

EXERCISES

CHAPTER 3

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1. Reducted

Definition Some rules for reference.

$$\begin{array}{c} \frac{x : \sigma \in \Gamma \quad \Gamma \text{ is a } \lambda 2 \text{ context}}{\Gamma \vdash x : \sigma} \text{ (T-Var)} \qquad \frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau} \text{ (T-App)} \\[10pt] \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma. M : \sigma \rightarrow \tau} \text{ (T-Abst)} \qquad \frac{\alpha \in \mathbb{T}_2 \quad \forall \tau \in \text{FV } \alpha, \Gamma \vdash \tau : *}{\alpha : * \in \Gamma} \text{ (T-Form)} \\[10pt] \frac{\Gamma \vdash M : \Pi_{\alpha: *} . A \quad \Gamma \vdash B : *}{\Gamma \vdash MB : A[\alpha := B]} \text{ (T2-App)} \qquad \frac{\Gamma, \alpha : * \vdash M : A}{\Gamma \vdash \lambda \alpha : * . M : \Pi_{\alpha: *} . A} \text{ (T2-Abst)} \end{array}$$

In this document, convention is that all type judgements in a proof tree, unless stated otherwise, is derived from a single and unique $\lambda 2$ context per tree. Multiple conclusions might be drawn on a single line from usage of the same inference rule for compactness. Eg:

ex 1. $\alpha, \beta : *$ **T-Form**

Is shorthand for

ex 1. $\Gamma \vdash \alpha : *$ **T-Form**

ex 2. $\Gamma \vdash \beta : *$ **T-Form**

Problem

(3.1) How many $\lambda 2$ contexts consisting of four and only four declarations

- (1) $\Gamma \vdash \alpha : *$ (2) $\Gamma \vdash \beta : *$
 (3) $\Gamma \vdash f : \alpha \rightarrow \beta$ (4) $\Gamma \vdash x : \alpha$

Solution. The last two declarations depend on the first two. Therefore this is an easy combinatorics problem: $2! \times 2! = 4$ contexts:

- 1 – 2 – 3 – 4 1 – 2 – 4 – 3
 2 – 1 – 3 – 4 2 – 1 – 4 – 3

Problem

(3.2) Give a full derivation in $\lambda 2$ to show the following type term is legal:

$$M \equiv \lambda \alpha, \beta, \gamma : *. \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx)$$

Solution.

1.	$\alpha : *$	Bound
2.	$\beta : *$	Bound
3.	$\gamma : *$	Bound
4.	$f : \alpha \rightarrow \beta$	Bound
5.	$g : \beta \rightarrow \gamma$	Bound
6.	$x : \alpha$	Bound
7.	$\alpha, \beta, \gamma : *$	T-Form
8.	$\alpha \rightarrow \beta, \beta \rightarrow \gamma : *$	T-Form
9.	$f : \alpha \rightarrow \beta, x : \alpha$	T-Var
10.	$fx : \beta$	8,8 T-App
11.	$g : \beta \rightarrow \gamma$	T-Var
12.	$g(fx) : \gamma$	11,10 T-App
13.	$\lambda x : \alpha. g(fx) : \alpha \rightarrow \gamma$	12 T-Abst
14.	$\lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) : (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	13 T-Abst
15.	$\lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) : (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	14 T-Abst

16.	$\frac{\lambda\gamma : *. \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) : \Pi\gamma : *. (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma}{\lambda\beta, \gamma : *. \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) : \Pi\beta, \gamma : *. (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma}$	15 T2-Abst
17.		16 T2-Abst
18.		
	$\lambda\alpha, \beta, \gamma : *. \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) : \Pi\alpha, \beta, \gamma : *. (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	17 T2-Abst

Problem

(3.3 a) Given M in 3.2, and a context Γ such that

$$\Gamma \vdash \text{nat} : *$$

$$\Gamma \vdash \text{bool} : *$$

$$\Gamma \vdash \text{succ} : \text{nat} \rightarrow \text{nat}$$

$$\Gamma \vdash \text{even} : \text{nat} \rightarrow \text{bool}$$

Prove $M \text{ nat nat bool succ even}$ is legal.

Solution. Proof by deriving the term's type.

Proof.

1.	$M : \Pi\alpha, \beta, \gamma : *. (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	T-Var
2.	$\text{nat}, \text{bool} : *$	T-Form
3.	$M \text{ nat} : \Pi\beta, \gamma : *. (\text{nat} \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \text{nat} \rightarrow \gamma$	2,1 T2-App
4.	$M \text{ nat nat} : \Pi\gamma : *. (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \gamma) \rightarrow \text{nat} \rightarrow \gamma$	2,3 T2-App
5.		
	$M \text{ nat nat bool} : (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \text{bool}) \rightarrow \text{nat} \rightarrow \text{bool}$	2,3 T2-App
6.	$\text{succ} : \text{nat} \rightarrow \text{nat}, \text{even} : \text{nat} \rightarrow \text{bool}$	T-Var
7.	$M \text{ nat nat bool succ} : (\text{nat} \rightarrow \text{bool}) \rightarrow \text{nat} \rightarrow \text{bool}$	6,5 T-App
8.	$M \text{ nat nat bool succ even} : \text{nat} \rightarrow \text{bool}$	6,7 T-App

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Problem

(3.3 b.i) Prove $\lambda x : \mathbf{nat}. \text{even} (\text{succ } x)$ via 3.3 a.

Solution. The result of beta reduction on the term in 3.3 a is what we are proving.

Proof.

$$\begin{aligned}
 & M \text{ nat nat bool succ even} \\
 & \equiv (\lambda \alpha, \beta, \gamma, f, g. \lambda x : \alpha. g(fx)) \text{ nat nat bool succ even} \\
 & \xrightarrow[\beta]{\text{}} (\lambda f : \text{nat} \rightarrow \text{nat}. \lambda g : \text{nat} \rightarrow \text{bool}. \lambda x : \text{nat}. g(f(x))) \text{ succ even} \\
 & \xrightarrow[\beta]{\text{}} (\lambda x : \text{nat}. \text{even} (\text{succ } x))
 \end{aligned}$$

By the subject reduction lemma, $\lambda x : \mathbf{nat}. \text{even} (\text{succ } x) : \mathbf{nat} \rightarrow \mathbf{bool}$, thus is legal. ■

Problem

(3.3 b.ii) Prove $\lambda x : \mathbf{nat}. \text{even} (\text{succ } x)$ via derivation in the context provided in 3.3 a.

Solution.

Proof.

1.	$\text{nat}, \text{bool} : *$	T-Form
2.	$x : \text{nat}$	Bound
3.	$\text{succ} : \text{nat} \rightarrow \text{nat}$	T-Var
4.	$x : \text{nat}$	T-Var
5.	$\text{succ } x : \text{nat}$	3,4 T-App
6.	$\text{even} : \text{nat} \rightarrow \text{bool}$	T-Var
7.	$\text{even} (\text{succ } x) : \text{bool}$	6,5 T-App
8.	$\lambda x : \text{nat}. \text{even} (\text{succ } x) : \text{nat} \rightarrow \text{bool}$	7 T-Abst

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