Excercises

Chapter 3

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1. Reducted

Definition Some rules for reference.

$$\frac{x : \sigma \in \Gamma \qquad \Gamma \text{ is a } \lambda 2 \text{ context}}{\Gamma \vdash x : \sigma} \text{(T-Var)} \qquad \frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash M\ N : \tau} \text{(T-App)}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma.\ M : \sigma \to \tau} \text{(T-Abst)} \qquad \frac{\alpha \in \mathbb{T}_2 \qquad \text{FV } \alpha \subseteq \text{dom } \Gamma}{\alpha : * \in \Gamma} \text{(T-Form)}$$

$$\frac{\Gamma \vdash M : \Pi_{\alpha:*}.\ A \qquad \Gamma \vdash B : *}{\Gamma \vdash M\ B : A\ [\alpha := B]} \text{(T2-App)} \qquad \frac{\Gamma, \alpha : * \vdash M : A}{\Gamma \vdash \lambda \alpha : *\ .\ M : \Pi_{\alpha:*} : A} \text{(T2-Abst)}$$

In this document, convention is that all type judgements in a proof tree, unless stated otherwise, is derived from a single and unique $\lambda 2$ context per tree. Multiple conclusions might be drawn on a single line from usage of the same inference rule for compactness. Eg:

ex 1. $\alpha, \beta : *$ **T-Form**

Is shorthand for

ex 1. $\Gamma \vdash \alpha : *$ **T-Form**

ex 2. $\Gamma \vdash \beta : *$ **T-Form**

Problem

(3.1) How many $\lambda 2$ contexts consisting of four and only four declarations

$$(1)\ \Gamma \vdash \alpha : * \qquad (2)\ \Gamma \vdash \beta : *$$
$$(3)\ \Gamma \vdash f : \alpha \to \beta \quad (4)\ \Gamma \vdash x : \alpha$$

Solution. The last two declarations depende on the first two. Therefore this is an easy combinatorics problem: $2! \times 2! = 4$ contexts:

$$1-2-3-4 \quad 1-2-4-3$$
$$2-1-3-4 \quad 2-1-4-3$$

Problem

(3.2) Give a full derivation in $\lambda 2$ to show the following type term is legal:

$$M \equiv \lambda \alpha, \beta, \gamma : * . \lambda f : \alpha \to \beta . \lambda g : \beta \to \gamma . \lambda x : \alpha .\ g\ (f\ x)$$

Solution.

1.	$\alpha : *$	**Bound**
2.	$\beta : *$	**Bound**
3.	$\gamma : *$	**Bound**
4.	$f : \alpha \to \beta$	**Bound**
5.	$g : \beta \to \gamma$	**Bound**
6.	$x : \alpha$	**Bound**
7.	$\alpha, \beta, \gamma : *$	**T-Form**
8.	$\alpha \to \beta, \beta \to \gamma : *$	**T-Form**
9.	$f : \alpha \to \beta, x : \alpha$	**T-Var**
10.	$f\ x : \beta$	**8,8 T-App**
11.	$g : \beta \to \gamma$	**T-Var**
12.	$g\ (f\ x) : \gamma$	**11,10 T-App**
13.	$\lambda x : \alpha .\ g\ (f\ x) : \alpha \to \gamma$	**12 T-Abst**
14.	$\lambda g : \beta \to \gamma . \lambda x : \alpha .\ g\ (f\ x) : (\beta \to \gamma) \to \alpha \to \gamma$	**13 T-Abst**
15.	$\lambda f : \alpha \to \beta . \lambda g : \beta \to \gamma . \lambda x : \alpha .\ g\ (f\ x)$ $: (\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$	**14 T-Abst**

16. $\lambda\gamma : * .\lambda f : \alpha \to \beta.\lambda g : \beta \to \gamma.\lambda x : \alpha.\ g\ (f\ x)$
$\quad : \Pi\gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **15 T2-Abst**

17. $\lambda\beta, \gamma : * .\lambda f : \alpha \to \beta.\lambda g : \beta \to \gamma.\lambda x : \alpha.\ g\ (f\ x)$
$\quad : \Pi\beta, \gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **16 T2-Abst**

18. $\lambda\alpha, \beta, \gamma : * .\lambda f : \alpha \to \beta.\lambda g : \beta \to \gamma.\lambda x : \alpha.\ g\ (f\ x)$
$\quad : \Pi\alpha, \beta, \gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **17 T2-Abst**

Problem

(3.3 a) Given M in 3.2, and a context Γ such that

$$\Gamma \vdash \texttt{nat} : *$$

$$\Gamma \vdash \texttt{bool} : *$$

$$\Gamma \vdash \text{succ} : \texttt{nat} \to \texttt{nat}$$

$$\Gamma \vdash \text{even} : \texttt{nat} \to \texttt{bool}$$

Prove M `nat nat bool` succ even is legal.

Solution. Proof by deriving the term's type.

Proof.

1. $M : \Pi\alpha, \beta, \gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **T-Var**
2. $\texttt{nat}, \texttt{bool} : *$ **T-Form**
3. $M\ \texttt{nat} : \Pi\beta, \gamma : * .(\texttt{nat} \to \beta) \to (\beta \to \gamma) \to \texttt{nat} \to \gamma$ **2,1 T2-App**
4. $M\ \texttt{nat nat} : \Pi\gamma : * .(\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \gamma) \to \texttt{nat} \to \gamma$ **2,3 T2-App**
5. $M\ \texttt{nat nat bool} : (\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \texttt{bool}) \to \texttt{nat} \to \texttt{bool}$ **2,3 T2-App**
6. $\text{succ} : \texttt{nat} \to \texttt{nat}, \text{even} : \texttt{nat} \to \texttt{bool}$ **T-Var**
7. $M\ \texttt{nat nat bool}\ \text{succ} : (\texttt{nat} \to \texttt{bool}) \to \texttt{nat} \to \texttt{bool}$ **6,5 T-App**
8. $M\ \texttt{nat nat bool}\ \text{succ even} : \texttt{nat} \to \texttt{bool}$ **6,7 T-App**

■

Problem

(3.3 b.i) Prove $\lambda x : \texttt{nat}.\ \text{even}\ (\text{succ}\ x)$ via 3.3 a.

Solution. The result of beta reduction on the term in 3.3 a is what we are proving.

Proof.

$$
\begin{aligned}
& M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \text{succ}\ \text{even} \\
\equiv\ & (\lambda \alpha, \beta, \gamma, f, g\ .\lambda x : \alpha.\ g\ (f\ x))\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \text{succ}\ \text{even} \\
\underset{\beta}{\twoheadrightarrow}\ & (\lambda f : \texttt{nat} \to \texttt{nat}.\lambda g : \texttt{nat} \to \texttt{bool}.\lambda x : \texttt{nat}.\ g\ (f\ (x)))\ \text{succ}\ \text{even} \\
\underset{\beta}{\to}\ & (\lambda x : \texttt{nat}.\ \text{even}\ (\text{succ}\ x))
\end{aligned}
$$

By the subject reduction lemma, $\lambda x : \texttt{nat}.\ \text{even}\ (\text{succ}\ x)) : \texttt{nat} \to \texttt{bool}$, thus is legal. ■

Problem

(3.3 b.ii) Prove $\lambda x : \texttt{nat}.\ \text{even}\ (\text{succ}\ x)$ via derivation in the context provided in 3.3 a.

Solution.

Proof.

1.	$\texttt{nat}, \texttt{bool} : *$	**T-Form**
2.	$x : \texttt{nat}$	**Bound**
3.	$\text{succ} : \texttt{nat} \to \texttt{nat}$	**T-Var**
4.	$x : \texttt{nat}$	**T-Var**
5.	$\text{succ}\ x : \texttt{nat}$	**3,4 T-App**
6.	$\text{even} : \texttt{nat} \to \texttt{bool}$	**T-Var**
7.	$\text{even}\ (\text{succ}\ x) : \texttt{bool}$	**6,5 T-App**
8.	$\lambda x : \texttt{nat}.\ \text{even}\ (\text{succ}\ x) : \texttt{nat} \to \texttt{bool}$	**7 T-Abst**

■

Problem

(3.4) Give a shorthanded (omit T-Var and T-Form) derivation in $\lambda 2$ to show the following term is valid in $\Gamma \equiv \texttt{nat} : *, \texttt{bool} : *$

$$(\lambda\alpha, \beta : * .\lambda f : \alpha \to \alpha.\lambda g : \alpha \to \beta.\lambda x : \alpha.\ g\ (f\ (f\ x)))\ \texttt{nat}\ \texttt{bool}$$

Solution.

Proof.

1.	$\alpha, \beta : *$	**Bound**
2.	$f : \alpha \to \alpha$	**Bound**
3.	$g : \alpha \to \beta$	**Bound**
4.	$x : \alpha$	**Bound**
5.	$f\ x : \alpha$	*,* **T-App**
6.	$f\ (f\ x) : \alpha$	*,5 **T-App**
7.	$g\ (f\ (f\ x)) : \beta$	*,6 **T-App**
8.	$\lambda x : \alpha.\ g\ (f\ (f\ x)) : \alpha \to \beta$	**7 T-Abst**
9.	$\lambda g : \alpha \to \beta.\lambda x : \alpha.\ g\ (f\ (f\ x)) : (\alpha \to \beta) \to \alpha \to \beta$	**8 T-Abst**
	$\lambda f : \alpha \to \alpha.\lambda g : \alpha \to \beta.\lambda x : \alpha.\ g\ (f\ (f\ x))$	
10.	$\quad : (\alpha \to \alpha) \to (\alpha \to \beta) \to \alpha \to \beta$	**9 T-Abst**
11.	$\lambda\alpha, \beta : * .\lambda f : \alpha \to \alpha.\lambda g : \alpha \to \beta.\lambda x : \alpha.\ g\ (f\ (f\ x))$	
	$\quad : \Pi\alpha, \beta : * .(\alpha \to \alpha) \to (\alpha \to \beta) \to \alpha \to \beta$	**10 T2-Abst**
12.	$(\lambda\alpha, \beta : * .\lambda f : \alpha \to \alpha.\lambda g : \alpha \to \beta.\lambda x : \alpha.\ g\ (f\ (f\ x)))\ \texttt{nat}$	
	$\quad : \Pi\beta : * .(\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \beta) \to \texttt{nat} \to \beta$	*,11 **T2-App**
13.	$(\lambda\alpha, \beta : * .\lambda f : \alpha \to \alpha.\lambda g : \alpha \to \beta.\lambda x : \alpha.\ g\ (f\ (f\ x)))\ \texttt{nat}\ \texttt{bool}$	
	$\quad : \Pi\beta : * .(\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \texttt{bool}) \to \texttt{nat} \to \texttt{bool}$	*,12 **T2-App**

■

Problem

(3.5 a) Let $\bot \equiv \Pi\alpha : * .\alpha$. Prove $\bot$ is legal.

Solution. Here a notion called kind checking is introduced. This has not yet been discussed in this book (?

Proof.

1.	$\alpha : *$	**Bound**
2.	$\vdash \alpha : *$	**T-Form**
3.	$\Pi\alpha : * . \alpha : * \to *$	**T2-Abst**

■

Problem

(3.5 b) Consider the context $\Gamma \equiv \beta : *, x : \bot$. Find an inhabitant of type β under Γ.

Solution. $x\ \beta$ is. Because x is of second-order type, it must be parametric to a type, thus x is of form $\lambda\alpha : * \,.\, M$ where $\Gamma, \alpha : * \vdash M : \alpha$.

Proof.

1.	$x : \Pi\alpha : * . \alpha$	**T-Var**
2.	$\beta : *$	**T-Form**
3.	$x\ \beta : \beta$	**1,2 T2-App**

■

Problem

(3.5 c) Give three inhabitants of $\beta \to \beta$ in β-nf under Γ in 3.5 b.

Solution.
1. $\lambda y : \beta.\ y$.

Proof.

1.	$y : \beta$	**Bound**
2.	$\vdash y : \beta$	**T-Var**
3.	$\lambda y : \beta.\ y : \beta \to \beta$	**2 T-Abst**

■

2. $\lambda y : \beta.\ x\ \beta$.

Proof.

1.	$y : \beta$	**Bound**
2.	$\quad x : \Pi\alpha : * .\alpha$	**T-Var**
3.	$\quad \beta : *$	**T-Form**
4.	$\quad x\ \beta : \beta$	**2,3 T2-App**
5.	$\lambda y : \beta.\ x\ \beta : \beta \to \beta$	**4 T-Abst**

■

3. $\lambda y : \beta.\ x\ (\beta \to \beta)\ y.$

Proof.

1.	$y : \beta$	**Bound**
2.	$\quad x : \Pi\alpha : * .\alpha$	**T-Var**
3.	$\quad \beta \to \beta : *$	**T-Form**
4.	$\quad x\ (\beta \to \beta) : \beta \to \beta$	**2,3 T2-App**
5.	$\quad y : \beta$	**T-Var**
6.	$\quad x\ (\beta \to \beta)\ y : \beta$	**4,5 T-App**
7.	$\lambda y : \beta.\ x\ (\beta \to \beta)\ y : \beta \to \beta$	**5 T-Abst**

■

Problem

(3.5 d) Prove that the following terms inhabit the same type in Γ:

$$\lambda f : \beta \to \beta \to \beta.\ f\ (x\ \beta)(x\ \beta)$$

$$x\ ((\beta \to \beta \to \beta) \to \beta)$$

Solution. We simply derive the types.

First Term.

1.	$f : \beta \to \beta \to \beta$	**Bound**
2.	$\quad f : \beta \to \beta \to \beta$	**T-Var**
3.	$\quad x : \Pi\alpha : * .\alpha$	**T-Var**
4.	$\quad \beta : *$	**T-Form**
5.	$\quad x\ \beta : \beta$	**3,4 T2-App**
6.	$\quad f\ (x\ \beta) : \beta \to \beta$	**2,5 T-App**

7.	$f\ (x\ \beta)(x\ \beta) : \beta$	**6,5 T-App**
8.	$\lambda f : \beta \to \beta \to \beta.\ f\ (x\ \beta)(x\ \beta) : (\beta \to \beta \to \beta) \to \beta$	**6 T-Abst**

■

Second Term.

1.	$(\beta \to \beta \to \beta) \to \beta : *$	**T-Form**
2.	$x : \Pi\alpha : * .\alpha$	**T-Var**
3.	$x\ ((\beta \to \beta \to \beta) \to \beta) : (\beta \to \beta \to \beta) \to \beta$	**2,1 T2-App**

■

The two terms were shown to both inhabit $(\beta \to \beta \to \beta) \to \beta$.

Problem

(3.6 a) Find inhabitant of type

$$\Pi\alpha, \beta : * .(\texttt{nat} \to \alpha) \to (\alpha \to \texttt{nat} \to \beta) \to \texttt{nat} \to \beta$$

In context $\Gamma \equiv \texttt{nat} : *$.

Solution.

$$\lambda\alpha, \beta : * .\lambda x : \texttt{nat} \to \alpha.\lambda y : (\alpha \to \texttt{nat} \to \beta).\lambda z : \texttt{nat}.\ y\ (x\ z)\ z$$

Proof.

1.	$\alpha, \beta : *$	**Bound**
2.	$\texttt{nat} \to \alpha : *$	**T-Form**
3.	$\quad x : \texttt{nat} \to \alpha$	**Bound**
4.	$\quad \alpha \to \texttt{nat} \to \beta : *$	**T-Form**
5.	$\quad\quad y : \alpha \to \texttt{nat} \to \beta$	**Bound**
6.	$\quad\quad \texttt{nat} : *$	**Bound**
7.	$\quad\quad\quad z : \texttt{nat}$	**Bound**
8.	$\quad\quad\quad y : \alpha \to \texttt{nat} \to \beta$	**T-Var**
9.	$\quad\quad\quad x : \texttt{nat} \to \alpha$	**T-Var**
10.	$\quad\quad\quad z : \texttt{nat}$	**T-Var**
11.	$\quad\quad\quad x\ z : \alpha$	**9,10 T-App**
12.	$\quad\quad\quad y\ (x\ z) : \texttt{nat} \to \beta$	**8,11 T-App**
13.	$\quad\quad\quad y\ (x\ z)\ z : \beta$	**12,10 T-App**

14. $\lambda z : \texttt{nat}.\ y\ (x\ z)\ z : \texttt{nat} \to \beta$ — **13 T-Abst**

$\lambda y : \alpha \to \texttt{nat} \to \beta . \lambda z : \texttt{nat}.\ y\ (x\ z)\ z$

15. $\quad : (\alpha \to \texttt{nat} \to \beta) \to \texttt{nat} \to \beta$ — **14 T-Abst**

$\lambda x : \texttt{nat} \to \alpha.\ y : \alpha \to \texttt{nat} \to \beta . \lambda z : \texttt{nat}.\ y\ (x\ z)\ z$

16. $\quad : (\texttt{nat} \to \alpha) \to (\alpha \to \texttt{nat} \to \beta) \to \texttt{nat} \to \beta$ — **15 T2-Abst**

17.

$\lambda \alpha, \beta : * .\ x : \texttt{nat} \to \alpha.\ y : \alpha \to \texttt{nat} \to \beta . \lambda z : \texttt{nat}.\ y\ (x\ z)\ z$

$: \Pi \alpha, \beta : * . (\texttt{nat} \to \alpha) \to (\alpha \to \texttt{nat} \to \beta) \to \texttt{nat} \to \beta$ — **16 T2-Abst**

■

Problem

(3.6 b) Find inhabitant of type

$$\Pi \delta : * . ((\alpha \to \gamma) \to \delta) \to (\alpha \to \beta) \to (\beta \to \gamma) \to \delta$$

In context $\Gamma \equiv \alpha : *, \beta : *, \gamma : *$

Solution.

$$\lambda \delta : * . \lambda x : (\alpha \to \gamma) \to \delta . \lambda y : \alpha \to \beta . \lambda z : \beta \to \gamma.\ x\ (\lambda u : \alpha.\ z\ (y\ u))$$

A derivation in shorthand will be given (omitting T-Form / T-Var)

Proof.

1. $\delta : *$ — **Bound**
2. $\quad x : (\alpha \to \gamma) \to \delta$ — **Bound**
3. $\quad\quad y : \alpha \to \beta$ — **Bound**
4. $\quad\quad\quad z : \beta \to \gamma$ — **Bound**
5. $\quad\quad\quad\quad u : \alpha$ — **Bound**
6. $\quad\quad\quad\quad\quad y\ u : \beta$ — ***,* T-App**
7. $\quad\quad\quad\quad\quad z\ (y\ u) : \gamma$

***,6 T-App**

8. $\quad\quad\quad\quad \lambda u : \alpha.\ z\ (y\ u) : \alpha \to \gamma$ — **7 T-Abst**
9. $\quad\quad\quad\quad x\ (\lambda u : \alpha.\ z\ (y\ u)) : \delta$ — **8 T-Abst**
10. $\quad\quad\quad \lambda z : \beta \to \gamma.\ x\ (\lambda u : \alpha.\ z\ (y\ u)) : (\beta \to \gamma) \to \delta$ — **9 T-Abst**

11. $\lambda y : \alpha \to \beta . \lambda z : \beta \to \gamma .\ x\ (\lambda u : \alpha .\ z\ (y\ u))$
$\quad : (\alpha \to \beta) \to (\beta \to \gamma) \to \delta$

10 T-Abst

12. $\lambda x : (\alpha \to \gamma) \to \delta . \lambda y : \alpha \to \beta . \lambda z : \beta \to \gamma .\ x\ (\lambda u : \alpha .\ z\ (y\ u))$
$\quad : ((\alpha \to \gamma) \to \beta) \to (\alpha \to \beta) \to (\beta \to \gamma) \to \delta$

11 T-Abst

13.

$$\lambda \delta : * . \lambda x : (\alpha \to \gamma) \to \delta . \lambda y : \alpha \to \beta . \lambda z : \beta \to \gamma .\ x\ (\lambda u : \alpha .\ z\ (y\ u))$$
$$: \Pi \delta : * . ((\alpha \to \gamma) \to \delta) \to (\alpha \to \beta) \to (\beta \to \gamma) \to \delta$$

12 T2-Abst

■

Problem

(3.6 c) Find inhabitant of type

$$\Pi \alpha, \beta, \gamma : * . (\alpha \to (\beta \to \alpha) \to \gamma) \to \alpha \to \gamma$$

In the empty context

Solution.

$$\lambda \alpha, \beta, \gamma : * . \lambda f : (\alpha \to (\beta \to \alpha) \to \gamma) . \lambda x : \alpha .\ f\ x\ (\lambda u : \beta .\ x)$$

Proof.

1.	α, β, γ	**Bound**
2.	$f : \alpha \to (\beta \to \alpha) \to \gamma$	**Bound**
3.	$x : \alpha$	**Bound**
4.	$f\ x : (\beta \to \alpha) \to \gamma$	*,* **T-App**
5.	$u : \beta$	**Bound**
6.	$x : \alpha$	**T-Var**
7.	$\lambda u : \beta .\ x : \beta \to \alpha$	**6 T-Abst**
8.	$f\ x\ (\lambda u : \beta .\ x) : \gamma$	**4,7 T-App**
9.	$\lambda x : \alpha .\ f\ x\ (\lambda u : \beta .\ x) : \alpha \to \gamma$	**8 T-Abst**
10.	$\lambda f : \alpha \to (\beta \to \alpha) \to \gamma . \lambda x : \alpha .\ f\ x\ (\lambda u : \beta .\ x)$ $: (\alpha \to (\beta \to \alpha) \to \gamma) \to \alpha \to \gamma$	**9 T-Abst**

11.

$\lambda\alpha, \beta, \gamma : * . \lambda f : \alpha \to (\beta \to \alpha) \to \gamma . \lambda x : \alpha .\ f\ x\ (\lambda u : \beta .\ x)$
$: \Pi\alpha, \beta, \gamma : * . (\alpha \to (\beta \to \alpha) \to \gamma) \to \alpha \to \gamma$ **10 T2-Abst**

■

Problem

(3.7) Let $\bot \equiv \Pi\alpha : * . \alpha$ and context $\Gamma \equiv \alpha : *, \beta : *, x : \alpha \to \bot, f : (\alpha \to \alpha) \to \alpha$. Give a derivation that succesively calculate an inhabitant of α and β, both in context Γ.

Solution. Have $M : \alpha := f\ (\lambda n : \alpha .\ n)$. Then $\Gamma \vdash x\ M\ \beta : \beta$.

Typing M.

1. $f : (\alpha \to \alpha) \to \alpha$ **T-Var**
2. $n : \alpha$ **Bound**
3. $\lfloor n : \alpha \rfloor$ **T-Var**
4. $\lambda n : \alpha .\ n : \alpha \to \alpha$ **3 T-Abst**
5. $f\ (\lambda n : \alpha .\ n) : \alpha$ **1,4 T-App**

■

Typing $x\ M\ \beta$.

1. $M : \alpha$ **T-Var**
2. $x : \alpha \to \Pi\alpha : * . \alpha$ **T-Var**
3. $M\ x : \Pi\alpha : * . \alpha$ **2,1 T-App**
4. $M\ x\ \beta : \beta$ **3,* T2-App**

■

Problem

(3.8) Recall $K \equiv \lambda x\, y\, .\, x \in \Lambda$ from untyped lambda calculus. Consider the following types

$$T_1 \equiv \Pi\alpha, \beta : * .\alpha \to \beta \to \alpha \quad T_2 \equiv \Pi\alpha : * .\alpha \to (\Pi\beta : * .\beta \to \alpha)$$

Find inhabitants of both type $t_1 : T_1$ and $t_2 : T_2$ under the empty context, which may be considered the closed $\lambda 2$ form of $K \in \Lambda_{\mathbb{T}2}$.

Solution.

$$\lambda\alpha, \beta : * .\lambda x : \alpha.\lambda y : \beta.\, x$$

$$\lambda\alpha : * .\lambda x : \alpha.\lambda\beta : * .\lambda y : \beta.\, x$$

First Form.

1.	$\alpha, \beta : *$	**Bound**
2.	$\quad x : \alpha$	**Bound**
3.	$\quad\quad y : \beta$	**Bound**
4.	$\quad\quad\quad x : \alpha$	**T-Var**
5.	$\quad\quad \lambda y : \beta.\, x : \beta \to \alpha$	**4 T-Abst**
6.	$\quad \lambda x : \alpha.\lambda y : \beta.\, x : \alpha \to \beta \to \alpha$	**5 T-Abst**
7.	$\lambda\alpha, \beta : * .\lambda x : \alpha.\lambda y : \beta.\, x : \Pi\alpha, \beta : * .\alpha \to \beta \to \alpha$	**5 T2-Abst**

■

Second Form.

1.	$\alpha : *$	**Bound**
2.	$\quad x : \alpha$	**Bound**
3.	$\quad\quad \beta : *$	**Bound**
4.	$\quad\quad\quad y : \beta$	**Bound**
5.	$\quad\quad\quad\quad x : \alpha$	**T-Var**
6.	$\quad\quad\quad \lambda y : \beta.\, x : \beta \to \alpha$	**5 T-Abst**
7.	$\quad\quad \lambda\beta : * .\lambda y : \beta.\, x : \Pi\beta : * .\beta \to \alpha$	**6 T2-Abst**
8.	$\quad \lambda x : \alpha.\lambda\beta : * .\lambda y : \beta.\, x : \alpha \to (\Pi\beta : * .\beta \to \alpha)$	**7 T-Abst**
9.	$\lambda\alpha : * .\lambda x : \alpha.\lambda\beta : * .\lambda y : \beta.\, x : \Pi\alpha : * .\alpha \to (\Pi\beta : * .\beta \to \alpha)$	**8 T2-Abst**

■

Problem

(3.9) Pretype the combinator

$$S \equiv \lambda x\ y\ z\ .\ x\ z\ (y\ z)$$

In closed form (typable in an empty context) in $\Lambda_{\mathbb{T}2}$.

Solution.

$$S \equiv \lambda\alpha, \beta, \gamma : * .\lambda x : \alpha \to \beta \to \gamma.\lambda y : \alpha \to \beta.\lambda z : \alpha.\ x\ z\ (y\ z)$$

Proof.

1.	$\alpha, \beta, \gamma : *$	**Bound**
2.	$x : \alpha \to \beta \to \gamma$	**Bound**
3.	$y : \alpha \to \beta$	**Bound**
4.	$z : \alpha$	**Bound**
5.	$x\ z : \beta \to \gamma$	$^*,^*$ **T-App**
6.	$y\ x : \beta$	$^*,^*$ **T-App**
7.	$x\ z\ (y\ x) : \gamma$	**5,6 T-App**
8.	$\lambda z : \alpha.\ x\ z\ (y\ x) : \alpha \to \gamma$	**7 T-Abst**
9.	$\lambda y : \alpha \to \beta.\lambda z : \alpha.\ x\ z\ (y\ x) : (\alpha \to \beta) \to \alpha \to \gamma$	**8 T-Abst**
	$\lambda x : \alpha \to \beta \to \gamma.\lambda y : \alpha \to \beta.\lambda z : \alpha.\ x\ z\ (y\ x)$	
10.	$: (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$	**9 T-Abst**
11.		
	$\lambda\alpha, \beta, \gamma : * .\lambda x : \alpha \to \beta \to \gamma.\lambda y : \alpha \to \beta.\lambda z : \alpha.\ x\ z\ (y\ x)$	
	$: \Pi\alpha, \beta, \gamma : * .(\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$	**10 T2-Abst**

■

Problem

(3.10 a) Consider the term

$$M \equiv \lambda x : \Pi\alpha : * .\alpha \to \alpha.\ x\ (\sigma \to \sigma)(x\ \sigma)$$

Prove that M is legal.

Solution. For a term to be legal there must exist a context so that the term could be typed. Here, a witness context is $\Gamma \equiv \sigma : *$.

Proof.

1.	$x : \Pi\alpha : * .\alpha \to \alpha$	**Bound**
2.	$\quad x\ (\sigma \to \sigma) : (\sigma \to \sigma) \to (\sigma \to \sigma)$	*,* **T2-App**
3.	$\quad x\ \sigma : \sigma \to \sigma$	*,* **T2-App**
4.	$\quad x\ (\sigma \to \sigma)(x\ \sigma) : \sigma \to \sigma$	**2,3 T-App**
5.		

$$\lambda x : \Pi\alpha : * .\alpha \to \alpha.\ x\ (\sigma \to \sigma)(x\ \sigma)$$
$$: (\Pi\alpha : * .\alpha \to \alpha) \to \sigma \to \sigma \qquad \textbf{4 T-Abst}$$

■

Problem

(3.10 b) Find a term N such that $M\ N$ is legal and may be considered to be a way to add type information to $(\lambda x\ .\ x\ x)(\lambda y\ .\ y)$

Solution.

$$M\ \sigma\ N \equiv (\lambda x : \Pi\alpha : * .\alpha \to \alpha.\ x\ (\sigma \to \sigma)(x\ \sigma))\sigma(\lambda y : \sigma.\ y)$$

Is the same as $(\lambda x\ .\ x\ x)(\lambda y\ .\ y)$ modulo type annotations.

Proof.

1.	$M : (\Pi\alpha : * .\alpha \to \alpha) \to \sigma \to \sigma$	**T-Var**
2.	$M\ \sigma : (\sigma \to \sigma) \to \sigma \to \sigma$	**1,* T2-App**
3.	$y : \sigma$	**Bound**
4.	$\quad y : \sigma$	**T-Var**
5.	have $N := \lambda y : \sigma.\ y : \sigma \to \sigma$	**4 T-Abst**
6.	$M\ \sigma\ N : \sigma \to \sigma$	**2,5 T-Abst**

■

Problem

(3.11) Recall $\bot \equiv \Pi\alpha : * .\alpha$ from 3.5. Type and prove the following term legal:

$$\lambda x : \bot.\ x\ (\bot \to \bot \to \bot)(x\ (\bot \to \bot)\ x)(x\ (\bot \to \bot \to \bot)\ x\ x)$$

Solution.

Proof. The type $\bot \to \bot$ is closed and well formed. Therefore, the term is legal.

1.	$\bot : * \equiv \Pi\alpha : * .\alpha$	**T-Form**
2.	$x : \bot$	**Bound**
3.	$x\ (\bot \to \bot \to \bot) : \bot \to \bot \to \bot$	*,* **T2-App**
4.	$x\ (\bot \to \bot) : \bot \to \bot$	*,* **T2-App**
5.	$x\ (\bot \to \bot)\ x : \bot$	**4,* T-App**
6.	$x\ (\bot \to \bot \to \bot)(x\ (\bot \to \bot)\ x) : \bot \to \bot$	**3,5 T-App**
7.	$x\ (\bot \to \bot \to \bot)\ x : \bot \to \bot$	**3,* T-App**
8.	$x\ (\bot \to \bot \to \bot)\ x\ x : \bot$	**7,* T-App**
9.	$x\ (\bot \to \bot \to \bot)(x\ (\bot \to \bot)\ x)(x\ (\bot \to \bot \to \bot)\ x\ x) : \bot$	**6,8 T-App**
10.	$\lambda x : \bot .\ x\ (\bot \to \bot \to \bot)(x\ (\bot \to \bot)\ x)$ $(x\ (\bot \to \bot \to \bot)\ x\ x) : \bot \to \bot$	**9 T-Abst**

■

Problem

(3.12) Given the Polymorphic Church Numerals:

$$\begin{aligned}
\texttt{nat} \in \mathbb{T}_2 &:= \Pi\alpha : * .(\alpha \to \alpha) \to \alpha \to \alpha \\
\overline{0} &\equiv \lambda\alpha : * .\lambda f : \alpha \to \alpha .\lambda x : \alpha .\ x : \texttt{nat} \\
\overline{1} &\equiv \lambda\alpha : * .\lambda f : \alpha \to \alpha .\lambda x : \alpha .\ f\ x : \texttt{nat} \\
\overline{2} &\equiv \lambda\alpha : * .\lambda f : \alpha \to \alpha .\lambda x : \alpha .\ f\ (f\ x) : \texttt{nat} \\
\text{succ} &\equiv \lambda n : \texttt{nat} .\lambda\beta : * .\lambda f : \beta \to \beta .\lambda x : \beta .\ f\ (n\ \beta\ f\ x)
\end{aligned}$$

Prove that

$$\text{succ}\ \overline{0} =_{\beta} \overline{1}$$

$$\text{succ}\ \overline{1} =_{\beta} \overline{2}$$

Solution.

$$
\begin{aligned}
&\text{succ } \overline{0} \\
&\equiv (\lambda n : \mathtt{nat}.\lambda\beta : *.\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ (n\ \beta\ f\ x))(\lambda\alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ x) \\
&\underset{\beta}{\to} (\lambda\beta : *.\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ ((\lambda\alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ x)\beta\ f\ x)) \\
&\underset{\beta_{\mathrm{T2}}}{\to} (\lambda\beta : *.\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ ((\lambda f : \beta \to \beta.\lambda x : \beta.\ x)\ f\ x)) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda\beta : *.\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ x) \overset{\beta\to\alpha}{\underset{\alpha_{\mathrm{T2}}}{\equiv}} \overline{1}
\end{aligned}
$$

$$
\begin{aligned}
&\text{succ } \overline{1} \\
&\equiv (\lambda n : \mathtt{nat}.\lambda\beta : *.\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ (n\ \beta\ f\ x))(\lambda\alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ f\ x) \\
&\underset{\beta}{\to} (\lambda\beta : *.\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ ((\lambda\alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ f\ x)\beta\ f\ x)) \\
&\underset{\beta_{\mathrm{T2}}}{\to} (\lambda\beta : *.\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ ((\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ x)\ f\ x)) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda\beta : *.\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ (f\ x)) \overset{\beta\to\alpha}{\underset{\alpha_{\mathrm{T2}}}{\equiv}} \overline{2}
\end{aligned}
$$

Problem

(3.13 a) We define addition in Polymorphic Church Numerals as

$$\text{add} \equiv \lambda m, n : \mathtt{nat}.\lambda\alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \mathtt{nat}.\ m\ \alpha\ f\ (n\ \alpha\ f\ x)$$

Show that

$$\text{add } \overline{1}\quad \overline{1} \underset{\beta}{=} \overline{2}$$

Solution.

$$
\begin{aligned}
&\text{add } \overline{1}\quad \overline{1} \\
&\equiv (\lambda m, n : \mathtt{nat}.\lambda\alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ m\ \alpha\ f\ (n\ \alpha\ f\ x))\overline{1}\quad \overline{1} \\
&\underset{\beta}{\twoheadrightarrow} \lambda\alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\overline{1}\alpha\ f\ \left(\overline{1}\alpha\ f\ x\right) \\
&\equiv \lambda\alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\overline{1}\alpha\ f\ ((\lambda\alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ f\ x)\alpha\ f\ x) \\
&\underset{\beta}{\twoheadrightarrow} \lambda\alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\overline{1}\alpha\ f\ (f\ x) \\
&\equiv \lambda\alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.(\lambda\alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ f\ x)\alpha\ f\ (f\ x) \\
&\underset{\beta}{\twoheadrightarrow} \lambda\alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ f\ (f\ x) \underset{\alpha}{\equiv} \overline{2}
\end{aligned}
$$

Problem

(3.13 b) Find a term mul simulates multiplication on `nat`.

Solution.

$$\text{mul} \equiv \lambda m, n : \texttt{nat}.\lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ m\ \alpha(n\ \alpha\ f)\ x$$

Proof. We derive the type first to prove a legal term.

1.	$m, n : \texttt{nat}$	**Bound**
2.	$\quad \alpha : *$	**Bound**
3.	$\quad\quad f : \alpha \to \alpha$	**Bound**
4.	$\quad\quad\quad x : \alpha$	**Bound**
5.	$\quad\quad\quad\quad m\ \alpha : (\alpha \to \alpha) \to \alpha \to \alpha$	*,* **T2-App**
6.	$\quad\quad\quad\quad n\ \alpha : (\alpha \to \alpha) \to \alpha \to \alpha$	*,* **T2-App**
7.	$\quad\quad\quad\quad n\ \alpha\ f : \alpha \to \alpha$	**6,* T-App**
8.	$\quad\quad\quad\quad m\ \alpha(n\ \alpha\ f) : \alpha \to \alpha$	**5,7 T-App**
9.	$\quad\quad\quad\quad m\ \alpha(n\ \alpha\ f)\ x : \alpha$	**8,* T-App**
10.	$\quad\quad\quad \lambda x : \alpha.\ m\ \alpha(n\ \alpha\ f)\ x : \alpha \to \alpha$	**9 T-Abst**
11.	$\quad\quad \lambda f : \alpha \to \alpha.\lambda x : \alpha.\ m\ \alpha(n\ \alpha\ f)\ x : (\alpha \to \alpha) \to \alpha \to \alpha$	**10 T-Abst**
12.	$\quad \lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ m\ \alpha(n\ \alpha\ f)\ x : \texttt{nat}$	**11 T2-Abst**
13.		

$$\lambda m, n : \texttt{nat}.\lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ m\ \alpha(n\ \alpha\ f)\ x$$
$$: \texttt{nat} \to \texttt{nat} \to \texttt{nat} \qquad \textbf{12 T-Abst}$$

This proves that the term does indeed produce a natural number from two. Next let's prove that

$$\forall\ \overline{n}, \overline{m} : \texttt{nat} \quad \text{mul}\ \overline{n}\ \ \overline{m} = \overline{n \times m}$$

It could be proven by induction that

$$\forall\ \overline{a} : \texttt{nat} \quad \overline{a} \equiv \lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ f^a\ x$$

$$\begin{aligned}
\text{mul}\ \overline{n}\ \ \overline{m} &\equiv (\lambda m, n : \texttt{nat}.\lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ m\ \alpha(n\ \alpha\ f)\ x)(\overline{n})(\overline{m}) \\
&\underset{\beta}{\twoheadrightarrow} \lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ \overline{n}\ \alpha(\overline{m}\ \alpha\ f)\ x \\
&\underset{\beta}{\twoheadrightarrow} \lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ \overline{n}\ \alpha(\lambda u : \alpha.\ f^m\ u)\ x \\
&\underset{\beta}{\twoheadrightarrow} \lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.(\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ f^n\ x)(\lambda u : \alpha.\ f^m\ u)\ x \\
&\underset{\beta}{\twoheadrightarrow} \lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.(\lambda u : \alpha.\ f^m\ u)^n\ x
\end{aligned}$$

By induction this can be further beta-reduced to

$$\underset{\beta}{\twoheadrightarrow} \lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ f^{mn}\ x \equiv \overline{m\ n}$$

■

Problem

(3.14) We present the Church-Encoded Boolean:

$$\texttt{bool} \in \mathbb{T}_2 := \Pi\alpha : * .\alpha \to \alpha \to \alpha$$
$$\text{true} \equiv \lambda\alpha : * .\lambda x, y : \alpha.\ x$$
$$\text{false} \equiv \lambda\alpha : * .\lambda x, y : \alpha.\ y$$

Construct a $\lambda 2$ term neg such that neg true $\underset{\beta}{=}$ false and neg false $\underset{\beta}{=}$ true.

Solution.

$$\text{neg} \equiv \lambda b : \texttt{bool}.\lambda\alpha : * .\ b\ \alpha(\text{false}\ \alpha)(\text{true}\ \alpha)$$

Neg True.

$$\begin{aligned}\text{neg true} &\equiv (\lambda b : \texttt{bool}.\lambda\alpha : * .\ b\ \alpha(\text{false}\ \alpha)(\text{true}\ \alpha))\ \text{true}\\ &\underset{\beta}{\twoheadrightarrow} \lambda\alpha : * .(\lambda x, y : \alpha.\ x)(\text{false}\ \alpha)(\text{true}\ \alpha)\\ &\underset{\beta}{\to} \lambda\alpha : * .\ \text{false}\ \alpha \underset{\eta}{\to} \text{false}\end{aligned}$$

■

Neg False.

$$\begin{aligned}\text{neg false} &\equiv (\lambda b : \texttt{bool}.\lambda\alpha : * .\ b\ \alpha(\text{false}\ \alpha)(\text{true}\ \alpha))\ \text{false}\\ &\underset{\beta}{\twoheadrightarrow} \lambda\alpha : * .(\lambda x, y : \alpha.\ y)(\text{false}\ \alpha)(\text{true}\ \alpha)\\ &\underset{\beta}{\to} \lambda\alpha : * .\ \text{true}\ \alpha \underset{\eta}{\to} \text{true}\end{aligned}$$

■

Problem

(3.15) Define

$$M \equiv \lambda u, v : \texttt{bool}.\lambda\beta : * .\lambda x, y : \beta.\ u\ \beta(v\ \beta\ x\ y)(v\ \beta\ y\ y)$$

And reduce M true true, M true false, M false true, M false false, and decide which logical operator is represented by M.

Solution.

$$
\begin{aligned}
M\ \text{true}\ \text{true} &\equiv (\lambda u, v : \texttt{bool}.\lambda\beta : * .\lambda x, y : \beta.\ u\ \beta(v\ \beta\ x\ y)(v\ \beta\ y\ y))\ \text{true}\ \text{true} \\
&\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.\ \text{true}\ \beta(\text{true}\ \beta\ x\ y)(\text{true}\ \beta\ y\ y) \\
&\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.(\text{true}\ \beta\ x\ y) \\
&\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.\ x \underset{\alpha}{\equiv} \text{true}
\end{aligned}
$$

$$
\begin{aligned}
M\ \text{true}\ \text{false} &\equiv (\lambda u, v : \texttt{bool}.\lambda\beta : * .\lambda x, y : \beta.\ u\ \beta(v\ \beta\ x\ y)(v\ \beta\ y\ y))\ \text{true}\ \text{false} \\
&\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.\ \text{true}\ \beta(\text{false}\ \beta\ x\ y)(\text{false}\ \beta\ y\ y) \\
&\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.(\text{false}\ \beta\ x\ y) \\
&\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.\ y \underset{\alpha}{\equiv} \text{false}
\end{aligned}
$$

$$
\begin{aligned}
M\ \text{false}\ \text{true} &\equiv (\lambda u, v : \texttt{bool}.\lambda\beta : * .\lambda x, y : \beta.\ u\ \beta(v\ \beta\ x\ y)(v\ \beta\ y\ y))\ \text{false}\ \text{true} \\
&\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.\ \text{false}\ \beta(\text{true}\ \beta\ x\ y)(\text{true}\ \beta\ y\ y) \\
&\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.(\text{true}\ \beta\ y\ y) \\
&\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.\ y \underset{\alpha}{\equiv} \text{false}
\end{aligned}
$$

$$
\begin{aligned}
M\ \text{false}\ \text{false} &\equiv (\lambda u, v : \texttt{bool}.\lambda\beta : * .\lambda x, y : \beta.\ u\ \beta(v\ \beta\ x\ y)(v\ \beta\ y\ y))\ \text{false}\ \text{false} \\
&\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.\ \text{false}\ \beta(\text{false}\ \beta\ x\ y)(\text{false}\ \beta\ y\ y) \\
&\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.(\text{false}\ \beta\ y\ y) \\
&\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.\ y \underset{\alpha}{\equiv} \text{false}
\end{aligned}
$$

Therefore M is equivalent to logical AND.

Problem

(3.16) Find $\lambda 2$ term representing the logical OR, XOR, IMP.

Solution.

$$
\begin{aligned}
\text{OR} &\equiv \lambda u, v : \texttt{bool}.\lambda\beta : * .\lambda x, y : \beta.\ u\ \beta\ x\ (v\ \beta\ x\ y) \\
\text{XOR} &\equiv \lambda u, v : \texttt{bool}.\lambda\beta : * .\lambda x, y : \beta.\ u\ \beta(v\ \beta\ y\ x)(v\ \beta\ x\ y) \\
\text{IMP} &\equiv \lambda u, v : \texttt{bool}.\lambda\beta : * .\lambda x, y : \beta.\ u\ \beta(v\ \beta\ x\ y)\ x
\end{aligned}
$$

All of them could be checked by finite enumeration over $\texttt{bool} \times \texttt{bool}$.

Problem

(3.17) Find isZero : $\texttt{nat} \to \texttt{bool}$ such that $\forall\, n : \texttt{nat}$, isZero $n \underset{\beta}{=}$ false except when $n \equiv \overline{0}$.

Solution.

$$\text{isZero} \equiv \lambda n : \texttt{nat}.\ n\ \texttt{bool}\ (\lambda u : \texttt{bool}.\ \text{false})\ \text{true}$$

Proof.

$$\begin{aligned}
\text{isZero}\ \overline{0} &\equiv (\lambda n : \texttt{nat}.\ n\ \texttt{bool}\ (\lambda u : \texttt{bool}.\ \text{false})\ \text{true})\overline{0} \\
&\underset{\beta}{\twoheadrightarrow} (\lambda \alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ x)\ \texttt{bool}\ (\lambda u : \texttt{bool}.\ \text{false})\ \text{true} \\
&\underset{\beta}{\twoheadrightarrow} (\lambda f : \texttt{bool} \to \texttt{bool}.\lambda x : \texttt{bool}.\ x)(\lambda u : \texttt{bool}.\ \text{false})\ \text{true} \\
&\underset{\beta}{\twoheadrightarrow} \text{true}
\end{aligned}$$

■

By induction it could be proven that any other natural numbers must be applied $\lambda u : \texttt{bool}.$ false to the body, making the result false, except for $\overline{0}$, where the function $f : \alpha \to \alpha$ never got applied.

Problem

(3.18 a) Define type

$$\texttt{tree} \equiv \Pi \alpha : * .(\texttt{bool} \to \alpha) \to (\texttt{bool} \to \alpha \to \alpha \to \alpha) \to \alpha$$

Then we construct an inhabitant

$$\lambda \alpha. * .\lambda u : \texttt{bool} \to \alpha.\lambda v : \texttt{bool} \to \alpha \to \alpha \to \alpha.\ M$$

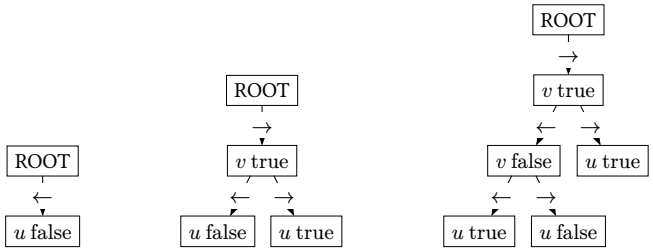
Is a node of a binary tree. Sketch graphs of trees where M is

$$u\ \text{false}$$
$$v\ \text{true}\ (u\ \text{false})(u\ \text{true})$$
$$v\ \text{true}\ (u\ \text{true})(v\ \text{false}\ (u\ \text{true})(u\ \text{false}))$$

Solution. A binary tree is usually defined as this:

```
inductive Tree (α : Type) where
  | leaf (value : α) : Tree α
  | node (left right : Tree α) : Tree α
```

With two constructors: a leaf or a node. Here α is the type of the payload at each node. There are two constructors: u is the left constructor, taking a `bool` value for the direction of the node. The v term is the node constructor, taking a `bool` as the direction, two α-typed terms as it's children.



Problem

(3.18 b) Give a $\lambda 2$ term, which, given input a polymorphic boolean p and two trees s and t, delivers the combined tree with p on top, left subtree s and right subtree t.

Solution.

$$\begin{aligned}\text{leaf} := \lambda p : \texttt{bool}.\lambda s, t : \texttt{tree}.\\ \lambda\alpha.\lambda u\ .\ \texttt{bool} \to \alpha.\lambda v : \texttt{bool} \to \alpha \to \alpha \to \alpha.\\ v\ p\ (s\ \alpha\ u\ v)(t\ \alpha\ u\ v)\end{aligned}$$

Proof. We suppose

$$\begin{aligned}s &\equiv \lambda\beta : * .\lambda u_s : \texttt{bool} \to \beta.\lambda v_s : \texttt{bool} \to \beta \to \beta \to \beta.\ S\\ t &\equiv \lambda\gamma : * .\lambda u_t : \texttt{bool} \to \gamma.\lambda v_t : \texttt{bool} \to \gamma \to \gamma \to \gamma.\ T\end{aligned}$$

We want

$$\text{leaf}\ p\ s\ t \underset{\beta}{=} \lambda\alpha : * .\lambda u : \texttt{bool} \to \alpha.\lambda v : \texttt{bool} \to \alpha \to \alpha \to \alpha.\ v\ p\ S\ T$$

For compactness we denote

$$\begin{aligned}\alpha : * \vdash \tau_{\text{mkleaf}} &\equiv \texttt{bool} \to \alpha\\ \alpha : * \vdash \tau_{\text{mknode}} &\equiv \texttt{bool} \to \alpha \to \alpha \to \alpha\end{aligned}$$

By beta reduction we have

$$
\begin{aligned}
\text{leaf}\, p\ s\ t &\equiv (\lambda p : \texttt{bool}.\lambda s, t : \texttt{tree}.\ldots)\ p\ s\ t \\
&\underset{\beta}{\twoheadrightarrow} (\lambda\alpha : * .\lambda u\ .\tau_{\text{mkleaf}}.\lambda v : \tau_{\text{mknode}}.\ v\ p\ (s\ \alpha\ u\ v)(t\ \alpha\ u\ v)) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda\alpha : * .\lambda u\ .\tau_{\text{mkleaf}}.\lambda v : \tau_{\text{mknode}}.\ v\ p\ (s\ \alpha\ u\ v)(t\ \alpha\ u\ v)) \\
&\underset{\beta}{\twoheadrightarrow} \lambda\alpha : * .\lambda u\ .\tau_{\text{mkleaf}}.\lambda v : \tau_{\text{mknode}}.\ v\ p\ S\ T
\end{aligned}
$$

■

Problem

(3.19) If $\Gamma \vdash L : \sigma$, then Γ is a valid $\lambda 2$ context.

Solution. The definition of "valid" here would be taken as relative to a judgement as the fact to be able to derive the judgement. Thus, it meant a complete inference path could be made using only statements and judgements derived from the context. Now proof by induction on inference rule that deducted $L : \sigma$.

Case 1 : T-Var. The premise is that Γ is a $\lambda 2$ context. ■

Case 2 : T-App. Therefore $L \equiv M\ N$ for some $M, N \in \Lambda_{\mathbb{T}2}$. Therefore,

$$\Gamma \vdash M : \tau \to \sigma \qquad \Gamma \vdash N : \tau$$

for some $\tau \in \mathbb{T}_2$. By the inductive hypothesis on any premise Γ is a valid $\lambda 2$ context. ■

Case 3 : T-Abst. Therefore $L \equiv \lambda x : \alpha.\ N$ such that

$$\Gamma, x : \alpha \vdash N : \beta$$

for some $\alpha, \beta \in \mathbb{T}_2$ such that $\sigma \equiv \alpha \to \beta$. By the inductive hypothesis, $\Gamma, x : \alpha$. By the recursive definition of $\lambda 2$ contexts, for some valid context Δ, $\forall\ n \in \text{dom}\ \Delta$, n could not depende on statement declared after n in the context. Therefore, no statement in Γ could depend on $x : \alpha$. Therefore, Γ is a valid context. ■

Case 4 : T-Form. The premise is that Γ is a valid $\lambda 2$ context. ■

Case 5 : T2-App. Therefore $L \equiv N\ B$ for some $N, B \in \mathbb{V}_2$ such that

$$\Gamma \vdash N : \Pi\alpha : * .\sigma \qquad \Gamma \vdash B : *$$

By the inductive hypothesis on the any premise Γ is a valid $\lambda 2$ context. ■

Case 6 : T2-Abst. Therefore $L \equiv \lambda\alpha : *\ .\ M$ for some $M \in \Lambda_{\mathbb{T}2}$ such that

$$\Gamma, \alpha : * \vdash M : \beta$$

and $\sigma \equiv \Pi\alpha : * \,.\beta$. By reasoning in *Case 3* no statement in the context could depend on any statement before the latter's declaration. Therefore no statement in Γ could depend on $\alpha : *$, making it a valid context. ■

Problem

(3.20) Prove the free variable lemma for $\lambda 2$.

$$\Gamma \vdash L : \sigma \Rightarrow \mathrm{FV}\ L \subseteq \mathrm{dom}\ \Gamma$$

Solution. Proof by induction on inference rules that deduced $L : \sigma$. The only rule not considered is T-Form since all terms apparent in $\mathbb{T}_2$.

Case 1 : T-Var. Therefore L is the only free variable in L. By the generation lemma $L : \sigma \in \Gamma$, so $\{L\} \subseteq \mathrm{dom}\ \Gamma$ ■

Case 2 : T-App. Therefore by the generation lemma $L \equiv M\ N$ for some $M, N \in \Lambda_{\mathbb{T}2}$ such that

$$\Gamma \vdash M : \tau \to \sigma \qquad \Gamma \vdash N : \tau$$

For some type τ. By the inductive hypothesis $\mathrm{FV}\ M \subseteq \Gamma$ and $\mathrm{FV}\ N \subseteq \mathrm{dom}\ \Gamma$. Therefore $\mathrm{FV}\ L = (\mathrm{FV}\ M) \cup (\mathrm{FV}\ N) \subseteq \mathrm{dom}\ \Gamma$. ■

Case 3 : T-Abst. Therefore by the generation lemma $L \equiv \lambda x : \alpha.\ M$ for some $M \in \Lambda_{\mathbb{T}2}$ such that

$$\Gamma, x : \alpha \vdash M : \beta$$

and $\sigma \equiv \alpha \to \beta$. By the inductive hypothesis $\mathrm{FV}\ M \subseteq \mathrm{dom}\ \Gamma \cup \{x\}$. Therefore

$$\mathrm{FV}\ L = \mathrm{FV}\ M \setminus \{x\} \subseteq (\mathrm{dom}\ \Gamma \cup \{x\}) \setminus \{x\} = \mathrm{dom}\ \Gamma$$

■

Case 4 : T2-App. Therefore $L \equiv B$ for some $N, B \in \mathbb{V}_2$ such that

$$\Gamma \vdash N : \Pi\alpha : * \,.\sigma \qquad \Gamma \vdash B : *$$

By the inductive hypothesis $\mathrm{FV}\ N \subseteq \mathrm{dom}\ \Gamma$. Since $B \in \mathbb{T}_2$ then $\mathrm{FV}\ B = \emptyset$. Therefore $\mathrm{FV}\ L = \mathrm{FV}\ N \subseteq \mathrm{dom}\ \Gamma$. ■

Case 5 : T2-Abst. Therefore $L \equiv \lambda\alpha : *\ .\ M$ for some $M \in \Lambda_{\mathbb{T}2}$ such that

$$\Gamma, \alpha : * \vdash M : \beta$$

and $\sigma \equiv \Pi\alpha : * \,.\beta$. Because $\alpha \in \mathbb{T}_2$ so $\alpha \notin \mathrm{FV}\ M$ and $\mathrm{FV}\ L = \mathrm{FV}\ M$, thus $\mathrm{FV}\ M = \mathrm{FV}\ L \subseteq \Gamma$. ■

Problem

(3.21) Give a recursive definition for $\mathrm{FTV} : \mathbb{T}_2 \cup \Lambda_{\mathbb{T}2} \to \mathbb{V}_2$

Solution. Here $\alpha \in \mathbb{V}_2$, $A, B \in \mathbb{T}_2$, $x \in V$ and $M \in \Lambda_{\mathbb{T}2}$.

Form	**Value**
$\mathrm{FTV}\ \alpha$	$\{\alpha\}$
$\mathrm{FTV}\ x$	$\emptyset$
$\mathrm{FTV}\ (A \to B)$	$\mathrm{FTV}(A) \cup \mathrm{FTV}(B)$
$\mathrm{FTV}\ (\Pi\alpha : * \,.\, A)$	$\mathrm{FTV}(A) \setminus \{\alpha\}$
$\mathrm{FTV}\ (M\ N)$	$\mathrm{FTV}(M) \cup \mathrm{FTV}(N)$
$\mathrm{FTV}\ (\lambda x : A \,.\, M)$	$\mathrm{FTV}(A) \cup \mathrm{FTV}(M)$
$\mathrm{FTV}\ (M\ B)$	$\mathrm{FV}(M) \cup \mathrm{FTV}(B)$
$\mathrm{FTV}\ (\lambda\alpha : * \,.\, M)$	$\mathrm{FTV}(M) \setminus \{\alpha\}$

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Completed Dec 16 10:11 pm.