

EXERCISES

CHAPTER 5

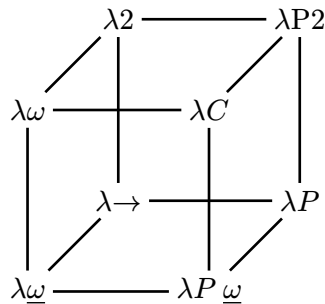
SEAN LI ¹

1. Reducted

Reference - Calculus of Constructions

$$\begin{array}{c}
 \frac{}{\emptyset \vdash * : \square} \text{Sort} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \text{Var} \qquad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \text{Weak} \\
 \\
 \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A . B : s_2} \text{Form} \\
 \\
 \frac{\Gamma \vdash M : \Pi x : A . B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B [x := N]} \text{App} \\
 \\
 \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A . B : s}{\Gamma \vdash \lambda x : A . M : \Pi x : A . B} \text{Abst} \\
 \\
 \frac{\Gamma \vdash A : B \quad B \stackrel{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \text{Conv}
 \end{array}$$

The λ -Cube



$\lambda \rightarrow$	$(*, *)$			
$\lambda 2$	$(*, *)$	$(\square, *)$		
$\lambda \omega$	$(*, *)$		(\square, \square)	
λP	$(*, *)$			$(*, \square)$
$\lambda \omega$	$(*, *)$	$(\square, *)$	(\square, \square)	
$\lambda 2P$	$(*, *)$	$(\square, *)$		$(*, \square)$
$\lambda P \omega$	$(*, *)$		(\square, \square)	$(*, \square)$
λC	$(*, *)$	$(\square, *)$	(\square, \square)	$(*, \square)$

Problem

(6.1 a) Give a complete derivation in tree format showing that

$$\perp \equiv \Pi \alpha : * . \alpha$$

is legal in λC .

Solution. Here we will show that there exists $s \in \text{sort}$ and Γ such that $\Gamma \vdash \perp : s$.

Proof.

$$\frac{\frac{\vdash * : \square}{\alpha : * \vdash \alpha : *} \text{Var}}{\vdash \Pi \alpha : * . \alpha : *} \text{Form}$$

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Problem

(6.1 a) Give a complete derivation in tree format showing that $\perp \rightarrow \perp$ is legal in λC where

$$\perp \equiv \Pi \alpha : * . \alpha$$

Solution. Here we will show that there exists $s \in \text{sort}$ and Γ such that $\Gamma \vdash \perp \rightarrow \perp : s$.

Proof.

$$\frac{(6.1 \text{ a}) \frac{\vdash \perp : *}{\vdash \perp : *} \quad \frac{(6.1 \text{ a}) \frac{\vdash \perp : *}{\vdash \perp : *} \text{Weak}}{x : \perp \vdash \perp : *} \text{Form}}{\vdash \Pi x : \perp . \perp : *} \text{Form}$$

■

Problem

(6.1 c) To which systems of the λ -cube does \perp belong? And $\perp \rightarrow \perp$?

Solution. The set of (s_1, s_2) pairs in formation rules of the derivation of \perp is $\{(\square, *)\}$. The minimal system corresponding is $\lambda 2$. The same for $\perp \rightarrow \perp$. Therefore \perp and $\perp \rightarrow \perp$ belongs to $\lambda 2$, $\lambda \omega$, λP and λC .

Problem

(6.2) Given context $\Gamma \equiv S : *, P : S \rightarrow *, A : *$. Prove by means of a flag derivation that the following expression is inhabited in λC with respect to Γ :

$$(\Pi x : S . (A \rightarrow P x)) \rightarrow A \rightarrow \Pi y : S . P y$$

Solution. The inhabitant is

$$M \equiv \lambda u : (\Pi x : S . (A \rightarrow P x)) . \lambda v : A . \lambda y : S . u y v$$

Proof.

1.	$S : *, P : S \rightarrow *, A : *$	
2.	$u : \Pi x : S . (A \rightarrow P x)$	
3.	$v : A$	
4.	$y : S$	
5.	$u y : A \rightarrow P y$	2,4 App
6.	$u y v : P y$	5,3 App
7.	$\lambda y : S . u y v : \Pi y : S . P y$	6 Abst
8.	$\lambda v : A . \lambda y : S . u y v : A \rightarrow \Pi y : S . P y$	7 Abst
9.	$\lambda u : \Pi x : S . (A \rightarrow P x) . \lambda v : A . \lambda y : S . u y v : \Pi x : S . (A \rightarrow P x) \rightarrow A \rightarrow \Pi y : S . P y$	8 Abst

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Problem

(6.3 a) Let \mathcal{J} be a judgement

$$\mathcal{J} \equiv S : *, P : S \rightarrow * \vdash \lambda x : S . (P x \rightarrow \perp) : S \rightarrow *$$

Derive \mathcal{J} in λC with shorthand flag notation.

Solution.

1.	$S : *, P : S \rightarrow *$	
2.	$x : S$	
3.	$P x : *$	1,2 App
4.	$\perp : *$	Weak from 6.1 a
5.	$P x \rightarrow \perp : *$	3,4 Form

6. $\lambda x : S . P x \rightarrow \perp : S \rightarrow *$ **5 Abst**

Problem

(6.3 b) Determine the (s_1, s_2) pairs corresponding to all Π abstractions occurring in \mathcal{J} .

Solution.

Abstraction	Line Number	(s_1, s_2)
$P : S \rightarrow *$	1	$(*, \square)$
$\perp \equiv \Pi\alpha : * . \alpha$	4	$(\square, *)$
$P x \rightarrow \perp$	5	$(\square, *)$
$\lambda x : S . P x \rightarrow \perp : S \rightarrow *$	6	$(*, \square)$

Problem

(6.3 c) What is the ‘smallest’ system in the λ -cube to which \mathcal{J} belongs?

Solution. There are $(*, *) - \lambda \rightarrow$ pairs, $(*, \square) - \lambda P$ pairs, and $(\square, *) - \lambda 2$. Therefore the minimal system \mathcal{J} belongs to is $\lambda P 2$.

Problem

(6.4 a) Let $\Gamma \equiv S : *, Q : S \rightarrow S \rightarrow *$ and

$$M \equiv (\Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp)) \rightarrow \Pi z : S . (Q z z \rightarrow \perp)$$

Derive $\Gamma \vdash M : *$ and determine the smallest subsystem to which this judgement belongs.

Solution.

1.	$S : *, Q : S \rightarrow S \rightarrow *$	
2.	$x : S$	
3.	$y : S$	
4.	$Q x : S \rightarrow *$	1,2 App
5.	$Q x y : *$	4,3 App
6.	$z : Q x y$	
7.	$Q y : S \rightarrow *$	1,3 App
8.	$Q y x : *$	7,2 App
9.	$t : Q y x$	
10.	$\perp : *$	Weak from 6.1 a
11.	$Q y x \rightarrow \perp : *$	8,10 Form
12.	$Q x y \rightarrow Q y x \rightarrow \perp : *$	5,11 Form
13.	$\Pi y : S . Q x y \rightarrow Q y x \rightarrow \perp : *$	1,12 Form
14.	$\Pi x, y : S . Q x y \rightarrow Q y x \rightarrow \perp : *$	1,13 Form
15.	$a : (\Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp))$	
16.	$z : S$	
17.	$Q z : S \rightarrow *$	1,16 App
18.	$Q z z : *$	17,16 App
19.	$b : Q z z$	
20.	$\perp : *$	Weak from 6.1 a
21.	$Q z z \rightarrow \perp : *$	18,20 Form
22.	$\Pi z : S . Q z z \rightarrow \perp : *$	1,21 Form
23.	$\Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp) \rightarrow \Pi z : S . Q z z \rightarrow \perp : *$	14,22 Form

Here's a table of all Π s that appeared

Abstraction	Line Number	(s_1, s_2)
$S \rightarrow *$	1 / 4 / 7 / 17	$(*, \square)$
$S \rightarrow S \rightarrow *$	1	$(*, \square)$
\perp	10 / 11 / 12 / 13 / 14 / 15 / 20 / 21 / 22 / 23	$(\square, *)$
$Q y x \rightarrow \perp$	11 / 12 / 13 / 14 / 15 / 23	$(*, *)$
$Q x y \rightarrow Q y x \rightarrow \perp$	12 / 13 / 14 / 15 / 23	$(*, *)$
$\Pi y : S . Q x y \rightarrow Q y x \rightarrow \perp$	13 / 14 / 23	$(*, *)$

$\Pi x, y : S . Q x y \rightarrow Q y x \rightarrow \perp$	14 / 23	$(*, *)$
$Q z z \rightarrow \perp$	21 / 22 / 23	$(*, *)$
$\Pi z : S . Q z z \rightarrow \perp$	22 / 23	$(*, *)$
$\Pi x, y : S . Q x y \rightarrow Q y x \rightarrow \perp \rightarrow$ $\Pi z : S . Q z z \rightarrow \perp$	23	$(*, *)$

There are $(*, *) - \lambda \rightarrow$ pairs, $(*, \square) - \lambda P$ pairs, and $(\square, *) - \lambda 2$ pairs. Therefore the minimal system available is $\lambda P2$.

Problem

(6.4 b) Prove in λC that M is inhabited in context Γ .

Solution. A shorthand derivation is given below:

Proof.

1. $S : *, Q : S \rightarrow S \rightarrow *$
2. $h : \Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp)$
3. $z : S$
4. $a : Q z z$
5. $\alpha : *$
6. $h z : \Pi y : S . (Q z y \rightarrow Q y z \rightarrow \perp)$ **2,3 App**
7. $h z z : Q z z \rightarrow Q z z \rightarrow \perp$ **6,3 App**
8. $h z z a : Q z z \rightarrow \perp$ **7,4 App**
9. $h z z a a : \Pi \alpha : * . \alpha$ **8,4 App**
10. $h z z a a \alpha : \alpha$ **9,5 App**
11. $\lambda \alpha : * . h z z a a \alpha : \Pi \alpha : * . \alpha$ **10 Abst**
12. $\lambda a : Q z z \lambda \alpha : * . h z z a a \alpha : Q z z \rightarrow \perp$ **11 Abst**
13. $\lambda z : S . \lambda a : Q z z \lambda \alpha : * . h z z a a \alpha : \Pi z : S . Q z z \rightarrow \perp$ **12 Abst**
14. $\lambda h : \Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp)$
 $\lambda z : S . \lambda a : Q z z \lambda \alpha : * . h z z a a \alpha$
 $: \Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp) \rightarrow \Pi z : S . Q z z \rightarrow \perp$ **13 Abst**

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Problem

(6.4 c) We may consider Q to be a relation on set S . Moreover by PAT we may see $A \rightarrow \perp$ as the negation $\neg A$ of prop A . How can M then be interpreted by the PAT paradigm?

Solution. By a direct type-to-proposition translation we have

$$M \equiv \forall x, y \in S, (Q(x, y) \Rightarrow \neg Q(y, x)) \Rightarrow \forall z \in S, (\neg Q(z, z))$$

It expresses the fact if Q is asymmetric then it is irreflexive.

Problem

(5.6 a) Let

$$\mathcal{J} \equiv S : * \vdash \lambda Q : S \rightarrow S \rightarrow *. \lambda x : S. Q x x : (S \rightarrow S \rightarrow *) \rightarrow S \rightarrow *$$

Give a shorthand derivation of \mathcal{J} and determine the smallest subsystem to which \mathcal{J} belongs.

Solution.

1.	$S : *$	
2.	$Q : S \rightarrow S \rightarrow *$	
3.	$x : S$	
4.	$Q x : S \rightarrow *$	2,3 App
5.	$Q x x : *$	4,3 App
6.	$\lambda x : S. Q x x : S \rightarrow *$	5 Abst
7.	$\lambda Q : S \rightarrow S \rightarrow *. \lambda x : S. Q x x : (S \rightarrow S \rightarrow *) \rightarrow S \rightarrow *$	6 Abst

Abstraction	Line Number	(s_1, s_2)
$S \rightarrow *$	2 / 4 / 6 / 7	$(*, \square)$
$S \rightarrow S \rightarrow *$	2 / 7	$(*, \square)$
$(S \rightarrow S \rightarrow *) \rightarrow (S \rightarrow *)$	7	(\square, \square)

The judgement contains $(*, \square) - \lambda P$ pairs and $(\square, \square) - \lambda \omega$ pairs. Therefore the minimal system \mathcal{J} belongs to is $\lambda P \underline{\omega}$.

Problem

(6.5 b) In \mathcal{J} of 6.5 a, we may consider the variable Q as expressing a relation on set S . How could you describe the subexpression $\lambda x : S . Q x x$ in this settings? And what is then the interpretation of the judgement \mathcal{J} ?

Solution. By a informal translation, the term meant “Given a relation Q over set S and an arbitrary element of S , return whether if $Q(x, x)$ holds”.

Problem

(6.6 a & b) Let

$$M \equiv \lambda S : * . \lambda P : S \rightarrow * . \lambda x : S . (P x \rightarrow \perp)$$

Prove M legal and determine its type. What is the smallest system in which the λ -cube in which M may occur?

Solution.

1.	$* : \square$	Sort
2.	$S : *$	
3.	$a : S$	
4.	$\begin{array}{ l} * : \square \end{array}$	
5.	$S \rightarrow * : \square$	2,3 Form
6.	$P : S \rightarrow *$	
7.	$x : S$	
8.	$P x : *$	6,7 App
9.	$a : P x$	
10.	$\begin{array}{ l} \perp : * \end{array}$	Weak from 6.1 a
11.	$P x \rightarrow \perp : *$	8,10 Form
12.	$\lambda x : S . P x \rightarrow \perp : S \rightarrow *$	11,5 Abst
13.	$S \rightarrow * : \square$	5,5 Weak
14.	$(S \rightarrow *) \rightarrow S \rightarrow * : \square$	5,13 Form
15.	$\lambda P : S \rightarrow * . \lambda x : S . P x \rightarrow \perp : (S \rightarrow *) \rightarrow S \rightarrow *$	12,5 Abst
16.	$\Pi S : * . (S \rightarrow *) \rightarrow S \rightarrow * : \square$	1,14 Form

17.

$\lambda S : * . \lambda P : S \rightarrow * . \lambda x : S . P x \rightarrow \perp$
 $: \Pi S : * . (S \rightarrow *) \rightarrow S \rightarrow *$

15,16 Abst

Abstraction	Line Number	(s_1, s_2)
$S \rightarrow *$	5 / 12 / 13 / 14 / 15	$(*, \square)$
$\perp \equiv \Pi \alpha : * . \alpha$	10 / 11	$(\square, *)$
$P x \rightarrow \perp$	11 / 12	$(*, *)$
$(S \rightarrow *) \rightarrow S \rightarrow *$	14 / 15	$(\square, *)$
$\Pi S : * . (S \rightarrow *) \rightarrow S \rightarrow *$	16 / 17	$(\square, *)$

The derivation contains $(*, *) - \lambda \rightarrow$ pairs, $(*, \square) - \lambda P$ pairs, and $(\square, *) - \lambda 2$ pairs. Therefore the minimal system in which M is legal is $\lambda P2$.

Problem

(6.6 c) How could you interpret the constructor M from 6.6 a, if $A \rightarrow \perp$ encodes $\neg A$?

Solution. Converting into mathematical function notation,

$$M(S, P, x) = \neg P(x) \text{ where } S \in \text{set}, P \subseteq S, x \in S$$

M constructs the negation of a predicate P over a set S applied to x , an element of S . An inhabitant of M would prove the negation.

Problem

(6.7 a) Given

$$\Gamma \equiv S : *, Q : S \rightarrow S \rightarrow *$$

We define under Γ terms

$$M_1 \equiv \lambda x, y : S . \Pi R : S \rightarrow S \rightarrow * . ((\Pi z : S . R z z) \rightarrow R x y)$$

$$M_2 \equiv \lambda x, y : S . \Pi R : S \rightarrow S \rightarrow * .$$

$$((\Pi u, v : S . (Q u v \rightarrow R u v)) \rightarrow R x y)$$

Give an inhabitant of $\Pi a : S . M_1 a a$ and a shorthand derivation proving your answer.

Solution. By β -reduction

$$\begin{aligned}
& \Pi a : S . M_1 a a \\
& \equiv \Pi a : S . (\lambda x, y : S . \Pi R : S \rightarrow S \rightarrow * . ((\Pi z : S . R z z) \rightarrow R x y)) a a \\
& \xrightarrow[\beta]{\gg} \Pi a : S . \Pi R : S \rightarrow S \rightarrow * . ((\Pi z : S . R z z) \rightarrow R a a)
\end{aligned}$$

One such term

$$M \equiv \lambda a : S . \lambda R : S \rightarrow S \rightarrow * . (\lambda h : (\Pi z : S . R z z) . h a)$$

Is an inhabitant.

Proof.

1.	$S : *, Q : S \rightarrow S \rightarrow *$	
2.	$a : S$	
3.	$b : S$	
4.	$c : S$	
5.	$* : \square$	Sort
6.	$S \rightarrow * : \square$	1,5 Form
7.	$S \rightarrow S \rightarrow * : \square$	1,6 Form
8.	$R : S \rightarrow S \rightarrow *$	
9.	$z : S$	
10.	$R z : S \rightarrow *$	8,9 App
11.	$R z z : *$	10,9 App
12.	$\Pi z : S . R z z : *$	1,11 Form
13.	$h : \Pi z : S . R z z$	
14.	$R a : S \rightarrow *$	8,2 App
15.	$R a a : *$	14,2 App
16.	$h a : R a a$	13,2 App
17.	$\Pi z : S . R z z \rightarrow R a a : *$	12,15 Form
18.	$\lambda h : \Pi z : S . R z z . h a : (\Pi z : S . R z z) \rightarrow R a a$	16,17 Abst
19.	$\Pi R : S \rightarrow S \rightarrow * . (\Pi z : S . R z z) \rightarrow R a a : *$	7,17 Form
20.	$\lambda R : S \rightarrow S \rightarrow * . \lambda h : \Pi z : S . R z z . h a : \Pi R : S \rightarrow S \rightarrow * . (\Pi z : S . R z z) \rightarrow R a a$	18,19 Abst
21.	$\Pi a : S . \Pi R : S \rightarrow S \rightarrow * . (\Pi z : S . R z z) \rightarrow R a a : *$	1,19 Form
22.	$\lambda a : S . \lambda R : S \rightarrow S \rightarrow * . \lambda h : \Pi z : S . R z z . h a : \Pi a : S . \Pi R : S \rightarrow S \rightarrow * . (\Pi z : S . R z z) \rightarrow R a a$	20,21 Abst

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Problem

(6.7 b) Under Γ of 6.7 a, give an inhabitant of $\Pi a, b : S . (Q a b \rightarrow M_2 a b)$ and a shorthand derivation proving your answer.

Solution. By β -reduction

$$\begin{aligned}
 & \Pi a, b : S . (Q a b \rightarrow M_2 a b) \\
 \equiv & \Pi a, b : S . Q a b \rightarrow (\lambda x, y : S . \Pi R : S \rightarrow S \rightarrow * . \\
 & (\Pi u, v : S . (Q u v \rightarrow R u v)) \rightarrow R x y) a b \\
 \xrightarrow[\beta]{} & \Pi a, b : S . Q a b \rightarrow (\Pi R : S \rightarrow S \rightarrow * . \Pi u, v : S . (Q u v \rightarrow R u v) \rightarrow R a b)
 \end{aligned}$$

One such term

$$\begin{aligned}
 M \equiv & \lambda a, b : S . \lambda h : Q a b . \lambda R : S \rightarrow S \rightarrow * . \lambda r : (\Pi u, v : S . Q u v \rightarrow R u v) . \\
 & r a b h
 \end{aligned}$$

is an inhabitant.

Note: this proof would be too long with all formation rules included. For now, legality of abstraction types is assumed. Since we are omitting many lines, line labels are also removed from rule labels.

Proof.

1.	$S : *, Q : S \rightarrow S \rightarrow *$	
2.	$a : S$	
3.	$b : S$	
4.	$h : Q a b$	
5.	$R : S \rightarrow S \rightarrow *$	
6.	$r : \Pi u, v : S . Q u v \rightarrow R u v$	
7.	$r a : \Pi v : S . Q a v \rightarrow R a v$	App
8.	$r a b : Q a b \rightarrow R a b$	App
9.	$r a b h : R a b$	App
10.	$\lambda r : \Pi u, v : S . Q u v \rightarrow R u v . r a b h$ $: (\Pi u, v : S . Q u v \rightarrow R u v) \rightarrow R a b$	Abst
	$\lambda R : S \rightarrow S \rightarrow * .$ $\lambda r : \Pi u, v : S . Q u v \rightarrow R u v . r a b h$ $: \Pi R : S \rightarrow S \rightarrow * .$ $(\Pi u, v : S . Q u v \rightarrow R u v) \rightarrow R a b$	Abst

12.	$ \begin{array}{l} \lambda h : Q\ a\ b.\lambda R : S \rightarrow S \rightarrow * . \\ \lambda r : \Pi u, v : S . Q\ u\ v \rightarrow R\ u\ v . r\ a\ b\ h \\ : Q\ a\ b \rightarrow \Pi R : S \rightarrow S \rightarrow * . \\ (\Pi u, v : S . Q\ u\ v \rightarrow R\ u\ v) \rightarrow R\ a\ b \end{array} $	Abst
13.	$ \begin{array}{l} \lambda b : S . \lambda h : Q\ a\ b.\lambda R : S \rightarrow S \rightarrow * . \\ \lambda r : \Pi u, v : S . Q\ u\ v \rightarrow R\ u\ v . r\ a\ b\ h \\ : \Pi b : S . Q\ a\ b \rightarrow \Pi R : S \rightarrow S \rightarrow * . \\ (\Pi u, v : S . Q\ u\ v \rightarrow R\ u\ v) \rightarrow R\ a\ b \end{array} $	Abst
14.	$ \begin{array}{l} \lambda a, b : S . \lambda h : Q\ a\ b.\lambda R : S \rightarrow S \rightarrow * . \\ \lambda r : \Pi u, v : S . Q\ u\ v \rightarrow R\ u\ v . r\ a\ b\ h \\ : \Pi a, b : S . Q\ a\ b \rightarrow \Pi R : S \rightarrow S \rightarrow * . \\ (\Pi u, v : S . Q\ u\ v \rightarrow R\ u\ v) \rightarrow R\ a\ b \end{array} $	Abst

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