Excercises

Chapter 3

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1. Reducted

Definition Some rules for reference.

$$\frac{x : \sigma \in \Gamma \qquad \Gamma \text{ is a } \lambda 2 \text{ context}}{\Gamma \vdash x : \sigma}\ (\text{T-Var}) \qquad \frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}\ (\text{T-App})$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma . M : \sigma \to \tau}\ (\text{T-Abst}) \qquad \frac{\alpha \in \mathbb{T}_2 \qquad \forall \tau \in \text{FV } \alpha, \Gamma \vdash \tau : *}{\alpha : * \in \Gamma}\ (\text{T-Form})$$

$$\frac{\Gamma \vdash M : \Pi_{\alpha:*}.A \qquad \Gamma \vdash B : *}{\Gamma \vdash MB : A[\alpha := B]}\ (\text{T2-App}) \qquad \frac{\Gamma, \alpha : * \vdash M : A}{\Gamma \vdash \lambda \alpha : * . M : \Pi_{\alpha:*} : A}\ (\text{T2-Abst})$$

In this document, convention is that all type judgements in a proof tree, unless stated otherwise, is derived from a single and unique $\lambda 2$ context per tree. Multiple conclusions might be drawn on a single line from usage of the same inference rule for compactness. Eg:

ex 1. $\alpha, \beta : *$ **T-Form**

Is shorthand for

ex 1. $\Gamma \vdash \alpha : *$ **T-Form**

ex 2. $\Gamma \vdash \beta : *$ **T-Form**

Problem

(3.1) How many $\lambda 2$ contexts consisting of four and only four declarations

$$(1)\ \Gamma \vdash \alpha : * \qquad (2)\ \Gamma \vdash \beta : *$$
$$(3)\ \Gamma \vdash f : \alpha \to \beta \qquad (4)\ \Gamma \vdash x : \alpha$$

Solution. The last two declarations depende on the first two. Therefore this is an easy combinatorics problem: $2! \times 2! = 4$ contexts:

$$\begin{array}{cc} 1-2-3-4 & 1-2-4-3 \\ 2-1-3-4 & 2-1-4-3 \end{array}$$

Problem

(3.2) Give a full derivation in $\lambda 2$ to show the following type term is legal:

$$M \equiv \lambda \alpha, \beta, \gamma : * .\lambda f : \alpha \to \beta . \lambda g : \beta \to \gamma . \lambda x : \alpha . g(fx)$$

Solution.

1.	$\alpha : *$	**Bound**
2.	$\beta : *$	**Bound**
3.	$\gamma : *$	**Bound**
4.	$f : \alpha \to \beta$	**Bound**
5.	$g : \beta \to \gamma$	**Bound**
6.	$x : \alpha$	**Bound**
7.	$\alpha, \beta, \gamma : *$	**T-Form**
8.	$\alpha \to \beta, \beta \to \gamma : *$	**T-Form**
9.	$f : \alpha \to \beta, x : \alpha$	**T-Var**
10.	$fx : \beta$	**8,8 T-App**
11.	$g : \beta \to \gamma$	**T-Var**
12.	$g(fx) : \gamma$	**11,10 T-App**
13.	$\lambda x : \alpha . g(fx) : \alpha \to \gamma$	**12 T-Abst**
14.	$\lambda g : \beta \to \gamma . \lambda x : \alpha . g(fx) : (\beta \to \gamma) \to \alpha \to \gamma$	**13 T-Abst**
15.	$\lambda f : \alpha \to \beta . \lambda g : \beta \to \gamma . \lambda x : \alpha . g(fx)$ $: (\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$	**14 T-Abst**

16. $\lambda\gamma : * .\lambda f : \alpha \to \beta.\lambda g : \beta \to \gamma.\lambda x : \alpha.g(fx)$
$: \Pi\gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **15 T2-Abst**

17. $\lambda\beta, \gamma : * .\lambda f : \alpha \to \beta.\lambda g : \beta \to \gamma.\lambda x : \alpha.g(fx)$
$: \Pi\beta, \gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **16 T2-Abst**

18.

$\lambda\alpha, \beta, \gamma : * .\lambda f : \alpha \to \beta.\lambda g : \beta \to \gamma.\lambda x : \alpha.g(fx)$
$: \Pi\alpha, \beta, \gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **17 T2-Abst**

Problem

(3.3 a) Given M in 3.2, and a context Γ such that

$$\Gamma \vdash \texttt{nat} : *$$

$$\Gamma \vdash \texttt{bool} : *$$

$$\Gamma \vdash \text{succ} : \texttt{nat} \to \texttt{nat}$$

$$\Gamma \vdash \text{even} : \texttt{nat} \to \texttt{bool}$$

Prove M $\texttt{nat nat bool}$ succ even is legal.

Solution. Proof by deriving the term's type.

Proof.

1. $M : \Pi\alpha, \beta, \gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **T-Var**
2. $\texttt{nat}, \texttt{bool} : *$ **T-Form**
3. $M\ \texttt{nat} : \Pi\beta, \gamma : * .(\texttt{nat} \to \beta) \to (\beta \to \gamma) \to \texttt{nat} \to \gamma$ **2,1 T2-App**
4. $M\ \texttt{nat}\ \texttt{nat} : \Pi\gamma : * .(\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \gamma) \to \texttt{nat} \to \gamma$ **2,3 T2-App**
5.

$M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool} : (\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \texttt{bool}) \to \texttt{nat} \to \texttt{bool}$ **2,3 T2-App**

6. $\text{succ} : \texttt{nat} \to \texttt{nat}, \text{even} : \texttt{nat} \to \texttt{bool}$ **T-Var**
7. $M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \text{succ} : (\texttt{nat} \to \texttt{bool}) \to \texttt{nat} \to \texttt{bool}$ **6,5 T-App**
8. $M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \text{succ even} : \texttt{nat} \to \texttt{bool}$ **6,7 T-App**

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Problem

(3.3 b.i) Prove $\lambda x : \texttt{nat}.\ \mathrm{even}\ (\mathrm{succ}\ x)$ via 3.3 a.

Solution. The result of beta reduction on the term in 3.3 a is what we are proving.

Proof.

$$\begin{aligned}
& M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \mathrm{succ}\ \mathrm{even} \\
\equiv\ & (\lambda \alpha, \beta, \gamma, f, g.\lambda x : \alpha.g(fx))\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \mathrm{succ}\ \mathrm{even} \\
\underset{\beta}{\twoheadrightarrow}\ & (\lambda f : \texttt{nat} \to \texttt{nat}.\lambda g : \texttt{nat} \to \texttt{bool}.\lambda x : \texttt{nat}.g(f(x)))\ \mathrm{succ}\ \mathrm{even} \\
\underset{\beta}{\to}\ & (\lambda x : \texttt{nat}.\ \mathrm{even}\ (\mathrm{succ}\ x))
\end{aligned}$$

By the subject reduction lemma, $\lambda x : \texttt{nat}.\ \mathrm{even}\ (\mathrm{succ}\ x)) : \texttt{nat} \to \texttt{bool}$, thus is legal. ∎

Problem

(3.3 b.ii) Prove $\lambda x : \texttt{nat}.\ \mathrm{even}\ (\mathrm{succ}\ x)$ via derivation in the context provided in 3.3 a.

Solution.

Proof.

1.	$\texttt{nat}, \texttt{bool} : *$	**T-Form**
2.	$x : \texttt{nat}$	**Bound**
3.	│ $\mathrm{succ} : \texttt{nat} \to \texttt{nat}$	**T-Var**
4.	│ $x : \texttt{nat}$	**T-Var**
5.	│ $\mathrm{succ}\ x : \texttt{nat}$	**3,4 T-App**
6.	│ $\mathrm{even} : \texttt{nat} \to \texttt{bool}$	**T-Var**
7.	│ $\mathrm{even}\ (\mathrm{succ}\ x) : \texttt{bool}$	**6,5 T-App**
8.	$\lambda x : \texttt{nat}.\ \mathrm{even}\ (\mathrm{succ}\ x) : \texttt{nat} \to \texttt{bool}$	**7 T-Abst**

∎

Problem

(3.4) Give a shorthanded (omit T-Var and T-Form) derivation in $\lambda 2$ to show the following term is valid in $\Gamma \equiv \texttt{nat} : *, \texttt{bool} : *$

$$(\lambda \alpha, \beta : * . \lambda f : \alpha \to \alpha . \lambda g : \alpha \to \beta . \lambda x : \alpha . g(f(fx)))\ \texttt{nat}\ \texttt{bool}$$

Solution.

Proof.

1. $\alpha, \beta : *$ **Bound**
2. $f : \alpha \to \alpha$ **Bound**
3. $g : \alpha \to \beta$ **Bound**
4. $x : \alpha$ **Bound**
5. $fx : \alpha$ ***,* T-App**
6. $f(fx) : \alpha$ ***,5 T-App**
7. $g(f(fx)) : \beta$ ***,6 T-App**
8. $\lambda x : \alpha . g(f(fx)) : \alpha \to \beta$ **7 T-Abst**
9. $\lambda g : \alpha \to \beta . \lambda x : \alpha . g(f(fx)) : (\alpha \to \beta) \to \alpha \to \beta$ **8 T-Abst**
10. $\lambda f : \alpha \to \alpha . \lambda g : \alpha \to \beta . \lambda x : \alpha . g(f(fx)) : (\alpha \to \alpha) \to (\alpha \to \beta) \to \alpha \to \beta$ **9 T-Abst**
11. $\lambda \alpha, \beta : * . \lambda f : \alpha \to \alpha . \lambda g : \alpha \to \beta . \lambda x : \alpha . g(f(fx)) : \Pi \alpha, \beta : * . (\alpha \to \alpha) \to (\alpha \to \beta) \to \alpha \to \beta$ **10 T2-Abst**
12. $(\lambda \alpha, \beta : * . \lambda f : \alpha \to \alpha . \lambda g : \alpha \to \beta . \lambda x : \alpha . g(f(fx)))\ \texttt{nat} : \Pi \beta : * . (\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \beta) \to \texttt{nat} \to \beta$ ***,11 T2-App**
13. $(\lambda \alpha, \beta : * . \lambda f : \alpha \to \alpha . \lambda g : \alpha \to \beta . \lambda x : \alpha . g(f(fx)))\ \texttt{nat}\ \texttt{bool} : \Pi \beta : * . (\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \texttt{bool}) \to \texttt{nat} \to \texttt{bool}$ ***,12 T2-App**

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