# 1 Which of the relations above do not represent functions? Justify your answer.

## 1.1 Relation $R_2$

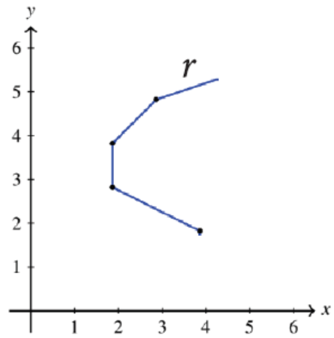
$$
\begin{aligned}
R_2 &= \Big\{(1,2),(2,2),(3,3),(4,4),(2,3),(5,2)\Big\} \\
\implies & \Big((2,2) \in R_2\Big) \land \Big((2,3) \in R_2\Big) \\
\implies & \exists x \exists y_1 \exists y_2 \Big(\big(x,y_1\big) \in R_2\Big) \land \Big(\big(x,y_2\big) \in R_2\Big) \land \Big(y_1 \neq y_2\Big) \\
\implies & \exists x \exists y_1 \exists y_2 \Big(\big(x,y_1\big) \in R_2\Big) \land \Big(\big(x,y_2\big) \in R_2\Big) \land \neg\Big(y_1 = y_2\Big) \\
\implies & \exists x \exists y_1 \exists y_2 \Big(\big(x,y_1\big) \in R_2 \land \big(x,y_2\big) \in R_2\Big) \land \neg\Big(y_1 = y_2\Big) \\
\implies & \exists x \exists y_1 \exists y_2 \neg\Big(\big(x,y_1\big) \in R_2 \land \big(x,y_2\big) \in R_2\Big) \lor \Big(y_1 = y_2\Big) \\
\implies & \exists x \exists y_1 \exists y_2 \Big(\big(x,y_1\big) \in R_2 \land \big(x,y_2\big) \in R_2\Big) \implies \Big(y_1 = y_2\Big) \\
\implies & \neg \forall x \forall y_1 \forall y_2 \Big(\big(x,y_1\big) \in R_2 \land \big(x,y_2\big) \in R_2\Big) \implies \Big(y_1 = y_2\Big) \\
\implies & \neg\Big(R_2 : D_{R_2} \to R_{R_2}\Big) \implies R_2 \text{ is not a function.}
\end{aligned}
$$

## 1.2 Relation $Q$

$$
Q \quad = \quad \begin{tikzcd}[row sep=tiny]
a \arrow[r] & 1 \\
b \arrow[r] & 2 \\
c \arrow[r] \arrow[dr] & 3 \\
& 4
\end{tikzcd}
$$

$$
\begin{aligned}
\implies & \Big((c,3) \in Q\Big) \land \Big((c,4) \in Q\Big) \\
\implies & \exists x \exists y_1 \exists y_2 \Big(\big(x,y_1\big) \in Q\Big) \land \Big(\big(x,y_2\big) \in Q\Big) \land \Big(y_1 \neq y_2\Big) \\
\implies & \exists x \exists y_1 \exists y_2 \Big(\big(x,y_1\big) \in Q\Big) \land \Big(\big(x,y_2\big) \in Q\Big) \land \neg\Big(y_1 = y_2\Big) \\
\implies & \exists x \exists y_1 \exists y_2 \Big(\big(x,y_1\big) \in Q \land \big(x,y_2\big) \in Q\Big) \land \neg\Big(y_1 = y_2\Big) \\
\implies & \exists x \exists y_1 \exists y_2 \neg\Big(\big(x,y_1\big) \in Q \land \big(x,y_2\big) \in Q\Big) \lor \Big(y_1 = y_2\Big) \\
\implies & \exists x \exists y_1 \exists y_2 \Big(\big(x,y_1\big) \in Q \land \big(x,y_2\big) \in Q\Big) \implies \Big(y_1 = y_2\Big) \\
\implies & \neg \forall x \forall y_1 \forall y_2 \Big(\big(x,y_1\big) \in Q \land \big(x,y_2\big) \in Q\Big) \implies \Big(y_1 = y_2\Big) \\
\implies & \neg\Big(Q : D_Q \to R_Q\Big) \implies Q \text{ is not a function.}
\end{aligned}
$$

## 1.3 Relation $r$




$$\implies \Big((2,3) \in r\Big) \wedge \Big((2,4) \in r\Big)$$

$$\implies \exists x \exists y_1 \exists y_2 \ \Big(\big(x, y_1\big) \in r\Big) \wedge \Big(\big(x, y_2\big) \in r\Big) \wedge \Big(y_1 \neq y_2\Big)$$

$$\implies \exists x \exists y_1 \exists y_2 \ \Big(\big(x, y_1\big) \in r\Big) \wedge \Big(\big(x, y_2\big) \in r\Big) \wedge \neg\Big(y_1 = y_2\Big)$$

$$\implies \exists x \exists y_1 \exists y_2 \ \Big(\big(x, y_1\big) \in r \wedge \big(x, y_2\big) \in r\Big) \wedge \neg\Big(y_1 = y_2\Big)$$

$$\implies \exists x \exists y_1 \exists y_2 \ \neg\Big(\big(x, y_1\big) \in r \wedge \big(x, y_2\big) \in r\Big) \vee \Big(y_1 = y_2\Big)$$

$$\implies \exists x \exists y_1 \exists y_2 \ \Big(\big(x, y_1\big) \in r \wedge \big(x, y_2\big) \in r\Big) \implies \Big(y_1 = y_2\Big)$$

$$\implies \neg \forall x \forall y_1 \forall y_2 \ \Big(\big(x, y_1\big) \in r \wedge \big(x, y_2\big) \in r\Big) \implies \Big(y_1 = y_2\Big)$$

$$\implies \neg\Big(r : D_r \to R_r\Big) \implies \ r \text{ is not a function.}$$

# 2 Assume that each of the functions listed above is onto. Find the domain and range of each function.

## 2.1 Function $R_1$

$$D_{R_1} = \{a, b, c, m\}$$

$$R_{R_1} = \{1, 2, 3, 5\}$$

## 2.2 Function $I$

$$\begin{aligned} I(x) &= x^2 - 1 \\ D_I &= \mathbb{R} \end{aligned}$$

$$\begin{aligned} I(x) &= x^2 - 1 \\ x^2 &= I(x) + 1 \\ x &= \pm\sqrt{I(x) + 1} \\ I(x) + 1 &\geq 0 \\ I(x) &\geq -1 \\ R_I &= [-1, \infty) \end{aligned}$$

## 2.3 Function $J$

$$\begin{aligned} J(x) &= 2x + 3 \\ D_J &= \mathbb{R} \end{aligned}$$

$$\begin{aligned} J(x) &= 2x + 3 \\ 2x &= J(x) - 3 \\ x &= \frac{J(x) - 3}{2} \\ J(x) &\in \mathbb{R} \\ R_J &= \mathbb{R} \end{aligned}$$

## 2.4 Function $K$

$$\begin{aligned} K(x) &= \sqrt{x + 5} \\ x + 5 &\geq 0 \\ x &\geq -5 \\ D_K &= [-5, \infty) \end{aligned}$$

$$\begin{aligned} K(x) &= \sqrt{x + 5} \\ K(x)^2 &= x + 5 \\ x &= K(x)^2 - 5 \\ K(x)^2 - 5 &\in [-5, \infty) \\ K(x)^2 - 5 &\geq -5 \\ K(x)^2 &\geq 0 \\ K(x) &\geq 0 \\ R_K &= [0, \infty) \end{aligned}$$

## 2.5 Function $T$

$$T(x) = \begin{cases} x+3, & x<0 \\ 6-x, & x \geq 0 \end{cases}$$
$$D_T = \{x \mid x<0\} \cup \{x \mid x \geq 0\}$$
$$D_T = \mathbb{R}$$

$$T(x) = \begin{cases} x+3, & x<0 \\ 6-x, & x \geq 0 \end{cases}$$
$$R_{T_{x<0}} = \{x+3 \mid x<0\}$$
$$R_{T_{x<0}} = (-\infty, 3)$$
$$R_{T_{x\geq 0}} = \{6-x \mid x \geq 0\}$$
$$R_{T_{x\geq 0}} = (-\infty, 6]$$
$$R_T = R_{T_{x<0}} \cup R_{T_{x\geq 0}}$$
$$R_T = (-\infty, 6]$$

## 2.6 Function $V$

$$V(x) = \frac{x^2+2}{x^2-5}$$
$$x^2-5 \neq 0$$
$$x^2 \neq 5$$
$$x \neq \pm\sqrt{5}$$
$$D_V = \mathbb{R} \setminus \{\sqrt{5}, -\sqrt{5}\}$$

$$V(x) = \frac{x^2+2}{x^2-5}$$
$$V(x)x^2 - 5V(x) = x^2+2$$
$$V(x)x^2 - x^2 = 5V(x)+2$$
$$x^2\Big(V(x)-1\Big) = 5V(x)+2$$
$$x^2 = \frac{5V(x)+2}{V(x)-1}$$
$$x = \pm\sqrt{\frac{5V(x)+2}{V(x)-1}}$$
$$\frac{5V(x)+2}{V(x)-1} \geq 0$$
$$\begin{cases} 5V(x)+2 \geq 0 \\ V(x)-1>0 \end{cases} \vee \begin{cases} 5V(x)+2 \leq 0 \\ V(x)-1<0 \end{cases}$$
$$\begin{cases} V(x) \geq -\frac{2}{5} \\ V(x)>1 \end{cases} \vee \begin{cases} V(x) \leq -\frac{2}{5} \\ V(x)<1 \end{cases}$$
$$\Big(V(x)>1\Big) \vee \Big(V(x) \leq -\frac{2}{5}\Big)$$
$$V(x) \in (1, \frac{-2}{5}]$$
$$R_V = (1, -\frac{2}{5}]$$

## 2.7 Function $\alpha$

$$D_\alpha = \{a, b, c, d, e, f\}$$

$$R_\alpha = \{1, 2, 3, 4\}$$

## 2.8 Function $\beta$

$$D_\beta = \{1, 2, 3, 4\} \qquad R_\beta = \{2, a, 3\}$$

## 2.9 Function $f$

$$D_f = [1, 6] \qquad R_f = [2, 6]$$

## 2.10 Function $f + g$

$$D_{f+g} = [1, 6] \qquad R_{f+g} = [4, 5]$$

# 3 Which of the functions listed above are not one-to-one? Justify your answer.

## 3.1 Function $I$

$$
\begin{aligned}
I(x) &= x^2 - 1 \\
\implies & \Big((1,0) \in I\Big) \wedge \Big(( -1,0) \in I\Big) \\
\implies & \exists x_1 \exists x_2 \exists y \, \Big((x_1, y) \in I\Big) \wedge \Big((x_2, y) \in I\Big) \wedge \Big(x_1 \neq x_2\Big) \\
\implies & \exists x_1 \exists x_2 \exists y \, \Big((x_1, y) \in I\Big) \wedge \Big((x_2, y) \in I\Big) \wedge \neg\Big(x_1 = x_2\Big) \\
\implies & \exists x_1 \exists x_2 \exists y \, \Big((x_1, y) \in I \wedge (x_2, y) \in I\Big) \wedge \neg\Big(x_1 = x_2\Big) \\
\implies & \exists x_1 \exists x_2 \exists y \, \neg\Big((x_1, y) \in I \wedge (x_2, y) \in I\Big) \vee \Big(x_1 = x_2\Big) \\
\implies & \exists x_1 \exists x_2 \exists y \, \Big((x_1, y) \in I \wedge (x_2, y) \in I\Big) \implies \Big(x_1 = x_2\Big) \\
\implies & \neg \forall x_1 \forall x_2 \forall y \, \Big((x_1, y) \in I \wedge (x_2, y) \in I\Big) \implies \Big(x_1 = x_2\Big) \\
\implies & \neg\Big(I : D_I \rightarrowtail R_I\Big) \implies \text{ $I$ is not a one-to-one function.}
\end{aligned}
$$

## 3.2 Function $T$

$$
\begin{aligned}
& T(x) = \begin{cases} x + 3, & x < 0 \\ 6 - x, & x \geq 0 \end{cases} \\
\implies & \Big(( -1,2) \in T\Big) \wedge \Big((4,2) \in T\Big) \\
\implies & \exists x_1 \exists x_2 \exists y \, \Big((x_1, y) \in T\Big) \wedge \Big((x_2, y) \in T\Big) \wedge \Big(x_1 \neq x_2\Big) \\
\implies & \exists x_1 \exists x_2 \exists y \, \Big((x_1, y) \in T\Big) \wedge \Big((x_2, y) \in T\Big) \wedge \neg\Big(x_1 = x_2\Big) \\
\implies & \exists x_1 \exists x_2 \exists y \, \Big((x_1, y) \in T \wedge (x_2, y) \in T\Big) \wedge \neg\Big(x_1 = x_2\Big) \\
\implies & \exists x_1 \exists x_2 \exists y \, \neg\Big((x_1, y) \in T \wedge (x_2, y) \in T\Big) \vee \Big(x_1 = x_2\Big) \\
\implies & \exists x_1 \exists x_2 \exists y \, \Big((x_1, y) \in T \wedge (x_2, y) \in T\Big) \implies \Big(x_1 = x_2\Big) \\
\implies & \neg \forall x_1 \forall x_2 \forall y \, \Big((x_1, y) \in T \wedge (x_2, y) \in T\Big) \implies \Big(x_1 = x_2\Big) \\
\implies & \neg\Big(T : D_T \rightarrowtail R_T\Big) \implies \text{ $T$ is not a one-to-one function.}
\end{aligned}
$$

## 3.3 Function $V$

$$
V(x) = \frac{x^2 + 2}{x^2 - 5}
$$

$$\implies \Big((\sqrt{2},0) \in V\Big) \wedge \Big(( -\sqrt{2},\ 0) \in V\Big)$$

$$\implies \exists x_1 \exists x_2 \exists y \ \Big((x_1,y) \in V\Big) \wedge \Big((x_2,y) \in V\Big) \wedge \Big(x_1 \neq x_2\Big)$$

$$\implies \exists x_1 \exists x_2 \exists y \ \Big((x_1,y) \in V\Big) \wedge \Big((x_2,y) \in V\Big) \wedge \neg\Big(x_1 = x_2\Big)$$

$$\implies \exists x_1 \exists x_2 \exists y \ \Big((x_1,y) \in V \wedge (x_2,y) \in V\Big) \wedge \neg\Big(x_1 = x_2\Big)$$

$$\implies \exists x_1 \exists x_2 \exists y \ \neg\Big((x_1,y) \in V \wedge (x_2,y) \in V\Big) \vee \Big(x_1 = x_2\Big)$$

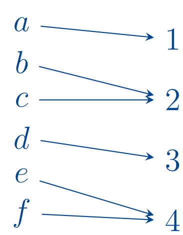
$$\implies \exists x_1 \exists x_2 \exists y \ \Big((x_1,y) \in V \wedge (x_2,y) \in V\Big) \implies \Big(x_1 = x_2\Big)$$

$$\implies \neg \forall x_1 \forall x_2 \forall y \ \Big((x_1,y) \in V \wedge (x_2,y) \in V\Big) \implies \Big(x_1 = x_2\Big)$$

$$\implies \neg\Big(V : D_V \rightarrowtail R_V\Big) \implies V \text{ is not a one-to-one function.}$$

## 3.4 Function $\alpha$

$$\alpha \quad = $$




$$\implies \Big((e,4) \in \alpha\Big) \wedge \Big((f,\ 4) \in \alpha\Big)$$

$$\implies \exists x_1 \exists x_2 \exists y \ \Big((x_1,y) \in \alpha\Big) \wedge \Big((x_2,y) \in \alpha\Big) \wedge \Big(x_1 \neq x_2\Big)$$

$$\implies \exists x_1 \exists x_2 \exists y \ \Big((x_1,y) \in \alpha\Big) \wedge \Big((x_2,y) \in \alpha\Big) \wedge \neg\Big(x_1 = x_2\Big)$$

$$\implies \exists x_1 \exists x_2 \exists y \ \Big((x_1,y) \in \alpha \wedge (x_2,y) \in \alpha\Big) \wedge \neg\Big(x_1 = x_2\Big)$$

$$\implies \exists x_1 \exists x_2 \exists y \ \neg\Big((x_1,y) \in \alpha \wedge (x_2,y) \in \alpha\Big) \vee \Big(x_1 = x_2\Big)$$

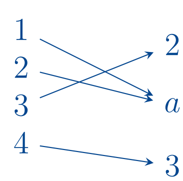
$$\implies \exists x_1 \exists x_2 \exists y \ \Big((x_1,y) \in \alpha \wedge (x_2,y) \in \alpha\Big) \implies \Big(x_1 = x_2\Big)$$

$$\implies \neg \forall x_1 \forall x_2 \forall y \ \Big((x_1,y) \in \alpha \wedge (x_2,y) \in \alpha\Big) \implies \Big(x_1 = x_2\Big)$$

$$\implies \neg\Big(\alpha : D_\alpha \rightarrowtail R_\alpha\Big) \implies \alpha \text{ is not a one-to-one function.}$$

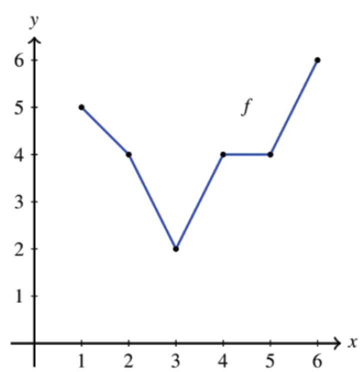
## 3.5 Function $\beta$

$$\beta \quad = $$



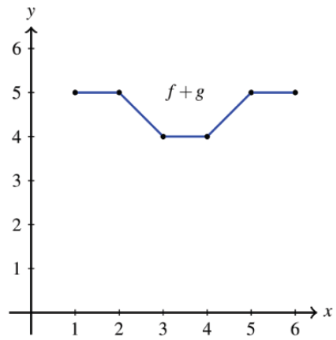
$$\implies \Big(\big(1,a\big) \in \beta\Big) \land \Big(\big(2,a\big) \in \beta\Big)$$
$$\implies \exists x_1 \exists x_2 \exists y \ \Big(\big(x_1,y\big) \in \beta\Big) \land \Big(\big(x_2,y\big) \in \beta\Big) \land \Big(x_1 \neq x_2\Big)$$
$$\implies \exists x_1 \exists x_2 \exists y \ \Big(\big(x_1,y\big) \in \beta\Big) \land \Big(\big(x_2,y\big) \in \beta\Big) \land \neg\Big(x_1 = x_2\Big)$$
$$\implies \exists x_1 \exists x_2 \exists y \ \Big(\big(x_1,y\big) \in \beta \land \big(x_2,y\big) \in \beta\Big) \land \neg\Big(x_1 = x_2\Big)$$
$$\implies \exists x_1 \exists x_2 \exists y \ \neg\Big(\big(x_1,y\big) \in \beta \land \big(x_2,y\big) \in \beta\Big) \lor \Big(x_1 = x_2\Big)$$
$$\implies \exists x_1 \exists x_2 \exists y \ \Big(\big(x_1,y\big) \in \beta \land \big(x_2,y\big) \in \beta\Big) \implies \Big(x_1 = x_2\Big)$$
$$\implies \neg \forall x_1 \forall x_2 \forall y \ \Big(\big(x_1,y\big) \in \beta \land \big(x_2,y\big) \in \beta\Big) \implies \Big(x_1 = x_2\Big)$$
$$\implies \neg\Big(\beta : D_\beta \rightarrowtail R_\beta\Big) \implies \beta \text{ is not a one-to-one function.}$$

## 3.6 Relation $f$




$$\implies \Big(\big(2,4\big) \in f\Big) \land \Big(\big(4,4\big) \in f\Big)$$
$$\implies \exists x \exists y_1 \exists y_2 \ \Big(\big(x,y_1\big) \in f\Big) \land \Big(\big(x,y_2\big) \in f\Big) \land \Big(y_1 \neq y_2\Big)$$
$$\implies \exists x \exists y_1 \exists y_2 \ \Big(\big(x,y_1\big) \in f\Big) \land \Big(\big(x,y_2\big) \in f\Big) \land \neg\Big(y_1 = y_2\Big)$$
$$\implies \exists x \exists y_1 \exists y_2 \ \Big(\big(x,y_1\big) \in f \land \big(x,y_2\big) \in f\Big) \land \neg\Big(y_1 = y_2\Big)$$
$$\implies \exists x \exists y_1 \exists y_2 \ \neg\Big(\big(x,y_1\big) \in f \land \big(x,y_2\big) \in f\Big) \lor \Big(y_1 = y_2\Big)$$
$$\implies \exists x \exists y_1 \exists y_2 \ \Big(\big(x,y_1\big) \in f \land \big(x,y_2\big) \in f\Big) \implies \Big(y_1 = y_2\Big)$$
$$\implies \neg \forall x \forall y_1 \forall y_2 \ \Big(\big(x,y_1\big) \in f \land \big(x,y_2\big) \in f\Big) \implies \Big(y_1 = y_2\Big)$$
$$\implies \neg\Big(f : D_f \to R_f\Big) \implies f \text{ is not a one-to-one function.}$$

### 3.7 Relation $f + g$




$$
\begin{aligned}
&\implies \Big((3,4) \in f+g\Big) \land \Big((4,4) \in f+g\Big) \\
&\implies \exists x \exists y_1 \exists y_2 \ \Big((x,y_1) \in f+g\Big) \land \Big((x,y_2) \in f+g\Big) \land \Big(y_1 \neq y_2\Big) \\
&\implies \exists x \exists y_1 \exists y_2 \ \Big((x,y_1) \in f+g\Big) \land \Big((x,y_2) \in f+g\Big) \land \neg\Big(y_1 = y_2\Big) \\
&\implies \exists x \exists y_1 \exists y_2 \ \Big((x,y_1) \in f+g \land (x,y_2) \in f+g\Big) \land \neg\Big(y_1 = y_2\Big) \\
&\implies \exists x \exists y_1 \exists y_2 \ \neg\Big((x,y_1) \in f+g \land (x,y_2) \in f+g\Big) \lor \Big(y_1 = y_2\Big) \\
&\implies \exists x \exists y_1 \exists y_2 \ \Big((x,y_1) \in f+g \land (x,y_2) \in f+g\Big) \implies \Big(y_1 = y_2\Big) \\
&\implies \neg \forall x \forall y_1 \forall y_2 \ \Big((x,y_1) \in f+g \land (x,y_2) \in f+g\Big) \implies \Big(y_1 = y_2\Big) \\
&\implies \neg\Big(f : D_{f+g} \to R_{f+g}\Big) \implies \ f+g \text{ is not a one-to-one function.}
\end{aligned}
$$