Relations

Dr. Constantinos Constantinides, P.Eng.

Department of Computer Science and Software Engineering
Concordia University Montreal, Canada

January 20, 2021

Binary relations over sets

- A *binary relation* R between sets A and B (sometimes written as $R : A \leftrightarrow B$) is defined as $R \subseteq A \times B$.
- Given $A = \{1, 2\}$ and $B = \{2, 3\}$, and $R =$ "is less than", then
  $A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$, and
  $R = \{(1, 2), (1, 3), (2, 3)\}$.
- We can also observe that $R \in \mathcal{P}(A \times B)$.
- The statement $(a, b) \in R$ is written as aRb, or $R(a, b)$.

Binary relations over sets /cont.

- We view a binary relation R as associating every ordered pair (a, b) the value true, if $(a, b) \in R$, or the value false if $(a, b) \notin R$.
- It is, thus, evident that $<$ (less than) is a binary relation, whereas $+$ (addition) or $-$ (subtraction) are not binary relations.

Binary relations over sets /cont.

- Using sets of ordered pairs is just one example we can express relations.
- Other methods include words, directed graphs or matrices.
- For example (Noodles likes Deborah), or likes(Noodles, Deborah), is an element of the binary relation *likes* over the set of all people.

Properties of relations

- A relation R on a set A is:

1 Reflexive when $\forall a \in A : aRa$,
e.g. "divides", "is equal to."

2 Irreflexive (or anti-reflexive) when no element of A is related to itself, i.e. $\forall a \in A : \neg(aRa)$,
e.g. "is greater than", "is not equal to."

- Note that "not reflexive" does not mean "irreflexive."
- For example, the relation "likes" (between people) is not reflexive since not everybody likes themselves, but it is not irreflexive either since not everybody dislikes themselves either.

Properties of relations /cont.

3. Symmetric, when $\forall a, b \in A : aRb \rightarrow bRa$,
   e.g. "is a blood relative of", "is married to."
4. Asymmetric, when $\forall a, b \in A : aRb \rightarrow \neg(bRa)$,
   e.g. "is father of", "is less than."

- A relation cannot be both symmetric and asymmetric.
- Note that "asymmetric" does not mean "not symmetric."
- A relation can be neither symmetric nor asymmetric, e.g. "likes."

Properties of relations /cont.

5. Antisymmetric when
$\forall a, b \in A : (aRb \wedge bRa) \rightarrow a = b$.

   Alternatively, R is antisymmetric if whenever $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$

   e.g. $\leq$ is antisymmetric, but "likes" is not.

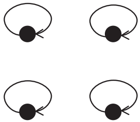
Properties of relations /cont.

6 Transitive when $\forall a, b, c \in A : (aRb \wedge bRc) \rightarrow aRc$,
e.g. "is ancestor of."

- A relation is intransitive iff it is not transitive.
- For example, the relation "likes" (between people) is intransitive.

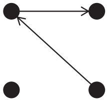
Visualization of properties of binary relations

Reflexive



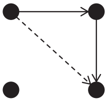
Every vertex has a self-loop.

Irreflexive



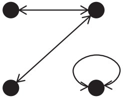
No vertex has a self-loop.

Transitive



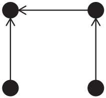
Non-solid edge is present by logical deduction.

Symmetric



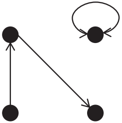
Every edge is bidirectional.

Asymmetric



No edge is bidirectional.

Antisymmetric



No edge is bidirectional, unless it is a self-loop.

Example 1: Relation "likes"

- We have seen above that the proposition *Noodles likes Deborah* defines the binary relation *likes* over the set of all people. The relation *likes* is

(a) not reflexive (not everybody likes themselves),

(b) not symmetric (not everybody's feelings are reciprocated), and

(c) not transitive.

Example 2: Relation “is son of”

- The relation *is son of* is

(a) irreflexive, as no person can be his own son,

(b) not symmetric, and

(c) not transitive.

Domain and range in binary relations

- A binary relation R can be modeled as a set of ordered pairs.
- The *domain* of R, *dom* R, is the set of all first elements of ordered pairs.
- The *range* of R, *ran* R, is the set of all second elements of ordered pairs.
- For example, for $R = \{(0,1),(0,2),(1,1),(3,5)\}$, then
  dom $R = \{0,1,3\}$, and
  ran $R = \{1,2,5\}$

Inverse of binary relations

- Any binary relation R has an *inverse* relation (denoted by R^{-1}, or $R^{\sim}$, also called *converse* relation) which is obtained by changing the order of the elements in the relation.
- For a relation R on a set A

$$R^{-1} = \{(b, a) \in A \times A \mid (a, b) \in R\}$$

- A binary relation over a set is equal to its inverse if and only if it is symmetric.

Example: Inverse relation

- For $R = \{(0,1),(0,2),(1,1),(3,5)\}$, then $R^{-1} = \{(1,0),(2,0),(1,1),(5,3)\}$, *dom* $R^{-1} = \{1,2,5\}$, and *ran* $R^{-1} = \{0,1,3\}$
- We observe that *dom* R = *ran* R^{-1}, and *ran* R = *dom* R^{-1}.

Equivalence relations

- **Definition**: Any relation that is reflexive, symmetric and transitive is called an *equivalence relation*,
  e.g. the relation "is equal to" on the set of numbers.

Example 3: Relation "has the same birthday as"

- For any $a, b, c \in A$, the relation R: "has the same birthday as" is

1. Reflexive: $\forall a \in A : aRa$,
2. Symmetric: $\forall a, b \in A : aRb \rightarrow bRa$, and
3. Transitive: $\forall a, b, c \in A : (aRb \wedge bRc) \rightarrow aRc$.

Partition

- A *partition* of a set A is a collection, P, of non-empty subsets that satisfy the following:

1. $A = A_1 \cup A_2 \cup \cdots \cup A_n$, and
2. $A_i \cap A_j = \varnothing, \text{ for } i \neq j.$
   where the sets (i.e. the elements of P) are called the *blocks* (also: *parts*, or *cells*) of the partition.

Partial order

- **Definition**: Any relation that is reflexive, antisymmetric and transitive is called a *partial order*,
  e.g. the relation "is less than or equal to" on the set of real numbers.
- A set for which a partial order is defined is called a *poset*.
- A *total order* on a set A is a partial ordering in which every pair of elements is related.

Predecessors and successors in posets

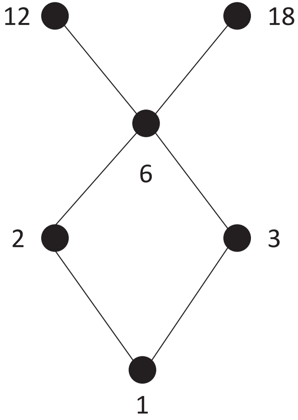
- If R is a partial order on a set A and xRy, $x \neq y$, we call x a *predecessor* of y (and subsequently we call y a *successor* of x).
- An element may possess many predecessors.
- If x is a predecessor of y and there is no z for which xRz and zRy, then we call x the *immediate predecessor* of y, denoted by $x < y$.

Hasse diagrams

- A *Hasse diagram* is a graph where vertices represent the elements of a poset A, and whenever $x < y$, vertex x is placed below vertex y and the vertices are joined by an edge.

Example: Hasse diagram

- Given the relation “is divisor of” on the set $A = \{1, 2, 3, 6, 12, 18\}$, the corresponding Hasse diagram is shown:



Minimal and maximal elements in Hasse diagrams

- In any poset there are *minimal elements* (ones without predecessors) and *maximal elements* (ones without successors).
- In the example, the poset has one minimal element (1) and two maximal elements $(12, 18)$.

Composition of binary relations

- Given two binary relations, we can form a new one by a process called a *composition*.
- For example, given "is brother of (x, y)" and "is parent of (y, z)", we can combine the two to form "is uncle of (x, z)."
- Formally, if $R \subseteq X \times Y$, and $S \subseteq Y \times Z$, then their *composition*, denoted by $R \circ S$, is the relation $R \circ S = \{(x, z) \in X \times Z \mid (\exists y \in Y : (x, y) \in R \wedge (y, z) \in S\}$
- In other words, the tuple (x, z) is an element of the new composition relation, if there exists a $y \in Y$ such that $(x, y) \in R \wedge (y, z) \in S$.

Relational override

- The *override* of R with S, denoted by $R \oplus S$ is obtained by adding to S all those ordered pairs from R whose first coordinates are not in the domain of S.
- Given the relations
  $R = \{(0,1),(0,2),(2,3)\}$, and $S = \{(0,1),(1,3),(3,0)\}$.
  Then
  $R \oplus S = \{(0,1),(1,3),(3,0),(2,3)\}$,
  i.e. the ordered pair $(2,3)$ has been added to S since
  $2 \notin dom\ S$.