EXCERCISES

CHAPTER 1

SEAN LI [1]

1. Reducted

Problem

(1.1) Simplify notation of the following terms

(a) $(\lambda x .(((x\ z)\ y)(x\ x)))$

(b) $((\lambda x .(\lambda y .(\lambda z .(z\ ((x\ y)\ z)))))(\lambda u\ .\ u))$

Solution.

(a) $\lambda x .(x\ z\ y)(x\ x)$

(b) $(\lambda x\ y\ z\ .\ x\ (x\ y\ z))(\lambda u\ .\ u)$

Problem

(1.2) Find the alpha equivalent terms to

$$\lambda x\ .\ x\ (\lambda x\ .\ x)$$

In

(a) $\lambda y\ .\ y\ (\lambda x\ .\ x)$

(b) $\lambda y\ .\ y\ (\lambda x\ .\ y)$

(c) $\lambda y\ .\ y\ (\lambda x\ .\ y)$

Solution. Only (a).

Problem

(1.3) Prove

$$\lambda x \,.\, x \,(\lambda z \,.\, y) \underset{\alpha}{=} \lambda z \,.\, z \,(\lambda z \,.\, y)$$

Solution.

Proof. By definition of alpha equivalence

$$M \underset{\alpha}{=} N \iff \exists \varphi, M^{\varphi} \underset{\alpha}{\to} N \wedge \text{FR } M = \text{FR } N$$

The witness of φ is substituting bound variable x with z, and z is not a free variable in the term, thus the two terms are alpha equivalent.

$$\lambda x \,.\, x \,(\lambda z \,.\, y)^{x \to z} \underset{\alpha}{\to} \lambda z \,.\, z \,(\lambda z \,.\, y)$$

■

Problem

(1.4) Consider the following term:

$$U := (\lambda z \,.\, z \, x \, z)((\lambda y \,.\, x \, y) \, x)$$

1. Find Sub U
2. Draw tree rep of U
3. Find FV U
4. Find alpha equivalent terms to U from below and point out which of those follows the Barendregt convention:

$$(a) \; (\lambda y \,.\, y \, x \, y)((\lambda z \,.\, x \, z) \, x)$$
$$(b) \; (\lambda x \,.\, x \, y \, x)((\lambda z \,.\, y \, z) \, y)$$
$$(c) \; (\lambda y \,.\, y \, x \, y)((\lambda y \,.\, x \, y) \, x)$$
$$(d) \; (\lambda v \,.(v \, x) \, v)((\lambda u \,.\, u \, v) \, x)$$

1. Find Sub U.

Solution.

$\text{Sub } U =$

$$\{(\lambda z \,.\, z\,x\,z)((\lambda y \,.\, x\,y)\,x), (\lambda z \,.\, z\,x\,z), ((\lambda y \,.\, x\,y)\,x)\} \cup$$
$$\{(\lambda y \,.\, x\,y), (y), (\lambda z \,.\, x\,z), (x)\} \cup$$
$$\{(\lambda y \,.\, x), (y)\} \cup \{(\lambda z \,.\, x), (z)\} \cup \{(y), (x)\}$$
$$= \{(\lambda z \,.\, z\,x\,z)(\lambda y \,.\, x\,y)\,x, (\lambda z \,.\, z\,x\,z), (\lambda y \,.\, x\,y)\,x,$$
$$(\lambda y \,.\, x\,y), (\lambda z \,.\, x\,z), (\lambda y \,.\, x), (\lambda z \,.\, x), y, x\}$$

2. Draw a tree rep of U.

Solution.

3. Find FV U

Solution.

$$\text{FV } U = \text{FV } (\lambda y \,.\, y\,x\,y) \cup \text{FV } (\lambda z \,.\, x\,z)\,x$$
$$= (\text{FV } y\,x\,y) \setminus \{y\} \cup (\text{FV } \lambda z \,.\, x\,z) \cup \{x\}$$
$$= (\text{FV } y\,x) \setminus \{y\} \cup (\text{FV } x\,z) \setminus \{z\} \cup \{x\}$$
$$= \{x\}$$

4. Find an alpha-equivalent term.

Solution.

$$(a) \underset{\alpha}{=} (c) \underset{\alpha}{=} U$$

Only (a) follows the Barendregt convention.

Problem

(1.5) Give the results of the following substitutions

$$(a)\ (\lambda x \,.\, y\,(\lambda y \,.\, x\,y))[y := \lambda z \,.\, z\,x]$$
$$(b)\ ((x\,y\,z)[x := y])[y := z]$$
$$(c)\ ((\lambda x \,.\, x\,y\,z)[x := y])[y := z]$$
$$(d)\ (\lambda y \,.\, y\,y\,x)[x := y\,z]$$

Solution.

(a) $(\lambda v\,.(\lambda z\,.\,z\,x)(\lambda u\,.\,v\,u))$

(b) $(y\,y\,z)[y := z] = z\,z\,z$

(c) $(\lambda x\,.\,x\,y\,z)[y := z] = (\lambda x\,.\,x\,z\,z)$

(d) $(\lambda u\,.\,u\,u\,(y\,z))$

Problem

(1.6)

$$\neg\left(\forall\,M\,L\,N \in \Lambda, M\,[x := N, y := L] \underset{\alpha}{\equiv} M\,[x := N][y := L]\right)$$

Solution.

Proof. Because RHS $= M\,[x := N][y := L] = M\,[x := N\,[y := L]][y := L]$, if $y \in$ FV N, then what x gets substituted with will have y substituted for L, which is completely different with LHS. ∎
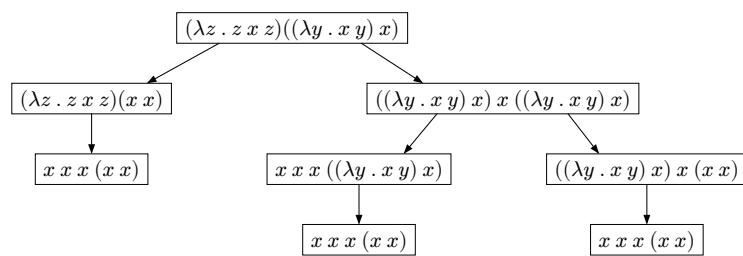
Problem

(1.7) Find all available redexes in

$$U := (\lambda z\,.\,z\,x\,z)((\lambda y\,.\,x\,y)\,x)$$

And all reduction pathes to the β-normal form.

Solution. The first redex is the term as an application itself; another the second term in the application.

(1.8) Show that

$$(\lambda x \,.\, x\, x)\, y \underset{\beta}{\neq} (\lambda x\, y \,.\, y\, x)\, x\, x$$

Solution. By Collorary 1.9.9, it suffices to prove the hypothesis with a proof of a common normal reduced form from LHS and RHS not existing.

Contradiction. By definition of $\underset{\beta}{=}$, there exists The set of all terms attainable from β -reduction on $(\lambda x \,.\, x\, x)\, y$ and $(\lambda x\, y \,.\, y\, x)\, x\, x$ do not intersect. Therefore,

$$\neg\Big(\exists L \in \Lambda, (\lambda x \,.\, x\, x)\, y \underset{\beta}{\twoheadrightarrow} L \wedge (\lambda x\, y \,.\, y\, x)\, x\, x \underset{\beta}{\twoheadrightarrow} L\Big) \implies \neg\Big((\lambda x \,.\, x\, x)\, y \underset{\beta}{=} (\lambda x\, y \,.\, y\, x)\, x\, x\Big)$$

∎

(1.9) Define the combinators

$$K := \lambda x\, y \,.\, x$$
$$S := \lambda x\, y\, z \,.\, x\, z\, (y\, z)$$

Prove that

$$\forall\, P\, Q \in \Lambda, K\, P\, Q \underset{\beta}{\twoheadrightarrow} P$$

$$\forall\, P\, Q\, R \in \Lambda, S\, P\, Q\, R \underset{\beta}{\twoheadrightarrow} P\, R\, (Q\, R)$$

Solution.

Proof.

$$K\, P\, Q = (\lambda x\, y \,.\, x)\, P\, Q \underset{\beta}{\rightarrow} (\lambda y \,.\, x)[x := P]\, Q \underset{\beta}{\rightarrow} P\, [y := Q] = P$$

$$S\, P\, Q\, R = (\lambda x\, y\, z \,.\, x\, z\, (y\, z)) \underset{\beta}{\twoheadrightarrow} (x\, z\, (y\, z))[x := P][y := Q][z := R] = P\, R\, (Q\, R)$$

∎

(1.10) We define the church numerals

$$\text{zero} := \lambda f\, x\, .\, x$$
$$\text{one} := \lambda f\, x\, .\, f\, x$$
$$\text{two} := \lambda f\, x\, .\, f\, f\, x$$
$$...$$
$$\text{num}_n := \lambda f\, x\, .\, f^n\, x$$

And operations

$$\text{add} := \lambda n\, m\, f\, x\, .\, m\, f\, (n\, f\, x)$$
$$\text{mul} := \lambda n\, m\, f\, x\, .\, m\, (n\, f)\, x$$

Show

(a) add one one $\twoheadrightarrow_\beta$ two

(b) add one one $\neq_\beta$ mul one zero

Solution.

(a) add one one $= (\lambda n\, m\, f\, x\, .\, m\, f\, (n\, f\, x))(\lambda f\, x\, .\, f\, x)(\lambda f\, x\, .\, f\, x)$

$\twoheadrightarrow_\beta (\lambda f\, x\, .(\lambda f\, x\, .\, f\, x)\, f\, ((\lambda f\, x\, .\, f\, x)\, f\, x))$

$\twoheadrightarrow_\beta (\lambda f\, x\, .(\lambda x\, .\, f\, x)\, f\, x)$

$\twoheadrightarrow_\beta (\lambda f\, x\, .\, f\, f\, x) = \text{two}$

(b) mul one one $= (\lambda n\, m\, f\, x\, .\, m\, (n\, f)\, x)(\lambda f\, x\, .\, f\, x)(\lambda f\, x\, .\, f\, x)$

$\twoheadrightarrow_\beta \lambda f\, x\, .(\lambda f\, x\, .\, f\, x)((\lambda f\, x\, .\, f\, x)\, f)\, x$

$\twoheadrightarrow_\beta \lambda f\, x\, .\, f\, x = \text{one}$

Because no intermediate form in the beta reduction process of the two terms are α-equivalent, by collorary 1.9.9 the two terms are not β-equivalent.

Problem

(1.11) We define

$$\text{succ} := \lambda m\, f\, x\,.\, f\,(m\, f\, x)\ \text{s.t. }\forall \text{num}_n, \text{succ num}_n = \text{num}_{n+1}$$

Prove

$$\text{succ zero} \underset{\beta}{=} \text{one}$$

$$\text{succ one} \underset{\beta}{=} \text{two}$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$\text{succ zero} = (\lambda m\, f\, x\,.\, f\,(m\, f\, x))(\lambda f\, x\,.\, x)$$

$$\underset{\beta}{\to} (\lambda f\, x\,.\, f\,((\lambda f\, x\,.\, x)\, f\, x))$$

$$\underset{\beta}{\twoheadrightarrow} (\lambda f\, x\,.\, f\, x) = \text{one}$$

The path succ zero $\underset{\beta}{\twoheadrightarrow}$ one derived above is the witness of a reduction chain from LHS to RHS.

$$\text{succ one} = (\lambda m\, f\, x\,.\, f\,(m\, f\, x))(\lambda f\, x\,.\, f\, x)$$

$$\underset{\beta}{\to} (\lambda f\, x\,.\, f\,((\lambda f\, x\,.\, f\, x)\, f\, x))$$

$$\underset{\beta}{\twoheadrightarrow} (\lambda f\, x\,.\, f\,(f\, x)) = \text{two}$$

The path succ one $\underset{\beta}{\twoheadrightarrow}$ two derived above is the witness of a reduction chain from LHS to RHS. ∎

Problem

(1.12) We define the λ-terms $\top_\lambda$ (true) and $\bot_\lambda$ (false) and $\neg_\lambda$ (not) by:

$$\top_\lambda := \lambda x\, y\,.\, x \quad \bot_\lambda := \lambda x\, y\,.\, y$$

$$\neg_\lambda := \lambda a\,.\, a\, \bot_\lambda \top_\lambda$$

Show that

$$\neg_\lambda(\neg_\lambda \top_\lambda) \underset{\beta}{=} \top_\lambda$$

$$\neg_\lambda(\neg_\lambda \bot_\lambda) \underset{\beta}{=} \bot_\lambda$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$\neg_\lambda(\neg_\lambda \top_\lambda) = \neg_\lambda((\lambda a . a \perp_\lambda \top_\lambda)(\lambda x \, y . x))$$
$$\underset{\beta}{\twoheadrightarrow} \neg_\lambda((\lambda x \, y . x)\perp_\lambda \top_\lambda)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda a . a \perp_\lambda \top_\lambda)\perp_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda x \, y . y)\perp_\lambda \top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} \top_\lambda$$

$$\neg_\lambda(\neg_\lambda \perp_\lambda) = \neg_\lambda((\lambda a . a \perp_\lambda \top_\lambda)(\lambda x \, y . y))$$
$$\underset{\beta}{\twoheadrightarrow} \neg_\lambda((\lambda x \, y . x)\perp_\lambda \top_\lambda)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda a . a \perp_\lambda \top_\lambda)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda x \, y . x)\perp_\lambda \top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} \perp_\lambda$$

∎

Problem

(1.13) Define

$$\text{iszero} := \lambda m . m \, (\lambda x . \perp_\lambda)\top_\lambda$$

Prove

$$\text{iszero zero} \underset{\beta}{\twoheadrightarrow} \top_\lambda$$

$$\forall \, n \in \mathbb{N}^+, \text{iszero num}_n \underset{\beta}{\twoheadrightarrow} \perp_\lambda$$

Solution.

$$\text{iszero zero} = (\lambda m . m \, (\lambda x . \perp_\lambda)\top_\lambda)(\lambda f \, x . x)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda f \, x . x)(\lambda x . \perp_\lambda)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} \top_\lambda$$

$$\text{iszero num}_n = (\lambda m \,.\, m \,(\lambda x \,.\bot_\lambda)\top_\lambda)(\lambda f\, x \,.\, f^n\, x)$$
$$\twoheadrightarrow_\beta (\lambda f\, x \,.\, f^n\, x)(\lambda x \,.\bot_\lambda)\top_\lambda$$
$$\twoheadrightarrow_\beta (\lambda x \,.\bot_\lambda)\big((\lambda x \,.\bot_\lambda)^{n-1}\top_\lambda\big) \twoheadrightarrow_\beta \bot_\lambda$$

Problem

(1.14) If-else can be modeled as

$$\text{ifelse} = \lambda x\, t\, f \,.\, x\, t\, f$$

Where when x, then t, else f. Prove correctness by applying $\top_\lambda$ and $\bot_\lambda$ on ifelse.

Solution.

$$\text{ifelse } \top_\lambda = (\lambda x\, t\, f \,.\, x\, t\, f)\top_\lambda$$
$$\twoheadrightarrow_\beta (\lambda t\, f \,.(\lambda x\, y \,.\, x)\, t\, f) \twoheadrightarrow_\beta (\lambda t\, f \,.\, t)$$

$$\text{ifelse } \top_\lambda = (\lambda x\, t\, f \,.\, x\, t\, f)\bot_\lambda$$
$$\twoheadrightarrow_\beta (\lambda t\, f \,.(\lambda x\, y \,.\, y)\, t\, f) \twoheadrightarrow_\beta (\lambda t\, f \,.\, f)$$

By applying the results to any two values, the correct corresponding value returns, ex, for ifelse $\top_\lambda$, t is always returned.

Problem

(1.15) Prove that $\Omega := (\lambda x \,.\, x\, x)(\lambda x \,.\, x\, x)$ does not have a β-nf.

Solution. Firstly let's prove Ω.

Proof. Induction on Ω's only reduction path proves that every $\Omega \twoheadrightarrow_\beta \Omega_i = \Omega$. For the base case because Ω has one and only one redux, it could only reduce to Ω_1 which is equivalent to itself. For the inductive step, $\Omega_i = \Omega$, therefore $\Omega_i \to_\beta \Omega_{i+1}$ is still Ω.

By definition, a term having a β-nf requires the existence of a form in β-nf such that the term can reduce to. By induction, Ω only reduces to Ω, and Ω is not in β-nf because it contains β-redex. Therefore, Ω can never reduce to a β-nf, thus it does not have a β-nf. ∎

Problem

(1.16) Let M be a λ-term with the following properties:

- M has a β-nf.
- There exists an infinite reduction path $M \equiv M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \dots$ on M.

Prove that every M_i has a β-nf, and give an example of M.

Solution. An example would be $(\lambda x\, y\, .\, y)\Omega$. Reduction can go on infinitely by reducing on Ω, but the β-nf of the term is $\lambda y\, .\, y$

Proof. Denote β-nf of M as M'. For any form in the reduction path, $M \underset{\beta}{\twoheadrightarrow} M_i$. In conjunction with $M \underset{\beta}{\twoheadrightarrow} M'$, by the Church-Rosser theorem, there exists L such that $M_i \underset{\beta}{\twoheadrightarrow} L$ and $M' \underset{\beta}{\twoheadrightarrow} L$. Because M' is in β-nf, L can only be M', thus $M_i \underset{\beta}{\twoheadrightarrow} M'$, so M_i is capable of reducing to M', a β-nf. Therefore, any form in the reduction path has a β-nf. ∎

Problem

(1.17) If $M\ N$ is strongly normalizing, then both M and N are strongly normalizing.

Solution.

Proof. If M is not strongly normalizing, then there exists a reduction path $M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \dots$ Therefore, $M\ N$ would have had a reduction path $M\ N \underset{\beta}{\to} M_1\ N \underset{\beta}{\to} \dots$ that is infinite, which contradicts with $M\ N$ being strongly normalizing. Vice versa for N. ∎

Problem

(1.18) Let $L, M, N \in \Lambda$ such that $L \underset{\beta}{=} M$ and $L \underset{\beta}{\twoheadrightarrow} N$. Moreover, N is in β-nf. Prove that $M \underset{\beta}{\twoheadrightarrow} N$.

Solution. Collorary 1.9.9.

(1.19) Define

$$U := \lambda z\, x\, .\, x\, (z\, z\, x) \quad \text{and} \quad Z := U\, U$$

Prove Z is a fixed point combinator.

Solution. Proving $\forall\, L \in \Lambda, L\, (Z\, L) \twoheadrightarrow_{\beta} Z\, L$.

Proof.

$$Z\, L = (\lambda z\, x\, .\, x\, (z\, z\, x))(\lambda z\, x\, .\, x\, (z\, z\, x))\, L$$
$$\twoheadrightarrow_{\beta} L\,((\lambda z\, x\, .\, x\, (z\, z\, x))(\lambda z\, x\, .\, x\, (z\, z\, x))\, L) \twoheadrightarrow_{\beta} L\, (Z\, L)$$

∎

(1.20) Solve for $M \in \Lambda$ in each equation:

$$M \underset{\beta}{=} \lambda x\, y\, .\, x\, M\, y$$
$$M\, x\, y\, z \underset{\beta}{=} x\, y\, z\, M$$

Solution. By the property of the Y combinator:

$$f\, (Y\, f) = Y\, f$$

The first equation can be remodeled as

$$M \underset{\beta}{=} L\, M \text{ where } L = \lambda m\, x\, y\, .\, x\, m\, y$$

Solving for fixed point of L :

$$M \equiv Y\, L \equiv L\, (Y\, L)$$
$$= (\lambda x\, .\, L\, (x\, x))(\lambda x\, .\, L\, (x\, x))$$
$$= (\lambda x\, .(\lambda m\, u\, v\, .\, u\, m\, v)(x\, x))(\lambda x\, .(\lambda m\, u\, v\, .\, u\, m\, v)(x\, x))$$
$$\twoheadrightarrow_{\beta} (\lambda x\, .(\lambda u\, v\, .\, u\, (x\, x)\, v))(\lambda x\, .(\lambda u\, v\, .\, u\, (x\, x)\, v))$$

The second equation can be η-reduced on both sides:

$$M\, x\, y\, z \underset{\beta}{=} x\, y\, z\, M$$
$$M \underset{\beta}{=} \lambda x\, y\, z\, .\, x\, y\, z\, M$$

11.

Remodeling equation:

$$M = N\,M \text{ where } M = \lambda m\,x\,y\,z\,.\,x\,y\,z\,m$$

then

$$
\begin{aligned}
M &\equiv Y\,N \equiv N\,(Y\,N) \\
&= (\lambda x\,.\,N\,(x\,x))(\lambda x\,.\,N\,(x\,x)) \\
&= (\lambda x\,.(\lambda m\,u\,y\,z\,.\,u\,y\,z\,m)(x\,x))(\lambda x\,.(\lambda m\,u\,y\,z\,.\,u\,y\,z\,m)(x\,x)) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda x\,.(\lambda u\,y\,z\,.\,u\,y\,z\,(x\,x)))(\lambda x\,.(\lambda u\,y\,z\,.\,u\,y\,z\,(x\,x)))
\end{aligned}
$$