Curves, points, incidences and covering

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Abstract

Motivated by the concept of arboricity or linear arboricity in an undirected graph that seeks to find structures, like non-crossing trees or paths, that covers edges of the graph, we pose certain problems related to points and curves in geometry. Specifically, given a point set, mostly a grid in our case, we seek upper and lower bounds on the number of curves that are needed to *cover* the point set. We say a curve *covers* a point if the curve passes through the point. We consider such coverings by monotonic curves, lines, orthoconvex curves, circles, etc. We also study a problem that is converse of the covering problem – if a set of n^2 points in $\mathbb{R}^2$ is covered by n lines then can we say something about the configuration of the points?

1 Statement of the problem

Let S_P^n denote the set of continuous curves in $\mathbb{R}^n$ satisfying property P, i.e.

$$S_P^n := \{f : [0,1] \to \mathbb{R}^n \mid f \text{ is continuous and } f \text{ satisfies property} P\}$$

.

Let C be a finite set of points in $\mathbb{R}^n$. Then we define the following:

Definition 1.1. *Given $F \subseteq S_P^n$, we say that F* ***covers*** *C if for all $x \in C$, there exists $f \in F$ such that $x \in F$.*

We study the following problem:

Problem 1.1. *Minimize $|F|$, subject to the following conditions:*

(i) F covers C,

(ii) $F \subseteq S_P^n$ and $|F| < \infty$.

Let $m(n, P, C)$ denote the minimum value of the above problem.

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2 Covering by monotonic curves

In this section we study the case where the property P is monotonicity.

Definition 2.1. *Let $f : [0,1] \to \mathbb{R}^n$ be a curve and suppose $f(t) = (f_1(t), \ldots, f_n(t))$ for $t \in [0,1]$. Then f is called **monotonic** if it satisfies the following property:*

$$t_1 \leq t_2 \ \Rightarrow\ f_i(t_1) \leq f_i(t_2) \text{ for each } i = 1, \ldots, n.$$

Given a finite subset C of $\mathbb{R}^n$, we define the poset $\mathcal{C}_P := (C, \leq_P)$ as follows. For $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y := (y_1, \ldots, y_n) \in \mathbb{R}^n$, we define $x \leq_P y$ if $x_i \leq y_i$ for $i = 1, \ldots, n$. We prove the following proposition.

Proposition 2.1. *Let $w(\mathcal{C}_P)$ denote the size of the largest antichain, called the width, of $\mathcal{C}_P$. Then*

$$m(n, P, C) = w(\mathcal{C}_P).$$

Proof. Let $x_i \in \mathcal{C}_P$ for $i = 1, \ldots, r$. Then note that $x_1 \leq_P \cdots \leq_P x_r$ is a chain if and only if $x_1, \ldots, x_r$ lie on the same curve (which is monotonic). Therefore $m(n, P, C)$ equals the number of chains in the minimal chain decomposition of $\mathcal{C}_P$, which by Dilworth's theorem equals the size of the largest antichain of $\mathcal{C}_P$. Hence $m(n, P, C) = w(\mathcal{C}_P)$. $\square$

Note that the poset $\mathcal{C}_P$ can be decomposed into $w(\mathcal{C}_P)$ many disjoint chains. Therefore the points in C can be covered by $m(n, P, C)$ many monotonic curves such that *no two curves intersect at a point of C*. From another viewpoint, if two monotonic curves intersect at a point, then one can always perturb one of the curves so that the new curves remain monotonic and cover the same points but they do not intersect.

2.1 Covering grid points

Here we consider the special case when $C = [k_1] \times \cdots \times [k_n]$, where $k_i \in \mathbb{N}$ for each i and $[k_i] := \{1, \ldots, k_i\}$. It is known that $\mathcal{C}_P$ is *graded* and *Sperner*. In fact, $\mathcal{C}_P$ is a Peck poset (i.e. rank symmetric, rank unimodal and Sperner) since it is a product of Peck posets, namely chains (see Theorem 6.2.1 of [5] for a proof of the fact that product of Peck posets is Peck). Therefore $w(\mathcal{C}_P)$ equals the size of the largest rank of $\mathcal{C}_P$, i.e.

$$w(\mathcal{C}_P) = \max_m A_m = A_{\lfloor (k_1 + \cdots + k_n + n)/2 \rfloor},$$

where, A_m equals the number of solutions of the equation $x_1 + \cdots + x_n = m$ such that $x_i \in [k_i]$ for each $i = 1, \ldots, n$. We mention the following two special cases.

(i) When $n = 2$ (i.e. C is a 2×2 grid),

$$w(\mathcal{C}_P) = \max_m A_m = \min\{k_1, k_2\}.$$

(ii) When $k_1 = \cdots = k_n = 2$ (i.e. $\mathcal{C}_P$ is a Boolean lattice),

$$w(\mathcal{C}_P) = \max_m A_m = A_{\lfloor n + \frac{n}{2} \rfloor} = \binom{n}{\lfloor n/2 \rfloor}.$$

2.2 Covering with multiplicity

Theorem 2.1. *Let m be the minimum number of monotonic curves required to cover a finite set of points $C \subseteq \mathbb{R}^n$. Let m_k be the minimum number of monotonic curves required to cover C such that every point is covered at least k times. Then $m_k = mk$.*

Proof. Let $C^{(i)} := \{x^{(i)} : x \in C\}$, for $i = 1, \ldots, k$ be k disjoint copies of C. Let $S := \cup_{i=1}^k C^{(i)}$. We define a partial order $\leq$ on S as follows: $x^{(i)} \leq y^{(j)}$ if and only if $x \leq_P y$. Then we have that $m_k = w(S)$. Now note that $\{x_1, \ldots, x_t\}$ is an antichain of C if and only if $\{x_i^{(j)} : i = 1 \ldots, t; j = 1, \ldots, k\}$ is an antichain of S. Therefore it follows that $w(S) = k \cdot w(\mathcal{C}_P)$ and hence by previous proposition $m_k = km$. $\square$

3 Covering by lines

In this section we consider the case where the property P is linearity and C is finite. Note that, for any two points there exists a line covering them. Therefore $m(n.P, C) \leq \frac{|C|}{2}$ (the equality is achieved for any generic/random set C). Now let $\ell(C)$ denote the maximum number of points in C any line can cover. Then we have $m(n, P, C) \geq |C|/\ell(C)$. Therefore we get

$$\frac{|C|}{\ell(C)} \leq m(n, P, C) \leq \frac{|C|}{2}. \tag{3.0.1}$$

3.1 Covering grid points

Here we consider the case when $C = [k_1] \times \cdots \times [k_n]$. We prove the following:

Claim. $\ell(C) = \max\{k_1, \ldots, k_n\}$.

Proof. Let $M := \max\{k_1, \ldots, k_n\}$. First we show that $\ell(C) \leq M$ by induction on n. The base case $n = 1$ is obvious. Now we proceed to the induction step. Let L be a line segment that lies inside the rectangular parallelopiped $[1, k_1] \times \cdots \times [1, k_n]$. Then L has length at most $\sqrt{\sum_{i=1}^{n}(k_i-1)^2}$. Now let $x := (x_1, \ldots, x_n)$ and $y := (y_1, \ldots, y_n)$ be two distinct points of C lying on L. If $x_i = y_i$ for some i, then L lies inside a lower dimensional rectangular parallelopiped and therefore by induction hypothesis L covers at most $\max\{k_j \mid j \neq i\} \leq M$ many points. So let us assume $x_i \neq y_i$ for all $i = 1, \ldots, n$. Then the distance between x and y is at least $\sqrt{n}$. Suppose L covers total t points of C. Then we have

$$(t-1)\sqrt{n} \;\leq\; \sqrt{\sum_{i=1}^{n}(k_i-1)^2} \;\leq\; \sqrt{n} \cdot \max\{k_1-1, \ldots, k_n-1\}$$

and this implies $t \leq \max\{k_1, \ldots, k_n\}$. Therefore we conclude that $\ell(C) \leq M$. On the other hand, there clearly exist lines covering M points, namely the lines parallel to the coordinate axis i_0. where $M = k_{i_0}$. Hence we have shown that $\ell(C) = M$. $\square$

So the previous claim implies that

$$m(n,P,C) \geq \frac{\prod_{i=1}^{n} k_i}{M} \geq \min\left\{\prod_{i\neq 1} k_i, \ldots, \prod_{i\neq n} k_i\right\} := N.$$

On the other hand $m(n,P,C) \leq N$, since there clearly exist an explicit covering of C by N many lines (namely by the lines parallel to the coordinate axis i_0). Therefore we get that

$$m(n,P,C) = \min\left\{\prod_{i\neq 1} k_i, \ldots, \prod_{i\neq n} k_i\right\}.$$

Next we mention the following two special cases.

(i) For $C = [k_1] \times [k_2]$ (2×2 grid),

$$m(n,P,C) = \min\{k_1, k_2\}.$$

(ii) For $C = \{0,1\}^n$ (Hypercube),

$$m(n,P,C) = 2^{n-1}.$$

Note that, for Hypercube both inequalities of (3.0.1) become tight.

3.2 Covering with multiplicity

Let m_k denote the minimum number of lines required to cover a set $C \subseteq \mathbb{R}^n$ such that every point is covered at least k times. Then $m_k \geq k|C|/\ell(C)$. The equality is achieved for grid.

3.3 On the converse of the covering problem

We have seen that $n \times n$ grid can be covered by n lines. Here we look at the converse question, namely, if a set of n^2 points in $\mathbb{R}^2$ is covered by n lines then can we say something about the configuration of the points?

Suppose a set of n^2 points is covered by n lines. Then there exists a line containing $\Omega(n)$ points, since otherwise the total number of points is less than n^2. Now if this line contains $o(n^2)$ points then there exists another line containing $\Omega(n)$ points. By continuing this, we can say that there exists a set of lines each containing $\Omega(n)$ points such that the total number of points in the union of these lines is $\Theta(n^2)$.

Now the following question seems natural. If a set of n^2 points is covered by n lines, then does there exist a subset of points of size $\Theta(n^2)$ which can be put inside a grid of size $\Theta(n^2)$, possibly after applying a projective transformation? We show that the answer is no.

Theorem 3.1. *There exists a finite set of n^2 points in $\mathbb{R}^2$ which can be covered with n lines but no subset of the points of size $\Omega(n^2)$ can be contained in a projective transformation of a rectangular grid of size $o(n^3)$.*

Proof. Given any two distinct points $p, p' \in \mathbb{R}^2$, we denote by $\ell(p, p')$ the unique line in $\mathbb{R}^2$ that contains both p and p'. By an $s \times t$ grid, we mean a point set that can be obtained by a projective transform f of the set $\{1, 2, \ldots, t\} \times \{1, 2, \ldots, s\}$. By a "horizontal line" of the grid, we mean a line $\ell(f(1, j), f(t, j))$ for some $j \in \{1, 2, \ldots, s\}$, and by a "vertical line" of the grid, we mean a line $\ell(f(i, 1), f(i, s))$, for some $i \in \{1, 2, \ldots, t\}$. The "size" of an $s \times t$ grid is st, i.e. the number of points in it. Note that every horizontal line of a grid intersects every vertical line of the grid (since there is a point of the grid that is contained in both of them).

For each $i \in \{1, 2, \ldots, n\}$, let L_i denote the line with equation $y = i$ and let $\mathcal{L} = \{L_i\}_{1 \le i \le n}$. Let $\mathcal{P}$ be the set of points defined as follows. Define P_1 to be some set of n distinct points from the line L_1. For each $1 < i \le n$, we define P_i to be a set of n distinct points from L_i that do not lie on any of the lines in $\{\ell(p, p') \colon p \neq p' \text{ and } p, p' \in \bigcup_{1 \le j \le i-1} P_j\}$. Let $\mathcal{P} = \bigcup_{1 \le i \le n} P_i$. Let $m = |\mathcal{P}|$. Note that we have $|\mathcal{L}| = n$. Clearly, $m = I(\mathcal{P}, \mathcal{L}) = n^2$. We claim that for any $\mathcal{P}' \subseteq \mathcal{P}$ such that $|\mathcal{P}'| = \Omega(m) = \Omega(n^2)$, any grid that contains all the points of $\mathcal{P}'$ has size $\Omega(n^3)$.

Note that by our construction, if any line contains two points $p, p' \in \mathcal{P}$ such that $p \in P_i$ and $p' \in P_j$, where $i \neq j$, then p and p' are the only points in $\mathcal{P}$ that are contained in that line. This implies that the following property is satisfied by $\mathcal{P}$ and $\mathcal{L}$.

(*) Any line in $\mathbb{R}^2$ that contains more than two points in $\mathcal{P}$ belongs to $\mathcal{L}$.

Since every line in $\mathcal{L}$ contains exactly n points of $\mathcal{P}$, we then have another property.

(+) Any line in $\mathbb{R}^2$ contains at most n points in $\mathcal{P}$.

Let $\mathcal{P}' \subseteq \mathcal{P}$ be such that $|\mathcal{P}'| = \Omega(n^2)$. Consider any grid $\mathcal{G}$ that contains all the points of $\mathcal{P}'$. Let $\mathcal{G}$ be an $s \times t$ grid. Let $h_1, h_2, \ldots, h_s$ denote the horizontal lines of $\mathcal{G}$ and let $v_1, v_2, \ldots, v_t$ denote the vertical lines of $\mathcal{G}$. Suppose for the sake of contradiction that there exist $i \in \{1, 2, \ldots, s\}$ and $j \in \{1, 2, \ldots, t\}$ such that both the lines h_i and v_j contain at least 3 points of $\mathcal{P}$ each. Then by property (*), h_i and v_j are both lines in $\mathcal{L}$. But as h_i and v_j intersect, they are two lines in $\mathcal{L}$ that intersect, which is a contradiction, since the lines in $\mathcal{L}$ are all parallel to each other. Thus, we can conclude without loss of generality that for each $i \in \{1, 2, \ldots, s\}$, the horizontal line h_i of $\mathcal{G}$ contains at most two points from $\mathcal{P}$, and hence at most two points from $\mathcal{P}'$. Since every point in $\mathcal{P}'$ is contained in at least one horizontal line of $\mathcal{G}$, we have that $s \geq |\mathcal{P}'|/2$ and therefore $s = \Omega(n^2)$. By property (+), each vertical line of $\mathcal{G}$ can contain at most n points of $\mathcal{P}'$, and therefore, $t \geq |\mathcal{P}'|/n$, which implies that $t = \Omega(n)$. Thus the size of the grid $\mathcal{G}$ is $st = \Omega(n^3)$.

□

Remark 1. *The above construction also provides a counter-example to the Conjecture 1.16 in [9].*

4 Covering by skew lines

We say that a line is *skew* if it is not parallel to x or y axis. We look at the question of covering $n \times n$ grid by minimum number of skew lines.

Note that the boundary of the $n \times n$ grid contains $4n - 4$ points. Now any skew line can contain at most 2 points from the boundary. So we need at least $2n - 2$ skew lines to cover the grid. Also note that the $n \times n$ can be covered by $2n - 2$ skew lines (consider the $2n - 3$ lines parallel to off-diagonal except the ones which pass through bottom-left and top-right corner and these two corners are covered by the main diagonal).

Open problem: What is the minimum number of skew hyperplanes required to cover the d-dimensional hypercube?

Current (2021) best known lower bound for the above problem is $\Omega(n^{0.51})$ by Yehuda and Yehudayoff ([11]).

5 Covering by algebraic curves

The minimum number of algebraic curves of degree k required to cover the $n \times n$ grid is equal to n/k. The lower bound follows from Combinatorial Nullstellensatz theorem (alon) and the upper bound follows by covering by lines and then considering a set of k lines as one curve of degree k.

By a result of Bombieri ([4]), an *irreducible* algebraic curve of degree k can contain at most $On^{1/k})$ points from $n \times n$ grid and hence minimum number of irreducible algebraic curves of degree k to cover the $n \times n$ grid is at least $\Omega(n^{2-1/k})$.

6 Covering by circles

A circle contains at most $O(n^\epsilon)$ points from an $n \times n$ grid for every $\epsilon > 0$ (see e.g. [6]). Therefore minimum number of circles required to cover $n \times n$ grid is $\Omega(n^{2-\epsilon})$, for every $\epsilon > 0$.

Regarding upper bound, note that there is a covering of the $n \times n$ grid by $O(n^2/\sqrt{\log n})$ circles. This is obtained by choosing a corner of the grid and drawing all concentric circles such that each of them is incident to at least one grid point. The number of such circles is $O(n^2/\sqrt{\log n})$ by a well known theorem of Ramanujan and Landau ([3],[7]).

7 Covering by convex curves

If C is a convex closed curve, then we say that it intersects non-trivially with a horizontal grid line if it contains more than two points from the line. Note that, any convex closed curve can intersect at most two horizontal grid lines non-trivially. This follows from the following lemma.

Lemma 7.1. *If a convex closed curve intersects a horizontal grid line non-trivially then it must lie entirely on one side of that line.*

Proof. Suppose the curve intersects a horizontal line at three points p, q, r, where q lies in the interior of line segment $[p, r]$. Since the curve is convex, there exists a line L through q such that the curve lies entirely on one side of L (Hyperplane seperation theorem). Now if L is different from the horizontal line then p and r lie on different sides of L. But since the curve lies on one side of L, it can not pass through both p and r, a contradiction. Therefore L is same as the horizontal line and the curve lies entirely on one side of this line. $\square$

Theorem 7.1. *The points of the $n \times n$ grid cannot be covered with less than $n/2$ convex closed curves.*

Proof. Suppose for the sake of contradiction that $C_1, C_2, \ldots, C_k$ are convex closed curves such that they together cover every point of the $n \times n$ grid and that $k < n/2$. Then since there are n horizontal grid lines, and by the observation above, each C_i can have a non-trivial intersection with at most 2 horizontal grid lines, we can conclude that there is some horizontal grid line such that no curve in $C_1, C_2, \ldots, C_k$ has a non-trivial intersection with that line. Now consider the points on that horizontal line. There are n points on this line. Each curve in $C_1, C_2, \ldots, C_k$ can cover at most two points from that line, since none of them intersects non-trivially with this horizontal line. But then since $k < n/2$, there must be some point on this horizontal line that is not covered by any curve in $C_1, C_2, \ldots, C_k$, which is a contradiction. $\square$

Almost same argument can be used to get an answer for $m \times n$ grid and this will be $\min\{m/2, n/2\}$.

Definition 7.1. *In $\mathbb{R}^n$, we say that a closed convex hypersurface intersects a hyperplane non-trivially if it intersects the hyperplane in at least $n+1$ points such that one of these $n+1$ points lie in the interior of the covex hull of the rest of the n points.*

With this definition the same argument (as in 2d case) will go through. So finally we have the following theorem by inductive argument.

Theorem 7.2. *The minimum number of closed convex hypersurfaces required to cover the $k_1 \times \cdots \times k_n$ grid is* $\min\{k_1/2 \ldots, k_n/2\}$.

7.1 Covering by strictly convex curves

Any strictly convex curve can contain $O(n^{2/3})$ points of an $n \times n$ grid by a theorem of Andrews ([2]). Therefore, we need $\Omega(n^{4/3})$ strictly convex curves to cover an $n \times n$ grid.

8 Covering by orthoconvex curves

If the boundary of orthogonal convex-hull (of a set of points) is a simple closed curve then we call it an "orthoconvex" curve. By "inner corner" of an orthoconvex curve we mean a point where the curve turns by 270 degrees.

If an orthoconvex curve (with k inner corners) covers a set of points then there is also an orthoconvex curve (with k inner corners) covering the same points which is not self-intersecting and all the corners are grid points. This can be done by pushing the sides/edges of the curve "outwards" (instead of inwards which corresponds to taking orthoconvex hull) till we hit a grid line.So w.l.o.g. we may impose following assumptions 'non-self-intersecting' and 'corners are grid points'.

In the following, by"curve", we mean an orthoconvex curve having at most one inner corner. We say that a curve "hits" a (horizontal or vertical) grid line if the curve has a non-trivial intersection with that grid line (i.e. the curve follows that grid line for some distance, rather than just crossing it). We say that a collection of curves C hits a (horizontal or vertical) grid line if there is some curve in C that hits that grid line. Given a collection of curves C, we say that a grid point is "exposed" (by C) if the grid point is not covered by any curve in C, but it lies on a horizontal grid line and a vertical grid line both of which are hit by C. Given a collection of curves C, a "corner" of C is a corner of the (minimum size) bounding box of C. So every collection C of curves has exactly 4 corners. If a corner of C is an exposed grid point, then we call it an "exposed corner". We say that a sequence of curves $c_1, c_2, \ldots, c_t$ is "good" if for every $i \in \{2, 3, \ldots, t\}$, c_i hits a grid line that is hit by $\{c_1, c_2, \ldots, c_{i-1}\}$. Clearly, every prefix of a good sequence is also a good sequence.

Lemma 8.1. *Let $c_1, c_2, \ldots, c_t$ be a good sequence of curves. Then $\{c_1, c_2, \ldots, c_t\}$ either: (a) hits at most 5t grid lines, or (b) hits $5t+1$ grid lines and has an exposed corner.*

Proof. We prove this by induction on t. It is not difficult to see that the lemma is true when $t = 1$. Let $i > 1$ and suppose that the lemma is true for the good sequence $c_1, c_2, \ldots, c_{i-1}$. Let $C = \{c_1, c_2, \ldots, c_{i-1}\}$. Then either C hits (a) at most $5i-5$ grid lines, or (b) hits $5i-4$ grid lines and has an exposed corner. In case (a), since the curve c_i can hit at most 5 grid lines that are not hit by C (recall that c_i hits at least one grid line that is also hit by C), we have that $C \cup \{c_i\}$ can hit at most $5i$ grid lines, and we are done. Next, let us consider case (b). Note that if c_i is a rectangle, then it can hit at most 3 grid lines that are not hit by C, and therefore, $C \cup \{c_i\}$ hits at most $5i-1$ grid lines, and we are done. So we can assume that c_i is not a rectangle. Also, if there are two grid lines that are hit by both C and c_i, then $C \cup \{c_i\}$ hits at most $5i$ grid lines, and we are done. So we can assume that c_i hits exactly one grid line that is hit by C, and therefore $C \cup \{c_i\}$ hits exactly $5i+1$ grid lines. In this case, we have to show that one of the corners of $C \cup \{c_i\}$ is exposed. Let B be the bounding box of $C \cup \{c_i\}$. Let g_0, g_1, g_2, g_3 be the grid lines on which the top, right, bottom, and left borders of B lie. Clearly, each of g_0, g_1, g_2, g_3 is hit by either C or c_i or both. Since c_i hits exactly one grid line that is hit by C, we have that at most one of g_0, g_1, g_2, g_3 is hit by both C and c_i. This implies that C and $\{c_i\}$ do not have shared corners. Note that a corner v of C is exposed, and a corner v' of $\{c_i\}$ is exposed. If each of g_0, g_1, g_2, g_3 is hit by C, then v is an exposed corner of $C \cup \{c_i\}$ (observe that v cannot be covered by c_i, because if it is, it has to be a corner of $\{c_i\}$, which would mean that C and $\{c_i\}$ have a shared corner) and we are done. Similarly, if each of g_0, g_1, g_2, g_3 is hit by c_i, then v' is an exposed corner of $C \cup \{c_i\}$ and we are again done. Thus we can assume that neither C nor c_i hits all the grid lines g_0, g_1, g_2, g_3. Recall that all grid lines except at most one in g_0, g_1, g_2, g_3 are hit by exactly one of C or c_i. Then there exists some $j \in \{0, 1, 2, 3\}$ such that one of $g_j, g_{j+1 \mod 4}$ is hit by C and not by c_i, and the other is hit by c_i and not by C. Then the grid point that is contained in both the grid lines g_j and $g_{j+1 \mod 4}$ is an exposed corner of $C \cup \{c_i\}$. This completes the proof.

□

Theorem 8.1. *If m orthoconvex curves with at most one inner corner cover the $n \times n$ grid, then $m \geq 2n/5$.*

Proof. Let C be a collection of m curves that cover the $n \times n$ grid. For two curves c and $d \in C$, we say that cRd if there is a grid line that is hit by both c and d. Let R^* be the transitive closure of R. Clearly, R^* is an equivalence relation. Let $S_1, S_2, \ldots, S_p$ be the equivalence classes of R^*.

CLAIM 1. For each $i \in \{1, 2, \ldots, p\}$, S_i does not expose any grid point.

Suppose for some $i \in \{1, 2, ..., p\}$, S_i exposes a grid point v. That is, v is not covered by S_i, but both the horizontal grid line as well as the vertical grid line that contains v are hit by S_i. Since C covers the whole grid, there is a curve $c \in C$ that covers v. As S_i does not cover v, we have that $c \in C - S_i$. As c covers v, c hits either the horizontal grid line containing v or the vertical grid line containing v. Since both these grid lines are hit by S_i, it follows that there exists some $d \in S_i$ such that c and d hit a common grid line. Then dRc, which implies that $c \in S_i$, which is a contradiction. This proves the claim.

CLAIM 2. The curves of each S_i can be arranged in a good sequence.

Let G be the graph with vertex set S_i and edge set R restricted to S_i. By enumerating the curves of S_i in the order in which they are visited by a graph traversal algorithm starting from an arbitrary vertex, we get a sequence of the curves in S_i such that before a curve c is encountered in the sequence, we encounter some curve d such that dRc (except for the first curve in the sequence). This sequence is clearly a good sequence of the curves in S_i. This proves the claim.

By Lemma and Claim 1, we know that for each $i \in 1, 2, ..., p$, S_i hits at most $5|S_i|$ grid lines. Thus the total number of grid lines that are hit by C is at most $5(|S_1|+|S_2|+\cdots+|S_p|) = 5|C| = 5m$. Since the curves in C have to hit $2n$ grid lines, we then have $5m \geq 2n$, which gives $m \geq 2n/5$. This proves the theorem. □

Note that the inequality of the above theorem is tight for $n = 5$. It is easy to check that 5×5 can be covered by 2 curves.

As a consequence of the above theorem we also get the following theorem on orthoconvex curves with at most 2 inner corners.

Theorem 8.2. *We need at least $2n/7$ orthoconvex curves with at most two inner corners to cover $n \times n$ grid*

Proof. Decompose each such cure into an orthoconvex curves with at most one inner corner and a rectangle. Now apply previous theorem. □

9 Covering by non-congruent curves

In the following we say that two curves are "non-congruent" if they are not translates of each other. We denote the maximum number of incidences between m points and n curves satisfying property P by $I_P(m, n)$.

Proposition 9.1. *Suppose $n \times n$ grid is covered by a set S of non-linear, non-congruent curves such that the set of curves $S + \mathbb{Z}^2$ has property P. Then $I_P(4n^2, |S|n^2) \geq n^4$.*

Proof. We consider the collection of curves obtained by translating S by x for all x in the $n \times n$ grid. This is our new set of curves. We also translate the grid by all such x, which gives our new set of points. Now note that, for this new collection of points and curves, we have $4n^2$ points, $|S|n^2$ curves and at least n^4 incidences. Therefore we have proved the proposition. □

In the following we say that a set of curves has k *degrees of freedom and multiplicity type* s if any two curves intersect at most k points and for any k points there are at most s curves passing through all of of them. Let $I_{k,s}(m,n)$ denote the maximum number of incidences between m points and n curves satisfying the above property.

Theorem 9.1. *Suppose the $n \times n$ grid is covered by a set S of non-linear, non-congruent curves such that $S+\mathbb{Z}^2$ has 2 degrees of freedom and multiplicity type c (where c is a constant w.r.t. n). Then $|S| = \Omega(n^2)$.*

Proof. Applying the proposition we have that $I_{2,c}(4n^2, |S|n^2) \geq n^4$. By a result of Pach and Sharir ([8]) we have that $I_{2,c}(m,n) = O(m^{2/3}n^{2/3} + m + n)$. Plugging this in the previous inequality and cancelling n^2 from both sides we obtain $n^2 = O(|S|^{2/3}(n^{2/3} + |S|^{1/3}))$. Now since $|S| \leq n^2$ we get $n^2 = O(|S|^{2/3}n^{2/3})$ and from this we directly obtain $|S| = \Omega(n^2)$. $\square$

9.1 Covering by circles of different radii

Let $I_C(m,n)$ denote the maximum number of incidences between m points and n circles. The following conjecture is well known (see e.g. [10]).

Conjecture 9.1. $I_C(m,n) = O(m^{2/3}n^{2/3}\log^c(mn) + m + n)$ *for some constant c.*

We will show that the above conjecture implies the following conjecture on covering.

Conjecture 9.2. *If the $n \times n$ grid is covered by m circles such that no two of them have equal radius, then $m = \Omega(n^2/\log^c(n))$ for some constant c.*

Proposition 9.2. *The former conjecture implies the later.*

Proof. Plugging in the bound of $I_C(m,n)$ of the former conjecture in the proposition and cancelling n^2 from both sides we obtain $n^2 = O(m^{2/3}(n^{2/3}\log^c(mn^4)+m^{1/3}))$. Now since $m \leq n^2$ we get $n^2 = O(m^{2/3}n^{2/3}\log^c(n^6))$ and from this we directly obtain the later conjecture. $\square$

Regarding upper bound, note that there is a covering of the $n \times n$ grid by $O(n^2/\sqrt{\log n})$ circles of different radii. This is obtained by choosing a corner of the grid and drawing all concentric circles such that each of them is incident to at least one grid point. The number of such circles is $O(n^2/\sqrt{\log n})$ by a well known theorem of Ramanujan and Landau ([3],[7]).

10 Covering by small curves

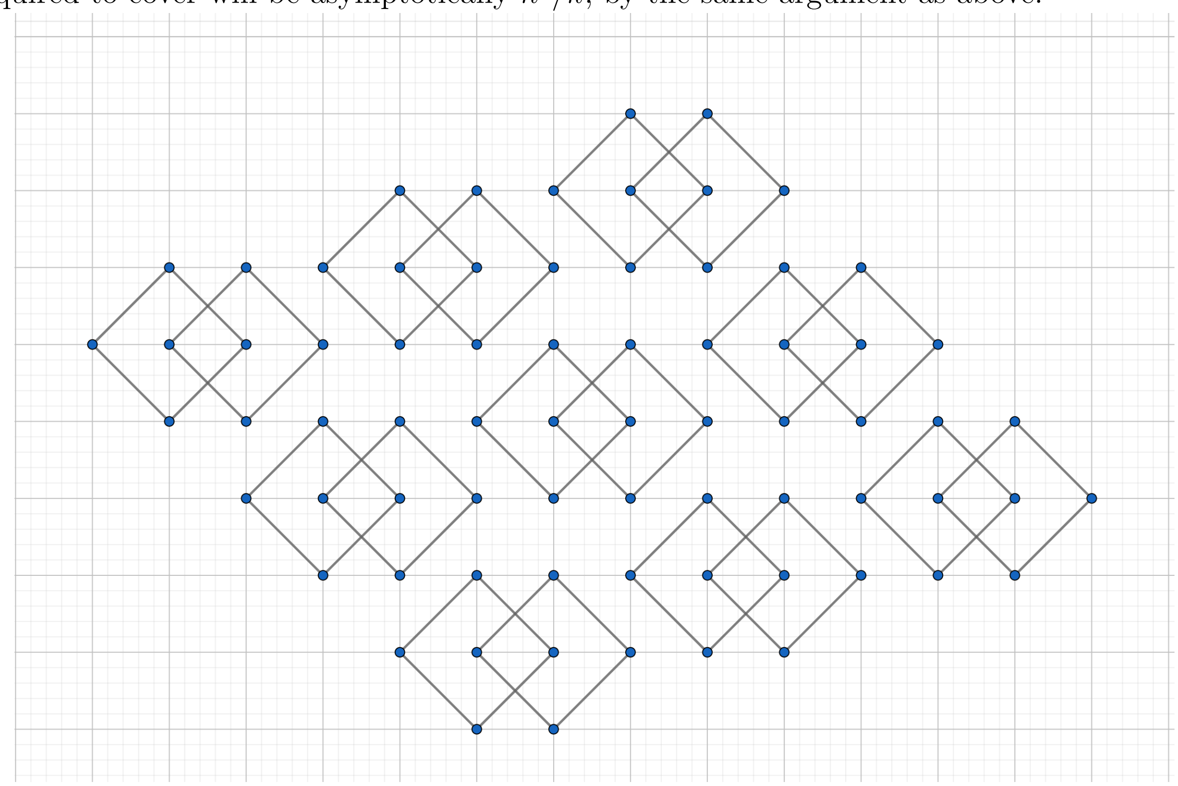
The infinite grid $\mathbb{Z} \times \mathbb{Z}$ can be tiled with unit-circles/diamonds and also with smallest L curves (see the attached images). From this one can deduce the following.

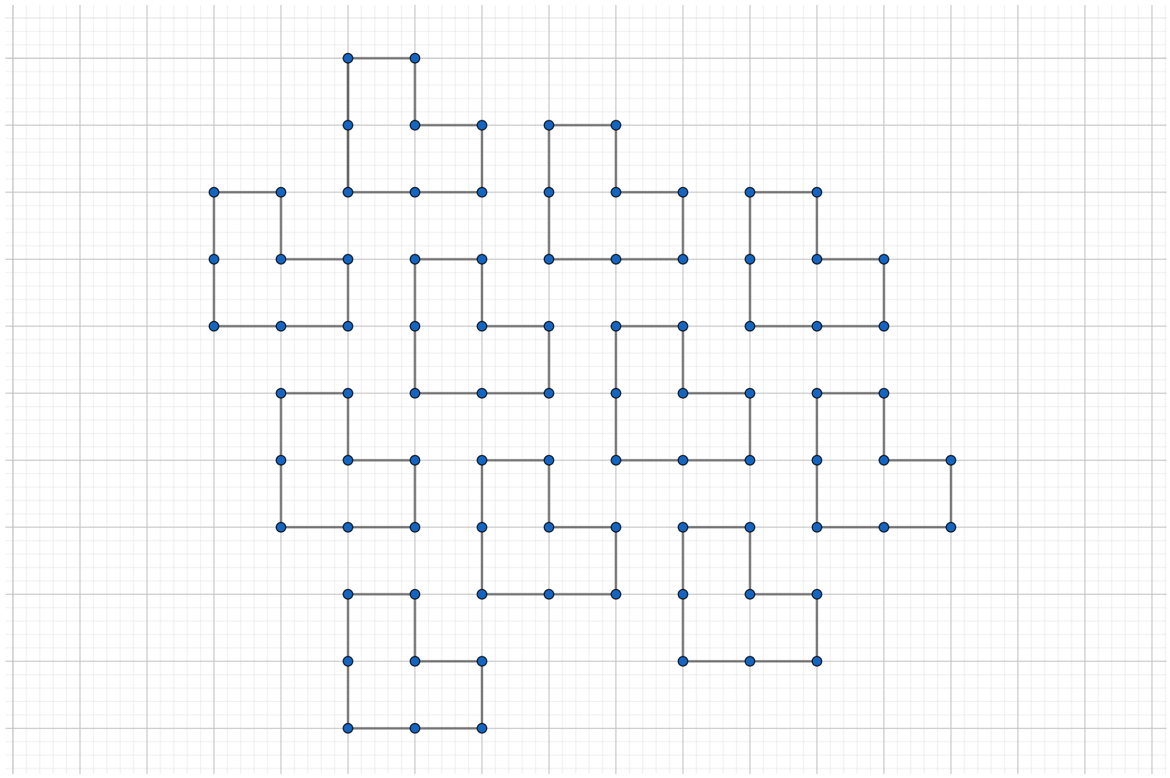
Theorem 10.1. *The $n \times n$ grid can be covered by $(n^2/4 + 2n)$ unit-circles/diamonds and $(n^2/8 + n)$ smallest L curves.*

Proof. Let me argue for the diamond case, the argument for L is essentially same. Let S be the set of all diamonds tiling $\mathbb{Z} \times \mathbb{Z}$. Let C be a minimal subset of S which covers the $n \times n$ grid. Then union of curves in C can contain at most $n^2 + 8n$ grid points (since there can be at most $2n$ extra grid points for each of the 4 boundaries). Now since every curve in C covers exactly 4 points and no point is shared by any two curves, we get that size of C is at most $(n^2 + 8n)/4 = n^2 + 2n$. □

Also clearly we need at least $n^2/4$ unit-circles and $n^2/8$ smallest L to cover the n×n grid. So as a corollary we get that, minimum number of unit-circles and L's required to cover the n×n grid is equal to $n^2/4$ and $n^2/8$ respectively, ignoring lower order terms (i.e. when n is large).

More generally, suppose we have a fixed "small" curve containing at most, say, k grid points (where k is constant w.r.t. n) and we want to cover the n×n grid by the copies of this curve. If $\mathbb{Z} \times \mathbb{Z}$ admits a tiling by translates of this curve (it will be interesting to ask for which curves such tiling exists), then we can conclude that the minimum number of curves required to cover will be asymptotically n^2/k, by the same argument as above.



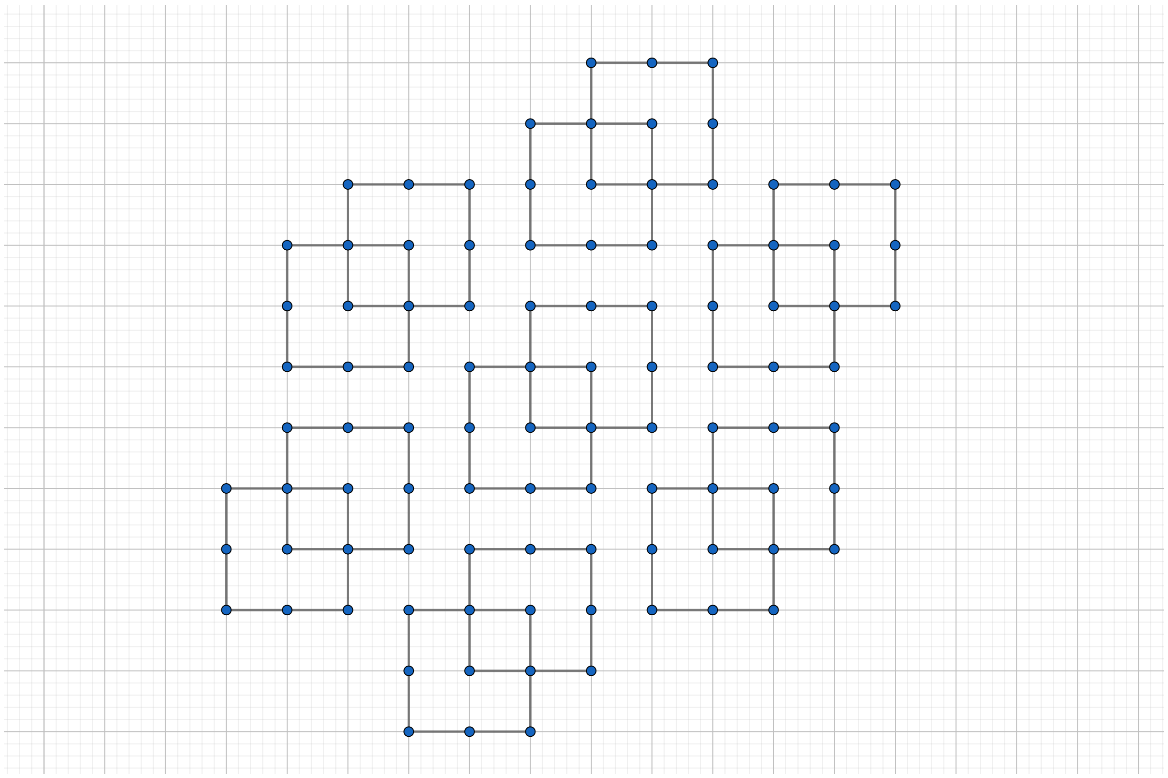


Similarly, one can also show that that minimum number of squares of length 2 needed to cover the n×n grid is equal to $n^2/7$ (asymptotically).

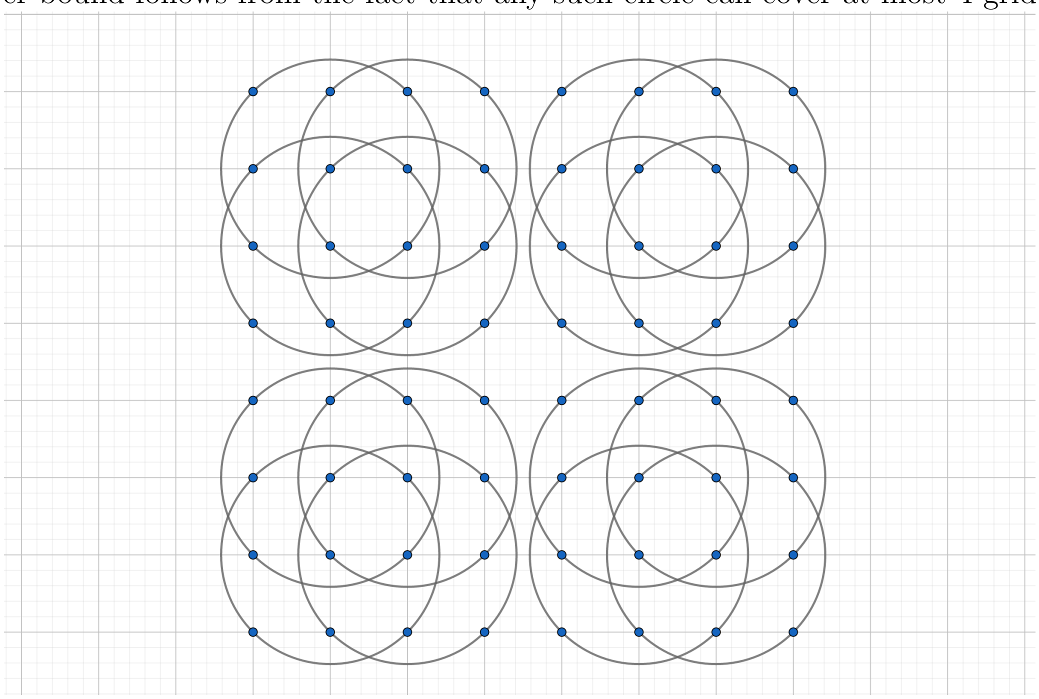
The upper bound follows from the "tile like" covering of $\mathbb{Z} \times \mathbb{Z}$ as in the attached image. For lower bound we argue as follows.

Theorem 10.2. *If $n \times n$ grid is covered by m squares of length 2, then $m \geq n^2/7$.*

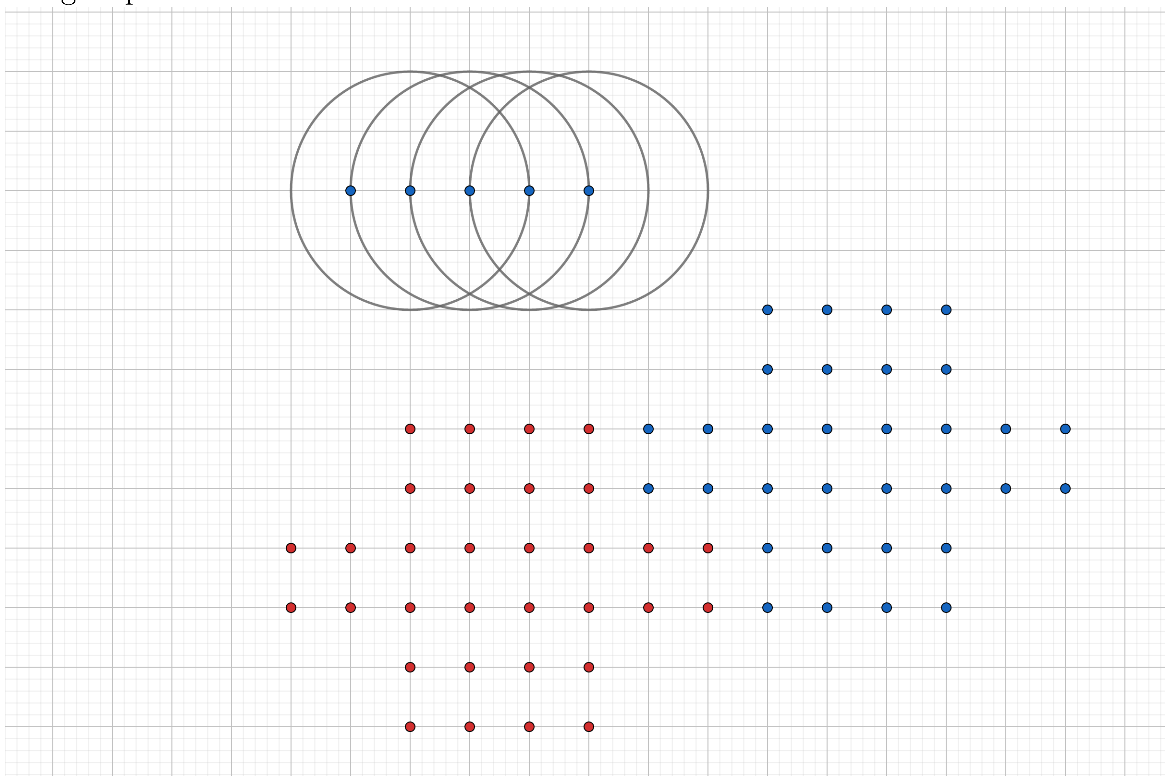
Proof. W.l.o.g we can assume that the squares have integral corners. Let us take a covering by m such squares and let C be the set of these squares. Then we define the graph G whose vertex set is C and c, c' in C are connected by an edge if the center of c is covered by c'. Let X be a connected component of G. By choosing a spanning tree of X and applying breadth-first search we can arrange vertices of X in a sequence $(c_1, c_2, \ldots, c_t)$ such that c_i is adjacent to c_j for some $j < i$, i.e. for each $i > 1$, center of c_i is covered by some curve appearing before c_i in the sequence. Now by induction on t, one can easily show that the curves $c_1, c_2, \ldots, c_t$ together can cover at most $7t$ grid points. Then summing over all connected components, we get that the curves in C together can cover at most $7m$ grid points. So we must have that $7m >= n^2$. □



For circles with radius $\sqrt(2)$, the answer is $n^2/4$. Since every 4×4 grid can be covered by 4 such circles (see attached image), we have the upper bound $4(n/4)^2 = n^2/4$. And the lower bound follows from the fact that any such circle can cover at most 4 grid points.



For circles of radius 2 also we have a tiling of $\mathbb{Z} \times \mathbb{Z}$ (see attached image). So we get an asymptotic upper bound $n^2/4$. This is also a lower bound since any such circle can cover at most 4 grid points.



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