

# Geometric Covering Number: Covering Points with Curves

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**Abstract.** Given a point set, mostly a grid in our case, we seek upper and lower bounds on the number of curves that are needed to *cover* the point set. We say a curve *covers* a point if the curve passes through the point. We consider such coverings by monotonic curves, lines, orthoconvex curves, circles, etc. We also study a problem that is converse of the covering problem – if a set of  $n^2$  points in the plane is covered by  $n$  lines then can we say something about the configuration of the points?

**Keywords:** Discrete geometry, Incidence, Covering

## 1 Introduction

Let  $S$  be a set of curves satisfying some fixed property (e.g., circle, convex curves, etc.) and  $P$  be a set of points in  $\mathbb{R}^d$ . A curve  $c \in S$  *covers* a point  $p \in P$  if  $p$  lies on the curve  $c$ . We say that  $S$  covers  $P$  if all points in  $P$  are covered by the union of all members of  $S$ . We will be interested in the minimum cardinality of  $S$ , satisfying the given property, that covers  $P$  (where the point set  $P$  is fixed).

To start with, let  $P$  be a set of points in  $\mathbb{R}^2$  in general position and the goal is to figure out the number of simple curves needed to cover  $P$ . The solution is trivial – sort the points based on their  $x$ -coordinates and join them from left to right; i.e., we need just one simple curve to cover  $P$ . As we move from a simple curve with no restrictions whatsoever, to a straight line, the problem becomes hard and deserves non-trivial solutions [17,5,1,19]. This obviously gives rise to a natural question about what happens to this problem if we consider point sets with some special configuration, like grids vis-a-vis different kinds of simple curves like circles, convex curves, orthoconvex curves, etc. To bring the variety of different point sets and curves under a unifying framework, we propose the following definition of *geometric covering number*.

**Definition 1** (*Geometric covering number*) *The geometric covering number of a point set  $P$  in  $\mathbb{R}^d$  with respect to a curve type  $C$  (like circle, convex curve, orthoconvex curve, etc.), denoted as  $\mathcal{G}_C(P, d)$ , is the minimum number of curves of type  $C$  needed to cover all points in  $P$ . A curve covers a point if the point lies on the curve. If the dimension is understood, we just write  $\mathcal{G}_C(P)$  instead of  $\mathcal{G}_C(P, d)$ .*

The notion of covering a point set with different geometric structures have been studied in the literature [7,16,9,11,13]. The common theme running through all such problems is about figuring out the minimum number of structures, e.g., trees, paths, line segments, etc., needed to form a cover of the point set. Given a set of points, a *covering path* is a polygonal path that visits all the points and similarly a *covering tree* is a tree whose edges are line segments that jointly cover all the points. Covering paths and trees for planar grids have been studied in [16], where bounds on the minimum number of line segments of such paths and trees are given. Analogous questions on covering paths and trees for higher dimensional grids have been studied in [11]. Given a set  $S$  of  $n$  points in the plane, the problem of finding the smallest number  $l$  of straight lines needed to cover all  $n$  points in  $S$  have been studied in [13], where bounds on the time complexity of this problem in terms of  $n$  and  $l$  (assuming  $l$  to be small) is given.

On the other hand, incidence problems in geometry [20,21] studies questions about finding the maximum possible number of pairs  $(p, \ell)$  such that  $p$  is a point belonging to a set of points and  $\ell$  is a line belonging to a set of lines and  $p$  lies on  $\ell$ . Incidence between points and other geometric structures like circles, planes, algebraic curves, etc. have also been studied. We do not intend to go into all of them as an interested reader can find them in [20,21]. On the other hand, researchers have studied the problems of *point line cover*, or its more general form of *point curve cover* [17,5,1,19]. These problems consist of a set  $P$  of  $n$  points on the plane and a positive integer  $k$ , and the question is whether there exists a set of at most  $k$  lines/hyperplanes/curves which cover all points in  $P$ . They are computationally hard problems, motivated from SET COVER, and the effort has been mostly in parametrized complexity where researchers focussed on finding tight kernels [8] for the problems [17,5,1,19].

*Notations:* We will use  $[x]$  to denote the set of natural numbers  $\{1, 2, \dots, x\}$ .  $P$  will denote a set of  $n$  points in dimension  $d$ . Unless otherwise stated,  $P$  will be finite.

*Organization of the paper:* In this paper, we study the notion of geometric covering number for a few types of curves. For most of the cases, our point set is a grid that we want to cover with a particular kind of curve. For completeness sake, we start with lines, the simplest curve, covering a finite grid in Section 2. We also investigate a converse question of covering in Section 2.2. Very simply put, the converse question deals with the following notion – if there is a guarantee that some lines cover an “unknown” point set, then can we say something about the configuration of the point set? From lines, we move onto monotone curves in Section 3. Section 4 considers three types of closed curves – circles, convex curves, and orthoconvex curves. Finally, Section 5 sums up the findings in this work. The Appendix is in Section 6 where we have put all the missing proofs and remarks. We feel our work will motivate studying the *geometric covering number* for more point set and curve pairs.

*Our contributions:* Two of our major contributions in this paper are the following. As a converse to the covering by lines problem, we show in Theorem 4 that for a set  $P$  of  $n^2$  points covered by  $n$  lines, it's not true that there always exists a subset of  $P$  of size  $\Theta(n^2)$  that can be put inside a grid of size  $\Theta(n^2)$ , possibly after a projective transformation. Regarding covering by orthoconvex curves, we proved in Theorem 15 that at least  $2n/5$  (which is achieved for  $n = 5$ ) orthoconvex curves with at most one inner corner and  $2n/7$  curves with at most two inner corners are required to cover an  $n \times n$  grid (Theorem 18). We also make the following observations regarding covering by other types of curves that are not very difficult to obtain. We noted in Proposition 10 that the answer to question of covering a grid by minimum number of monotonic curves can be obtained by applying Dilworth's Theorem on posets. For algebraic curves, the answer (Theorem 7) came as a consequence of the Combinatorial Nullstellensatz. For circles, the existing results in the literature imply very close upper and lower bounds (as noted in Proposition 11) and the case of convex curves is settled by an easy argument in Theorem 13.

## 2 Covering by lines and its converse problem

In the first part of this section, we consider covering grids by lines (the bounds are easy to obtain; we include it for the sake of completeness). In the next part, we consider a “converse” question – if a set of  $n^2$  points in  $\mathbb{R}^2$  is covered by  $n$  lines, then can we say something about the configuration of the points?

### 2.1 Covering by lines

Note that for any two points there exists a line covering them. Therefore,  $\mathcal{G}_C(P) \leq \frac{|P|}{2}$  (the equality is achieved for any set of points in general position). Now let  $\ell(P)$  denote the maximum number of points in  $P$  any line can cover. Then we have  $\mathcal{G}_C(P) \geq |P|/\ell(P)$ . Therefore, we get  $\frac{|P|}{\ell(P)} \leq \mathcal{G}_C(P) \leq \frac{|P|}{2}$ . Now we consider the case when  $P = [k_1] \times \cdots \times [k_d]$ . We state the following whose proof is in Appendix 6.1:

**Proposition 2**  $\ell(P) = \max\{k_1, \dots, k_d\}$ .

Proposition 2 implies that  $\mathcal{G}_C(P) \geq \frac{\prod_{i=1}^d k_i}{\ell(P)} \geq \min\left\{\prod_{i \neq 1} k_i, \dots, \prod_{i \neq d} k_i\right\} := N$ . On the other hand,  $\mathcal{G}_C(P) \leq N$  since there clearly exists an explicit covering of  $P$  by  $N$  lines (namely, by the lines parallel to the coordinate axis  $i_0$ , where  $\ell(P) = k_{i_0}$ ). Therefore, we get that  $\mathcal{G}_C(P) = \min\left\{\prod_{i \neq 1} k_i, \dots, \prod_{i \neq d} k_i\right\}$ .

**Remark 3 (Skew lines)** We say that a line is skew if it is not parallel to  $x$  or  $y$ -axis. We look at the question of covering an  $n \times n$  grid by the minimum number of skew lines.

*Note that the boundary of the  $n \times n$  grid contains  $4n - 4$  points. Now any skew line can contain at most 2 points from the boundary. So we need at least  $2n - 2$  skew lines to cover the grid. Also note that the  $n \times n$  grid can be covered by  $2n - 2$  skew lines (consider the  $2n - 3$  lines parallel to the off-diagonal except the ones which pass through the bottom-left and top-right corners and these two corners are covered by the main diagonal).*

*It is an open problem to find the minimum number of skew hyperplanes required to cover the  $d$ -dimensional hypercube. Current (2023) best known lower bound for the above problem is  $d/2$ , as observed in [22] (see Proposition 1.3).*

## 2.2 On the converse of the covering problem

Since  $d = 2$  for an  $n \times n$  grid, from the above discussion, it can be covered using  $n$  lines. Here we look at the converse question, namely, if a set of  $n^2$  points in  $\mathbb{R}^2$  is covered by  $n$  lines then can we say something about the configuration of the points?

Suppose a set of  $n^2$  points is covered by  $n$  lines. Then there exists a line containing  $\Omega(n)$  points, since otherwise the total number of points is less than  $n^2$ . Now if this line contains  $o(n^2)$  points, then there exists another line containing  $\Omega(n)$  points. By continuing this, we can say that there exists a set of lines each containing  $\Omega(n)$  points such that the total number of points in the union of these lines is  $\Theta(n^2)$ .

Now the following question seems natural. If a set  $P$  of  $n^2$  points is covered by  $n$  lines, then does there always exist a subset of  $P$  of size  $\Theta(n^2)$  which can be put inside a grid of size  $\Theta(n^2)$ , possibly after applying a projective transformation? We show that the answer is no.

**Theorem 4** *There exists a finite set  $P$  of  $n^2$  points in  $\mathbb{R}^2$  which can be covered with  $n$  lines but no subset of  $P$  of size  $\Omega(n^2)$  can be contained in a projective transformation of a rectangular grid of size  $o(n^3)$ .*

*Proof.* Given any two distinct points  $p, p' \in \mathbb{R}^2$ , we denote by  $\ell(p, p')$  the unique line in  $\mathbb{R}^2$  that contains both  $p$  and  $p'$ . By an  $s \times t$  grid, we mean a point set that can be obtained by a projective transform  $f$  of the set  $[t] \times [s]$ . By a “horizontal line” of the grid, we mean a line  $\ell(f(1, j), f(t, j))$  for some  $j \in [s]$ , and by a “vertical line” of the grid, we mean a line  $\ell(f(i, 1), f(i, s))$ , for some  $i \in [t]$ . The “size” of an  $s \times t$  grid is  $st$ , i.e., the number of points in it. Note that every horizontal line of a grid intersects every vertical line of the grid (since there is a point of the grid that is contained in both of them).

For each  $i \in [n]$ , let  $L_i$  denote the line with equation  $y = i$  and let  $\mathcal{L} = \{L_i\}_{1 \leq i \leq n}$ . Let  $\mathcal{P}$  be the set of points defined as follows. Define  $P_1$  to be some set of  $n$  distinct points from the line  $L_1$ . For each  $1 < i \leq n$ , we define  $P_i$  to be a set of  $n$  distinct points from  $L_i$  that do not lie on any of the lines formed by points on other lines, i.e. in  $\{\ell(p, p') : p \neq p' \text{ and } p, p' \in \bigcup_{1 \leq j \leq i-1} P_j\}$ . Let

$\mathcal{P} = \bigcup_{1 \leq i \leq n} P_i$ . Let  $m = |\mathcal{P}|$ . Note that we have  $|\mathcal{L}| = n$ . We claim that for any  $\mathcal{P}' \subseteq \mathcal{P}$  such that  $|\mathcal{P}'| = \Omega(m) = \Omega(n^2)$ , any grid that contains all the points of  $\mathcal{P}'$  has size  $\Omega(n^3)$ .

Note that by our construction, if any line contains two points  $p, p' \in \mathcal{P}$  such that  $p \in P_i$  and  $p' \in P_j$ , where  $i \neq j$ , then  $p$  and  $p'$  are the only points in  $\mathcal{P}$  that are contained in that line. This implies that the following property is satisfied by  $\mathcal{P}$  and  $\mathcal{L}$ .

(\*) Any line in  $\mathbb{R}^2$  that contains more than two points in  $\mathcal{P}$  belongs to  $\mathcal{L}$ .

Since every line in  $\mathcal{L}$  contains exactly  $n$  points of  $\mathcal{P}$ , we then have another property.

(+) Any line in  $\mathbb{R}^2$  contains at most  $n$  points in  $\mathcal{P}$ .

Let  $\mathcal{P}' \subseteq \mathcal{P}$  be such that  $|\mathcal{P}'| = \Omega(n^2)$ . Consider any grid  $\mathbb{G}$  that contains all the points of  $\mathcal{P}'$ . Let  $\mathbb{G}$  be an  $s \times t$  grid. Let  $h_1, h_2, \dots, h_s$  denote the horizontal lines of  $\mathbb{G}$  and let  $v_1, v_2, \dots, v_t$  denote the vertical lines of  $\mathbb{G}$ . Suppose for the sake of contradiction that there exist  $i \in [s]$  and  $j \in [t]$  such that both the lines  $h_i$  and  $v_j$  contain at least 3 points of  $\mathcal{P}$  each. Then by property (\*),  $h_i$  and  $v_j$  are both lines in  $\mathcal{L}$ . But as  $h_i$  and  $v_j$  intersect, they are two lines in  $\mathcal{L}$  that intersect, which is a contradiction, since the lines in  $\mathcal{L}$  are all parallel to each other (note that parallel lines under a projective transformation may not be parallel but they do not intersect at any of the  $s \times t$  grid points defined). Thus, we can conclude without loss of generality that for each  $i \in [s]$ , the horizontal line  $h_i$  of  $\mathbb{G}$  contains at most two points from  $\mathcal{P}$ , and hence at most two points from  $\mathcal{P}'$ . Since every point in  $\mathcal{P}'$  is contained in at least one horizontal line of  $\mathbb{G}$ , we have that  $s \geq |\mathcal{P}'|/2$  and therefore  $s = \Omega(n^2)$ . By property (+), each vertical line of  $\mathbb{G}$  can contain at most  $n$  points of  $\mathcal{P}'$ , and therefore,  $t \geq |\mathcal{P}'|/n$ , which implies that  $t = \Omega(n)$ . Thus the size of the grid  $\mathbb{G}$  is  $st = \Omega(n^3)$ . ◀

**Remark 5** *The above construction also provides a counter-example<sup>1</sup> to the Conjecture 1.17 as stated in [23]. The formal statement of the conjecture is: Consider sufficiently large positive integers  $m$  and  $n$  that satisfy  $m = O(n^2)$  and  $m = \Omega(\sqrt{n})$ . Let  $P$  be a set of  $m$  points and  $L$  be a set of  $n$  lines, both in  $\mathbb{R}^2$ , such that  $I(P, L) = \Theta(m^{2/3}n^{2/3})$  (the number of incidences). Then there exists a subset  $P' \subset P$  such that  $|P'| = \Theta(m)$  and  $P'$  is contained in a section of the integer lattice of size  $\Theta(m)$ , possibly after applying a projective transformation to it.*

### 2.3 Covering by algebraic curves

In this subsection, we address the question of covering a grid by algebraic curves. The answer comes as a direct application of the famous Combinatorial Nullstellensatz Theorem due to Noga Alon.

<sup>1</sup> This was communicated to Prof. Adam Sheffer who told us that this only exposes a typo in the statement of the conjecture which is more interesting and challenging when  $m = o(n^2)$ . Note that in our construction we have  $m = n^2$ .

**Lemma 6 (Combinatorial Nullstellensatz [2])** *Let  $f = f(x_1, \dots, x_d)$  be a polynomial in  $\mathbb{R}[x_1, \dots, x_d]$ . Suppose the degree  $\deg(f)$  of  $f$  is  $\sum_{i=1}^d t_i$  where each  $t_i$  is a non-negative integer, and suppose the coefficient of  $\prod_{i=1}^d x_i^{t_i}$  in  $f$  is non-zero. Then, if  $S_1, \dots, S_d$  are subsets of  $\mathbb{R}$  with  $|S_i| > t_i$ , there are  $s_1 \in S_1, s_2 \in S_2, \dots, s_d \in S_d$  so that  $f(s_1, \dots, s_d) \neq 0$ .*

**Theorem 7** *Suppose the  $n \times n$  grid is covered by  $m$  algebraic curves of degree at most  $k$ . Then  $m \geq n/k$ .*

*Proof.* Suppose  $m < n/k$ . Let the algebraic curves defined by  $f_1(x, y) = 0, \dots, f_m(x, y) = 0$  cover the  $n \times n$  grid, where  $\deg(f_i) \leq k$ . Then the polynomial  $f(x, y) := \prod_{i=1}^m f_i(x, y)$  vanishes at each grid point. Suppose  $\deg(f) = t_1 + t_2$  with the coefficient of  $x^{t_1}y^{t_2}$  in  $f$  being non-zero. Now note that  $t_i \leq t_1 + t_2 = \deg(f) \leq mk < n$ , for each  $i = 1, 2$ . So by Lemma 6, there exists a grid point  $(s_1, s_2)$  so that  $f(s_1, s_2) \neq 0$  and we arrive at a contradiction. Therefore, we conclude that  $m \geq n/k$ .  $\blacktriangleleft$

**Corollary 8**  $\mathcal{G}_C(P) = \lceil n/k \rceil$ , where  $P$  is an  $n \times n$  grid and  $C$  denotes algebraic curves of degree at most  $k$ .

*Proof.* The lower bound follows from the previous theorem and the upper bound follows from covering by lines and then considering a set of  $k$  lines as one curve of degree  $k$ .  $\blacktriangleleft$

See Appendix 6.2 for a discussion on irreducible algebraic curves.

### 3 Covering by monotonic curves

In this section, we consider the case when the curve is *monotonic*.

**Definition 9** *Let  $f : [0, 1] \rightarrow \mathbb{R}^d$  be a curve and suppose  $f(t) = (f_1(t), \dots, f_d(t))$  for  $t \in [0, 1]$ . Then  $f$  is called **monotonic** if it satisfies the following property:  $t_1 \leq t_2 \Rightarrow f_i(t_1) \leq f_i(t_2)$  for each  $i = 1, \dots, d$ .*

Given a finite subset  $P$  of  $\mathbb{R}^d$ , we define the poset  $\mathcal{P} := (P, \leq)$  as follows. For  $x := (x_1, \dots, x_d) \in \mathbb{R}^d$  and  $y := (y_1, \dots, y_d) \in \mathbb{R}^d$ , we define  $x \leq y$  if  $x_i \leq y_i$  for  $i = 1, \dots, d$ .

We say that two elements  $a$  and  $b$  of a poset  $P$  are *comparable* if either  $a \geq b$  or  $b \leq a$ . An *antichain* in a poset is a set of elements no two of which are comparable to each other, and a *chain* is a set of elements every two of which are comparable. A chain decomposition is a partition of the elements of the poset into disjoint chains. Size of an antichain is its number of elements, and the size of a chain decomposition is its number of chains.

**Proposition 10** *Let  $w(\mathcal{P})$  denote the size of the largest antichain, called the width, of  $\mathcal{P}$ . Then  $\mathcal{G}_C(P) = w(\mathcal{P})$ , where  $P$  is any point set and  $C$  denotes monotonic curves.*

*Proof.* Let  $x_i \in \mathcal{P}$  for  $i = 1, \dots, r$ . Then note that  $x_1 \leq \dots \leq x_r$  is a chain if and only if  $x_1, \dots, x_r$  lie on the same curve (which is monotonic). Therefore,  $\mathcal{G}_C(P)$  equals the number of chains in a chain decomposition of smallest size of  $\mathcal{P}$ , which by Dilworth's theorem [10] equals the size of the largest antichain of  $\mathcal{P}$ . Hence  $\mathcal{G}_C(P) = w(\mathcal{P})$ .  $\blacktriangleleft$

Note that the poset  $\mathcal{P}$  can be decomposed into  $w(\mathcal{P})$  many disjoint chains. Therefore, the points in  $P$  can be covered by  $\mathcal{G}_C(P)$  many monotonic curves such that *no two curves intersect at a point of  $P$* .

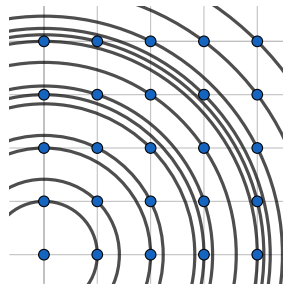
## 4 Covering by closed curves

In this section, we consider covering grids by circles, convex curves and ortho-convex curves. Notice that the curves need not be of the same size, e.g., when we are considering covering by circles, all the circles need not be of the same size.

### 4.1 Covering by circles and convex curves

*Covering by circles.* A circle contains at most  $O(n^\epsilon)$  points from an  $n \times n$  grid for every  $\epsilon > 0$  (see e.g. [14]). Therefore, the minimum number of circles required to cover an  $n \times n$  grid is  $\Omega(n^{2-\epsilon})$ , for every  $\epsilon > 0$ . Regarding upper bound, note that there is a covering of the  $n \times n$  grid by  $O(n^2/\sqrt{\log n})$  circles. This is obtained by choosing a corner of the grid and drawing all concentric circles such that each of them is incident to at least one grid point (see Figure 1). The number of such circles is  $O(n^2/\sqrt{\log n})$  by a well known theorem of Ramanujan and Landau ([4],[18]). The theorem says that the number of positive integers that are less than  $n$  that are the sum of two squares is  $\Theta(n/\sqrt{\log n})$ . We sum it up as the following.

**Proposition 11**  $\Omega(n^{2-\epsilon}) \leq \mathcal{G}_C(P) \leq O(n^2/\sqrt{\log n})$ , where  $P$  is an  $n \times n$  grid and  $C$  denotes circles.



**Fig. 1.** Covering of  $5 \times 5$  grid by circles

*Covering by convex curves.* A closed convex curve intersects non-trivially with a horizontal grid line if it contains more than two points from the line. Note that, any closed convex curve can intersect at most two horizontal grid lines non-trivially. This follows from the following lemma whose proof is in Appendix 6.3.

**Lemma 12** *If a closed convex curve intersects a horizontal grid line non-trivially, then it must lie entirely on one side of that line.*

**Theorem 13** *The points of the  $n \times n$  grid cannot be covered with less than  $n/2$  closed convex curves, i.e.  $\mathcal{G}_C(P) \geq n/2$  where  $P$  is an  $n \times n$  grid and  $C$  denotes closed convex curves.*

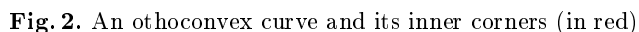
*Proof.* Suppose, for the sake of contradiction, that  $C_1, C_2, \dots, C_k$  are  $k$  closed convex curves such that they together cover every point of the  $n \times n$  grid and that  $k < n/2$ . Then, since there are  $n$  horizontal grid lines, and by Lemma 12 above, each  $C_i$  can have a non-trivial intersection with at most 2 horizontal grid lines, we can conclude that there is some horizontal grid line such that no curve in  $C_1, C_2, \dots, C_k$  has a non-trivial intersection with that line. Now consider the points on that horizontal line. There are  $n$  points on this line. Each curve in  $C_1, C_2, \dots, C_k$  can cover at most two points from that line and none of them intersects non-trivially with this horizontal line. But then, since  $k < n/2$ , there must be some point on this horizontal line that is not covered by any curve in  $C_1, C_2, \dots, C_k$ , which is a contradiction. ◀

Almost same argument can be used to get an answer for an  $m \times n$  grid and this will be  $\min\{\lceil m/2 \rceil, \lceil n/2 \rceil\}$ .

## 4.2 Covering by orthoconvex curves

A set  $K \subseteq \mathbb{R}^2$  is defined to be *orthogonally convex* if, for every line  $\ell$  that is parallel to one of standard basis vectors  $(1, 0)$  or  $(0, 1)$ , the intersection of  $K$  with  $\ell$  is empty, a point, or a single segment. The *orthogonal convex hull* of a point set  $P \subseteq \mathbb{R}^2$  is the intersection of all connected orthogonally convex supersets of  $P$ . If the boundary of orthogonal convex hull (of a set of points) is a simple closed curve then we call it an *orthoconvex* curve. An orthoconvex curve has only two types of angles, namely  $90^\circ$  and  $270^\circ$ . By *inner corner* of an orthoconvex curve, we mean a point where the curve turns by  $270^\circ$ . See Figure 2 for an example of an orthoconvex where the red points are its inner corners.

If an orthoconvex curve (with  $k$  inner corners) covers a set of points, then there is also an orthoconvex curve (with  $k$  inner corners) covering the same points which is not self-intersecting and all the corners are grid points. This can be done by pushing the sides/edges of the curve “outwards” (instead of inwards which corresponds to taking orthoconvex hull) until we hit a grid line. So w.l.o.g., we may impose the following assumptions of ‘non-self-intersecting’ and ‘corners are grid points’.

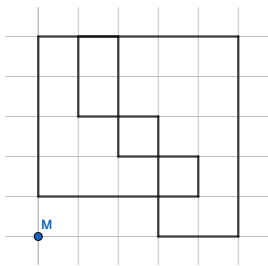


**Lemma 14** *Let  $c_1, c_2, \dots, c_t$  be a good sequence of curves. Then  $\{c_1, c_2, \dots, c_t\}$  either: (a) hits at most  $5t$  grid lines, or (b) hits  $5t + 1$  grid lines and has an exposed corner.*

(See Figure 4 for an illustration of case (b), where as Figure 5 shows an example of case (a))

*Proof.* We prove this by induction on  $t$ . It is not difficult to see that the lemma is true when  $t = 1$ . Let  $i > 1$  and suppose that the lemma is true for the good sequence  $c_1, c_2, \dots, c_{i-1}$ . Let  $C = \{c_1, c_2, \dots, c_{i-1}\}$ . Then either  $C$  hits (a) at most  $5i - 5$  grid lines, or (b) hits  $5i - 4$  grid lines and has an exposed corner.

In case (a), since the curve  $c_i$  can hit at most 5 grid lines that are not hit by  $C$  (recall that  $c_i$  hits at least one grid line that is also hit by  $C$ ), we have that  $C \cup \{c_i\}$  can hit at most  $5i$  grid lines, and we are done. Next, let us consider case (b). Note that if  $c_i$  is a rectangle, then it can hit at most 3 grid lines that are not hit by  $C$  (note that, a rectangle has four sides and  $c_i$  hits at least one grid line that is also hit by  $C$ ), and therefore,  $C \cup \{c_i\}$  hits at most  $5i - 4 + 3 = 5i - 1$  grid lines, and we are done. So we can assume that  $c_i$  is not a rectangle. Also, if there are two grid lines that are hit by both  $C$  and  $c_i$ , then  $C \cup \{c_i\}$  hits at most  $5i$  grid lines, and we are done. So we can assume that  $c_i$  hits exactly one grid line that is hit by  $C$ , and therefore,  $C \cup \{c_i\}$  hits exactly  $5i + 1$  grid lines. In this case, we have to show that one of the corners of  $C \cup \{c_i\}$  is exposed. Let  $B$  be the bounding box of  $C \cup \{c_i\}$ . Let  $g_0, g_1, g_2, g_3$  be the grid lines on which the top, right, bottom, and left borders of  $B$  lie. Clearly, each of  $g_0, g_1, g_2, g_3$  is hit by either  $C$  or  $c_i$  or both. Since  $c_i$  hits exactly one grid line that is hit by  $C$ , we have that at most one of  $g_0, g_1, g_2, g_3$  is hit by both  $C$  and  $c_i$ . This implies that  $C$  and  $\{c_i\}$  do not have shared corners. Note that a corner  $v$  of  $C$  is exposed, and a corner  $v'$  of  $\{c_i\}$  is exposed. If each of  $g_0, g_1, g_2, g_3$  is hit by  $C$ , then  $v$  is an exposed corner of  $C \cup \{c_i\}$  (observe that  $v$  cannot be covered by  $c_i$ , because if it is, it has to be a corner of  $\{c_i\}$ , which would mean that  $C$  and  $\{c_i\}$  have a shared corner) and we are done. Similarly, if each of  $g_0, g_1, g_2, g_3$  is hit by  $c_i$ , then  $v'$  is an exposed corner of  $C \cup \{c_i\}$  and we are again done. Thus we can assume that neither  $C$  nor  $c_i$  hits all the grid lines  $g_0, g_1, g_2, g_3$ . Recall that all grid lines except at most one in  $g_0, g_1, g_2, g_3$  are hit by exactly one of  $C$  or  $c_i$ . Then there exists some  $j \in \{0, 1, 2, 3\}$  such that one of  $g_j, g_{j+1 \bmod 4}$  is hit by  $C$  and not by  $c_i$ , and the other is hit by  $c_i$  and not by  $C$ . Then the grid point that is contained in both the grid lines  $g_j$  and  $g_{j+1 \bmod 4}$  is an exposed corner of  $C \cup \{c_i\}$ . This completes the proof.  $\blacktriangleleft$



**Fig. 4.** Two curves that hit 11 grid lines and has an exposed corner (M)

**Theorem 15** *If  $m$  orthoconvex curves with at most one inner corner cover the  $n \times n$  grid, then  $m \geq 2n/5$ .*

*Proof.* Let  $C$  be a collection of  $m$  curves that cover the  $n \times n$  grid. For two curves  $c$  and  $d \in C$ , we say that  $cRd$  if there is a grid line that is hit by both  $c$  and  $d$ . Let  $R^*$  be the transitive closure of  $R$ . Clearly,  $R^*$  is an equivalence relation. Let  $S_1, S_2, \dots, S_p$  be the equivalence classes of  $R^*$ . We need the following claims for the proof.

**Claim 16** *For each  $i \in [p]$ ,  $S_i$  does not expose any grid point.*

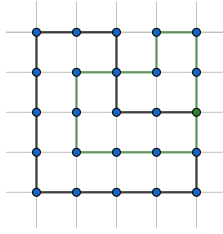
*Proof.* Suppose for some  $i \in [p]$ ,  $S_i$  exposes a grid point  $v$ . That is,  $v$  is not covered by  $S_i$ , but both the horizontal grid line as well as the vertical grid line that contains  $v$  are hit by  $S_i$ . Since  $C$  covers the whole grid, there is a curve  $c \in C$  that covers  $v$ . As  $S_i$  does not cover  $v$ , we have that  $c \in C - S_i$ . As  $c$  covers  $v$ ,  $c$  hits either the horizontal grid line containing  $v$  or the vertical grid line containing  $v$ . Since both these grid lines are hit by  $S_i$ , it follows that there exists some  $d \in S_i$  such that  $c$  and  $d$  hit a common grid line. Then  $dRc$ , which implies that  $c \in S_i$ , which is a contradiction. This proves the claim. ◀

**Claim 17** *The curves of each equivalence class  $S_i$  can be arranged in a good sequence.*

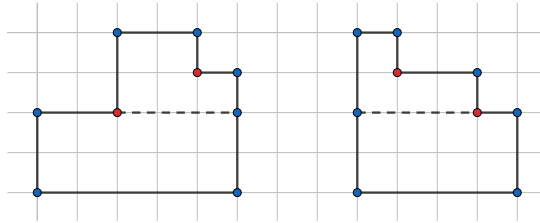
*Proof.* Let  $G$  be the graph with vertex set  $S_i$  and edge set  $R$  restricted to  $S_i$ . By enumerating the curves of  $S_i$  in the order in which they are visited by a graph traversal algorithm starting from an arbitrary vertex, we get a sequence of the curves in  $S_i$  such that before a curve  $c$  is encountered in the sequence, we encounter some curve  $d$  such that  $dRc$  (except for the first curve in the sequence). This sequence is clearly a good sequence of the curves in  $S_i$ . This proves the claim. ◀

By Lemma 14 and Claims 16 and 17, we know that for each  $i \in [p]$ ,  $S_i$  hits at most  $5|S_i|$  grid lines. Thus the total number of grid lines that are hit by  $C$  is at most  $5(|S_1| + |S_2| + \dots + |S_p|) = 5|C| = 5m$ . If the curves in  $C$  hit  $2n$  grid lines, we then have  $5m \geq 2n$ , which gives  $m \geq 2n/5$ . Otherwise, suppose that the collection  $C$  of  $m$  curves, where  $m \leq 2n/5$ , hits less than  $2n$  grid lines. That is, there is some (horizontal or vertical) grid line that is not hit by any curve in  $C$ . Then every curve in  $C$  can cover at most two points on this grid line (if it covers more than two, then the curve hits this grid line). So at most  $2m \leq 4n/5$  points on this grid line can be covered by the collection of curves  $C$ , which means that some points on this grid line are not covered by any curve in  $C$ , which is a contradiction. So we conclude that  $m \geq 2n/5$  and this proves the theorem. ◀

Note that, the inequality of the above theorem is tight for  $n = 5$  since the  $5 \times 5$  grid can be covered by 2 curves (shown in Figure 5). As a consequence of the above theorem, we also get the following theorem on orthoconvex curves with at most 2 inner corners.



**Fig. 5.** Covering of  $5 \times 5$  grid by two orthoconvex curves (with at most one inner corner)



**Fig. 6.** Decomposition of orthoconvex curves with 2 inner corners

**Theorem 18** *We need at least  $2n/7$  orthoconvex curves with at most two inner corners to cover an  $n \times n$  grid*

*Proof.* Suppose we have a covering by  $m$  such curves. Note that we can decompose each orthoconvex curve with two inner corners into an orthoconvex curve with at most one inner corner and a rectangle (see Figure 6). Hence we obtain a covering by  $m$  orthoconvex curves with at most one inner corner and  $m$  rectangles. These  $m$  orthoconvex curves with at most one inner corner can together hit at most  $5m$  grid lines (see proof of Theorem 15) and the rectangles together hit at most  $2m$  extra grid lines (since each rectangle hit at most two extra grid lines). So the total number of grid lines hit by our original curves is at most  $7m$ . Since the curves have to hit  $2n$  grid lines (by the same reasoning as in proof of Theorem 15), we then have  $7m \geq 2n$ . Hence, we conclude that  $m \geq 2n/7$ . ◀

See Appendix 6.4 for a remark on covering by orthoconvex curves.

## 5 Conclusion and discussion

In this paper, we mainly discussed the problem of covering a grid (mostly planar) by minimum number of curves of various types. An interesting open problem in this direction is to cover the hypercube by minimum number of skew hyperplanes. We leave it as an open problem to figure out what happens when there are more inner corners for covering by an orthoconvex curve. Lastly, we mention that in this article we only considered 1-fold covering where every grid point was covered at least once. But, in general, we could ask analogous questions for  $r$ -fold covering (i.e., every point is covered at least  $r$  times) for  $r \geq 2$ .

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## 6 Appendix

### 6.1 Proof of Proposition 2

*Proof.* Let  $M := \max\{k_1, \dots, k_d\}$ . First we show that  $\ell(P) \leq M$  by induction on  $d$ . The base case  $d = 1$  is obvious. Now we proceed to the induction step. Let  $L$  be a line segment that lies inside the rectangular parallelepiped  $[1, k_1] \times \dots \times [1, k_d]$ . Then  $L$  has length at most  $\sqrt{\sum_{i=1}^d (k_i - 1)^2}$ . Now let  $x := (x_1, \dots, x_d)$  and  $y := (y_1, \dots, y_d)$  be two distinct points of  $P$  lying on  $L$ . If  $x_i = y_i$  for some  $i$ , then  $L$  lies inside a lower dimensional rectangular parallelepiped and therefore, by induction hypothesis,  $L$  covers at most  $\max\{k_j \mid j \neq i\} \leq M$  many points. So let us assume  $x_i \neq y_i$  for all  $i = 1, \dots, d$ . Then the distance between  $x$  and  $y$  is at least  $\sqrt{d}$ . Suppose  $L$  covers a total of  $t$  points of  $P$ . Then we have

$$(t-1)\sqrt{d} \leq \sqrt{\sum_{i=1}^d (k_i - 1)^2} \leq \sqrt{d} \cdot \max\{k_1 - 1, \dots, k_d - 1\}$$

and this implies  $t \leq \max\{k_1, \dots, k_d\}$ . Therefore, we conclude that  $\ell(P) \leq M$ . On the other hand, there clearly exist lines covering  $M$  points, namely the lines parallel to the coordinate axis  $i_0$ , where  $M = k_{i_0}$ . Hence, we have shown that  $\ell(P) = M$ .  $\blacktriangleleft$

### 6.2 A remark on irreducible algebraic curves

**Remark 19 (Irreducible algebraic curves)** *By a result of Bombieri and Pila [6], an irreducible algebraic curve of degree  $k$  can contain at most  $O(n^{1/k})$  points from an  $n \times n$  grid and hence, the minimum number of irreducible algebraic curves of degree  $k$  to cover the  $n \times n$  grid is at least  $\Omega(n^{2-1/k})$ .*

Using the same reasoning as in the previous theorem and corollary, one also has the following result on covering the  $n_1 \times \dots \times n_d$  grid by algebraic hypersurfaces.

**Theorem 20** *The minimum number of algebraic hypersurfaces of degree at most  $k$  needed to cover the  $n_1 \times \dots \times n_d$  grid is equal to  $\lceil n/k \rceil$ , i.e.,  $\mathcal{G}_C(P) = \lceil n/k \rceil$ , where  $P$  is an  $n_1 \times \dots \times n_d$  grid and  $C$  denotes algebraic hypersurfaces of degree at most  $k$ .*

### 6.3 Proof of Lemma 12

*Proof.* Suppose the curve intersects a horizontal line at three points  $p, q, r$ , where  $q$  lies in the interior of line segment  $[p, r]$ . Since the curve is convex, there exists a line  $L$  through  $q$  such that the curve lies entirely on one side of  $L$  (hyperplane separation theorem). Now if  $L$  is different from the horizontal line, then  $p$  and  $r$  lie on different sides of  $L$ . But since the curve lies on one side of  $L$ , it can not pass through both  $p$  and  $r$ , a contradiction. Therefore,  $L$  is same as the horizontal line and the curve lies entirely on one side of this line. ◀

### 6.4 Remark on covering by orthoconvex curves

**Remark 21** *We think that the bound  $2n/7$  of Theorem 18 is probably not tight. So a natural problem is to obtain a tight bound for covering by orthoconvex curves with at most 2 inner corners. The next natural follow up question would be: what happens for orthoconvex curves with at most  $k$  inner corners for  $k = 3, 4$  etc. It seems our arguments for  $k = 1, 2$  can not be extended to these cases to obtain non-trivial bounds and hence require new ideas. Another question of interest is to find the minimum number of general orthoconvex curves (with no restrictions on the number of inner corners) required to cover an  $n \times n$  grid. One can check that for  $n = 4, 5, 6, 7, 8, 9$  and  $10$ , the  $n \times n$  grid can be covered by 2, 2, 2, 3, 3, 3 and 4 orthoconvex curves, respectively. To us, the general problem of orthoconvex curves seems difficult. Note that we have obvious lower and upper bounds of  $\lceil (n+1)/4 \rceil$  and  $\lfloor n/2 \rfloor$  respectively, since, any orthoconvex curve can contain at most  $4n - 4$  grid points (the number of grid points on the boundary of an  $n \times n$  grid) and on the other hand, an  $n \times n$  grid can be covered by  $\lfloor n/2 \rfloor$  orthoconvex curves. Any improvement over these bounds would be interesting.*