

# EXERCISES

## CHAPTER 3

SEAN LI <sup>1</sup>

### 1. Reducted

---

**Definition** Some rules for reference.

$$\begin{array}{c} \frac{x : \sigma \in \Gamma \quad \Gamma \text{ is a } \lambda 2 \text{ context}}{\Gamma \vdash x : \sigma} \text{ (T-Var)} \qquad \frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau} \text{ (T-App)} \\[10pt] \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma. M : \sigma \rightarrow \tau} \text{ (T-Abst)} \qquad \frac{\alpha \in \mathbb{T}_2 \quad \forall \tau \in \text{FV } \alpha, \Gamma \vdash \tau : *}{\alpha : * \in \Gamma} \text{ (T-Form)} \\[10pt] \frac{\Gamma \vdash M : \Pi_{\alpha: *} . A \quad \Gamma \vdash B : *}{\Gamma \vdash MB : A[\alpha := B]} \text{ (T2-App)} \qquad \frac{\Gamma, \alpha : * \vdash M : A}{\Gamma \vdash \lambda \alpha : * . M : \Pi_{\alpha: *} . A} \text{ (T2-Abst)} \end{array}$$

In this document, convention is that all type judgements in a proof tree, unless stated otherwise, is derived from a single and unique  $\lambda 2$  context per tree. Multiple conclusions might be drawn on a single line from usage of the same inference rule for compactness. Eg:

ex 1.  $\alpha, \beta : *$  **T-Form**

Is shorthand for

ex 1.  $\Gamma \vdash \alpha : *$  **T-Form**

ex 2.  $\Gamma \vdash \beta : *$  **T-Form**

---

### Problem

(3.1) How many  $\lambda 2$  contexts consisting of four and only four declarations

- (1)  $\Gamma \vdash \alpha : *$       (2)  $\Gamma \vdash \beta : *$   
 (3)  $\Gamma \vdash f : \alpha \rightarrow \beta$     (4)  $\Gamma \vdash x : \alpha$

*Solution.* The last two declarations depend on the first two. Therefore this is an easy combinatorics problem:  $2! \times 2! = 4$  contexts:

- 1 – 2 – 3 – 4    1 – 2 – 4 – 3  
 2 – 1 – 3 – 4    2 – 1 – 4 – 3

### Problem

(3.2) Give a full derivation in  $\lambda 2$  to show the following type term is legal:

$$M \equiv \lambda \alpha, \beta, \gamma : *. \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx)$$

*Solution.*

1.	$\alpha : *$	<b>Bound</b>
2.	$\beta : *$	<b>Bound</b>
3.	$\gamma : *$	<b>Bound</b>
4.	$f : \alpha \rightarrow \beta$	<b>Bound</b>
5.	$g : \beta \rightarrow \gamma$	<b>Bound</b>
6.	$x : \alpha$	<b>Bound</b>
7.	$\alpha, \beta, \gamma : *$	<b>T-Form</b>
8.	$\alpha \rightarrow \beta, \beta \rightarrow \gamma : *$	<b>T-Form</b>
9.	$f : \alpha \rightarrow \beta, x : \alpha$	<b>T-Var</b>
10.	$fx : \beta$	<b>8,8 T-App</b>
11.	$g : \beta \rightarrow \gamma$	<b>T-Var</b>
12.	$g(fx) : \gamma$	<b>11,10 T-App</b>
13.	$\lambda x : \alpha. g(fx) : \alpha \rightarrow \gamma$	<b>12 T-Abst</b>
14.	$\lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) : (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	<b>13 T-Abst</b>
15.	$\lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) : (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	<b>14 T-Abst</b>

16.	$\frac{\lambda\gamma : *. \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) : \Pi\gamma : *. (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma}{\lambda\beta, \gamma : *. \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) : \Pi\beta, \gamma : *. (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma}$	15 T2-Abst
17.		16 T2-Abst
18.		
	$\lambda\alpha, \beta, \gamma : *. \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) : \Pi\alpha, \beta, \gamma : *. (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	17 T2-Abst

### Problem

(3.3 a) Given  $M$  in 3.2, and a context  $\Gamma$  such that

$$\Gamma \vdash \text{nat} : *$$

$$\Gamma \vdash \text{bool} : *$$

$$\Gamma \vdash \text{succ} : \text{nat} \rightarrow \text{nat}$$

$$\Gamma \vdash \text{even} : \text{nat} \rightarrow \text{bool}$$

Prove  $M \text{ nat nat bool succ even}$  is legal.

*Solution.* Proof by deriving the term's type.

*Proof.*

1.	$M : \Pi\alpha, \beta, \gamma : *. (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	T-Var
2.	$\text{nat}, \text{bool} : *$	T-Form
3.	$M \text{ nat} : \Pi\beta, \gamma : *. (\text{nat} \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \text{nat} \rightarrow \gamma$	2,1 T2-App
4.	$M \text{ nat nat} : \Pi\gamma : *. (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \gamma) \rightarrow \text{nat} \rightarrow \gamma$	2,3 T2-App
5.		
	$M \text{ nat nat bool} : (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \text{bool}) \rightarrow \text{nat} \rightarrow \text{bool}$	2,3 T2-App
6.	$\text{succ} : \text{nat} \rightarrow \text{nat}, \text{even} : \text{nat} \rightarrow \text{bool}$	T-Var
7.	$M \text{ nat nat bool succ} : (\text{nat} \rightarrow \text{bool}) \rightarrow \text{nat} \rightarrow \text{bool}$	6,5 T-App
8.	$M \text{ nat nat bool succ even} : \text{nat} \rightarrow \text{bool}$	6,7 T-App

■

### Problem

(3.3 b.i) Prove  $\lambda x : \mathbf{nat}. \text{even} (\text{succ } x)$  via 3.3 a.

*Solution.* The result of beta reduction on the term in 3.3 a is what we are proving.

*Proof.*

$$\begin{aligned}
 & M \text{ nat nat bool succ even} \\
 & \equiv (\lambda \alpha, \beta, \gamma, f, g. \lambda x : \alpha. g(fx)) \text{ nat nat bool succ even} \\
 & \xrightarrow[\beta]{\text{}} (\lambda f : \text{nat} \rightarrow \text{nat}. \lambda g : \text{nat} \rightarrow \text{bool}. \lambda x : \text{nat}. g(f(x))) \text{ succ even} \\
 & \xrightarrow[\beta]{\text{}} (\lambda x : \text{nat}. \text{even} (\text{succ } x))
 \end{aligned}$$

By the subject reduction lemma,  $\lambda x : \mathbf{nat}. \text{even} (\text{succ } x) : \mathbf{nat} \rightarrow \mathbf{bool}$ , thus is legal. ■

### Problem

(3.3 b.ii) Prove  $\lambda x : \mathbf{nat}. \text{even} (\text{succ } x)$  via derivation in the context provided in 3.3 a.

*Solution.*

*Proof.*

1.	$\text{nat}, \text{bool} : *$	<b>T-Form</b>
2.	$x : \text{nat}$	<b>Bound</b>
3.	$\text{succ} : \text{nat} \rightarrow \text{nat}$	<b>T-Var</b>
4.	$x : \text{nat}$	<b>T-Var</b>
5.	$\text{succ } x : \text{nat}$	<b>3,4 T-App</b>
6.	$\text{even} : \text{nat} \rightarrow \text{bool}$	<b>T-Var</b>
7.	$\text{even} (\text{succ } x) : \text{bool}$	<b>6,5 T-App</b>
8.	$\lambda x : \text{nat}. \text{even} (\text{succ } x) : \text{nat} \rightarrow \text{bool}$	<b>7 T-Abst</b>

■

### Problem

(3.4) Give a shorthand (omit T-Var and T-Form) derivation in  $\lambda 2$  to show the following term is valid in  $\Gamma \equiv \text{nat} : *, \text{bool} : *$

$$(\lambda \alpha, \beta : *. \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx))) \text{ nat } \text{bool}$$

*Solution.*

*Proof.*

1.	$\alpha, \beta : *$	<b>Bound</b>
2.	$f : \alpha \rightarrow \alpha$	<b>Bound</b>
3.	$g : \alpha \rightarrow \beta$	<b>Bound</b>
4.	$x : \alpha$	<b>Bound</b>
5.	$fx : \alpha$	<b>*, * T-App</b>
6.	$f(fx) : \alpha$	<b>*, 5 T-App</b>
7.	$g(f(fx)) : \beta$	<b>*, 6 T-App</b>
8.	$\lambda x : \alpha. g(f(fx)) : \alpha \rightarrow \beta$	<b>7 T-Abst</b>
9.	$\lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx)) : (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$	<b>8 T-Abst</b>
	$\lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx))$	
10.	$: (\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$	<b>9 T-Abst</b>
11.		
	$\lambda \alpha, \beta : *. \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx))$	
	$: \Pi \alpha, \beta : *. (\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$	<b>10 T2-Abst</b>
12.		
	$(\lambda \alpha, \beta : *. \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx))) \text{ nat}$	
	$: \Pi \beta : *. (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta$	<b>*, 11 T2-App</b>
13.		
	$(\lambda \alpha, \beta : *. \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx))) \text{ nat } \text{bool}$	
	$: \Pi \beta : *. (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \text{bool}) \rightarrow \text{nat} \rightarrow \text{bool}$	<b>*, 12 T2-App</b>

■

### Problem

(3.5 a) Let  $\perp \equiv \Pi \alpha : *. \alpha$ . Prove  $\perp$  is legal.

*Solution.* Here a notion called kind checking is introduced. This has not yet been discussed in this book (?)

*Proof.*

1.  $\alpha : *$  **Bound**
2.  $\frac{}{\alpha : *}$  **T-Form**
3.  $\Pi \alpha : *. \alpha : * \rightarrow *$  **T-Form**

■

### Problem

(3.5 b) Consider the context  $\Gamma \equiv \beta : *, x : \perp$ . Find an inhabitant of type  $\beta$  under  $\Gamma$ .

*Solution.*  $x\beta$  is. Because  $x$  is of second-order type, it must be parametric to a type, thus  $x$  is of form  $\lambda \alpha : *. M$  where  $\Gamma, \alpha : * \vdash M : \alpha$ .

*Proof.*

1.  $x : \Pi \alpha : *. \alpha$  **T-Var**
2.  $\beta : *$  **T-Form**
3.  $x\beta : \beta$  **1,2 T2-App**

■

### Problem

(3.5 c) Give three inhabitants of  $\beta \rightarrow \beta$  in  $\beta$ -nf under  $\Gamma$  in 3.5 b.

*Solution.*

1.  $\lambda y : \beta. y$ .

*Proof.*

1.  $y : \beta$  **Bound**
2.  $\frac{}{y : \beta}$  **T-Var**
3.  $\lambda y : \beta. y : \beta \rightarrow \beta$  **2 T-Abst**

■

2.  $\lambda y : \beta. x\beta$ .

*Proof.*

1.	$y : \beta$	<b>Bound</b>
2.	$x : \Pi\alpha : * . \alpha$	<b>T-Var</b>
3.	$\beta : *$	<b>T-Form</b>
4.	$x\beta : \beta$	<b>2,3 T2-App</b>
5.	$\lambda y : \beta. x\beta : \beta \rightarrow \beta$	<b>4 T-Abst</b>

■

3.  $\lambda y : \beta. x(\beta \rightarrow \beta)y.$

*Proof.*

1.	$y : \beta$	<b>Bound</b>
2.	$x : \Pi\alpha : * . \alpha$	<b>T-Var</b>
3.	$\beta \rightarrow \beta : *$	<b>T-Form</b>
4.	$x(\beta \rightarrow \beta) : \beta \rightarrow \beta$	<b>2,3 T2-App</b>
5.	$y : \beta$	<b>T-Var</b>
6.	$x(\beta \rightarrow \beta)y : \beta$	<b>4,5 T-App</b>
7.	$\lambda y : \beta. x(\beta \rightarrow \beta)y : \beta \rightarrow \beta$	<b>5 T-Abst</b>

■

### Problem

(3.5 d) Prove that the following terms inhabit the same type in  $\Gamma$ :

$$\lambda f : \beta \rightarrow \beta \rightarrow \beta. f(x\beta)(x\beta)$$

$$x((\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta)$$

*Solution.* We simply derive the types.

*First Term.*

1.	$f : \beta \rightarrow \beta \rightarrow \beta$	<b>Bound</b>
2.	$f : \beta \rightarrow \beta \rightarrow \beta$	<b>T-Var</b>
3.	$x : \Pi\alpha : * . \alpha$	<b>T-Var</b>
4.	$\beta : *$	<b>T-Form</b>
5.	$x\beta : \beta$	<b>3,4 T2-App</b>
6.	$f(x\beta) : \beta \rightarrow \beta$	<b>2,5 T-App</b>

7.  $\frac{}{f(x\beta)(x\beta) : \beta}$  **6,5 T-App**
8.  $\lambda f : \beta \rightarrow \beta \rightarrow \beta. f(x\beta)(x\beta) : (\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$  **6 T-Abst**

■

*Second Term.*

1.  $(\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta : *$  **T-Form**
2.  $x : \Pi \alpha : *. \alpha$  **T-Var**
3.  $x((\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta) : (\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$  **2,1 T2-App**

■

The two terms were shown to both inhabit  $(\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$ .

### Problem

(3.6 a) Find inhabitant of type

$$\Pi \alpha, \beta : *. (\text{nat} \rightarrow \alpha) \rightarrow (\alpha \rightarrow \text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta$$

In context  $\Gamma \equiv \text{nat} : *$ .

*Solution.*

$$\lambda \alpha, \beta : *. \lambda x : \text{nat} \rightarrow \alpha. \lambda y : (\alpha \rightarrow \text{nat} \rightarrow \beta). \lambda z : \text{nat}. y(xz)z$$

*Proof.*

1.  $\alpha, \beta : *$  **Bound**
2.  $\text{nat} \rightarrow \alpha : *$  **T-Form**
3.  $x : \text{nat} \rightarrow \alpha$  **Bound**
4.  $\alpha \rightarrow \text{nat} \rightarrow \beta : *$  **T-Form**
5.  $y : \alpha \rightarrow \text{nat} \rightarrow \beta$  **Bound**
6.  $\text{nat} : *$  **Bound**
7.  $z : \text{nat}$  **Bound**
8.  $y : \alpha \rightarrow \text{nat} \rightarrow \beta$  **T-Var**
9.  $x : \text{nat} \rightarrow \alpha$  **T-Var**
10.  $z : \text{nat}$  **T-Var**
11.  $xz : \alpha$  **9,10 T-App**
12.  $y(xz) : \text{nat} \rightarrow \beta$  **8,11 T-App**
13.  $y(xz)z : \beta$  **12,10 T-App**

14.	$\lambda z : \mathbf{nat}. y(xz)z : \mathbf{nat} \rightarrow \beta$	<b>13 T-Abst</b>
15.	$\lambda y : \alpha \rightarrow \mathbf{nat} \rightarrow \beta. \lambda z : \mathbf{nat}. y(xz)z$ $: (\alpha \rightarrow \mathbf{nat} \rightarrow \beta) \rightarrow \mathbf{nat} \rightarrow \beta$	<b>14 T-Abst</b>
16.	$\lambda x : \mathbf{nat} \rightarrow \alpha. y : \alpha \rightarrow \mathbf{nat} \rightarrow \beta. \lambda z : \mathbf{nat}. y(xz)z$ $: (\mathbf{nat} \rightarrow \alpha) \rightarrow (\alpha \rightarrow \mathbf{nat} \rightarrow \beta) \rightarrow \mathbf{nat} \rightarrow \beta$	<b>15 T2-Abst</b>
17.		
	$\lambda \alpha, \beta : *. x : \mathbf{nat} \rightarrow \alpha. y : \alpha \rightarrow \mathbf{nat} \rightarrow \beta. \lambda z : \mathbf{nat}. y(xz)z$ $: \Pi \alpha, \beta : *. (\mathbf{nat} \rightarrow \alpha) \rightarrow (\alpha \rightarrow \mathbf{nat} \rightarrow \beta) \rightarrow \mathbf{nat} \rightarrow \beta$	<b>16 T2-Abst</b>

■

### Problem

(3.6 b) Find inhabitant of type

$$\Pi \delta : *. ((\alpha \rightarrow \gamma) \rightarrow \delta) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta$$

In context  $\Gamma \equiv \alpha : *, \beta : *, \gamma : *$

*Solution.*

$$\lambda \delta : *. \lambda x : (\alpha \rightarrow \gamma) \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x(\lambda u : \alpha. z(yu))$$

A derivation in shorthand will be given (omitting T-Form / T-Var)

*Proof.*

1.	$\delta : *$	<b>Bound</b>
2.	$x : (\alpha \rightarrow \gamma) \rightarrow \delta$	<b>Bound</b>
3.	$y : \alpha \rightarrow \beta$	<b>Bound</b>
4.	$z : \beta \rightarrow \gamma$	<b>Bound</b>
5.	$u : \alpha$	<b>Bound</b>
6.	$yu : \beta$	<b>*, T-App</b>
7.	$z(yu) : \gamma$	<b>*,6 T-App</b>
8.	$\lambda u : \alpha. z(yu) : \alpha \rightarrow \gamma$	<b>7 T-Abst</b>
9.	$x(\lambda u : \alpha. z(yu)) : \delta$	<b>8 T-Abst</b>
10.	$\lambda z : \beta \rightarrow \gamma. x(\lambda u : \alpha. z(yu)) : (\beta \rightarrow \gamma) \rightarrow \delta$	<b>9 T-Abst</b>
	$\lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x(\lambda u : \alpha. z(yu))$	
11.	$: (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta$	<b>10 T-Abst</b>

12.  $\lambda x : (\alpha \rightarrow \gamma) \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x(\lambda u : \alpha. z(yu))$  **11 T-Abst**  
 13.  $: ((\alpha \rightarrow \gamma) \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta$
- $\lambda \delta : *. \lambda x : (\alpha \rightarrow \gamma) \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x(\lambda u : \alpha. z(yu))$   
 $: \Pi \delta : *. ((\alpha \rightarrow \gamma) \rightarrow \delta) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta$  **12 T2-Abst**

■

### Problem

(3.6 c) Find inhabitant of type

$$\Pi \alpha, \beta, \gamma : *. (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$$

In the empty context

*Solution.*

$$\lambda \alpha, \beta, \gamma : *. \lambda f : (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma). \lambda x : \alpha. fx(\lambda u : \beta. x)$$

*Proof.*

- |     |  |                  |
|-----|--|------------------|
| 1.  | $\alpha, \beta, \gamma$  | <b>Bound</b>     |
| 2.  | $f : \alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma$   | <b>Bound</b>     |
| 3.  | $x : \alpha$   | <b>Bound</b>     |
| 4.  | $fx : (\beta \rightarrow \alpha) \rightarrow \gamma$   | <b>*, T-App</b>  |
| 5.  | $u : \beta$  | <b>Bound</b>     |
| 6.  | $x : \alpha$   | <b>T-Var</b>     |
| 7.  | $\lambda u : \beta. x : \beta \rightarrow \alpha$  | <b>6 T-Abst</b>  |
| 8.  | $fx(\lambda u : \beta. x) : \gamma$  | <b>4,7 T-App</b> |
| 9.  | $\lambda x : \alpha. fx(\lambda u : \beta. x) : \alpha \rightarrow \gamma$   | <b>8 T-Abst</b>  |
| 10. | $\lambda f : \alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma. \lambda x : \alpha. fx(\lambda u : \beta. x)$<br>$: (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$ | <b>9 T-Abst</b>  |
| 11. |  |                  |
- $\lambda \alpha, \beta, \gamma : *. \lambda f : \alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma. \lambda x : \alpha. fx(\lambda u : \beta. x)$   
 $: \Pi \alpha, \beta, \gamma : *. (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$  **10 T2-Abst**

■

### Problem

(3.7) Let  $\perp \equiv \Pi\alpha : * . \alpha$  and context  $\Gamma \equiv \alpha : *, \beta : *, x : \alpha \rightarrow \perp, f : (\alpha \rightarrow \alpha) \rightarrow \alpha$ . Give a derivation that succesively calculate an inhabitant of  $\alpha$  and  $\beta$ , both in context  $\Gamma$ .

*Solution.* Have  $M : \alpha := f(\lambda n : \alpha. n)$ . Then  $\Gamma \vdash xM\beta : \beta$ .

*Typing  $M$ .*

1.  $f : (\alpha \rightarrow \alpha) \rightarrow \alpha$     **T-Var**
2.  $n : \alpha$     **Bound**
3.  $\frac{}{n : \alpha}$     **T-Var**
4.  $\lambda n : \alpha. n : \alpha \rightarrow \alpha$     **3 T-Abst**
5.  $f(\lambda n : \alpha. n) : \alpha$     **1,4 T-App**

■

*Typing  $xM\beta$ .*

1.  $M : \alpha$     **T-Var**
2.  $x : \alpha \rightarrow \Pi\alpha : * . \alpha$     **T-Var**
3.  $Mx : \Pi\alpha : * . \alpha$     **2,1 T-App**
4.  $Mx\beta : \beta$     **3,\* T2-App**

■

### Problem

(3.8) Recall  $K \equiv \lambda xy. x \in \Lambda$  from untyped lambda calculus. Consider the following types

$$T_1 \equiv \Pi\alpha, \beta : * . \alpha \rightarrow \beta \rightarrow \alpha \quad T_2 \equiv \Pi\alpha : * . \alpha \rightarrow (\Pi\beta : * . \beta \rightarrow \alpha)$$

Find inhabitants of both type  $t_1 : T_1$  and  $t_2 : T_2$  under the empty context, which may be considered the closed  $\lambda 2$  form of  $K \in \Lambda_{\mathbb{T}2}$ .

*Solution.*

$$\lambda\alpha, \beta : * . \lambda x : \alpha. \lambda y : \beta. x$$

$$\lambda\alpha : * . \lambda x : \alpha. \lambda\beta : * . \lambda y : \beta. x$$

*First Form.*

1.	$\alpha, \beta : *$	<b>Bound</b>
2.	$x : \alpha$	<b>Bound</b>
3.	$y : \beta$	<b>Bound</b>
4.	$x : \alpha$	<b>T-Var</b>
5.	$\lambda y : \beta. x : \beta \rightarrow \alpha$	<b>4 T-Abst</b>
6.	$\lambda x : \alpha. \lambda y : \beta. x : \beta \rightarrow \alpha$	<b>5 T-Abst</b>
7.	$\lambda \alpha, \beta : *. \lambda x : \alpha. \lambda y : \beta. x : \beta \rightarrow \alpha \rightarrow \beta \rightarrow \alpha$	<b>5 T2-Abst</b>

■

*Second Form.*

1.	$\alpha : *$	<b>Bound</b>
2.	$x : \alpha$	<b>Bound</b>
3.	$\beta : *$	<b>Bound</b>
4.	$y : \beta$	<b>Bound</b>
5.	$x : \alpha$	<b>T-Var</b>
6.	$\lambda y : \beta. x : \beta \rightarrow \alpha$	<b>5 T-Abst</b>
7.	$\lambda \beta : *. \lambda y : \beta. x : \beta \rightarrow \alpha$	<b>6 T2-Abst</b>
8.	$\lambda x : \alpha. \lambda \beta : *. \lambda y : \beta. x : \beta \rightarrow \alpha \rightarrow (\Pi \beta : *. \beta \rightarrow \alpha)$	<b>7 T-Abst</b>
9.	$\lambda \alpha : *. \lambda x : \alpha. \lambda \beta : *. \lambda y : \beta. x : \beta \rightarrow \alpha \rightarrow (\Pi \alpha : *. \alpha \rightarrow (\Pi \beta : *. \beta \rightarrow \alpha))$	<b>8 T2-Abst</b>

■

### Problem

(3.9) Pretype the combinator

$$S \equiv \lambda x y z. x z (y z)$$

In closed form (typable in an empty context) in  $\Lambda_{T2}$ .

*Solution.*

$$S \equiv \lambda \alpha, \beta, \gamma : *. \lambda x : \alpha \rightarrow \beta \rightarrow \gamma. \lambda y : \alpha \rightarrow \beta. \lambda z : \alpha. x z (y z)$$

*Proof.*

1.	$\alpha, \beta, \gamma : *$	<b>Bound</b>
2.	$x : \alpha \rightarrow \beta \rightarrow \gamma$	<b>Bound</b>
3.	$y : \alpha \rightarrow \beta$	<b>Bound</b>

4.			$z : \alpha$	<b>Bound</b>
5.			$xz : \beta \rightarrow \gamma$	<b>*,* T-App</b>
6.			$yx : \beta$	<b>*,* T-App</b>
7.			$xz(yx) : \gamma$	<b>5,6 T-App</b>
8.			$\lambda z : \alpha.xz(yx) : \alpha \rightarrow \gamma$	<b>7 T-Abst</b>
9.			$\lambda y : \alpha \rightarrow \beta.\lambda z : \alpha.xz(yx) : (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$	<b>8 T-Abst</b>
10.			$\lambda x : \alpha \rightarrow \beta \rightarrow \gamma.\lambda y : \alpha \rightarrow \beta.\lambda z : \alpha.xz(yx)$ $: (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$	<b>9 T-Abst</b>
11.				
			$\lambda \alpha, \beta, \gamma : *. \lambda x : \alpha \rightarrow \beta \rightarrow \gamma.\lambda y : \alpha \rightarrow \beta.\lambda z : \alpha.xz(yx)$ $: \Pi \alpha, \beta, \gamma : *. (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$	<b>10 T2-Abst</b>

■

### Problem

(3.10 a) Consider the term

$$M \equiv \lambda x : \Pi \alpha : *. \alpha \rightarrow \alpha.x(\sigma \rightarrow \sigma)(x\sigma)$$

Prove that  $M$  is legal.

*Solution.* For a term to be legal there must exist a context so that the term could be typed. Here, a witness context is  $\Gamma \equiv \sigma : *$ .

*Proof.*

1.	$x : \Pi \alpha : *. \alpha \rightarrow \alpha$	<b>Bound</b>
2.	$x(\sigma \rightarrow \sigma) : (\sigma \rightarrow \sigma) \rightarrow (\sigma \rightarrow \sigma)$	<b>*,* T2-App</b>
3.	$x\sigma : \sigma \rightarrow \sigma$	<b>*,* T2-App</b>
4.	$x(\sigma \rightarrow \sigma)(x\sigma) : \sigma \rightarrow \sigma$	<b>2,3 T-App</b>
5.	$\lambda x : \Pi \alpha : *. \alpha \rightarrow \alpha.x(\sigma \rightarrow \sigma)(x\sigma) : (\Pi \alpha : *. \alpha \rightarrow \alpha) \rightarrow \sigma \rightarrow \sigma$	<b>4 T-Abst</b>

■

### Problem

(3.10 b) Find a term  $N$  such that  $MN$  is legal and may be considered to be a way to add type information to  $(\lambda x.xx)(\lambda y.y)$

*Solution.*

$$M\sigma N \equiv (\lambda x : \Pi\alpha : * . \alpha \rightarrow \alpha . x(\sigma \rightarrow \sigma)(x\sigma))\sigma(\lambda y : \sigma . y)$$

Is the same as  $(\lambda x . xx)(\lambda y . y)$  modulo type annotations.

*Proof.*

1.  $M : (\Pi\alpha : * . \alpha \rightarrow \alpha) \rightarrow \sigma \rightarrow \sigma$  **T-Var**
2.  $M\sigma : (\sigma \rightarrow \sigma) \rightarrow \sigma \rightarrow \sigma$  **1,\* T2-App**
3.  $y : \sigma$  **Bound**
4.  $\boxed{y : \sigma}$  **T-Var**
5. have  $N := \lambda y : \sigma . y : \sigma \rightarrow \sigma$  **4 T-Abst**
6.  $M\sigma N : \sigma \rightarrow \sigma$  **2,5 T-Abst**

■

### Problem

(3.11) Recall  $\perp \equiv \Pi\alpha : * . \alpha$  from 3.5. Type and prove the following term legal:

$$\lambda x : \perp . x(\perp \rightarrow \perp \rightarrow \perp)(x(\perp \rightarrow \perp)x)(x(\perp \rightarrow \perp \rightarrow \perp)xx)$$

*Solution.*

*Proof.* The type  $\perp \rightarrow \perp$  is closed and well formed. Therefore, the term is legal.

1.  $\perp : * \equiv \Pi\alpha : * . \alpha$  **T-Form**
2.  $x : \perp$  **Bound**
3.  $\boxed{x(\perp \rightarrow \perp \rightarrow \perp) : \perp \rightarrow \perp \rightarrow \perp}$  **\*,\* T2-App**
4.  $\boxed{x(\perp \rightarrow \perp) : \perp \rightarrow \perp}$  **\*,\* T2-App**
5.  $\boxed{x(\perp \rightarrow \perp)x : \perp}$  **4,\* T-App**
6.  $\boxed{x(\perp \rightarrow \perp \rightarrow \perp)(x(\perp \rightarrow \perp)x) : \perp \rightarrow \perp}$  **3,5 T-App**
7.  $\boxed{x(\perp \rightarrow \perp \rightarrow \perp)x : \perp \rightarrow \perp}$  **3,\* T-App**
8.  $\boxed{x(\perp \rightarrow \perp \rightarrow \perp)xx : \perp}$  **7,\* T-App**
9.  $\boxed{x(\perp \rightarrow \perp \rightarrow \perp)(x(\perp \rightarrow \perp)x)(x(\perp \rightarrow \perp \rightarrow \perp)xx) : \perp}$  **6,8 T-App**
10.  $\lambda x : \perp . x(\perp \rightarrow \perp \rightarrow \perp)(x(\perp \rightarrow \perp)x)(x(\perp \rightarrow \perp \rightarrow \perp)xx) : \perp$  **9 T-Abst**

■

### Problem

(3.12) Given the Polymorphic Church Numerals:

$$\mathbf{nat} \in \mathbb{T}_2 := \Pi\alpha : *. (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$$

$$\bar{0} \equiv \lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. x : \mathbf{nat}$$

$$\bar{1} \equiv \lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f x : \mathbf{nat}$$

$$\bar{2} \equiv \lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f(fx) : \mathbf{nat}$$

$$\mathbf{succ} \equiv \lambda n : \mathbf{nat}. \lambda\beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f(n\beta f x)$$

Prove that

$$\mathbf{succ} \bar{0} \underset{\beta}{=} \bar{1}$$

$$\mathbf{succ} \bar{1} \underset{\beta}{=} \bar{2}$$

*Solution.*

$$\begin{aligned} & \mathbf{succ} \bar{0} \\ & \equiv (\lambda n : \mathbf{nat}. \lambda\beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f(n\beta f x))(\lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. x) \\ & \xrightarrow{\beta} (\lambda\beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f((\lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. x)\beta f x)) \\ & \xrightarrow{\beta_{\mathbb{T}_2}} (\lambda\beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f((\lambda f : \beta \rightarrow \beta. \lambda x : \beta. x)fx)) \\ & \xrightarrow[\beta]{\beta \rightarrow \alpha} (\lambda\beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. fx) \underset{\alpha_{\mathbb{T}_2}}{\equiv} \bar{1} \\ \\ & \mathbf{succ} \bar{1} \\ & \equiv (\lambda n : \mathbf{nat}. \lambda\beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f(n\beta f x))(\lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f x) \\ & \xrightarrow{\beta} (\lambda\beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f((\lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f x)\beta f x)) \\ & \xrightarrow{\beta_{\mathbb{T}_2}} (\lambda\beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f((\lambda f : \beta \rightarrow \beta. \lambda x : \beta. f x)fx)) \\ & \xrightarrow[\beta]{\beta \rightarrow \alpha} (\lambda\beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f(fx)) \underset{\alpha_{\mathbb{T}_2}}{\equiv} \bar{2} \end{aligned}$$

### Problem

(3.13 a) We define addition in Polymorphic Church Numerals as

$$\text{add} \equiv \lambda m, n : \mathbf{nat}. \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. m \alpha f (n \alpha f x)$$

Show that

$$\text{add } \bar{1} \quad \bar{1} \underset{\beta}{=} \bar{2}$$

*Solution.*

$$\begin{aligned} & \text{add } \bar{1} \quad \bar{1} \\ & \equiv (\lambda m, n : \mathbf{nat}. \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. m \alpha f (n \alpha f x)) \bar{1} \quad \bar{1} \\ & \xrightarrow[\beta]{\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. \bar{1} \alpha f (\bar{1} \alpha f x)} \\ & \equiv \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. \bar{1} \alpha f ((\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. f x) \alpha f x) \\ & \xrightarrow[\beta]{\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. \bar{1} \alpha f (f x)} \\ & \equiv \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. (\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. f x) \alpha f (f x) \\ & \xrightarrow[\beta]{\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. f (f x)} \underset{\alpha}{=} \bar{2} \end{aligned}$$

### Problem

(3.13 b) Find a term `mul` simulates multiplication on `nat`.

*Solution.*

$$\text{mul} \equiv \lambda m, n : \mathbf{nat}. \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. m \alpha (n \alpha f) x$$

*Proof.* We derive the type first to prove a legal term.

1.	$m, n : \mathbf{nat}$	<b>Bound</b>
2.	$\alpha : *$	<b>Bound</b>
3.	$f : \alpha \rightarrow \alpha$	<b>Bound</b>
4.	$x : \alpha$	<b>Bound</b>
5.	$m \alpha : (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$	<b>*, T2-App</b>
6.	$n \alpha : (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$	<b>*, T2-App</b>
7.	$n \alpha f : \alpha \rightarrow \alpha$	<b>6, T-App</b>
8.	$m \alpha (n \alpha f) : \alpha \rightarrow \alpha$	<b>5,7 T-App</b>

9.	$\frac{\frac{\frac{}{m\alpha(n\alpha f)x : \alpha}}{\lambda x : \alpha. m\alpha(n\alpha f)x : \alpha \rightarrow \alpha}}{\lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m\alpha(n\alpha f)x : (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha}$	<b>8,* T-App</b>
10.		<b>9 T-Abst</b>
11.		<b>10 T-Abst</b>
12.	$\frac{\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m\alpha(n\alpha f)x : \mathbf{nat}}{\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m\alpha(n\alpha f)x : \mathbf{nat}}$	<b>11 T2-Abst</b>
13.		

$\lambda m, n : \mathbf{nat}. \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m\alpha(n\alpha f)x$   
 $: \mathbf{nat} \rightarrow \mathbf{nat} \rightarrow \mathbf{nat}$  **12 T-Abst**

This proves that the term does indeed produce a natural number from two. Next let's prove that

$$\forall \bar{n}, \bar{m} : \mathbf{nat} \quad \text{mul } \bar{n} \ \bar{m} = \overline{n \times m}$$

It could be proven by induction that

$$\forall \bar{a} : \mathbf{nat} \quad \bar{a} \equiv \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f^a x$$

$$\begin{aligned} \text{mul } \bar{n} \ \bar{m} &\equiv (\lambda m, n : \mathbf{nat}. \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m\alpha(n\alpha f)x)(\bar{n})(\bar{m}) \\ &\xrightarrow[\beta]{\rightarrow} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. \bar{n}\alpha(\bar{m}\alpha f)x \\ &\xrightarrow[\beta]{\rightarrow} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. \bar{n}\alpha(\lambda u : \alpha. f^m u)x \\ &\xrightarrow[\beta]{\rightarrow} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. (\lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f^n x)(\lambda u : \alpha. f^m u)x \\ &\xrightarrow[\beta]{\rightarrow} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. (\lambda u : \alpha. f^m u)^n x \end{aligned}$$

By induction this can be further beta-reduced to

$$\xrightarrow[\beta]{\rightarrow} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f^{mn} x \equiv \overline{mn}$$

■

### Problem

(3.14) We present the Church-Encoded Boolean:

$$\mathbf{bool} \in \mathbb{T}_2 := \Pi \alpha : *. \alpha \rightarrow \alpha \rightarrow \alpha$$

$$\mathbf{true} \equiv \lambda \alpha : *. \lambda x, y : \alpha. x$$

$$\mathbf{false} \equiv \lambda \alpha : *. \lambda x, y : \alpha. y$$

Construct a  $\lambda 2$  term  $\mathbf{neg}$  such that  $\mathbf{neg} \ \mathbf{true} \equiv_{\beta} \mathbf{false}$  and  $\mathbf{neg} \ \mathbf{false} \equiv_{\beta} \mathbf{true}$ .

*Solution.*

$$\mathbf{neg} \equiv \lambda b : \mathbf{bool}. \lambda \alpha : *. b\alpha(\mathbf{false} \ \alpha)(\mathbf{true} \ \alpha)$$

*Neg True.*

$$\begin{aligned}
\text{neg true} &\equiv (\lambda b : \text{bool} . \lambda \alpha : * . b \alpha (\text{false } \alpha) (\text{true } \alpha)) \text{ true} \\
&\xrightarrow[\beta]{\rightarrow} \lambda \alpha : * . (\lambda x, y : \alpha . x) (\text{false } \alpha) (\text{true } \alpha) \\
&\xrightarrow[\beta]{\rightarrow} \lambda \alpha : * . \text{false } \alpha \xrightarrow[\eta]{\rightarrow} \text{false}
\end{aligned}$$

■

*Neg False.*

$$\begin{aligned}
\text{neg false} &\equiv (\lambda b : \text{bool} . \lambda \alpha : * . b \alpha (\text{false } \alpha) (\text{true } \alpha)) \text{ false} \\
&\xrightarrow[\beta]{\rightarrow} \lambda \alpha : * . (\lambda x, y : \alpha . y) (\text{false } \alpha) (\text{true } \alpha) \\
&\xrightarrow[\beta]{\rightarrow} \lambda \alpha : * . \text{true } \alpha \xrightarrow[\eta]{\rightarrow} \text{true}
\end{aligned}$$

■

### Problem

(3.15) Define

$$M \equiv \lambda u, v : \text{bool} . \lambda \beta : * . \lambda x, y : \beta . u \beta (v \beta x y) (v \beta y y)$$

And reduce  $M \text{ true true}$ ,  $M \text{ true false}$ ,  $M \text{ false true}$ ,  $M \text{ false false}$ , and decide which logical operator is represented by  $M$ .

*Solution.*

$$\begin{aligned}
M \text{ true true} &\equiv (\lambda u, v : \text{bool} . \lambda \beta : * . \lambda x, y : \beta . u \beta (v \beta x y) (v \beta y y)) \text{ true true} \\
&\xrightarrow[\beta]{\rightarrow} \lambda \beta : * . \lambda x, y : \beta . \text{true } \beta (\text{true } \beta x y) (\text{true } \beta y y) \\
&\xrightarrow[\beta]{\rightarrow} \lambda \beta : * . \lambda x, y : \beta . (\text{true } \beta x y) \\
&\xrightarrow[\beta]{\rightarrow} \lambda \beta : * . \lambda x, y : \beta . x \equiv_{\alpha} \text{true}
\end{aligned}$$

$$\begin{aligned}
M \text{ true false} &\equiv (\lambda u, v : \text{bool} . \lambda \beta : * . \lambda x, y : \beta . u \beta (v \beta x y) (v \beta y y)) \text{ true false} \\
&\xrightarrow[\beta]{\rightarrow} \lambda \beta : * . \lambda x, y : \beta . \text{true } \beta (\text{false } \beta x y) (\text{false } \beta y y) \\
&\xrightarrow[\beta]{\rightarrow} \lambda \beta : * . \lambda x, y : \beta . (\text{false } \beta x y) \\
&\xrightarrow[\beta]{\rightarrow} \lambda \beta : * . \lambda x, y : \beta . y \equiv_{\alpha} \text{false}
\end{aligned}$$

$$\begin{aligned}
M \text{ false true} &\equiv (\lambda u, v : \text{bool}. \lambda \beta : *. \lambda x, y : \beta. u \beta (v \beta x y) (v \beta y y)) \text{ false true} \\
&\xrightarrow[\beta]{\rightarrow} \lambda \beta : *. \lambda x, y : \beta. \text{false } \beta (\text{true } \beta x y) (\text{true } \beta y y) \\
&\xrightarrow[\beta]{\rightarrow} \lambda \beta : *. \lambda x, y : \beta. (\text{true } \beta y y) \\
&\xrightarrow[\beta]{\rightarrow} \lambda \beta : *. \lambda x, y : \beta. y \equiv_{\alpha} \text{false}
\end{aligned}$$

$$\begin{aligned}
M \text{ false false} &\equiv (\lambda u, v : \text{bool}. \lambda \beta : *. \lambda x, y : \beta. u \beta (v \beta x y) (v \beta y y)) \text{ false false} \\
&\xrightarrow[\beta]{\rightarrow} \lambda \beta : *. \lambda x, y : \beta. \text{false } \beta (\text{false } \beta x y) (\text{false } \beta y y) \\
&\xrightarrow[\beta]{\rightarrow} \lambda \beta : *. \lambda x, y : \beta. (\text{false } \beta y y) \\
&\xrightarrow[\beta]{\rightarrow} \lambda \beta : *. \lambda x, y : \beta. y \equiv_{\alpha} \text{false}
\end{aligned}$$

Therefore  $M$  is equivalent to logical AND.

### Problem

(3.16) Find  $\lambda 2$  term representing the logical OR, XOR, IMP.

*Solution.*

$$\begin{aligned}
\text{OR} &\equiv \lambda u, v : \text{bool}. \lambda \beta : *. \lambda x, y : \beta. u \beta x (v \beta x y) \\
\text{XOR} &\equiv \lambda u, v : \text{bool}. \lambda \beta : *. \lambda x, y : \beta. u \beta (v \beta y x) (v \beta x y) \\
\text{IMP} &\equiv \lambda u, v : \text{bool}. \lambda \beta : *. \lambda x, y : \beta. u \beta (v \beta x y) x
\end{aligned}$$

All of them could be checked by finite enumeration over  $\text{bool} \times \text{bool}$ .