SOEN 331: Introduction to Formal Methods for Software Engineering

Tutorial exercises on Temporal logic

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Problem 1

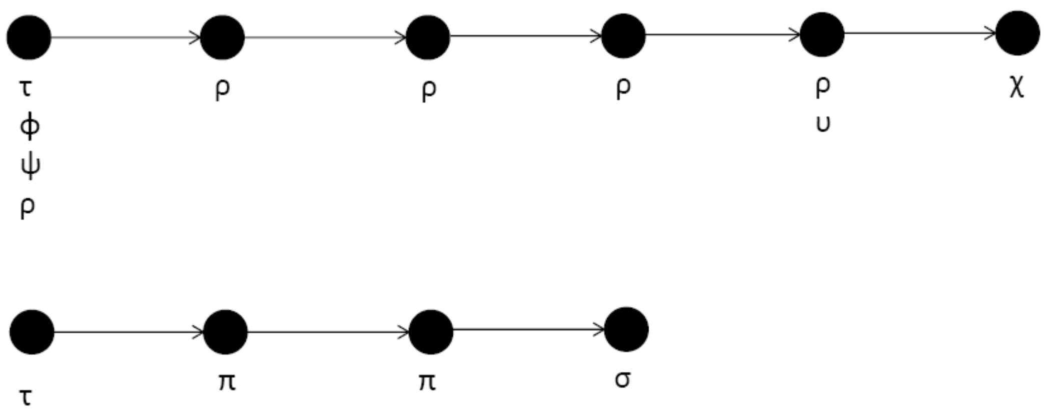
The behavior of a program is expressed by the following temporal formula:

$$\Box \left[\begin{array}{c} \textbf{start} \rightarrow \neg(\phi \oplus \psi) \\ \\ \textbf{start} \rightarrow \tau \\ \\ \tau \wedge (\neg\phi \wedge \neg\psi) \rightarrow \bigcirc(\pi \ \mathcal{U} \ \sigma) \\ \\ \tau \wedge \bigcirc\pi \rightarrow \bigcirc^3\sigma \\ \\ \tau \wedge \phi \rightarrow \upsilon \ \mathcal{R} \ \rho \\ \\ \tau \wedge \psi \wedge \bigcirc\rho \rightarrow \bigcirc^4\upsilon \\ \\ \rho \wedge \bigcirc(\rho \wedge \upsilon) \rightarrow \bigcirc^2\chi \end{array} \right]$$

1. Visualize all models of behavior.
2. Specify conditions (models of behavior), if any exist, under which the program can terminate. If none exist, please indicate so.

Solution:

The behavior is shown below:



There exist only two models of behavior. In both of them the program terminates:

$$\langle\ (\tau \wedge \phi \wedge \psi \wedge \rho), \rho, \rho, \rho, (\rho \wedge \upsilon), \chi\ \rangle$$

and

$$\langle\ \tau, \pi, \pi, \sigma\ \rangle$$

Problem 2

Interpret (do not visualize) each of the following temporal expressions:

1. $\bigcirc\Box\phi \rightarrow \Diamond\psi$

   If ϕ becomes an invariant at time $= i + 1$, then ψ eventually becomes true.

2. $\bigcirc^2\Box\phi \rightarrow \bigcirc^2\Box(\psi \rightarrow \chi)$

   If ϕ becomes an invariant at time $= i + 2$, then from that time whenever ψ becomes true, then χ also becomes true at the same time.

3. $\bigcirc(\phi \wedge \psi) \rightarrow \bigcirc^2\Box\chi$

   If both ϕ and ψ become true at time $= i + 1$, then χ becomes an invariant at time $i + 2$.

4. $(\phi \vee \psi) \rightarrow \bigcirc\chi \; \mathcal{W} \; \tau$

   If either ϕ, or ψ or both are true at time $= i$, then χ becomes true at time $= i + 1$ and it will remain true unless τ becomes true. (Note that there is no guarantee that τ will become true.)

5. $(\neg\phi \wedge \neg\psi) \rightarrow \tau \; \mathcal{U} \; \chi$

   We can simplify the LHS of the expression, using De Morgan's law, as $\neg(\phi \vee \psi)$. If ϕ and ψ are both false at $time = i$, then τ is true at time $= i$ and remains true until χ becomes true. Note that the expression imposes a guarantee that χ will eventually become true.

6. $\neg(\phi \oplus \psi) \rightarrow \Diamond\tau$

   If ϕ and ψ have the same truth value at time $= i$, then eventually τ will become true.

7. $(\chi \oplus \Box\tau) \rightarrow \bigcirc\Diamond\Box\omega$

   If either χ is true at time $= i$, or τ is an invariant (starting from time $= i$), then starting from time $= i + 1$ ω will eventually become an invariant.

8. $(\chi \wedge \bigcirc\psi) \rightarrow \bigcirc(\pi \;\mathcal{R}\; \tau)$

   If χ is true at time $= i$ and ψ is true at time $= i+1$, then starting from time $= i+1$ τ becomes true and remains true until and including the time where π becomes true. Note that π is not guaranteed to eventually become true.

9. $\Box(\phi \rightarrow \bigcirc\psi \;\mathcal{W}\; \sigma)$

   It is always the case that whenever ϕ is true, then ψ becomes true at the next time and remains true unless σ becomes true. Note that σ may never become true.

10. $\Box(\phi \rightarrow \Diamond\tau)$

    It is always the case that whenever ϕ becomes true, then eventually τ becomes true.

Problem 3

Provide a description and visualization of each of the following expressions:

1. $\Box\phi \rightarrow \Diamond\psi$

   If ϕ is an invariant, then ψ becomes true eventually.

2. $\Box\phi \rightarrow \bigcirc\Box\Diamond\psi$

   If ϕ is an invariant, the ψ will be true infinitely often, starting from the next moment.

3. $(\phi \wedge \bigcirc\psi) \rightarrow \Diamond\Box\tau$

   If phi is true at time $= i$ and ψ is true at time $= i + 1$, then eventually τ becomes true and stays true.

4. $\Box((\psi \wedge \bigcirc\chi) \rightarrow \bigcirc\tau)$

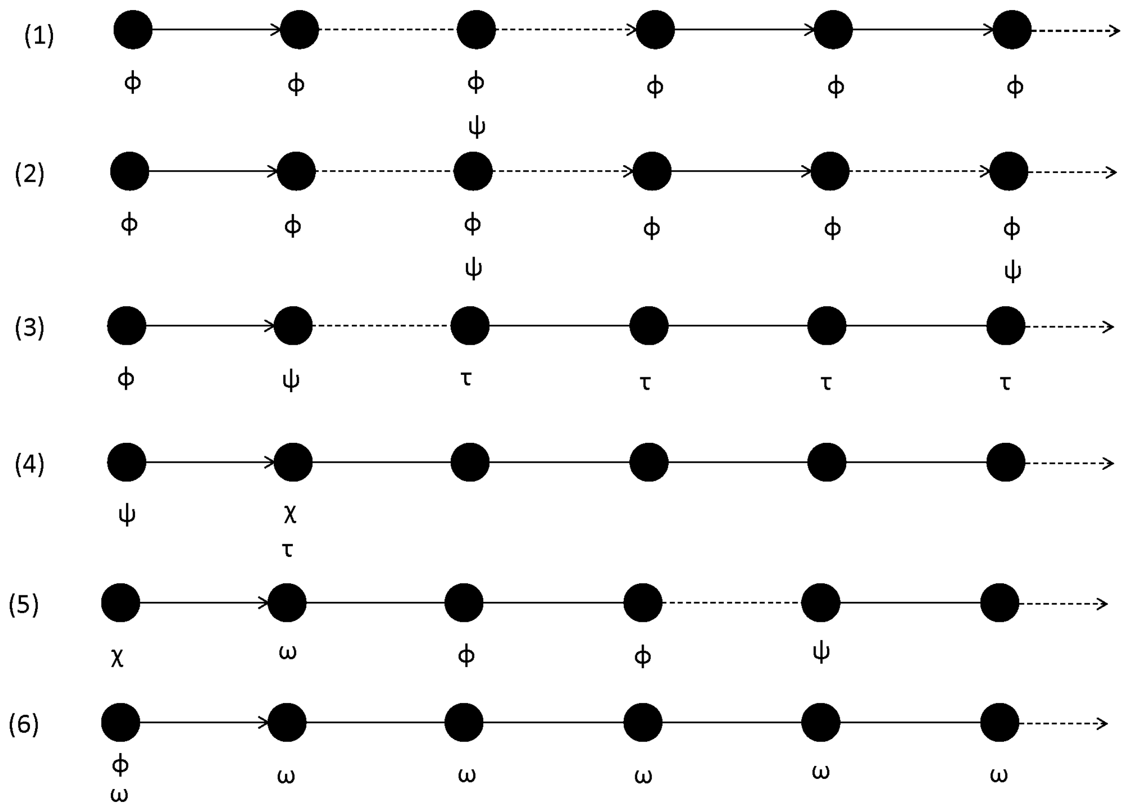
   It is always the case that if ψ is true at time $= i$ and if χ is true at time $= i + 1$, then τ is true at time $= i + 1$.

5. $(\chi \wedge \bigcirc\omega) \rightarrow \bigcirc^2(\phi\ \mathcal{U}\ \psi)$

   If χ is true at time $= i$ and if ω is true at time $= i + 1$, then at time $= i + 2$ ϕ becomes true and stays true until ψ becomes true.

6. $(\phi \oplus \psi) \rightarrow \Box\omega$

   If one of ϕ or ψ are true at time $= i$, then ω becomes at invariant at time $= i$.



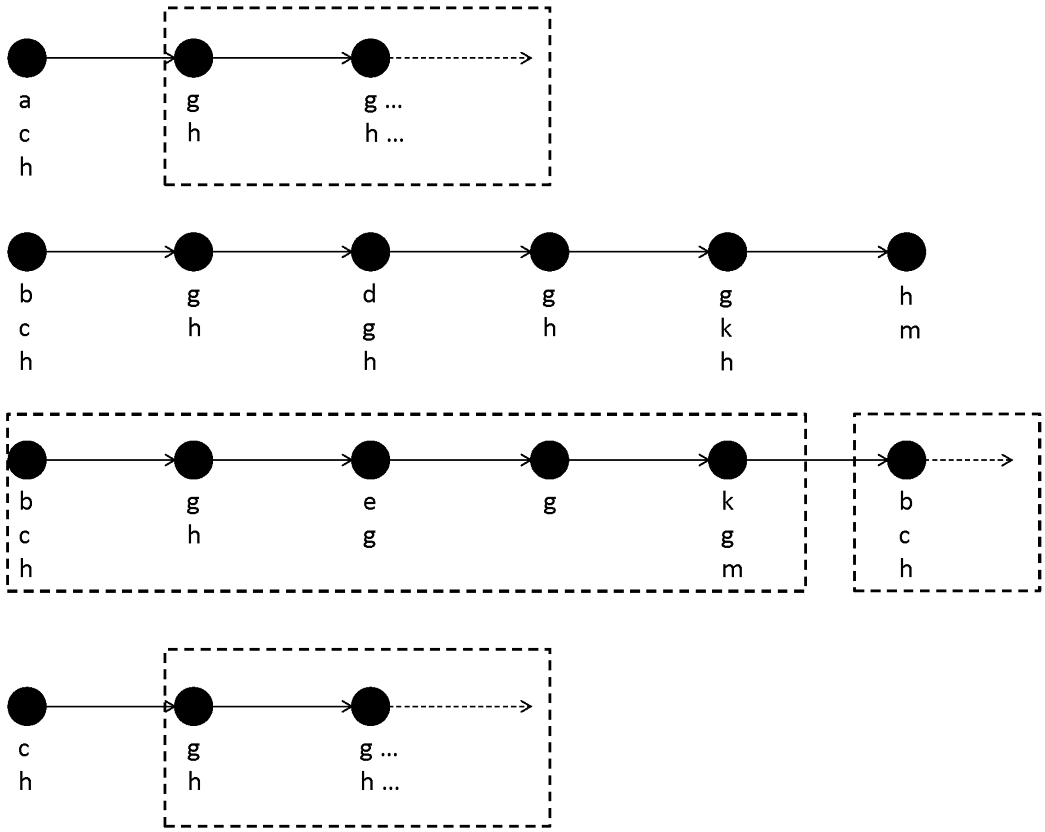
Also: Similar visualization for ψ instead of ϕ.

Problem 4

The behavior of a program is expressed by the following temporal formula:

$$\Box \left[\begin{array}{c} 1.\ \ \textbf{start} \rightarrow \neg a \vee \neg b \\ \\ 2.\ \ \textbf{start} \rightarrow c \\ \\ 3.\ \ b \wedge c \rightarrow \bigcirc^2 (d \oplus e) \\ \\ 4.\ \ a \vee c \rightarrow \bigcirc (k\ \mathcal{R}\ g) \\ \\ 5.\ \ (d \vee e) \rightarrow \bigcirc^2 k \\ \\ 6.\ \ c \rightarrow (h\ \mathcal{W}\ (e \wedge g)) \\ \\ 7.\ \ (d \wedge g \wedge h) \rightarrow \bigcirc^3 m \\ \\ 8.\ \ d \rightarrow m\ \mathcal{R}\ h \\ \\ 9.\ \ e \wedge \bigcirc^2 (k \wedge g) \rightarrow \bigcirc^2 m \\ \\ 10.\ \ (e \wedge g) \rightarrow \bigcirc^3 c \\ \\ 11.\ \ k \wedge m \rightarrow \bigcirc h \\ \\ 12.\ \ (e \wedge \bigcirc^2 k) \rightarrow \bigcirc^3 b \end{array} \right]$$

1. Visualize all models of behavior.



2. Specify conditions (model of behavior), if any exist, under which the program can terminate.

   There exists one model whereby the program terminates, given by the following expression:

   $\langle\ (b \wedge c \wedge h), (g \wedge h), (d \wedge g \wedge h), (g \wedge h), (g \wedge k \wedge h), (h \wedge m)\ \rangle$

3. For the expressions below, indicate (true/false) whether there exists a model where the expression holds. When true, cross reference your particular model:

PROPERTY	**TRUE/FALSE**
$(a \wedge c) \rightarrow \Diamond\Box(g \wedge h)$	**true (1)**
$h\ \mathcal{U}\ m$	**false**
$h\ \mathcal{U}\ (k \wedge g)$	**false**
$(b \wedge c) \rightarrow \Box\Diamond(b \wedge c)$	**true (3)**
$(k \wedge \bigcirc(k \wedge g)) \rightarrow \bigcirc m$	**false**
$h\ \mathcal{S}\ c$	**true (4)**
$((g \wedge h) \wedge \bigcirc d) \rightarrow \bigcirc^2(g \wedge h)$	**true (2)**
$e\ \mathcal{R}\ h$	**false**
The program has the following stability property: $\Diamond\Box(b \wedge\ c \wedge h)$	**false**
The program has the following recurrence property: $\Box\Diamond(b \wedge\ c \wedge h)$	**true (3)**
$(g \wedge h)$ is an invariant property of the program.	**false**
There is a guarantee that $(g \wedge k \wedge h)$	**true (2)**
The program has the following property: $(b \wedge c \wedge h) \rightarrow \Diamond(b \wedge c \wedge h)$.	**true (3)**
The program has the following precedence property: $(b \wedge c \wedge h) \rightarrow ((g \wedge h)\ \mathcal{U}\ b \wedge c \wedge h))$	**false**

Problem 5

Provide a description and visualization of each of the following expressions:

1. $\Box(\phi \rightarrow \bigcirc^2\psi)$

2. $\Box\phi \rightarrow \bigcirc\psi$

3. $\phi \rightarrow \bigcirc\Diamond\Box\psi$

4. $(\phi \wedge \bigcirc\psi) \rightarrow \bigcirc^2\Diamond\Box\omega$

5. $\Box((\phi \wedge \bigcirc\psi) \rightarrow \bigcirc^2\Diamond\Box\omega)$

6. $(\phi \wedge \bigcirc\psi) \rightarrow \tau\ \mathcal{R}\ \upsilon$

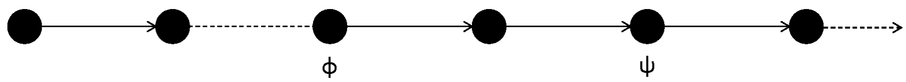
7. $(\phi \wedge \bigcirc\psi) \rightarrow \bigcirc(\tau\ \mathcal{R}\ \upsilon)$

8. $(\phi \wedge \bigcirc\psi) \rightarrow \bigcirc(x\ \mathcal{U}\ \tau)$

9. $(\phi \wedge \Box\psi) \rightarrow \bigcirc^2\Diamond\omega$

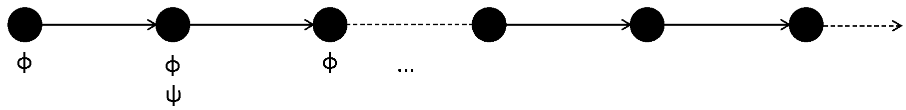
10. $(\phi \wedge \bigcirc^2\psi) \rightarrow \bigcirc^2\Box\omega$

(1)



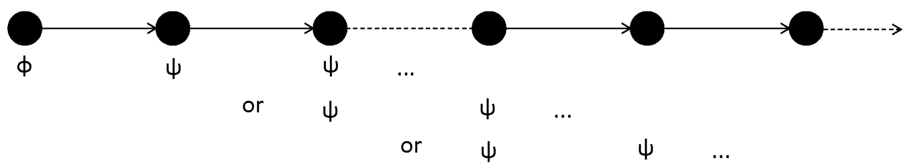
Whenever ϕ becomes true, then ψ becomes true in 2 moments of time.

(2)



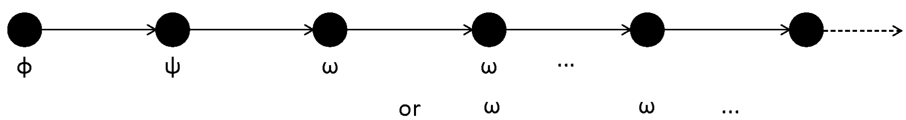
If ϕ is an invariant, then ψ becomes true at time i+1.

(3)



From i+1 onwards, ψ could become an invariant.

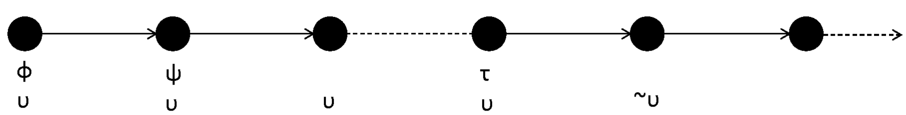
(4)



From i+2 onwards, ω can become an invariant anywhere on the path.

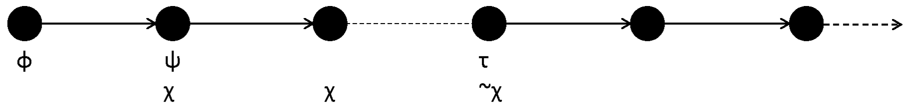
(5) As above, but entire sequence <ϕ, ψ, [...] ω, ...> can appear anywhere on the path.

(6)

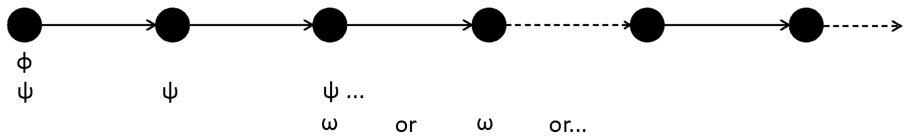


(7) As above, but υ starts being true at time i+1.

(8)

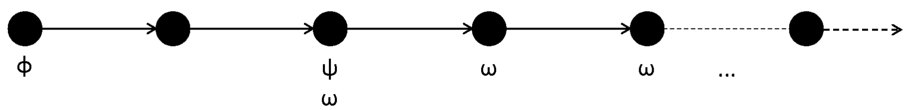


(9)



ψ is globally true (or: an invariant property)

(10)



ω becomes an invariant from *i + 2*

Problem 6

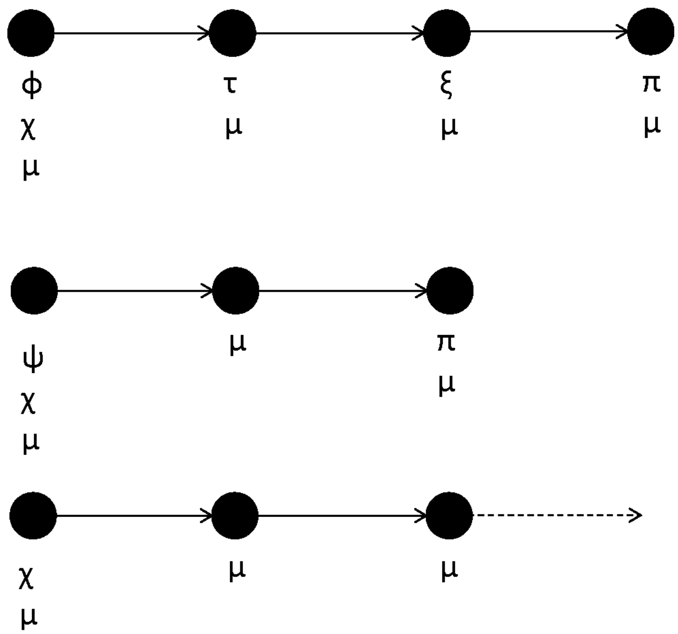
The behavior of a program is expressed by the following temporal formula:

$$\Box \begin{bmatrix} \textbf{start} \rightarrow \neg\phi \vee \neg\psi \\ \\ \textbf{start} \rightarrow \chi \\ \\ \phi \wedge \chi \rightarrow \bigcirc(\tau \; \mathcal{W} \; \xi) \\ \\ \chi \rightarrow (\pi \; \mathcal{R} \; \mu) \\ \\ \psi \wedge \bigcirc\mu \rightarrow \bigcirc^2\pi \\ \\ \tau \wedge \mu \rightarrow \bigcirc\xi \\ \\ \xi \wedge \mu \rightarrow \bigcirc\pi \end{bmatrix}$$

1. Visualize all models of behavior.
2. Specify conditions (models of behavior), if any exist, under which the program can terminate. If none exist, please indicate so.

Solution:

The behavior is shown below:



There exist two models of behavior whereby the program terminates, given by the following expressions:

$$\langle\ (\phi \wedge \chi \wedge \mu), (\tau \wedge \mu), (\xi \wedge \mu), (\pi \wedge \mu)\ \rangle$$

and

$$\langle\ (\psi \wedge \chi \wedge \mu), \mu, (\pi \wedge \mu)\ \rangle$$

Problem 7

Interpret each of the following temporal expressions:

1. $\Box\phi \rightarrow \bigcirc\psi$

   If ϕ is an invariant, then ψ becomes true at time $i+1$.

2. $\Box\phi \rightarrow \bigcirc\Diamond\Box\psi$

   If ϕ is an invariant, then from time $i+1$, ψ will eventually become an invariant.

3. $(\phi \wedge \bigcirc\psi) \rightarrow \bigcirc\Box\chi$

   If ϕ is true at time $= i$ and ψ is true at time $= i+1$, then χ becomes an invariant at time $i+1$.

4. $(\phi \wedge \psi) \rightarrow \phi \ \mathcal{W} \ \tau$

   If ϕ and ψ are both true at time $= i$, then ϕ remains true unless τ becomes true. (Note that there is no guarantee that τ will become true.)

5. $(\neg\phi \vee \neg\psi) \rightarrow \bigcirc(\tau \ \mathcal{U} \ \chi)$

   The LHS of the expression can be rewritten as $\neg(\phi \wedge \psi)$. If either ϕ is true at $time = i$, or ψ is true at $time = i$, or if both ϕ and ψ are false at $time = i$, then τ becomes true at time $i+1$ and remains true until χ becomes true. Note that there is a guarantee that χ will eventually become true.

6. Is the previous expression semantically equivalent to

   $(\phi \oplus \psi) \rightarrow \bigcirc(\tau \ \mathcal{U} \ \chi)$? Explain your reasoning.

   No. The current expression forces one of ϕ or ψ to be true.

7. $(\chi \oplus \Box\psi) \rightarrow \Box\Diamond\omega$

   If either χ is true at time i, or ψ is an invariant (starting from time i), then ω is true infinitely often.

8. $(\Box\chi \wedge \bigcirc\psi) \rightarrow \bigcirc(\pi \; \mathcal{R} \; \tau)$

   If χ is an invariant (starting from time i), and ψ is true at time $i+1$, then τ becomes true at time $i+1$ and remains true until and including the time where π becomes true. Note that π is not guaranteed to eventually become true.

9. $\phi \rightarrow \bigcirc(\psi \wedge \chi \; \mathcal{U} \; \tau)$

   If ϕ is true at time i, then ψ becomes true at time $i+1$ and χ also becomes true at time $i+1$ and stays true until (but not including) the time when τ becomes true. Note that τ is guaranteed to eventually become true.