Research Statement

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My research interest lies in Combinatorics, Discrete Geometry and Optimization. Currently I am looking at the following problems.

1 Covering Points by Curves

We consider the problem of covering a finite set of points (given in certain configuration) in euclidean space by minimum number of continuous curves (satisfying certain property). We have been able to establish the following results on covering *integer grid points*.

1. Minimum number of *straight lines* required to cover the $k_1 \times \cdots \times k_n$ grid is equal to

$$\min\left\{\prod_{i\neq 1} k_i, \ldots, \prod_{i\neq n} k_i\right\}.$$

2. Minimum number of *monotonic curves* required to cover the $k_1 \times \cdots \times k_n$ grid is equal to

$$A_{\lfloor (k_1+\cdots+k_n+n)/2 \rfloor},$$

where A_m denotes the number of integral solutions of the equation $x_1 + \cdots + x_n = m$ such that $1 \le x_i \le k_i$ for each $i = 1, \ldots, n$.

3. Minimum number of *closed convex curves* required to cover the $k \times l$ grid is equal to

$$\min\{\lceil k/2 \rceil, \lceil l/2 \rceil\}.$$

More generally, the minimum number of *closed convex hyper-surfaces* required to cover the $k_1 \times \cdots \times k_n$ grid is equal to $\min\{\lceil k_1/2 \rceil, \ldots, \lceil k_n/2 \rceil\}$.

Currently we are looking at the problem of covering grid points in plane by *ortho-convex* curves. In general we have the upper bound $n/2$ and lower bound $n/4$ on the minimum number of curves required to cover the $n \times n$ grid. It seems natural to consider covering by ortho-convex curves of maximum length (this is w.l.o.g.) such that they only *intersect* *transversely*. We show that such coverings exist at least for every even n (in addition to $n = 5, 6, 7, 8, 9, 10$). If we only consider covering by such curves, we found that the minimum number of curves required is at least $(1 - 1/\sqrt{2})n$ (upto leading order). We are looking into the question on how much we loose by considering such restricted covering.

A related and possibly simpler problem would be to consider the covering of $n \times n$ grid by orthoconvex curves with at most one inner corner ($270°$ vertex). These L shaped (or their rotations) curves are the simplest orthoconvex curves. Note that, there are only polynomially many ($O(n^6)$) such curves and so one can check in polynomial time whether a constant number of such curves cover the grid. So it seems that this case should be easier to handle. For 5×5 grid we need 2 such curves and for 6×6 we need at least 3. It is easy to get the lower bound $n/3$ on the minimumm number of such curves required to cover $n \times n$ grid. But we could deduce a slightly better lower bound $4n/11$. More specifically, if m is the minimum number of curves required to cover the $n \times n$ grid by orthoconvex curves with at most one inner corner, then $m \geq 4n/11$.

Besides ortho-convex curves, we are also looking at some other related problems, namely, covering grid points by convex curves with 'no linear part' and covering grid points by circles.

2 Degree Sequences and Partitions

Let $H = ([n], E)$ be a k-uniform hypergraph with *vertices* $[n] = \{1, 2, \ldots, n\}$ and *edges* of size k i.e. $E \subseteq \binom{V}{k}$. The degree d_i of a vertex i is the number of edges containing it, i.e. $d_i = \#\{S \in E : i \in S\}$. The sequence $d_H := \{d_1, d_2, \ldots, d_n\}$ is called the *degree sequence* of H and the *degree partition* of H is obtained by rearranging d_H in weakly decreasing order.

We define the *polytope of degree sequences* ($DS(n, k)$) to be the convex-hull of degree sequences of all k-uniform hypergraphs on $[n]$ and similarly the *polytope of degree partitions* ($DP(n, k)$) to be the convex-hull of degree partitions of all k-uniform hypergraphs on $[n]$. Then we are interested in the study of these polytopes.

The case $k = 2$ (*graphs*) has been extensively studied. The extreme points, edges and facets of $DS(n, 2)$ have been determined (see [1]). Stanley in [2] obtained detailed information on $DS(n, 2)$ including generating functions for all face numbers, volume, and number of lattice points. In particular, he "counted" the number of distinct degree sequences of graphs on n vertices. In 2006, the vertices, edges and facets of $DP(n, 2)$ were determined in [3]. We are interested in the question of finding the number of lattice points in $DP(n, 2)$, which is open.

Next we look at the polytope of degree sequences with sum of coordinates equal to a fixed integer. Let H_e denote the hyperplane $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 2e\}$, for every positive integer e. Then we look at the hyperplane sections $DS_e(n, 2) := DS(n, 2) \cap H_e$. Using "majorization" and Birkhoff polytope ideas we could determine the extreme points of $DS_e(2, n)$. We believe that a direct proof is also possible. Let a_e denote the number of extreme points of $DS_e(n,2)$ which are distinct *upto the rearrangement* of coordinates. Then, we could show that the sequence $\{a_0, a_1, a_2, \ldots\}$ is *unimodal* (i.e. it increases up to a certain point and then decreases).

Now we come to the general case $k \geq 2$ (hypergraphs). The polytope $DS(n,k)$ was studied in [4], where its extreme points and facets were determined and a characterization of adjacency of extreme points was also obtained. Some other interesting information regarding this polytope was obtained in [5]. We are interested in counting the number of lattice points in $DS(n,k)$. This polytope is an example of a class of special kind of polytopes known as *zonotopes* (Minkowski sum of line segments) for which there is a linear-algebraic answer for the number of lattice points. We hope to find a combinatorial answer similar to the case of graphs (which was obtained in [2]). Finally, almost nothing is known about the polytope $DP(n,k)$ for general k. We believe that the extreme points of $DP(n,k)$ can be shown to be a certain class of objects associated to a certain poset.

3 Determinants of Path Generating Functions

Let $\mathcal{P}_n^+(l,k) := \sum_P w(P)$, where P runs over all paths from $(0,l)$ to (n,k) consisting of steps from $\{(1,0),(1,1),(1,-1)\}$ which never run below the x-axis, and where $w(P)$ is the product of all weights of the steps of P, where the weights of the steps are defined by $w((1,0)) = x+y$, $w((1,1)) = 1$, and $w((1,-1)) = xy$. It was proved in [6] that

$$\det_{0\leq i,j\leq n-1}(\mathcal{P}_{i+j}^+(0,k)) = \begin{cases} (-1)^{n_1\binom{k+1}{2}}(xy)^{(k+1)^2\binom{n_1}{2}} & \text{if } n = n_1(k+1); \\ 0 & \text{if } n \not\equiv 0 \pmod{k+1}. \end{cases}$$

and an analogous identity for $\det_{0\leq i,j\leq n-1}(\mathcal{P}_{i+j+1}^+(0,k))$ was also given.

Note that, these determinants are common weighted generalizations of the classical Henkel determinants

$$\det_{0\leq i,j\leq n-1}(C_{i+j}), \quad \det_{0\leq i,j\leq n-1}(C_{i+j+1}), \quad \det_{0\leq i,j\leq n-1}(M_{i+j}), \quad \text{and} \quad \det_{0\leq i,j\leq n-1}(M_{i+j+1}),$$

where $C_n := \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan Number and $M_n := \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_k$ is the n-th Motzkin Number. Note that C_n counts the number of lattice paths from $(0,0)$ to $(2n,0)$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$, which never run below the x-axis, and that M_n counts the number of lattice paths from $(0,0)$ to $(n,0)$ consisting of up-steps $(1,1)$, level steps $(1,0)$, and down-steps $(1,-1)$, which never run below the x-axis. So if we specialize $x = -y = \sqrt{-1}$, then $\mathcal{P}_{2n}^+(0,0)$ reduces to C_n and on the other hand, if we specialize $x = \frac{1}{2}(1+\sqrt{-3})), y = \frac{1}{2}(1-\sqrt{-3})$, then $\mathcal{P}_n^+(0,0)$ reduces to M_n.

The above two determinantal identities were evaluated using row operations. We could deduce the first identity identitiy for $k = 0,1$ and the second identity for $k = 0$ combinatorially using Lindström-Gessel-Viennot lemma. For general k, combinatorial proofs of the above identities is open.

Christian Krattenthaler posed the problem of evaluation of the determinant

$$\det_{0\le i,j\le n-1}(\mathcal{P}^{+}_{i+j-1}(0,k))$$

and he suggested that similar techniques as in [6] might work. Again for $k=0$, we could evaluate this determinant combinatorially which is given by the following

$$\det_{0\le i,j\le n-1}(\mathcal{P}^{+}_{i+j-1}(0,0)) = -(xy)^{\binom{n-1}{2}}\frac{y^{n-1}-x^{n-1}}{y-x}.$$

We are interested in the evaluation of this determinant for general k.

In [7], Krattenthaler and Yaqubi gave formulas for the following determinants

$$\det_{0\le i,j\le n-1}\left(\sum_{l\ge 0}\mathcal{P}^{+}_{i+j}(0,l)\right) \quad \text{and} \quad \det_{0\le i,j\le n-1}\left(\sum_{l\ge 1}\mathcal{P}^{+}_{i+j+1}(0,l)\right).$$

We are interested in the evaluation of the following analogous determinant

$$\det_{0\le i,j\le n-1}\left(\sum_{l\ge 1}\mathcal{P}^{+}_{i+j-1}(0,l)\right),$$

which we might be able to derived using similar techniques as in [7] (suggested by Christian Krattenthaler).

References

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