Synopsis

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In this thesis we study two problems, one on characterizing line graphs of hypergraphs and another on finding alternating trails in edge coloured graphs.

1 Characterizing line graphs

Finite graphs and hypergraphs have been extensively studied. Quite often one needs to consider an edge-to-vertex dual called a "line graph" (Whitney 1932, Krausz 1943). Given a simple graph $G(V, E)$ where V is a finite set of vertices and $E \subseteq \binom{V}{2}$, its *line graph* is $L_G(V, E)$, where $L_G(V) = E(G) =: \{e_1, e_2, \ldots, e_m\}$ and $\{e_i, e_j\} \in L_G(E)$ if the intersection of the two edges e_i and e_j is non empty. For an example, consider Figure 1, where Figure 1b is the line graph of Figure 1a.

Figure 1

It is a natural question to ask: Given a simple graph $G_1(V, E)$, is it isomorphic to line graph of a simple graph $G_2(V, E)$? If so, how many such graphs $G_2(V, E)$ are there?

It is easy to see that the 3-claw graph (Figure 2a) is not the line graph of any simple graph. The line graph of a 3-claw is a triangle (Figure 2b) and the line graph of a triangle is also a triangle. It turns out that this is the only case where the line graph L_G does not uniquely determine G. It is also clear that any graph that contains a 3-claw as an induced subgraph cannot be the line graph of a simple graph. In 1968, Beineke in [1] characterized L_G by a finite list of forbidden induced subgraphs (such a characterization is called "finite characterization"). In particular he showed that any graph with at least 4 vertices, that does not contain the list of 9 forbidden subgraphs in Figure 3 is the line graph of a unique (upto isomorphism) simple graph. In 1973, Bermond and Mayer in [2]

(a)

(b)

Figure 2

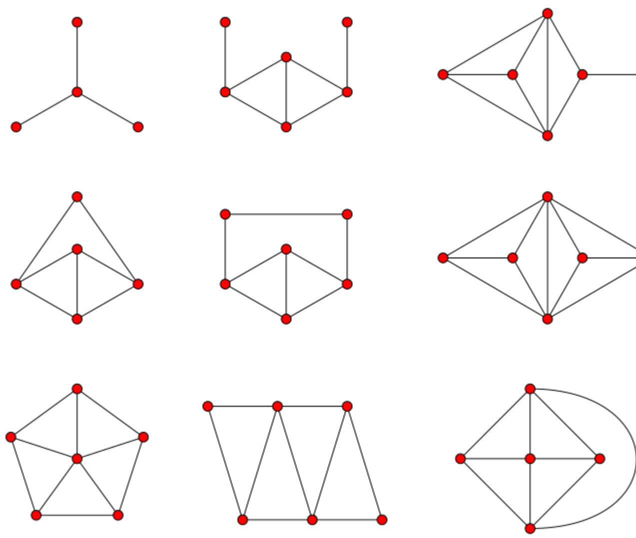

Figure 3

obtained finite characterization for finite graphs with multiple edges. Krausz in [3] also gave a characterization of line graphs. He showed that there exists a collection of *cliques* (complete graphs) $C_1, C_2, \dots, C_k$ in the line graph L_G such that vertices of G are in one-to-one correspondence with the cliques $C_1, C_2, \dots, C_k$ and two vertices in G lie on an edge if the corresponding cliques intersect. For example, Figure 4b gives the clique partition (4 different cliques are indicated by different types of lines) of the line graph of Figure 4a.

(a)

(b)

Figure 4

We now consider the k-uniform hypergraph generalization of this problem.

Let $H = (X, E)$ be a simple hypergraph, where X is a finite set (called *vertices*) and E

is a collection of subsets (called *edges* or hyper-edges) of X. We say that H is k-*uniform* if $E \subseteq \binom{X}{k}$. We define *line graph* of a hypergraph $H = (X, E)$ (denoted by $L(H)$) to be a graph where $V(L(H)) = E$ and $E(L(H))$ is the set of all unordered pairs $\{e, e'\}$ of distinct elements of E such that $e \cap e' \neq \emptyset$ in H. Let L_k denote the set of line graphs of k-uniform hypergraphs. It is a classical problem to characterize these families. It seems that this problem is very difficult. A simpler case seems to be the class of "linear hypergraphs".

A hypergraph H is called *linear* hypergraph if every pair of distinct vertices of H is in at most one edge of H (i.e. *pair degree* is at most 1). It may be observed that 2-uniform linear hypergraphs are simple graphs. An example of a linear hypergraph and its line graph is given in Figure 5.

Figure 5

As in the case of graphs, it may be observed that $k+1$-claw cannot be the line graph of a k-uniform linear hypergraph. It may be observed that the graph in Figure 6a is also forbidden. This idea can be extended to construct an infinite family (Figure 6b) that is forbidden.

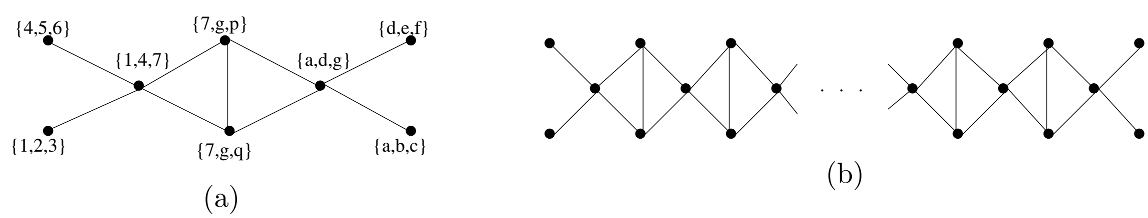

Figure 6

In 1977, Lovász in [5] showed that for $k \geq 3$, the class L_k^l (line graphs of k-uniform linear hypergraphs) has no finite forbidden graph characterization. Niak et al. ([5]) in 1982 obtained finite characterization for L_3^l with $\delta \geq 69$, where δ represents minimum vertex degree in graph. Skums et al. ([6]) in 2009 obtained finite characterization for L_3^l with $\delta \geq 16$. Metelsky ([7]) in 2017 proved that for $k \geq 4$ and $a \in \mathbb{Z}_{>0}$, L_k^l with $\delta > a$ has no finite characterization.

In this work we consider *non linear* hypergraphs with pair degree of the vertices bounded by some integer greater than 1. We show that a finite characterization is possible if we consider line graphs with a certain minimum *edge-degree* (where edge-degree of an edge in a graph is the number of distinct triangles containing that edge). We show this for more general families than L_k^l. For a hypergraph $H = (X, E)$ we define

$$\Delta_2(H) = \max_{\{x,y\}\subseteq X} d_H(\{x, y\}),$$

where $d_H(\{x, y\})$ is the number of edges in H containing x, y. Let $L_k^{(p)}$ denote the set of line graphs of k-uniform hypergraphs with $\Delta_2(H) \leq p$. Note that $L_k^{(1)} = L_k^l$, since H is linear if $\Delta_2(H) = 1$.

By observing some of the infinite families of the forbidden graphs we define three forbidden families.

1. $\mathcal{F}_1(p, k)$ denote the set of graphs G of order $pk^2 + 3$ with two non-adjacent vertices u, v such that $N(\{v, w\}) = V(G) \setminus \{v, w\}$,
2. $\mathcal{F}_2(p, k)$ denote the set of graphs G of order $pk^2 + (p - 2)k + 3$ containing a maximal clique K of size $pk^2 + (p - 2)k + 2$ and a $v \notin K$ such that v is adjacent to at least $pk + 1$ vertices of K,
3. $\mathcal{F}_3(p, k)$ denote the set of graphs G of order less than $2(pk^2 + (p - 2)k + 2)$ containing a pair of distinct maximal cliques K_1, K_2 of size $pk^2 + (p - 2)k + 2$ such that $|V(K_1) \cap V(K_2)| \geq p + 1$.

It may be checked that these families are indeed forbidden. We show the converse. In particular we prove the following theorem.

Theorem 1.1. *There is a polynomial $f(k, p)$ of degree at most 4 with the property that, given any pair k, p, there exists a finite family $\mathcal{F}(k, p)$ of forbidden graphs such that any graph G with minimum edge-degree at least $f(k, p)$ is a member of $L_k^{(p)}$ if and only if G has no induced subgraph isomorphic to a member of $\mathcal{F}(k, p)$.*

Let $\mathcal{F}(p, k) = \mathcal{F}_1(p, k) \bigcup \mathcal{F}_2(p, k) \bigcup \mathcal{F}_3(p, k) \bigcup\{k + 1\text{-claw}\}$. We prove our theorem with this $\mathcal{F}(p, k)$ and $f(k, p) = pk^3 + (p - 3)k + 1$. To show the other direction we use the idea of "clique partition" for line graphs due to Krausz. This is given by the following characterization of line graphs: A graph $G \in L_k^{(p)}$ if and only if there is a set of cliques $\mathcal{K} : K_1, K_2, \ldots, K_r$ of G such that (1) every edge belongs to at least one member of $\mathcal{K}$, (2) every vertex belongs to at most k members of $\mathcal{K}$, (3) if K_i, K_j are distinct elements of $\mathcal{K}$, then $|V(K_i) \cap V(K_j)| \leq p$. To prove our theorem we define define $\mathcal{K}$ to be the set of all maximal cliques in G of size at least $pk^2 + (p - 2)k + 2$ and show that show that $\mathcal{K}$ satisfies the conditions (1), (2) and (3).

This is a joint work with Amitava Bhattacharya, Alosyus Godinho and Navin M Singhi.

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2 Balanced decomposition of colored graphs

One of the central topics in combinatorial optimization is network flows. It has multiple formulations. We present the formulation that motivated this work. A directed graph $D(V,E)$ has finite set of vertices V, finite set of edge labels $\{e_1, e_2, \ldots, \ldots e_m\}$ and each edge label corresponds to an element of $V \times V - \Delta$ (where $\Delta := \{(v,v) : v \in V\}$). We refer to the first coordinate of an edge as *start* and second coordinate as *end* vertex. Note that "multiple" edges are allowed in this definition.

A *network* is a quadruple $N(D,c,s,t)$, where $D = D(V,E)$ is a a directed graph, $c : E(D) \to \mathbb{Q}_{\geq 0}$ is a map, s a special vertex call *source* and t a special vertex called *sink*. By a *circuit* we mean $C = (v_0, e_1, v_1, e_2, v_2, \ldots, e_m, v_0)$, $\quad m \geq 0$, where $v_i \in V$ are distinct for all i, $e_j = (v_{j-1}, v_j)$ for all $j < m$, $e_m = (v_{m-1}, v_0)$. For undirected graphs the $e_j = \{v_{j-1}, v_j\}$, for all $j < m$, $e_m = \{v_{m-1}, v_0\}$.

An equivalent form of the Flow Theorem is the Hoffman Circulation Theorem. It may be stated as below.

Theorem 2.1. *(Hoffman) Let $D = (V,E)$ be a directed graph and let $\mathscr{C}$ be its collection of directed circuits. Let $u, \ell : E \to \mathbb{Q}_{\geq 0}$ satisfy $u \geq \ell \geq 0$; then the following are equivalent:*

1. *there exists* $\alpha : \mathscr{C} \to \mathbb{Q}_{\geq 0}$ *such that* $u \geq \sum_{c \in \mathscr{C}} \alpha(C) f_C \geq \ell$;

2. *for each* $X \subseteq V, u(\partial^+(X) \geq \ell(\partial^+(V - X))$.

where $f_C : E \to \mathbb{Q}_{\geq 0}$ *is the characteristic function on* C*, i.e.* $f_C(e) = 1$ *if* $e \in C$*, 0 otherwise and* $\partial^+(S) := \{(x, y) \in E : x \in S and\ y \notin S\}$*, for every* $S \subseteq V$.

For example, for the graph in Figure 7a with u, ℓ as mentioned, the assignment α as in Figure 7b satisfies both conditions of the above theorem.

Figure 7

Seymour in [1] considered undirected version of this Theorem. Given an undirected graph $G(V, E)$, and u, ℓ maps from $E \to \mathbb{Q}_{\geq 0}$ when can we write it as a "sum of circuits" that is: There exists $\alpha : \mathscr{C} \to \mathbb{Q}_{\geq 0}$ such that $u \geq \sum_{c \in \mathscr{C}} \alpha(C) f_C \geq \ell$, where $\mathscr{C}$ is the set of all circuits in $G(V, E)$.

Let $V = V_1 \cup V_2$ be a partition of the vertex set. Let $B \subseteq E(G)$ be the set of edges that have one end point in V_1 and another in V_2. This set B is call a *cut*. For any graph that can be written as a sum of circuits, it is clearly necessary that for every cut B and every edge $e \in B$ the following inequality holds (see Figure 8):

$$\ell(e) \leq \sum_{f \in B - \{e\}} u(f).$$

Seymour proved that it is sufficient.

Theorem 2.2. *(Seymour) Let* $G = (V, E)$ *be a simple graph and let* $\mathscr{C}$ *be its collection of circuits. Let* $u, \ell : E \to \mathbb{Q}_{\geq 0}$ *satisfy* $u \geq \ell \geq 0$ *and* f_C *denote the characteristic function on* C*; then the following are equivalent:*

1. *there exists* $\alpha : \mathscr{C} \to \mathbb{Q}_{\geq 0}$ *such that* $u \geq \sum_{c \in \mathscr{C}} \alpha(C) f_C \geq \ell$;

2. *for each cut* B *and each* $e \in B$; $u(B - \{e\}) \geq \ell(e)$.

For example, the graph in Figure 8 can be written as sum of circuits as in Figure 9.

We consider this question in 2-colored weighted graphs. Let $G(V, E)$ be a graph (the graph may have multiple edges) with $\mathcal{C} : E \to \{R, B\}$ (the edges are colored red or blue)

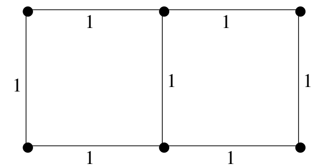


Figure 8

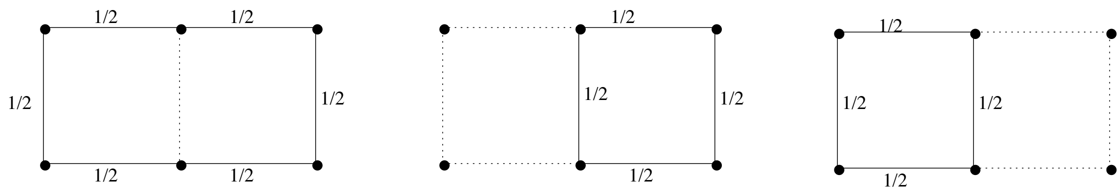


Figure 9

and two functions $u, \ell : E \to \mathbb{Q}_{\geq 0}$. In this case there is no direction on the edges but one may define "balance at a vertex". More formally, at every vertex the sum of weights of the blue edges is equal to the sum of weights of the red edges. This is similar to the notion of *conservation of flow*, where at every vertex inflow is equal to outflow. Motivated by the network flow results and the Seymour's Theorem on "Sums of Circuits" we consider the following natural question.

Question 2.1. *Given $G(V, E)$ a graph (may have multiple edges) with $\mathcal{C} : E \to \{R, B\}$, two functions $\ell, u : E \to \mathbb{Q}_{\geq 0}$ with $\ell(e) \leq u(e)$ does there exist $w : E \to \mathbb{Q}_{\geq 0}$ with*

1. *$\ell(e) \leq w(e) \leq u(e)$ and*
2. *for all vertices $v \in V$, $\sum_{e \in E_R(v)} w(e) = \sum_{e \in E_B(v)} w(e)$, where were $E_R(v)$, $E_B(v)$ denotes the set of R, B colored edges incident on v respectively.*

The key step in proving Hoffman circulation theorem is characterization of reachability from a vertex u to a vertex v by a directed path. In particular if a vertex v is reachable from vertex u then there is a simple algorithm that can produce such a path. If v is not reachable from u then there exists a partition of $V = V_1 \cup V_2$ such that $u \in V_1$, $v \in V_2$ and there are no edges are there starting from a vertex in V_1 and ending in a vertex in V_2. A similar theory is also needed to solve the above Question.

Let $G = (V, E)$ be a graph with vertex set V and edge set $E \subseteq \binom{V}{2}$. The *coloring* of the edges is given by the map $\mathcal{C} : E \to C$. Let $S \subseteq V$, called *terminals*. An *alternating trail* connecting $s, t \in V$ is defined as the sequence

$$W = (v_0 = s, e_1, v_1, e_2, v_2, \ldots, e_m, v_m = t), \qquad m \geq 0,$$

where $v_i \in V$ for all i, $e_j \in E$ for all j, e_js are distinct, and $\mathcal{C}(e_j) \neq \mathcal{C}(e_{j+1})$ for each $j = 1, \ldots, m-1$. The alternating trail W is called *closed* if $v_0 = v_m$ and $\mathcal{C}(e_m) \neq \mathcal{C}(e_1)$.

We are interested in the question: When can we find alternating trail connecting distinct terminals? Next we define Tutte Sets which are the obstacles to such trails.

A subset $A \subseteq (V - S)$ is a *Tutte set* when

(i) each component of $G - A$ has at most one terminal.

(ii) A can be written as a disjoint union $A = \dot{\bigcup}_{c \in C} A(c)$

such that conditions (a), (b), and (c) below hold.

A vertex $u \in A$ is said to have *color* c if $u \in A(c)$. An edge $e \in E$ is said to be *mismatched* if e connects a vertex $u \in A$ with a vertex $v \in V - A$ and e is different from the color of u, or e connects two vertices $u, v \in A$ and e is different from both the colors of u and of v.

Conditions (a), (b), and (c) are as follows:

(a) if H is a component of $G - A$ containing a terminal, then there is no mismatched edge with an endpoint in H.

(b) if H is a component of $G - A$ containing no terminals, then there is at most one mismatched edge with an endpoint in H.

(c) there are no mismatched edges with both endpoints in A.

Figure 10 gives an example of Tutte set.

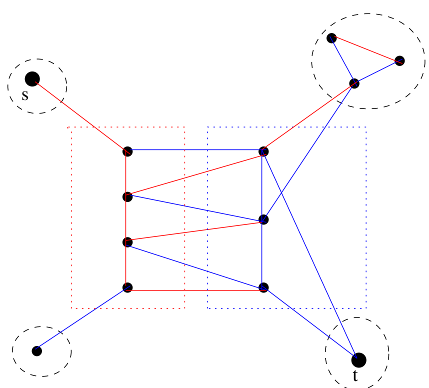
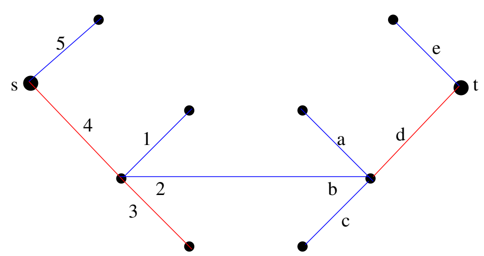


Figure 10

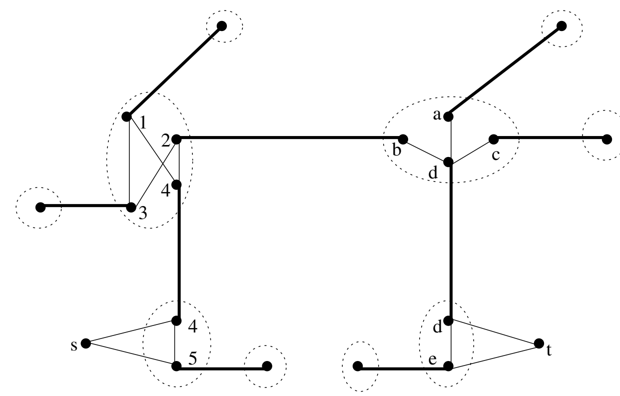
Then leads to the following theorem.

Theorem 2.3. *There is no alternating trail connecting distinct terminals in G if and only if there is a Tutte Set in G.*

Suppose there is a Tutte set A in G. Let T be an alternating $s-t$ trail in G. Then from the definition of Tutte set it follows that, the first time T enters A, it must be via an edge that is not mismatched and every time T leaves A, it must be via a mismatched edge. Thus T can never reach the destination component containing t, a contradiction. For example, in Figure 10 one can never reach from s to t via an alternating trail. To prove the converse we reduce this problem to the special case of finding maximum matching in a graph. For this we define a new graph $G' = (V', E')$ together with a matching M such that G has an alternating trail if and only if G' has an M-augmenting path. This construction is due to Jácint Szabó (private communication). See Figure 11 for an example. Then if G has no alternating trail, the matching M in G' is maximum. Using Edmonds' algorithm (see [5]), we get a "maximal blossom forest" w.r.t. M in G'. From this maximal blossom forest in G' we extract our desired Tutte set of G. The converse part can also be proved using a colored generalization of Edmonds' blossom forest algorithm, but the above approach is shorter.



(a)



(b)

Figure 11

Let $G = (V, E)$ be a bi-coloured graph with colouring $c : E \to \{R, B\}$. Then we define:

- *Alternating cone* ($\mathcal{A}(G, c)$) to be the set of all assignments of non-negative real weights to every edge so that at every vertex, the sum of weights of the incident red edges equals the sum of weights of the incident blue edges.

- *Cone of closed alternating trails* ($\mathcal{T}(G, c)$) to be the cone generated by the characteristic vectors of closed alternating trails (CAT) in (G, c).

- *Cycle cone* ($\mathcal{L}(G, c)$) to be the cone generated by the characteristic vectors of the cycles in G.

The cone $\mathcal{A}(G, c)$ (namely its dimension, extreme rays, facets etc.) was studied in [2]. Note that $\mathcal{T}(G, c) \subseteq \mathcal{A}(G, c) \cap \mathcal{L}(G, c)$: Clearly every characteristic vector of CAT is in

$\mathcal{A}(G,c)$ and if we ignore the colours, the edge-set of the CAT is a disjoint union of the edge-sets of some cycles. Conversely, let $w \in \mathcal{A}(G,c) \cap \mathcal{L}(G,c)$. Then w satisfies:

1. *balance condition*: $\sum_{e \in E_R(v)} w(e) = \sum_{e \in E_B(v)} w(e)$, at each vertex $v \in V$ (where were $E_R(v), E_B(v)$) denotes the set of R, B colored edges incident on v respectively),

2. *cut condition*: $w(e) \leq w(B - \{e\})$ for each cut B and for each edge e in B (where a *cut* is the set of edges between X and $V \setminus X$ for some $\emptyset \neq X \subsetneq V$).

Then we want to show that $w \in \mathcal{T}(G,c)$. This follows from the following theorem.

Theorem 2.4. *Let $G(V,E)$ be a 2-colored simple graph with coloring c and let $\mathcal{T}$ be the collection of closed alternating trails in G. Let $w : E \to \mathbb{Q}_{\geq 0}$ satisfies the balance and the cut condition. Then we can find, in polynomial time, an assignment $\alpha : \mathcal{T} \to \mathbb{Q}_{\geq 0}$ such that $\sum_{T \in \mathcal{T}} \alpha(T) f_T = w$, where $f_T : \mathcal{T} \to \mathbb{Q}_{\geq 0}$ denotes characteristic function on T (i.e. $f_T(e) = 1$ if $e \in \mathcal{T}$ and 0 otherwise).*

In particular, as a corollary, we get that $\mathcal{T}(G,c) = \mathcal{A}(G,c) \cap \mathcal{L}(G,c)$.

The uncolored version of the above theorem was considered in [6]. In [3] it was shown that a balanced sum of cycles is a fractional sum of balanced subgraphs.

The main ideas of the proof of the theorem are as follows.

First, we try to find a *tight cut* i.e. a cut D with an edge (called *tight edge*) $e \in D$ satisfying $w(e) = w(D \setminus \{e\})$. This is done by finding minimum cut for the graph $(V, E \setminus \{e\})$ for every edge e. This takes time $O(|E||V|^3)$.

Suppose we found a tight cut D between X and $V \setminus X$ for some $X \subseteq V$. Then we define two edge-weighted bi-coloured graphs $(G_X(e), c_1, w_1)$ and $(G_{V \setminus X}(e), c_2, w_2)$ and divide our problem into two subproblems for these graphs with fewer vertices. We recursively solve the problem for $(G_X(e), c_1, w_1)$ and $(G_{V \setminus X}(e), c_2, w_2)$ and combine their solution for a solution of the overall problem.

Now suppose no such tight cut is found. Then we find a CAT as follows. For every edge $e = \{u, v\}$, we consider the graph with edges $(E \setminus \{e\}) \cup \{\{u, s\}, \{v, t\}\}$ where s, t are new vertices and the new edges $\{u, s\}$ and $\{v, t\}$ have the same colour as e. We try to find an alternating $s - t$ trail for this new graph. If we have found such a trail then this gives us a CAT in G. Otherwise we have a Tutte set. Then while traversing the Tutte set using alternating edges we will eventually repeat a vertex since it has finitely many vertices and therefore we have found a CAT. This can be done in $O(|V|^3)$ time.

After we have chosen a closed alternating trail T, we subtract the maximum we can from the weights of each edge of T and we recursively solve the problem for resulting graph. Let $w_t := w - t$. Then we need to find maximum t such that w_t is *non-negative* and satisfies the *balance* and *cut* condition. Since T is CAT, balance condition is automatically satisfied. Let $t_1 := \min\{w(e) : e \in T\}$. To ensure non-negativity, we should have $t \leq t_1$. If w_{t_1} satisfies the cut condition then t_1 is the required maximum value. Suppose w_{t_1} does not satisfy the cut condition, then we find maximum t which does not violet cut condition. To find this maximum value we need to solve the following ratio minimization problem.

$$\text{minimize } \frac{a_0 + a_1x_1 + \cdots + a_nx_n}{b_0 + b_1x_1 + \cdots + b_nx_n} \quad \text{subject to} \quad x \in F,$$

where $a_0 = -2w(e), b_0 = -2$; for each edge f, $a_f = w(f)$ and $b_f = 1$ if $f \in T$, $b_f = 0$ otherwise. F is the set of $|E|$-dimensional 0-1 vectors such that edges corresponding to 1 form a cut.

Let θ be an unknown parameter. Define $c_j := a_j - \theta b_j$ for each $j \geq 1$. Then we consider the following linear problem

$$\text{minimize } c_1x_1 + \cdots + c_nx_n \quad \text{subject to} \quad x \in F.$$

Suppose we can find $\theta = \theta_0$ for which the above minimum equals $\theta_0 b_0 - a_0$. Then one can show that this is also the required minimum for the previous problem. Megiddo in [4] showed that such a θ_0 can be found efficiently. Note that we only need to restrict $\theta \in [0, t_1]$. This implies that $c_j \geq 0$ for all j. So the above linear problem is exactly the *min-cut problem* with capacities c_j and it is well known that this can be solved efficiently. Since we can find min-cuts in $O(|V|^3)$ time, we can find min-ratio cuts in $O(|V|^6)$ time. We end with the following conjecture.

Conjecture 1. *If a graph is balanced, all edge weights are even and satisfies the cut condition then it is integral sum of closed alternating trails.*

This is a joint work with Amitava Bhattacharya, Uri N. Peled and Murali K. Srinivasan.

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