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AN AXIOMATIZATION OF THE GINI COEFFICIENT

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This paper deals with the axiomatization of the Gini coefficient in the general case, i.e. it characterizes its cardinal properties in ranking income distributions whose aggregate incomes and populations are not necessarily the same. An axiomatization of an index asymptotically equal to the Gini coefficient is also given. The asymptotic equivalence of the two axiomatic systems is used to highlight some properties of the Gini coefficient.

Key words: Income inequality; Gini coefficient.

1. Introduction

The Gini coefficient is one of the indices most frequently used to evaluate income inequality. It has been axiomatized by several authors, including Sen (1974, pp. 398–401). Sen's approach is to characterize the social welfare function implicit in the ranking of income distributions generated by the Gini coefficient, in the pure distribution case; that is, when comparing income distributions whose aggregate incomes and populations are the same. Our result can be viewed as complementing his as it gives an axiomatization in the more general context of the comparison of income distributions whose total incomes and populations are not necessarily the same. The work of Takayama (1979), Kakwani (1980), Blackorby and Donaldson (1978), and Donaldson and Weymark (1980) should also be mentioned.

As the extension to varying populations does not lend itself easily to a social welfare interpretation, we mostly eschew such an interpretation and concentrate on the logical aspects of the derivation of the index from a set of axioms. An advantage of this is that the results are, mutatis mutandis, of relevance to the measurement of the inequality of other socio-economic variables. Inequality indices are commonly used to compare distributions with different total incomes and populations. It is the case when different communities or the same community at different times are compared; empirical studies typically perform such comparisons. The practitioner interested in the properties or the axiomatic derivation of the indices he is using will find little guidance in the existing literature, almost exclusively concerned as it is with the pure distribution case. Although some of our axioms (i.e. Axioms T and E) refer to the pure distribution case and have a clear interpretation from the welfare point of view, we have also taken into account considerations which transcend the

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pure distribution case, such as 'consistency' when comparing distributions with different populations (Axiom S) and comparability of income distributions with the same population but different total incomes (Axiom CPC).

It is the purpose of this paper to investigate how an eclectic set of axioms expressing those different considerations precipitates the Gini coefficient, and to discuss the conflicts and complementarities which arise.

The Gini coefficient is usually stated under the form

$$G = 1 + 1/n - \frac{2}{n^2 \bar{y}} \sum_{i=1}^{n} y_i(n+1-i) \tag{1}$$

where y_i is i's income and $\bar{y} = \sum_{i=1}^{n} y_i/n$, $y_1 \leq y_2 \cdots \leq y_n$.

In the justification or derivation of the index, the system of weights applied to the individual incomes in (1) is at the center of the picture. Sen assumes what is tantamount to the equidistance of two successive weights and we follow suit. The procedure is admittedly somewhat arbitrary although, following Borda, a case can be made for it in terms of insufficiency of reason, see Sen (1976, p. 222). Equidistance is not enough though to determine the ordering implied by a particular system of weights [$(n+1-i)$ in the above formulation]. In the pure distribution case, the terms other than the individual incomes and the weights in (1) are constant and then the equidistance of the weights is enough to determine an unique ranking of the income vectors [i.e. $(2n+3-4i)$ or $(-i)$ would give the same ranking]; such is not the case in the more general context we are interested in here; that is when comparing income vectors with any population and total income. We intend to show that in such a case, the specific weighting system of (1) is less arbitrary than is usually thought and that once equidistance in the pure distribution case is postulated, the particular ranking given by the Gini coefficient follows from additional considerations which are meaningful.

Another way of looking at the Gini coefficient and rationalizing the weight's equidistance is in terms of pair-wise income comparisons. Let us first define the mean difference as:

$$\Delta = [1/n^2] \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j|; \tag{2}$$

that is the average of the absolute differences between all pairs of incomes (Kendall and Stuart, 1977, p. 46). Then (Sen, 1973, p. 31):

$$G = \Delta/2\bar{y} = \frac{1}{2\bar{y}n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} |y_i - y_j| = 1 - \frac{1}{\bar{y}n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{Min}(y_i, y_j). \tag{3}$$

This way of formulating the Gini coefficient leads to the following illuminating paraphrase: "Suppose the welfare level of any pair of individuals is equated to the welfare level of the worse-off person of the two. Then, if the total welfare of the group is identified with the sum of the welfare levels of all pairs, we get the welfare function underlying the Gini coefficient" (Sen, 1973, p. 33).

In the same vein, G. Pyatt has characterized the Gini coefficient as the outcome of a statistical game. For each individual we select at random some income from the population of incomes. If the chosen income is larger than the individual's income, he can retain the value selected, otherwise he keeps his income. For individual i the expected gain is equal to

$$1/n \sum_{j=1}^{n} \operatorname{Max}(0, y_i - y_j) \quad \forall i. \tag{4}$$

Then the Gini coefficient can be interpreted as the "average gain to be expected if each individual has the choice of being himself or some other member of the population drawn at random, expressed as a proportion of the average level of income" (G. Pyatt, 1976, p. 244).

In (2) through (4) the pair-wise comparisons include the comparison of i's income with itself. This we can call pair-wise comparisons with repetition following Kendall and Stuart (1977, p. 46), who call (2) the mean difference with repetition and (5) the mean difference without repetition. Alternatively, the comparisons can be done without repetition; then the number of comparisons is $n(n-1)$. This gives as a variant of (2):

$$\Delta' = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{i=1}^{n} |y_i - y_j|. \tag{5}$$

Likewise, noting that (1) can be written as

$$G = -\sum_{i=1}^{n} y_i(n+1-2i)/\bar{y}n^2,$$

we get the variant of the Gini coefficient without repetition:

$$G' = -\sum_{i=1}^{n} y_i(n+1-2i)/\bar{y}n(n-1). \tag{6}$$

Clearly the distinction between G and G' is unimportant if n is large; nevertheless, once the equidistance of the weights in the pure distribution case is postulated, it takes quite different considerations to precipitate G and G', respectively, as we will see.

2. Two axiomatizations of the Gini coefficient

Let us define: Y is an income vector with $y_i \geq 0, i = 1, \ldots, n$ and $\sum_{i=1}^{n} y_i > 0$. We assume that Y is always ordered such that $y_1 \leq y_2 \leq \cdots \leq y_n$. $I(Y)$ is some function of Y.

$$g(Y,k) = (y'_1, y'_2, \ldots, y'_{kn})$$

with $y'_{(i-1)k+j} = y_i$, $i = 1, \ldots, n$, $j = 1, \ldots, k$, $k \in \mathbb{N}$, $\mathbb{N}$ is the set of natural numbers.

In other words, $g(Y,k)$ is the income vector obtained from Y by replicating its population k times.

$$fi(Y,r) = Y'' = (y_1'', y_2'', \ldots, y_n'')$$

with $y_j'' = y_j, j \neq i, i+1, y_i'' = y_i + r, y_{i+1}'' = y_{i+1} - r; 0 < r \leq (y_{i+1} - y_i)/2, i = 1, 2, \ldots, n-1$.

In other words, $fi(Y,r)$ is the income vector resulting from an order preserving transfer being performed between two successive income holders in Y.

$$D(n,S) = \left\{ Y = (y_1, y_2, \ldots, y_n) \geq 0: \sum_{i=1}^{n} y_i = S > 0 \right\};$$

i.e., $D(n,S)$ is the set of nonnegative vectors of dimension n and such that the sum of the components is the same.

We propose the following axioms.

Axiom of Equidistance (Axiom E)

$$I(Y) = -\sum_{i=1}^{n} y_i w_i(n, \mathcal{Y})$$

with $w_i(n,\mathcal{Y}) - w_{i-1}(n,\mathcal{Y}) = w_j(n,\mathcal{Y}) - w_{j-1}(n,\mathcal{Y}) \quad \forall\, i,j \leq n$.

Any function $I(Y)$ can be written without loss of generality as $-\sum_{i=1}^{n} y_i w_i(Y)$ where the sign has been chosen for convenience. Axiom E states the weight equidistance rule in the pure distribution case; it implies that an order preserving equalizing transfer between any two successive people has the same effect on $I(Y)$.

Transfer axiom (Axiom T)

$$I(Y) > I(fi(Y,r)) \quad \forall i < n, \quad \forall Y.$$

Axiom T is a form of the familiar Pigou–Dalton condition that equalizing transfers should decrease inequality. The axiom also refers to the pure distribution case. Axioms T and E together impose on $I(Y)$ the ranking properties of the Gini coefficient in the pure distribution case. Together they are equivalent to Sen's system in Sen (1974).

Population Symmetry axiom (Axiom S)

$$I(g(Y,k)) = I(Y) \quad \forall Y.$$

S is rather familiar. It requires that if two or more identical populations were pooled together, the level of inequality of the whole and of the parts should be the same. Last we have:

Constant Population Comparability axiom (Axiom CPC)

$$\mathrm{Min}\{I(Y)\colon\ Y\in D(n,S)\}=\mathrm{Min}\{I(Y)\colon\ Y\in D(n,S')\}$$

and

$$\mathrm{Max}\{I(Y)\colon\ Y\in D(n,S)\}=\mathrm{Max}\{I(Y)\colon\ Y\in D(n,S')\}\quad \forall\, S,S',n.$$

Axiom CPC is not familiar but it is argued that it embodies a property for an inequality index that anyone involved with comparing income vectors whose total incomes are different will consider a prerequisite. Axiom CPC requires that the range of the index should be the same over the redistribution of any total income over a given number of people. Let us assume for concreteness that axiom T is also postulated. Then, given n and $\bar{y}$, the index will reach its largest value if someone receives all the income and everyone else has zero income. Likewise, the index will reach its minimum for a perfectly egalitarian distribution. Such a distribution cannot be 'improved upon' by further redistribution. Therefore it should not be considered as being more nor less unequal that the perfectly egalitarian redistribution of some other total income over the same people, if one is contemplating passing judgment over redistributive equality at constant population, but with different total incomes. Axiom CPC together with Axiom T imply for example that the perfectly egalitarian distribution of before tax income exhibits the same degree of inequality than the perfectly egalitarian distribution of after tax income. Such a property seems highly congenial with what we intuitively understand with inequality.

Note that Axiom CPC is weaker than the so-called mean-invariance property which requires that $I(\lambda Y)=I(Y)$ for all $\lambda>0$. This later property is convenient in that it makes the measurement of inequality unit-free; it also embodies the distributional judgment that if everyone's income was increased proportionately, the level of inequately would remain the same, a distributional judgment that one might not want to make. See Dalton (1920) and Kolm (1976) for a discussion. Axiom CPC can be viewed as the mean-invariance property confined to the most extreme distributions only. The spirit of Axiom CPC is entirely different though. It expresses neither a concern for the arbitrariness of the unit system nor a value judgment; it is merely a pragmatic comparability rule.

We now prove that the ranking of the income vectors with any population and any total income by any function $I(Y)$ which satisfies our four axioms is the one given by the Gini coefficient. We will in fact prove a slightly stronger statement by showing that $I(Y)$ and the Gini coefficient are then cardinally equivalent. It is convenient to establish two lemmas first.

Lemma 2.1. *If $I(Y)$ satisfies axioms* T, E *and* CPC, *then it is of the form*

$$I(Y)=-\sum_{i=1}^{n}y_i v_i(n)/\bar{y}$$

where

$$v_i(n) = v_1(n) - (i-1)s(n);$$

$v_1(n)$ and $s(n)$ are functions of n and $s(n) > 0$ for all n.

Proof. By Axiom E, $I(Y) = -\sum_{i=1}^{n} y_i w_i(n, \bar{y})$ with

$$w_i(n,\bar{y}) - w_{i+1}(n,\bar{y}) = t(n,\bar{y}) \quad \forall\, i<n;\ n>1.$$

By Axiom T, $t(n,\bar{y})>0$, $n>1$ $\forall Y$. Then $w_i(n,\bar{y})$ can be written as $w_i(n,\bar{y}) = w_1(n,\bar{y}) - (i-1)t(n,\bar{y})$ and

$$I(Y) = -\sum_{i=1}^{n} y_i[w_1(n,\bar{y}) - (i-1)t(n,\bar{y})]. \tag{7}$$

By Axiom T the minimum value of $I(Y)$ for all Y of dimension n and given $\bar{y}$ is for a perfectly egalitarian distribution. Then,

$$I(Y) = -n\bar{y}[w_1(n,\bar{y}) - \tfrac{1}{2}(n-1)t(n,\bar{y})]. \tag{8}$$

By Axiom T the maximum value of $I(Y)$ for all Y of dimension n and given $\bar{y}$ is for a distribution such that someone receives all the income and everyone else has zero income. Then,

$$I(Y) = -n\bar{y}[w_1(n,\bar{y}) - (n-1)t(n,\bar{y})]. \tag{9}$$

Axiom CPC and (8) imply, for all Y, Y',

$$n\bar{y}[w_1(n,\bar{y}) - \tfrac{1}{2}(n-1)t(n,\bar{y})] = n\bar{y}'[w_1(n,\bar{y}') - \tfrac{1}{2}(n-1)t(n,\bar{y}')]. \tag{10}$$

Axiom CPC and (9) imply, for all Y, Y',

$$n\bar{y}[w_1(n,\bar{y}) - (n-1)t(n,\bar{y})] = n\bar{y}'[w_1(n,\bar{y}') - (n-1)t(n,\bar{y}')]. \tag{11}$$

(10) minus (11) implies

$$\bar{y}(n-1)t(n,\bar{y}) - \tfrac{1}{2}\bar{y}(n-1)t(n,\bar{y}) = \bar{y}'(n-1)t(n,\bar{y}') - \tfrac{1}{2}\bar{y}'(n-1)t(n,\bar{y}').$$

This implies $t(n,\bar{y})/t(n,\bar{y}') = \bar{y}'/\bar{y}$. So $t(n,\bar{y}) = s(n)/\bar{y}$ $\forall Y$ where $s(n)$ is some function of n. Then (9) implies

$$\bar{y}w_1(n,\bar{y}) - (n-1)s(n) = \bar{y}'w_1(n,\bar{y}') - (n-1)s(n),$$

so $w_1(n,\bar{y})/w_1(n,\bar{y}') = \bar{y}'/\bar{y}$. Then $w_1(n,\bar{y}) = v(n)/\bar{y}$, $\forall Y$ where $v(n)$ is some function of n. Then (7) is

$$I(Y) = -\sum_{i=1}^{n} y_i[v_1(n) - (i-1)s(n)]/\bar{y}.$$

As $t(n,\bar{y})>0$ and $s(n) = t(n,\bar{y})\bar{y}$, we have $s(n)>0$.

As can easily be checked, the members of the family of indices generated by the lemma are mean invariant.

Lemma 2.2. *$v_i(n) = v_1(n) - (i-1)s(n)$, with $n \in \mathbb{N}$, $i = 1, 2, \ldots, n$, and $v_1(n)$ and $s(n)$ any function of n that satisfies*

$$v_1(n) = (i-1)s(n) = \sum_{j=1}^{k} [v_1(kn) - ((i-1)k + j - 1)s(kn)] \qquad \forall k \in \mathbb{N}, \quad \forall i, \quad \forall n \tag{12}$$

iff

$$v_i(n) = (2s(2)/n^2)[(v_1(1)/2s(2) + 1)n + 1 - 2i].$$

Proof. (12) can be written as

$$v_1(n) - (i-1)s(n) = k[v_1(kn) - \tfrac{1}{2}((2i-1)k - 1)s(kn)]. \tag{13}$$

If in (13) we take $n = 2$, then all $e = kn$ constitute, for all $k \in N$, the set of positive even numbers and (13) implies

$$v_1(2) - (i-1)s(2) = (e/2)[v_1(e) - \tfrac{1}{2}((2i-1)e/2 - 1)s(e)], \quad i = 1, 2. \tag{14}$$

If $i = 1$, (14) is

$$v_1(2) = (e/2)[v_1(e) - \tfrac{1}{4}(e-2)s(e)]. \tag{15}$$

If $i = 2$, (14) is

$$v_1(2) - s(2) = (e/2)[v_1(e) - \tfrac{1}{4}(3e-2)s(e)]. \tag{16}$$

(15) minus (16) gives

$$s(e) = [2/e]^2 s(2) \quad \forall e. \tag{17}$$

Now, if in (13) we take $n = d$ where d is any odd number larger than 1 and $k = 2$, then $e = dk$ is some positive even number. Then (13) implies

$$v_1(d) - (i-1)s(d) = 2[v_1(e) - \tfrac{1}{2}(4i-3)s(e)] = 2[v_1(e) - \tfrac{1}{2}(4i-3)(2/e)^2 s(2)] \tag{18}$$

for $i = 1, 2, \ldots, d$, by (6). If $i = 1$, (18) implies

$$v_1(d) = 2[v_1(e) - \tfrac{1}{2}(2/e)^2 s(2)]. \tag{19}$$

If $i = 2$, (18) implies

$$v_1(d) - s(d) = 2[v_1(e) - \tfrac{5}{2}(2/e)^2 s(2)]. \tag{20}$$

(19) minus (20) gives

$$s(d) = (2/d)^2 s(2) \quad \forall d. \tag{21}$$

(17) and (21) imply

$$s(n) = (2/n)^2 s(2) \quad \forall n > 1. \tag{22}$$

If in (13) we take $n=1$ and $kn=n$, we get via (22)

$$v_1(1)=n\left[v_1(n)-\frac{n-1}{2}s(n)\right]=n\left[v_1(n)-\frac{2(n-1)}{n^2}s(2)\right]\quad \forall n.$$

This gives $v_1(n)=v_1(1)/n+(n-1)2s(2))/n^2$ $\forall n$. This and (13) imply

$$v_i(n)=\frac{1}{n^2}[nv_1(1)+(n-1)2s(2)-(i-1)4s(2)]$$

$$=\frac{2s(2)}{n^2}\left[\left(\frac{v_1(1)}{2s(2)}+1\right)n+1-2i\right]\quad \forall i,\quad \forall n. \tag{23}$$

This proves necessity; sufficiency is easy to establish.

Theorem 2.3. *If $I(Y)$ satisfies Axioms* E, T, S *and* CPC, *then it is cardinally equivalent to the Gini coefficient.*

Proof. If $I(Y)$ satisfies Axioms E, T and CPC, then by Lemma 2.1 we have

$$I(Y)=-\sum_{i=1}^{n} y_i w_i(n,\bar{y}), \tag{24}$$

with

$$w_i(n,\bar{y})=v_i(n)/\bar{y}=[v_1(n)-(i-1)s(n)]/\bar{y}$$

where $v_1(n)$ and $s(n)$ are any functions of n with $s(n)>0$ $\forall n$. If $I(Y)$ also satisfies Axiom S, then if $Y'=g(Y,k)$, $I(Y)=I(Y')$ and, as $\bar{y}=\bar{y}'$, we have

$$\sum_{i=1}^{n} y_i[v_1(n)-(i-1)s(n)]=\sum_{i=1}^{kn} y_i'[v_1(kn)-(i-1)s(kn)]. \tag{25}$$

The r.h.s. of (25) can be written as

$$\sum_{i=1}^{n}\sum_{j=1}^{k} y'_{(i-1)k+j}[v_1(kn)-((i-1)k+j-1)s(kn)].$$

As, for $j=1,2,\ldots,k$, $y'_{(i-1)k+j}=y_i$ $\forall i$, then this is equal to

$$\sum_{i=1}^{n} y_i\sum_{j=1}^{k}[v_1(kn)-((i-1)k+j-1)s(kn)].$$

Then (25) implies

$$\sum_{i=1}^{n} y_i[v_1(n)-(i-1)s(n)]=\sum_{i=1}^{n} y_i\sum_{j=1}^{k}[v_1(kn)-((i-1)k+j-1)s(kn)]. \tag{26}$$

As (26) is to hold if the value of any individual income y_i varies, it implies

$$v_1(n)-(i-1)s(n)=\sum_{j=1}^{k}[v_1(kn)-((i-1)k+j-1)s(kn)]\quad \forall i. \tag{27}$$

Then, by Lemma 2.2,

$$I(Y) = -\sum_{i=1}^{n} y_i \frac{2s(2)}{n^2\bar{y}} [((v_1(1)/2s(2))+1)n+1-2i]$$

$$= -v_1(1) - 2s(2) - \frac{2s(2)}{n^2\bar{y}} \sum_{i=i}^{n} y_i(1-2i), \tag{28}$$

which has the same cardinal properties regardless of the value of $v_1(1)$ and the permissible value (>0, by Axiom T) of $s(2)$. If we take $v_1(1)=0$ and $s(2)=1/2$, then (28) is

$$I(Y) = -\sum_{i=1}^{n} y_i(n+1-2i)/n^2\bar{y}, \tag{29}$$

which is the Gini coefficient.

To paraphrase the theorem a bit: by Lemma 2.2, the only systems of weights of the form $v_i(n)-(i-1)s(n)$, dictated by Axiom E, to satisfy the properties postulated by Axiom S are of the form

$$v_i(n) = K'(Kn+1-2i)/n^2\bar{y}$$

where K and K' are any constant. This is easily seen if we write (23) with $v_1(1)/2s(2)+1=K$ and $2s(2)=K'$. This gives

$$I(Y) = -K' \sum_{i=1}^{n} y_i \frac{Kn+1-2i}{n^2\bar{y}}, \quad \text{with } K'>0, \tag{30}$$

for the inequality measure.

It turns out that whatever positive (by Axiom T) value we choose for $s(2)$ and therefore K', and whatever value we choose for $v_1(1)$ and therefore K, all the members of the family described by (30) are cardinally equivalent and the Gini coefficient is a member of the family.

Note that the mean difference does not have the corresponding property in the sense that to multiply (30) by $2\bar{y}$ (as $G=\Delta/2\bar{y}$) gives a family of 'generalized mean differences' of the form

$$-K'' \sum_{i=1}^{n} y_i(Kn+1-2i)/n^2$$

(which satisfies Axioms T, E and S but of course not the second part of CPC) whose ranking properties depend on the value of K, unless one confines oneself to the pure distribution case. This illustrates that the role of Axiom CPC is more intricate than merely to introduce the property of mean invariance.

Theorem 2.4. *The Gini coefficient satisfies Axioms* E, T, S *and* CPC.

Proof. It is straightforward to prove that the Gini coefficient satisfies the 4 axioms.

We will now consider a different system of axioms. From the comparability point of view, one might have in mind a stronger requirement than axiom CPC. One might indeed want to postulate that the range of an inequality index is to be the same over the redistribution of any total income not only between a given number of people but between any number of people. This would allow the comparison of redistributive inequality in England and India, for example, in the sense that the range of the index applied to both countries would be the same. Formally, one might want to postulate the following axiom.

Strong Comparability axiom (Axiom SC)

$$\mathrm{Min}\{I(Y)\colon Y \in D(n,S)\} = \mathrm{Min}\{I(Y)\colon Y \in D(n',S')\}$$

and

$$\mathrm{Max}\{I(Y)\colon Y \in D(n,S)\} = \mathrm{Max}\{I(Y)\colon Y \in D(n',S')\} \quad \forall n, n', S, S'.$$

We note two simple general propositions.

Proposition 2.5. *If any function $I(Y)$ satisfies Axioms* T, S *and* (*the first part of*) CPC, *then it satisfies the first part of Axiom* SC.

Proof. By Axiom T an egalitarian distribution gives the lowest value of the index for the redistribution of a given total income over a given number of people. Such a distribution is the n-replication of some unidimensional 'distribution'. For all such distributions $I(Y)$ has the same value by Axiom CPC. By Axiom S, $I(Y)$ has the same value for all egalitarian distributions, for all n and $\bar{y}$.

Proposition 2.6. *If any function $I(Y)$ satisfies Axioms* T *and* S, *then it cannot satisfy* (*the second part of*) *Axiom* SC.

Proof. Suppose Y has dimension n and everyone has zero income but one person; Y' is a distribution with $2n$ people, $2n-1$ of which have zero income and one who receives twice the total income of Y, Y'' is a distribution with $2n$ people, $2n-2$ of which receive zero income and two who receive each the total income of Y. Axiom S implies $I(Y) = I(Y'')$; Axiom T implies $I(Y') > I(Y'')$; therefore, it cannot be, as Axiom SC would imply, that $I(Y) = I(Y')$.

The Gini coefficient thus satisfies the first but not the second part of Axiom SC. It is easy to check that the upper bound of G is $(n-1)/n$. G satisfies the second part of Axiom SC asymptotically in the sense that its upper bound converges to a constant as n increases. Clearly if G was multiplied by $n/(n-1)$, it would satisfy Axiom SC. To transform the Gini coefficient in that fashion gives G', the Gini coefficient 'without repetition' introduced in Section 1 (Eq. (6)). G', on the other hand, does not satisfy Axiom S. It can be axiomatized by replacing Axiom CPC by SC and dropping Axiom S from the axiomatic system of G, as the following theorem establishes.

Theorem 2.7. *If $I(Y)$ satisfies Axioms* E, T *and* SC, *then it is cardinally equivalent to G'*

Proof. As Axiom SC implies Axiom CPC, we can use the result of Lemma 2.1

$$I(Y) = -\sum_{i=1}^{n} y_i[v_1(n) - (i-1)s(n)]/\bar{y} \tag{31}$$

where $v_1(n)$ and $s(n)$ are functions of n and $s(n)>0$. If someone receives all the income, then, by Axiom SC, $I(Y)$ must be a constant. This gives

$$-n[v_1(n) - (n-1)s(n)] = C, \tag{32}$$

where C is some constant. If everyone receives the same income, by Axiom SC, $I(Y)$ must be some constant (smaller than C by Axiom T). This gives

$$-n[v_1(n) - (n-1)s(n)/2] = C' \tag{33}$$

where C' is some constant. (33) minus (32) gives $n(n-1)s(n)/2 = C - C' > 0$. Let us write $C - C' = 2L$, then $s(n) = L/n(n-1) > 0$.

Then (32) implies $v_1(n) = M/n$ where M is some constant, and (31) gives

$$I(Y) = -\sum_{i=1}^{n} y_i[M/n - (i-1)L/n(n-1)]/\bar{y} = -M + L\sum_{i=1}^{n} y_i \frac{(i-1)}{\bar{y}n(n-1)},$$

whose cardinal properties are the same for all M and strictly positive L. By Axiom T, $L>0$. If one takes $M=1$ and $L=1/2$, one obtains

$$I(Y) = -1 - \sum_{i=1}^{n} y_i \frac{(2-2i)}{\bar{y}n(n-1)} = -\left[\sum_{i=1}^{n} y_i \frac{(n-1)}{\bar{y}n(n-1)} + \sum_{i=1}^{n} y_i \frac{(2-2i)}{\bar{y}n(n-1)}\right]$$
$$= -\sum_{i=1}^{n} y_i \frac{(n+1-2i)}{\bar{y}n(n-1)}$$

Theorem 2.8. *G' satisfies Axioms* E, T *and* SC.

Proof. It is straightforward to prove that G' satisfies Axioms E, T and SC.

Although G' does not satisfy Axiom S exactly, it satisfies it asymptotically in the sense that as n increases, G' converges to a function which satisfies Axiom S.

3. Conclusion

In any empirical work where n is 'large', G and G' are equivalent and the two pairs of Theorems 2.3, 2.4 and 2.7, 2.8 offer two different routes to justify or gain insight into the ranking properties of the Gini coefficient. Axiom SC introduces a convenient comparability property but is somewhat objectionable in itself; it implies for example that if $n=2$ and someone (i.e. 50% of the population) receives all the income, the level of inequality should be the same as when, say, $n=100$ and someone monopolizes the whole income (i.e. 1% of the population). This violates the common sense embodied into Axiom S. One way of looking at the set of 4 theorems presented above is to see in it a demonstration of the proposition that for an inequality index based on a rank order system of weights in the pure distribution case, such as described by Axiom E, the extension to the case of a variable population and/or total income does not create in practice any conflict between the desirability, from a distributional judgment point of view, of having Axiom S satisfied and the convenience, from a pragmatic comparability point of view, of having Axiom SC satisfied.

Let us state such a property formally.

Property P. *$I(Y)$ satisfies both Axioms* S *and* SC, *at least asymptotically.*

To illustrate that Property P is hardly ubiquitous among inequality indices we conclude by having a look at some frequently used inequality indices with Property P in mind. First, from such a point of view, $G-G'$ appears quite different from the inequality measures obtained by specifying the ranking properties in the pure distribution case with a system of weights based on the difference from the mean. Indeed we can write the variance as $V=-\sum_{i=1}^{n} y_i w_i(Y)$, with $w_i(Y)=(\bar{y}-y_i)/n$.

Of our axioms, V satisfies Axioms T, S and the first part of CPC, like the mean difference (2) which is in a sense its counterpart in the rank order version of the weights. The variance does not satisfy Property P, of course.

By dividing V by $\bar{y}^2$ we get the 'relative' inequality measure corresponding to the variance; the square of the coefficient of variation

$$C^2=-\sum_{i=1}^{n} y_i(\bar{y}-y_i)/n\bar{y}^2,$$

which satisfies Axioms T, S and both parts of CPC. If everyone has the same income, then $C^2=0$; if someone receives all the income, then $C^2=(n-1)$. Like G, C^2 does not satisfy Axiom SC exactly; unlike it it does not satisfy it asymptotically either and therefore does not have Property *P*. If one wanted to transform C^2 in such a way that it satisfies Axiom SC, one would have to multiply it by $1/(n-1)$. This gives an index which can be written as

$$C^{*2}=-1/(n-1)-\sum_{i=1}^{n} y_i^2/\bar{y}^2 n(n-1).$$

If everyone has the same income, $C^{*2}=0$; if someone receives all the income, $C^{*2}=1$. Therefore, C^{*2} satisfies Axiom SC. It violates Axiom S 'very badly' though: for a large n, if two people equally share all the income and everyone else has zero income – for an even n the duplication of some income distribution with a single non-zero income – then the value of C^{*2} is approximately equal to 1/2.

Herfindalh's index is

$$H = \sum_{i=1}^{n} y_i^2/\bar{y}^2 n^2.$$

It satisfies Axiom SC asymptotically as its upper bound is 1 and its lower bound is $1/n$. For a large n, it is approximately equal to the second term of C^{*2}, as can easily be established and behaves in a way similar to C^{*2}'s with respect to Axiom S.

Theil's measure is

$$T = \sum_{i=1}^{?} (y_i/n\bar{y})\ln(y_i/\bar{y}).$$

It satisfies Axiom S but if someone receives all the income, its value is $\ln(n)$ and it does not therefore satisfy Property P.

The relative mean deviation: $\sum_{i=1}^{n} |\bar{y}-y_i|/\bar{y}n$, satisfies Property P as can easily be checked. It is quite unsuitable as an inequality index as it does not satisfy Axiom T.

Atkinson's index

$$A = 1 - \frac{1}{\bar{y}}\left[\frac{1}{n}\sum_{i=1}^{n} y_i^{1-\varepsilon}\right]^{1/(1-\varepsilon)} \quad \text{for } 0<\varepsilon<1,$$

is the only familiar inequality index to satisfy Axiom T and Property P, like $G-G'$. It satisfies Axiom S exactly and its range is $[0, 1-(1/n)^{1/(1-\varepsilon)}]$, which means that it satisfies Axiom SC asymptotically and thus Property P.

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