Animation for Computer Games
COMP 477/6311

Prof. Tiberiu Popa

Rotations - Quaternions

Transformations

$$f\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

- 4x4 matrices using homogeneous coordinates

Transformations

- What does it mean to have a good representation?
  1. **Complete**: to represent all transformations that are needed (i.e. represent translations, rotations, scale, etc.)
  2. **Efficient** application of the transformation (i.e. we need to transform potentially hundreds of thousands of points, need to do this fast!!!
  3. **Composition**: the ability to combine multiple transformation into one (i.e. remember the skeleton: the transformation of a bone is the composition of all transformations of its parent – we'd like to be able to compose all those transformation into one
  4. **Interpolation**: the ability to compute in-between transformations (i.e. when we have to compute the in-between frames in keyframe animation

$$f\left(\begin{matrix} x \\ y \\ z \\ w \end{matrix}\right) = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

Transformations

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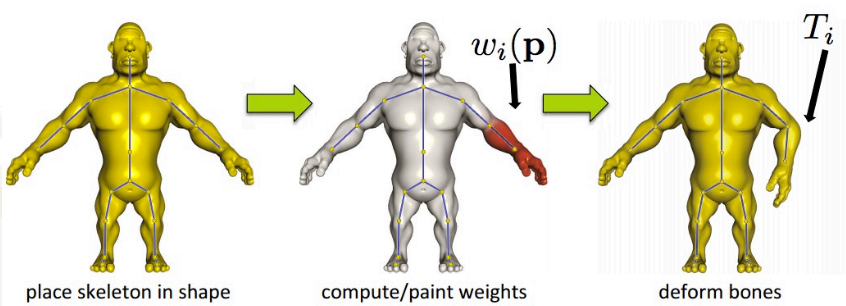
- 4x4 matrices using homogeneous coordinates
- Interpolation is critical for keyframe animation

$$\mathbf{p}' = \sum_{i \in B} w_i(\mathbf{p}) T_i \mathbf{p}$$



Rotations

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- 4x4 matrices represent many different types of transformations
- What type of 4x4 matrices are in fact rotations?

Rotations

A rotation around an arbitrary axis that intersects the origin:
If axis does not interest origin → we compose it with translations like we saw before

$$\begin{pmatrix} R_1^1 & R_1^2 & R_1^3 & 0 \\ R_2^1 & R_2^2 & R_2^3 & 0 \\ R_3^1 & R_3^2 & R_3^3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}(\mathbf{u},\theta) = \begin{bmatrix} u_x^2 + \cos\theta(1-u_x^2) & u_x u_y(1-\cos\theta) - u_z \sin\theta & u_x u_z(1-\cos\theta) + u_y \sin\theta & 0 \\ u_x u_y(1-\cos\theta) + u_z \sin\theta & u_y^2 + \cos\theta(1-u_y^2) & u_y u_z(1-\cos\theta) - u_x \sin\theta & 0 \\ u_x u_z(1-\cos\theta) - u_y \sin\theta & u_y u_z(1-\cos\theta) + u_x \sin\theta & u_z^2 + \cos\theta(1-u_z^2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotations

A rotation around an arbitrary axis that intersects the origin:

R submatrix is orthogonal!!!

We found our well defined mathematical definition of a rotation matrix using matrices

Every such matrix represents a unique rotation → from matrix it is possible to compute the axis and angle

$$\begin{pmatrix} R_1^1 & R_1^2 & R_1^3 & 0 \\ R_2^1 & R_2^2 & R_2^3 & 0 \\ R_3^1 & R_3^2 & R_3^3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Rotations

How about interpolating matrix element in a piece-wise fashion:

What is the transformation: $\frac{1}{2}R + \frac{1}{2}\hat{R}$?

In LBS it is element-wise scalar combinations?

Is this ok? Why or why not?

$$R=\begin{pmatrix} R_1^1 & R_1^2 & R_1^3 & 0 \\ R_2^1 & R_2^2 & R_2^3 & 0 \\ R_3^1 & R_3^2 & R_3^3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \hat{R} = \begin{pmatrix} \hat{R}_1^1 & \hat{R}_1^2 & \hat{R}_1^3 & 0 \\ \hat{R}_2^1 & \hat{R}_2^2 & \hat{R}_2^3 & 0 \\ \hat{R}_3^1 & \hat{R}_3^2 & \hat{R}_3^3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotations

What is the transformation: $\frac{1}{2}R + \frac{1}{2}\hat{R}$?
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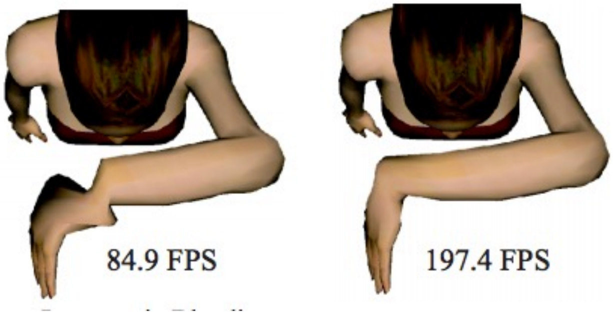


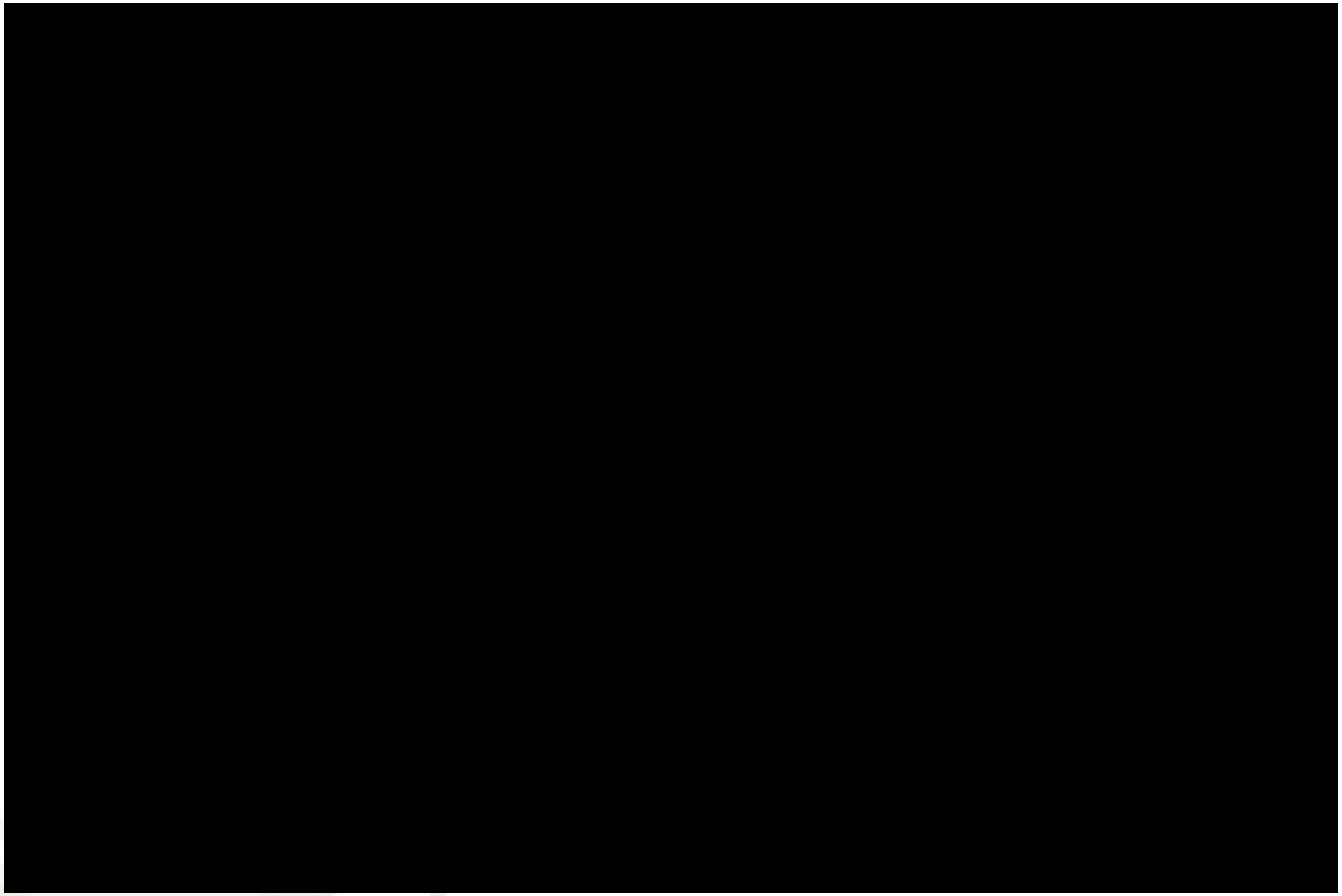
[Kavan et al]

Used in practice because of simplicity, but not ok

- Artifacts
- Mathematically incorrect

$$R=\begin{pmatrix} R_1^1 & R_1^2 & R_1^3 & 0 \\ R_2^1 & R_2^2 & R_2^3 & 0 \\ R_3^1 & R_3^2 & R_3^3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \hat{R} = \begin{pmatrix} \hat{R}_1^1 & \hat{R}_1^2 & \hat{R}_1^3 & 0 \\ \hat{R}_2^1 & \hat{R}_2^2 & \hat{R}_2^3 & 0 \\ \hat{R}_3^1 & \hat{R}_3^2 & \hat{R}_3^3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotations



https://www.youtube.com/watch?v=94USA9yMzAw

Rotations

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Rotations

Other representations/parameterization:

- Axis angle
- Euler angles
- Quaternions

Spoiler alert!!! – quaternions will prevail

Rotations – Axis/angle

Axis → unit vector, angle → scalar

How do we apply a transformation to a point?

Is it a good representation?

Axis/Angle

- What does it mean to have a good representation?

1. **Complete**: to represent all rotations
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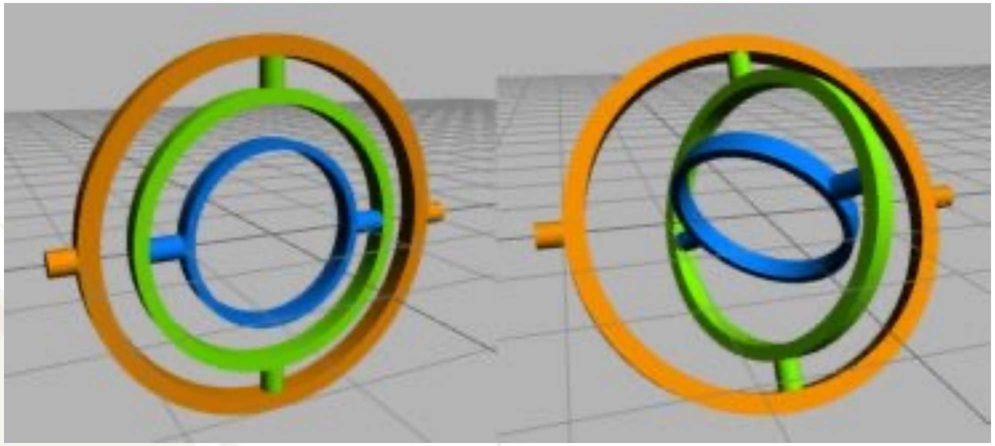
Rotations – Euler Angles

Any rotations can be decomposed as 3 rotations over the canonical axes: X, Y and Z (i.e. pitch, roll and yaw)

Classical representation in aviation/robotics

How do we apply a transformation to a point?

Is it a good representation?



Euler Angles

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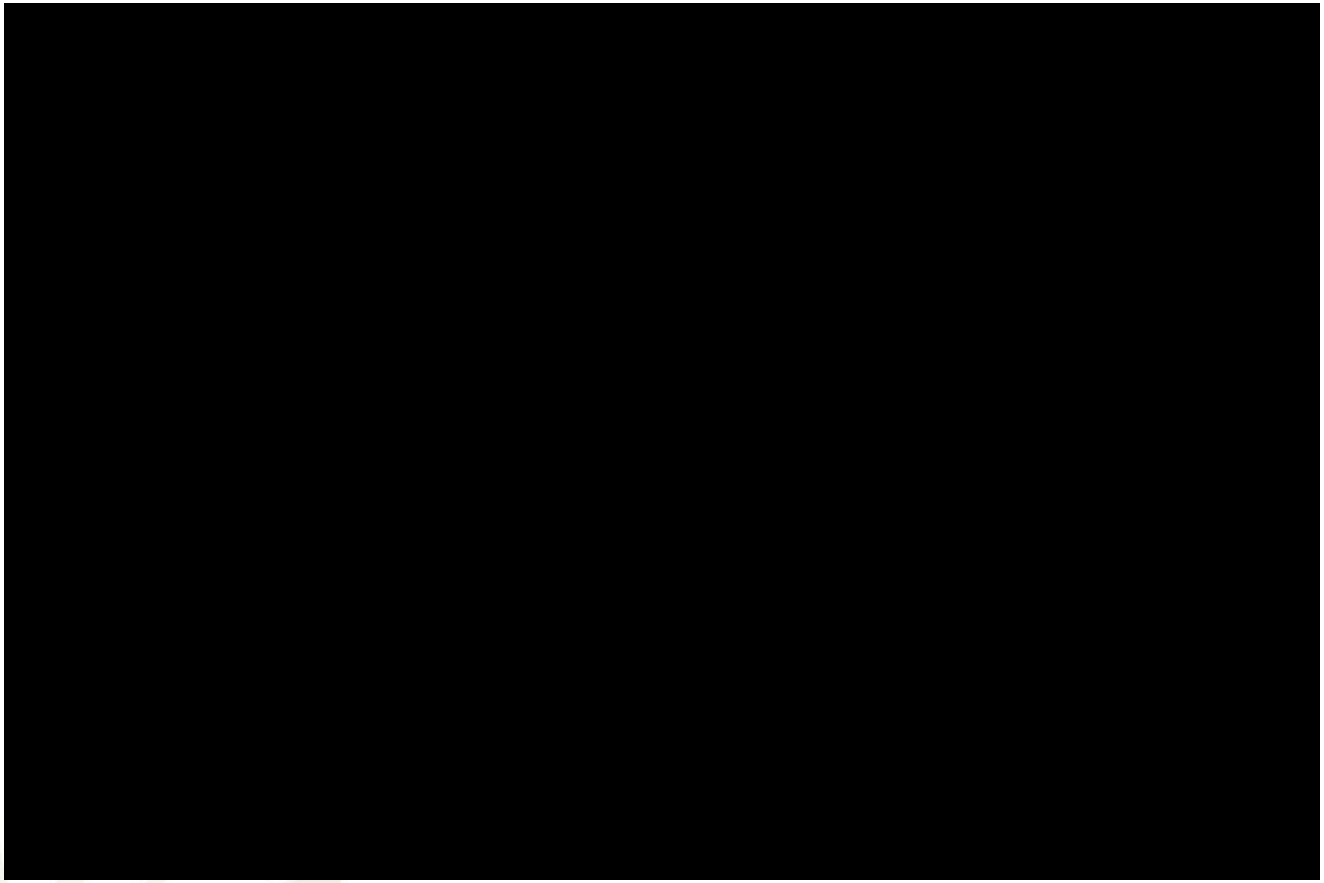
Rotations



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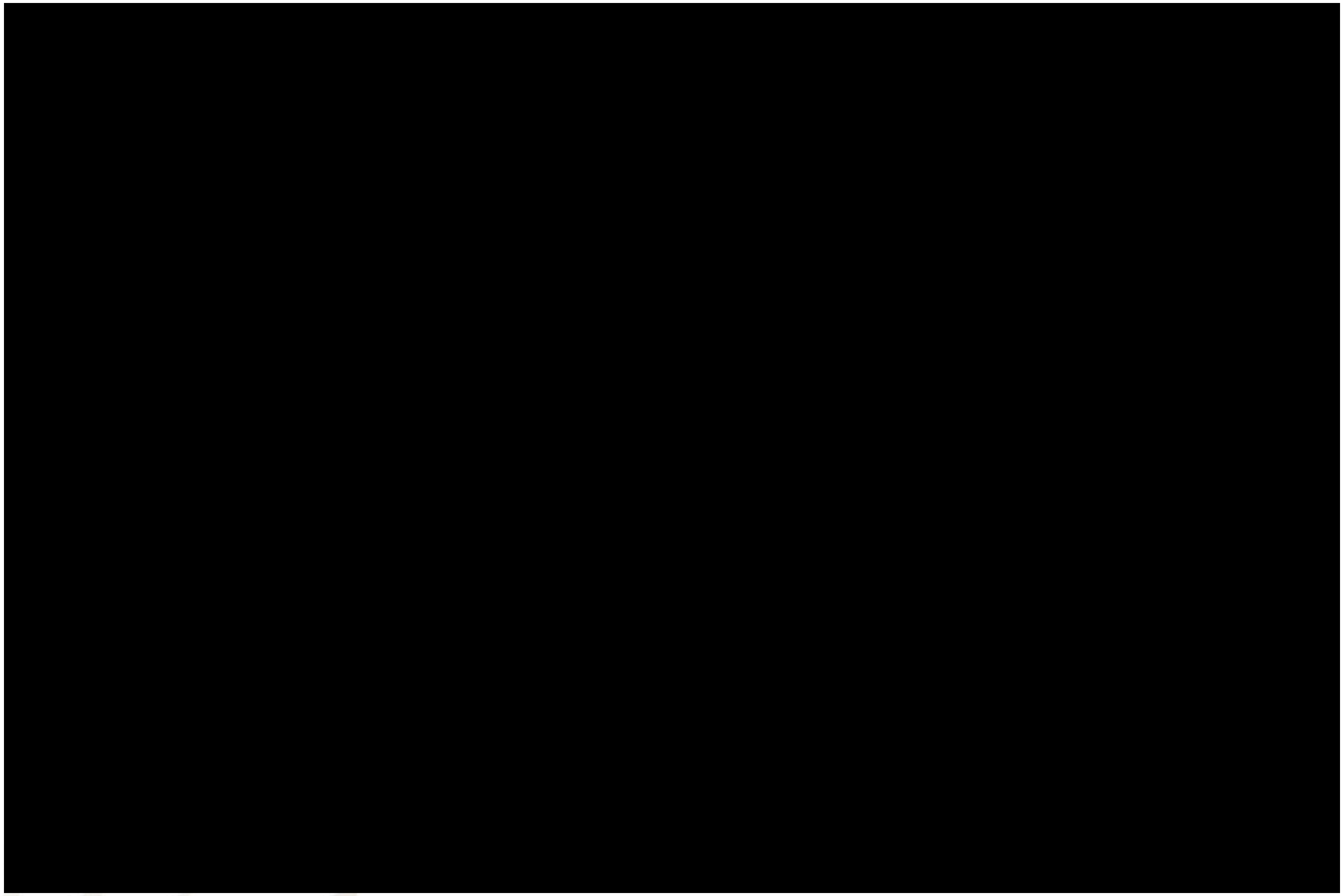
Rotations



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