

# Predicate logic

Dr. Constantinos Constantinides, P.Eng.

Department of Computer Science and Software Engineering  
Concordia University Montreal, Canada

December 31, 2021

# Introduction

- ▶ Much like propositional logic, predicate logic uses symbols to represent knowledge.
- ▶ Propositional logic does not allow us to say things like “*All objects of this class have this particular property*”, or “*Some objects of this class have this property*”, etc.

## Introduction /cont.

- ▶ Let  $P(x)$  denote the statement "*x is an odd number.*"
- ▶ The statement cannot be given a truth value since the value for  $x$  is not yet specified. We say that variable  $x$  is a *free variable* and it constitutes the *subject* of the statement, whereas "*is an odd number*" refers to a property of the subject and is called the *predicate*.
- ▶  $P(x)$  is called a *propositional function*, as each choice of  $x$  produces a proposition.

## Introduction /cont.

- ▶ **Definition:** A *propositional function* is a statement containing one or more free variables. The statement becomes a proposition once values are assigned to all its free variables.
- ▶ In  $P(x)$  : “ $x$  is an odd number.”,  $P(5)$  becomes a proposition by setting  $x = 5$  and its value is true, whereas the proposition  $P(6)$  is false.
- ▶ Sometimes, the entire  $P(x)$  is referred to as a predicate.

## Using predicates to define sets: Untyped set comprehension

- ▶  $\{x \mid x < 5\}$  is the set of all values of  $x$  for which  $x < 5$ .
- ▶ This notation is an example of *untyped set comprehension*.
- ▶ We read “the set of values  $x$  such that  $x < 5$ .”

## Using predicates to define sets: Untyped set comprehension /cont.

- ▶ A set comprehension can be modeled by a decision procedure where for an input variable it gives a result of true or false, e.g.  $2 \in \{x \mid x < 5\}$  is true.

## Using predicates to define sets: Typed set comprehension

- ▶ A *typed set comprehension* is of the form  $\{x \in X \mid p(x)\}$  where  $p(x)$  is a predicate with free variable  $x$ .
- ▶ For example,  $\{x \in \mathbb{N} \mid x < 5\}$  represents the set of natural numbers less than five.
- ▶ Is  $2.5 \in \{x \in \mathbb{N} \mid x < 5\}$  true? No, since  $2.5 \notin \mathbb{N}$ .

## Set replacement

- ▶ An extension to set comprehension is to follow the declaration and predicate by a formula, e.g.  
 $\{x \in \mathbb{N} \mid x < 5 \bullet x^2\}$  refers to a set of numbers which are squares of numbers less than 5, i.e.  $\{0, 1, 4, 9, 16\}$ .
- ▶ Each element in the set  $\{x \in \mathbb{N} \mid x < 5\}$  is replaced by its square so that a new set is formed.



# Quantifiers

- ▶ As its name suggests, a *quantifier* is an operator that states that a predicate is true for a given quantity of objects.
- ▶ The expression “for all”, denoted by the symbol  $\forall$ , is called the *universal quantifier*.
- ▶ The expression “there exists”, denoted by the symbol  $\exists$  is called the *existential quantifier*.

# Universal quantification

- ▶ **Definition:** The *universal quantification* of  $P(x)$  is the statement “ $P(x)$  is true for all values of  $x$ .” Symbolically this is expressed as  $\forall x P(x)$ .
- ▶ The statement  $\forall x P(x)$  is true when  $P(x)$  is true for all  $x$ . It is false when there is at least one  $x$  for which  $P(x)$  is false.
- ▶ Its negation becomes  $\neg \forall x P(x)$  and it is true when there is an  $x$  for which  $P(x)$  is false, or  $\exists x \neg P(x)$ .

## An initial example of a universal quantification

- ▶ In the domain of animals, how do we express “All cats are mammals”?
- ▶ We can rephrase the statement as “If  $x$  is a cat, then  $x$  is a mammal”:

$$\forall x (cat(x) \rightarrow mammal(x))$$

## Example: Universal quantification

- ▶ How do we express “Only dogs bark”?
- ▶ We can rephrase the statement as “It barks only if it is a dog”, or “If it barks, then it is a dog”:

$$\forall x(barks(x) \rightarrow dog(x))$$

# Existential quantification

- ▶ **Definition:** The *existential quantification* of  $P(x)$  is the statement “There exists an element  $x$  such that  $P(x)$  is true.” Symbolically this is expressed as  $\exists x P(x)$ .
- ▶ The statement  $\exists x P(x)$  is true when there is at least one  $x$  for which  $P(x)$  is true.
- ▶ It is false when  $P(x)$  is false for all  $x$ . Its negation becomes  $\neg \exists x P(x)$  and it is true when for every  $x$ ,  $P(x)$  is false, or  $\forall x \neg P(x)$ .
- ▶ In existential quantification we can make a distinction between “there exists at least one” (as defined above) and “there exists exactly one”, symbolically expressed as  $\exists!$ .

## Example: Combining quantifiers

- ▶ Consider the following statement: “For every integer number  $x$ , there is a successor integer number  $y$ .”
- ▶ We can denote the successor property with the predicate  $P(x, y) : y = x + 1$  and write the statement symbolically as

$$\forall x \exists y P(x, y)$$

## Bound variables

- ▶ When we assign a value to a free variable, then the variable becomes a *bound variable*.
- ▶ Another way for a free variable to become bound is when we can apply a quantifier to it, such as “*There exists an  $x$  such that  $x > 0$ .*”
- ▶ We say that the variable  $x$  is bound by the quantifier “there exists.”

## Precedence rules in the presence of quantifiers

- ▶ Both universal and existential quantifiers have a higher precedence than the rest of the connectives, but they have a lower precedence than the negation operator.
- ▶ Consider the predicates  $p(x) = \text{"x is living."}$  and  $q(x) = \text{"x is dead"}$ :
  - 1  $\forall x(p(x) \vee q(x))$  is interpreted as everything is either living or dead.
  - 2  $\forall x p(x) \vee q(x)$  is interpreted as everything is living or  $x$  is dead.



# Negation

- ▶ Consider the statement *“All birds are white”* or

$$\forall x P(x)$$

where the predicate  $P(x)$  stands for *“bird is white.”*

- ▶ To negate the statement we can say *“It is not true (or: not the case) that all birds are white”* or *“Not all birds are white.”*

## Negation /cont.

- ▶ Note that it would be incorrect to say “*All birds are not white*” as this would claim that there are no white birds which is clearly false.
- ▶ The negated statement can be written symbolically as

$$\neg[\forall xP(x)]$$

- ▶ An alternative way to express the negated statement is to say “*There is at least one bird that is not white*”, written symbolically as

$$\exists x[\neg P(x)]$$

## Negation /cont.

- ▶ Note that the process of negating the statement corresponds to negating the predicate and changing the quantifier. In other words

$$\neg[\forall x P(x)] \equiv \exists x[\neg P(x)]$$

- ▶ Similarly, to negate a predicate with the existential quantifier, we can write

$$\neg[\exists x P(x)] \equiv \forall x[\neg P(x)]$$

## Negation - Summary

| Negated statement      | Equivalent statement   | When is it true?                           | When is it false?                         |
|------------------------|------------------------|--|---|
| $\neg[\forall x P(x)]$ | $\exists x[\neg P(x)]$ | There is an $x$ for which $P(x)$ is false. | $P(x)$ is true for every $x$ .            |
| $\neg[\exists x P(x)]$ | $\forall x[\neg P(x)]$ | For every $x$ , $P(x)$ is false.           | There is an $x$ for which $P(x)$ is true. |

## Nested quantifiers

- ▶ A quantifier is called *nested* if it occurs within the scope of another. For quantifiers of the same type, the order does not matter.
- ▶ However, the order in which quantifiers of different types are placed is important.
- ▶  $\forall x \exists y P(x, y)$  reads “**For every  $x$  there is a  $y$  for which  $P(x, y)$  is true.**”
- ▶ In other words, no matter which  $x$  we choose, there must be a value of  $y$  (possibly depending on the choice of  $x$ ) for which  $P(x, y)$  is true.
- ▶  $\exists y \forall x P(x, y)$  reads “**There is an  $y$  that makes  $P(x, y)$  true for every  $x$ .**”

## Example 1: Nested quantifiers

- ▶ Consider the predicate  $loves(x, y)$  denoting “ $x$  loves  $y$ ”. We can express the following predicates using nested quantifiers:
- ▶  $\forall x \exists y loves(x, y)$  reads “Everyone loves someone”, i.e. no matter which  $x$  we choose, there must always be some  $y$  that makes  $loves(x, y)$  true.
- ▶  $\exists y \forall x loves(x, y)$  reads “There is someone who is loved by everybody”, i.e. there is some particular  $y$  for which  $loves(x, y)$  is true, regardless of the choice of  $x$ .

## Example 1: Nested quantifiers /cont.

- ▶ If  $\exists y \forall x P(x, y)$  is true, then  $\forall x \exists y P(x, y)$  is also true, e.g. if “There is someone who is loved by everybody”, then we can safely conclude that “Everyone loves someone.”
- ▶ However, if  $\forall x \exists y P(x, y)$  is true, e.g. “Everyone loves someone”, then it is not necessary that  $\exists y \forall x \text{ loves}(x, y)$  is true, e.g. it is not necessary that “There is someone who is loved by everybody.”

## Example 1: Nested quantifiers

- ▶ We can also translate the following sentences into formal statements:
- ▶ “Someone loves someone.”  $\exists x \exists y \text{ loves}(x, y)$ , or  $\exists y \exists x \text{ loves}(x, y)$
- ▶ “Everyone loves someone.”  $\forall x \exists y \text{ loves}(x, y)$
- ▶ “Everyone is loved by someone.”  $\forall y \exists x \text{ loves}(x, y)$
- ▶ “There is someone who loves everyone.”  $\exists x \forall y \text{ loves}(x, y)$
- ▶ “There is someone who is loved by everybody.”  $\exists y \forall x \text{ loves}(x, y)$
- ▶ “Everyone loves everyone.”  $\forall x \forall y \text{ loves}(x, y)$ , or  $\forall y \forall x \text{ loves}(x, y)$



## Example 2: Nested quantifiers

- ▶ Consider the predicate  $B(p, r)$  denoting the predicate “*Person  $p$  has booked room  $r$ .*” and the sentence “*No room is booked by more than one person.*”
- ▶ If no room is booked by more than one person, then the predicates  $B(p, r)$  and  $B(q, r)$  cannot both be true unless  $p$  and  $q$  denote the same person.
- ▶ Symbolically this can be expressed as

$$\forall p \forall q \forall r [(B(p, r) \wedge B(q, r)) \rightarrow (p = q)]$$

## Example 3: Nested quantifiers

- Consider the following: “Every airline  $x$  flies to exactly one city  $y$ .” This can be formulated as:

$$\forall x \exists! y (airline(x) \wedge city(y) \wedge flies(x, y))$$

## Example 4: Nested quantifiers

- ▶ Let  $sendMessage(x, y)$  be the statement “ $x$  has sent a message to  $y$ ” where the domain is all students in class. Note that  $x$  and  $y$  can be the same person.
- ▶  $\exists x \exists y P(x, y)$ : There is some student who has sent a message to some student.
- ▶  $\forall x \forall y P(x, y)$ : Every student in the class has sent a message to every student in the class.

## Example 4: Nested quantifiers /cont.

- ▶  $\exists y \forall x P(x, y)$ : Recall that this reads "**There in a  $y$  that makes  $P(x, y)$  true for every  $x$ .**" There is a student in class who has been sent a message by every student in class.
- ▶  $\exists x \forall y P(x, y)$ : This is similar to the above, only the roles have been changed: There is some student who has sent a message to every student in the class.

## Example 4: Nested quantifiers /cont.

- ▶  $\forall x \exists y P(x, y)$ : Recall that this reads “**For every**  $x$  **there is a**  $y$  **for which**  $P(x, y)$  **is true.**” Every student in class has sent a message to some (at least one) student in class.
- ▶  $\forall y \exists x P(x, y)$ : This is similar to the above, only the roles have been changed: Every student in class has been sent a message from some (at least one) student in class.

## Example 5: Nested quantifiers

- ▶ Consider the statement “Everyone has exactly one movie that he or she likes.”
- ▶ Let us express the statement in formal logic by introducing the predicate  $likes(x, y)$  to represent the statement “ $x$  likes movie  $y$ .”
- ▶ Let us start with  $\forall x \exists y likes(x, y)$  which reads “**For every  $x$  there is a  $y$  for which  $likes(x, y)$  is true**”, or “Everyone likes some movie.”
- ▶ This does not fully capture the original statement because it does not exclude the possibility that there may be more than one movie that a person likes.

## Example 5: Nested quantifiers /cont.

- ▶ To complete the statement we need to add that no other movie (i.e. no movie which is not  $y$ ) is liked by  $x$ , or  $\forall z((z \neq y) \rightarrow \neg \text{likes}(x, z))$ .
- ▶ Putting everything together we have

$$\forall x \exists y (\text{likes}(x, y) \wedge \forall z ((z \neq y) \rightarrow \neg \text{likes}(x, z)))$$

## Negating nested quantifiers

The negation of nested quantifiers follows the same rules as with single quantifiers. The statement

$$\forall x \exists y P(x, y)$$

reads “For every  $x$ , there is a  $y$  for which  $P(x, y)$  is true.” The negated statement would read “There is an  $x$  such that  $P(x, y)$  is false for every  $y$ ”, or

$$\exists x \forall y \neg P(x, y)$$

The statement

$$\exists x \forall y P(x, y)$$

reads “There is an  $x$  for which  $P(x, y)$  is true for every  $y$ .” The negated statement would read “For every  $x$  there is a  $y$  for which  $P(x, y)$  is false”, or

$$\forall x \exists y \neg P(x, y)$$



## Summary of quantifications of two variables

| Statement  | When is it true   | When is it false   |
|--|---|--|
| $\forall x \forall y P(x, y)$<br>$\forall y \forall x P(x, y)$ | $P(x, y)$ is true for every pair $(x, y)$ .                 | There is a pair $(x, y)$ for which $P(x, y)$ is false.       |
| $\forall x \exists y P(x, y)$                                  | For every $x$ there is a $y$ for which $P(x, y)$ is true.   | There is an $x$ such that $P(x, y)$ is false for every $y$ . |
| $\exists x \forall y P(x, y)$                                  | There is an $x$ for which $P(x, y)$ is true for every $y$ . | For every $x$ there is a $y$ for which $P(x, y)$ is false.   |
| $\exists x \exists y P(x, y)$<br>$\exists y \exists x P(x, y)$ | There is a pair $(x, y)$ , for which $P(x, y)$ is true.     | $P(x, y)$ is false for every pair $(x, y)$ .                 |

# Equivalences

$$\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$$

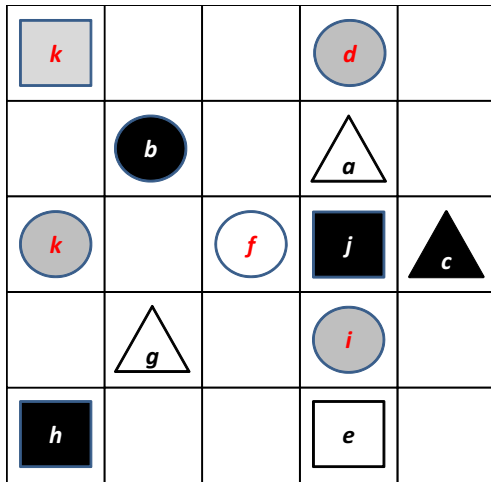
$$\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$$

$$\exists x (P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \exists x Q(x)$$

$$\exists x (P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$$

$$\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$$

## Example: Formalizing sentences from a natural language with Tarski's world

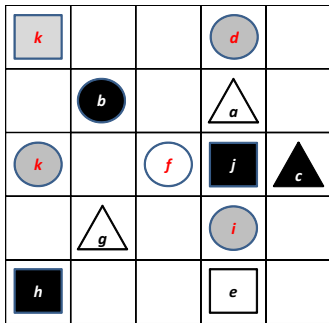


## Example: Formalizing sentences from a natural language with Tarski's world /cont.

We adopt the following predicates:

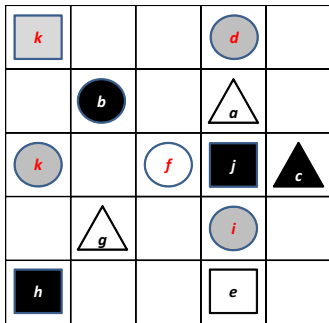
- ▶ *square* ( $x$ ) indicates that  $x$  is a square.
- ▶ *circle* ( $x$ ) indicates that  $x$  is a circle.
- ▶ *triangle* ( $x$ ) indicates that  $x$  is a triangle.
- ▶ *black* ( $x$ ) indicates that  $x$  is a black.
- ▶ *gray* ( $x$ ) indicates that  $x$  is a gray.
- ▶ *white* ( $x$ ) indicates that  $x$  is a white.
- ▶ *aboveOf* ( $x, y$ ) indicates that  $x$  is above  $y$ , perhaps in a different column.
- ▶ *rightOf* ( $x, y$ ) indicates that  $x$  is to the right of  $y$ , perhaps in a different row.
- ▶ *sameColor* ( $x, y$ ) indicates that  $x$  and  $y$  have the same color.

## Example: Formalizing sentences from a natural language with Tarski's world /cont.



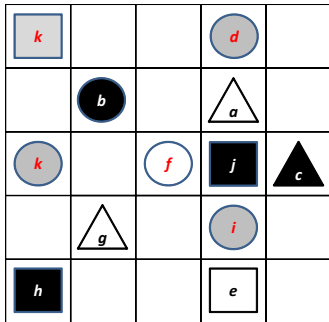
- ▶ All squares are black:  $\forall x (square(x) \rightarrow black(x))$ .  
False, since  $e$  and  $k$  are both squares and non-black. Note that to prove false one counter-example would be enough.
- ▶ Everything white is a triangle.  $\forall x (white(x) \rightarrow triangle(x))$ .  
False, since  $e$  and  $f$  are both white and not triangles.

## Example: Formalizing sentences from a natural language with Tarski's world /cont.



- There is a square that lies to the left of  $d$ .  
 $\exists x (square(x) \wedge rightOf(d, x))$ .  
True:  $h$  and  $k$  both lie to the left of  $d$ .
- There is a black circle.  $\exists x (circle(x) \wedge black(x))$ . True:  $b$ .

## Example: Formalizing sentences from a natural language with Tarski's world /cont.



- ▶ All circles are above  $g$ .  $\forall x (circle(x) \rightarrow aboveOf(x, g))$ .  
False, since  $i$  is a circle and it is not above  $g$ .
- ▶ For every square, there exists a circle of the same color.

$$\forall x (square(x) \rightarrow \exists y (circle(y) \wedge sameColour(x, y)))$$

This is true.

# Categorical propositions

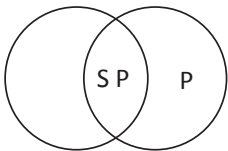
- ▶ A *categorical proposition* (or *categorical statement*) is a proposition that asserts or denies that all or some of the members of one category (the *subject term*) are included in another (the *predicate term*).
- ▶ For example, the categorical proposition “*All birds are white*” asserts that all members of category *birds* are included in category *being white*.
- ▶ The category of the subject (*birds*) and refers to what the proposition is about, whereas the category of the predicate (*being white*) refers to what the proposition affirms (or denies) about the subject.



# Standard forms of categorical propositions

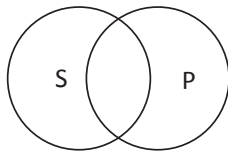
- ▶ Aristotle identified four primary distinct types of categorical proposition and gave them standard forms (referred to as A, E, I, and O).
- ▶ For the subject category S, and the predicate category P, the four standard forms are:
  - ▶ All S are P. (A form)
  - ▶ No S are P. (E form)
  - ▶ Some S are P. (I form)
  - ▶ Some S are not P. (O form)

## Visualizing categorical propositions



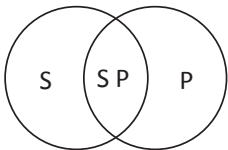
All S are P

A



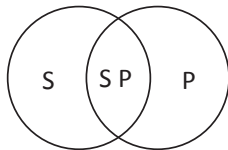
No S are P

E



Some S are P

I



Some S are not P

O

## Properties of categorical propositions: Quantity

- ▶ *Quantity* refers to the number of members of the subject class that are used in the proposition.
- ▶ If the proposition refers to all members of the subject class, it is *universal*.
- ▶ If the proposition does not employ all members of the subject class, it is *particular*.
- ▶ Categorical propositions A and E have a *universal quantity* as they make a claim about all members of the subject class, whereas propositions I and O have a *particular quantity* as they make a claim about some (at least one) member of the subject class.

## Properties of categorical propositions: Quality

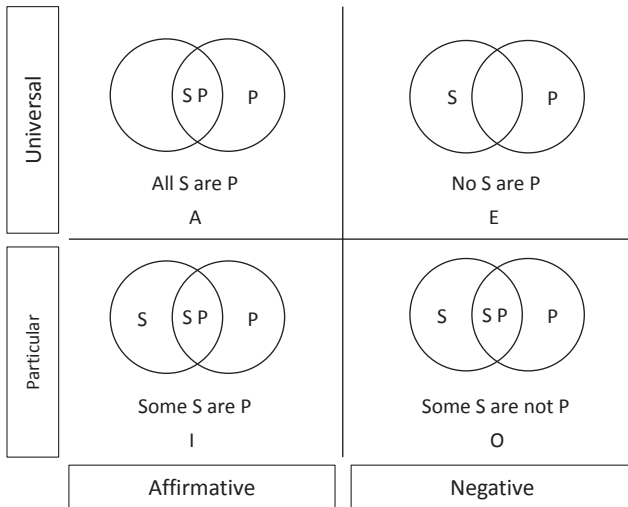
- ▶ *Quality* describes whether the proposition affirms or denies the inclusion of a subject within the class of the predicate.
- ▶ The two possible qualities are called *affirmative* and *negative*.
- ▶ Propositions A and I have an *affirmative quality* as they affirm class membership, whereas propositions E and O have *negative quality* as they deny class membership.

## Summary of categorical propositions

Letting S and P stand for subject and predicate respectively, we have four forms of categorical propositions as shown below:

| NAME | FORM             | TITLE                  |                        |
|------|------------------|------------------------|------------------------|
| A    | All S are P      | Universal affirmative  | $\forall xP(x)$        |
| E    | No S are P       | Universal negative     | $\forall x[\neg P(x)]$ |
| I    | Some S are P     | Particular affirmative | $\exists xP(x)$        |
| O    | Some S are not P | Particular negative    | $\exists x[\neg P(x)]$ |

# Visualizing categorical propositions with their properties



# Formalizing sentences from a natural language

- ▶ In the episode “Fidelity” of House M. D., Dr. Gregory House says: *“I don’t ask why patients lie, I just assume they all do.”*
- ▶ Consider the following list of categorical propositions where  $P(x)$  denotes the subject “ $x$  is a patient” and  $Q(x)$  denotes the predicate “ $x$  lies”:
  - ▶ “Every patient lies.”  $\forall x (P(x) \rightarrow Q(x))$  (All P are Q: Type A)
  - ▶ “No patient lies.”  $\forall x (P(x) \rightarrow \neg Q(x))$  (No P are Q: Type E)
  - ▶ “Some patient lies.”  $\exists x (P(x) \wedge Q(x))$  (Some P are Q: Type I)
  - ▶ “Not all patients lie.”  $\exists x (P(x) \wedge \neg Q(x))$  (Some P are not Q: Type O)

## Formalizing sentences from a natural language /cont.

- ▶ Consider the statement “**All** patients **are** honest.” What type is it? Is it A?
- ▶ No. Given the subject and predicate, the statement is of type E: “No patient lies.”
- ▶ Similarly, the statement “**No** patient **is** honest” is of type A because, given the subject and predicate it is interpreted as “All patients lie.”
- ▶ To determine the proper type, a statement should be interpreted w.r.t. the subject and predicate.



## Formalizing categorical propositions

- ▶ “Some patient lies.”  $\exists x (P(x) \wedge Q(x))$
- ▶ “No patient lies.”  $\forall x (P(x) \rightarrow \neg Q(x))$
- ▶ “All patients lie.”  $\forall x (P(x) \rightarrow Q(x))$
- ▶ “Not all patients lie.”  $\exists x (P(x) \wedge \neg Q(x))$
- ▶ “Every patient lies.”  $\forall x (P(x) \rightarrow Q(x))$
- ▶ “There is an honest patient.”  $\exists x (P(x) \wedge \neg Q(x))$
- ▶ “No patient is honest.”  $\forall x (P(x) \rightarrow Q(x))$
- ▶ “All patients are honest.”  $\forall x (P(x) \rightarrow \neg Q(x))$

## Formalizing sentences from a natural language /cont.

- ▶ We notice that each formalization satisfies one of the following two properties:
- ▶ The universal quantifier  $\forall x$  quantifies an implication.
- ▶ The existential quantifier  $\exists x$  quantifies a conjunction.

## Formalizing sentences from a natural language /cont.

- ▶ Consider the statement “Some patient lies” which was formalized as  $\exists x (P(x) \wedge Q(x))$ .
- ▶ Can we argue that the sentence can also be formalized as  $\exists x (P(x) \rightarrow Q(x))$ ?
- ▶ This would be equivalent to  $\forall x P(x) \rightarrow \exists x Q(x)$  which means “If everyone is a patient then someone lies” which does not convey the meaning of the original sentence.

# Contradictory categorical propositions

- ▶ **Definition:** A pair of categorical propositions are called *contradictories* if they have opposite truth values: they cannot both be true and cannot both be false.

## Contradictory categorical propositions: Example 1

- ▶ Consider the statement “Every person owns a house.” Its contradictory statement is “Not every person owns a house.”
- ▶ Given a subject category “x is a person” and predicate category “x owns a house”, then “Every person owns a house” is of type A.
- ▶ Its contradictory statement “Not every person owns a house” can be rephrased as “Some people do not own a house” which is of type O.
- ▶ Universal affirmations and particular denials are contradictory statements.

## Contradictory categorical propositions: Example 2

- ▶ Consider the statement “No people suffer from hunger.” Its contradictory statement is “Some people suffer from hunger.”
- ▶ Given a subject category “x is a person” and predicate category “x suffers from hunger”, then “No people suffer from hunger.” is of type E.
- ▶ Its contradictory statement “Some people suffer from hunger” is of type I.
- ▶ Universal denials and particular affirmations are contradictory statements.

## Contradictory categorical propositions: Example 3

- ▶ let  $P(x)$  to denote “ $x$  is a patient” and  $Q(x)$  to denote “ $x$  lies”
- ▶ “All patients lie” ( $\forall x (P(x) \rightarrow Q(x))$ ), and “There is an honest patient” (which can be interpreted as There is some patient who does not lie) ( $\exists x (P(x) \wedge \neg Q(x))$ ) is a pair of contradictory propositions.
- ▶ “No patient lies” ( $\forall x (P(x) \rightarrow \neg Q(x))$ ) and “Some patient lies” ( $\exists x (P(x) \wedge Q(x))$ ) is a pair of contradictory categorical propositions.

# Contrary categorical propositions

- ▶ **Definition:** A pair of categorical propositions are called *contraries* if they cannot both be true, but could both be false.



## Contrary categorical propositions: Example 1

- ▶ Consider the statement “All people are rich.” Its contrary statement is “No people are rich.”
- ▶ Given a subject category “x is a person” and predicate category “x is rich”, then “All people are rich” is of type A and “No people are rich” is of type E.
- ▶ A pair of universal statements are contraries.

## Contrary categorical propositions: Example 2

- ▶ “All patients lie” ( $\forall x (P(x) \rightarrow Q(x))$ ) and “No patient lies” ( $\forall x (P(x) \rightarrow \neg Q(x))$ ) are contrary categorical propositions.

# Subcontrary categorical propositions

- ▶ **Definition:** A pair of categorical propositions are called *subcontraries* if they cannot both be false but could both be true.

## Subcontrary categorical propositions: Example

- ▶ “Some patient lies” ( $\exists x (P(x) \wedge Q(x))$ ) and “There is an honest patient” (or There is some patient who does not lie) ( $\exists x (P(x) \wedge \neg Q(x))$ ) are subcontrary categorical propositions.

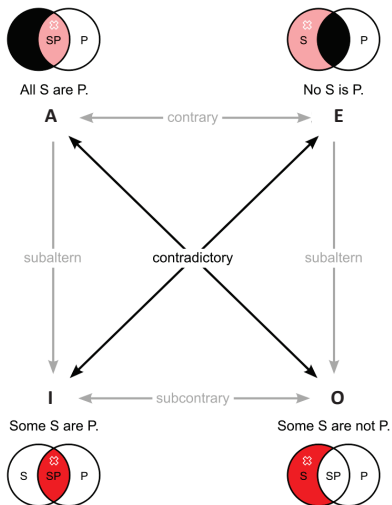
# Subaltern and superaltern categorical propositions

- ▶ **Definition:** Two categorical propositions are called *superaltern* and *subaltern* if the subaltern must be true if its superaltern is true (and subsequently the superaltern must be false if the subaltern is false).
- ▶ I is a subaltern of A (Some S are P, if All S are P).
- ▶ O is a subaltern of E (Some S are not P, if No S are P).

## Subaltern and superaltern categorical propositions: Example

- ▶ “Some patient lies” ( $\exists x (P(x) \wedge Q(x))$ ) is a subaltern of “All patients lie” ( $\forall x (P(x) \rightarrow Q(x))$ ).
- ▶ “There is an honest patient” ( $\exists x (P(x) \wedge \neg Q(x))$ ) is a subaltern of “No patient lies” ( $\forall x (P(x) \rightarrow \neg Q(x))$ ).

# Summary: The square of opposition



# Categorical syllogisms

- ▶ Recall from propositional logic that a syllogism is an inference in which one proposition necessarily follows from two others.
- ▶ A *categorical syllogism* consists of three parts (major premise, minor premise, and conclusion) all of which are *categorical propositions*.



## Categorical syllogisms /cont.

Some valid forms of categorical syllogisms are shown below:

*All M are P. All S are M. Therefore, All S are P.*

*All P are M. Some S are not M. Therefore, Some S are not P.*

*Some M are not P. All M are S. Therefore, Some S are not P.*

*All P are M. No M are S. Therefore, No S are P.*

*All M are P. Some S are M. Therefore, Some S are P.*

*No M are P. Some S are M. Therefore, Some S are not P.*

# Universal conditional statements

- ▶ A universal conditional statement has the form  $\forall x$  if  $P(x)$  then  $Q(x)$ , or  $\forall x(P(x) \rightarrow Q(x))$ .
- ▶ It can be proven that its negation has the form  $\exists x(P(x) \wedge \neg Q(x))$ .
- ▶ For the universal conditional statement above,
- ▶ its contrapositive statement is

$$\forall x(\neg Q(x) \rightarrow \neg P(x))$$

- ▶ its converse statement is

$$\forall x(Q(x) \rightarrow P(x))$$

- ▶ and its inverse is

$$\forall x(\neg P(x) \rightarrow \neg Q(x))$$