Unordered structures

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Introduction

- Unordered structures include sets and bags (or multisets).

Sets

- A *set* is a collection of objects, called its *elements* (also: *members*).
- If S is a set and x is an element in S, then we write $x \in S$.
- If x is not an element of S we write $x \notin S$.
- The set of no elements is called the *empty* set (also: *null* set), denoted by $\{\}$ or $\emptyset$.

Sets /cont.

- Sets have two characteristics:
  1. No element repetition is allowed.
  2. The ordering of the elements is not important.

Sets /cont.

- One way to define a set is by a method called *enumeration*. As the term suggests, this entails listing some or all elements of the set, separated by commas and enclosed within braces ($\{\ldots\}$), e.g.

$$Primary\ Colors = \{Red, Yellow, Blue\}$$

$$\mathbb{N} = \{0, 1, 2, ...\}$$

- We can also define a set by *comprehension*. This entails describing a property that all set elements must have, i.e.

$$\{x : S | P(x)\}$$

  where x denotes all elements of the set, S denotes the type of the elements, also called the *domain*, and $P(x)$ is a property that all elements must satisfy, defined by a predicate.
- For example, $\{1, 2, \ldots, 10\} = \{x : \mathbb{N}_1 | x \leq 10\}$

Important sets

- The empty set is one of the several important sets that have special notation.
- $\mathbb{N}$: The set of all natural numbers. To be specific whether or not zero is included, we can add a subscript or superscript 0 in the former case, and a subscript 1 or superscript * in the latter case, i.e.

  $$\mathbb{N}_0 = \mathbb{N}^0 = \{0, 1, 2, 3, \ldots\}$$

  and

  $$\mathbb{N}_1 = \mathbb{N}^* = \{1, 2, 3, \ldots\}$$

- $\mathbb{Z}$: The set of all integers. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- $\mathbb{Q}$: The set of rational numbers: $\{x : x = \frac{m}{n} \; for \; m, n : \mathbb{Z}\}$.
- $\mathbb{R}$: The set of real numbers.
- $\mathbb{C}$: The set of complex numbers.

Set equality

- Two sets are *equal* if they have the same elements.
- We denote the fact that two sets A and B are equal by $A = B$. If sets A and B are not equal, we write $A \neq B$.
- Note that since order is not important,

  $$\{a, b, c\} = \{c, a, b\}$$

  as opposed to

  $$\langle a, b, c\rangle \neq \langle c, a, b\rangle$$

- Note also that

  $$a \neq \{a\} \neq \{\{a\}\}$$

  since a is a single object, $\{a\}$ is a set with one element, namely a, whereas $\{\{a\}\}$ is a set with one element, namely the set $\{a\}$ which contains one element, a.

Set equality /cont.

- If A and B are sets and every element of A is also an element of B, then we say that A is a *subset* of B, denoted by $A \subset B$.
- It follows, from the definition, that every set is a subset of itself.
- It also follows that the empty set is a subset of any set A, i.e. $\emptyset \subset A$. We can use the notion of subsets to define set equality $A = B$ to mean $A \subset B$ *and* $B \subset A$.
- Given a set S, the *power set* of S, denoted by $P(S)$, is the set of all subsets of S, i.e.

$$P(S) : \{e | e \subseteq S\}$$

- For example, for $S = \{a, b, c\}$, $P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, S\}$.

Operations on sets

- The *union* of two sets A and B, denoted as $A \cup B$, is given by

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

- The *intersection* of two sets A and B, denoted as $A \cap B$, is given by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

- The *difference* between two sets A and B, denoted as $A \backslash B$ (or $A - B$), is given by

$$A \backslash B = \{x : x \in A \text{ and } x \notin B\}$$

- When $B \subset A$, then the set difference $B \backslash A$ is also called the *relative complement* of A with respect to B.
- The *symmetric difference* of two sets A and B, denoted as $A \oplus B$, is given by

$$\begin{aligned} A \oplus B &= \{x : x \in A \text{ or } x \in B \text{ but not both}\} \\ &= A \backslash B \ \cup \ B \backslash A \end{aligned}$$

Disjoint sets

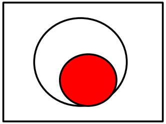
- Two sets A, B are called *disjoint* if and only if their intersection is empty, i.e.

$$A \cap B = \emptyset$$

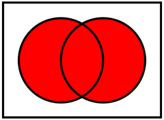
Set visualization: Venn diagrams

- We can illustrate the logical relation between a finite collection of sets by Venn diagrams whereby each set is represented as an simple closed curve drawn on a plane.
- Usually the curves are drawn as circles or rectangles.
- All closed curves exist inside the boundary of a rectangular region which represents the *universal set* which is a set that contains all sets.

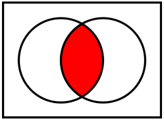
Set visualization: Venn diagrams /cont.



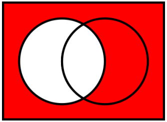
$A \subset B$
A (inside circle) is proper subset of B.



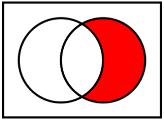
$A \cup B$
Union of A and B.



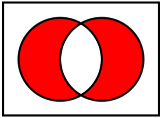
$A \cap B$
Intersection of A and B.



$A^c = U \setminus A$
Complement of A in U.



$B - A = A^c \cap B$
Difference between B (right) and A (left).



$A \oplus B$
Symmetric difference between A and B.

Algebra of sets

Identity laws	$A \cup \emptyset = A$	$A \cap U = A$
Domination laws	$A \cup U = U$	$A \cap \emptyset = \emptyset$
Idempotent laws	$A \cup A = A$	$A \cap A = A$
Associative laws	$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$
Commutative laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Complement laws	$A \cup \overline{A} = U$	$A \cap \overline{A} = \emptyset$

Set cardinality

- The *cardinality* of a set A, denoted by $|A|$ (or $\#$A) is a measure of how many elements A has, e.g. for the set *Primary Colors* defined above,

$$|PrimaryColors| = 3$$

- The *principle of inclusion and exclusion* provides a counting rule for the union of two sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Bags

- A *bag* (or *multiset*) is a structure which contains a collection of elements.
- Like a set, the ordering of the elements is not important in a bag. However, unlike a set, repetitions are allowed in a bag.
- Note that since order is not important and repetitions are allowed,

$$\{a, b, b, c\} = \{c, a, b, b\}$$

$$\{a, b, c\} \neq \{c, a, b, b\}$$