

Functions

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Functions: Some terminology

- ▶ For sets A and B , a *function* from A to B , denoted as

$$f : A \rightarrow B$$

is an assignment (mapping) of each element of (source set) A to exactly one element of (target set) B .

- ▶ For a unique $b \in B$ assigned by f to $a \in A$ we write

$$f(a) = b$$

- ▶ A is the *domain* of f , and B is the *codomain* of f .
- ▶ In $f(a) = b$, b is the *image* of a , and a is the *pre-image* of b .
- ▶ The set of all images is the *range* of f . The range, R , is the particular set of values in the codomain that the function actually maps elements of the domain to, i.e.
 $R \subseteq B$ of f .

Functions: Domain and codomain

- ▶ We have already seen propositional functions in predicate logic, e.g. $P(x)$: “ x is an island” is a function from objects to propositions, e.g.
 $P : \text{Object} \rightarrow \text{Proposition}.$
- ▶ For example: $P(\text{Montreal}) = \text{“Montreal is an island.”}$
- ▶ A propositional operator can be viewed as a function from ordered pairs of Boolean values to a Boolean value, e.g.
 $\wedge : \text{Boolean} \times \text{Boolean} \rightarrow \text{Boolean}.$
- ▶ Example: $\wedge((T, F)) = F.$

Partial functions

- ▶ A *partial function* from (source set) A to (target set) B denoted as

$$f : A \rightharpoonup B$$

is a function defined for some subset A' of A , i.e. it does not force the mapping for every element of A to an element of B , i.e.

$$\text{dom } f \subset A$$

as opposed to a *total function* where $\text{dom } f = A$.

One-to-one (injective) functions

- ▶ f is *one-to-one* (or *injective*) if *each* element of the codomain is mapped to by at most one element of the domain (never maps distinct elements of its domain to the same element of its codomain), i.e.

$$\forall a, b (a \neq b \rightarrow f(a) \neq f(b))$$

in the domain of f .

- ▶ Is $f(x) = x^2$ one-to-one? (Note: The domain is the set of integers.)
- ▶ No since $f(1) = f(-1) = 1$, but $1 \neq -1$.

Onto (surjective) functions

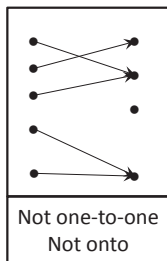
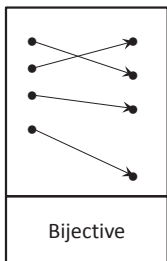
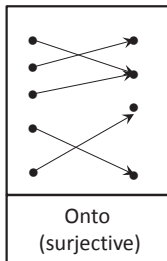
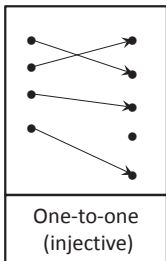
- ▶ f is *onto* (or *surjective*) if each element of the codomain is mapped to by at least one element of the domain, iff for every $b \in B$ there is an $a \in A$ with $f(a) = b$, i.e.

$$\forall b \exists a (f(a) = b)$$

Bijjective functions

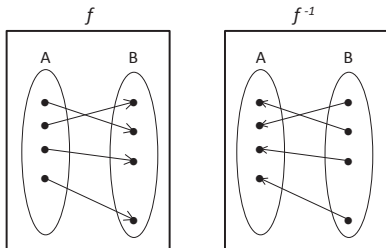
- ▶ f is *one-to-one correspondence* (or *bijection*) if it is both one-to-one and onto.

Visualization of function types



Inverse functions

- If f is a bijection such that $f : A \rightarrow B$, then there is a function from B to A that maps each element of B back to its corresponding element in A .



- This is called the inverse function for f , denoted by $f^{-1} : B \rightarrow A$.

Composition of functions

- ▶ If $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions, then $g \circ f$ is a composite relation between A and C such that

$$g \circ f : A \rightarrow C$$

given by

$$(g \circ f)(a) = g(f(a))$$

- ▶ Note that $\text{ran } f \subseteq \text{dom } g$.

Example: Composition of functions

- Consider the following:

$$\forall x \in \mathbb{N} \bullet f(x) = x + 1 \wedge g(x) = 5x$$

$$g(f(3)) = g(4) = 20, \text{ or } (g \circ f)(3) = 20$$

- Note that $\text{dom}(g \circ f) \subseteq \text{dom } f$.

Modeling functions by product sets

- ▶ For example, for function with $\text{dom } f = \{1, 2, 3, 4\}$ and $\forall x \in \text{dom } f \bullet f(x) = x(x - 2)$, we often write

$$f = \{(1, -1), (2, 0), (3, 3), (4, 8)\}$$

- ▶ Note on notation: Ordered pairs are often represented using *maplet notation*, e.g. the pair (x, y) is written as $x \mapsto y$.
- ▶ We often talk about the set which models a function as actually being the function itself.