

Wavelet transform & Multi-resolution processing

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Materials adapted from Dr. T. D. Bui

Reading

The content of the lecture corresponds to Chapter 7 of the textbook for the following sections

Page 481-483, 504-526

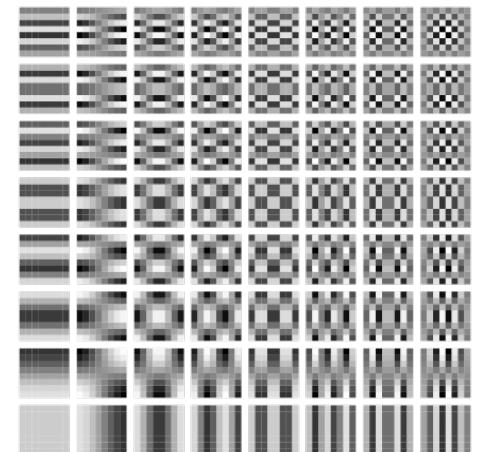
For related math operations and concepts, please refer to Page 464-470

What is a transform?

- A **transform** can be applied to a raw signal to convert it from one domain to another (e.g Fourier).
- Enables the identification of features that may not be as easily observed in the original domain.
- The original image is usually in the time/spatial domain

Transforms

- FT is probably the most popular transform.
- Many other transforms are used quite often by engineers and mathematicians
 - *Hilbert transform*
 - *Discrete Cosine Transform (DCT) -> JPEG format*
 - *Short-time Fourier transform (STFT)*
 - *Wavelet transform (we will see today)*
- Basis functions & coefficients



Basis functions for DCT

Fourier Transform (FT)

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\pi ft} dt$$

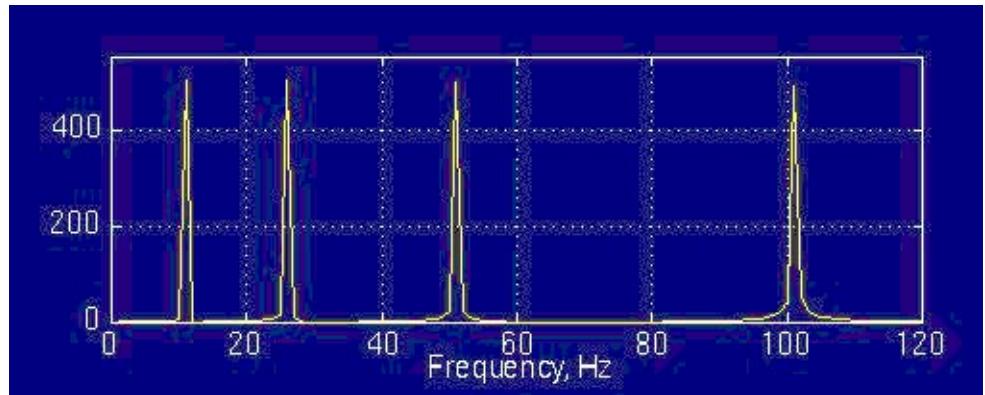
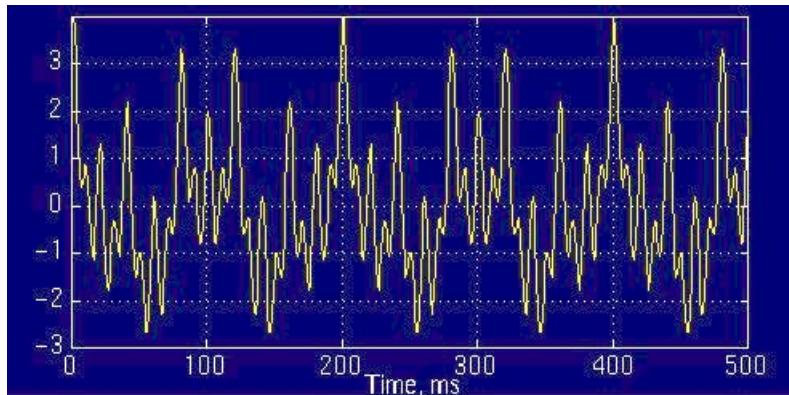
$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{j\pi ft} df$$

- No frequency information is available in the time-domain signal.
 - No time information is available in the Fourier transformed signal.
- ✓ Is it necessary to have both the time and the frequency information at the same time?

Stationary Signals

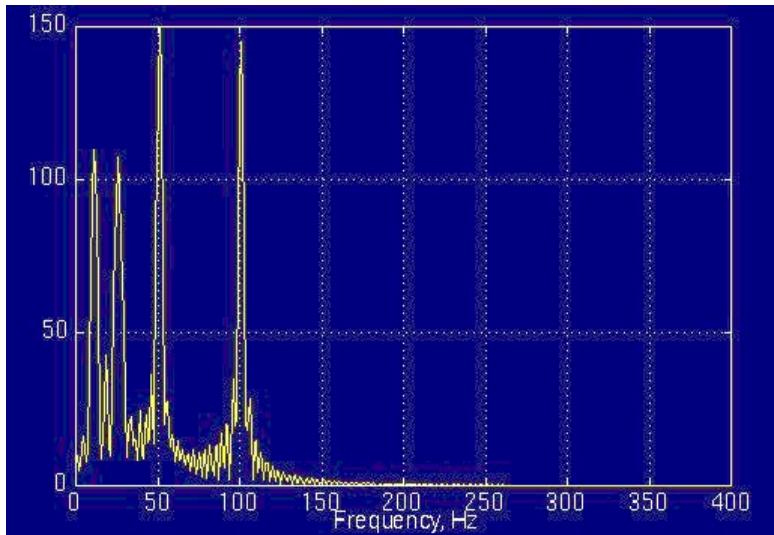
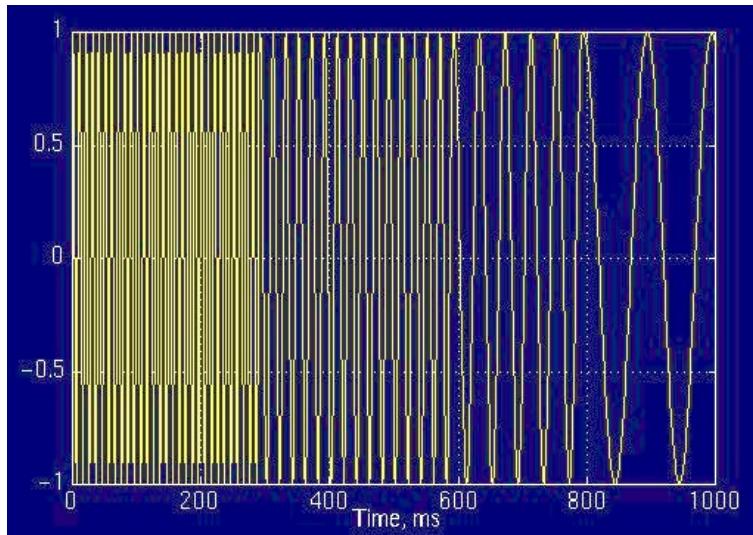
- Signals whose frequency content do not change in time:

$$x(t) = \cos(2\pi \cdot 10 \cdot t) + \cos(2\pi \cdot 25 \cdot t) + \cos(2\pi \cdot 50 \cdot t) + \cos(2\pi \cdot 100 \cdot t)$$

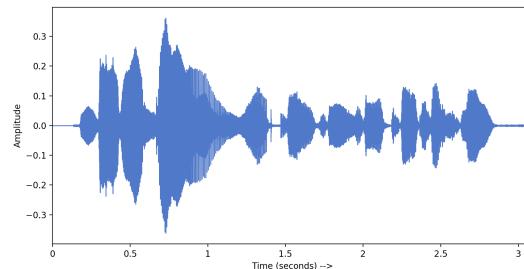


Multiple Freq Presented at Different Time

– non-stationary



The interval 0 to 300 ms has a **100 Hz** sinusoid, the interval 300 to 600 ms has a **50 Hz** sinusoid, the interval 600 to 800 ms has a **25 Hz** sinusoid, and finally the interval 800 to 1000 ms has a **10 Hz** sinusoid



Time-Frequency Relationship

- FT is not a suitable technique for **non-stationary** signals.
- FT gives what frequency components (spectral components) exist in the signal - Nothing more, nothing less.
- When the **time localization** of the spectral components are needed, a transform giving the *TIME-FREQUENCY REPRESENTATION* of the signal is needed.

Short-Time Fourier Transform

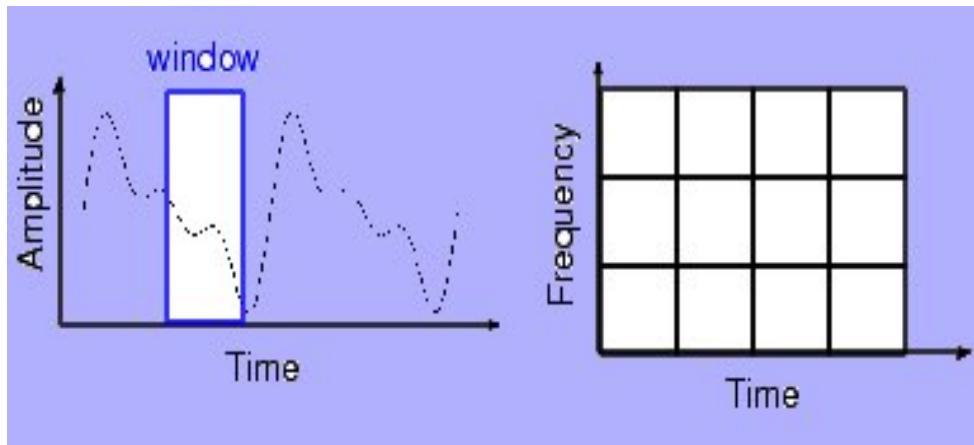
- In STFT, the signal is divided into small enough segments, where these segments of the signal can be assumed to be stationary.
- A window function "**w**" is chosen. The width of this window must be equal to the segment of the signal where its stationarity is valid.

STFT

$$\text{STFT}_x^{(\omega)}(t', f) = \int_t [x(t)\omega(t-t')]e^{-j2\pi ft}dt$$

$\omega(t)$: the window function

A function of time and frequency:



Drawbacks of STFT

- Unchanged window.
- Dilemma of resolution:

Narrow window => poor frequency resolution

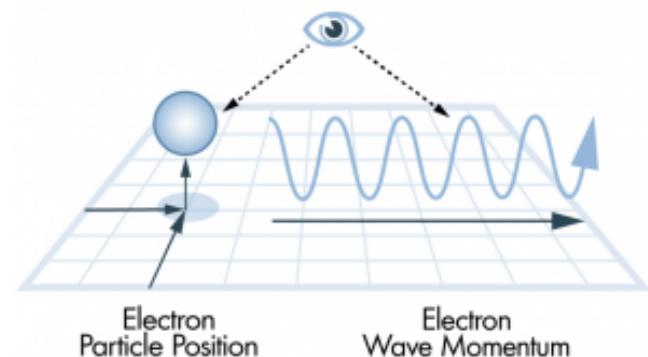
Wide window => poor time resolution

- Heisenberg Uncertainty Principle

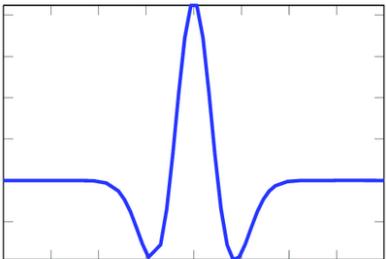
Cannot precisely know what frequency exists at what time intervals!

$$\Delta x \Delta p \geq \frac{h}{4\pi}$$

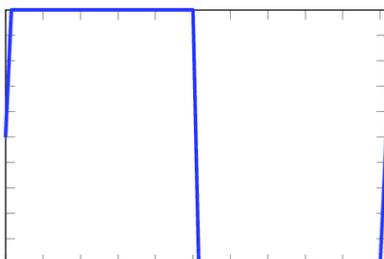
Uncertainty in position Uncertainty in momentum A really small number



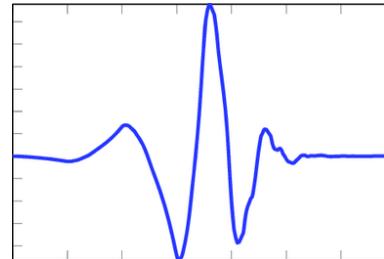
Wavelets



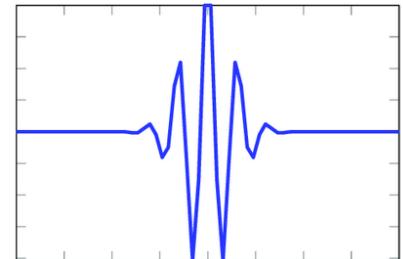
(e) Mexican Hat Wavelet



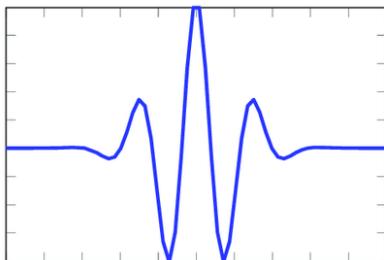
(f) Haar Wavelet



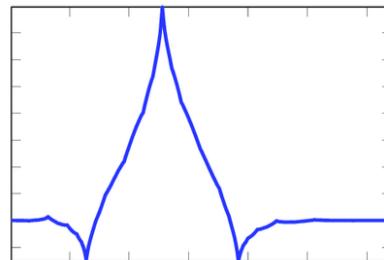
(a) db Wavelet



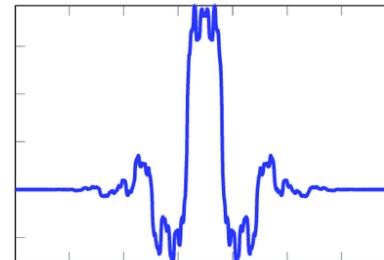
(b) Morlet Wavelet



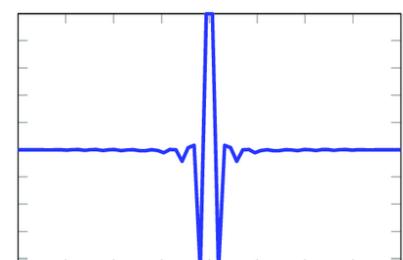
(g) Gaussian Wavelet



(h) Coiflet Wavelet



(c) Biorthogonal Wavelet



(d) Spline Wavelet

- A waveform with a *finite duration*, *average value of 0*, and *non-zero norm*
- Different types of wavelets exist
- Wavelet transform (WT)

Uncertainty Principle

- We cannot know exactly **what frequency exists at what time instance**, but we can only know **what frequency bands exist at what time intervals.**
- This is a problem of resolution, and it is the main reason why WT was invented:
 - ✓ *STFT gives a fixed resolution at all times*
 - ✓ *Whereas WT gives a variable resolution.*

The Continuous Wavelet Transform (CWT)

- The kernel functions used in Wavelet transform are all obtained from one prototype function, by scaling and translating the prototype function.
- This prototype is called the **Mother wavelet**

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)$$

Mother wavelet

Scale parameter

Translation parameter

$$CWT_x^{(\psi)}(a,b) = W(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \cdot \psi^*\left(\frac{t-b}{a}\right) dt$$

CWT of $f(t)$ at scale a and translation b

Normalization factor

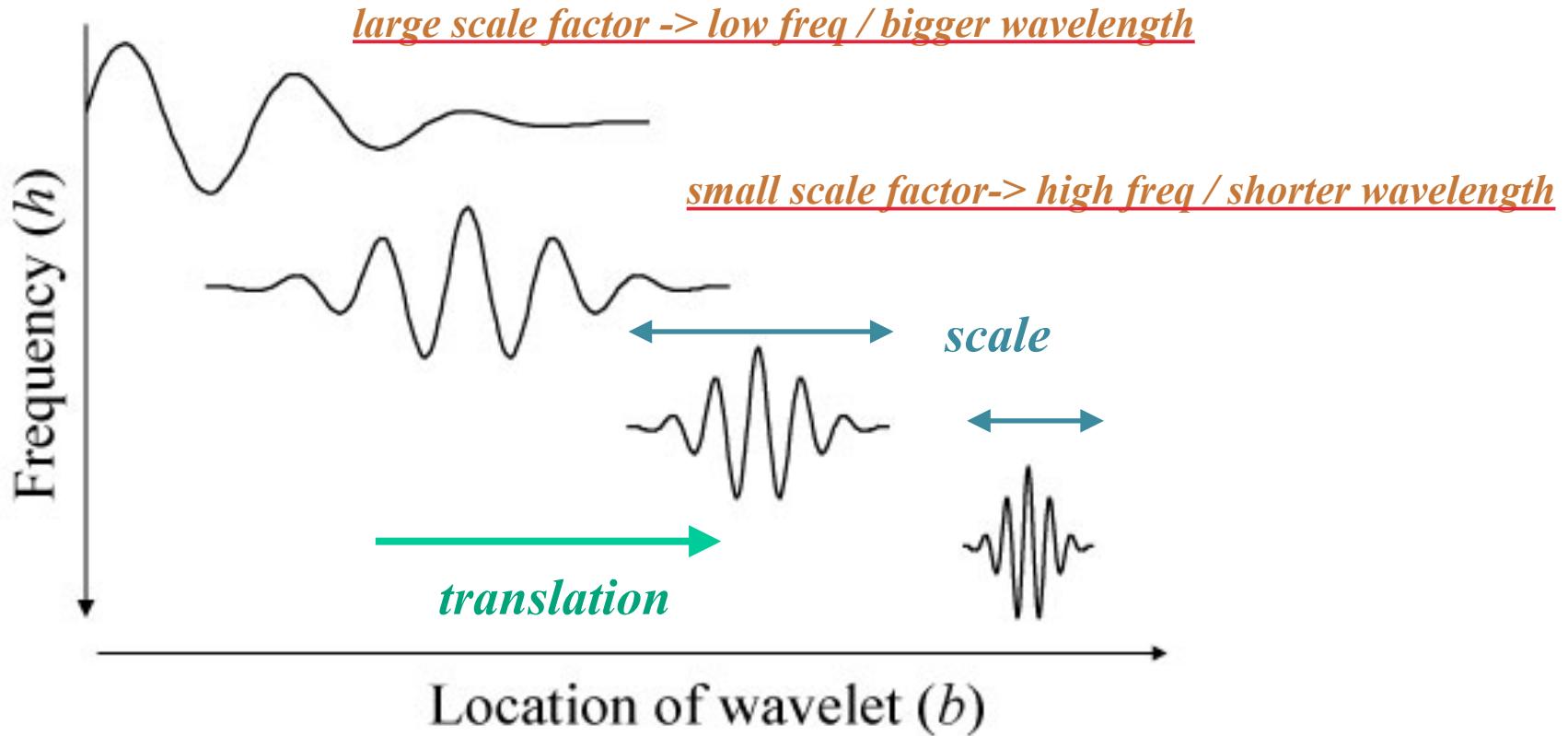
translation

Scaling

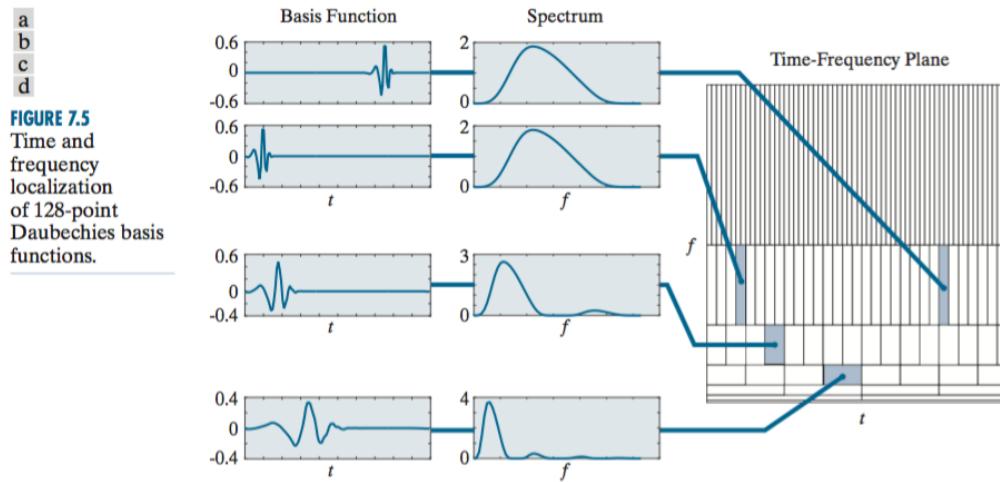
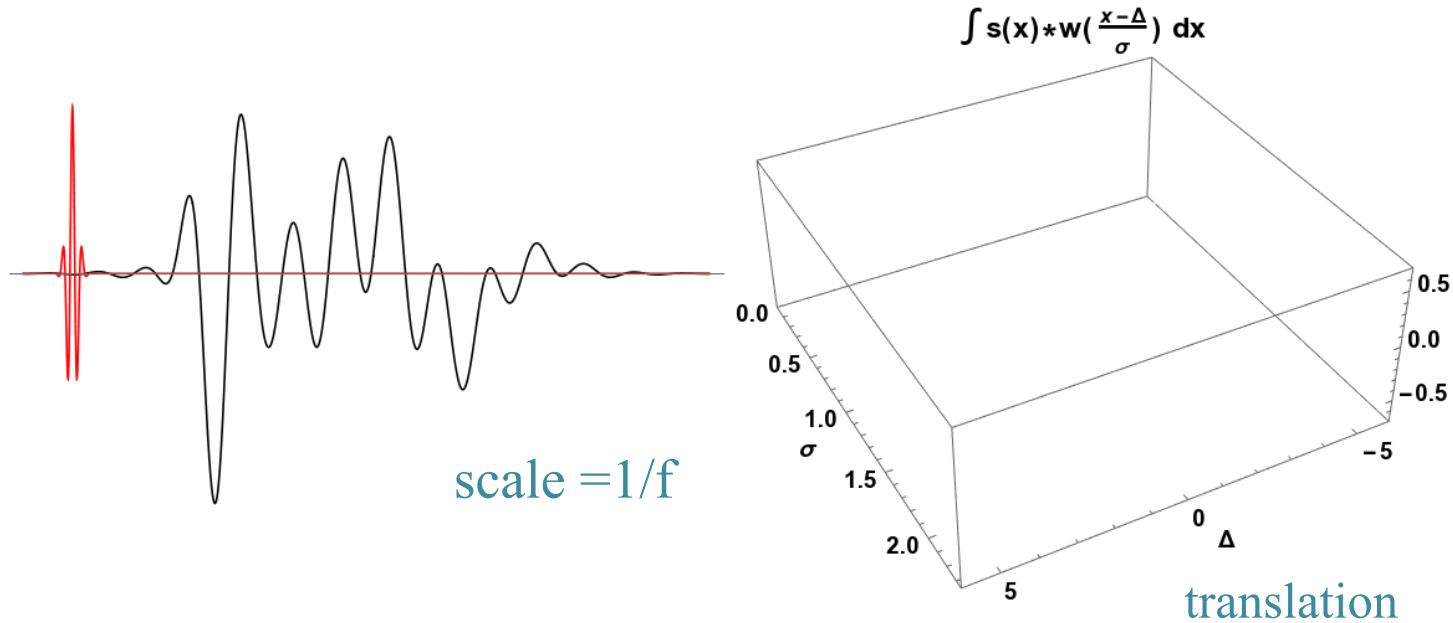
low scale → high frequency

a and b are continuous parameters

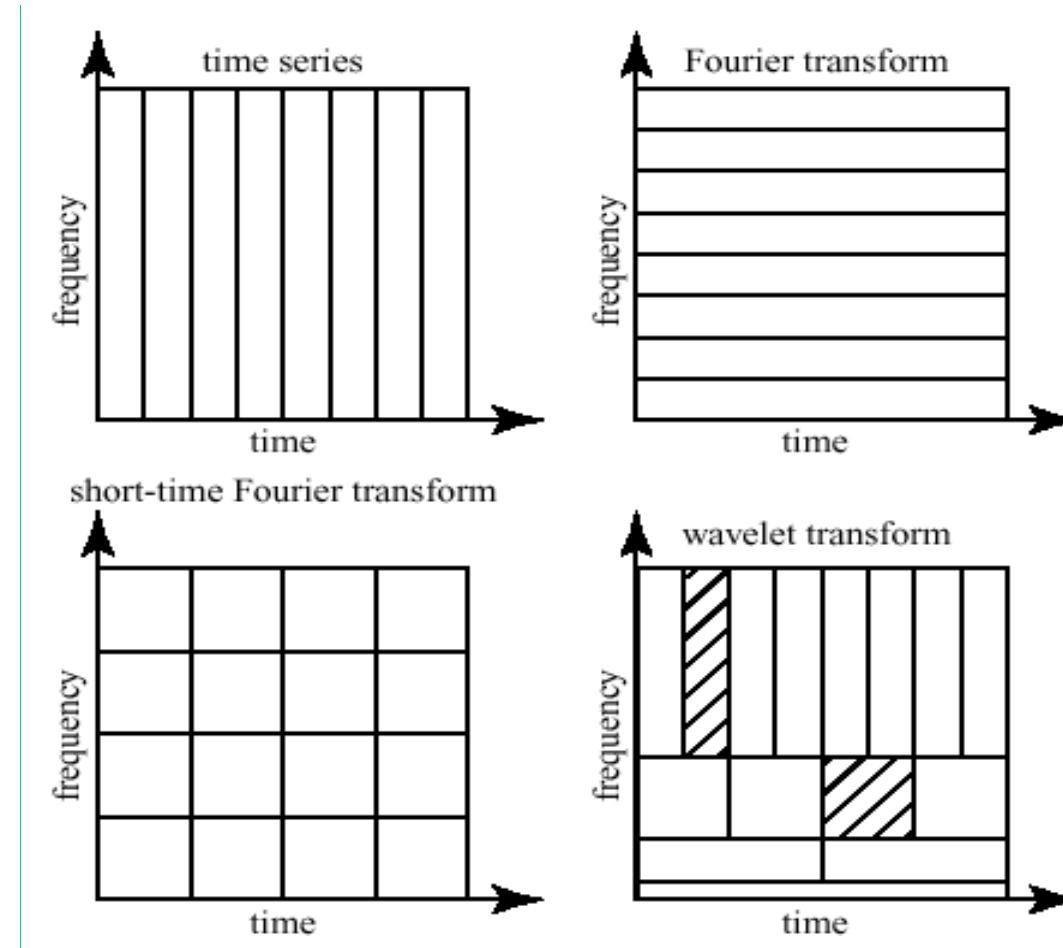
Wavelets



Wavelets



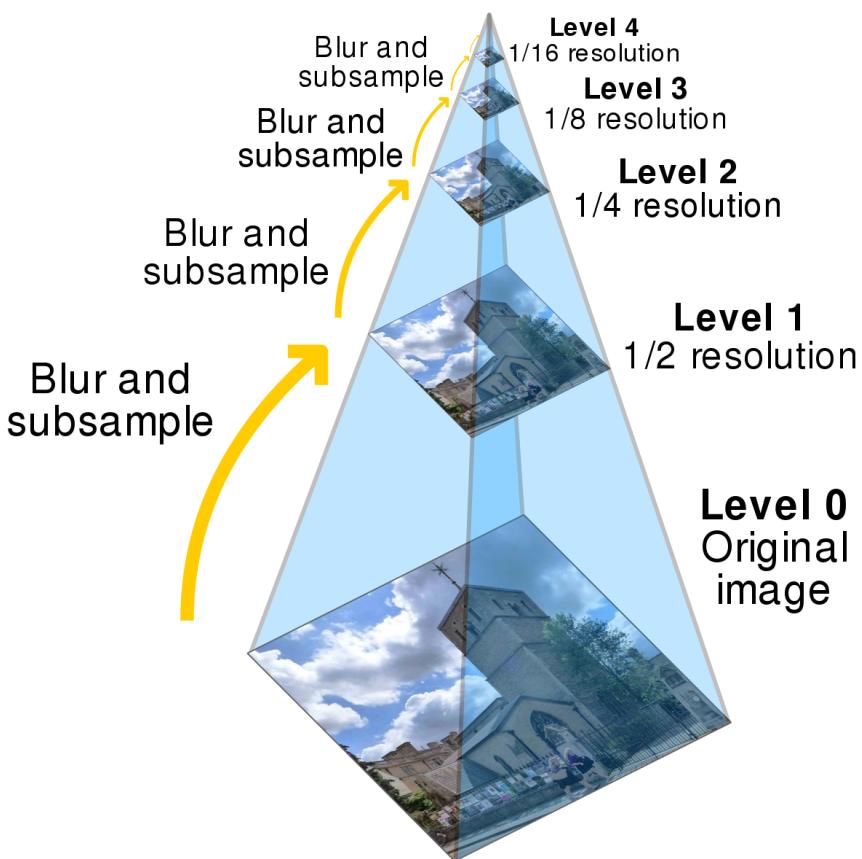
Comparison of Transformations



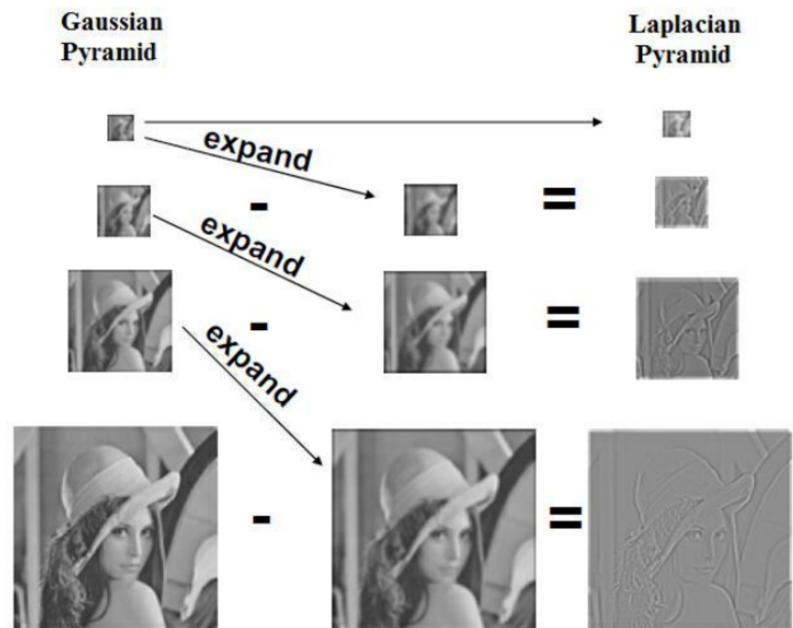
From http://www.cerm.unifi.it/EUcourse2001/Gunther_lecturenotes.pdf, p.10

Image Pyramid

Gaussian pyramid



Laplacian pyramid



Using Level defined as above then
Level is the *inverse to Scale*

Image Pyramid

Gaussian Pyramid

Original image
512x512



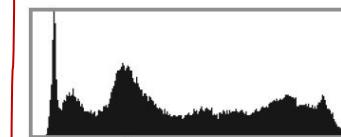
(9)

Laplacian Pyramid
using bilinear
interpolation

Three low resolution approximations

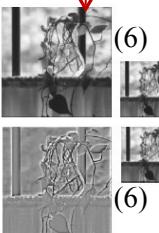
256x256

(8)



128x128

(7)



(7)

64x64

a
b



(8)

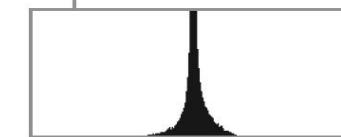


FIGURE 7.3
Two image pyramids and their histograms:
(a) an approximation pyramid;
(b) a prediction residual pyramid.

Image Pyramid

Scale space representation is very useful in image processing

- Extract image features such as edges at multiple scales
- Redundancy reduction and image modelling for
 - ✓ efficient coding
 - ✓ image enhancement/restoration
 - ✓ image analysis/synthesis

Discrete Wavelet transform (DWT)

$$D[a, b] = \frac{1}{\sqrt{b}} \sum_{m=0}^{p-1} f[t_m] \psi \left[\frac{t_m - a}{b} \right]$$

translation

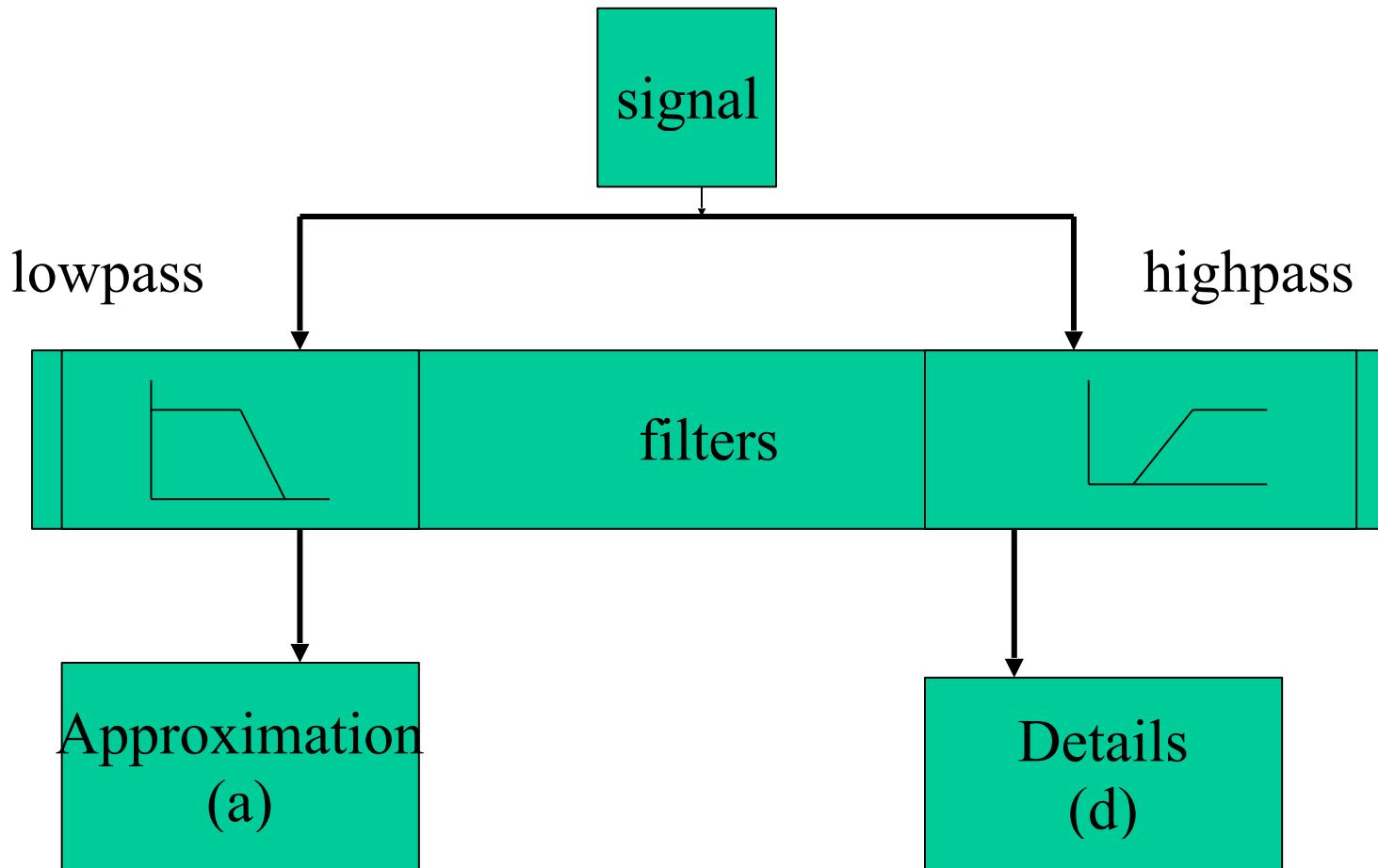
scaling factor

- Use a summation instead of integral
- Integer and fixed scaling (often in the multiple of 2)
- Integer translation: e.g., in the step of 1

DWT uses the wavelets, together with a single scaling function, to represent a function or image as a linear combination of the wavelets and scaling function.

We will see the scaling function later

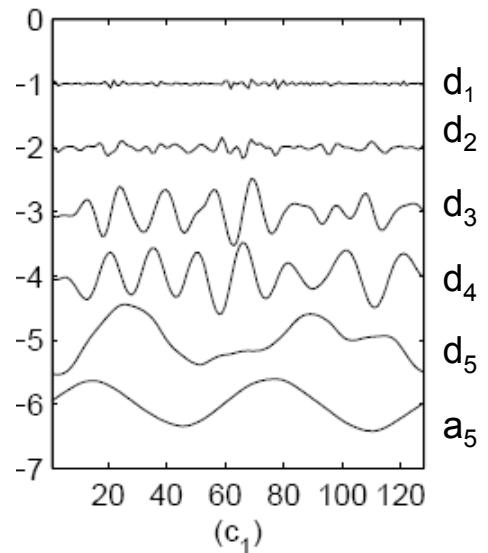
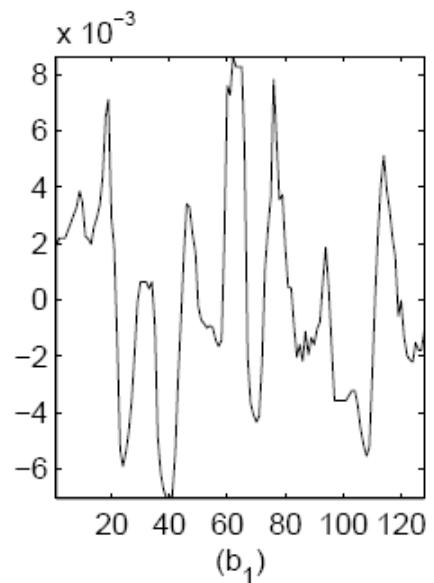
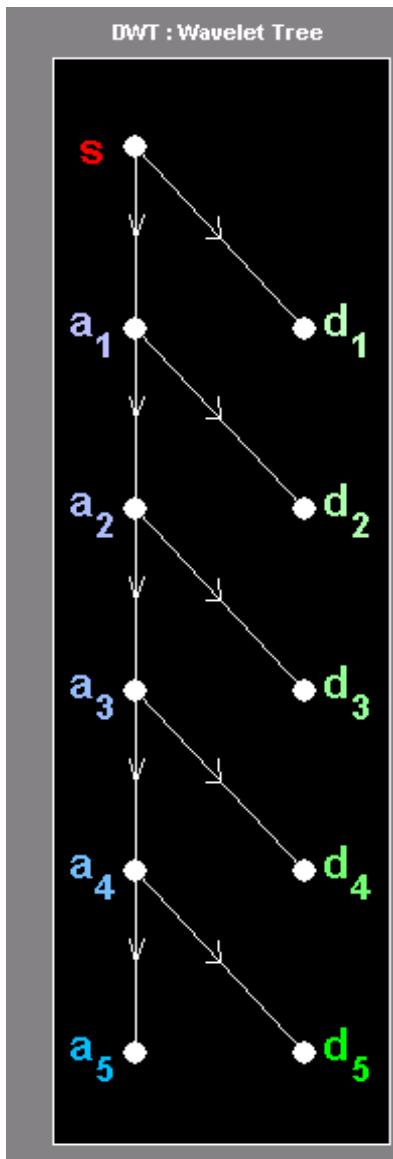
Discrete Wavelet transform (DWT)



(we will see more about this later)

Levels of decomposition

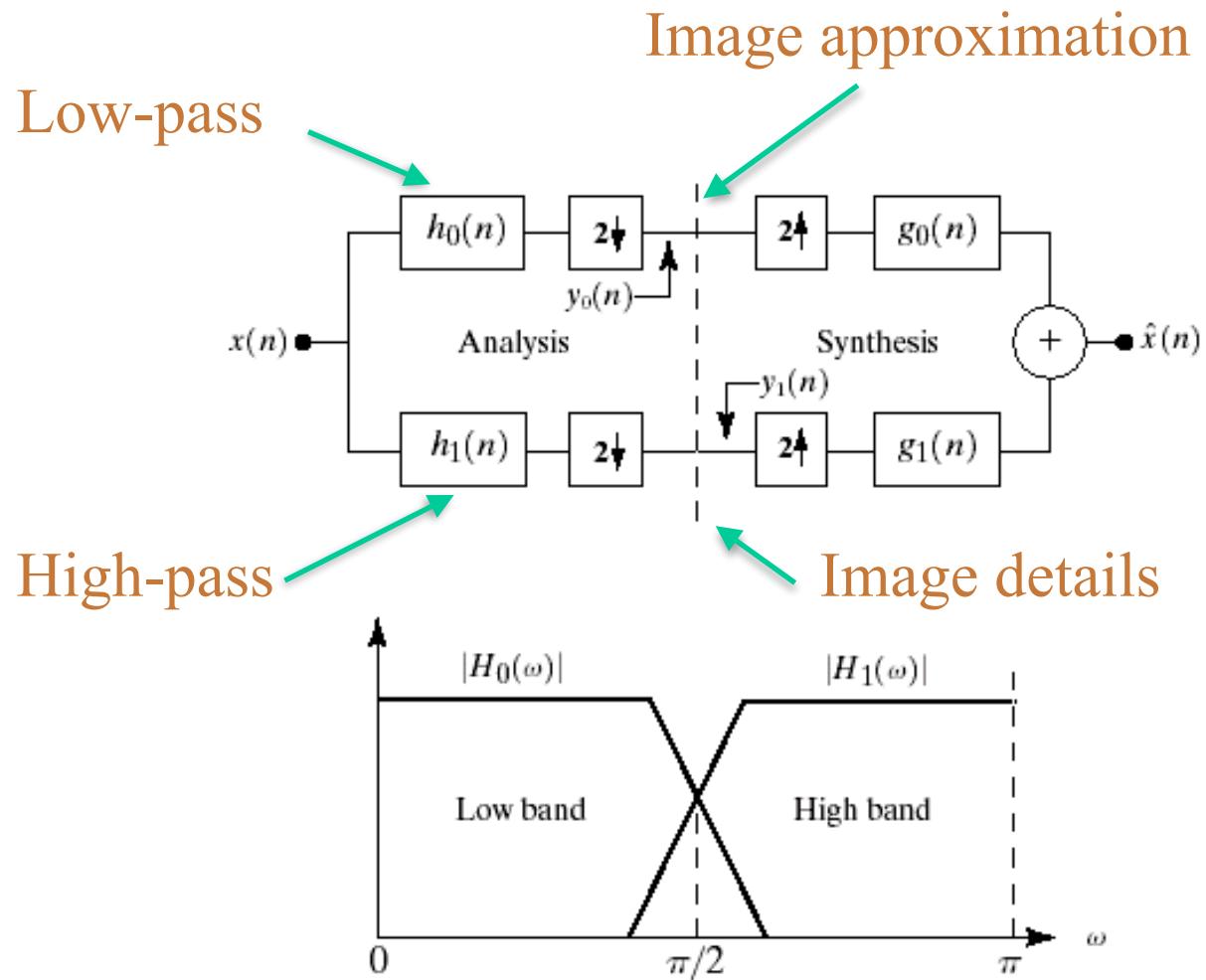
- Successively decompose the approximation
- Level 5 decomposition =
 $a_5 + d_5 + d_4 + d_3 + d_2 + d_1$
- No limit to the number of decompositions performed



Sub-band Coding

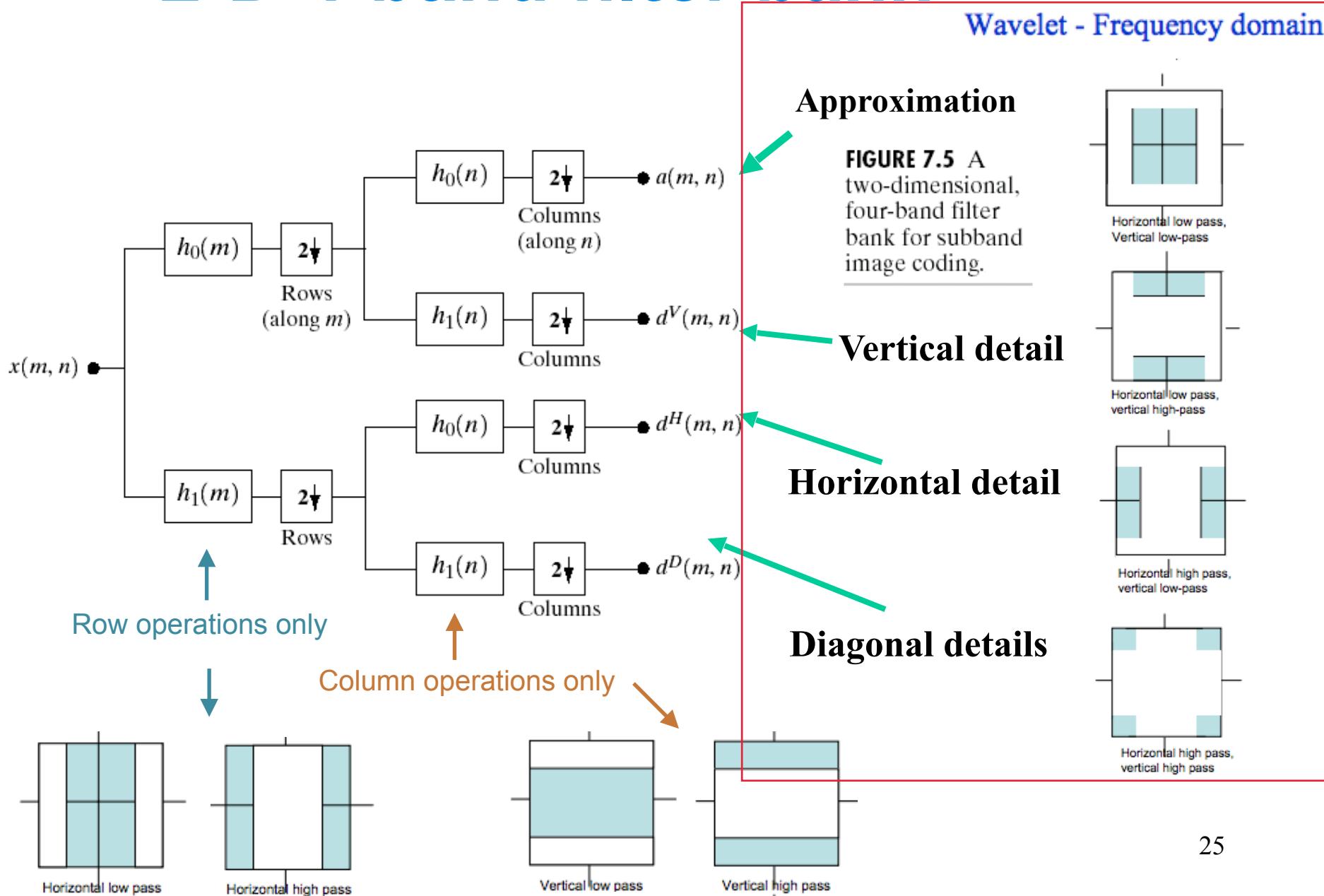
a
b

FIGURE 7.4 (a) A two-band filter bank for one-dimensional subband coding and decoding, and (b) its spectrum splitting properties.

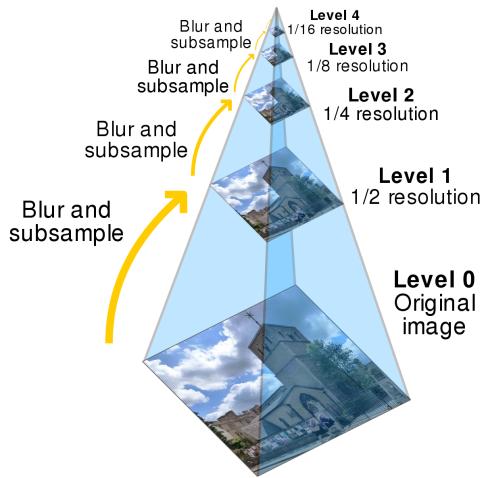


Need to pick the right filters for perfect reconstruction

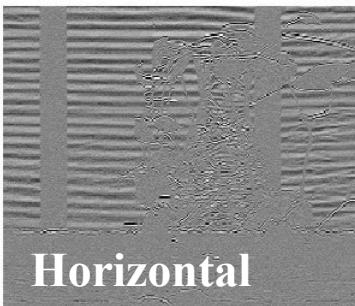
2-D 4-band filter bank



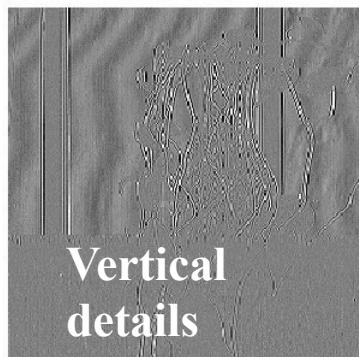
Sub-band filter bank



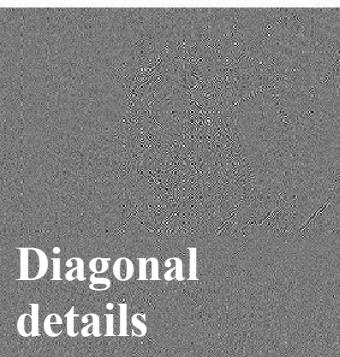
Approximation



Horizontal details

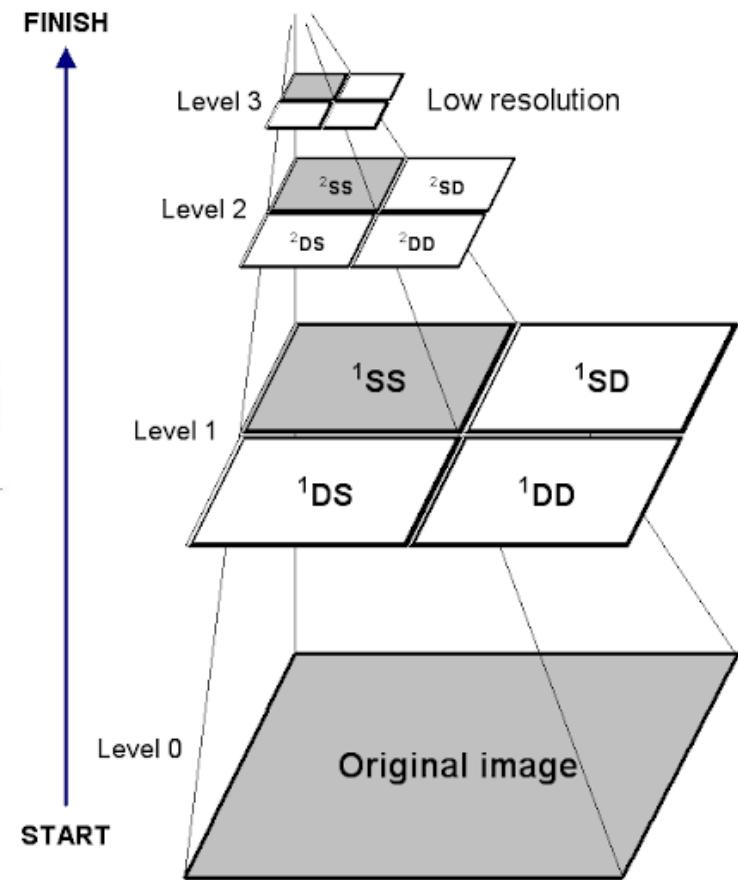


Vertical details

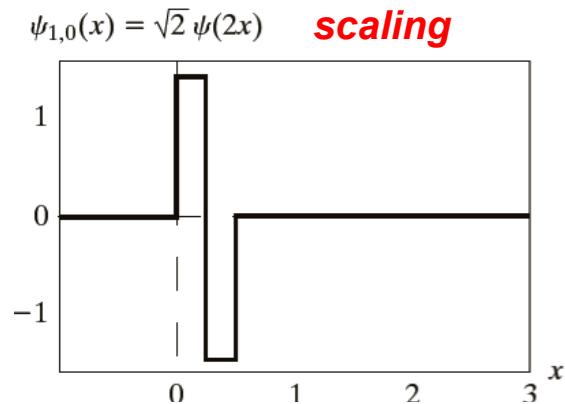
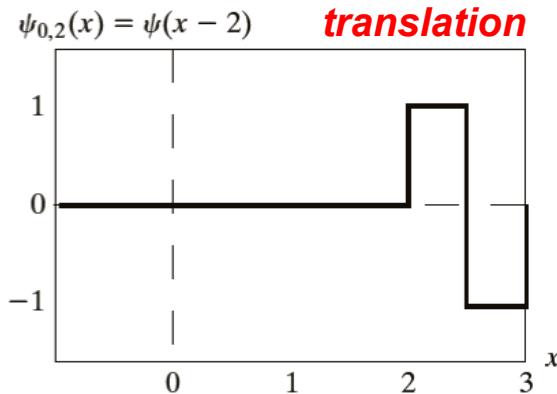
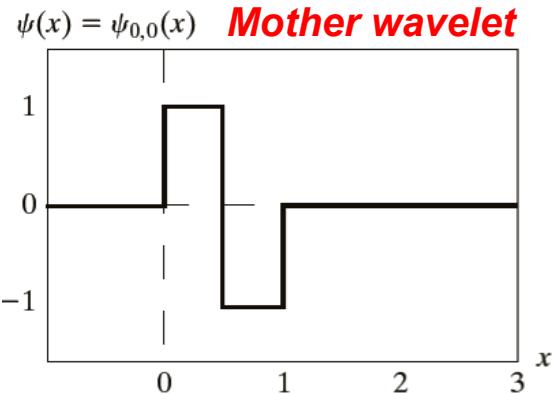


Diagonal details

FIGURE 7.7 A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.5.



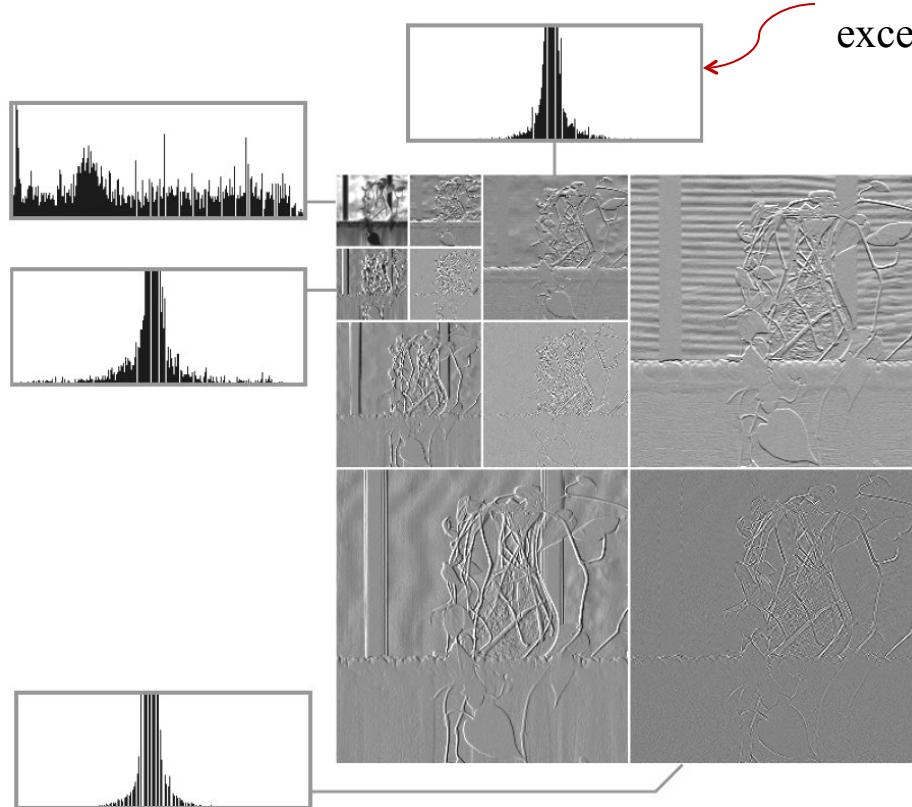
Haar Transform



a	b
c	d
e	f

FIGURE 7.14
Haar wavelet
functions in W_0
and W_1 .

Haar Transform



a
b c d

FIGURE 7.10
(a) A discrete wavelet transform using Haar \mathbf{H}_2 basis functions. Its local histogram variations are also shown. (b)–(d) Several different approximations (64×64 , 128×128 , and 256×256) that can be obtained from (a).

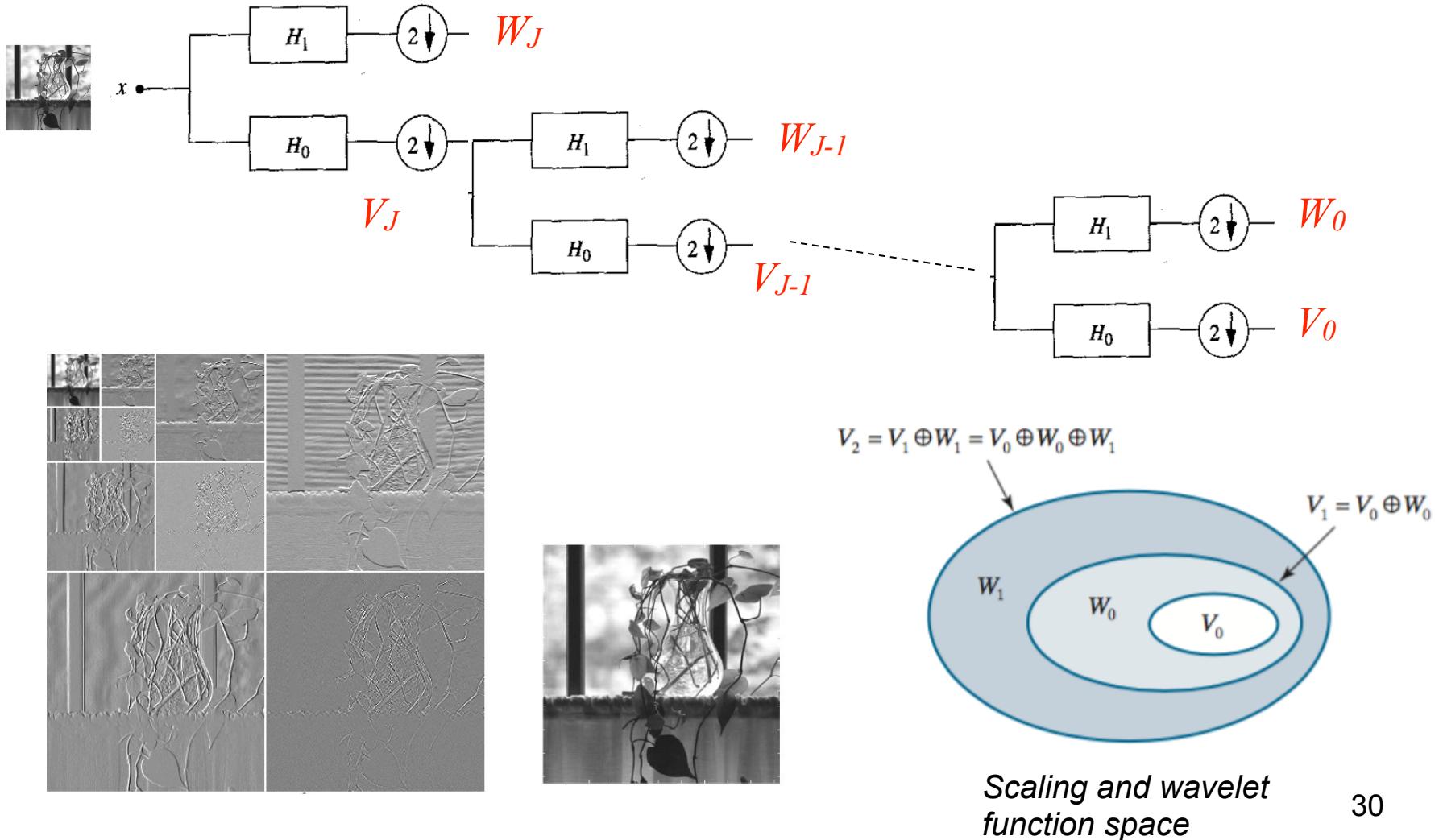




Take a break!

Multi Resolution Analysis

MRA provides a framework for examining functions at different scales



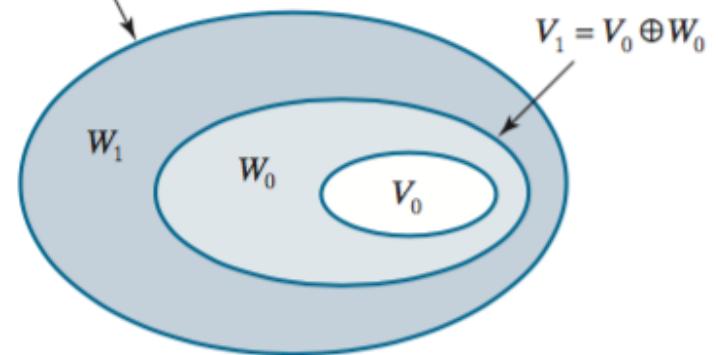
Multi Resolution Analysis

*Allows inspection of details
at different levels*

Definition:

A Multi-resolution analysis is a nested sequence of closed subspaces

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1$$



*Scaling and wavelet
function space*

$$V_j \subset L^2(R) \quad \text{for} \quad j \in \mathbb{Z}$$

$$\dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \dots$$

Multi Resolution Analysis

(1) The subspaces have a zero intersection

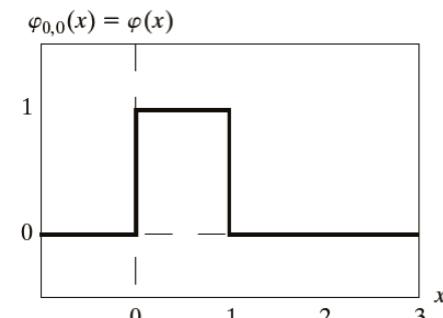
(2) The union is $L^2(R)$

(3) Two scale relations exist

$$f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1} \quad \forall j \in \mathbb{Z}$$

$$f(x) \in V_0 \Leftrightarrow f(x - k) \in V_0 \quad \forall k \in \mathbb{Z}$$

(4) Finally there exists a **scaling function** $\phi(x) \in V_0$ whose integer translations $\{\phi_{0,k} \mid k \in \mathbb{Z}\}$ constitute an orthonormal basis for V_0 .



MRA: Scaling Function

1. The scaling function $\phi_{j,k}(x)$:

$$\phi_{j,k}(x) = 2^{j/2} \phi(2^j x - k) \text{ for all } j, k \in \mathbb{Z}$$

$$\int \phi(x) dx = 1$$

k determines the position of $\phi_{j,k}(x)$ along the x -axis,
 j determines the scale (width) of $\phi_{j,k}(x)$, and $2^{j/2}$ the height
or amplitude. The shape of $\phi_{j,k}(x)$ change with j .

2. The refinement/dilation equation for the scaling function:

$$\varphi(x) = \sum_{k \in \mathbb{Z}} h_\varphi(k) \sqrt{2} \varphi(2x - k) \quad h_\varphi(k) = \langle \varphi(x), \sqrt{2} \varphi(2x - k) \rangle$$

i.e. the expansion functions of any subspace can be built
from the next higher resolution space

Fundamental Requirements of MRA

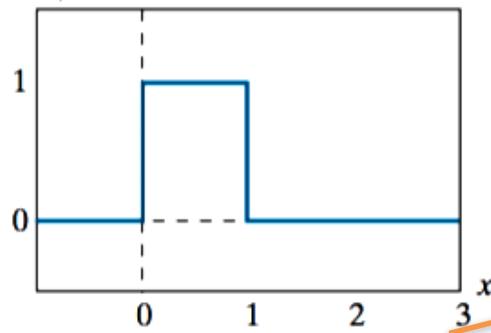
- The scaling function is orthogonal to its integer translate
- The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales
- The only function that is common to all V_j is $f(x) = 0$
- Any function can be represented with arbitrary precision

Haar Scaling Function

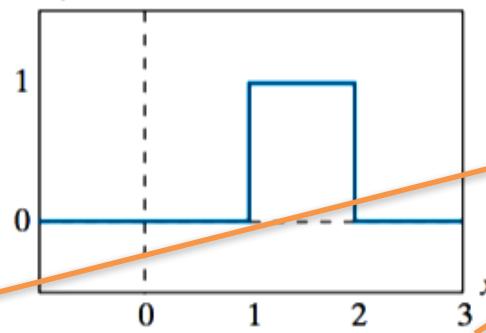
Dilation

$$\begin{aligned}\varphi(x) &= \frac{1}{\sqrt{2}}[\sqrt{2}\varphi(2x)] + \frac{1}{\sqrt{2}}[\sqrt{2}\varphi(2x-1)] \\ &= \varphi(2x) + \varphi(2x-1)\end{aligned}$$

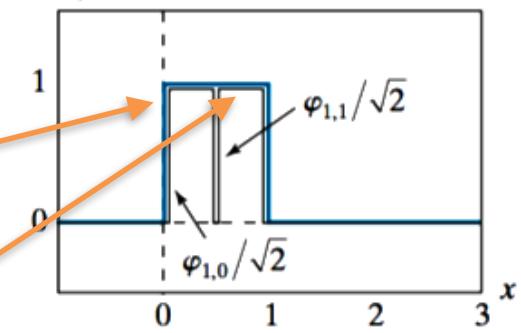
$$\varphi_{0,0}(x) = \varphi(x)$$



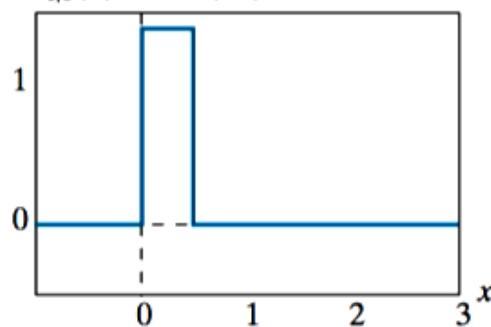
$$\varphi_{0,1}(x) = \varphi(x-1)$$



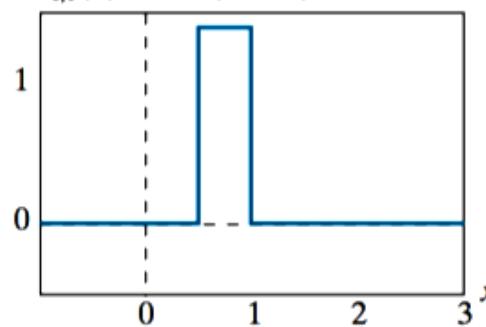
$$\varphi_{0,0}(x) \in V_1$$



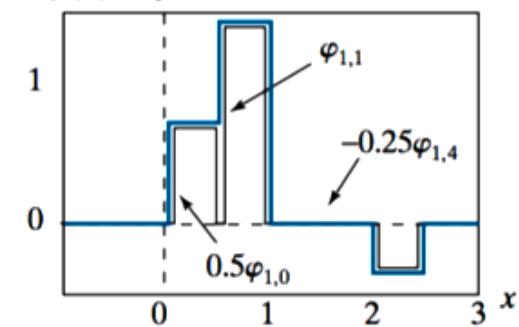
$$\varphi_{1,0}(x) = \sqrt{2}\varphi(2x)$$



$$\varphi_{1,1}(x) = \sqrt{2}\varphi(2x-1)$$



$$f(x) \in V_1$$



a b c
d e f

FIGURE 7.19 The Haar scaling function.

Haar Scaling Function

Dilation

$$\begin{aligned}\varphi(x) &= \frac{1}{\sqrt{2}} [\sqrt{2}\varphi(2x)] + \frac{1}{\sqrt{2}} [\sqrt{2}\varphi(2x-1)] \\ &= \varphi(2x) + \varphi(2x-1)\end{aligned}$$

Scaling function
coefficients:

$$\varphi(x) = \sum_{k \in \mathbb{Z}} h_\varphi(k) \sqrt{2} \varphi(2x - k)$$
$$\{h_\varphi(n) | n = 0, 1\} = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}$$

n is the integer position shift

MRA: Wavelet Functions

Similar to the Scaling Function

1. The wavelet functions satisfy:

$$f(x) \in W_j \Leftrightarrow f(2x) \in W_{j+1}$$

Thus the wavelet function ψ can be expressed as follows:

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k), \quad j = 0, 1, \dots \text{ and } k = 0, 1, \dots, 2^j - 1.$$

2. The dilation equation for the wavelet function:

$$\psi(x) = \sum h_\psi(k) \sqrt{2} \varphi(2x - k)$$

$$\varphi(x) = \sum_{k \in \mathbf{Z}} h_\varphi(k) \sqrt{2} \varphi(2x - k)$$

Complementary scaling and wavelet functions are orthogonal

$$\langle \varphi_{j_0,k}(x), \psi_{j_0,l}(x) \rangle = 0 \quad \text{for } k \neq l$$

Wavelet function coefficients:

$$h_\psi(k) = (-1)^k h_\varphi(1 - k)$$

Haar Wavelet Functions

$$h_\psi(k) = (-1)^k h_\varphi(1-k)$$

$$\{h_\varphi(n)|n=0,1\} = \{1/\sqrt{2}, 1/\sqrt{2}\}$$

$$\psi(x) = \sum_k h_\psi(k) \sqrt{2} \varphi(2x - k)$$

It follows that

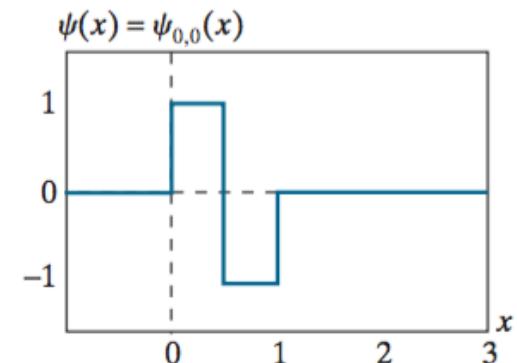
Haar wavelet
coefficients:

$$h_\psi(k) = \begin{cases} 2^{-1/2} & \text{for } k = 0 \\ -2^{-1/2} & \text{for } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

Haar wavelet:

$$\psi(x) = \phi(2x) - \phi(2x-1) \Rightarrow \psi(x) = \begin{cases} 1 & \text{for } 0 \leq x < \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$



Haar Wavelet Functions

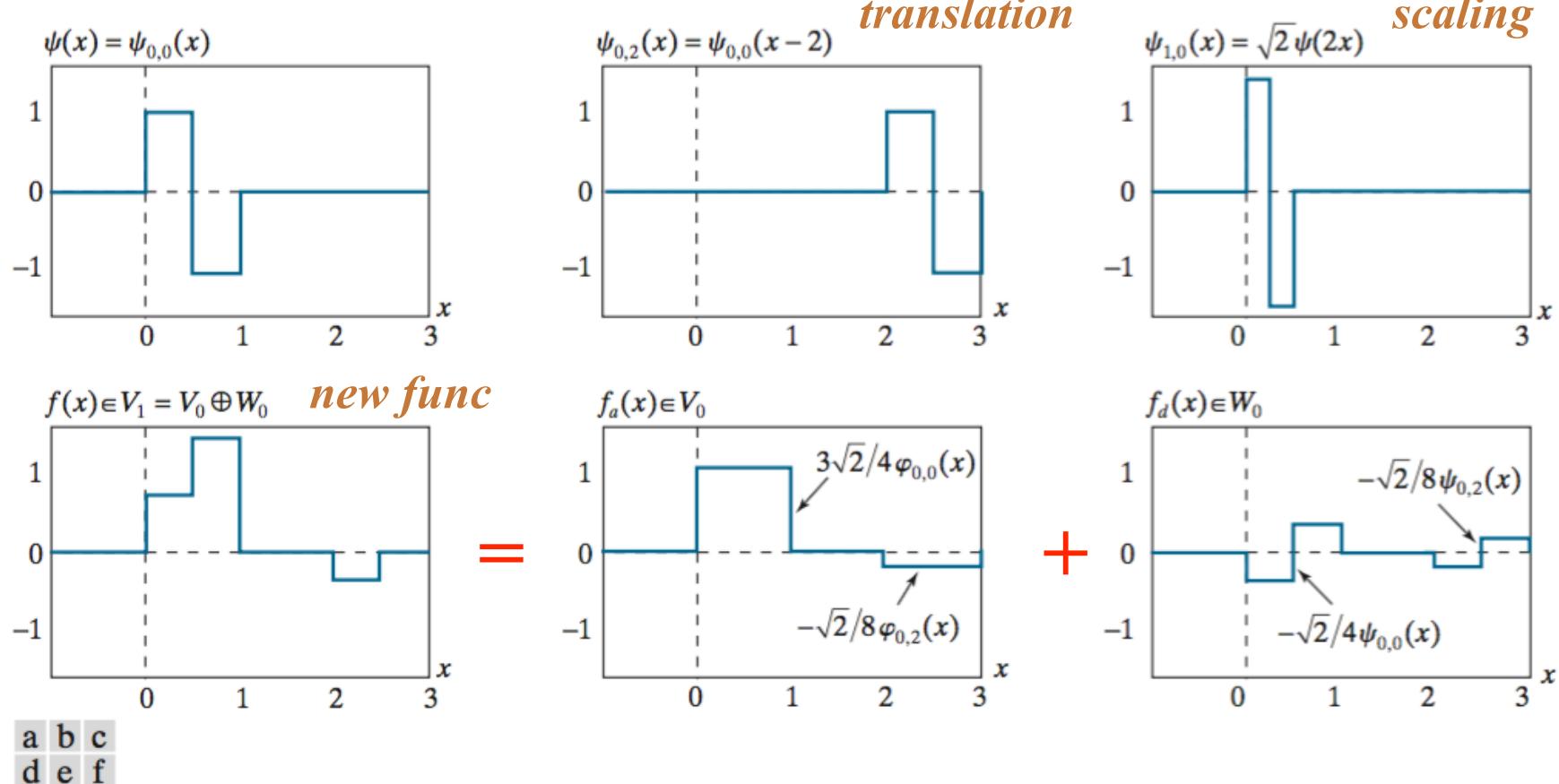


FIGURE 7.21 Haar wavelet functions.

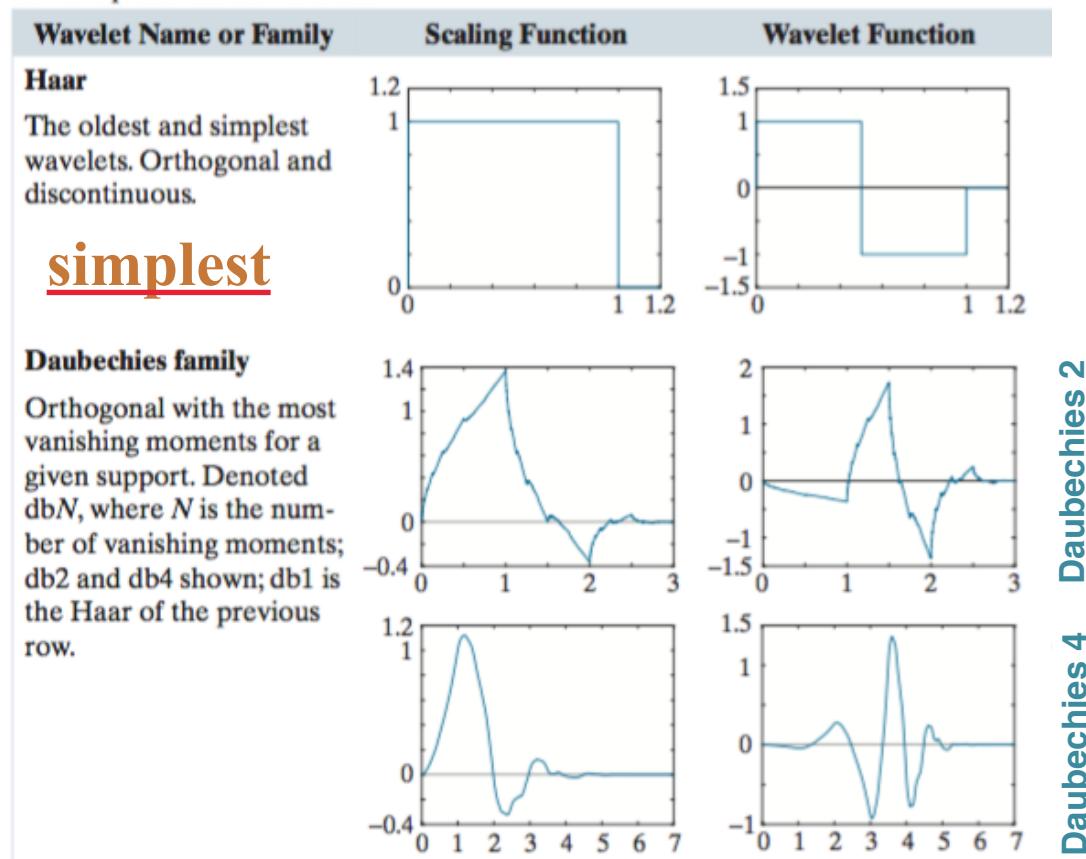
$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k), \quad j, k \in \mathbb{Z}$$

MRA: Series Expansion

By solving the dilation equation of the scaling function with increasing number of expansions, we are able to obtain other families of wavelets + scaling function

TABLE 7.1

Some representative wavelets.



Scaling function coefficients of Daubechies 4

$$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}$$

$$h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}$$

$$h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}$$

$$h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$$

More vanishing moments,
More detailed representation

Wavelet Series Expansion

Express a function with scaling and wavelet functions

$$f(x) = \sum_k c_{j_0}(k) \varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_k d_j(k) \psi_{j,k}(x)$$

*Approximation
Coefficient*

$$c_{j_0} = \langle f(x), \varphi_{j_0,k}(x) \rangle$$

*Detail
Coefficient*

$$d_j = \langle f(x), \psi_{j,k}(x) \rangle$$

Example 7.18 in the textbook:
expand the function y with Haar wavelets

$$y = \begin{cases} x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$y = \underbrace{\frac{1}{3} \varphi_{0,0}(x)}_{V_0} + \underbrace{\left[-\frac{1}{4} \psi_{0,0}(x) \right]}_{W_0} + \underbrace{\left[-\frac{\sqrt{2}}{32} \psi_{1,0}(x) - \frac{3\sqrt{2}}{32} \psi_{1,1}(x) \right]}_{W_1} + \dots$$

$V_1 = V_0 \oplus W_0$

$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1$

Discrete wavelet transform

In the discrete cases, where we have N-point discrete function and N is a power of 2 ($N=2^J$)

$$f(x) = \frac{1}{\sqrt{N}} \left[T_\varphi(0,0)\varphi(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} T_\psi(j,k)\psi_{j,k}(x) \right]$$

where

Approximation Coefficient

$$\boxed{T_\varphi(0,0)} = \langle f(x), \varphi_{0,0}(x) \rangle = \langle f(x), \varphi(x) \rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x) \varphi^*(x)$$

and

Detail Coefficient

$$\boxed{T_\psi(j,k)} = \langle f(x), \psi_{j,k}(x) \rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x) \psi_{j,k}^*(x)$$

Read Example 7.19 in the textbook with discrete Haar wavelets

Discrete wavelet transform

Read Example 7.19 in the textbook with discrete Haar wavelets

$$f(0) = 1, f(1) = 4, f(2) = -3, \text{ and } f(3) = 0. \quad N=4$$

$$\mathbf{t}_H = \mathbf{A}_H \mathbf{f} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -1.5\sqrt{2} \\ -1.5\sqrt{2} \end{bmatrix}$$

$$f(x) = \frac{1}{2} [T_\varphi(0,0)\varphi(x) + T_\psi(0,0)\psi_{0,0}(x) + T_\psi(1,0)\psi_{1,0}(x) + T_\psi(1,1)\psi_{1,1}(x)]$$

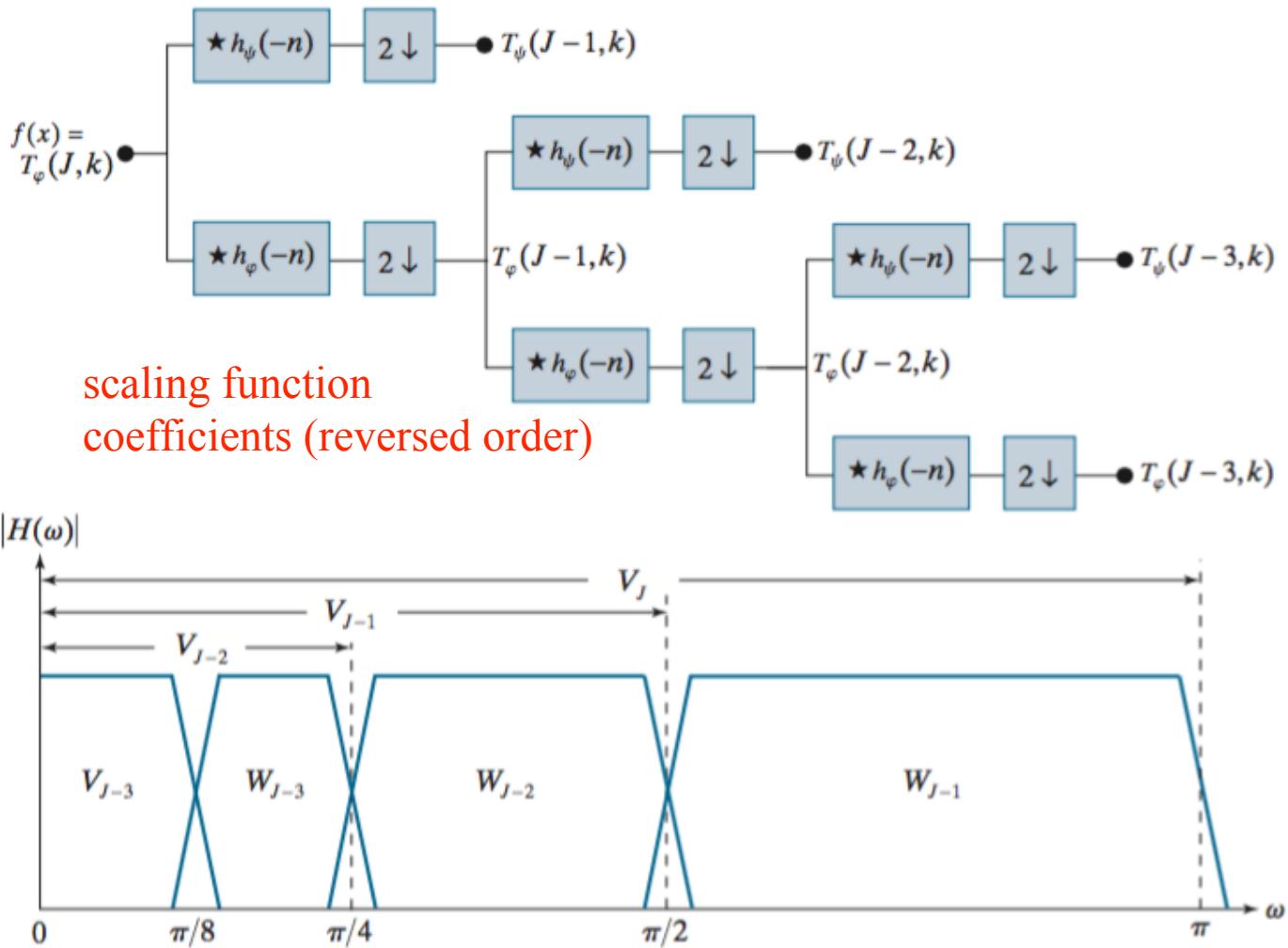
Fast wavelet transform (FWT)

wavelet coefficients (reversed order)

a
b

FIGURE 7.24

(a) A three-stage or three-scale FWT analysis filter bank and (b) its frequency-splitting characteristics. Because of symmetry in the DFT of the filter's impulse response, it is common to display only the $[0, \pi]$ region.



Note that with convolution, there is a need to reverse the order of the entries for the filter

Fast wavelet transform (FWT)

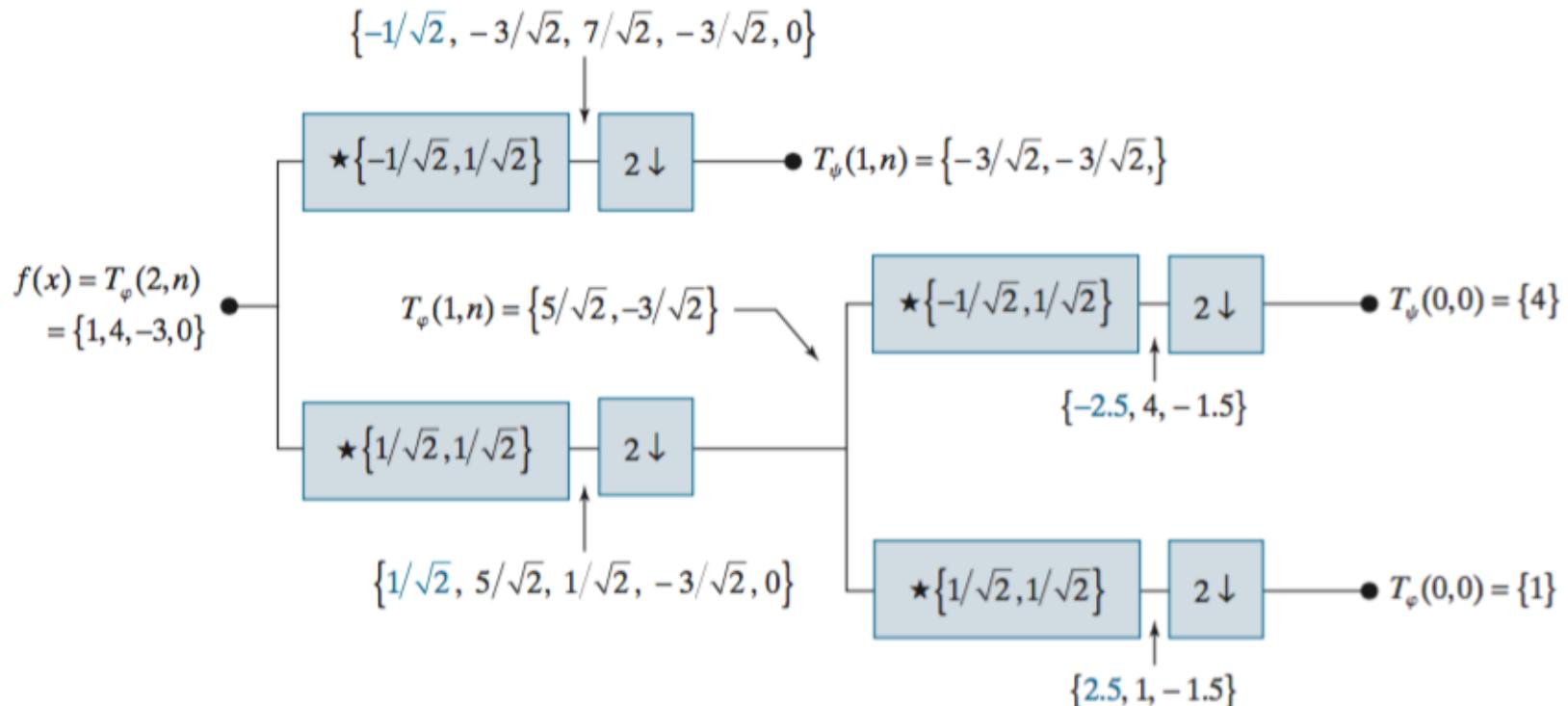


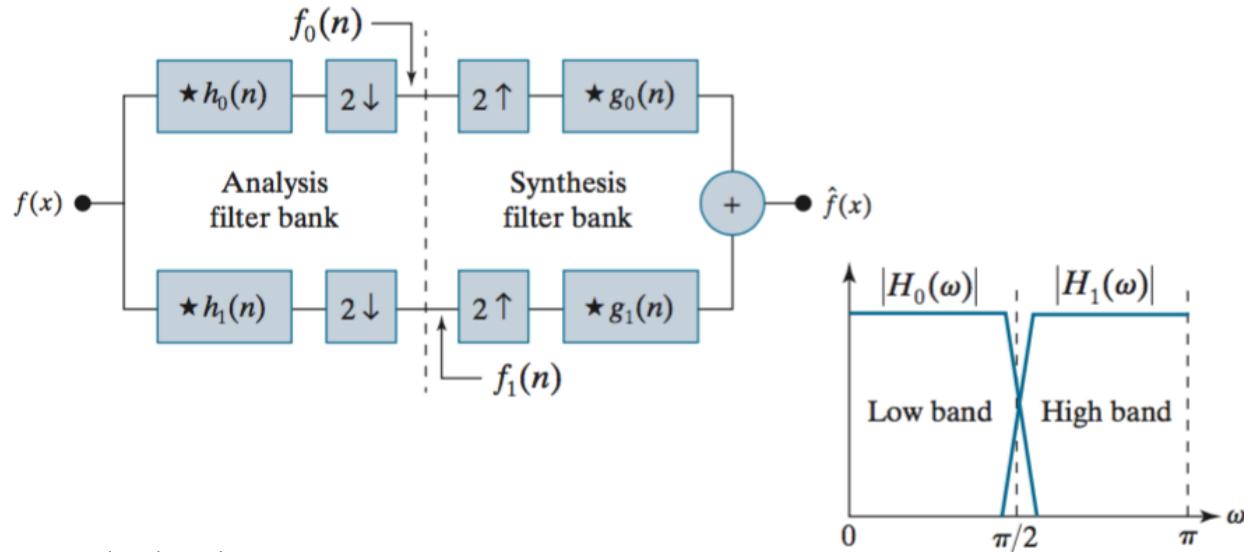
FIGURE 7.25 Computing a two-scale fast wavelet transform of sequence $\{1, 4, -3, 0\}$ using Haar scaling and wavelet coefficients.

Inverse Fast wavelet transform (IFWT)

a b

FIGURE 7.26

(a) A two-band digital filtering system for sub-band coding and decoding and (b) its spectrum-splitting properties.



Cross-modulation

$$g_0(n) = (-1)^n h_1(n)$$

$$g_1(n) = (-1)^n h_0(n)$$

$$g_1(n) = (-1)^n g_0(K-1-n)$$

$$h_0(n) = g_0(K-1-n)$$

$$h_1(n) = g_1(K-1-n)$$

Inverse Fast wavelet transform (IFWT)

Note how filter functions
change from FWT

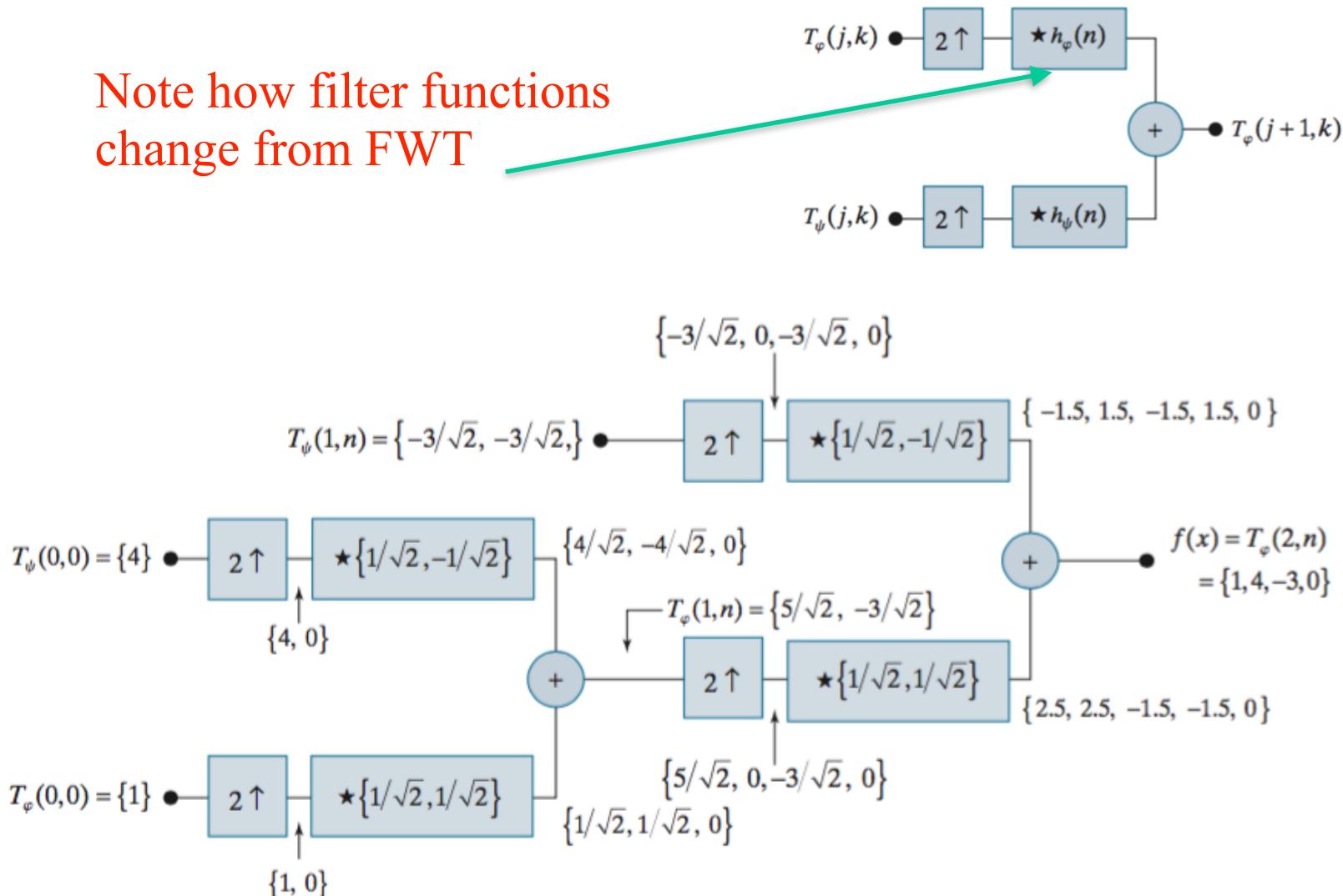
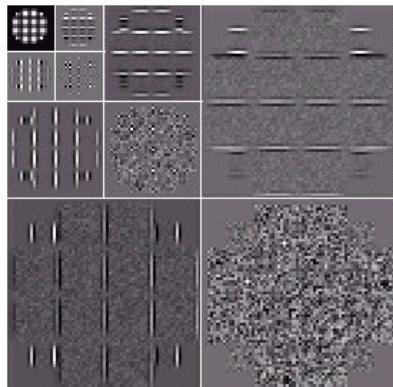
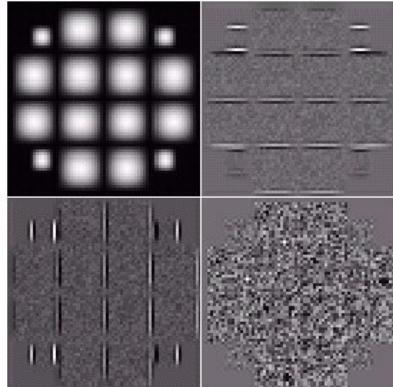
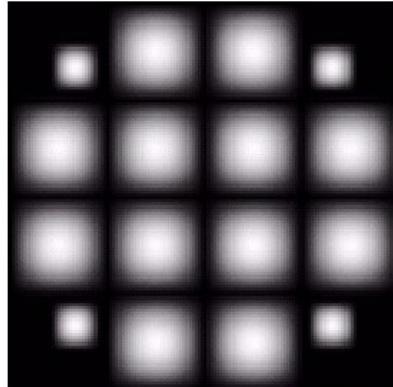


FIGURE 7.28 Computing a two-scale inverse fast wavelet transform of sequence $\{1, 4, -1.5\sqrt{2}, -1.5\sqrt{2}\}$ with Haar scaling and wavelet functions.

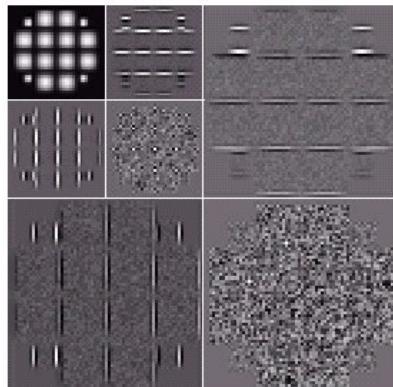
2-D Wavelet Transform



a
b
c
d

FIGURE 7.25

Computing a 2-D three-scale FWT:
(a) the original image;
(b) a one-scale FWT;
(c) a two-scale FWT;
and (d) a three-scale FWT.

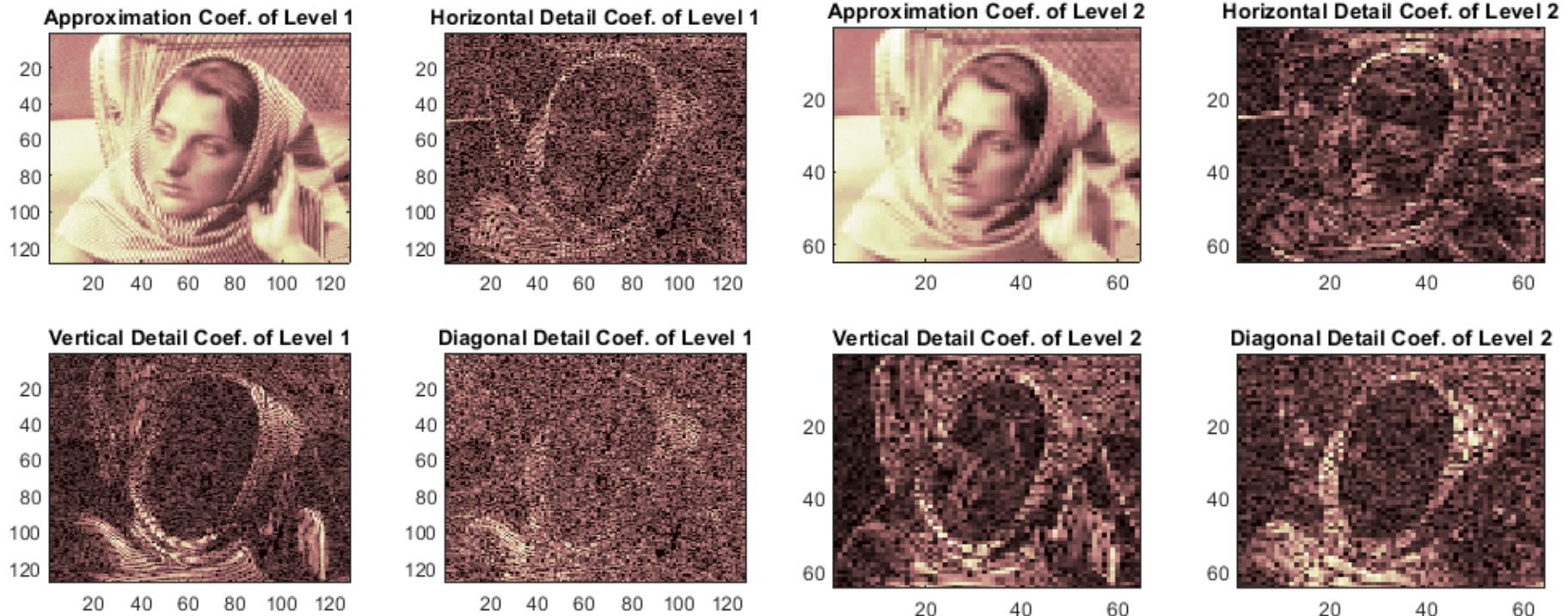


Wavelet denoising via synthesis



2-D DWT DEMO ON MATLAB

Results obtained with wavedec2() function



<https://www.mathworks.com/help/wavelet/ref/wavedec2.html>

Applications of Wavelets

- Pattern recognition
- Feature extraction
- Image processing
- Biometrics
- Financial analysis
- Perfect reconstruction
- Video compression
- Medical imaging
- etc

Summary

- Definition and benefits of wavelets
- Image Pyramid
- Filter banks: decomposition & synthesis
- Continuous & discrete wavelet transform
- Scaling and wavelet functions & properties
- Scaling and wavelet function space
- Different wavelet families (Haar, Daubechies)

Useful Link

- Matlab wavelet tool using guide
- <http://www.wavelet.org>
- <http://www.multires.caltech.edu/teaching/>
- <http://www-dsp.rice.edu/software/RWT/>
- www.multires.caltech.edu/teaching/courses/waveletcourse/sig95.course.pdf
- <http://www.amara.com/current/wavelet.html>

Reading

- Chapter 7
- Page 481-483, 504-526

For related math operations and concepts,
please refer to Page 464-470

Questions

- Why wavelet transform is needed?
- What are the basic properties of wavelets?
- Wavelet: scale vs. frequency; translation vs. time
- Image Pyramids: two types & why do we need them?
- Wavelets: scaling function vs. wavelet function
- Haar vs. Daubechies-4: which one gives better approximation of images?