On crossing families of pseudoline segments and triangles

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Abstract

In this paper we prove some results on crossing families. Pach and Solymosi gave a characterization of the planar point configurations that admits a perfect cross-matching. In this paper we extend their result to pseudoconfigutation of points. We also give lower bound on the number of pairwise crossing pseudoline segments in terms of the number of halving pseudolines. Lastly, we give an efficient algorithm that decides whether a set of points in general position in the plane admits a partition into vertex-disjoint pairwise star-crossing triangles.

1 Introduction

The study of "crossing families" of geometric objects started with the work of Aronov, Erdös, Goddard, Kleitman, Klugerman, Pach and Schulman [3], where they investigated the problem of finding a large size set of pairwise crossing segments for a given set of points in general position in the plane. The prevailing conjecture ([4]) is that any set of n points in general position in the plane determines at least $\Omega(n)$ parwise crossing segments. While this conjecture is still open, a major progress was made recently in [5] by Pach, Rubin and Tardos, where they show the existence of $n^{1-o(1)}$ pairwise crossing segments. We refer the reader to this paper and the references therein for more on this problem. A related problem of natural interest is to characterize the planar point configurations (of size, say $2n$) which has the maximum number (n) of pairwise crossing segments (in other words, it admits a "perfect cross-matching"). Pach and Solymosi in [6] gave such a characterization in terms of the number of halving lines, which in turn leads to a $O(n \log n)$-time algorithm to decide the existence of such a matching.

A natural and important generalization of a planar point configuration is the so called pseudoconfiguration of points, which is given by an n-element point-set in the plane and a

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collection of $\binom{n}{2}$ x-monotone continuous curves (called pseudolines) through every pair of points, such that, every pair of curves intersects at most once. Note that, if we take the curves to be straight lines then we have our normal point-configuration in general position (i.e. no three points are collinear). For more on pseudoline arrangements and pseudoconfiguration of points see [7],[8] and [9]. Understanding crossing patterns of edges of topological graphs has been explored in the literature [10, 11, 12, 13, 14, 15, 16], although there are very few tight results in this direction. Perhaps the best know theorem of this kind is the so-called *Crossing Lemma* [17], which states that any topological graph having n vertices and m edges with $m > 4n$ determines at least $\Omega(m^3/n^2)$ crossing pairs of edges. Now in the same sprit of [6] we may ask for a characterization of the pseudoconfiguration of points of size $2n$ that has the maximum number (n) of pairwise crossing pseudoline segments. We give such a characterization in Theorem 2.1 and this immediately gives us an efficient algorithm to decide the existence of a perfect cross-matching for pseudoline segments.

Crossing families of other geometric objects, e.g. paths, triangles, elbows, cycles etc. have also been explored in the literature [18, 19, 20], of which the case of triangles is particularly interesting. It was shown in [19] that any n points in general position span $\lfloor n/3 \rfloor$ vertex disjoint triangles that are pairwise crossing (meaning that their boundaries have pairwise non-empty intersections). Moreover, the authors also prove the following topological generalization: In any pseudolinear drawing of a complete graph K_{3n} one can find $(1 - o(1))n$ vertex-disjoint and pairwise crossing triangles. In this paper, we look at a special kind of crossing of triangles, which we call "star-crossing". We give (Theorem 3.1) a $O(n^2)$ time algorithm that checks whether a set of $3n$ points in general position in the plane admits a partition into vertex-disjoint pairwise star-crossing triangles. This algorithm uses the concept of centre-point of a point set (see [21]).

2 Crossing pseudolines

Given a set $P = \{p_1, p_2, \ldots, p_{2n}\}$ of $2n$ points in the plane, let A be a collection of continuous curves in the Euclidean plane satisfying the following properties: (i) Each curve in A is x-monotone nad unbounded in both directions; in other words, it intersects each vertical line in exactly one point, (ii) For every pair of points in P, there is a curve in A passing through both of them, and (iii) Every pair of curves in A meet at most once and cross at their intersection. The pair (P, A) is called a *pseudoconfiguration of points* and the elements of A are called *pseudolines*.

Let (P, A) be a pseudo-configuration of points. Given two points $p_i, p_j \in P$, the pseudoline in A that passes through p_i ans p_j will be denoted by p_ip_j and the portion of the pseudoline bounded by the end points p_i and p_j will be denoted by $\widetilde{p_ip_j}$, which will be called a *pseudoline segment*. The pseudoline p_ip_j is will be called a *halving pseudoline* of both side of p_ip_j contain precisely $n-1$ points. We say that two pseudoline segments *cross* if they have an interior point in common. If there are n pairwise crossing pseudoline segments $\widetilde{p_ip_j}$ whose end points belong to P, we say that there is a *perfect cross-matching*.

Theorem 2.1. *Let (P, A) be a pseudoconfiguration of points, where $|P| = 2n$. Then there is a perfect cross-matching if and only if there are precisely n halving pseudolines.*

Proof. Let $P = \{p_1, p_2, \ldots, p_{2n}\}$. Suppose first that there is a perfect cross-matching. Then there are n pairwise crossing pseudoline segments $\widetilde{p_{2i-1}p_{2i}}$, $1 \leq i \leq n$ (without loss of generality). Then each of the pseudolines $p_{2i-1}p_{2i}$, $1 \leq i \leq n$, is a halving pseudoline, because each of them seperates the two end points of all other pseudoline segments $\widetilde{p_{2j-1}p_{2j}}$. Next we show that there are no other halving pseudoline.

For the sake of contradiction, let us assume that (say) p_1p_3 is also a halving pseudoline. Without loss of generality, we may assume that p_{2i-1} is to the right of p_1p_2 and p_{2i} is to the left of p_1p_2, for every $2 \leq i \leq n$ (otherwise just rename the points). Suppose the puseudolines p_1p_2 and p_1p_3 subdivide the plane into four sectors S_1, S_2, S_3 and S_4 as in Figure 1. Now, p_4 lies either in S_3 or S_2. Suppose $p_4 \in S_3$ (Figure 1). Then p_4 and $\widetilde{p_1p_2}$ are on different sides of p_1p_3. Also, $\widetilde{p_3p_4}$ crosses $\widetilde{p_1p_2}$. Therefore, p_3p_4 must cross p_1p_3 at another point x different from p_3. But this contradicts the pseudoline property. Hence, p_4 lies in S_2.

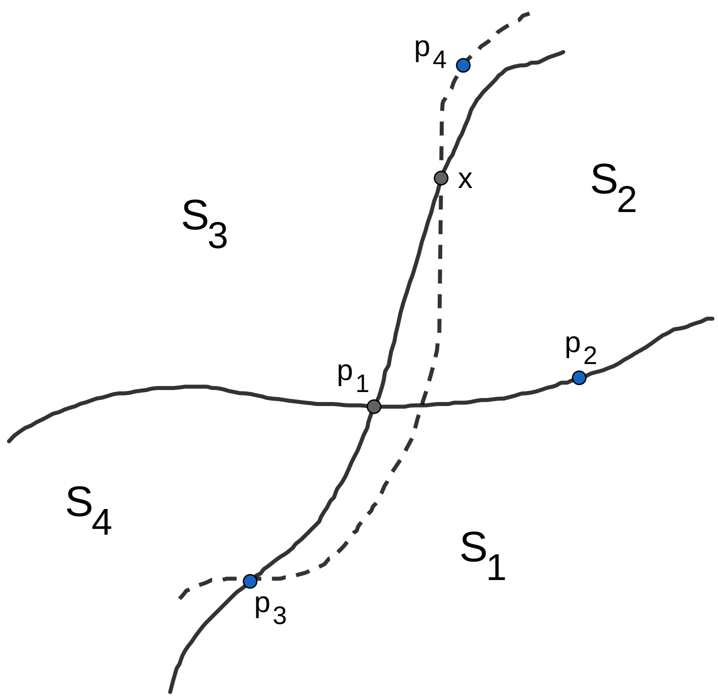

Figure 1: p_4 cannot lie in S_3

Claim 1. *For each $i \geq 3$, if p_{2i-1} lies to the left of p_1p_3 then p_{2i} lies to the right of p_1p_3.*

Proof. Since p_{2i-1} is to the right of p_1p_2, we have that $p_{2i-1} \in S_4$ (see Figure 2). Suppose p_{2i} lies to the left of p_1p_3. Then, $p_{2i} \in S_3$ (since p_{2i} lies to the left of p_1p_2). Now, p_{2i} and $\widetilde{p_1p_2}$ are on different sides of p_1p_3 and $\widetilde{p_{2i-1}p_{2i}}$ crosses $\widetilde{p_1p_2}$. But this implies that $p_{2i-1}p_{2i}$ must cross p_1p_3 at another point (say) x different from p_3, which is a contradiction. This completes the proof of the claim. $\square$

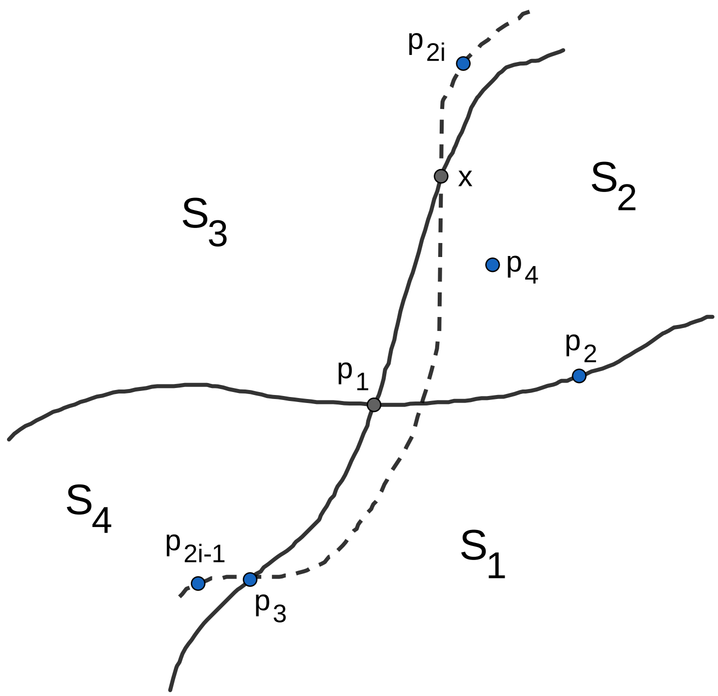

Figure 2: p_{2i} lies to the right of p_1p_3

Now, note that both p_2 and p_4 are on the right of p_1p_3. This, together with the Claim 1 implies that the number of points to the right of p_1p_3 exceeds the number of points to the left of it by at least 2. But, this contradicts our assumption that p_1p_3 is a halving pseudoline. This proves the 'only if' part of the theorem.

To prove the other direction, we will need the following.

Claim 2. *There exists at least one halving pseudoline through every point of P.*

Proof. Let $p \in P$ and $P = \{p, p_1, p_2, \ldots, p_N\}$ (note that, $N = 2n-1$). Now, the pseudolines $pp_1, pp_2, \ldots, pp_N$ divide the plane into sectors $S_1, S_2, \ldots, S_{2N}$ (see Figure 3). Let ℓ_i be the "right boundary" of S_i, for every $1 \leq i \leq 2N$. Note that ℓ_i is a oriented (towards the open end) "half-pseudoline". Without loss of generality, assume ℓ_i is a half partition of pp_i, for $i = 1, \ldots, N$. Note that, then ℓ_{N+i} is the other half partition of pp_i, for $i = 1, \ldots, N$. Now, when we move from ℓ_i to $\ell_{i+1 \pmod{2N}}$, the number of points on either side (right or left) of

$pp_{i+1 \pmod{2N}}$ will increase or decrease by 1 from that of pp_i. So when we move from ℓ_1 to $\ell N+1$, the pseudoline pp_1 is "rotated by 180 degrees" and the number of points on either side is interchanged. Therefore, for some $1 \leq i \leq N$, pp_i has equal number of points on both sides, i.e., pp_i is a halving pseudoline. This proves our claim. $\square$

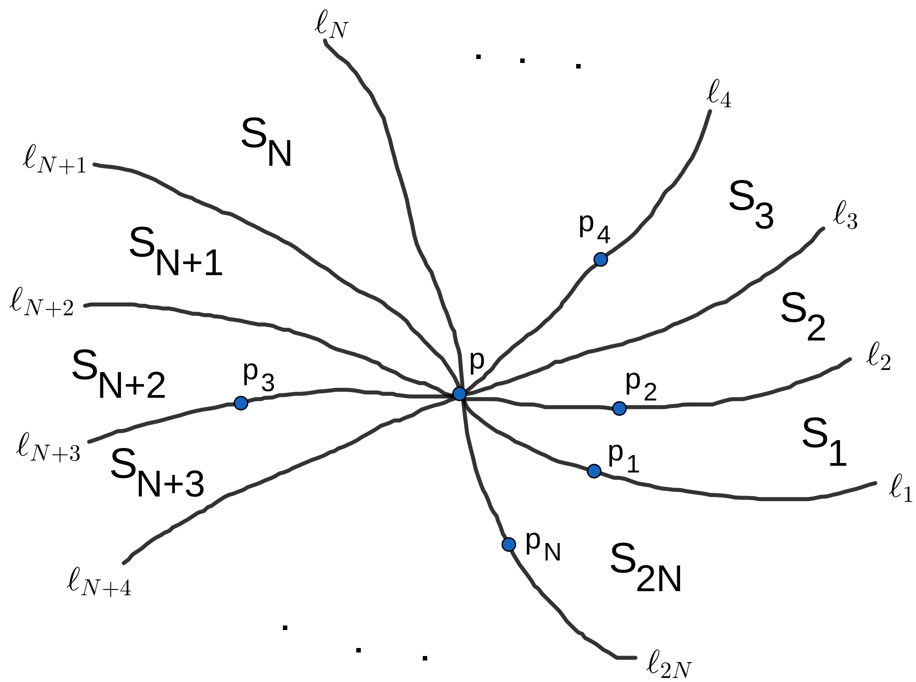

Figure 3: There is a halving pseudoline through p

Now suppose that there are precisely n halving pseudolines. By the above claim, there is at least one halving pseudoline through every point p_k and hence there must be exactly one such pseudoline. Let us assume, without loss of generality, that $\{p_{2i-1}p_{2i} : 1 \leq i \leq n\}$ is the complete list of halving pseudolines. We will show that the pseudoline segments $\widetilde{p_{2i-1}p_{2i}}$ $(1 \leq i \leq n)$ are pairwise crossing.

For the sake of contradiction, suppose $\widetilde{p_1p_2}$ and $\widetilde{p_3p_4}$ do not cross. Without loss of generality, let us assume that p_1 and p_2 have the same x-coordinate and p_2 is to the right of p_1 (otherwise just rotate the point set) and that $\widetilde{p_3p_4}$ lies to the right of p_1p_2 (otherwise rename the points). Let the pseudolines $\{p_1p_i : 2 \leq i \leq 2n\}$ divide the plane into sectors $S_1, S_2, \ldots, S_{2N}$ (as in Figure 4), where $(N = 2n-1)$. Let ℓ_i be the "right boundary" of S_i, for every $1 \leq i \leq 2N$ (where $p_2 \in \ell_1$). Note that ℓ_i is a oriented (towards the open end) "half-pseudoline". Let $\hat{\ell}_i$ denote the pseudoline that is the extension of ℓ_i. Note that, $\hat{\ell}_i = \hat{\ell}_{i+N \pmod{2N}}$.

First, let us assume that p_3p_4 intersects ℓ_1 (like in Figure 4). Let $k := \max\{i :$

p_3p_4 intersects $\ell_i\}$. Note that, points which lie to the right of p_3p_4 also lie to the right of $\hat{\ell}_k$. Now, there are $n-1$ points to the right of p_3p_4. Therefore, there are at least $n-1$ and 3 more points (p_1, p_2 and at least one of p_3 or p_4) on or to the right of $\hat{\ell}_k$. Hence there are at most $2n-(n+2)=n-2$ points to the left of $\hat{\ell}_k$. Now, note that, the number of points to the right of $\hat{\ell}_2$ is either equal or one more than that of $\hat{\ell}_1$. If they are equal then we already have a contradiction (since $\hat{\ell}_2$ can not be a halving pseudoline). Otherwise, the number of points to the right of $\hat{\ell}_2$ is equal to n. Now, as we continue to move from ℓ_2 to ℓ_{N+k}, the number of points to the right of $\hat{\ell}_{N+k}$ becomes $n-2$ (note that, the right side of $\hat{\ell}_{N+k}$ is same as the left side of $\hat{\ell}_k$) and during each such move tthe number of points on either side of a pseudoline increases or decreases by 1. Hence, there exists some pseudoline through p_1 other than p_1p_2 such that number of points to the right of it is $n-1$, which is a contradiction.

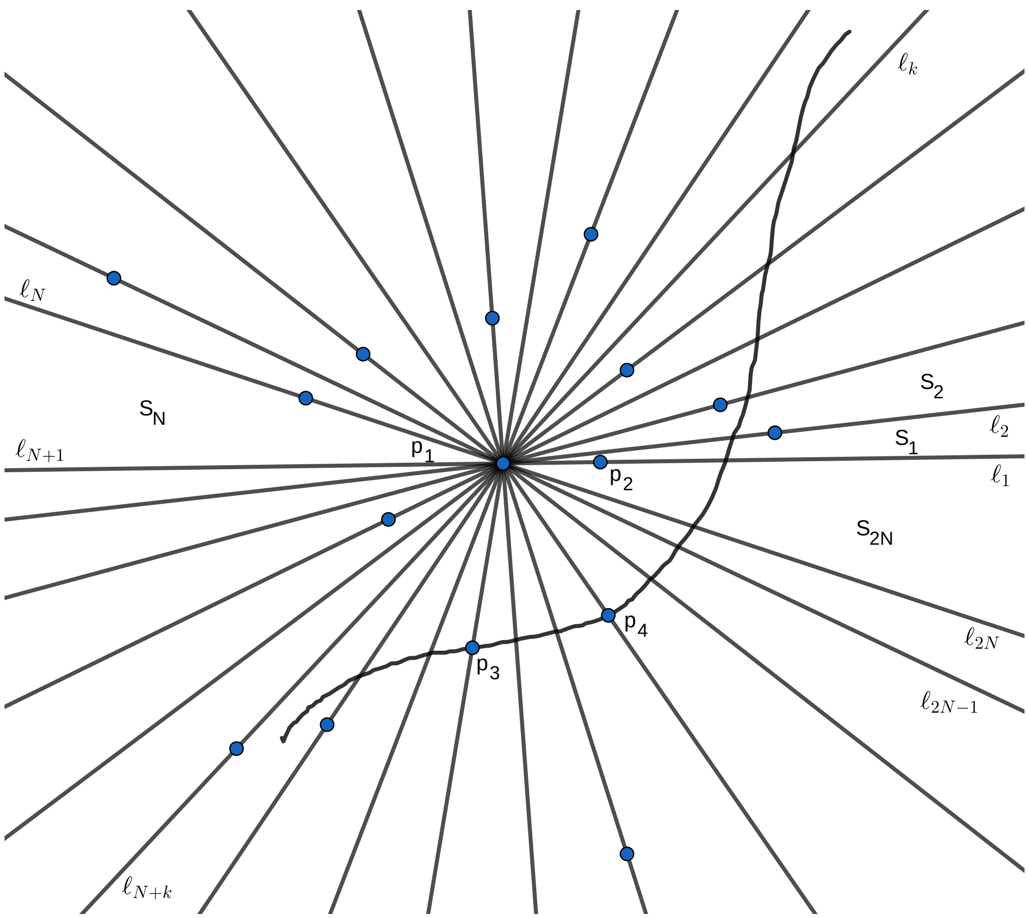

Figure 4: p_3p_4 intersects ℓ_1

Now suppose p_3p_4 does not intersect ℓ_1 (as in Figure 5). In that case, starting from ℓ_2 we only move till ℓ_k (instead of ℓ_{N+k}) and repeat the same argument as in the previous paragraph. This completes the proof of the theorem.

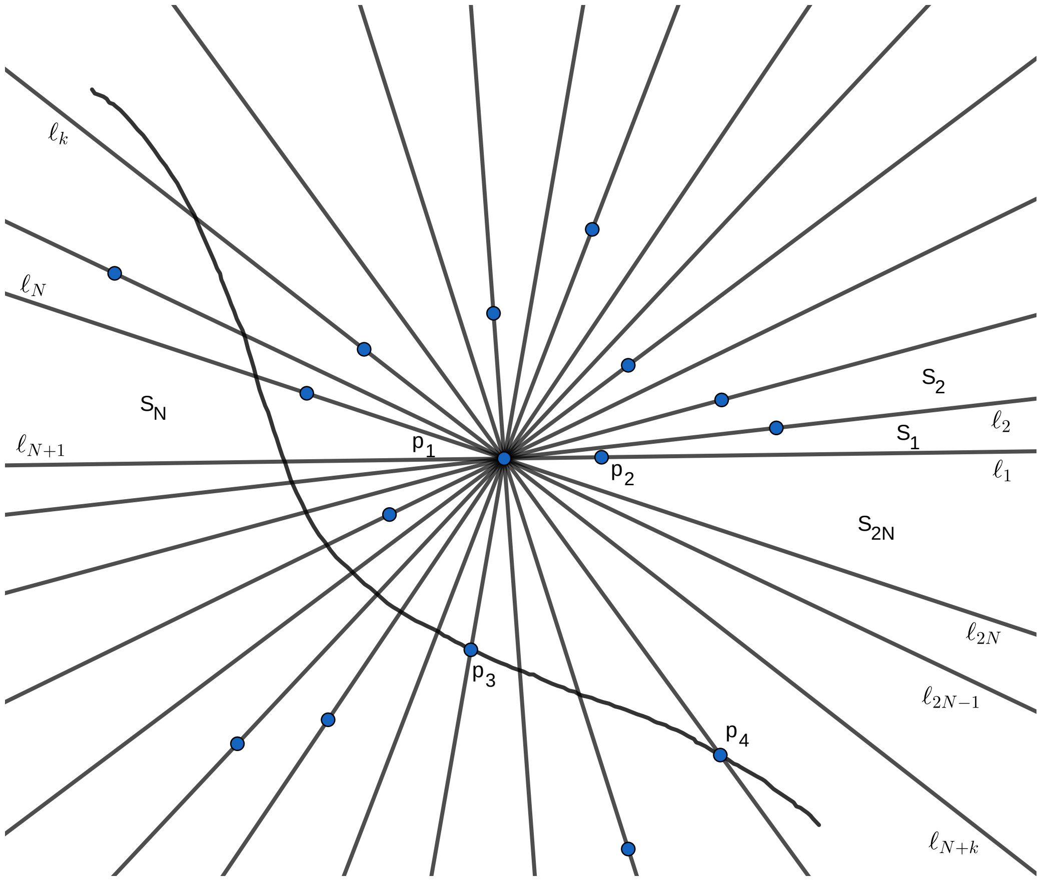

Figure 5: p_3p_4 does not intersect ℓ_1

$\square$

For the previous theorem, we had the situation where we had the minimum possible number (n) of halving pseudolines and we showed that this implies there are maximum possible number (n) of pairwise crossing pseudoline segments. Now, we may ask what happens if we had h (where $h > n$) many halving pseudolines; then can we give some lower bound on the maximum number of pairwise crossing pseudoline segments (in terms of h and n)? Our next theorem gives an answer to this question.

Given a pseudoconfiguration of points (P, A), we define a graph G, with vertex set P, as follows. We put an edge between two vertices p_i and p_j if p_ip_j is a halving pseudoline (w.r.t. A).

Lemma 2.1. *Every vertex in the G has an odd degree.*

Proof. Let p be vertex of G. Consider the halving pseudolines passing through p and let these be $\{pp_i : i = 1, \ldots, N\}$. Suppose they divide the plane into sectors $S_1, \ldots, S_{2N}$ (see Figure 6). Let ℓ_i be the oriented (towards the open end) "right boundary" of S_i and let

$\hat{\ell}_i$ be the pseudoline that is the extension of ℓ_i. Clearly $p_i \in \hat{\ell}_i$. Now, note that, $p_i \in \ell_i$ if and only if $p_{i+1 \pmod{2N}} \in \ell_{i+1+N \pmod{2N}}$, or equivalently, $p_i \in \ell_{i+N \pmod{2N}}$ if and only if $p_{i+1 \pmod{2N}} \in \ell_{i+1 \pmod{2N}}$ (this is because, each $\hat{\ell}_i$ is a halving pseudoline). This forces that N must be odd $\square$

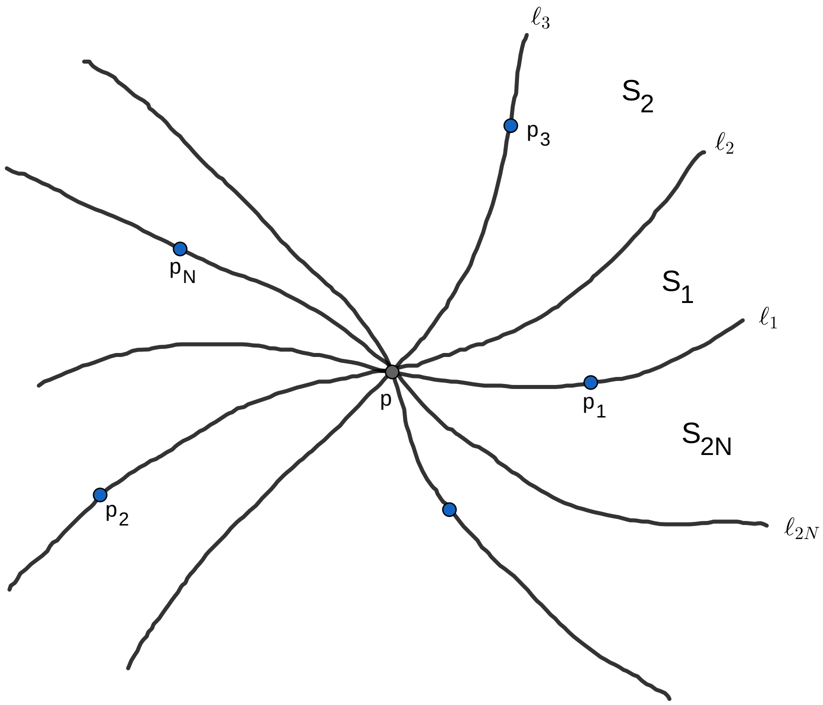

Figure 6: Degree of p is odd

Theorem 2.2. *Let (P, A) be a pseudoconfiguration of points, where $|P| = 2n$. Suppose there are h many halving pseudolines. Then, there are at least $3n - 2h$ many pairwise crossing pseudoline segments.*

Proof. Let G be the graph (on vertex set P) be as defined before. Then G has h edges. Suppose G has r disjoint edges and let G' the graph obtained by deleting these edges (and their vertices). Suppose G' has c many connected components. Now note that, G' has $2n - 2r$ vertices and by Lemma 2.1, every component of G' must have at least 4 vertices. This implies that $c \leq (2n - 2r)/4$. Now, the number of edges (which is h) of G must be at least $r + (2n - 2r - c)$. Therefore,

$$h \geq 2n - r - \frac{2n - 2r}{4} = \frac{3n - r}{2}$$

and, this implies that $r \geq 3n - 2h$.

Now, by the same argument as in the proof of the "if" part of Theorem 2.1, it follows that the pseudoline segments represented by the disjoint edges of the graph G must be pairwise crossing. Therefore, we get a set of $3n-2h$ many pairwise crossing pseudoline segments. $\square$

3 Crossing triangles

Definition 3.1. *We say that two triangles in the plane are* **star-crossing** *if the complement of their union has 8 (which is the maximum number possible) connected components. Equivalently, note that two triangles $\triangle abc$ and $\triangle pqr$ are star-crossing if and only if the points a, p, b, q, c, r are the consecutive vertices of a convex hexagon.*

Let $P = \{p_1, p_2, \ldots, p_{3n}\}$ be a set of $3n$ points in the plane in general position. Let $s(P)$ denote the maximum number of pairwise star-crossing triangles whose vertices belong to P and let $s(n) = \min_P s(P)$, where minimum is taken over all $3n$ element sets in general position in the plane. It is a natural question to ask about the asymptotic behaviour of $s(n)$.

Note that, if we have $3k$ points $p_1, \ldots, p_k, p'_1, \ldots, p'_k, p''_1, \ldots p''_k$ in convex position, then the family of triangles $\{\triangle p_i p'_i p''_i : i = 1, \ldots, k\}$ is pairwise star-crossing (which follows from the equivalent statement of the above definition). Now, by a theorem of Erdös and Szekeres ([1]) it is known that any n points in general position contain $\Omega(\log n)$ points in convex position. Hence, it follows that

$$s(n) = \Omega(\log n).$$

On the other hand, if we have a set $\{\triangle p_i p'_i p''_i : i = 1, \ldots, k\}$ of k pairwise star-crossing triangles, then the line-segments $\{\overline{p_i p'_i} : i = 1 \ldots, k\}$ are pairwise crossing. Now, it is known (see [2]) that, there exists a configuration of n points in the plane with at most $n/5$ pairwise crossing segments and hence at most $n/15$ pairwise star-crossing triangles. So we get the upper bound

$$s(n) \leq n/15.$$

It would be an interesting problem to improve these bounds on $s(n)$.

Partitioning a point set into pairwise star-crossing triangles

Note that $s(P) \leq n$ for every $3n$-element set P. And, $s(P) = n$ if and only if P can be partitioned into n vertex-disjoint pairwise star-crossing triangles. Below we give an $O(n^2)$-time algorithm which decides whether a set of $3n$ points satisfies this property, and, if so, finds such a partition.

Theorem 3.1. *There is a $O(n^2)$ time algorithm that decides whether a set of $3n$ points in general position in the plane admits a partition into vertex-disjoint pairwise star-crossing triangles and, if so, computes it.*

Proof. Our algorithm is as follows.

- *Step 1*: Compute a center point c of the given point set.
- *Step 2*: Label the $3n$ points $1, 2, \ldots, n, 1', 2', \ldots, n', 1'', 2'', \ldots, n''$ in counter-clockwise order around the center point c.
- *Step 3*: For every pair (i, j), check if the triangles $\triangle ii'i''$ and $\triangle jj'j''$ are star-crossing.

If *Step 3* is successful for every pair (i, j) declare that $\{\triangle 11'1'', \triangle 22'2'', \ldots, \triangle nn'n''\}$ is the desired partition, otherwise, we declare that no such partition exists. Note that the time-complexity of this algorithm is $O(n^2)$. Next we prove the correctness of our algorithm.

If the algorithm says yes, then clearly the point-set admits a partition into vertex-disjoint pairwise star-crossing triangles. Conversely, suppose the point-set admits a partition into n vertex-disjoint pairwise star-crossing triangles, say, $\{\triangle p_i p'_i p''_i : i = 1, \ldots, n\}$ We will show that our algorithm outputs exactly these triangles.

Claim 3. *The center point c lies in the interior of each triangle $\triangle p_i p'_i p''_i$ for $i = 1, \ldots, n$.*

Proof. Suppose c is outside a triangle $\triangle p_i p'_i p''_i$. Then there exists a line through c such that $\triangle p_i p'_i p''_i$ lies on one side of this line. Hence the all the points $p_i, \ldots, p_n, p'_1, \ldots, p'_n, p''_1, \ldots, p''_i$ lies on one side of this line. Therefore, the other side of the line must contain less than n points. But this is a contradiction to the fact that c is a center point. □

Without loss of generality, let us assume that the point p_1 is labelled 1 by our algorithm (Step 2) (otherwise just rename). We will show that the points $p_1, \ldots, p_n, p'_1, \ldots, p'_n, p''_1, \ldots p''_n$ are labelled $1, \ldots, n, 1', \ldots, n', 1'', \ldots, n''$ respectively by our algorithm. We prove this by induction on n. For the base case, note that the center point c lies in the interior of $\triangle p_1 p'_1 p''_1$ (by the above Claim) and this implies that the points p_1, p'_1 and p''_1 are labelled $1, 1'$ and $1''$ respectively. By induction hypothesis, let us assume that the points $p_1, \ldots, p_{n-1}, p'_1, \ldots, p'_{n-1}, p''_1, \ldots p''_{n-1}$ are labelled $1, \ldots, n-1, 1', \ldots, n-1', 1'', \ldots, n-1''$ respectively. That is, the points $p_1, \ldots, p_{n-1}$ are consecutive with respect to the counter-clockwise order around c. Without loss of generality, we may assume that p_n occurs between p_{n-1} and p'_1 in this order. Now, the points $p_i, p_n, p'_i, p'_n, p''_i, p''_n$ must be consecutive for each $i = 1, \ldots, n-1$ (note that the point c lies inside $\triangle p_i p'_i p''_i \cap \triangle p_n p'_n p''_n$). This forces the point p'_n to be be positioned between p'_{n-1} and p''_1 and p''_n to be positioned between p''_{n-1} and p_1. This completes the inductive step. Hence the algorithm outputs yes and the required triangles.

□

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