Animation for Computer Games

COMP 477/6311

Prof. Tiberiu Popa

# Elastic Objects

# A little bit of physics

# The Basics

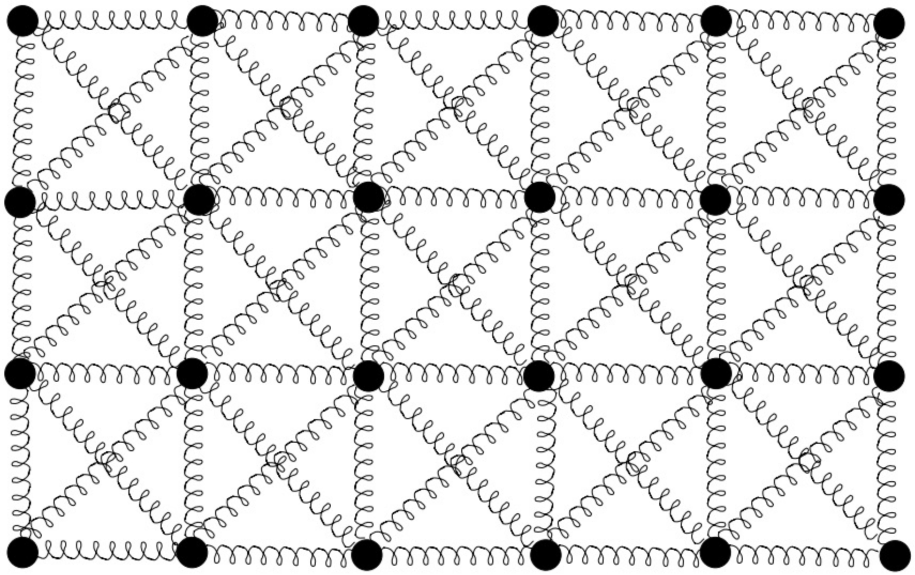
- Integration techniques are applied to all physics simulations
- Differences are in: forces, discretization
  - Rigid bodies
    - gravity
    - friction
  - Elastic bodies (e.g. cloth)
    - gravity
    - elastic forces
    - friction
  - Fluids
    - Navier-Stokes equations

# Elastic objects

- Elastic objects when deformed tend to return to return to the rest pose
- Internal forces push them towards the rest pose (stress)
- Deformation from rest-pose (strain)
- Hook's law: linear relationship between stress and strain
- A little abstract because we need a discretization

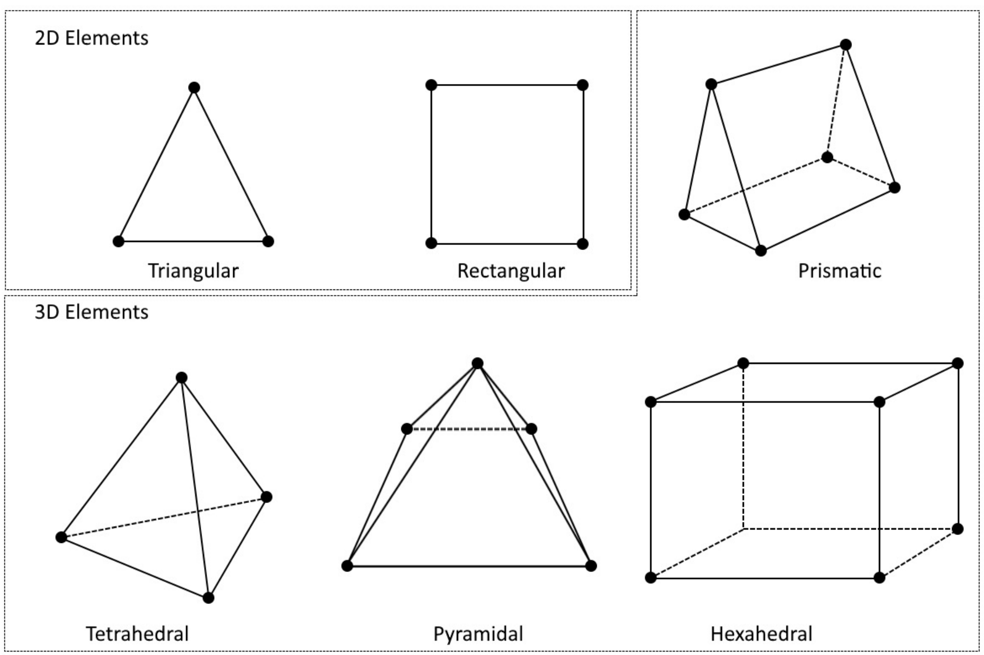
# Elastic objects

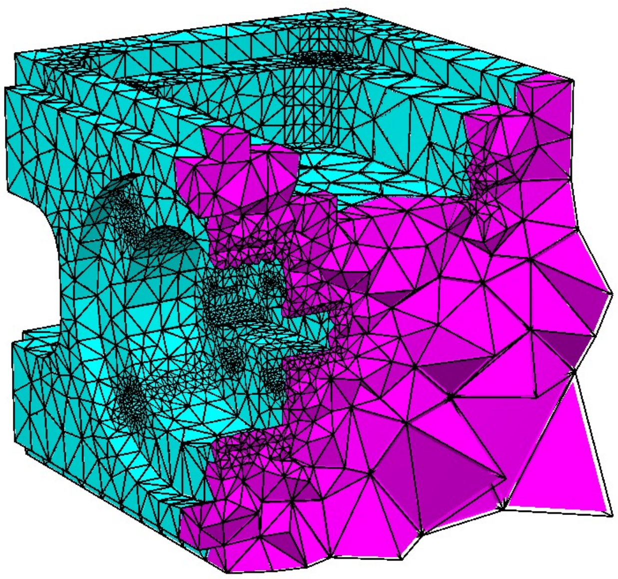
- Elastic objects are discretized in 2 ways:
  - Spring systems



# Elastic objects

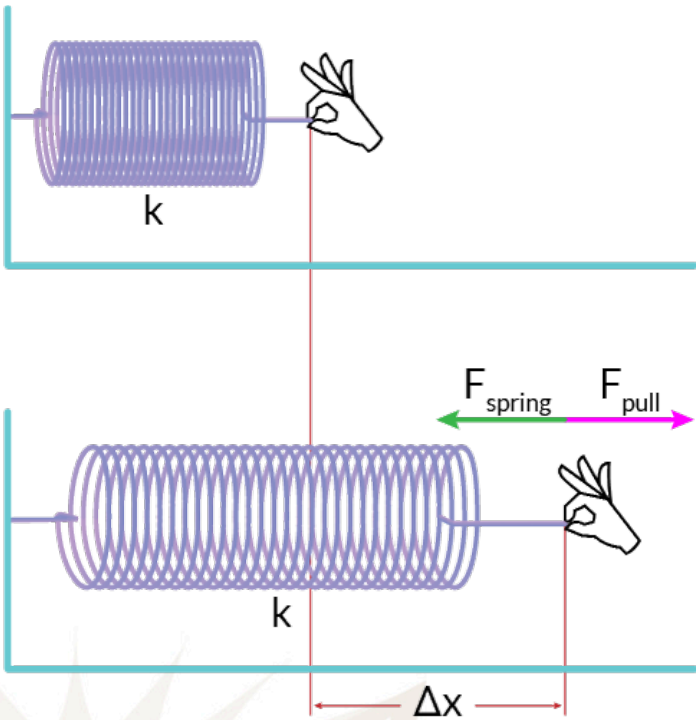
- Elastic objects are discretized in 2 ways:
  - Finite Element Methods (FEM)



# Springs

- Rather simple (1d model)




$$F^{spring} = F^{int} = -k \cdot \Delta x$$

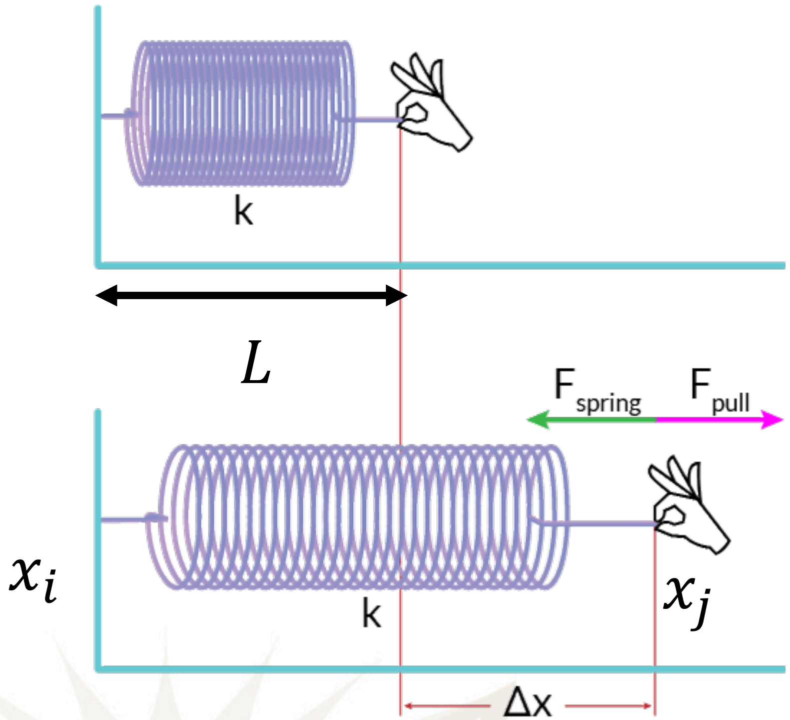
(Hooke's law)

$k$ → stiffness of the spring

https://sciencenotes.org/hookes-law-example-problem/

# Springs

- 1D models embedded in 2D or 3D




$$F^{spring} = F^{int} = -k \cdot \Delta x$$

(Hooke's law)

$$F^{int} = -k \cdot (\|x_i - x_j\| - L) \cdot \frac{x_j - x_i}{\|x_i - x_j\|}$$

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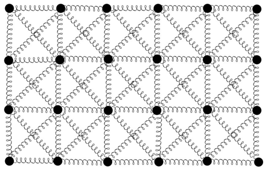
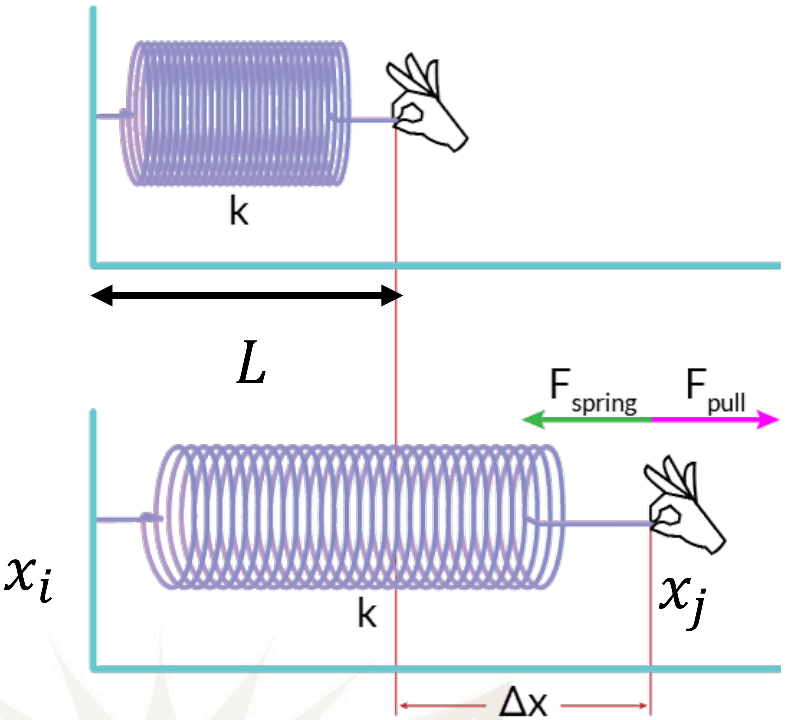
# Springs



- Other forces?
- Infinite motion?
- Internal friction




$$F^{damp} = F^{dp} = -\gamma \cdot v$$

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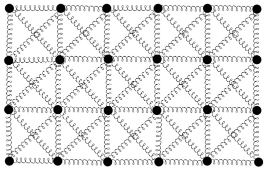
# Springs



- Plug in these forces together with other external forces:
  – Gravity
  – Wind

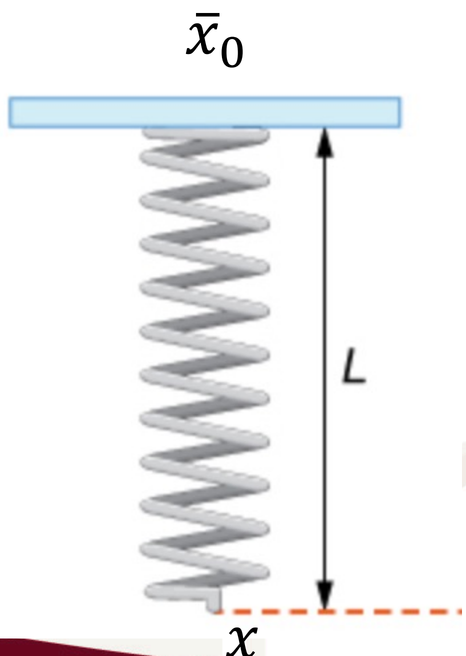
$$F^{dp} = -\gamma \cdot v$$

$$F^{int} = -k \cdot (\|x_i - x_j\| - L) \cdot \frac{x_j - x_i}{\|x_i - x_j\|}$$

- Use your favorite time integrator
  – Explicit solvers don't work so well on springs
  – Use implicit/backward Euler in your assignment

# An example

- A spring fixed at point $\bar{x}_0$, the other end-point $x$ is left mobile and has a mass of M; the spring starts in an undeformed configuration, initial velocity is 0; spring has a rest length of $L$; we have gravity as an external force



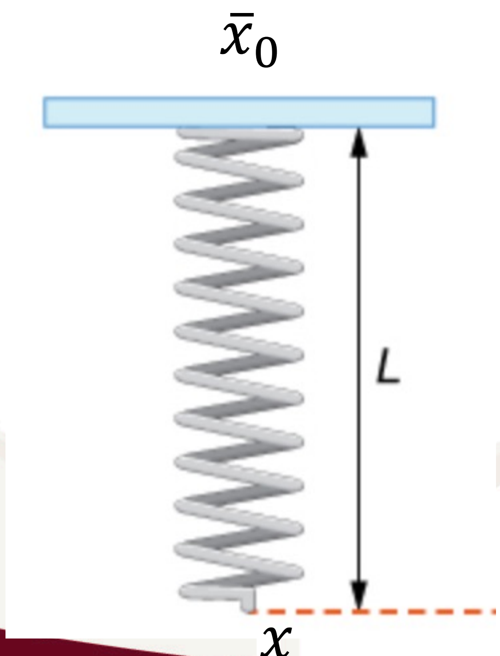

$$y_0 = \begin{pmatrix} \bar{x}_0 + \begin{pmatrix} 0 \\ -L \end{pmatrix} \\ 0 \end{pmatrix},$$

$$F^{dp} = -\gamma \cdot v$$

$$F^{int} = -k \cdot (\|x - \bar{x}_0\| - L) \cdot \frac{x - \bar{x}_0}{\|x - \bar{x}_0\|}$$

# An example

- Implicit Euler: $y_{k+1} = y_k + f(t_{k+1}, y_{k+1})\Delta t$ (eq 12)
- $f(\mathrm{t}, y) = f\left(t, \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}\right) = \begin{pmatrix} v(t) \\ M^{-1}F(t) \end{pmatrix}$ (eq 9)

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# An example

$$y_{k+1} = y_k + f(t_{k+1}, y_{k+1})\Delta t$$

$$f(\mathrm{t}, y) = f\left(t, \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}\right) = \begin{pmatrix} v(t) \\ M^{-1}F(t) \end{pmatrix}$$

Implicit Euler Scheme

$$F_{damp} = F_{dp} = -\gamma \cdot v$$

$$F^{int} = -k \cdot (\|x - \bar{x}_0\| - L) \cdot \frac{x - \bar{x}_0}{\|x - \bar{x}_0\|}$$

$G = -gM$ (gravity force)

Forces

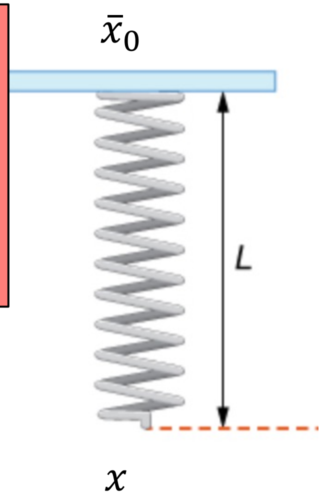
$$y_0 = \begin{pmatrix} \bar{x}_0 + \begin{pmatrix} 0 \\ -L \end{pmatrix} \\ 0 \end{pmatrix}$$

Initial values

$$\begin{pmatrix} x_{k+1} \\ v_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ v_k \end{pmatrix} + \begin{pmatrix} v_{k+1} \\ M^{-1}(F_{k+1}^{dp} + F_{k+1}^{int} - gM) \end{pmatrix} \Delta t =$$

$$= \begin{pmatrix} x_k \\ v_k \end{pmatrix} + \begin{pmatrix} v_{k+1} \\ M^{-1}\left(-\gamma \cdot v_{k+1} + -k \cdot (\|x_{k+1} - \bar{x}_0\| - L) \cdot \frac{x_{k+1} - \bar{x}_0}{\|x_{k+1} - \bar{x}_0\|} - gM\right) \end{pmatrix} \Delta t$$

Equation to solve




Non-linear equation

Newton-Raphson method (among others)

# Practical Aspects

- Multiple connected Springs (i.e. spring system)
- Some end-points are fixed, some can move (i.e. variable)
- $x, v \in \mathbb{R}^{N\times k}$ where $k$ is the dimension (i.e. 2 or 3) and $N$ is the number of variables
- If a spring is between a variable and a fixed end-point, force are contributing to the motion of the variable end-point
- If a spring is between 2 variable –end-points, internal force force is split between the 2 end-points
- This logic makes the differentiation quite difficult

# Springs

- Pros:
  - Simple to understand and implement
  - Fast (compared with FEM)
- Limitations:
  - Internal forces in one direction → difficult to model complex objects
  - Can model cloth but with some hacks

# Finite Element Methods (FEM)

- Generalized springs really → springs in all available directions
- Widely used in Physics simulation