Predicate logic

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Introduction

- Much like propositional logic, predicate logic uses symbols to represent knowledge.
- Propositional logic does not allow us to say things like *"All objects of this class have this particular property"*, or *"Some objects of this class have this property"*, etc.

Introduction /cont.

- Let $P(x)$ denote the statement *"x is an odd number."*
- The statement cannot be given a truth value since the value for x is not yet specified. We say that variable x is a *free variable* and it constitutes the *subject* of the statement, whereas *"is an odd number"* refers to a property of the subject and is called the *predicate*.
- $P(x)$ is called a *propositional function*, as each choice of x produces a proposition.

Introduction /cont.

- **Definition**: A *propositional function* is a statement containing one or more free variables. The statement becomes a proposition once values are assigned to all its free variables.
- In $P(x)$: *"x is an odd number."*, $P(5)$ becomes a proposition by setting $x = 5$ and its value is true, whereas the proposition $P(6)$ is false.
- Sometimes, the entire $P(x)$ is referred to as a predicate.

Using predicates to define sets: Untyped set comprehension

- $\{x \mid x < 5\}$ is the set of all values of x for which $x < 5$.
- This notation is an example of *untyped set comprehension*.
- We read "the set of values x such that $x < 5$."

Using predicates to define sets: Untyped set comprehension /cont.

- A set comprehension can be modeled by a decision procedure where for an input variable it gives a result of true or false, e.g. $2 \in \{x \mid x < 5\}$ is true.

Using predicates to define sets: Typed set comprehension

- A *typed set comprehension* is of the form $\{x \in X \mid p(x)\}$ where $p(x)$ is a predicate with free variable x.
- For example, $\{x \in \mathbb{N} \mid x < 5\}$ represents the set of natural numbers less than five.
- Is $2.5 \in \{x \in \mathbb{N} \mid x < 5\}$ true? No, since $2.5 \notin \mathbb{N}$.

Set replacement

- An extension to set comprehension is to follow the declaration and predicate by a formula, e.g.
  $\{x \in \mathbb{N} \mid x < 5 \bullet x^2\}$ refers to a set of numbers which are squares of numbers less than 5, i.e. $\{0, 1, 4, 9, 16\}$.
- Each element in the set $\{x \in \mathbb{N} \mid x < 5\}$ is replaced by its square so that a new set is formed.

Quantifiers

- As its name suggests, a *quantifier* is an operator that states that a predicate is true for a given quantity of objects.
- The expression "for all", denoted by the symbol $\forall$, is called the *universal quantifier*.
- The expression "there exists", denoted by the symbol $\exists$ is called the *existential quantifier*.

Universal quantification

- **Definition**: The *universal quantification* of $P(x)$ is the statement "P(x) is true for all values of x." Symbolically this is expressed as $\forall x P(x)$.
- The statement $\forall x P(x)$ is true when $P(x)$ is true for all x. It is false when there is at least one x for which $P(x)$ is false.
- Its negation becomes $\neg \forall x P(x)$ and it is true when there is an x for which $P(x)$ is false, or $\exists x \neg P(x)$.

An initial example of a universal quantification

- In the domain of animals, how do we express "All cats are mammals"?
- We can rephrase the statement as "If x is a cat, then x is a mammal":

$$\forall x(cat(x) \rightarrow mammal(x))$$

Example: Universal quantification

- How do we express "Only dogs bark"?
- We can rephrase the statement as "It barks only if it is a dog", or "If it barks, then it is a dog":

$$\forall x(barks(x) \rightarrow dog(x))$$

Existential quantification

- **Definition**: The *existential quantification* of $P(x)$ is the statement "There exists an element x such that P(x) is true." Symbolically this is expressed as $\exists x P(x)$.
- The statement $\exists x P(x)$ is true when there is at least one x for which $P(x)$ is true.
- It is false when $P(x)$ is false for all x. Its negation becomes $\neg \exists x P(x)$ and it is true when for every x, $P(x)$ is false, or $\forall x \neg P(x)$.
- In existential quantification we can make a distinction between "there exists at least one" (as defined above) and "there exists exactly one", symbolically expressed as $\exists!$.

Example: Combining quantifiers

- Consider the following statement: "For every integer number x, there is a successor integer number y."
- We can denote the successor property with the predicate $P(x, y) : y = x + 1$ and write the statement symbolically as

$$\forall x \exists y P(x, y)$$

Bound variables

- When we assign a value to a free variable, then the variable becomes a *bound variable*.
- Another way for a free variable to become bound is when we can apply a quantifier to it, such as *"There exists an x such that $x > 0$."*
- We say that the variable x is bound by the quantifier "there exists."

Precedence rules in the presence of quantifiers

- Both universal and existential quantifiers have a higher precedence than the rest of the connectives, but they have a lower precedence than the negation operator.
- Consider the predicates $p(x) =$ "x is living." and $q(x) =$ "x is dead":

1 $\forall x (p(x) \vee q(x))$ is interpreted as everything is either living or dead.

2 $\forall x p(x) \vee q(x)$ is interpreted as everything is living or x is dead.

Negation

- Consider the statement *"All birds are white"* or

$$\forall x P(x)$$

  where the predicate $P(x)$ stands for *"bird is white."*
- To negate the statement we can say *"It is not true (or: not the case) that all birds are white"* or *"Not all birds are white."*

Negation /cont.

- Note that it would be incorrect to say *"All birds are not white"* as this would claim that there are no white birds which is clearly false.
- The negated statement can be written symbolically as

$$\neg[\forall x P(x)]$$

- An alternative way to express the negated statement is to say *"There is at least one bird that is not white"*, written symbolically as

$$\exists x[\neg P(x)]$$

Negation /cont.

- Note that the process of negating the statement corresponds to negating the predicate and changing the quantifier. In other words
$$\neg[\forall x P(x)] \equiv \exists x[\neg P(x)]$$
- Similarly, to negate a predicate with the existential quantifier, we can write
$$\neg[\exists x P(x)] \equiv \forall x[\neg P(x)]$$

Negation - Summary

Negated statement	Equivalent statement	When is it true?	When is it false?
$\neg[\forall x P(x)]$	$\exists x[\neg P(x)]$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.
$\neg[\exists x\ P(x)]$	$\forall x[\neg P(x)]$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.

Nested quantifiers

- A quantifier is called *nested* if it occurs within the scope of another. For quantifiers of the same type, the order does not matter.
- However, the order in which quantifiers of different types are placed is important.
- $\forall x \exists y P(x, y)$ reads **"For every x there is a y for which $P(x, y)$ is true."**
- In other words, no matter which x we choose, there must be a value of y (possibly depending on the choice of x) for which $P(x, y)$ is true.
- $\exists y \forall x P(x, y)$ reads **"There in an y that makes $P(x, y)$ true for every x."**

Example 1: Nested quantifiers

- Consider the predicate $loves(x, y)$ denoting "x loves y". We can express the following predicates using nested quantifiers:
- $\forall x \exists y \; loves(x, y)$ reads "Everyone loves someone", i.e. no matter which x we choose, there must always be some y that makes $loves(x, y)$ true.
- $\exists y \forall x \; loves(x, y)$ reads "There is someone who is loved by everybody", i.e. there is some particular y for which $loves(x, y)$ is true, regardless of the choice of x.

Example 1: Nested quantifiers /cont.

- If $\exists y \forall x P(x, y)$ is true, then $\forall x \exists y P(x, y)$ is also true, e.g. if "There is someone who is loved by everybody", then we can safely conclude that "Everyone loves someone."
- However, if $\forall x \exists y P(x, y)$ is true, e.g. "Everyone loves someone", then it is not necessary that $\exists y \forall x\ loves(x, y)$ is true, e.g. it is not necessary that "There is someone who is loved by everybody."

Example 1: Nested quantifiers

- We can also translate the following sentences into formal statements:
- "Someone loves someone." $\exists x \exists y \; \mathit{loves}(x, y)$, or $\exists y \exists x \; \mathit{loves}(x, y)$
- "Everyone loves someone." $\forall x \exists y \; \mathit{loves}(x, y)$
- "Everyone is loved by someone." $\forall y \exists x \; \mathit{loves}(x, y)$
- "There is someone who loves everyone." $\exists x \forall y \; \mathit{loves}(x, y)$
- "There is someone who is loved by everybody." $\exists y \forall x \; \mathit{loves}(x, y)$
- "Everyone loves everyone." $\forall x \forall y \; \mathit{loves}(x, y)$, or $\forall y \forall x \; \mathit{loves}(x, y)$

Example 2: Nested quantifiers

- Consider the predicate $B(p, r)$ denoting the predicate *"Person p has booked room r."* and the sentence *"No room is booked by more than one person."*
- If no room is booked by more than one person, then the predicates $B(p, r)$ and $B(q, r)$ cannot both be true unless p and q denote the same person.
- Symbolically this can be expressed as

$$\forall p \forall q \forall r[(B(p, r) \wedge B(q, r)) \rightarrow (p = q)]$$

Example 3: Nested quantifiers

- Consider the following: "Every airline x flies to exactly one city y." This can be formulated as:

$$\forall x \exists! y (\mathit{airline}(x) \land \mathit{city}(y) \land \mathit{flies}(x, y))$$

Example 4: Nested quantifiers

- Let $sentMessage(x, y)$ be the statement "x has sent a message to y" where the domain is all students in class. Note that x and y can be the same person.
- $\exists x \exists y P(x, y)$: The is some student who has sent a message to some student.
- $\forall x \forall y P(x, y)$: Every student in the class has sent a message to every student in the class.

Example 4: Nested quantifiers /cont.

- $\exists y \forall x P(x, y)$: Recall that this reads **"There in a y that makes $P(x, y)$ true for every x."** There is a student in class who has been sent a message by every student in class.
- $\exists x \forall y P(x, y)$: This is similar to the above, only the roles have been changed: There is some student who has sent a message to every student in the class.

Example 4: Nested quantifiers /cont.

- $\forall x \exists y P(x, y)$: Recall that this reads **"For every** x **there is a** y **for which** $P(x, y)$ **is true."** Every student in class has sent a message to some (at least one) student in class.
- $\forall y \exists x P(x, y)$: This is similar to the above, only the roles have been changed: Every student in class has been sent a message from some (at least one) student in class.

Example 5: Nested quantifiers

- Consider the statement "Everyone has exactly one movie that he or she likes."
- Let us express the statement in formal logic by introducing the predicate $\mathit{likes}(x, y)$ to represent the statement "x likes movie y."
- Let us start with $\forall x \exists y\ \mathit{likes}(x, y)$ which reads **"For every x there is a y for which $\mathit{likes}(x, y)$ is true"**, or "Everyone likes some movie."
- This does not fully capture the original statement because it does not exclude the possibility that there may be more than one movie that a person likes.

Example 5: Nested quantifiers /cont.

- To complete the statement we need to add that no other movie (i.e. no movie which is not y) is liked by x, or $\forall z((z \neq y) \rightarrow \neg likes(x, z))$.
- Putting everything together we have

$$\forall x \exists y(likes(x, y) \wedge \forall z((z \neq y) \rightarrow \neg likes(x, z)))$$

Negating nested quantifiers

The negation of nested quantifiers follows the same rules are with single quantifiers. The statement

$$\forall x \exists y P(x, y)$$

reads "For every x, there is a y for which $P(x, y)$ is true." The negated statement would read "There is an x such that $P(x, y)$ is false for every y", or

$$\exists x \forall y \neg P(x, y)$$

The statement

$$\exists x \forall y P(x, y)$$

reads "There is an x for which $P(x, y)$ is true fo every y." The negated statement would read "For every x there is a y for which $P(x, y)$ is false", or

$$\forall x \exists y \neg P(x, y)$$

Summary of quantifications of two variables

Statement	When is it true	When is it false
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair (x, y).	There is a pair (x, y) for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There in an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair (x, y), for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair (x, y).

Equivalences

$$\forall x \forall y \; P(x,y) \equiv \forall y \forall x \; P(x,y)$$

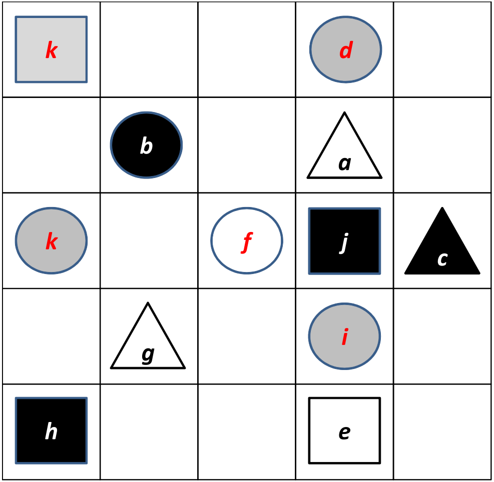
$$\exists x \exists y \; P(x,y) \equiv \exists y \exists x \; P(x,y)$$

$$\exists x \; (P(x) \to Q(x)) \equiv \forall x \; P(x) \to \exists x \; Q(x)$$

$$\exists x \; (P(x) \vee Q(x)) \equiv \exists x \; P(x) \;\; \vee \;\; \exists x \; Q(x)$$

$$\forall x \; (P(x) \wedge Q(x))) \equiv \forall x \; P(x) \;\; \wedge \;\; \forall x \; Q(x)$$

Example: Formalizing sentences from a natural language with Tarski's world

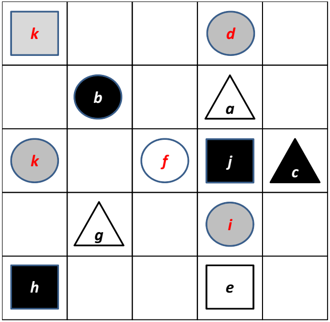


Example: Formalizing sentences from a natural language with Tarski's world /cont.

We adopt the following predicates:

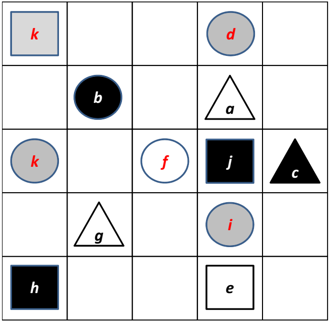
- *square* (x) indicates that x is a square.
- *circle* (x) indicates that x is a circle.
- *triangle* (x) indicates that x is a triangle.
- *black* (x) indicates that x is a black.
- *gray* (x) indicates that x is a gray.
- *white* (x) indicates that x is a white.
- *aboveOf* (x, y) indicates that x is above y, perhaps in a different column.
- *rightOf* (x, y) indicates that x is to the right of y, perhaps in a different row.
- *sameColor* (x, y) indicates that x and y have the same color.

Example: Formalizing sentences from a natural language with Tarski's world /cont.



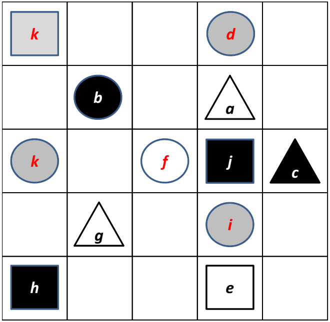
- All squares are black: $\forall x\ (square(x) \rightarrow black(x))$.
  False, since e and k are both squares and non-black. Note that to prove false one counter-example would be enough.
- Everything white is a triangle. $\forall x\ (white(x) \rightarrow triangle(x))$.
  False, since e and f are both white and not triangles.

Example: Formalizing sentences from a natural language with Tarski's world /cont.



- There is a square that lies to the left of d.
  $\exists x\ (square(x) \wedge rightOf(d, x))$.
  True: h and k both lie to the left of d.
- There is a black circle. $\exists x\ (circle(x) \wedge black(x))$. True: b.

Example: Formalizing sentences from a natural language with Tarski's world /cont.



- All circles are above g. $\forall x\ (circle(x) \to aboveOf(x, g))$.
  False, since i is a circle and it is not above g.
- For every square, there exists a circle of the same color.

$$\forall x\ (square(x) \to \exists y(circle(y) \land sameColour(x, y)))$$

This is true.

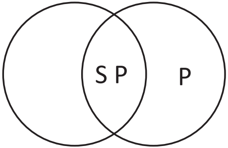
Categorical propositions

- A *categorical proposition* (or *categorical statement*) is a proposition that asserts or denies that all or some of the members of one category (the *subject term*) are included in another (the *predicate term*).
- For example, the categorical proposition *"All birds are white"* asserts that all members of category *birds* are included in category *being white*.
- The category of the subject (*birds*) and refers to what the proposition is about, whereas the category of the predicate (*being white*) refers to what the proposition affirms (or denies) about the subject.

Standard forms of categorical propositions

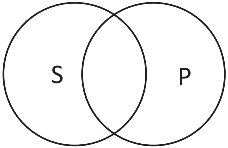
- Aristotle identified four primary distinct types of categorical proposition and gave them standard forms (referred to as A, E, I, and O).
- For the subject category S, and the predicate category P, the four standard forms are:
- All S are P. (A form)
- No S are P. (E form)
- Some S are P. (I form)
- Some S are not P. (O form)

Visualizing categorical propositions



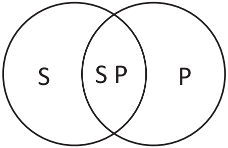
All S are P

A



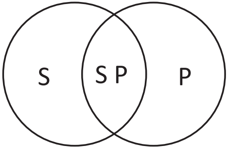
No S are P

E



Some S are P

I



Some S are not P

O

Properties of categorical propositions: Quantity

- *Quantity* refers to the number of members of the subject class that are used in the proposition.
- If the proposition refers to all members of the subject class, it is *universal*.
- If the proposition does not employ all members of the subject class, it is *particular*.
- Categorical propositions A and E have a *universal quantity* as they make a claim about all members of the subject class, whereas propositions I and O have a *particular quantity* as they make a claim about some (at least one) member of the subject class.

Properties of categorical propositions: Quality

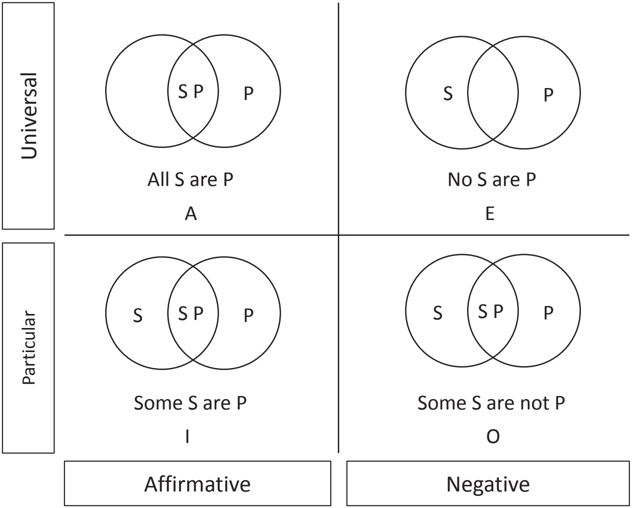
- *Quality* describes whether the proposition affirms or denies the inclusion of a subject within the class of the predicate.
- The two possible qualities are called *affirmative* and *negative*.
- Propositions A and I have an *affirmative quality* as they affirm class membership, whereas propositions E and O have *negative quality* as they deny class membership.

Summary of categorical propositions

Letting S and P stand for subject and predicate respectively, we have four forms of categorical propositions as shown below:

NAME	FORM	TITLE	
A	All S are P	Universal affirmative	$\forall x P(x)$
E	No S are P	Universal negative	$\forall x[\neg P(x)]$
I	Some S are P	Particular affirmative	$\exists x P(x)$
O	Some S are not P	Particular negative	$\exists x[\neg P(x)]$

Visualizing categorical propositions with their properties



Formalizing sentences from a natural language

- In the episode "Fidelity" of House M. D., Dr. Gregory House says: *"I don't ask why patients lie, I just assume they all do."*
- Consider the following list of categorical propositions where $P(x)$ denotes the subject "x is a patient" and $Q(x)$ denotes the predicate "x lies":
  - "Every patient lies." $\forall x\ (P(x) \rightarrow Q(x))$ (All P are Q: Type A)
  - "No patient lies." $\forall x\ (P(x) \rightarrow \neg Q(x))$ (No P are Q: Type E)
  - "Some patient lies." $\exists x\ (P(x) \wedge Q(x))$ (Some P are Q: Type I)
  - "Not all patients lie." $\exists x\ (P(x) \wedge \neg Q(x))$ (Some P are not Q: Type O)

Formalizing sentences from a natural language /cont.

- Consider the statement "**All** patients **are** honest." What type is it? Is is A?
- No. Given the subject and predicate, the statement is of type E: "No patient lies."
- Similarly, the statement "**No** patient **is** honest" is of type A because, given the subject and predicate it is interpreted as "All patients lie."
- To determine the proper type, a statement should be interpreted w.r.t. the subject and predicate.

Formalizing categorical propositions

- "Some patient lies." $\exists x\,(P(x) \wedge Q(x))$
- "No patient lies." $\forall x\,(P(x) \rightarrow \neg Q(x))$
- "All patients lie." $\forall x\,(P(x) \rightarrow Q(x))$
- "Not all patients lie." $\exists x\,(P(x) \wedge \neg Q(x))$
- "Every patient lies." $\forall x\,(P(x) \rightarrow Q(x))$
- "There is an honest patient." $\exists x\,(P(x) \wedge \neg Q(x))$
- "No patient is honest." $\forall x\,(P(x) \rightarrow Q(x))$
- "All patients are honest." $\forall x\,(P(x) \rightarrow \neg Q(x))$

Formalizing sentences from a natural language /cont.

- We notice that each formalization satisfies one of the following two properties:
- The universal quantifier $\forall x$ quantifies an implication.
- The existential quantifier $\exists x$ quantifies a conjunction.

Formalizing sentences from a natural language /cont.

- Consider the statement "Some patient lies" which was formalized as $\exists x \; (P(x) \wedge Q(x))$.
- Can we argue that the sentence can also be formalized as $\exists x \; (P(x) \to Q(x))$?
- This would be equivalent to $\forall x \; P(x) \to \exists x Q(x)$ which means "If everyone is a patient then someone lies" which does not convey the meaning of the original sentence.

Contradictory categorical propositions

- **Definition**: A pair of categorical propositions are called *contradictories* if they have opposite truth values: they cannot both be true and cannot both be false.

Contradictory categorical propositions: Example 1

- Consider the statement "Every person owns a house." Its contradictory statement is "Not every person owns a house."
- Given a subject category "x is a person" and predicate category "x owns a house", then "Every person owns a house" is of type A.
- Its contradictory statement "Not every person owns a house" can be rephrased as "Some people do not own a house" which is of type O.
- Universal affirmations and particular denials are contradictory statements.

Contradictory categorical propositions: Example 2

- Consider the statement "No people suffer from hunger." Its contradictory statement is "Some people suffer from hunger."
- Given a subject category "x is a person" and predicate category "x suffers from hunger", then "No people suffer from hunger." is of type E.
- Its contradictory statement "Some people suffer from hunger" is of type I.
- Universal denials and particular affirmations are contradictory statements.

Contradictory categorical propositions: Example 3

- let P(x) to denote "x is a patient" and Q(x) to denote "x lies"
- "All patients lie" ($\forall x\ (P(x) \rightarrow Q(x))$), and "There is an honest patient" (which can be interpreted as There is some patient who does not lie) ($\exists x\ (P(x) \wedge \neg Q(x))$) is a pair of contradictory propositions.
- "No patient lies" ($\forall x\ (P(x) \rightarrow \neg Q(x))$) and "Some patient lies" ($\exists x\ (P(x) \wedge Q(x))$) is a pair of contradictory categorical propositions.

Contrary categorical propositions

- **Definition**: A pair of categorical propositions are called *contraries* if they cannot both be true, but could both be false.

Contrary categorical propositions: Example 1

- Consider the statement "All people are rich." Its contrary statement is "No people are rich."
- Given a subject category "x is a person" and predicate category "x is rich", then "All people are rich" is of type A and "No people are rich" is of type E.
- A pair of universal statements are contraries.

Contrary categorical propositions: Example 2

- "All patients lie" ($\forall x\ (P(x) \to Q(x))$) and "No patient lies" ($\forall x\ (P(x) \to \neg Q(x))$) are contrary categorical propositions.

Subcontrary categorical propositions

- **Definition**: A pair of categorical propositions are called *subcontraries* if they cannot both be false but could both be true.

Subcontrary categorical propositions: Example

- "Some patient lies" ($\exists x\ (P(x) \wedge Q(x))$) and "There is an honest patient" (or There is some patient who does not lie) ($\exists x\ (P(x) \wedge \neg Q(x))$) are subcontrary categorical propositions.

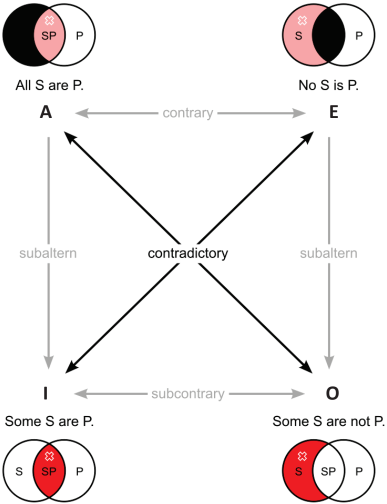
Subaltern and superaltern categorical propositions

- **Definition**: Two categorical propositions are called *superaltern* and *subaltern* if the subaltern must be true if its superaltern is true (and subsequently the superaltern must be false if the subaltern is false).
- I is a subaltern of A (Some S are P, if All S are P).
- O is a subaltern of E (Some S are not P, if No S are P).

Subaltern and superaltern categorical propositions: Example

- "Some patient lies" ($\exists x\ (P(x) \wedge Q(x))$) is a subaltern of "All patients lie" ($\forall x\ (P(x) \rightarrow Q(x))$).
- "There is an honest patient" ($\exists x\ (P(x) \wedge \neg Q(x))$) is a subaltern of "No patient lies" ($\forall x\ (P(x) \rightarrow \neg Q(x))$).

Summary: The square of opposition



Categorical syllogisms

- Recall from propositional logic that a syllogism is an inference in which one proposition necessarily follows from two others.
- A *categorical syllogism* consists of three parts (major premise, minor premise, and conclusion) all of which are *categorical propositions*.

Categorical syllogisms /cont.

Some valid forms of categorical syllogisms are shown below:

All M are P. All S are M. Therefore, *All S are P*.

All P are M. Some S are not M. Therefore, *Some S are not P*.

Some M are not P. All M are S. Therefore, *Some S are not P*.

All P are M. No M are S. Therefore, *No S are P*.

All M are P. Some S are M. Therefore, *Some S are P*.

No M are P. Some S are M. Therefore, *Some S are not P*.

Universal conditional statements

- A universal conditional statement has the form $\forall x$ *if* $P(x)$ *then* $Q(x)$, or $\forall x(P(x) \rightarrow Q(x))$.
- It can be proven that its negation has the form $\exists x(P(x) \wedge \neg Q(x))$.
- For the universal conditional statement above,
- its contrapositive statement is

$$\forall x(\neg Q(x) \rightarrow \neg P(x))$$

- its converse statement is

$$\forall x(Q(x) \rightarrow P(x))$$

- and its inverse is

$$\forall x(\neg P(x) \rightarrow \neg Q(x))$$