Sensor Data Fusion

Exercise 6

Prof. Dr.-Ing. Marcus Baum

Dr. Kolja Thormann

www.fusion.informatik.uni-goettingen.de

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN

DATA FUSION Lab

Today

- Lecture review
- Questions
- Homework 5 solution
- Example - Gaussian Joint Density
- Problem 6 - MMSE Estimator
- Homework 6 presentation

Questions

Why is it difficult to obtain the optimal Bayesian estimator?

- No closed-form solution (in general)
- complete prior and likelihood information required
- Kalman filter is optimal Bayesian filter if
  - Linearity is given
  - All distributions are Gaussian

Questions

What happens in the Kalman filter update for **H=I** if the noise of the prior is much higher than the measurement noise and vice versa?

Regard the update formula:

$$\hat{x}_{k+1} = \hat{x}_k + \underbrace{\mathbf{C}_k\mathbf{H}^\mathrm{T}(\overbrace{\mathbf{H}\mathbf{C}_k\mathbf{H}^\mathrm{T} + \mathbf{C}_{ee}}^{\mathbf{S}})^{-1}}_{\mathbf{K}}(y - \mathbf{H}\hat{x}_k)$$

With the prior noise $\mathbf{C}_k$ being much larger then the measurement noise $\mathbf{C}_{ee}$, $\mathbf{K}$ gets close to being the identity matrix, so that the state is subtracted and the update will only use the measurement to set the new state. Vice versa, $\mathbf{S}^{-1}$ would be close to zero, leaving only the prior state.

Which assumption about the different distributions are made to do recursive estimation in the Kalman filter?

We assume independence between the prior x and each measurement noise e_k as well as all the noises among each other, i.e., Cov$[x, e_k] = 0$ and Cov$[e_k, e_l] = 0$ with $k \neq l$.

$$p(x|y_1,\ldots,y_n) = \frac{p(y_1,\ldots,y_n|x)p(x)}{p(y_1,\ldots,y_n)} = \frac{\prod_{k=1}^{n}\big(p(y_k|x)\big)p(x)}{p(y_2,\ldots,y_n|y_1)p(y_1)} = \frac{\prod_{k=2}^{n}\big(p(y_k|x)\big)p(x|y_1)}{p(y_2,\ldots,y_n|y_1)}$$

Homework 5: Solution

Consider again the setup of the Silly Estimator exercise with $x \sim \mathcal{N}(1,1)$ and $y = x + e$ with $e \sim \mathcal{N}(0,1)$.

a) Write a function which calculates a possible value for x based on its distribution and then takes a measurement from that value using the measurement noise.

b) Consider the natural estimator $\theta_n(y) = y$. Calculate the empirical mean square error of 1000 runs using this estimator and compare it with the analytic solution.

c) Repeat the process with the silly estimator $\theta_s(y) = 0$.

see notebook for solution

From exercise 5:

$$\text{BMSE}(\theta_n(y)) = \sigma_e^2 = 1$$

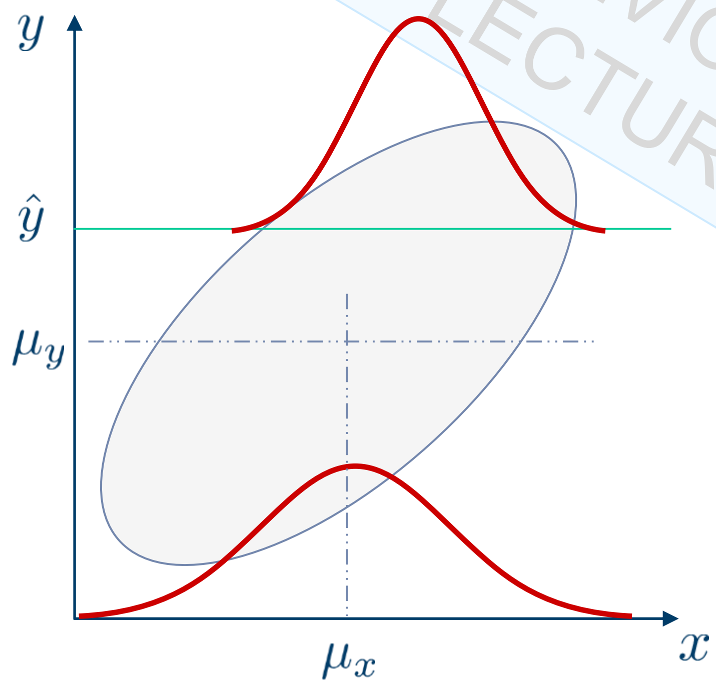
$$\text{BMSE}(\theta_s(y)) = \sigma_x^2 + \mu_x^2 = 2$$

Example – Gaussian Joint Density: Conditioning

PREVIOUS LECTURE

Gaussian joint density:

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \underbrace{\begin{bmatrix} \mathbf{C}_{xx} & \mathbf{C}_{xy} \\ \mathbf{C}_{yx} & \mathbf{C}_{yy} \end{bmatrix}}_{\mathbf{C}}\right)$$



$$\begin{aligned} \mu_x^+ &= \mu_x + \mathbf{C}_{xy}\mathbf{C}_{yy}^{-1}(\hat{y} - \mu_y) \\ \mathbf{C}^+ &= \mathbf{C}_{xx} - \mathbf{C}_{xy}\mathbf{C}_{yy}^{-1}\mathbf{C}_{yx} \end{aligned}$$

Conditioning on y:

$$p(x|y = \hat{y}) = \mathcal{N}(\mu_x^+, \mathbf{C}^+)$$

Example – Gaussian Joint Density: Conditioning

Remark 1:

$$\underbrace{\begin{bmatrix} \mathbf{I} & -\mathbf{C}_{xy}\mathbf{C}_{yy}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}}_{\mathbf{T}} \cdot \underbrace{\begin{bmatrix} \mathbf{C}_{xx} & \mathbf{C}_{xy} \\ \mathbf{C}_{yx} & \mathbf{C}_{yy} \end{bmatrix}}_{\mathbf{C}} \cdot \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{C}_{yy}^{-1}\mathbf{C}_{yx} & \mathbf{I} \end{bmatrix}}_{\mathbf{T}^T} = \underbrace{\begin{bmatrix} \mathbf{C}_{xx} - \mathbf{C}_{xy}\mathbf{C}_{yy}^{-1}\mathbf{C}_{yx} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{yy} \end{bmatrix}}_{\mathbf{D}}$$

Hence $\mathbf{C} = \mathbf{T}^{-1}\mathbf{D}\mathbf{T}^{-T}$ and $\mathbf{C}^{-1} = \mathbf{T}^T\mathbf{D}^{-1}\mathbf{T}$

Example – Gaussian Joint Density: Conditioning

- Conditioning:

$$p(x|y) = \frac{p(x,y)}{p(y)}$$

$$= \frac{\sqrt{(2\pi)^m}}{\sqrt{(2\pi)^{m+n}}} \frac{\sqrt{\det(\mathbf{C}_{yy})}}{\sqrt{\det(\mathbf{C})}} \frac{\exp\left(-\frac{1}{2}\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}^T \mathbf{C}^{-1} \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}\right)}{\exp\left(-\frac{1}{2}(y-\mu_y)^T \mathbf{C}_{yy}^{-1}(y-\mu_y)\right)}$$

- With Remark 1:

$$\det\{\mathbf{C}\} = \det\{\mathbf{C}_{xx} - \mathbf{C}_{xy}\mathbf{C}_{yy}^{-1}\mathbf{C}_{yx}\}\det\{\mathbf{C}_{yy}\}$$

Example – Gaussian Joint Density: Conditioning

As

$$
\begin{aligned}
&\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}^T \mathbf{C}^{-1} \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix} \\
&= \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}^T \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{C}_{yy}^{-1}\mathbf{C}_{yx} & \mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{C}_{xx} - \mathbf{C}_{xy}\mathbf{C}_{yy}^{-1}\mathbf{C}_{yx} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{yy} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \mathbf{I} & -\mathbf{C}_{xy}\mathbf{C}_{yy}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix} \\
&= \begin{bmatrix} x-\mu_x^+ \\ y-\mu_y \end{bmatrix}^T \begin{bmatrix} \mathbf{C}_{xx} - \mathbf{C}_{xy}\mathbf{C}_{yy}^{-1}\mathbf{C}_{yx} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{yy} \end{bmatrix}^{-1} \begin{bmatrix} x-\mu_x^+ \\ y-\mu_y \end{bmatrix} \\
&= (x-\mu_x^+)^T(\mathbf{C}_{xx} - \mathbf{C}_{xy}\mathbf{C}_{yy}^{-1}\mathbf{C}_{yx})^{-1}(x-\mu_x^+) + (y-\mu_y)^T\mathbf{C}_{yy}^{-1}(y-\mu_y)
\end{aligned}
$$

with $\mu_x^+ = \mu_x + \mathbf{C}_{xy}\mathbf{C}_{yy}^{-1}(y-\mu_y)$, the conditional distribution from the previous slide simplifies to the desired formula.

Problem 6 – MMSE Estimation

Consider a two-dimensional joint density

$$p(x,y) = \begin{cases} 2 & \text{for } [x,y] \in S \\ 0 & \text{otherwise} \end{cases}$$

where the set S is a triangle specified by the points $[0,0]$, $[1,0]$, and $[1,1]$.

a) What is the conditional distribution $p(x|y)$?

b) What is the MMSE estimator of x given y?

c) What is the conditional MSE associated to the MMSE estimator?

d) What is the unconditional MSE associated to the MMSE estimator?

If $x \sim \mathcal{U}(a,b)$, we have

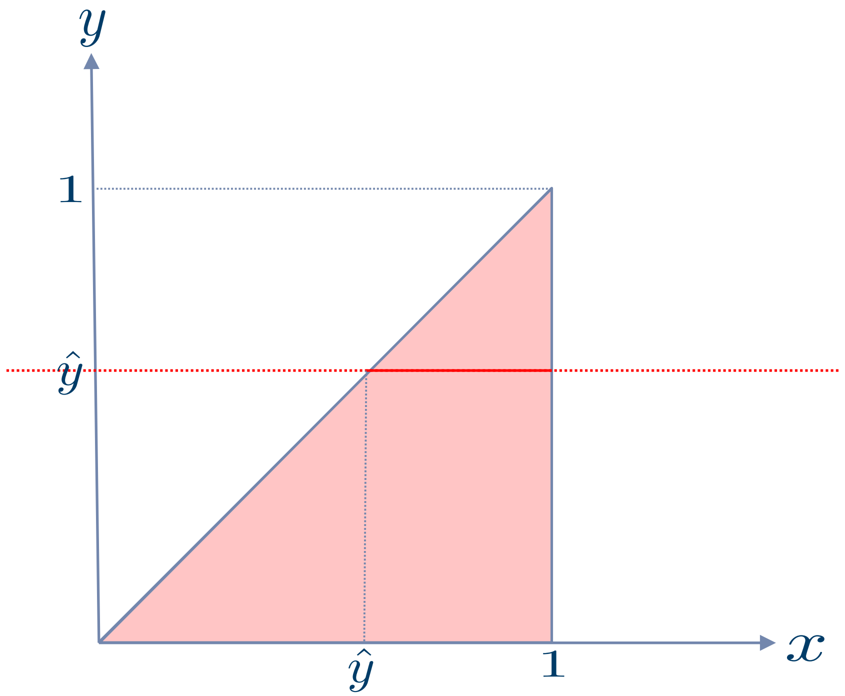
- $p(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$
- $\mathrm{E}[x] = \frac{1}{2}(a+b)$
- $\mathrm{Var}[x] = \frac{1}{12}(b-a)^2$

Problem 6: Solution

a) $p(x|y) = \begin{cases} \frac{1}{1-y} & \text{for } x \in [y, 1] \\ 0 & \text{otherwise} \end{cases}$

b) $\theta_y = \mathrm{E}[x|y] = \frac{1}{2}(y+1)$

c) $\mathrm{MSE}(\theta_y|y) = \mathrm{E}[(x - \mathrm{E}(x|y))^2|y] = \mathrm{Var}[x|y] = \frac{1}{12}(1-y)^2$



Problem 6: Solution

d)

$$
\begin{aligned}
\text{MSE}(\theta_y) &= \mathrm{E}[||x-\theta_y||^2] \\
&= \mathrm{E}_y[\mathrm{E}[||x-\mathrm{E}[x|y]||^2|y]] \\
&= \mathrm{E}_y[\text{MSE}(\theta_y|y)] \\
&= \mathrm{E}_y[\frac{1}{12}(1-y)^2] \\
&= \int \frac{1}{12}(1-y)^2 \boxed{p(y)} dy \\
&= \int_0^1 \frac{1}{12}(1-y)^2 2(1-y)dy \\
&= -\frac{1}{6}\frac{1}{4}(1-y)^4|_0^1 \\
&= \frac{1}{24}
\end{aligned}
$$

$$
p(x,y) = \begin{cases} 2 \ , & \text{for } y \in [0,1] \ , x \in [y,1] \\ 0 \ , & \text{otherwise} \end{cases}
$$

$$
\begin{aligned}
p(y) &= \int p(x,y)dx \\
&= \begin{cases} \int_y^1 2dx \ , & y \in [0,1] \\ 0 \ , & \text{otherwise} \end{cases} \\
&= \begin{cases} 2(1-y) \ , & y \in [0,1] \\ 0 \ , & \text{otherwise} \end{cases}
\end{aligned}
$$

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN

Assume a robot in 2D-space. Its position is modeled as a Gaussian random variable. The prior has $\hat{\mathbf{x}}_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$ and

$$\mathbf{C}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

a) A sensor measures the robot's true position. Formulate and implement a measurement equation assuming independent zero-mean Gaussian noise with

$$\mathbf{C}_{ee} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} .$$

b) Implement a function which samples a true position of $\mathbf{x}$ from the prior and then generates a measurement from the true position.

c) Implement the Kalman update formula to calculate the posterior distribution.

Homework 6

d) Now, assume the sensor will provide 5 measurements in a row. Use the Kalman filter update formulas to update the robot's state recursively.

e) Visualize the robot's covariance matrix as an ellipse and observe how it changes with each update.