Functions

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Functions: Some terminology

- For sets A and B, a *function* from A to B, denoted as

  $$f : A \to B$$

  is an assignment (mapping) of each element of (source set) A to exactly one element of (target set) B.
- For a unique $b \in B$ assigned by f to $a \in A$ we write

  $$f(a) = b$$

- A is the *domain* of f, and B is the *codomain* of f.
- In $f(a) = b$, b is the *image* of a, and a is the *pre-image* of b.
- The set of all images is the *range* of f. The range, R, is the particular set of values in the codomain that the function actually maps elements of the domain to, i.e.
  $R \subseteq B$ of f.

Functions: Domain and codomain

- We have already seen propositional functions in predicate logic, e.g. $P(x)$: "x is an island" is a function from objects to propositions, e.g.
  P : Object $\rightarrow$ Proposition.
- For example: $P(Montreal) =$ "Montreal is an island."
- A propositional operator can be viewed as a function from ordered pairs of Boolean values to a Boolean value, e.g.
  $\wedge$: Boolean $\times$ Boolean $\rightarrow$ Boolean.
- Example: $\wedge((T, F)) = F$.

Partial functions

- A *partial function* from (source set) A to (target set) B denoted as

  $$f : A \nrightarrow B$$

  is a function defined for some subset A' of A, i.e. it does not force the mapping for every element of A to an element of B, i.e.

  $$dom\, f \subset A$$

  as opposed to a *total function* where $dom\, f = A$.

One-to-one (injective) functions

- f is *one-to-one* (or *injective*) if *each* element of the codomain is mapped to by at most one element of the domain (never maps distinct elements of its domain to the same element of its codomain), i.e.

$$\forall a, b \, (a \neq b \rightarrow f(a) \neq f(b))$$

  in the domain of f.
- Is $f(x) = x^2$ one-to-one? (Note: The domain is the set of integers.)
- No since $f(1) = f(-1) = 1$, but $1 \neq -1$.

Onto (surjective) functions

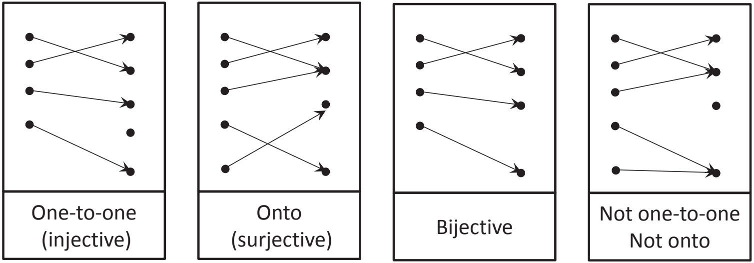
- f is *onto* (or *surjective*) if each element of the codomain is mapped to by at least one element of the domain, iff for every $b \in B$ there is an $a \in A$ with $f(a) = b$, i.e.

  $$\forall\, b\ \exists\, a\,(f(a) = b)$$

Bijective functions

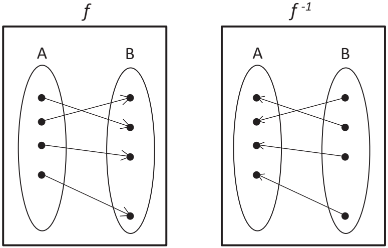
- f is *one-to-one correspondence* (or *bijection*) if it is both one-to-one and onto.

Visualization of function types



Inverse functions

- If f is a bijection such that $f : A \to B$, then there is a function from B to A that maps each element of B back to its corresponding element in A.



- This is called the inverse function for f, denoted by $f^{-1} : B \to A$.

Composition of functions

- If $f : A \to B$ and $g : B \to C$ are functions, then $g \circ f$ is a composite relation between A and C such that

  $$g \circ f : A \to C$$

  given by

  $$(g \circ f)(a) = g(f(a))$$

- Note that $ran\, f \subseteq dom\, g$.

Example: Composition of functions

- Consider the following:

$$\forall x \in \mathbb{N} \bullet f(x) = x + 1 \wedge g(x) = 5x$$

$$g(f(3)) = g(4) = 20, or (g \circ f)(3) = 20$$

- Note that $dom\,(g \circ f) \subseteq dom\,f$.

Modeling functions by product sets

- For example, for function with $dom\, f = \{1, 2, 3, 4\}$ and $\forall\, x \in dom\, f \bullet f(x) = x(x-2)$, we often write

  $$f = \{(1, -1), (2, 0), (3, 3), (4, 8)\}$$

- Note on notation: Ordered pairs are often represented using *maplet notation*, e.g. the pair (x, y) is written as $x \mapsto y$.
- We often talk about the set which models a function as actually being the function itself.