x) \varphi^*(x)$$

and

Detail Coefficient

$$\boxed{T_\psi(j,k)} = \langle f(x), \psi_{j,k}(x) \rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x) \psi_{j,k}^*(x)$$

Read Example 7.19 in the textbook with discrete Haar wavelets

# Discrete wavelet transform

Read Example 7.19 in the textbook with discrete Haar wavelets

$$f(0) = 1, f(1) = 4, f(2) = -3, \text{ and } f(3) = 0. \quad N=4$$

$$\mathbf{t}_H = \mathbf{A}_H \mathbf{f} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -1.5\sqrt{2} \\ -1.5\sqrt{2} \end{bmatrix}$$

$$f(x) = \frac{1}{2} [T_\varphi(0,0)\varphi(x) + T_\psi(0,0)\psi_{0,0}(x) + T_\psi(1,0)\psi_{1,0}(x) + T_\psi(1,1)\psi_{1,1}(x)]$$

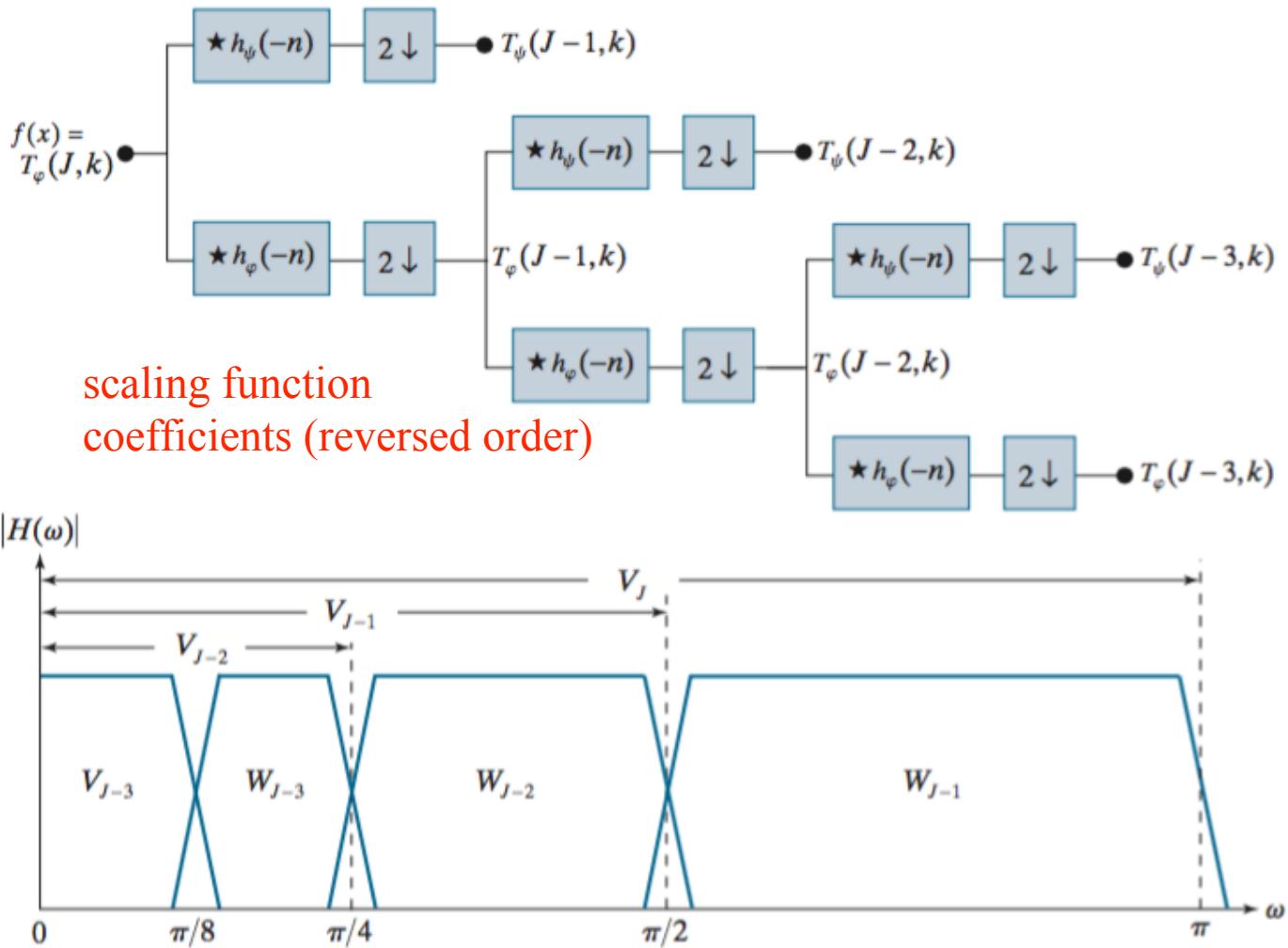
# Fast wavelet transform (FWT)

wavelet coefficients (reversed order)

a  
b

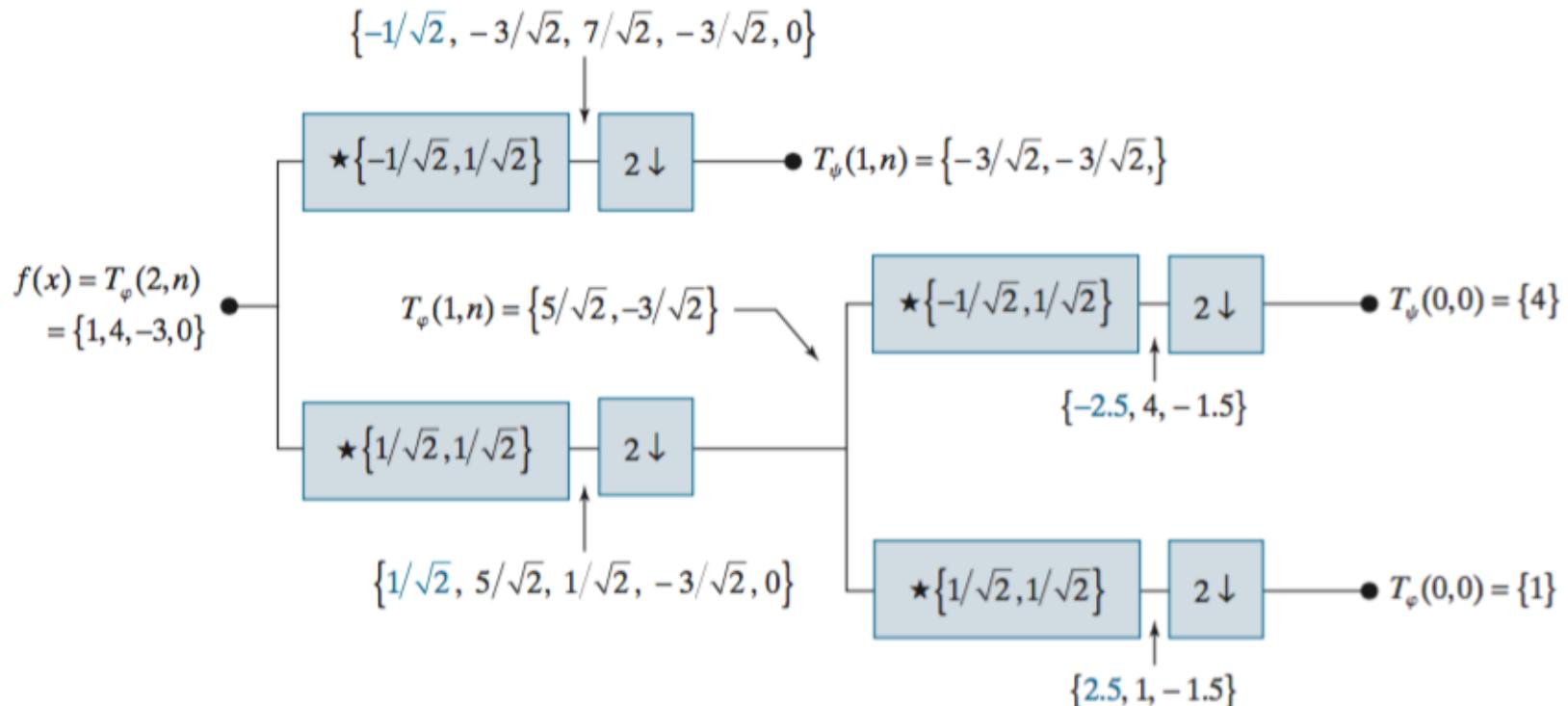
**FIGURE 7.24**

(a) A three-stage or three-scale FWT analysis filter bank and (b) its frequency-splitting characteristics. Because of symmetry in the DFT of the filter's impulse response, it is common to display only the  $[0, \pi]$  region.



Note that with convolution, there is a need to reverse the order of the entries for the filter

# Fast wavelet transform (FWT)



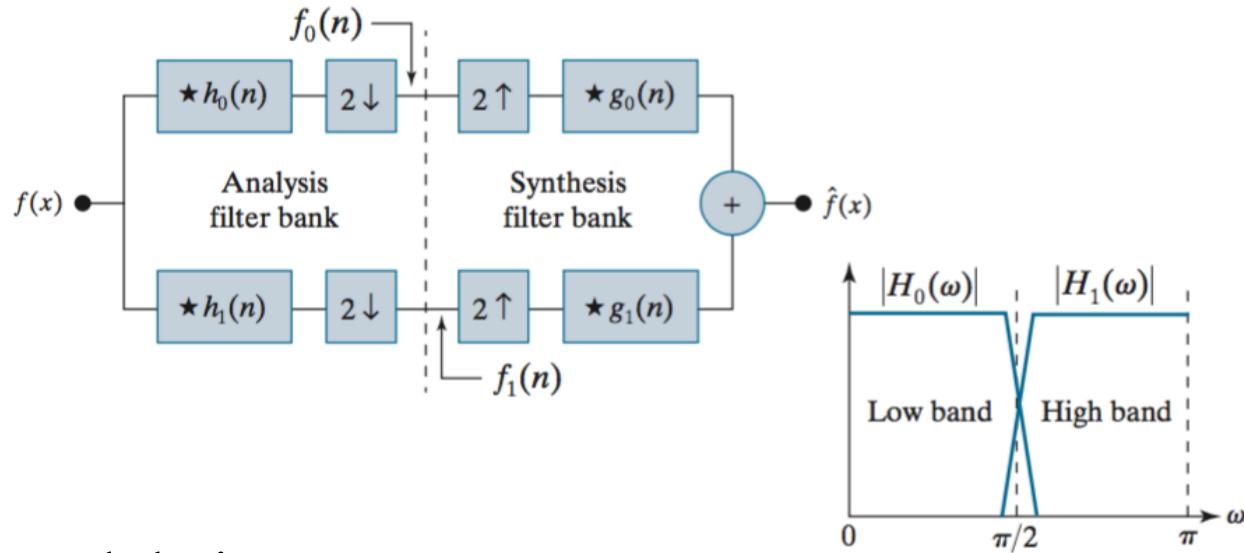
**FIGURE 7.25** Computing a two-scale fast wavelet transform of sequence  $\{1, 4, -3, 0\}$  using Haar scaling and wavelet coefficients.

# Inverse Fast wavelet transform (IFWT)

a b

**FIGURE 7.26**

(a) A two-band digital filtering system for sub-band coding and decoding and  
 (b) its spectrum-splitting properties.



Cross-modulation

$$g_0(n) = (-1)^n h_1(n)$$

$$g_1(n) = (-1)^n h_0(n)$$

$$g_1(n) = (-1)^n g_0(K-1-n)$$

$$h_0(n) = g_0(K-1-n)$$

$$h_1(n) = g_1(K-1-n)$$

# Inverse Fast wavelet transform (IFWT)

Note how filter functions  
change from FWT

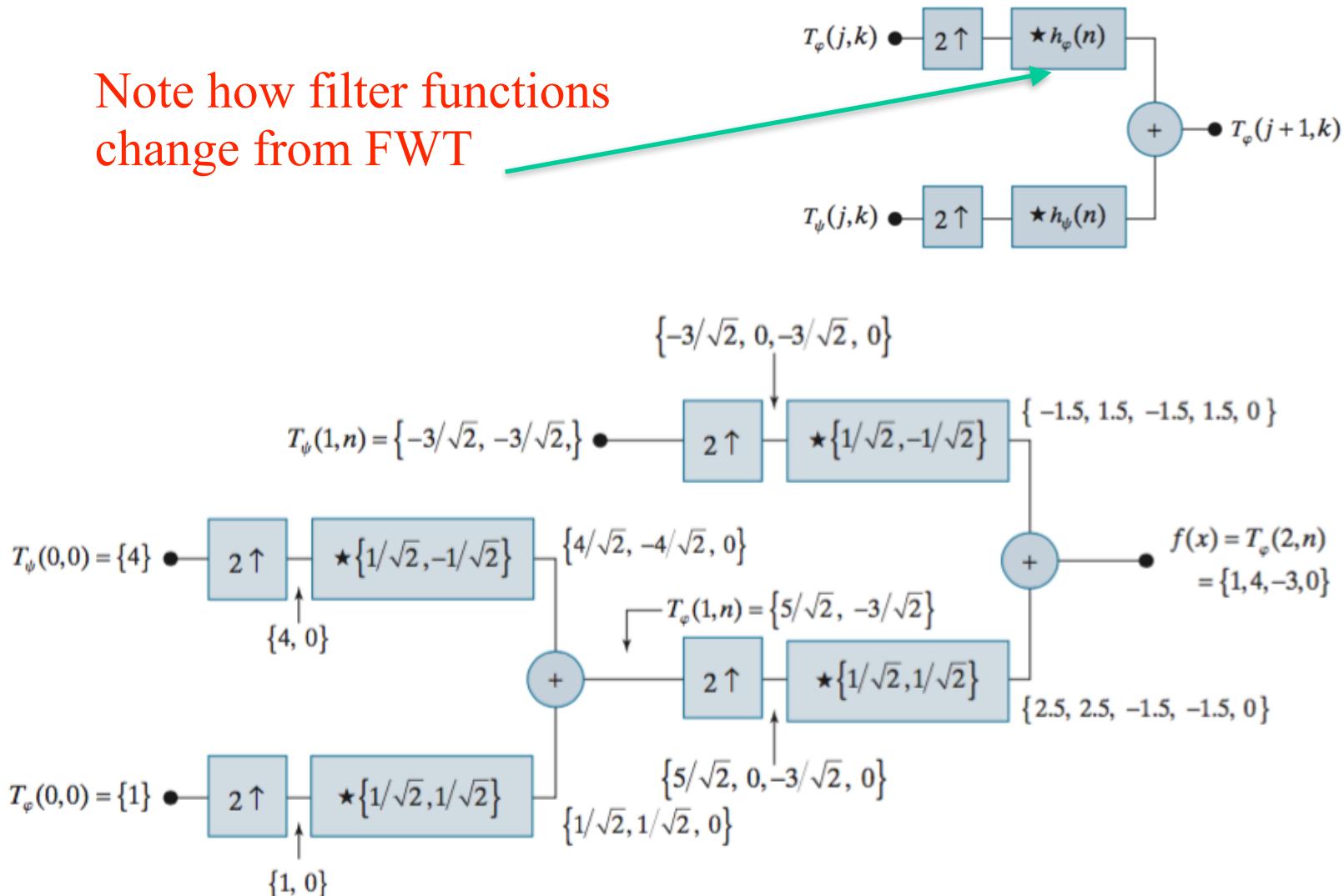
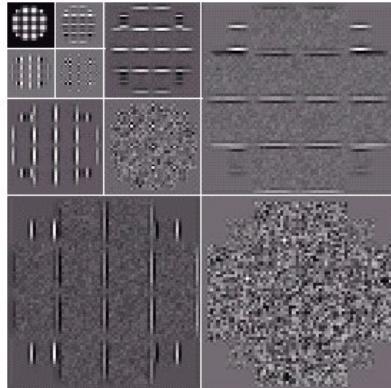
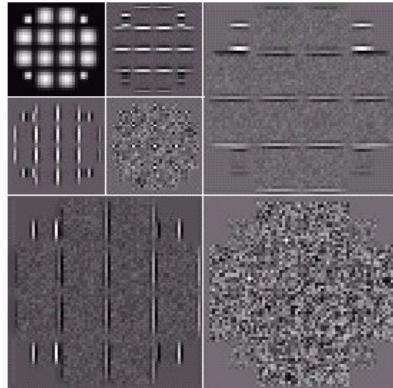
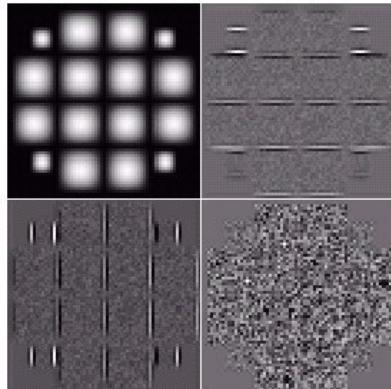
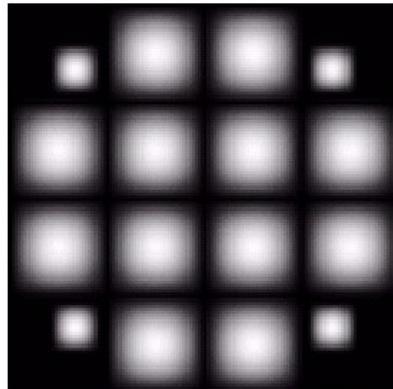


FIGURE 7.28 Computing a two-scale inverse fast wavelet transform of sequence  $\{1, 4, -1.5\sqrt{2}, -1.5\sqrt{2}\}$  with Haar scaling and wavelet functions.

# 2-D Wavelet Transform



a  
b  
c  
d

**FIGURE 7.25**

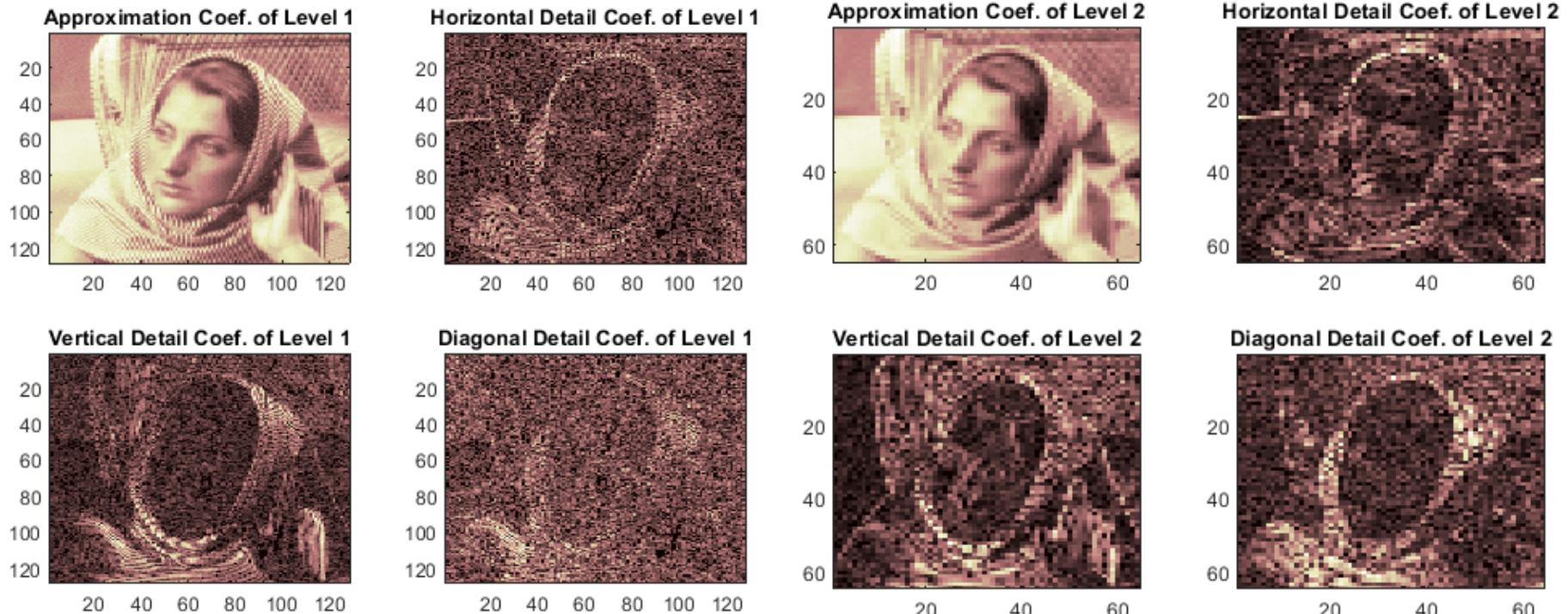
Computing a 2-D three-scale FWT:  
(a) the original image;  
(b) a one-scale FWT;  
(c) a two-scale FWT;  
and (d) a three-scale FWT.

## Wavelet denoising via synthesis



# 2-D DWT DEMO ON MATLAB

Results obtained with wavedec2() function



<https://www.mathworks.com/help/wavelet/ref/wavedec2.html>

# Applications of Wavelets

- Pattern recognition
- Feature extraction
- Image processing
- Biometrics
- Financial analysis
- Perfect reconstruction
- Video compression
- Medical imaging
- etc

# Summary

- Definition and benefits of wavelets
- Image Pyramid
- Filter banks: decomposition & synthesis
- Continuous & discrete wavelet transform
- Scaling and wavelet functions & properties
- Scaling and wavelet function space
- Different wavelet families (Haar, Daubechies)

# Useful Link

- Matlab wavelet tool using guide
- <http://www.wavelet.org>
- <http://www.multires.caltech.edu/teaching/>
- <http://www-dsp.rice.edu/software/RWT/>
- [www.multires.caltech.edu/teaching/courses/waveletcourse/sig95.course.pdf](http://www.multires.caltech.edu/teaching/courses/waveletcourse/sig95.course.pdf)
- <http://www.amara.com/current/wavelet.html>

# Reading

- Chapter 7
- Page 481-483, 504-526

For related math operations and concepts,  
please refer to Page 464-470

# Questions

- Why wavelet transform is needed?
- What are the basic properties of wavelets?
- Wavelet: scale vs. frequency; translation vs. time
- Image Pyramids: two types & why do we need them?
- Wavelets: scaling function vs. wavelet function
- Haar vs. Daubechies-4: which one gives better approximation of images?