Least Squares FIR

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August 2023

1 Formulas for Least Squares Deconvolution

1.1 Convolution in terms of linear algebra

Let M be the desired length of the FIR $\vec{h}$. And let N be the length of the input and output sequences $\vec{x}$ and $\vec{y}$.

Convolution can be described by the linear equation:

$$\vec{y} = C I_{m,n} \vec{h}. \tag{1}$$

C is the NxN circulant matrix made from $\vec{x}$, and $I_{m,n}$ is the MxN "identity matrix" that zero-pads $\vec{h}$.

I.e. Assuming $N = 5$ and $M = 3$,

$$C = \begin{bmatrix} x_0 & x_4 & x_3 & x_2 & x_1 \\ x_1 & x_0 & x_4 & x_3 & x_2 \\ x_2 & x_1 & x_0 & x_4 & x_3 \\ x_3 & x_2 & x_1 & x_0 & x_4 \\ x_4 & x_3 & x_2 & x_1 & x_0 \end{bmatrix}$$

$$I_{m,n} = \left[\frac{I}{0}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using a property of circulant matrices (Mentioned in Equation 4 in the Appendix), we can rewrite our linear system as:

$$\vec{y} = W^H X W I_{m,n} \vec{h} = A\vec{h}, \tag{2}$$

where X is the diagonal matrix with the DFT components of $\vec{x}$.

1.2 Normal Equation

Setting up the normal equation for the linear system from Equation 2, we get:

$$\begin{aligned} A^T\vec{y} &= A^H\vec{y} = (W^H X W I_{m,n})^H \vec{y} \\ &= I_{m,n}^T W^H X^H W \vec{y} \end{aligned}$$

$$\begin{aligned} A^T A &= (W^H X W I_{m,n})^H (W^H X W I_{m,n}) \\ &= (I_{m,n}^T W^H X^H W)(W^H X W I_{m,n} \\ &= I_{m,n}^T W^H X^H X W I_{m,n} \\ &= I_{m,n}^T U I_{m,n}, \end{aligned}$$

where U is the circulant matrix constructed from a vector $\vec{u}$. $\vec{u}$ has $diag(\vec{X})^H \vec{X}$ as its fourier components. Using the same propery of circulant matrices as before.

Note also that, since $\vec{u}$ has real fouriercomponents, it will have a certain symmetry, i.e. $\vec{u} = [u_0, u_1, u_2, u_3, ..., u_3, u_2, u_1]$. This will lead to U being a symmetric circulant matrix.

For instance with $N = 6$, it will have the form.

$$C = \begin{bmatrix} u_0 & u_1 & u_2 & u_3 & u_2 & u_1 \\ u_1 & u_0 & u_1 & u_2 & u_3 & u_2 \\ u_2 & u_1 & u_0 & u_1 & u_2 & u_3 \\ u_3 & u_2 & u_1 & u_0 & u_1 & u_2 \\ u_2 & u_3 & u_2 & u_1 & u_0 & u_1 \\ u_1 & u_2 & u_3 & u_2 & u_1 & u_0 \end{bmatrix}$$

As mentioned above, $A^T A = I_{m,n}^T U I_{m,n}$. This means that $A^T A$ is the matrix from the first MxM elements of U. This will be a symmetric Toeplitz matrix.

1.2.1 Normal Equations Algorithmically

$A^T\vec{y}$:

- $\vec{Y}$ = DFT($\vec{y}$), $\vec{X}$=DFT($\vec{x}$)
- $\vec{t1}$ = elementwise_multiplication(conjugate($\vec{X}$),$\vec{Y}$)
- $\vec{t2}$ = IDFT($\vec{t1}$)
- $A^T\vec{y}$ = first M elements of $\vec{t2}$

$A^T A$:

- $\vec{X}$=DFT($\vec{x}$)
- $\vec{t1}$ = elementwise_multiplication(conjugate($\vec{X}$),$\vec{X}$)
- $\vec{t2}$ = IDFT($\vec{t1}$)
- $\vec{c} = \vec{r}$ = first M elements of $\vec{t2}$
- (Optional) Add l2 regularization to c_0. $c_0 \rightarrow c_0 + \lambda$

$A^T A$ is, as mentioned earlier, a toeplitz matrix. They are uniquely represented by their first column and row: $\vec{c}$ and $\vec{r}$. In the symmetric case $\vec{c} = \vec{r}$.

Direct Toeplitz solvers can solve this system with $\mathcal{O}(M^2)$ complexity. It takes around 30s for $M \approx 100k$ using scipy. For larger systems, one can use the Preconditioned Conjugate Gradient Method, which can do an iteration with $\mathcal{O}(Mlog(M))$ complexity using FFTs.

1.3 Normal Equation Linear Convolution

Using the same Equation 1 as before, we can add a pertubation:

$$\vec{y} = [C + \epsilon(T - C)]I_{n,m}\vec{h} = [C + \epsilon\Delta]I_{n,m}\vec{h} = A\vec{h}, \tag{3}$$

where T is C with the upper right triangle set to zero(excluding the diagonal. This is a Toeplitz matrix.

$$A^T A = I_{m,n}(C^H C + \epsilon(\Delta^H C + C^H \Delta) + \epsilon^2 |\Delta|^2) I_{n,m} + \lambda I$$

Writing $\vec{h} = \vec{h_0} + \epsilon\vec{h_1} + \epsilon^2\vec{h_2} + ...$, we get the pertubation sequence:

- ϵ^0: $P_1\vec{h_0} = I_{m,n}C^H\vec{y}$
- ϵ^1: $P_1\vec{h_1} + P_2\vec{h_0} = I_{m,n}\Delta^H\vec{y}$
- ϵ^2: $P_1\vec{h_2} + P_2\vec{h_1} + P_3\vec{h_0} = 0$
- ϵ^3: $P_1\vec{h_3} + P_2\vec{h_2} + P_3\vec{h_1} = 0$
- ...

$$
\begin{aligned}
P_1 &= I_{m,n}C^H C I_{n,m} + \lambda I \\
P_2 &= I_{m,n}(\Delta^H C + C^H \Delta) I_{n,m} \\
P_3 &= I_{m,n}|\Delta|^2 I_{n,m}
\end{aligned}
$$

2 Appendix

2.1 DFT Matrix

Furthermore, W is the NxN DFT matrix. I.e. multiplying a column-vector $\vec{x}$ from the left with W, will give a vector $\vec{X}$ containing the DFT components of that vector. Multiplying the vector of DFT components from the left with $W^{-1} = W^H$ will then transform it back to the original vector.

2.2 Circulant Matrices in terms of DFT Matrix

It is a property of circulant matrices that they can be rewritten as:

$$C = W^H X W \tag{4}$$

,where

$$X = diag(\vec{X}) = \begin{bmatrix} X_0 & 0 & 0 & 0 & 0 \\ 0 & X_1 & 0 & 0 & 0 \\ 0 & 0 & X_2 & 0 & 0 \\ 0 & 0 & 0 & X_3 & 0 \\ 0 & 0 & 0 & 0 & X_4 \end{bmatrix}.$$

The circulant matrix is constructed from $\vec{x}$ and X contains its DFT components $\vec{X}$ on the diagonal.

2.3 Normal Equation

Let $\vec{y}$ be a length N vector, A be a NxM matrix and x be length M vector.

The least-squares solution of the linear system

$$\vec{y} = A\vec{x}$$

is

$$\vec{A^T b} = (A^T A)\vec{x}.$$

This can be derived from calculating the sum of squares error of the original linear system, taking the derivative wrt. $\vec{x}$ and setting to zero.

2.3.1 Normal Equations with regularization

If we change the sum of squares error function of the linear system to the following:

$$||A\vec{x} - b||^2 \rightarrow ||A\vec{x} - b||^2 + ||\Gamma\vec{x}||^2,$$

then $A^T A \rightarrow A^T A + \Gamma^T \Gamma$. This is sometimes called Tikhonov regularization.

Often Γ is a scalar multiplied with the identity matrix.This is sometimes called l2-regularization or Ridge Regression in a statistics context. This will have the effect of the normal equation favoring solutions with smaller absolute values of $\vec{x}$. This will increase numerical stability as we can get the least squares solution for underdetermined systems. It can also give more predictive power, when the linear system is based on noisy observations.

Other choices of Γ can be to try and exploit the structure of the data, and let Γ be a high-pass operator or low-pass operator, when $\vec{y}$ and A are based on regular time-series.