Wavelet transform & Multi-resolution processing

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Materials adapted from Dr. T. D. Bui

Reading

The content of the lecture corresponds to Chapter 7 of the textbook for the following sections

Page 481-483, 504-526

For related math operations and concepts, please refer to Page 464-470

What is a transform?

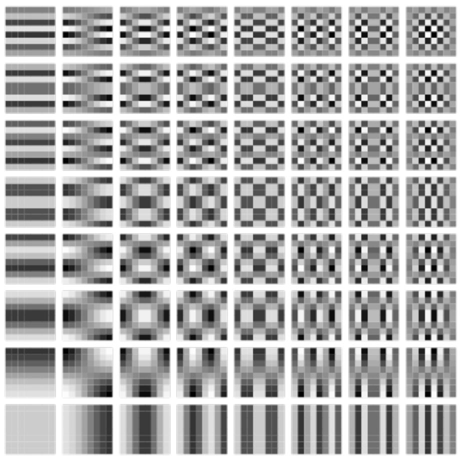
- A **transform** can be applied to a raw signal to convert it from one domain to another (e.g Fourier).

- Enables the identification of features that may not be as easily observed in the original domain.

- The original image is usually in the time/spatial domain

Transforms

- FT is probably the most popular transform.
- Many other transforms are used quite often by engineers and mathematicians
  - *Hilbert transform*
  - *Discrete Cosine Transform (DCT) -> JPEG format*
  - *Short-time Fourier transform (STFT)*
  - *Wavelet transform (we will see today)*
- Basis functions & coefficients



Basis functions for DCT

Fourier Transform (FT)

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-2j\pi ft} dt$$

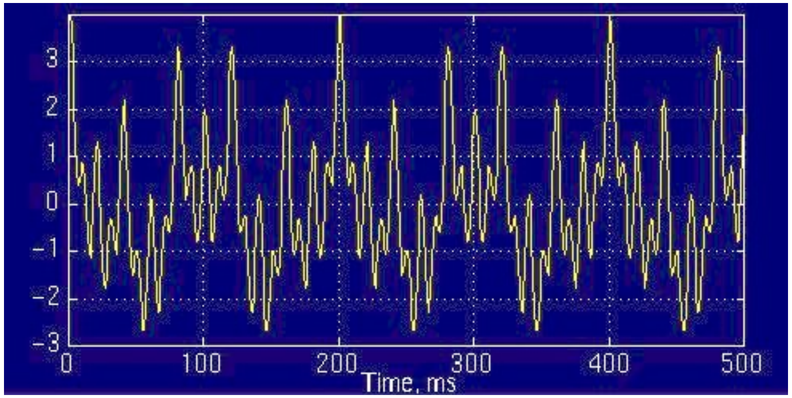
$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{2j\pi ft} df$$

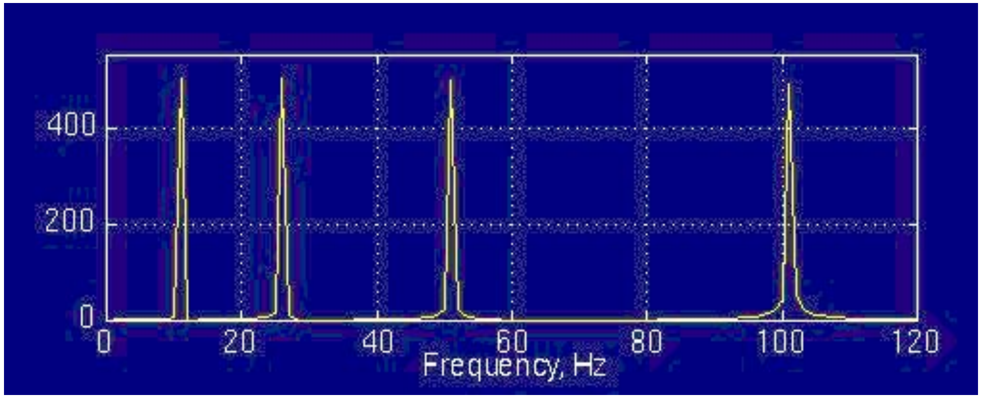
- No frequency information is available in the time-domain signal.
- No time information is available in the Fourier transformed signal.
  - ✓ Is it necessary to have both the time and the frequency information at the same time?

Stationary Signals

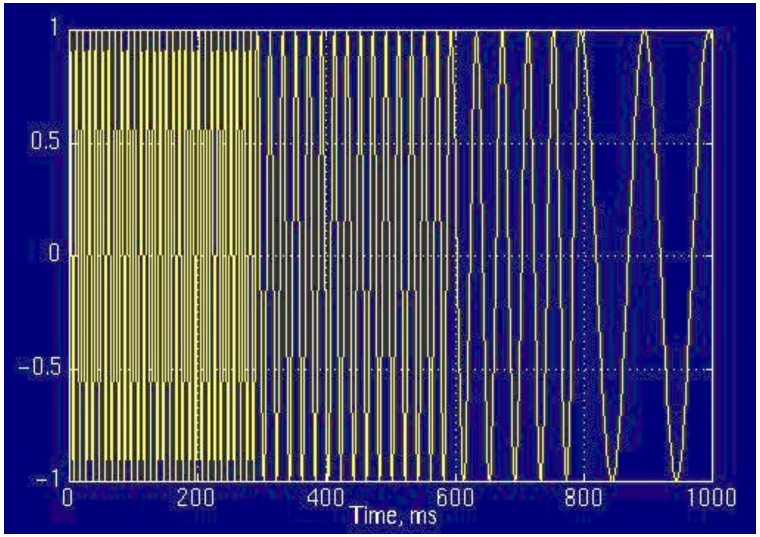
- Signals whose frequency content do not change in time:

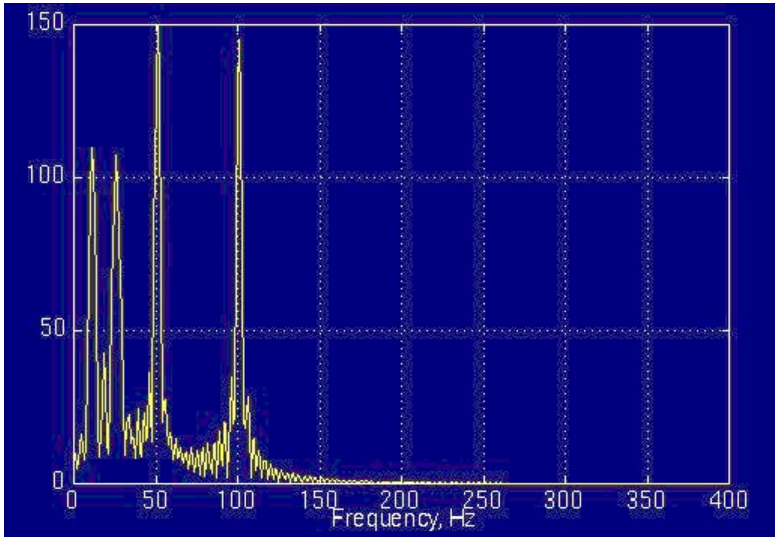
*x(t)=cos(2*pi*10*t)+cos(2*pi*25*t)+cos(2*pi*50*t)+cos(2*pi*100*t)*



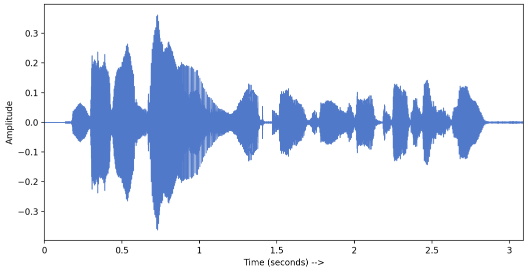


Multiple Freq Presented at Different Time *– non-stationary*





The interval 0 to 300 ms has a **100 Hz** sinusoid, the interval 300 to 600 ms has a **50 Hz** sinusoid, the interval 600 to 800 ms has a **25 Hz** sinusoid, and finally the interval 800 to 1000 ms has a **10 Hz** sinusoid



Time-Frequency Relationship

- FT is not a suitable technique for **non-stationary** signals.
- FT gives what frequency components (spectral components) exist in the signal - Nothing more, nothing less.
- When the **time localization** of the spectral components are needed, a transform giving the *TIME-FREQUENCY REPRESENTATION* of the signal is needed.

Short-Time Fourier Transform

- In STFT, the signal is divided into small enough segments, where these segments of the signal can be assumed to be stationary.

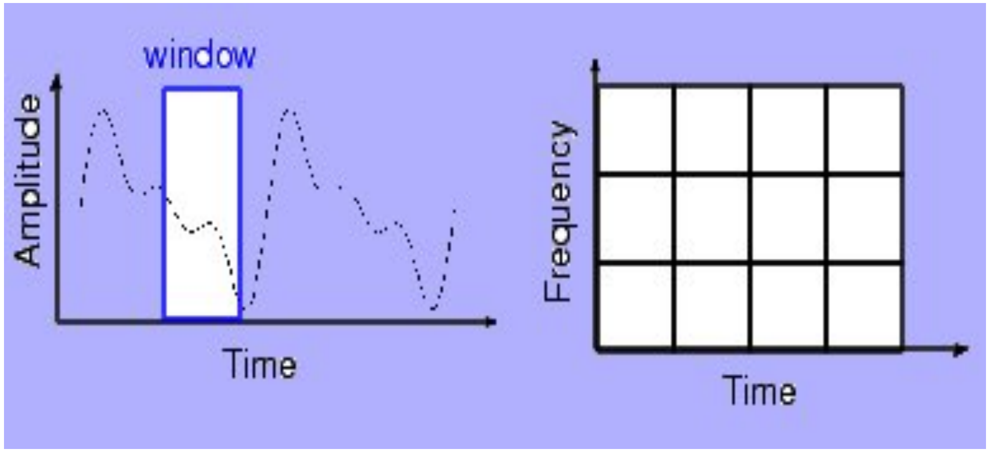
- A window function **"w"** is chosen. The width of this window must be equal to the segment of the signal where its stationarity is valid.

STFT

$$\mathrm{STFT}_X^{(\omega)}(t', f) = \int_t \left[x(t)\,\omega(t - t')\right] e^{-j2\pi ft}\, dt$$

$\omega(t)$: the window function

A function of time and frequency:



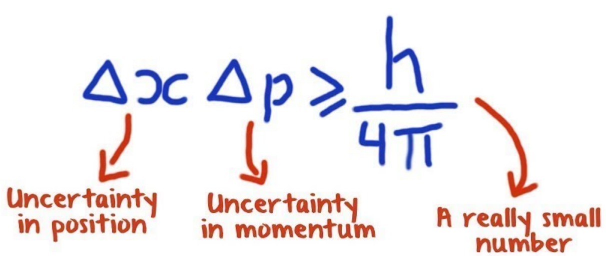
Drawbacks of STFT

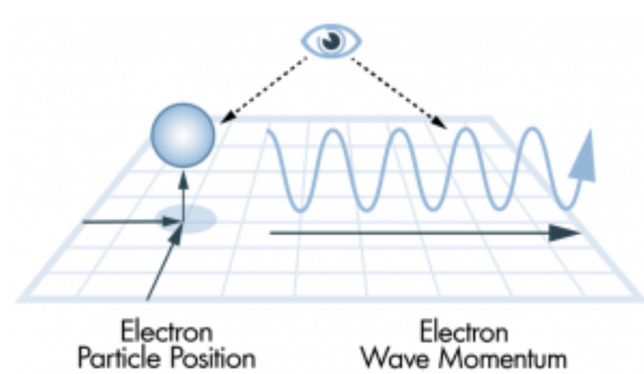
- Unchanged window.
- Dilemma of resolution:

  Narrow window => poor frequency resolution
  Wide window => poor time resolution

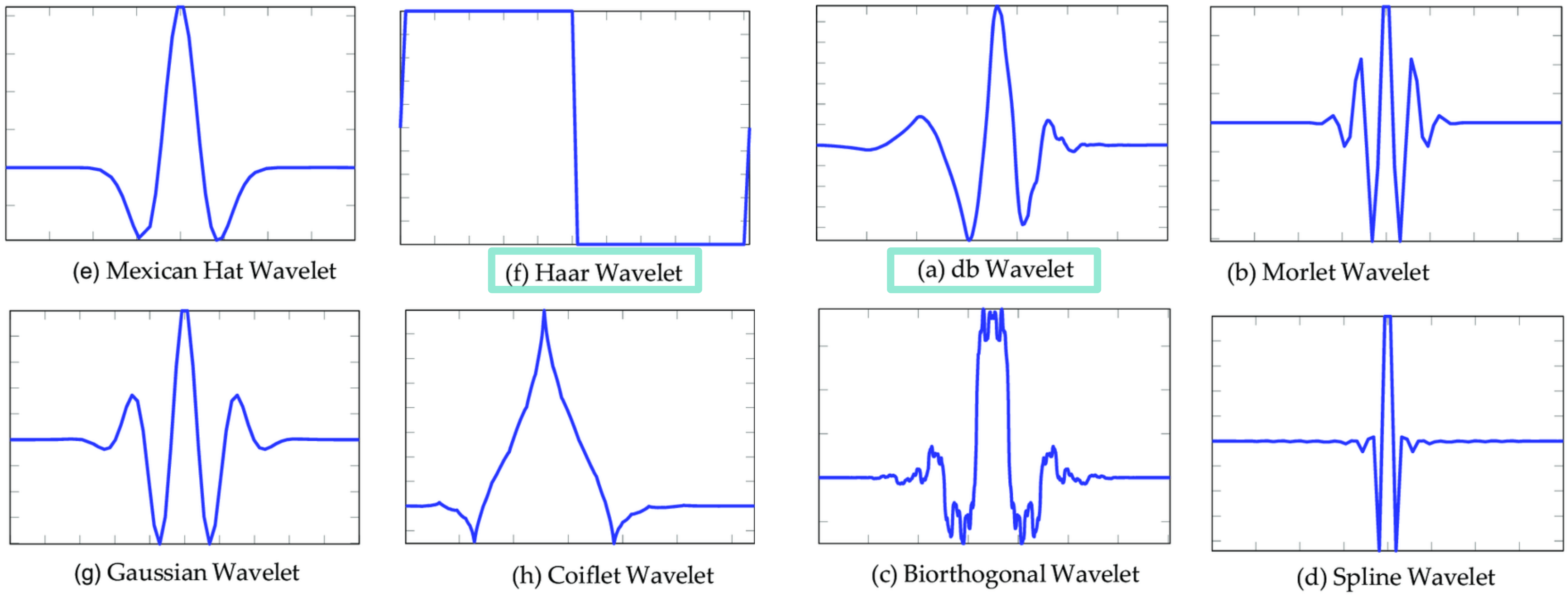
- Heisenberg Uncertainty Principle

  Cannot precisely know what frequency exists at what time intervals!





Wavelets



- A waveform with a *finite duration*, *average value of 0*, and *non-zero norm*
- Different types of wavelets exist
- Wavelet transform (WT)

Uncertainty Principle

- We cannot know exactly **what frequency exists** at **what time instance**, but we can only know **what frequency bands** exist at **what time intervals**.

- This is a problem of resolution, and it is the main reason why WT was invented:

  ✓ *STFT gives a fixed resolution at all times*

  ✓ *Whereas WT gives a variable resolution.*

The Continuous Wavelet Transform (CWT)

- The kernel functions used in Wavelet transform are all obtained from one prototype function, by scaling and translating the prototype function.
- This prototype is called the ***Mother wavelet***

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$$

Translation parameter

Scale parameter

$$CWT_x^{(\psi)}(a,b) = W(a,b) = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} f(t)\cdot\psi^*\left(\frac{t-b}{a}\right)dt$$

Mother wavelet

translation

CWT of $f(t)$ at scale a and translation b

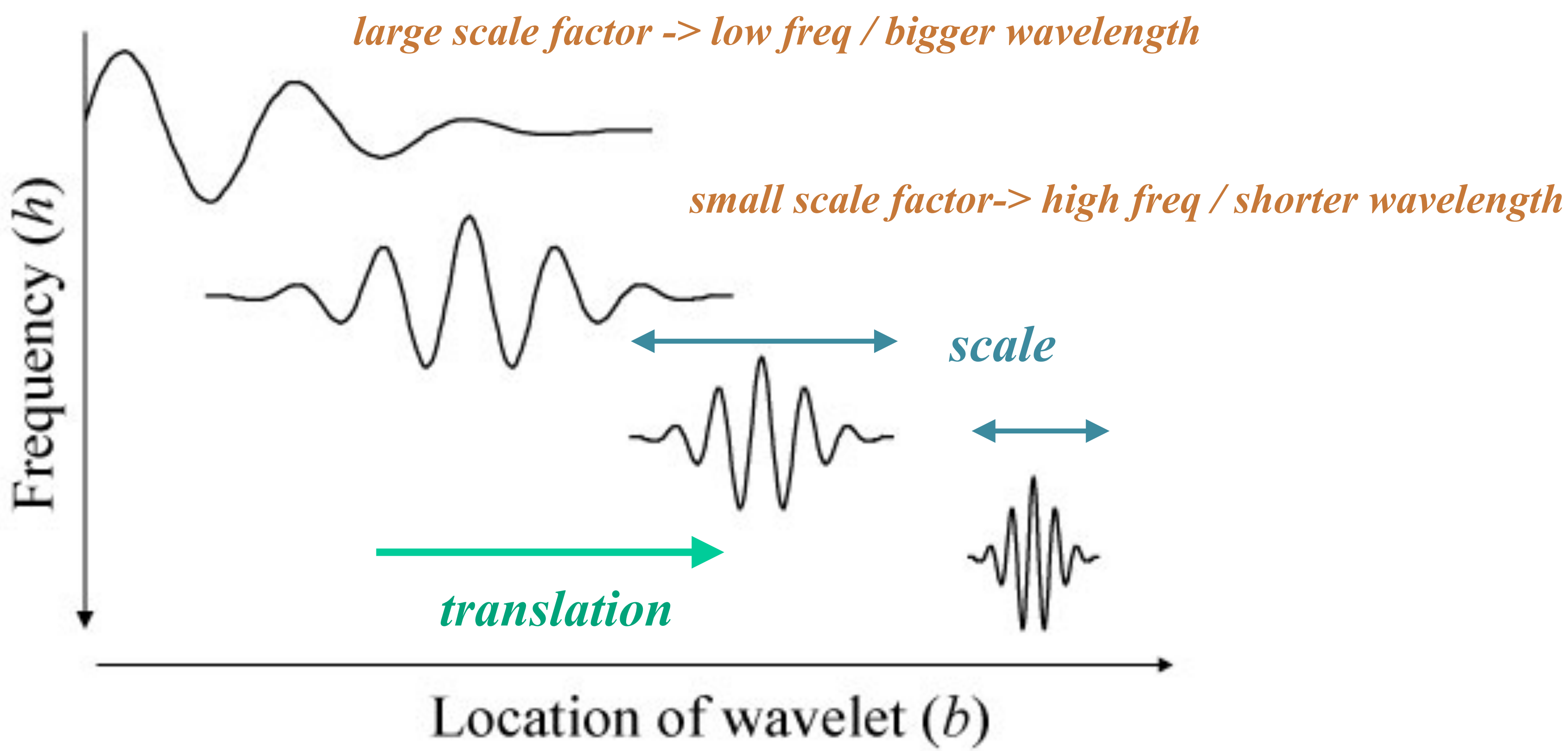
Normalization factor

Scaling

low scale ➔ high frequency

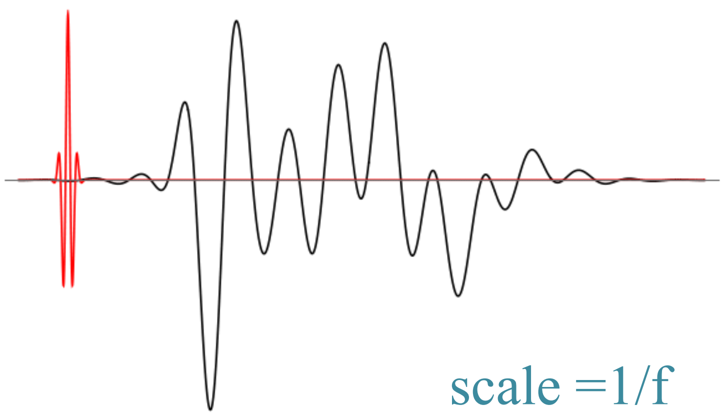
a and b are continuous parameters

Wavelets

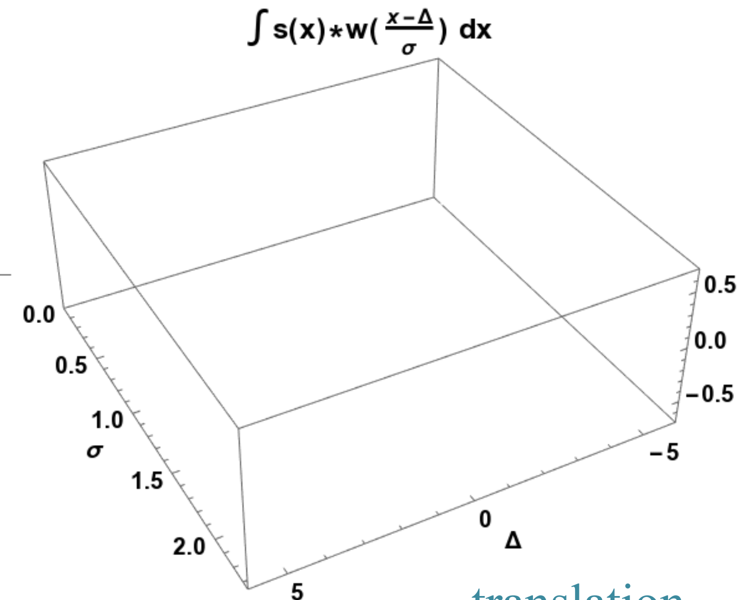


large scale factor -> low freq / bigger wavelength
small scale factor-> high freq / shorter wavelength
scale
translation
Frequency (h)
Location of wavelet (b)

Wavelets



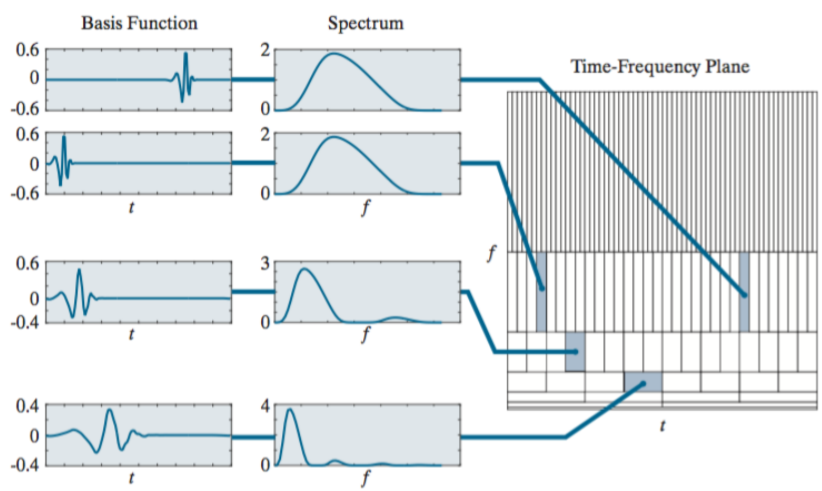
scale =1/f



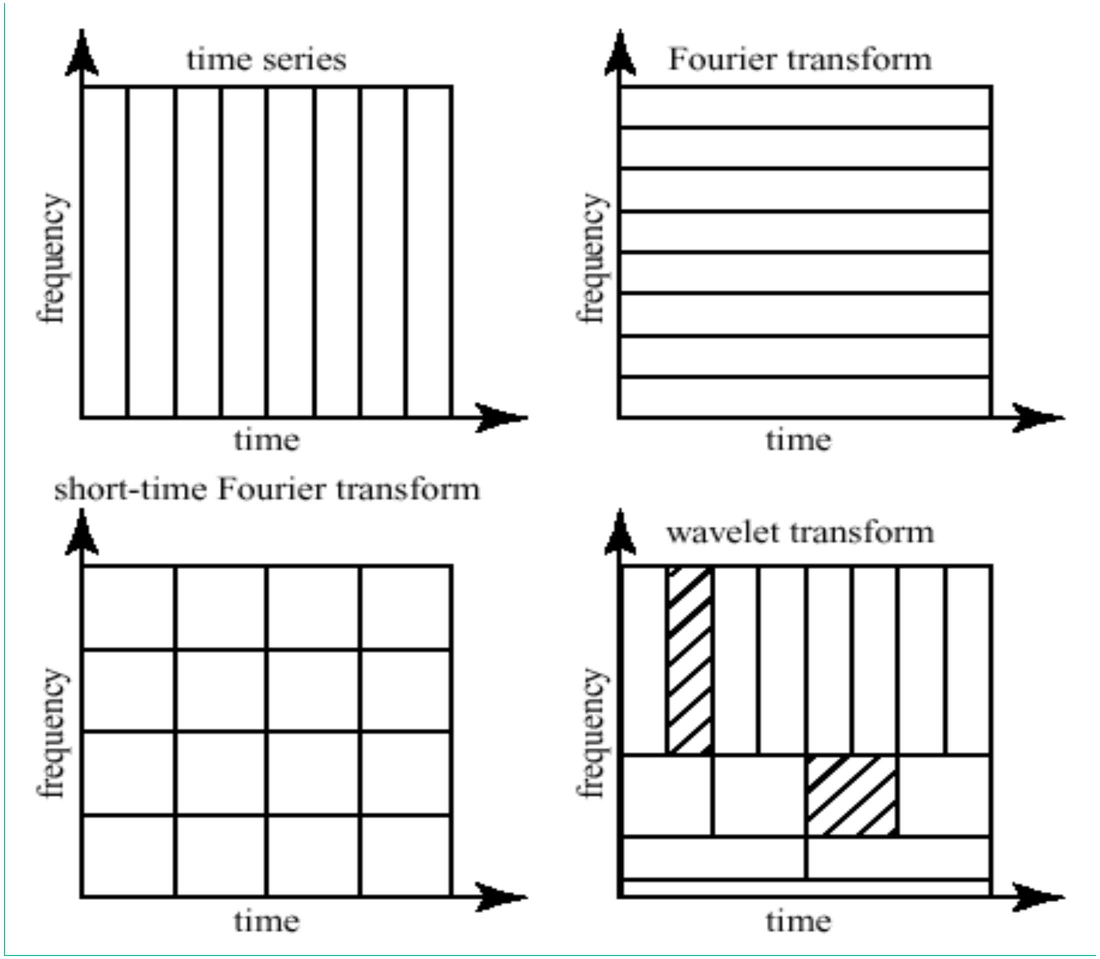
translation

a
b
c
d

FIGURE 7.5 Time and frequency localization of 128-point Daubechies basis functions.



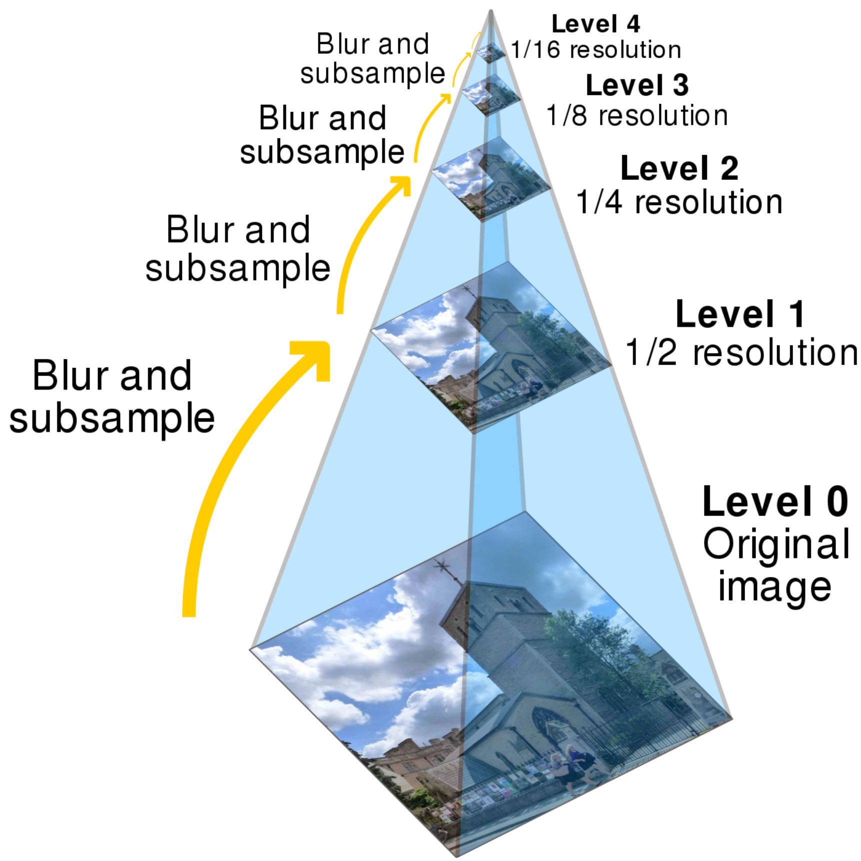
Comparison of Transformations



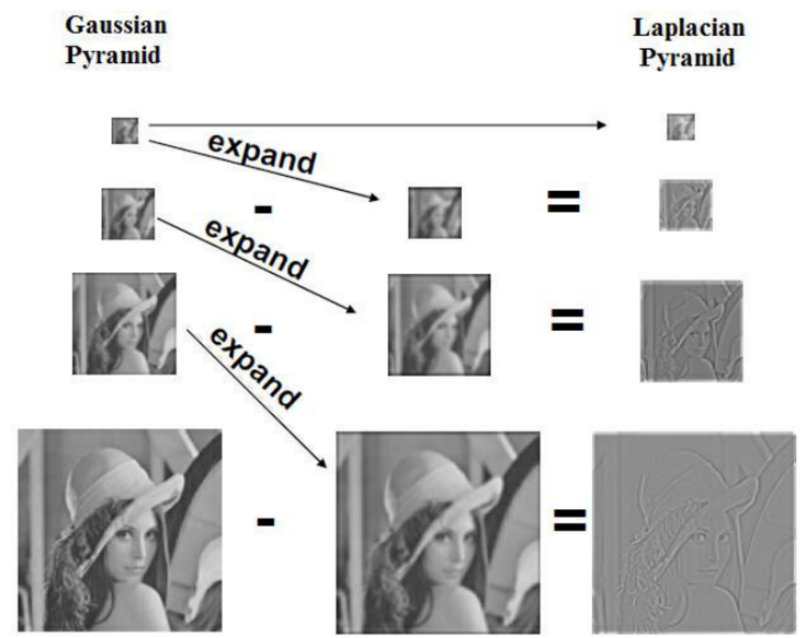
From http://www.cerm.unifi.it/EUcourse2001/Gunther_lecturenotes.pdf, p.10

Image Pyramid

Gaussian pyramid



Laplacian pyramid



Using Level defined as above then
Level is the ***inverse*** *to* Scale

Image Pyramid

Gaussian Pyramid

Laplacian Pyramid using bilinear interpolation

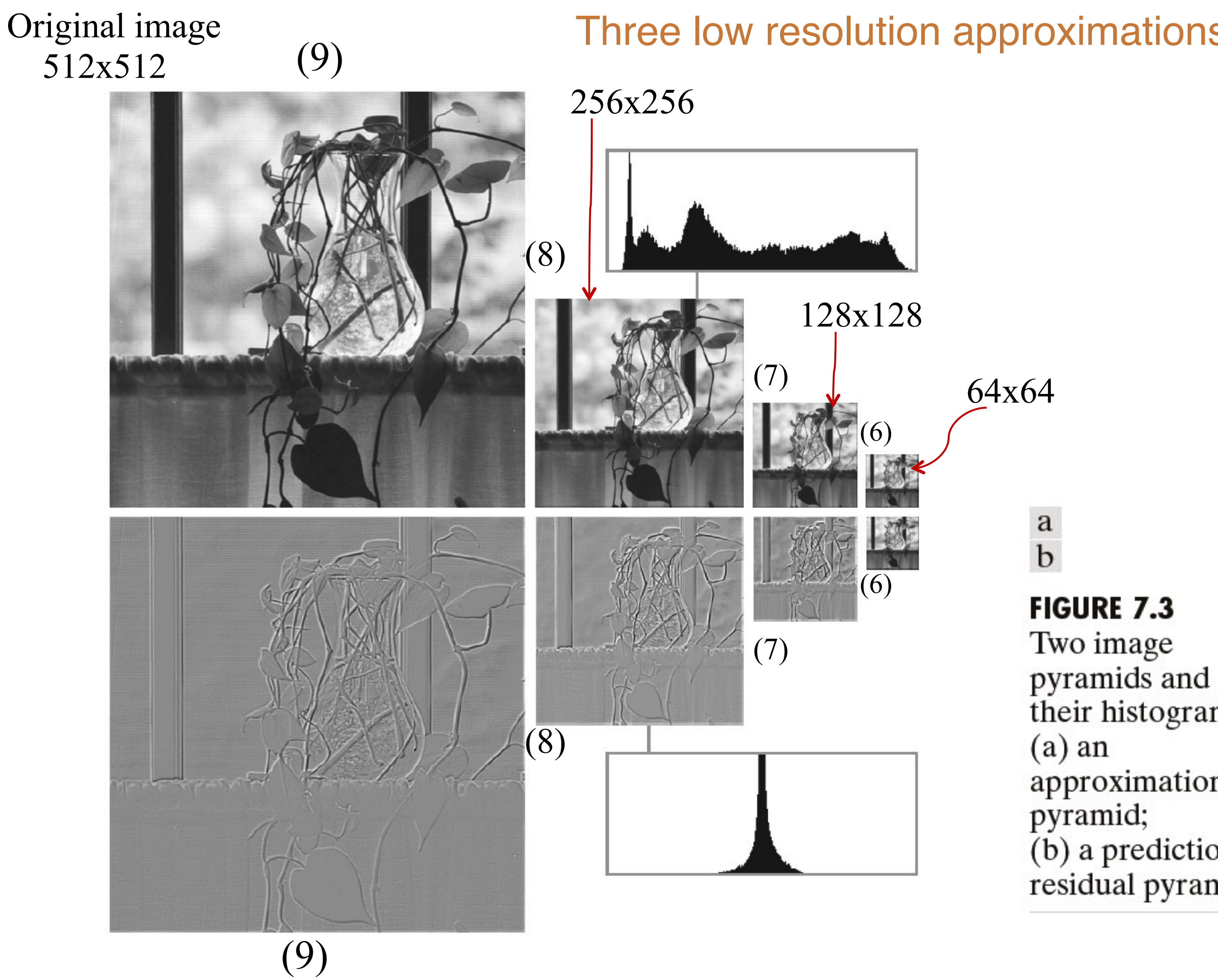


FIGURE 7.3 Two image pyramids and their histograms: (a) an approximation pyramid; (b) a prediction residual pyramid.

Image Pyramid

Scale space representation is very useful in image processing

- Extract image features such as edges at multiple scales
- Redundancy reduction and image modelling for
  - ✓ efficient coding
  - ✓ image enhancement/restoration
  - ✓ image analysis/synthesis

Discrete Wavelet transform (DWT)

translation

$$D[a,b] = \frac{1}{\sqrt{b}} \sum_{m=0}^{p-1} f[t_m] \psi\left[\frac{t_m - a}{b}\right]$$

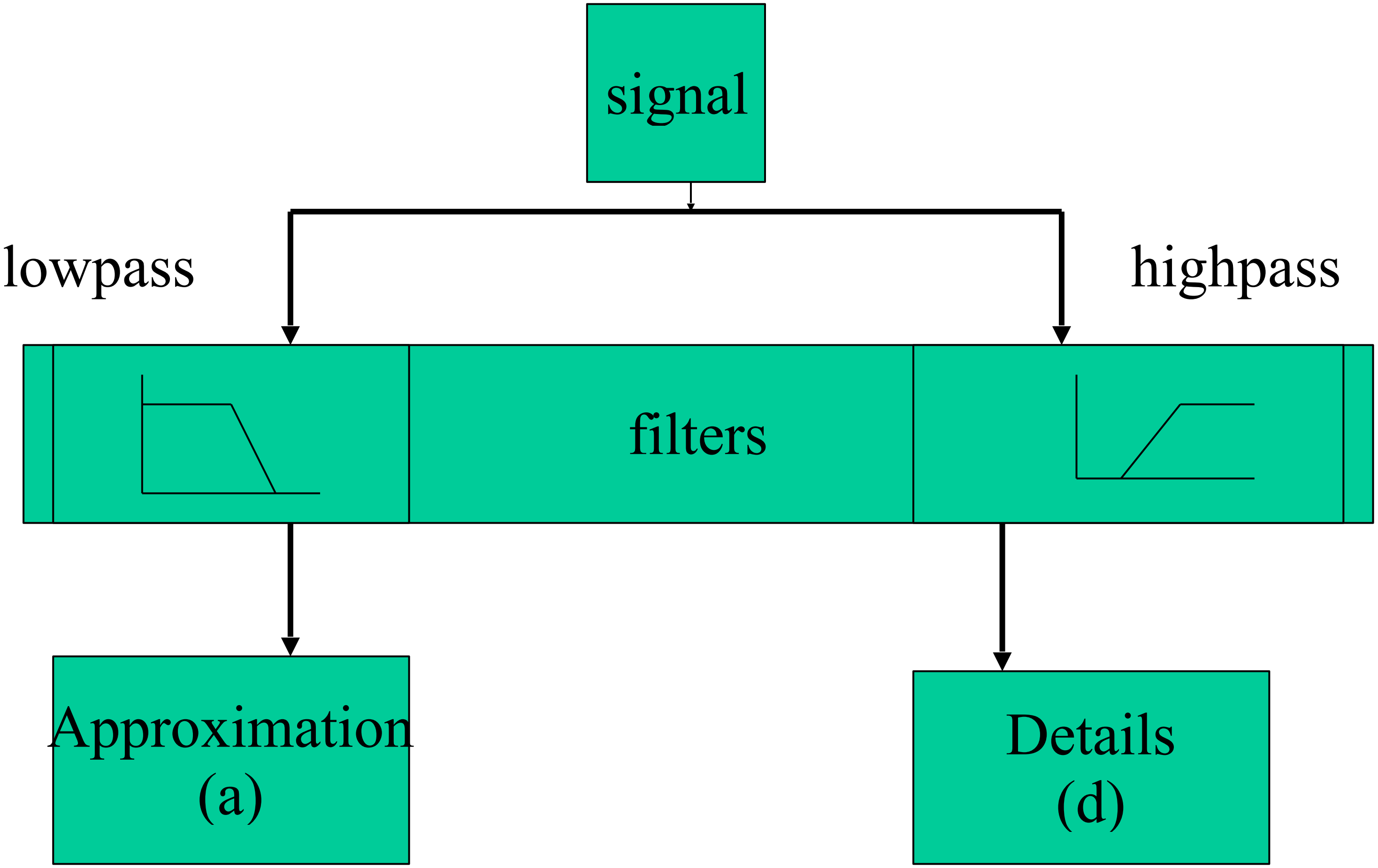
scaling factor

- Use a summation instead of integral
- Integer and fixed scaling (often in the multiple of 2)
- Integer translation: e.g., in the step of 1

DWT uses the wavelets, together with a single scaling function, to represent a function or image as a linear combination of the wavelets and scaling function.

We will see the scaling function later

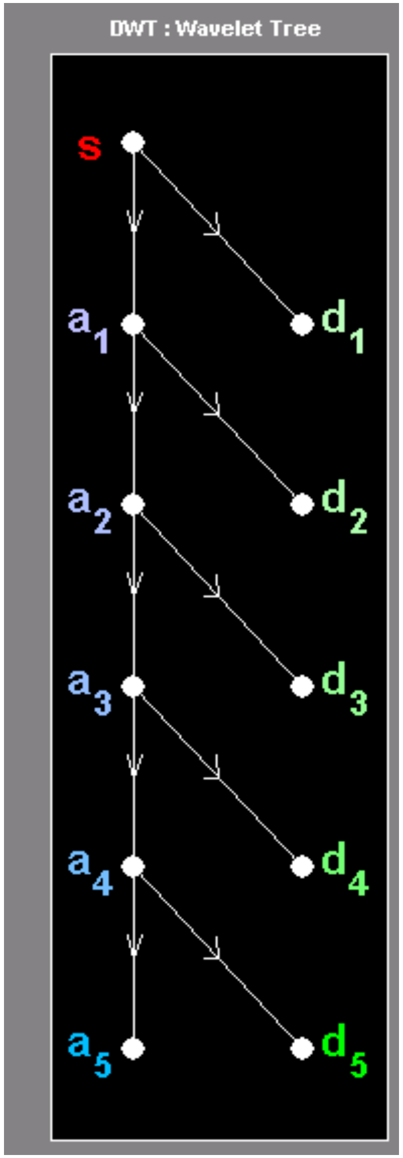
Discrete Wavelet transform (DWT)

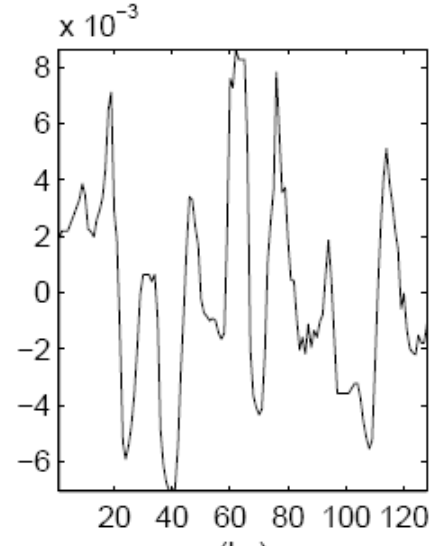


(we will see more about this later)

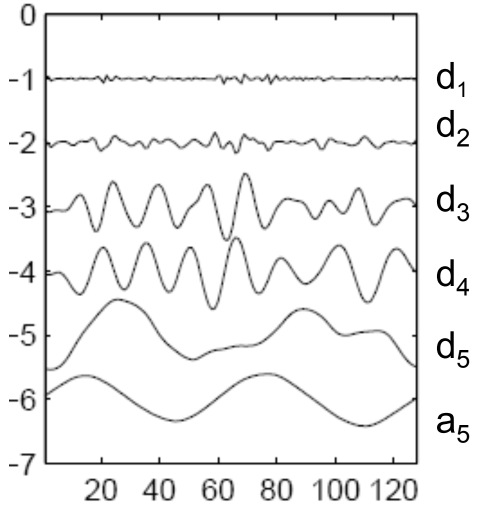
Levels of decomposition

- Successively decompose the approximation
- Level 5 decomposition = a5 + d5 + d4 + d3 + d2 + d1
- No limit to the number of decompositions performed





(b_1)

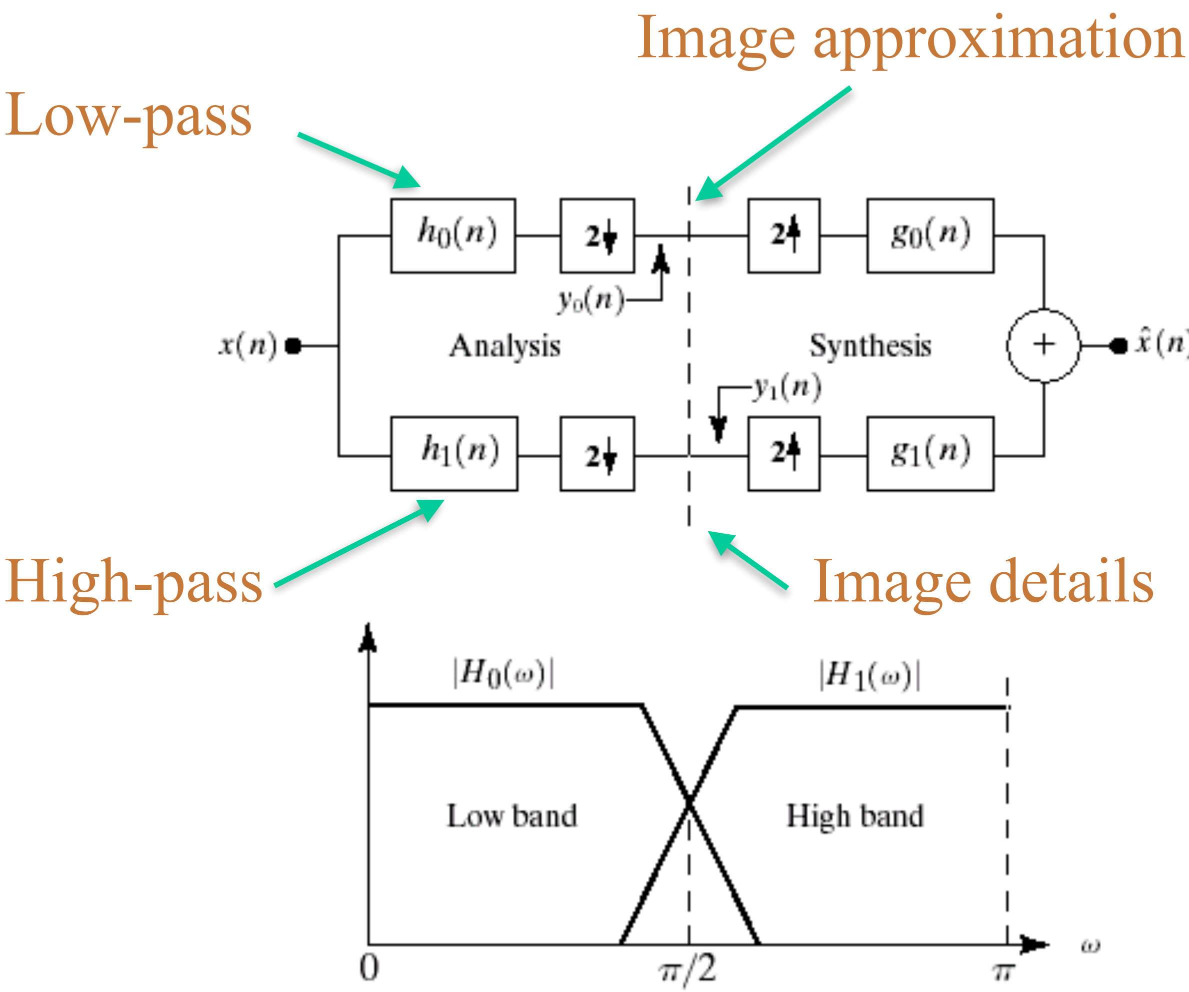


(c_1)

Sub-band Coding

a
b

FIGURE 7.4 (a) A two-band filter bank for one-dimensional subband coding and decoding, and (b) its spectrum splitting properties.



Need to pick the right filters for perfect reconstruction

2-D 4-band filter bank

Wavelet - Frequency domain

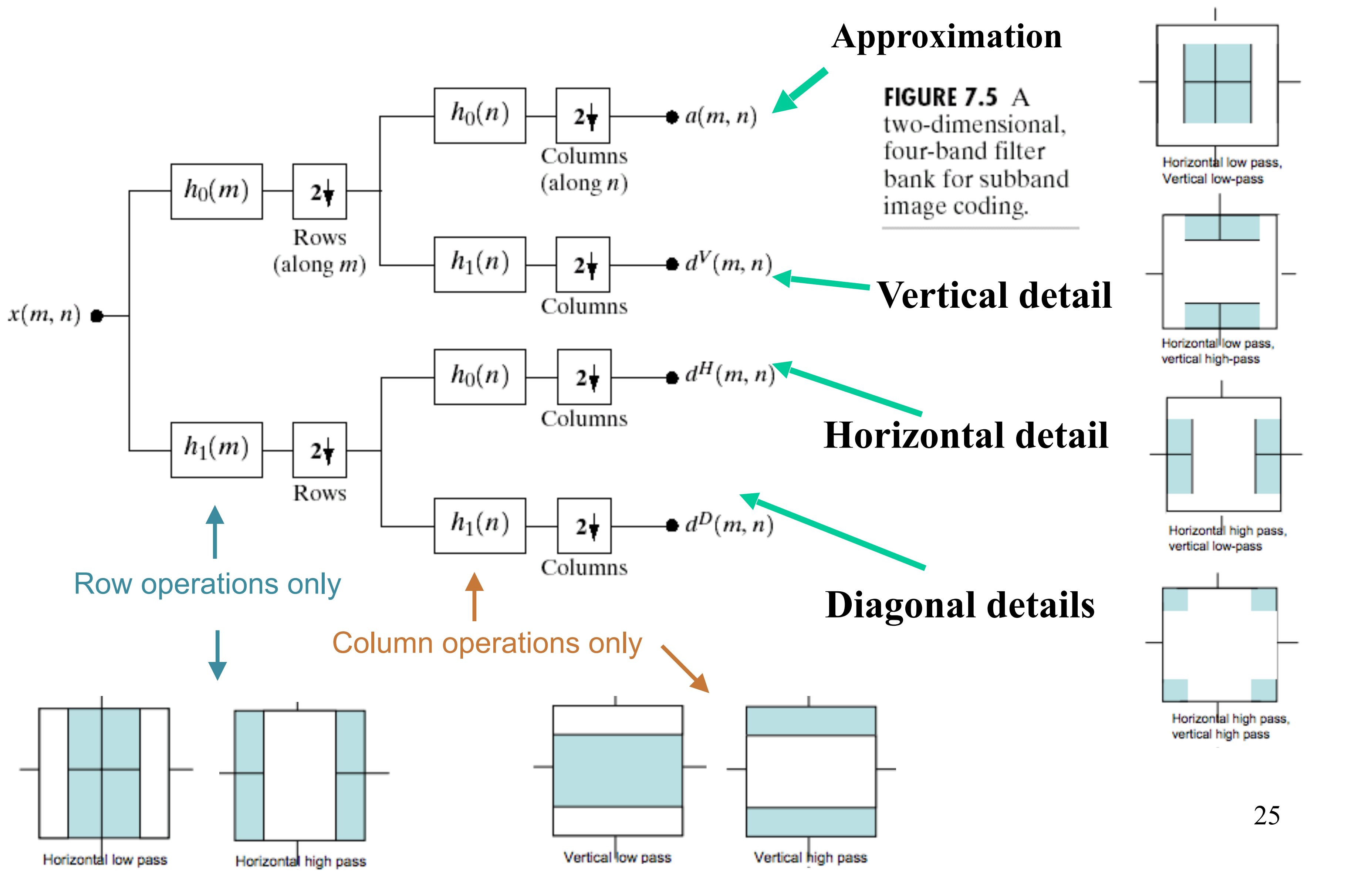
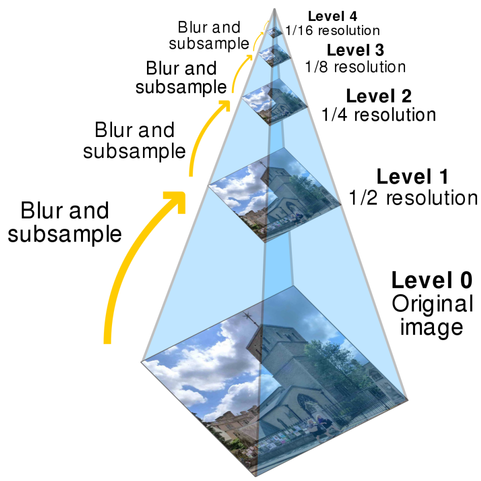


FIGURE 7.5 A two-dimensional, four-band filter bank for subband image coding.

Sub-band filter bank



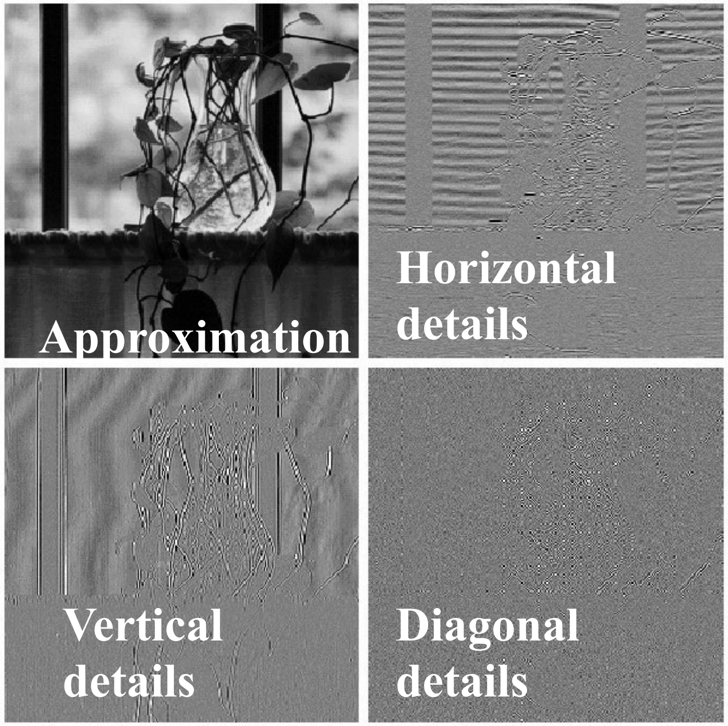
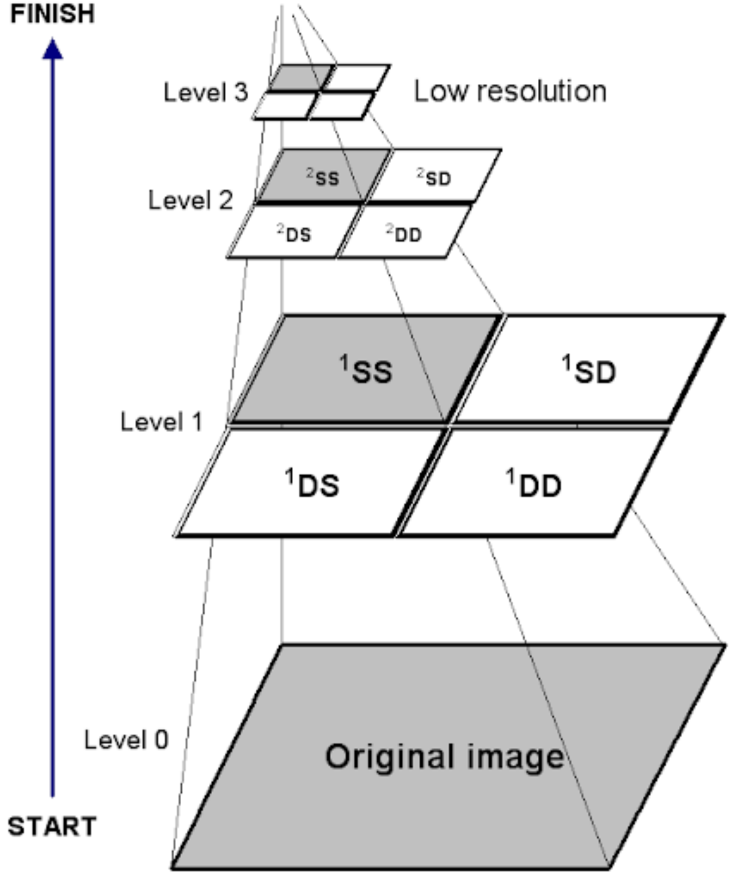
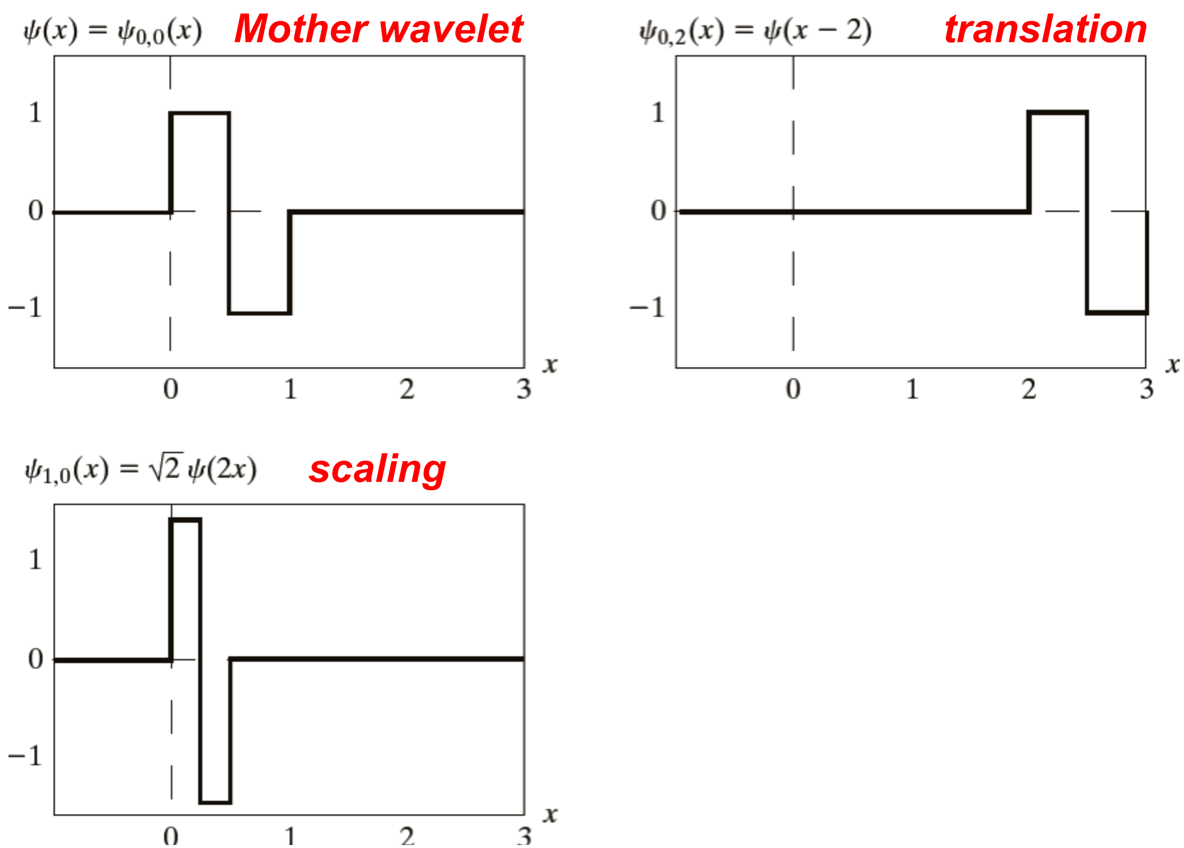


FIGURE 7.7 A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.5.



Haar Transform

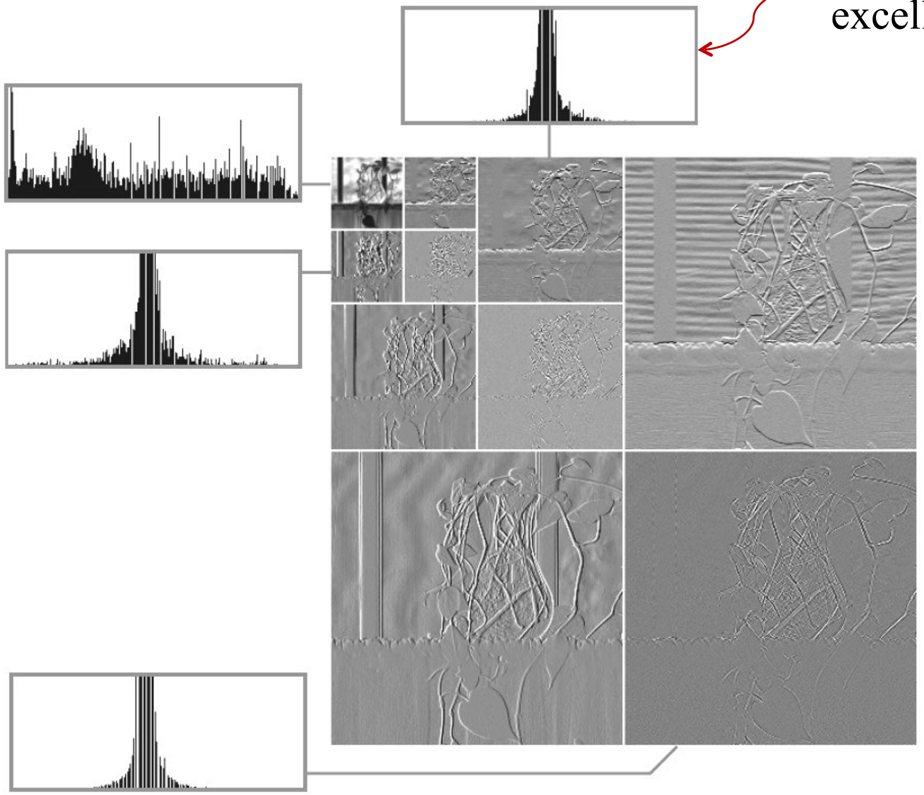


a	b
c	d
e	f

FIGURE 7.14 Haar wavelet functions in W_0 and W_1.

Haar Transform

A large number of pixels near zero make excellent candidate for compression

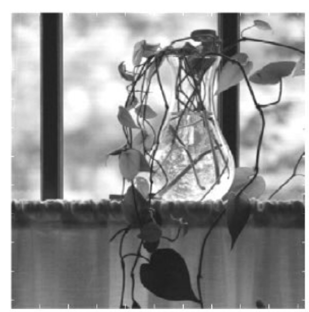


a
b c d

FIGURE 7.10
(a) A discrete wavelet transform using Haar $\mathbf{H}_2$ basis functions. Its local histogram variations are also shown. (b)–(d) Several different approximations (64×64, 128×128, and 256×256) that can be obtained from (a).

64x64

128x128



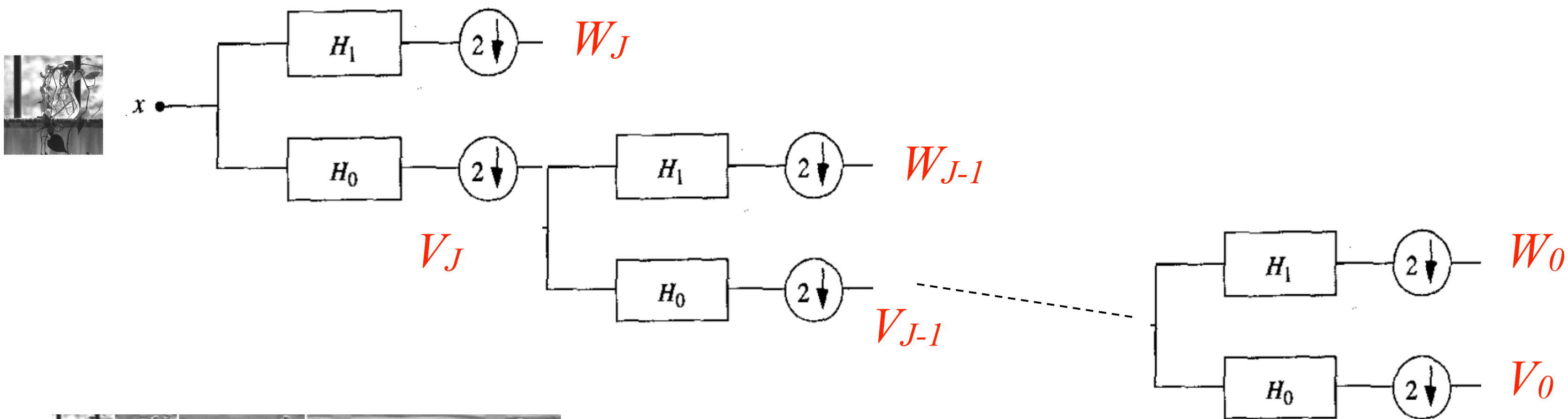
256x256

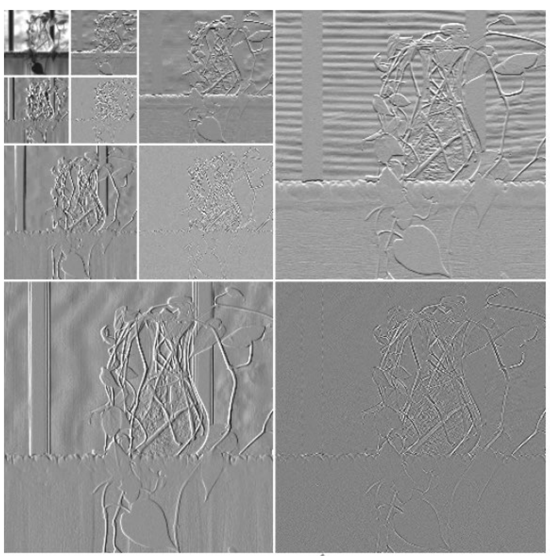
Original image is 512x512

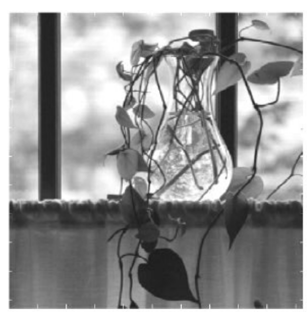
Take a break!

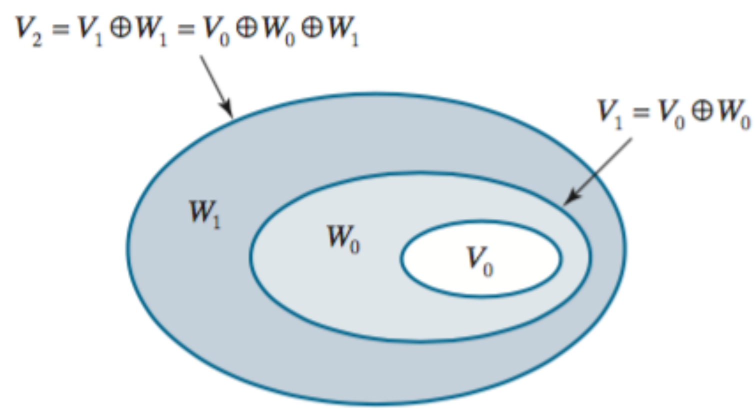
Multi Resolution Analysis

MRA provides a framework for examining functions at different scales









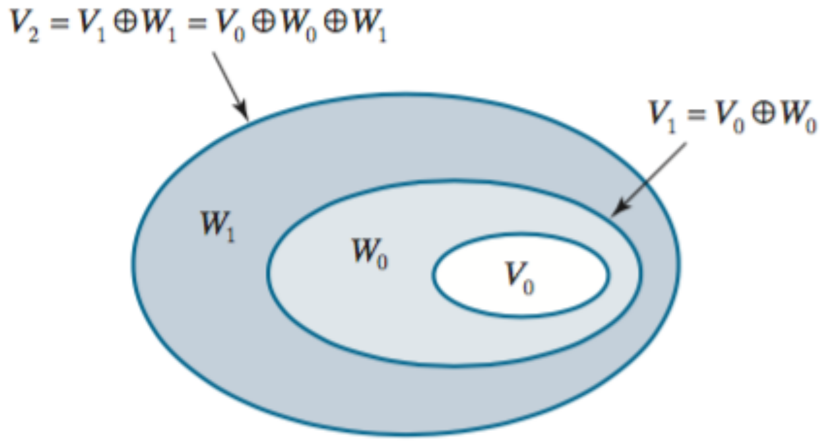
Scaling and wavelet function space

Multi Resolution Analysis

Allows inspection of details at different levels

Definition:

A Multi-resolution analysis is a nested sequence of closed subspaces



Scaling and wavelet function space

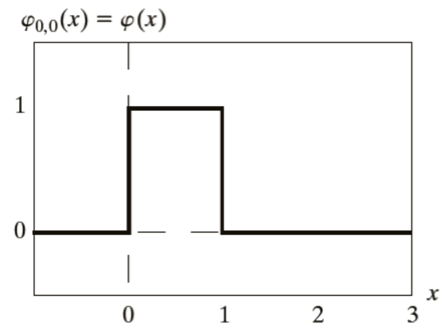
$$V_j \subset L^2(R) \quad for \quad j \in Z$$

$$\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots$$

Multi Resolution Analysis

(1) The subspaces have a zero intersection

(2) The union is $L^2(R)$



(3) Two scale relations exist

$$f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1} \quad \forall j \in Z$$

$$f(x) \in V_0 \Leftrightarrow f(x-k) \in V_0 \quad \forall k \in Z$$

(4) Finally there exists a scaling function $\phi(x) \in V_0$ whose integer translations $\{\phi_{0,k} \mid k \in Z\}$ constitute an orthonormal basis for V_0.

MRA: Scaling Function

$$\int \phi(x)dx = 1$$

1. The scaling function $\phi_{j,k}(x)$:

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) \ \ for\ all\ \ j,k \in Z$$

k determines the position of $\phi_{j,k}(x)$ along the *x*-axis,

j determines the scale (width) of $\phi_{j,k}(x)$, and $2^{j/2}$ the height or amplitude. The shape of $\phi_{j,k}(x)$ change with j.

2. The refinement/dilation equation for the scaling function:

$$\varphi(x) = \sum_{k \in \mathbf{Z}} h_{\varphi}(k)\sqrt{2}\varphi(2x - k) \qquad h_{\varphi}(k) = \left\langle \varphi(x), \sqrt{2}\varphi(2x - k) \right\rangle$$

i.e. the expansion functions of any subspace can be built from the next higher resolution space

Fundamental Requirements of MRA

- The scaling function is orthogonal to its integer translate
- The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales
- The only function that is common to all V_j is $f(x) = 0$
- Any function can be represented with arbitrary precision

Haar Scaling Function

Dilation

$$\varphi(x) = \frac{1}{\sqrt{2}}\left[\sqrt{2}\varphi(2x)\right] + \frac{1}{\sqrt{2}}\left[\sqrt{2}\varphi(2x-1)\right]$$
$$= \varphi(2x) + \varphi(2x-1)$$

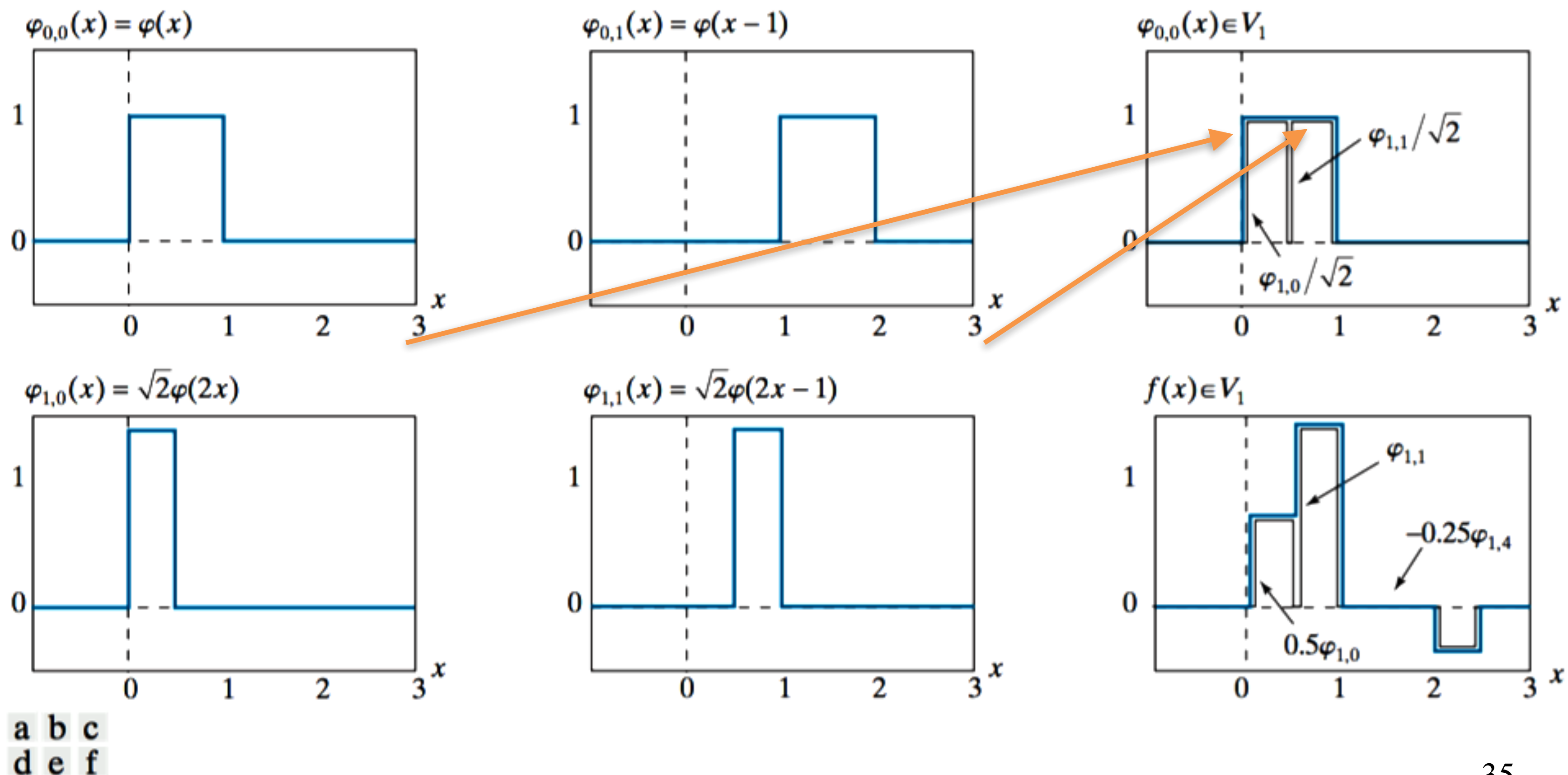


FIGURE 7.19 The Haar scaling function.

Haar Scaling Function

Dilation

$$\varphi(x) = \frac{1}{\sqrt{2}}\left[\sqrt{2}\varphi(2x)\right] + \frac{1}{\sqrt{2}}\left[\sqrt{2}\varphi(2x-1)\right]$$
$$= \varphi(2x) + \varphi(2x-1)$$

$$\varphi(x) = \sum_{k\in\mathbf{Z}} \boxed{h_\varphi(k)} \sqrt{2}\varphi(2x-k)$$

Scaling function coefficients:

$$\left\{h_\varphi(n) \middle| n = 0, 1\right\} = \left\{1/\sqrt{2}, 1/\sqrt{2}\right\}$$

n is the integer position shift

MRA: Wavelet Functions

Similar to the Scaling Function

1. The wavelet functions satisfy:

$$f(x) \in W_j \Leftrightarrow f(2x) \in W_{j+1}$$

Thus the wavelet function ψ can be expressed as follows:

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \;\; j = 0, 1, \ldots \text{ and } k = 0, 1, \ldots, 2^j - 1.$$

2. The dilation equation for the wavelet function:

$$\psi(x) = \sum h_\psi(k)\sqrt{2}\varphi(2x-k)$$

$$\varphi(x) = \sum_{k \in \mathbf{Z}} h_\varphi(k)\sqrt{2}\varphi(2x-k)$$

Complementary scaling and wavelet functions are orthogonal

$$\left\langle \varphi_{j_0,k}(x), \psi_{j_0,l}(x) \right\rangle = 0 \quad \text{for } k \neq l$$

Wavelet function coefficients:

$$\boxed{h_\psi(k) = (-1)^k h_\varphi(1-k)}$$

Haar Wavelet Functions

$$\boxed{h_\psi(k) = (-1)^k h_\varphi(1-k)}$$

$$\{h_\varphi(n) | n = 0, 1\} = \{1/\sqrt{2}, 1/\sqrt{2}\}$$

$$\psi(x) = \sum_k h_\psi(k)\sqrt{2}\varphi(2x-k)$$

It follows that

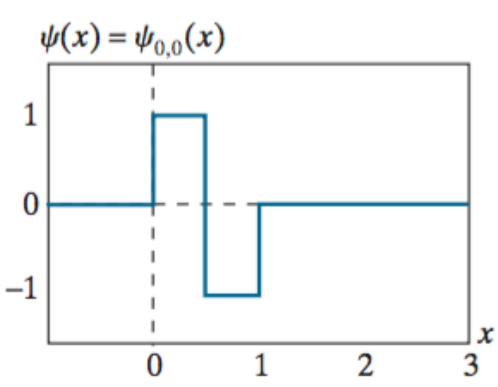
Haar wavelet coefficients:

$$h_\psi(k) = \begin{cases} 2^{-1/2} & for \quad k = 0 \\ -2^{-1/2} & for \quad k = 1 \\ 0 & otherwise \end{cases}$$

Therefore

Haar wavelet:

$$\psi(x) = \phi(2x) - \phi(2x-1) \quad \Rightarrow \quad \psi(x) = \begin{cases} 1 & for \quad 0 \leq x < \frac{1}{2} \\ -1 & for \quad \frac{1}{2} \leq x < 1 \\ 0 & otherwise \end{cases}$$



Haar Wavelet Functions

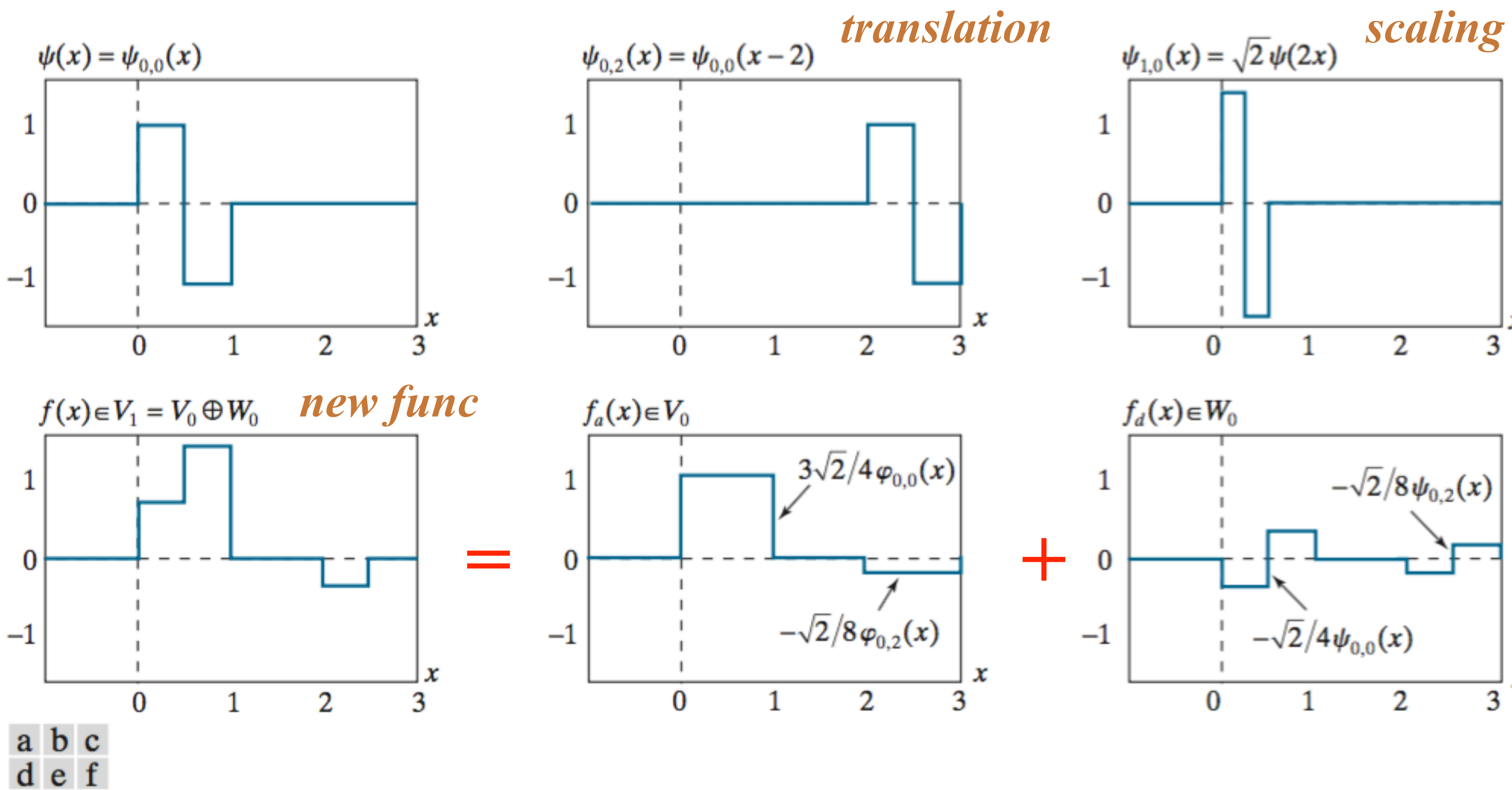


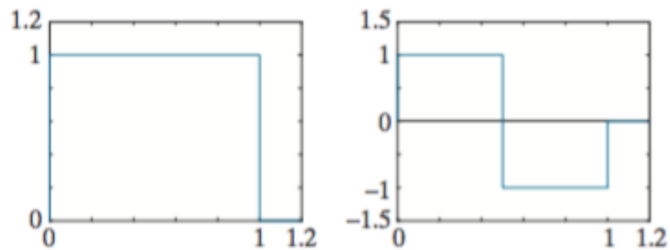
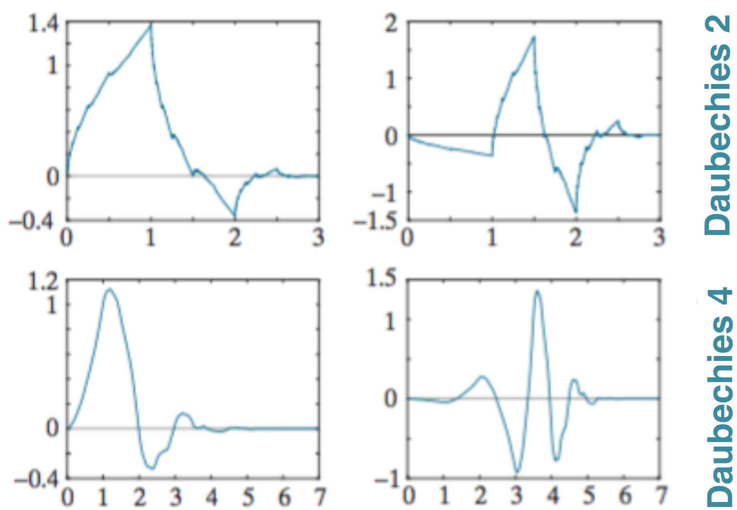
FIGURE 7.21 Haar wavelet functions.

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad j,k \in Z$$

MRA: Series Expansion

By solving the dilation equation of the scaling function with increasing number of expansions, we are able to obtain other families of wavelets + scaling function

TABLE 7.1
Some representative wavelets.

Wavelet Name or Family	Scaling Function	Wavelet Function
Haar The oldest and simplest wavelets. Orthogonal and discontinuous. **simplest**		
Daubechies family Orthogonal with the most vanishing moments for a given support. Denoted db$N$, where N is the number of vanishing moments; db2 and db4 shown; db1 is the Haar of the previous row.		

Scaling function coefficients of Daubechies 4

$$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}$$

$$h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}$$

$$h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}$$

$$h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$$

More vanishing moments,
More detailed representation

Wavelet Series Expansion

Express a function with scaling and wavelet functions

$$f(x) = \sum_k c_{j_0}(k)\varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty}\sum_k d_j(k)\psi_{j,k}(x)$$

Approximation Coefficient $c_{j_0} = \langle f(x), \varphi_{j_0,k}(x)\rangle$

Detail Coefficient $d_j = \langle f(x), \psi_{j,k}(x)\rangle$

Example 7.18 in the textbook:
expand the function y with Haar wavelets

$$y = \begin{cases} x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$y = \underbrace{\underbrace{\underbrace{\frac{1}{3}\varphi_{0,0}(x)}_{V_0} + \underbrace{\left[-\frac{1}{4}\psi_{0,0}(x)\right]}_{W_0}}_{V_1 = V_0 \oplus W_0} + \underbrace{\left[-\frac{\sqrt{2}}{32}\psi_{1,0}(x) - \frac{3\sqrt{2}}{32}\psi_{1,1}(x)\right]}_{W_1}}_{V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1} + \cdots$$

Discrete wavelet transform

In the discrete cases, where we have N-point discrete function and N is a power of 2 ($N=2^J$)

$$f(x)=\frac{1}{\sqrt{N}}\left[T_{\varphi}(0,0)\varphi(x)+\sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1}T_{\psi}(j,k)\psi_{j,k}(x)\right]$$

where

Approximation Coefficient

$$T_{\varphi}(0,0)=\left\langle f(x),\varphi_{0,0}(x)\right\rangle=\left\langle f(x),\varphi(x)\right\rangle=\frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}f(x)\varphi^{*}(x)$$

and

Detail Coefficient

$$T_{\psi}(j,k)=\left\langle f(x),\psi_{j,k}(x)\right\rangle=\frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}f(x)\psi_{j,k}^{*}(x)$$

Read Example 7.19 in the textbook with discrete Haar wavelets

Discrete wavelet transform

Read Example 7.19 in the textbook with discrete Haar wavelets

$f(0) = 1$, $f(1) = 4$, $f(2) = -3$, and $f(3) = 0$. $N=4$

$$\mathbf{t}_{\mathrm{H}} = \mathbf{A}_{\mathrm{H}}\mathbf{f} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -1.5\sqrt{2} \\ -1.5\sqrt{2} \end{bmatrix}$$

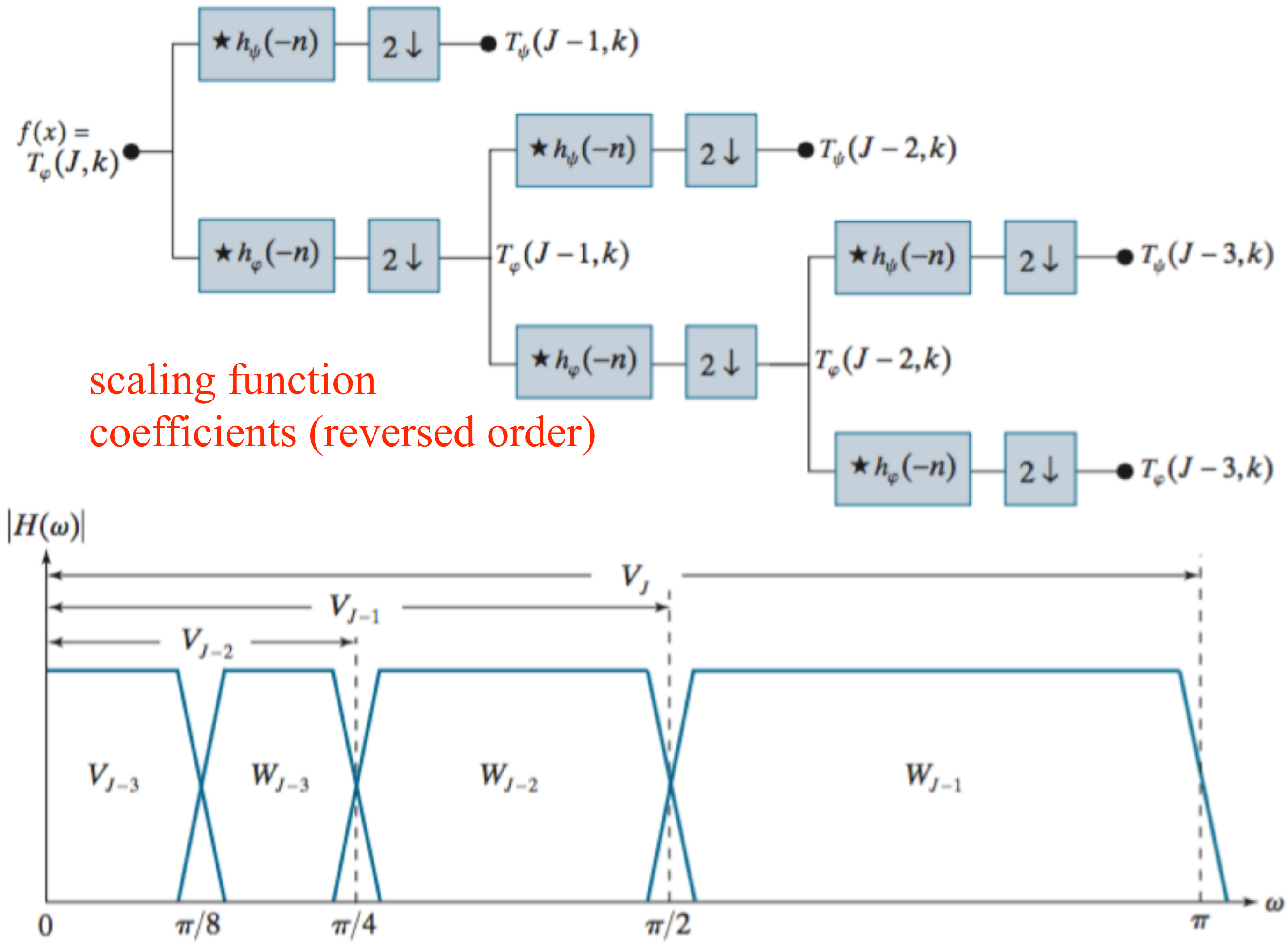
$$f(x) = \frac{1}{2}\left[T_\varphi(0,0)\varphi(x) + T_\psi(0,0)\psi_{0,0}(x) + T_\psi(1,0)\psi_{1,0}(x) + T_\psi(1,1)\psi_{1,1}(x)\right]$$

Fast wavelet transform (FWT)

wavelet coefficients (reversed order)

a
b

FIGURE 7.24
(a) A three-stage or three-scale FWT analysis filter bank and (b) its frequency-splitting characteristics. Because of symmetry in the DFT of the filter's impulse response, it is common to display only the $[0, \pi]$ region.



scaling function coefficients (reversed order)

Note that with convolution, there is a need to reverse the order of the entries for the filter

Fast wavelet transform (FWT)

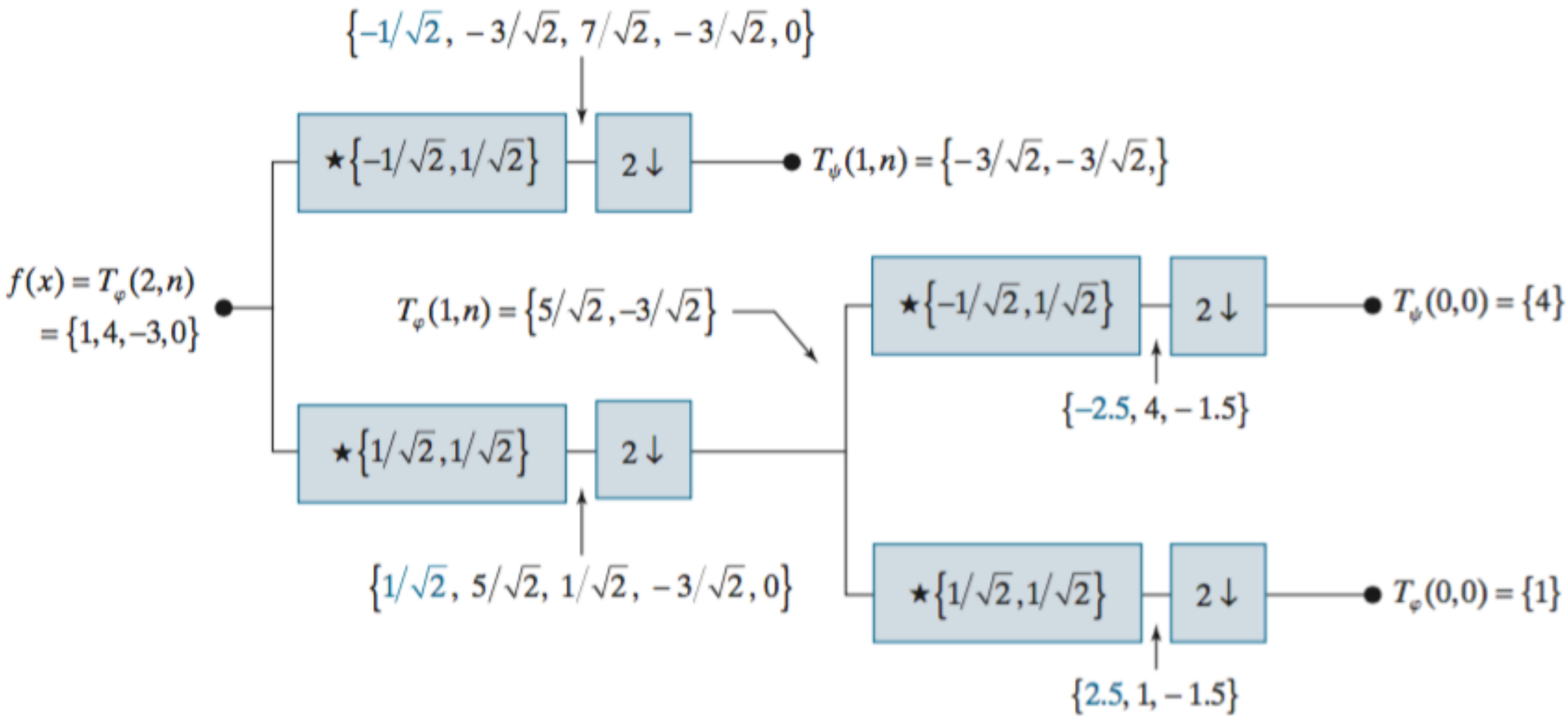


FIGURE 7.25 Computing a two-scale fast wavelet transform of sequence $\{1, 4, -3, 0\}$ using Haar scaling and wavelet coefficients.

Inverse Fast wavelet transform (IFWT)

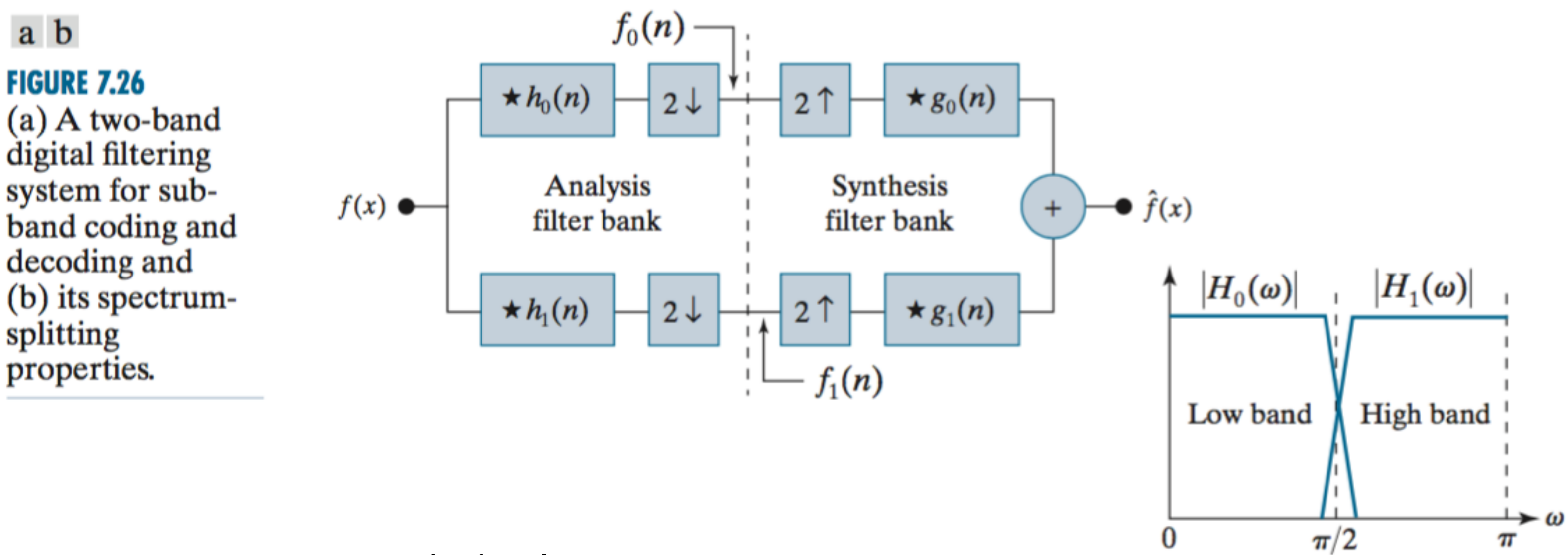


FIGURE 7.26
(a) A two-band digital filtering system for sub-band coding and decoding and (b) its spectrum-splitting properties.

Cross-modulation

$$\boxed{\begin{aligned} g_0(n) &= (-1)^n h_1(n) \\ g_1(n) &= (-1)^n h_0(n) \end{aligned}}$$

$$g_1(n) = (-1)^n g_0(K-1-n)$$
$$h_0(n) = g_0(K-1-n)$$
$$h_1(n) = g_1(K-1-n)$$

Inverse Fast wavelet transform (IFWT)

Note how filter functions change from FWT

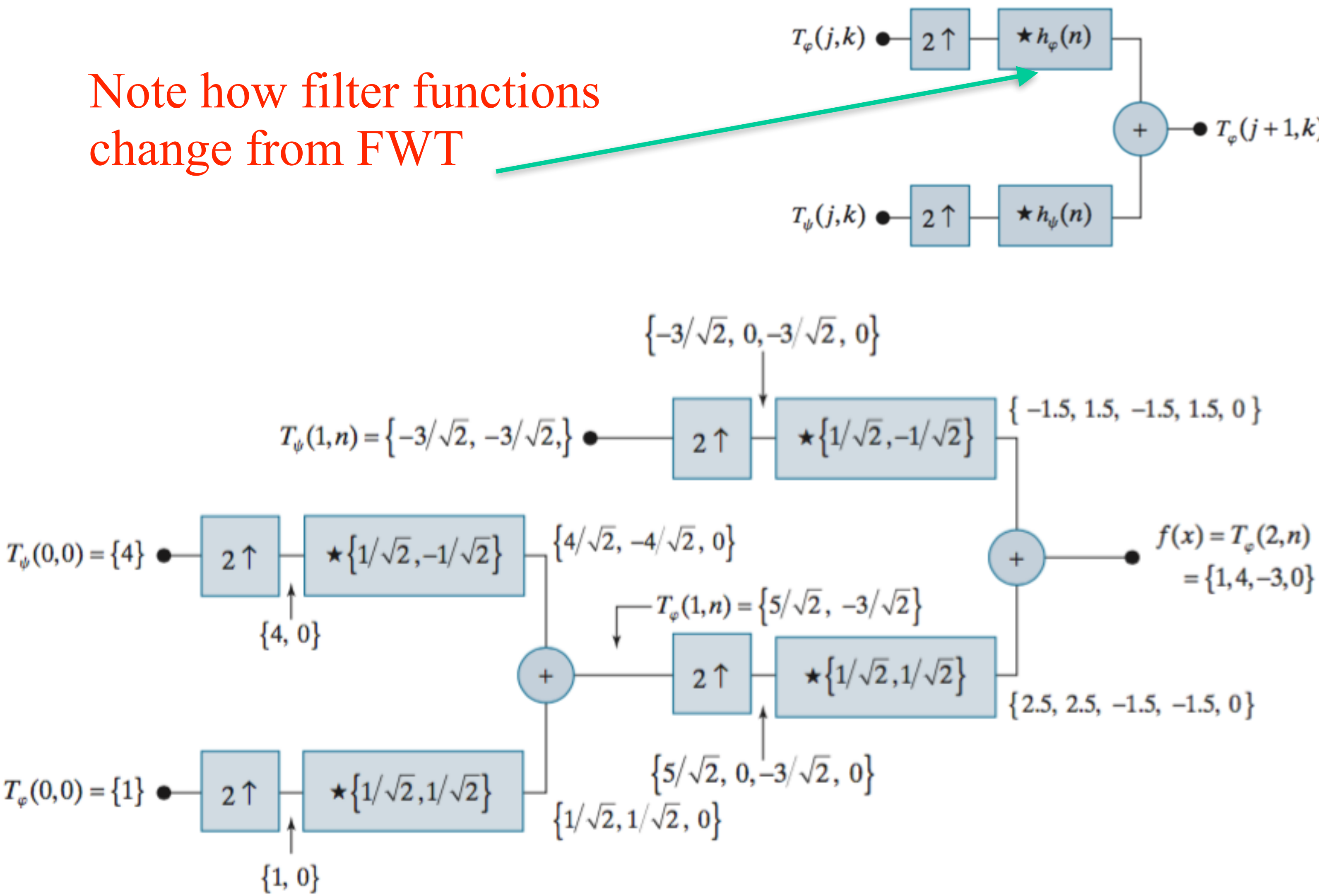
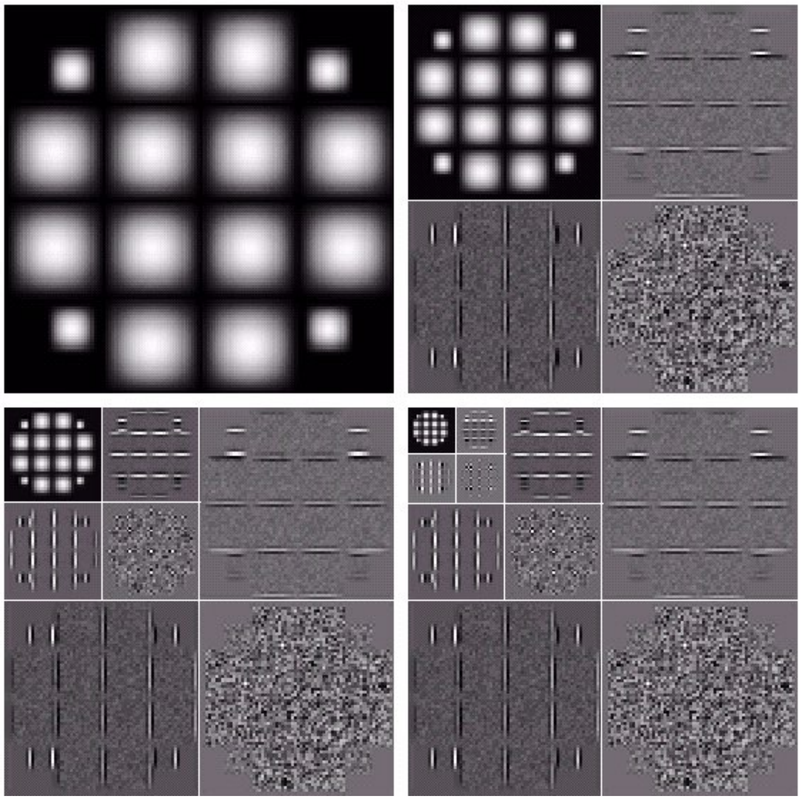


FIGURE 7.28 Computing a two-scale inverse fast wavelet transform of sequence $\{1,\ 4,\ -1.5\sqrt{2},\ -1.5\sqrt{2}\}$ with Haar scaling and wavelet functions.

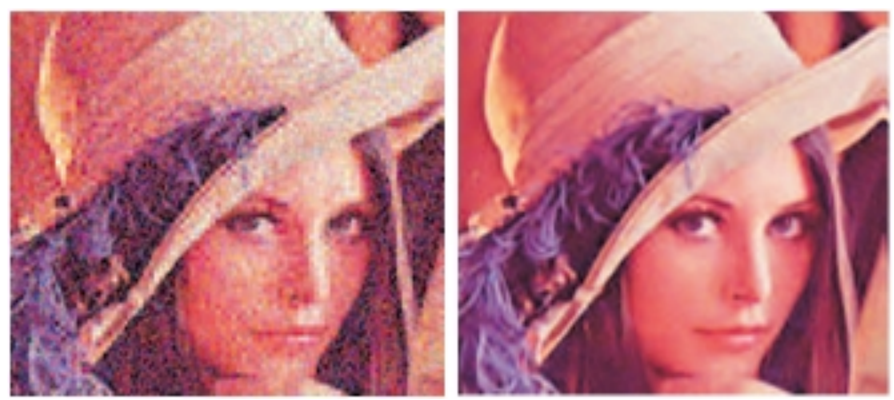
2-D Wavelet Transform



a b
c d

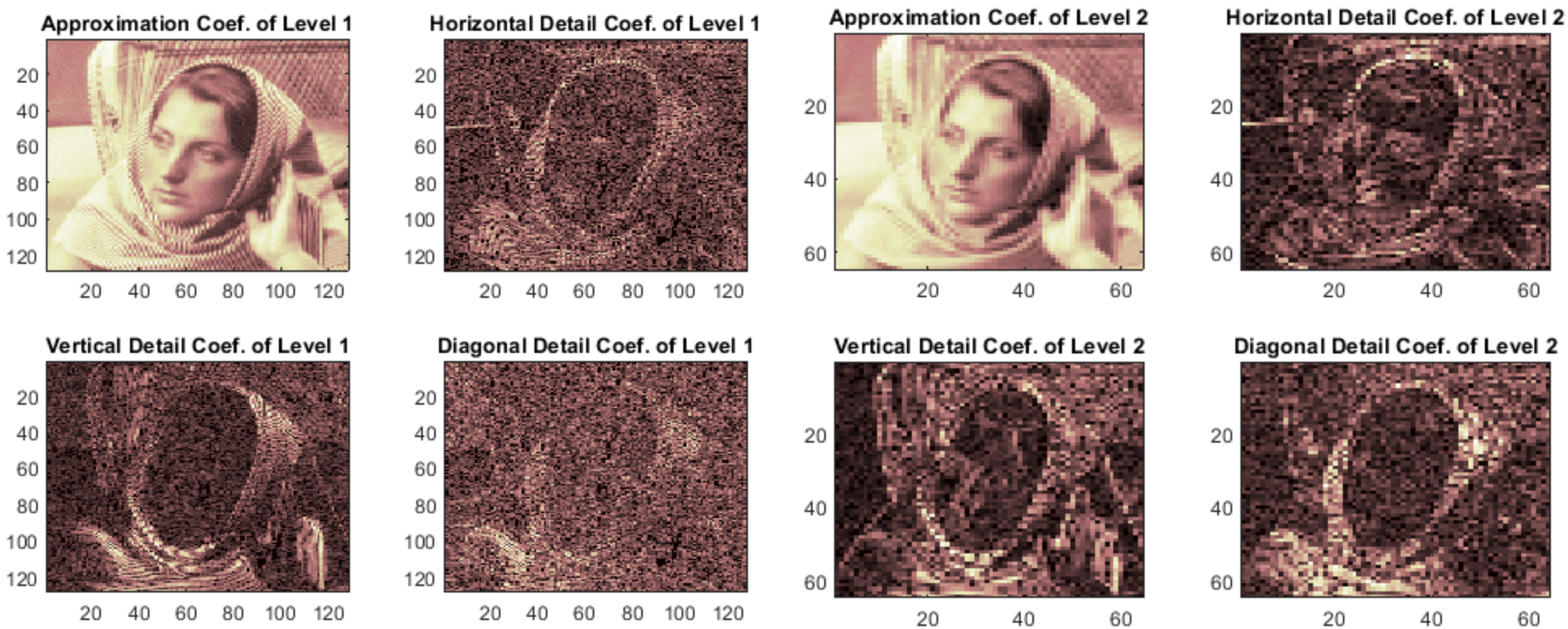
FIGURE 7.25 Computing a 2-D three-scale FWT: (a) the original image; (b) a one-scale FWT; (c) a two-scale FWT; and (d) a three-scale FWT.

Wavelet denoising via synthesis



2-D DWT DEMO ON MATLAB

Results obtained with wavedec2() function



https://www.mathworks.com/help/wavelet/ref/wavedec2.html

Applications of Wavelets

- Pattern recognition
- Feature extraction
- Image processing
- Biometrics
- Financial analysis
- Perfect reconstruction
- Video compression
- Medical imaging
- etc

Summary

- Definition and benefits of wavelets
- Image Pyramid
- Filter banks: decomposition & synthesis
- Continuous & discrete wavelet transform
- Scaling and wavelet functions & properties
- Scaling and wavelet function space
- Different wavelet families (Haar, Daubechies)

Useful Link

- Matlab wavelet tool using guide
- http://www.wavelet.org
- http://www.multires.caltech.edu/teaching/
- http://www-dsp.rice.edu/software/RWT/
- www.multires.caltech.edu/teaching/courses/waveletcourse/sig95.course.pdf
- http://www.amara.com/current/wavelet.html

Reading

- Chapter 7
- Page 481-483, 504-526

For related math operations and concepts, please refer to Page 464-470

Questions

- Why wavelet transform is needed?
- What are the basic properties of wavelets?
- Wavelet: scale vs. frequency; translation vs. time
- Image Pyramids: two types & why do we need them?
- Wavelets: scaling function vs. wavelet function
- Haar vs. Daubechies-4: which one gives better approximation of images?