Simulation Techniques in Finance
7% CVX, UNH, XOM Barrier Reverse Convertible

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Abstract

In this project, we attempt to use Stochastic Models, specifically Geometric Brownian Motion, to price options for the above mentioned Barrier Reverse Convertible product. We aim to improve our predictions for the prices of the product by using various variance reduction methods. The Greeks for the product are also calculated.

1 Product Description

1.1 Barrier Reverse Convertible

A barrier reverse convertible is a path dependent option whose payoff is affected by the event of the underlying asset's price reaching a certain threshold (Barrier B) at any point during the product's lifespan, and the final price of the Underlying asset (S_T) at maturity (T). This barrier is set by the Option issuer to be a percentage of the price of the Underlying (Strike Price K) on the start date of the option (Initial Fixing Date). This makes the product a path dependent option and thus requires us to simulate the entire path of the option in order for us to price it properly. Note that the Underlying might not be at the Strike price at the Initial Fixing date but that price is still used as a reference for calculating the payoff.

Our product has a lifespan of 18 months. The initial fixing date is 25 May 2021, first payment date is 2 June 2021, and final fixing is on 29 November 2022. The barrier is set at 60% of the price at initial fixing.

1.2 Payoff Function

Our product is priced on the performance of 3 underlying assets, Chevron Corporation ($CVX), United Health Group Inc. ($UNH) and Exxon Mobil Corp. ($XOM). The payoff of the product depends on any one of the assets touching the Barrier and being below the strike price, in which case the worst performing asset will be converted into shares of that asset at a predetermined conversion ratio. If that does not happen, then the full Note denomination ($1000 USD) will be redeemed at the end of the Final Fixing Date

Note that the actual price of the underlying share is not taken into consideration in the loss calculation but rather just the percentage of the price decrease. Therefore, all calculations will be done in terms of percentage rather than absolute underlying prices for ease of simulation and further calculation.

There is also a coupon payment (C) that is payed out at 7% p.a., or 1.75% per quarter. Thus a holder of this option is entitled to a maximum of 10.5% in coupon payments if they hold onto the option for the entire duration of the product.

The payoff for our product is defined for two different scenarios:

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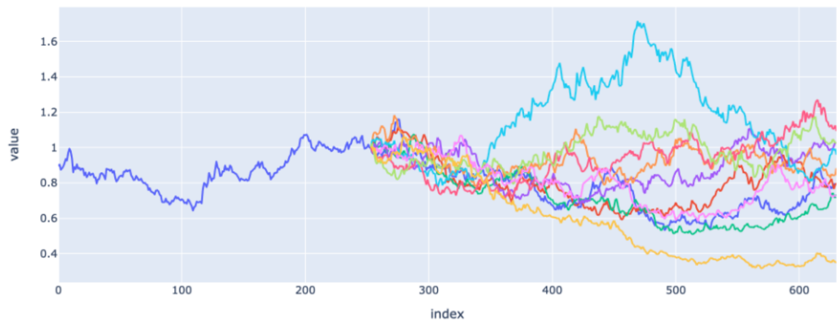

Figure 1: 10 price path simulations using GBM.

1. No barrier event is triggered, or the price of all underlying assets close at/above initial level: investors receive back 100% of the denomination (1000 USD) in cash, and coupon payment 1.75% per quarter.
2. A barrier event has been triggered and at least one underlying asset closes below its initial level: invested capital is converted into shares of the worst performing underlying asset, and investors receive coupon payment of 1.75% per quarter.

Mathematically,

$$\chi(\mathbf{S}) = 1000 \times \begin{Bmatrix} min(S_i), & \mathbb{1}_{\{S_{i_\tau} \le B\}} \wedge \mathbb{1}_{\{S_{i_T} < K\}} \\ 1, & \text{otherwise} \end{Bmatrix} + C(\tau) \tag{1}$$

where $\tau \in \{0 \dots T\}, \mathbf{S} = S_i, i = 1, 2, 3,$ and $C(\tau)$ denotes the coupon payment received when purchasing the product at time $t = T - \tau$

2 Modeling Underlying Assets

2.1 Geometric Brownian Motion

We use Multidimensional Geometric Brownian Motion as the simulation technique for our project. Geometric Brownian Motion is a stochastic process which satisfies the following stochastic differential equation under the physical measure $\mathbb{P}$:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{2}$$

where $W(t)$ is a Wiener process or Brownian Motion and the constants μ and σ are the percentage drift and percentage volatility respectively. In addition to this, we perform simulations in a risk-neutral world, under the assumption of the Martingale property. As such, mean log-returns μ in equation 2 is replaced with risk-free interest rate r. Since our product's payoff is path dependent, we use the equation 3 to simulate a price path of the ith asset at each time step t_j.

$$S_j^{(i)} = S_{j-1}^{(i)} \exp(r_{t_j} \mathbb{1} \Delta t - \frac{\text{diag}(\Sigma(t_{j-1}, t_j))\Delta t}{2} + A(t_{j-1}, t_j) Z_j^{(i)}) \tag{3}$$

where $\mathbb{1} = (1, \dots, 1)^\top \in \mathbb{R}^p$, $j = 1, \dots, m$, $i = 1, \dots, p$, $[Z_1^{(i)}, \dots, Z_m^{(i)}] \sim_{iid} N_p(0, I_p)$, and $\Sigma(t_{j-1}, t_j)$ is the covariance matrix, $A(t_{j-1} t_j) A(t_{j-1}, t_j)^\top = \Sigma(t_{j-1}, t_j)$. Figure 1 shows 10 simulated price paths for CVX.

We use a rolling window of the last year's (252 days) historical asset prices to calculate Σ at each time step t_j, to generate n simulated price paths. The payoff of each price path is calculated using equation 1.

Figure 2: 10 price path simulations with antithetic variates.

2.2 Antithetic Variates

Variance reduction is an indispensable tool when it comes to financial simulations. For this project, we implemented antithetic variates to reduce the variance of pricing paths. For each price path $S^{(i)}$ generated using $Z^{(i)}$ in the Monte-Carlo approach, its antithetic variate price path is defined as follows:

$$\tilde{S}_j^{(i)} = S_{j-1}^{(i)} \text{exp}(r_{t_j} \mathbb{1} \Delta t - \frac{\text{diag}(\Sigma(t_{j-1}, t_j))\Delta t}{2} - A(t_{j-1}, t_j) Z_j^{(i)}) \tag{4}$$

Figure 2 shows 10 simulated price paths generated using antithetic variates. Because of the negatively correlated movement of antithetic pairs, the overall variance of expected prices is reduced.

2.3 Empirical Martingale Correction (EMC)

We used the equation as used in the original research paper (Duan and Simonato [1995]), which seeks to correct for discretization errors in the simulation of the path of the asset. It works by scaling the t_jth price in relation to all the other $m-1$ paths and thus ensures that the newly generated paths follow the martingale property. This also has the benefit of reducing the variance of the simulated prices. The formula used is as below:

$$\begin{aligned} S^{*(i)}(t_0) &= S^{(i)}(t_0) = S^{(i)}(0) \\ Z^{(i)}(t_j) &= \frac{S^{*(i)}(t_{j-1}) S^{(i)}(t_j)}{S^{(i)}(t_{j-1})} \\ Z^{(0)}(t_j) &= \frac{e^{-rt_j}}{n} \sum_{i=1}^{n} Z^{(i)}(t_j) \\ S^{*(i)}(t_j) &= \frac{S(0) Z^{(i)}(t_j)}{Z^{(0)}(t_j)} \end{aligned} \tag{5}$$

Figure 4 shows the variance of price simulations generated with Monte Carlo, antithetic variates, and Empirical Martingale Correction at each time step. Note the drastic reduction in variance achieved by both antithetic variates and EMC, as compared to standard Monte Carlo.

3 Product Pricing

The product's value is defined as the present value of its expected payoff. For n simulated asset paths $\{S^{(i)}\}_{i=1}^{n}$ under the risk-neutral measure $\mathbb{Q}$, option price is calculated using the payoff as defined in equation 1:

Figure 3: 10 Simulated price paths with Empirical Martingale Correction

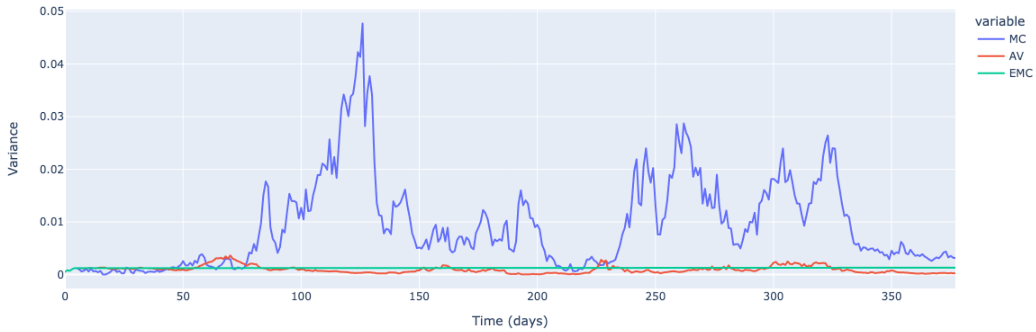

Figure 4: Comparison of variance reduction techniques for a sample simulated path.

$$\hat{f}(S,t) = \frac{1}{n}\sum_{i=1}^{n} e^{-r(T-t)}\chi(S^{(i)}) \tag{6}$$

Expected payoff is calculated at each time step t_j, using one year rolling window of historical asset prices to produce estimated option prices over time.

4 Backtesting

Figure 5 shows results of pricing our product in with 300 simulated price paths and various methods of variance reduction. Firstly, it can be seen that using antithetic variates results in a notable reduction of variance, as the green line has a lower fluctuation than standard Monte-Carlo as shown by the red line. In addition to this, it clear to see that EMC has produced simulations that are much closer to the actual option price, as compared to the other two methods. This is consistent with what we expected from the results, as Monte-Carlo methods do not follow the Martingale property exactly. Because payoff is calculated under the risk-neutral measure, this results in slightly inaccurate payoff calculations, especially for path dependent options such as this one.

It can be seen that price predictions begin to stabilize and converge to around 1000 dollars after the 100 day mark. This is roughly the discounted value of the denomination with coupon payment added. This pattern is very likely due to the performance of the underlying assets is fairly favorable (Figure 6), and as the product matures, the simulated paths become shorter and shorter. As such, the chances of a barrier event occurring become lower and lower as $t \to T$ and the expected payoff begins to approach $1000 + C(\tau)$, where $C(\tau)$ is the coupon payment received for buying the option at time $t = T - \tau$.

Figure 5: Estimated option prices derived from Monte-Carlo simulation, antithetic variates, and EMC (300 simulated paths).

Figure 6: Performance of underlying assets during lifetime of product.

The relatively large volatility in the first 100 days or so could be because the prices are calculated by using a fairly low number of simulations. Because the payoff of our product is path dependent, generating simulations of n paths has a complexity of $\mathcal{O}(n^2)$, which is very computationally intensive. As such, we were unable to perform these simulations at scale for this project. It stands to reason that setting a higher number of simulations (ie. $n = 100000$) would produce more stable predictions.

4.1 Results

Table 4.1 shows the mean squared and root mean squared error of each pricing method. While the Monte Carlo and antithetic variate simulations produced price estimates that were roughly equally accurate, EMC produced results that were far more favorable. This emphasizes the importance of correcting for the Martingale property when simulating in a risk-neutral measure. EMC proves to be a very effective tool for achieving this.

	MSE	RMSE
Monte Carlo	2654.66	51.52
Antithetic Variates	2726.16	52.21
Empirical Martingale Correction	884.91	29.74

Table 1: Backtest errors for each pricing method.

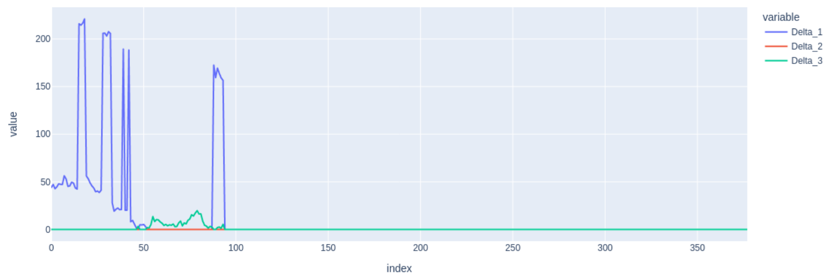

Figure 7: Delta

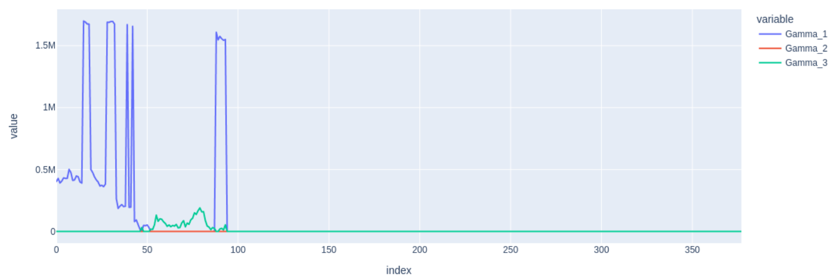

Figure 8: Gamma

4.2 Greeks

Figures 7 to 11 show the various Greek values of this product. They were calculated with the finite difference method since it cannot be directly calculated as in the Black-Scholes Model. We used a value of $h = 0.02$ as we feel it is sufficiently small relative to the size of the the other parameters of the model. We can see that the change in the option price only occurs for the first 100 days of the option's lifetime. This is in line with the explanation given previously with the product's price flatlining. We are unsure as to the cause of the high values for Gamma as well as the spikes in the values of Delta. We suspect some of it is due to the coupon payment dates and others could be due to the change in interest rates.

As for the remaining Greeks, they do have fairly consistent values that also flatline after 100 days. Theta is the only exception where there is some variation towards maturity.

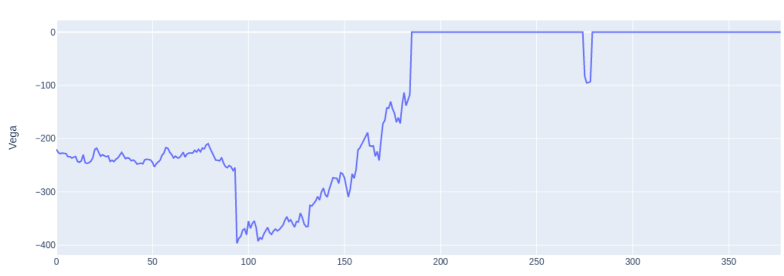

Figure 9: Vega

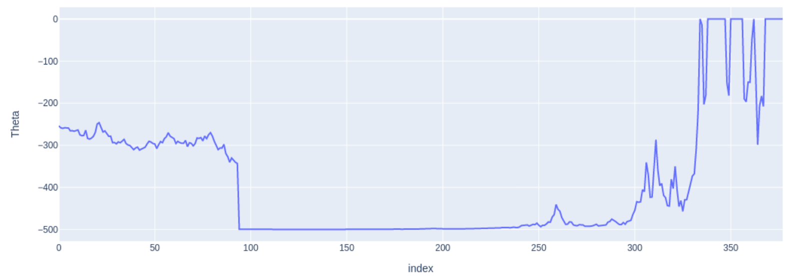

Figure 10: Theta

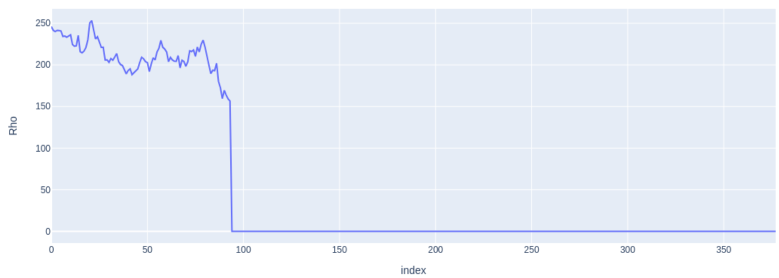

Figure 11: Rho

5 Conclusion

From this project, we have found that using stochastic processes can be a very powerful tool in financial simulations. The Monte Carlo method provides an excellent basis for generating simulations with Geometric Brownian Motion for asset prices. These asset prices have been used to value a structured product to a reasonable level of accuracy with a relatively low number of simulations. In addition to this, we have found that Empirical Martingale Correction is a very useful tool to preserve the Martingale property in Monte Carlo simulations. This is especially important for us, as we perform simulations under a risk-neutral measure. As the payoff of our product is path-dependent, using EMC led to a dramatic reduction in RMSE and improved our valuation estimates by a large margin.

Given more time and resources, it would be worthwhile to explore how variance reduction techniques such as antithetic variates could be used in conjunction with EMC, and if this would produce favorable results. It would also be worth generating many more simulated paths to help reduce fluctuations and variance in the expected payoff. Using other models for the simulation like the Heston Model might end up giving better results but will com ewith its own challenges like parameter tuning.

References

Jin-Chuan Duan and Jean-Guy Simonato. Empirical martingale simulation for asset prices. *Management Science*, 44:1218–1233, 1995.