Research Statement

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My research interest lies in Combinatorics, Discrete Geometry and Optimization. Currently I am working on the following problems.

1 Covering a point set by curves

We consider the problem of covering a finite set of points (given in certain configuration) in Euclidean space $\mathbb{R}^n$ by minimum number of continuous curves (satisfying certain property). We mostly focus on covering *integer grid points*. In the following, by a $k_1 \times \cdots \times k_n$ grid we mean the set $\{1, \ldots, k_1\} \times \{1, \ldots, k_2\} \times \cdots \times \{1, \ldots, k_n\}$ in $\mathbb{R}^n$.

1.1 Covering by lines

It is not too difficult to show that any line can contain at most $\max\{k_1, \ldots, k_n\}$ points from an $k_1 \times \cdots \times k_n$ grid in $\mathbb{R}^n$ and from this it follows that the minimum number of lines required to cover the $k_1 \times \cdots \times k_n$ grid is equal to

$$\min\left\{\prod_{i\neq 1} k_i, \ldots, \prod_{i\neq n} k_i\right\}.$$

The problem of covering by skew lines is more interesting. We say that a line or hyperplane is *skew* if it is not parallel to any of the coordinate axes. Note that any skew line in $\mathbb{R}^2$ can contain at most two points from the boundary of $n \times n$ grid and the boundary has $4n-4$ grid points. So we need at least $2n-2$ skew lines to cover the $n \times n$ grid. On the other hand, it is easy to find coverings of $n \times n$ grid by $2n-2$ skew lines (e.g. consider the $2n-3$ lines parallel to off-diagonal except the ones which pass through bottom-left and top-right corner and these two corners are covered by the main diagonal). We remark that it is an *open problem* to find the minimum number of skew hyperplanes that cover the d-dimensional hypercube. Recently the lower bound $\Omega(n^{0.51})$ was obtained by Yehuda and Yehudayoff ([4]).

We have seen that $n \times n$ grid can be covered by n lines. Next we look at the *converse question*, namely, if a set of n^2 points in $\mathbb{R}^2$ is covered by n lines then can we say something about the configuration of the points? The following question seems natural. If a set of n^2

points is covered by n lines, then does there exist a subset of points of size $\Theta(n^2)$ which can be put inside a grid of size $\Theta(n^2)$, possibly after applying a projective transformation? We prove the following theorem which shows that the answer is no.

Theorem 1.1 ([1]). *There exists a set of n^2 points in $\mathbb{R}^2$ which can be covered with n lines but no subset of the points of size $\Omega(n^2)$ can be contained in a projective transformation of a rectangular grid of size $o(n^3)$.*

We construct the points and lines in the above theorem as follows. We take n parallel lines and put n points on each line in such a way that no three points from three different lines are co-linear.

We remark that the above construction also provides a *counter-example* to the Conjecture 1.16 (thereby disproving it) in [3].

1.2 Covering by monotonic curves

A curve $f = (f_1, \dots, f_n) : [0, 1] \to \mathbb{R}^n$ is called *monotonic* if

$$t_1 \le t_2 \;\Rightarrow\; f_i(t_1) \le f_i(t_2) \text{ for each } i = 1, \dots, n.$$

Given a finite subset C of $\mathbb{R}^n$, we define the poset $\mathcal{C} := (C, \le)$ as follows. For $x := (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y := (y_1, \dots, y_n) \in \mathbb{R}^n$, we define $x \le y$ if $x_i \le y_i$ for $i = 1, \dots, n$. Then note that, covering C by monotonic curves is equivalent to decomposing $\mathcal{C}$ into a collection of chains. Therefore by *Dilworth's Theorem* we get that the minimum number of monotonic curves needed to cover C is equal to to the size of the largest antichain of $\mathcal{C}$, which we denote by $w(\mathcal{C})$ (called the *width* of the poset $\mathcal{C}$). Applying this to the grid we prove the following theorem.

Theorem 1.2 ([1]). *Minimum number of monotonic curves required to cover the $k_1 \times \cdots \times k_n$ grid is equal to*

$$\max_m A_m = A_{\lfloor (k_1 + \cdots + k_n + n)/2 \rfloor},$$

where, A_m equals the number of solutions of the equation $x_1 + \cdots + x_n = m$ such that $x_i \in [k_i]$ for each $i = 1, \dots, n$.

Two special cases of the above theorem are following. When $n = 2$ (i.e. C is a *bi-dimensional grid*), $\max_m A_m = \min\{k_1, k_2\}$. When $k_1 = \cdots = k_n = 2$ (i.e. C is a *hypercube*), $\max_m A_m = A_{\lfloor n + \frac{n}{2} \rfloor} = \binom{n}{\lfloor n/2 \rfloor}$.

Next we look at the problem of *covering with multiplicity*, i.e. covering a set C by minimum number (m_k) of monotonic curves such that every point is covered at least k times. Let $C^{(i)} := \{x^{(i)} : x \in C\}$, for $i = 1, \dots, k$ be k disjoint copies of C. Let $C_k := \cup_{i=1}^{k} C^{(i)}$. We define a partial order $\le$ on C_k as follows: $x^{(i)} \le y^{(j)}$ if and only if $x \le y$. Let us denote this poset by $\mathcal{C}_k$. Then note that covering C with multiplicity k is equivalent to chain decomposition of the poset $\mathcal{C}_k$. Therefore by Dilworth's theorem we get that $m_k = w(\mathcal{C}_k)$. Now it is not too difficult to show that $w(\mathcal{C}_k) = k \cdot w(\mathcal{C})$ and hence $m_k = k \cdot w(\mathcal{C})$.

1.3 Covering by convex and orthoconvex curves

It is easy to see that $m \times n$ grid can be covered by $\min\{\lceil m/2\rceil, \lceil n/2\rceil\}$ convex curves. We show that this is also the minimum number of convex curves required to cover the $m \times n$ grid. More generally, we show that the minimum number of closed convex hypersurfaces required to cover the $k_1 \times \cdots \times k_n$ grid is $\min\{\lceil k_1/2\rceil \ldots, \lceil k_n/2\rceil\}$. Covering by *strictly-convex* curves becomes more interesting. Using a result of Andrews ([2]), we get a *super-linear* lower bound $\Omega(n^{4/3})$ on the minimum number of strictly-convex curves required to cover the $n \times n$ grid.

If the boundary of *orthogonal convex-hull* (of a set of points) is a simple closed curve then we call it an "orthoconvex" curve. The $n \times n$ grid, for $n = 4, 5, 6, 7, 8, 9, 10$ can be covered by $2, 2, 2, 3, 3, 3, 4$ orthoconvex curves respectively. For general n, finding the minimum number of orthoconvex curves required to cover the $n \times n$ grid does not look easy. So we consider orthoconvex curves with *few corners*, which seem more tractable, By "inner corner" of an orthoconvex curve we mean a point where the curve turns by 270 degrees. We prove the following.

Theorem 1.3 ([1])**.** *If m orthoconvex curves with at most one inner corner cover the $n \times n$ grid, then $m \geq 2n/5$.*

It is easy to check that 5×5 grid can be covered by 2 orthoconvex curves with one inner corner. So the bound of the above theorem becomes tight for $n = 5$. Moreover, as a consequence of the above theorem we also deduce that we need at least $2n/7$ orthoconvex curves with at most *two inner corners* to cover $n \times n$ grid. However this bound is most likely not tight. Currently we trying to improve it and also trying to see what we can say for more general orthoconvex curves.

1.4 Covering by circles

From the fact that a circle contains at most $O(n^\epsilon)$ points from an $n \times n$ grid for every $\epsilon > 0$, we get that the minimum number of circles required to cover $n \times n$ grid is $\Omega(n^{2-\epsilon})$ for every $\epsilon > 0$. On the other hand, there is a covering of the $n \times n$ grid by $O(n^2/\sqrt{\log n})$ circles. This is obtained by choosing a corner of the grid and drawing all concentric circles such that each of them is incident to at least one grid point. The number of such circles is $O(n^2/\sqrt{\log n})$ by a well known theorem of Ramanujan and Landau on sums of two squares.

Next we look at the problem of covering $n \times n$ grid by *circles of different radii.* Note that the $n \times n$ grid can be covered by $O(n^2/\sqrt{\log n})$ circles of different radii. by the construction mentioned above. Now regarding lower bound, suppose that $n \times n$ grid is covered by m

circles of different radii. Now we translate every circle in the covering by a point on the $n \times n$ grid and moreover we translate the grid points as well. Note that, we now have $\Theta(n^2)$ points, mn^2 circles and n^4 point-circle incidences. Now from the *conjectured upper bound* $O(s^{2/3}t^{2/3}\log^c(st) + s + t)$ (where c is a constant) on the maximum number of incidences between s points and t circles, we obtain the following conjecture on covering.

Conjecture 1 ([1]). *If the $n \times n$ grid is covered by m circles such that no two of them have equal radius, then $m = \Omega(n^2/\log^c n)$ for some constant c.*

Let us remark that, using previous ideas one can obtain sharp bounds for covering by certain "non-congruent" curves. Suppose $n \times n$ grid is covered by a set S of *non-linear* curves such that *no two of them are translates of each other*. We also assume that the curves in $S + \mathbb{Z}^2$ have *two degrees of freedom* and *multiplicity type* c (i.e., if any two curves intersect at most 2 points and for any 2 points there are at most c curves passing through both of them), where c is a constant. Then using the previous argument of translation and point-line incidence like bounds (due to Pach and Sharir) for curves in $S + \mathbb{Z}^2$, we deduce that size of S must be $\Omega(n^2)$.

1.5 Covering by small curves

Note that, a circle of radius 1 (unit-circle) can cover at most 4 grid points. Therefore we need at least $n^2/4$ unit-circles to cover the $n \times n$ grid. To prove a matching (asymptotically, i.e. for large n) upper bound, we show that the infinite grid $\mathbb{Z} \times \mathbb{Z}$ can be *tiled* with unit-circles. Using similar ideas one can also show that minimum number of circles of radius $\sqrt{2}$ and 2 required to cover the $n \times n$ grid is equal to $n^2/4$ in both cases.

More generally, suppose we have a fixed "small" curve containing at most, say, k grid points (where k is constant w.r.t. n) and we want to cover the $n \times n$ grid by the copies of this curve. If $\mathbb{Z} \times \mathbb{Z}$ admits a tiling by translates of this curve (it will be interesting to ask for which curves such tiling exists), then we can conclude that the minimum number of curves required to cover will be asymptotically n^2/k.

Next we look at covering by small squares. We prove the following.

Theorem 1.4 ([1]). *If $n \times n$ grid is covered by m squares of side length 2, then $m \geq n^2/7$.*

The matching asymptotic upper bound follows from the "tile like" covering of $\mathbb{Z} \times \mathbb{Z}$ by squares of side length 2. We think that similar ideas could be used to study covering by other small squares.

[For more details and proofs of the results mentioned above, please refer to our paper [1]]

References

[1] A. Bishnu, M. Francis and P. Majumder, *Curves, points, incidences and covering*, https://majumder-pritam.github.io/covering21.pdf, in preparation.

[2] G. E. Andrews, *An asymptotic expression for the number of solutions of a general class of diophantine equations*, Trans. Amer. Math. Soc.99, 272–277 (1961).

[3] A. Sheffer, *Polynomial Methods and Incidence Theory*, https://faculty.baruch.cuny.edu/ASheffer/000book.pdf, to be published by Cambridge University Press, 2022.

[4] G. Yehuda and A. Yehhudayoff, *Slicing the hypercube is not easy*, arXiv:2102.05536.