Animation for Computer Games
COMP 477/6311

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Rotations - Quaternions

Quaternions

- Elegant notion for representing rotations
- Compact representation
- Mathematical element with a set of operators (e.g. multiplication)
- Efficient implementation
- Widely used for animation

Quaternions

FIGURE 1.1 *Sir William Rowan Hamilton, 4 August 1805–2 September 1865.* (History of Mathematics web pages of the University of St. Andrews, Scotland.)

Quaternions

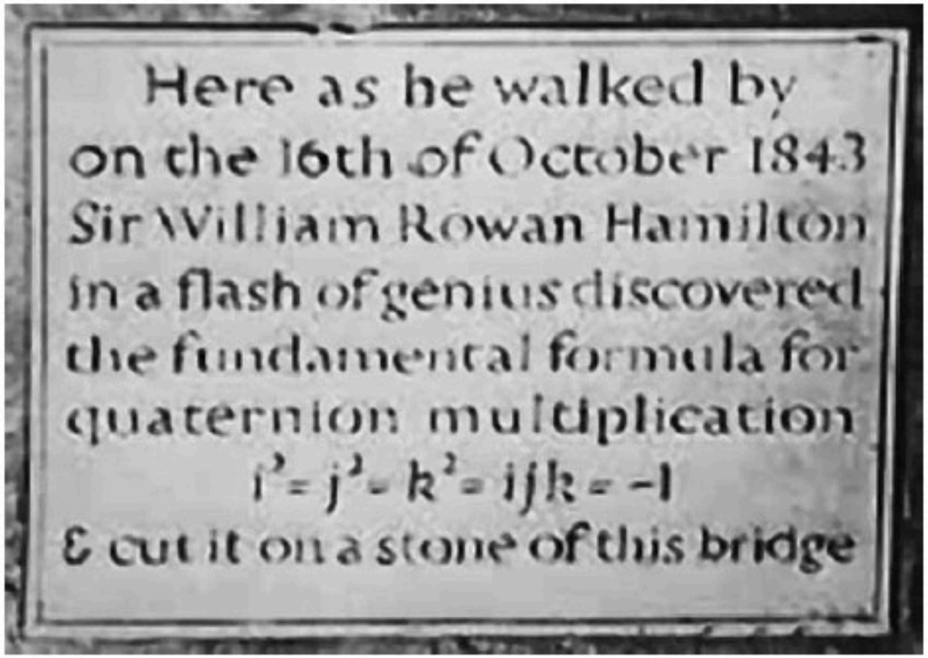


FIGURE 1.2 The plaque on Broome Bridge in Dublin, Ireland, commemorating the legendary location where Hamilton conceived of the idea of quaternions. In fact, Hamilton and his wife were walking on the banks of the canal beneath the bridge, and the plaque is set in a wall there. (Photograph courtesy University of Dublin.)

Definition

- A ***quaternion q*** is a "hypercomplex" number defined as:

$$\boldsymbol{q} = c + x\,i + y\,j + z\,k$$

- *c, y, x, z* are real numbers
- *i, j, k* are imaginary

- Notation:

$$\boldsymbol{q} = c + \boldsymbol{u}$$

- *c*: real part, ***u***: ***pure quaternion***

$$\boldsymbol{u} = x\,i + y\,j + z\,k$$

Basic Properties and Operators

- Addition

$$\boldsymbol{q}+\boldsymbol{q}'=(c+c')+(x+x')i+(y+y')j+(z+z')k$$

- Multiplication is similar to complex numbers

- Multiplication of complex operators

$$i^2 = j^2 = k^2 = -1$$

$$\boxed{ij=k, \quad ji=-k;} \quad jk=i, \quad kj=-i; \quad ki=j, \quad ik=-j$$

Quaternion multiplication is NOT commutative!

Basic Properties and Operators

- One-elements of addition and multiplication

$$0 = 0 + 0i + 0j + 0k$$

$$1 = 1 + 0i + 0j + 0k$$

Basic Properties and Operators

- Conjugate elements

$$\boxed{\boldsymbol{q} = c + \boldsymbol{u} \quad ; \quad \bar{\boldsymbol{q}} = c - \boldsymbol{u}}$$

- Absolute

$$\boldsymbol{q}\bar{\boldsymbol{q}} = c^2 + x^2 + y^2 + z^2$$

$$\boldsymbol{q}\bar{\boldsymbol{q}} = |\boldsymbol{q}|^2$$

- Inverse elements of
  - addition

$$-\boldsymbol{q} = -c - x\,i - y\,j - z\,k$$

  - multiplication $\boldsymbol{q}^{-1} = \dfrac{1}{|\boldsymbol{q}|^2}\bar{\boldsymbol{q}}$

Unit Quaternions

- Quaternions of length 1 are fundamental to encode transformations
- To the whiteboard

3D Rotation using Quaternions

- Represent point ***P*** as a pure quaternion

$$\boldsymbol{P} = (x, y, z) \qquad \boldsymbol{p} = 0 + \boldsymbol{v} = xi + yj + zk$$

- Rotation using quaternion operators

$$\boxed{\boldsymbol{R_q}(\boldsymbol{p}) = \boldsymbol{q}\boldsymbol{p}\bar{\boldsymbol{q}} \qquad \boldsymbol{q} = c + \boldsymbol{u} = \cos\theta + \sin\theta\, \boldsymbol{n}}$$

- Rotation of ***P*** along axis ***N*** by angle 2θ

Points and Rotation

- Recipe: Take ***p*** and ***q*** and compute

$$\boldsymbol{p}' = \boldsymbol{q}\boldsymbol{p}\overline{\boldsymbol{q}}$$

$$\boldsymbol{q} = cos(\theta / 2) + sin(\theta / 2)\boldsymbol{n}$$

$$\boldsymbol{n} = N_1 i + N_2 j + N_3 k$$

- Elegant implementation using operator overloading

Points and Translation

- Translation vector as a pure quaternion

$$\boldsymbol{p}' = \boldsymbol{p} + \boldsymbol{t}$$

- Sequences of rotations ***r*** and translations ***t*** using a transformation operator ***M***

$$\boldsymbol{p} \rightarrow \boldsymbol{p}' = \boldsymbol{M}_{(\boldsymbol{t},\boldsymbol{r})}(\boldsymbol{p}) = \boldsymbol{r}\boldsymbol{p}\bar{\boldsymbol{r}} + \boldsymbol{t}$$

- $\boldsymbol{M}_{(\boldsymbol{t},\boldsymbol{r})}$ denotes rotation-translation-operator
- $\boldsymbol{M}_{(0,\boldsymbol{r})}$ describes rotation and $\boldsymbol{M}_{(\boldsymbol{t},1)}$ translation

$$\boldsymbol{M}_{(\boldsymbol{t},\boldsymbol{r})} = \boldsymbol{M}_{(0,\boldsymbol{r})} \circ \boldsymbol{M}_{(\boldsymbol{t},1)}$$

Interpolation

Shoemake (1985) introduced quaternion interpolation to graphics.

- Same problem as matrices: quaternion combination is multiplicative
- To interpolate between q_0 and q_1, we need:

$$q = q_0^{1-t}\, q_1^t = q_0\left(q_0^{-1}\, q_1\right)^t$$

- Efficient computation:

$$q = \frac{\sin(1-t)\Omega}{\sin\Omega} q_0 + \frac{\sin t\Omega}{\sin\Omega} q_1$$

where

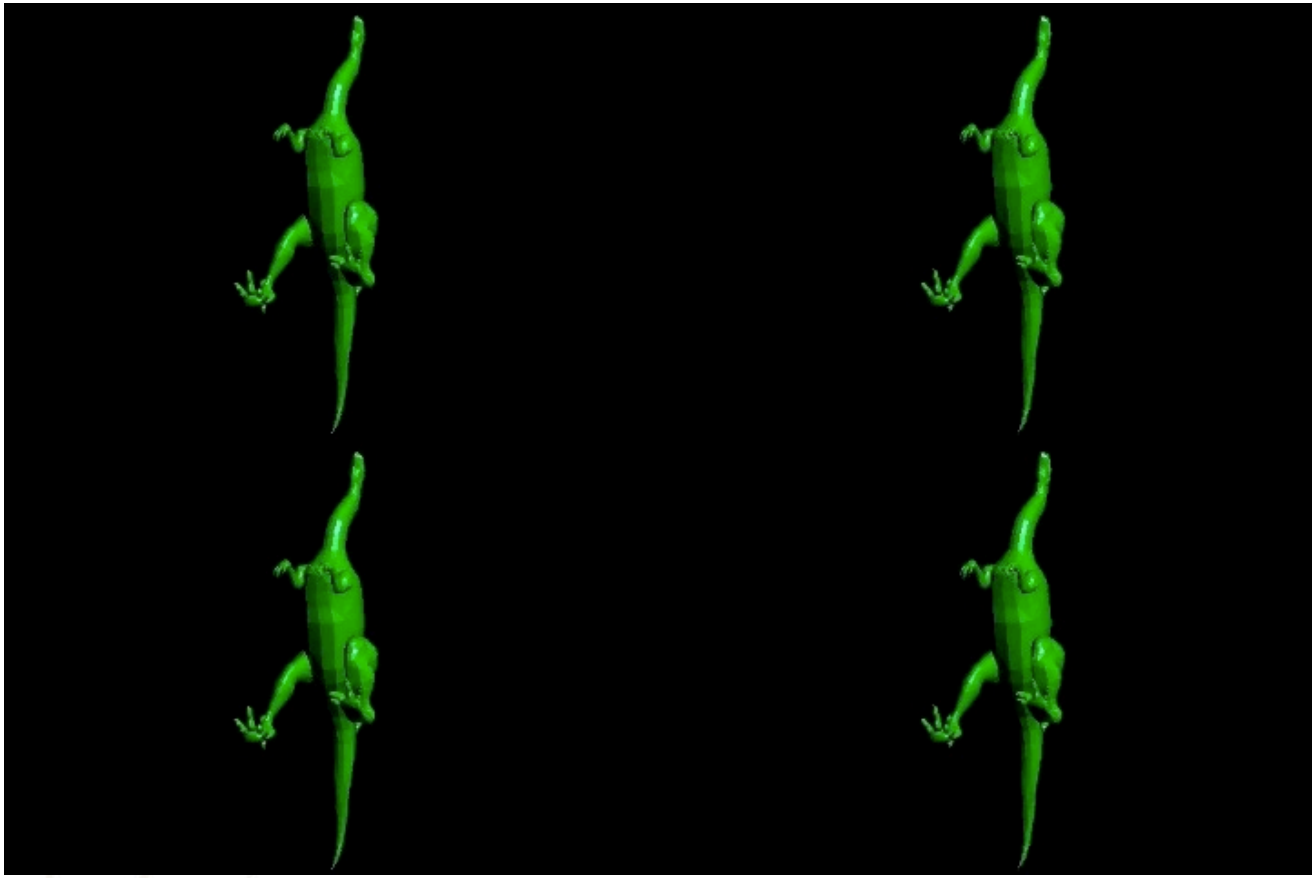
$$\cos\Omega = q_0 \cdot q_1$$

- For small angles, linear interpolation is good enough but faster:

$$q = (1-t)q_0 + tq_1$$

However, normalization may be required.

Rotations



https://www.youtube.com/watch?v=94USA9yMzAw

Conversions

The quaternion $[s;\ (x, y, z)]$ corresponds to the matrix $\mathbf{M}$ where

$$\mathbf{M} = \begin{bmatrix} 1-2(y^2+z^2) & 2(xy-sz) & 2(xz+sy) \\ 2(xy+sz) & 1-2(x^2+z^2) & 2(yz-sx) \\ 2(xz-sy) & 2(yz+sx) & 1-2(x^2+y^2) \end{bmatrix}$$

If

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

is a rotation matrix, the corresponding quaternion $[s;\ (x, y, z)]$ can be calculated using Shoemake's algorithm. ϵ is a small number, e.g., 10^{-6}.

$$\begin{aligned}
&s^2 := (1 + M_{11} + M_{22} + M_{33})/4 \\
&\text{if } s^2 > \epsilon\text{:} \\
&\qquad x := (M_{23} - M_{32})/4s \\
&\qquad y := (M_{31} - M_{13})/4s \\
&\qquad z := (M_{12} - M_{21)}/4s
\end{aligned}$$

else:

    $s := 0$

    $x^2 := -(M_{22} + M_{33})/2$

    if $x^2 > \epsilon$:

        $y := M_{12}/2x$

        $z := M_{13}/2x$

    else:

        $x := 0$

        $y^2 := (1 - M_{33})/2$

        if $y^2 > \epsilon$:

            $z := M_{23}/2y$

        else:

            $y := 0$

            $z := 1$

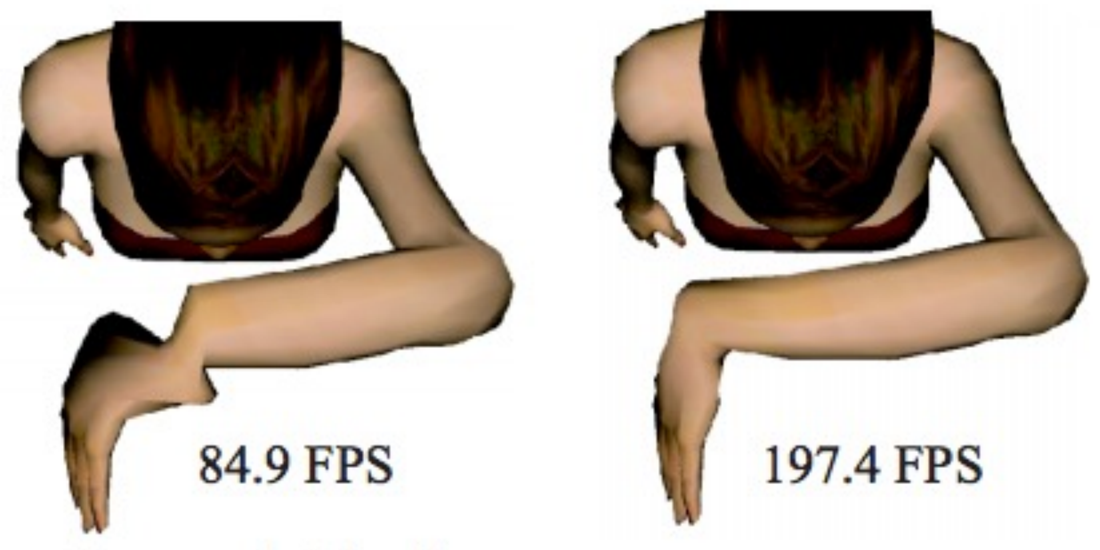
Let

$$q = (s, x, y, z).$$

Then

$$\begin{aligned}
\theta &= \text{atan2}\left(2(yz + xz),\ -x^2 - y^2 + z^2 + s^2\right) \\
\phi &= \sin^{-1}(-2(xz - ys)) \\
\psi &= \text{atan2}\left(2(xy + zs),\ x^2 - y^2 - z^2 + s^2\right)
\end{aligned}$$

LBS vs. Quaternions



[Kavan et al]

Linear Blend Skinning (LBS)