Relations

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Binary relations over sets

- A *binary relation* R between sets A and B (sometimes written as $R : A \leftrightarrow B$) is defined as $R \subseteq A \times B$.
- Given $A = \{1, 2\}$ and $B = \{2, 3\}$, and $R =$ "is less than", then
  $A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$, and
  $R = \{(1, 2), (1, 3), (2, 3)\}$.
- We can also observe that $R \in \mathcal{P}(A \times B)$.
- The statement $(a, b) \in R$ is written as aRb, or $R(a, b)$.

Binary relations over sets /cont.

- We view a binary relation R as associating every ordered pair (a, b) the value true, if $(a, b) \in R$, or the value false if $(a, b) \notin R$.
- It is, thus, evident that $<$ (less than) is a binary relation, whereas $+$ (addition) or $-$ (subtraction) are not binary relations.

Binary relations over sets /cont.

- Using sets of ordered pairs is just one example we can express relations.
- Other methods include words, directed graphs or matrices.
- For example (Noodles likes Deborah), or likes(Noodles, Deborah), is an element of the binary relation *likes* over the set of all people.

Properties of relations

- A relation R on a set A is:

1. Reflexive when $\forall a \in A : aRa$,
   e.g. "divides", "is equal to."
2. Irreflexive (or anti-reflexive) when no element of A is related to itself, i.e. $\forall a \in A : \neg(aRa)$,
   e.g. "is greater than", "is not equal to."

- Note that "not reflexive" does not mean "irreflexive."
- For example, the relation "likes" (between people) is not reflexive since not everybody likes themselves, but it is not irreflexive either since not everybody dislikes themselves either.

Check universal quantifier

Properties of relations /cont.

3 Symmetric, when $\forall a, b \in A : aRb \to bRa$,
e.g. "is a blood relative of", "is married to."

4 Asymmetric, when $\forall a, b \in A : aRb \to \neg(bRa)$,
e.g. "is father of", "is less than."

- A relation cannot be both symmetric and asymmetric.
- Note that "asymmetric" does not mean "not symmetric."
- A relation can be neither symmetric nor asymmetric, e.g. "likes."

asy = everybody
not
not asy = not
everybody

Properties of relations /cont.

5 Antisymmetric when
$\forall a, b \in A : (aRb \wedge bRa) \rightarrow a = b.$

Alternatively, R is antisymmetric if whenever
$(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$

e.g. $\leq$ is antisymmetric, but "likes" is not.

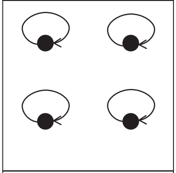
Properties of relations /cont.

6 Transitive when $\forall a, b, c \in A : (aRb \wedge bRc) \rightarrow aRc$, e.g. "is ancestor of."

- A relation is intransitive iff it is not transitive.
- For example, the relation "likes" (between people) is intransitive.

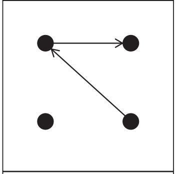
Visualization of properties of binary relations

Reflexive



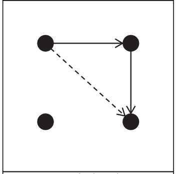
Every vertex has a self-loop.

Irreflexive



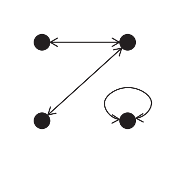
No vertex has a self-loop.

Transitive



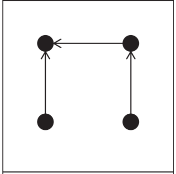
Non-solid edge is present by logical deduction.

Symmetric



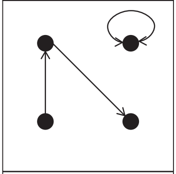
Every edge is bidirectional.

Asymmetric



No edge is bidirectional.

Antisymmetric



No edge is bidirectional, unless it is a self-loop.

Example 1: Relation “likes”

- We have seen above that the proposition *Noodles likes Deborah* defines the binary relation *likes* over the set of all people. The relation *likes* is

(a) not reflexive (not everybody likes themselves),

(b) not symmetric (not everybody’s feelings are reciprocated), and

(c) not transitive.

Example 2: Relation "is son of"

- The relation *is son of* is

(a) irreflexive, as no person can be his own son,

(b) not symmetric, and

(c) not transitive.

Domain and range in binary relations

- A binary relation R can be modeled as a set of ordered pairs.
- The *domain* of R, *dom* R, is the set of all first elements of ordered pairs.
- The *range* of R, *ran* R, is the set of all second elements of ordered pairs.
- For example, for $R = \{(0,1),(0,2),(1,1),(3,5)\}$, then
  dom $R = \{0,1,3\}$, and
  ran $R = \{1,2,5\}$

Inverse of binary relations

- Any binary relation R has an *inverse* relation (denoted by R^{-1}, or $R^{\sim}$, also called *converse* relation) which is obtained by changing the order of the elements in the relation.
- For a relation R on a set A

$$R^{-1} = \{(b, a) \in A \times A \mid (a, b) \in R\}$$

- A binary relation over a set is equal to its inverse if and only if it is symmetric.

Example: Inverse relation

- For $R = \{(0,1),(0,2),(1,1),(3,5)\}$, then
  $R^{-1} = \{(1,0),(2,0),(1,1),(5,3)\}$,
  $dom\ R^{-1} = \{1,2,5\}$, and
  $ran\ R^{-1} = \{0,1,3\}$
- We observe that
  $dom\ R = ran\ R^{-1}$, and
  $ran\ R = dom\ R^{-1}$.

Equivalence relations

- **Definition**: Any relation that is reflexive, symmetric and transitive is called an *equivalence relation*, e.g. the relation "is equal to" on the set of numbers.

Example 3: Relation "has the same birthday as"

- For any $a, b, c \in A$, the relation R: "has the same birthday as" is

1. Reflexive: $\forall a \in A : aRa$,
2. Symmetric: $\forall a, b \in A : aRb \rightarrow bRa$, and
3. Transitive: $\forall a, b, c \in A : (aRb \wedge bRc) \rightarrow aRc$.

Partition

- A *partition* of a set A is a collection, P, of non-empty subsets that satisfy the following:

1. $A = A_1 \cup A_2 \cup \cdots \cup A_n$, and
2. $A_i \cap A_j = \varnothing, \ \textit{for} \ i \neq j$.
   where the sets (i.e. the elements of P) are called the *blocks* (also: *parts*, or *cells*) of the partition.

Partial order

- **Definition**: Any relation that is reflexive, antisymmetric and transitive is called a *partial order*,
  e.g. the relation "is less than or equal to" on the set of real numbers.
- A set for which a partial order is defined is called a *poset*.
- A *total order* on a set A is a partial ordering in which every pair of elements is related.

Predecessors and successors in posets

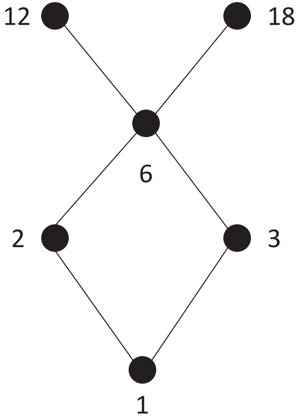
- If R is a partial order on a set A and xRy, $x \neq y$, we call x a *predecessor* of y (and subsequently we call y a *successor* of x).
- An element may possess many predecessors.
- If x is a predecessor of y and there is no z for which xRz and zRy, then we call x the *immediate predecessor* of y, denoted by $x < y$.

Hasse diagrams

- A *Hasse diagram* is a graph where vertices represent the elements of a poset A, and whenever $x < y$, vertex x is placed below vertex y and the vertices are joined by an edge.

Example: Hasse diagram

- Given the relation "is divisor of" on the set $A = \{1, 2, 3, 6, 12, 18\}$, the corresponding Hasse diagram is shown:



Minimal and maximal elements in Hasse diagrams

- In any poset there are *minimal elements* (ones without predecessors) and *maximal elements* (ones without successors).
- In the example, the poset has one minimal element (1) and two maximal elements $(12, 18)$.

Composition of binary relations

- Given two binary relations, we can form a new one by a process called a *composition*.
- For example, given "is brother of (x, y)" and "is parent of (y, z)", we can combine the two to form "is uncle of (x, z)."
- Formally, if $R \subseteq X \times Y$, and $S \subseteq Y \times Z$, then their *composition*, denoted by $R \circ S$, is the relation $R \circ S = \{(x, z) \in X \times Z \mid (\exists y \in Y : (x, y) \in R \wedge (y, z) \in S\}$
- In other words, the tuple (x, z) is an element of the new composition relation, if there exists a $y \in Y$ such that $(x, y) \in R \wedge (y, z) \in S$.

Relational override

- The *override* of R with S, denoted by $R \oplus S$ is obtained by adding to S all those ordered pairs from R whose first coordinates are not in the domain of S.
- Given the relations
  $R = \{(0,1),(0,2),(2,3)\}$, and $S = \{(0,1),(1,3),(3,0)\}$.
  Then
  $R \oplus S = \{(0,1),(1,3),(3,0),(2,3)\}$,
  i.e. the ordered pair $(2,3)$ has been added to S since
  $2 \notin dom\ S$.