Excercises

Chapter 5

Sean Li [1]

1. Reducted

Definition Some rules for reference.

λC Calculus Rules

$$\frac{}{\emptyset \vdash * : \square}\ \text{Sort} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}\ \text{Var} \qquad \frac{\Gamma \vdash A : B \qquad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}\ \text{Weak}$$

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi\, x : A \,.\, B : s_2}\ \text{Form}$$

$$\frac{\Gamma \vdash M : \Pi\, x : A \,.\, B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B\,[x := N]}\ \text{App}$$

$$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash \Pi\, x : A \,.\, B : s}{\Gamma \vdash \lambda x : A \,.\, M : \Pi\, x : A \,.\, B}\ \text{Abst}$$

$$\frac{\Gamma \vdash A : B \qquad B \overset{\beta}{=} B' \qquad \Gamma \vdash B' : s}{\Gamma \vdash A : B'}\ \text{Conv}$$

Problem

(6.1 a) Give a complete derivation in tree format showing that

$$\perp \equiv \Pi \alpha : * \,.\, \alpha$$

is legal in λC.

Solution. Here we will show that there exists $s \in$ sort and Γ such that $\Gamma \vdash \perp : s$.

Proof.

$$\dfrac{\vdash * : \square \qquad \dfrac{\vdash * : \square}{\alpha : * \vdash \alpha : *}\,\text{Var}}{\vdash \Pi \alpha : * \,.\alpha : *}\,\text{Form}$$

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Problem

(6.1 a) Give a complete derivation in tree format showing that $\perp \to \perp$ is legal in λC where

$$\perp \equiv \Pi \alpha : * \,.\alpha$$

Solution. Here we will show that there exists $s \in$ sort and Γ such that $\Gamma \vdash \perp \to \perp : s$.

Proof.

$$\dfrac{\text{(6.1 a)}\,\dfrac{}{\vdash \perp : *} \qquad \dfrac{\text{(6.1 a)}\,\dfrac{}{\vdash \perp : *} \qquad \text{(6.1 a)}\,\dfrac{}{\vdash \perp : *}}{x : \perp \vdash \perp : *}\,\text{Weak}}{\vdash \Pi\, x : \perp . \perp : *}\,\text{Form}$$

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Problem

(6.1 c) To which systems of the λ-cube does $\perp$ belong? And $\perp \to \perp$?

Solution. The set of (s_1, s_2) pairs in formation rules of the derivation of $\perp$ is $\{(\square, *)\}$. The minimal system corresponding is $\lambda 2$. The same for $\perp \to \perp$. Therefore $\perp$ and $\perp \to \perp$ belongs to $\lambda 2$, $\lambda\omega$, λP and λC.

Problem

(6.2) Given context $\Gamma \equiv S : *, P : S \to *, A : *$. Prove by means of a flag derivation that the following expression is inhabited in λC with respect to Γ:

$$(\Pi\, x : S \,.(A \to P\, x)) \to A \to \Pi\, y : S \,.\, P\, y$$

Solution. The inhabitant is

$$M \equiv \lambda u : (\Pi\, x : S\,.(A \to P\,x)).\lambda v : A\,.\lambda y : S\,.\,u\,y\,v$$

Proof.

$$\begin{array}{lll}
1. & S : *, P : S \to *, A : * & \\
2. & \quad u : \Pi\, x : S\,.(A \to P\,x) & \\
3. & \quad\quad v : A & \\
4. & \quad\quad\quad y : S & \\
5. & \quad\quad\quad\quad u\,y : A \to P\,y & \textbf{2,4 App} \\
6. & \quad\quad\quad\quad u\,y\,v : P\,y & \textbf{5,3 App} \\
7. & \quad\quad\quad \lambda y : S\,.\,u\,y\,v : \Pi\,y : S\,.\,P\,y & \textbf{6 Abst} \\
8. & \quad\quad \lambda v : A\,.\lambda y : S\,.\,u\,y\,v : A \to \Pi\,y : S\,.\,P\,y & \textbf{7 Abst} \\
 & \quad \lambda u : \Pi\, x : S\,.(A \to P\,x).\lambda v : A\,.\lambda y : S\,.\,u\,y\,v & \\
9. & \quad\quad : \Pi\, x : S\,.(A \to P\,x) \to A \to \Pi\,y : S\,.\,P\,y & \textbf{8 Abst}
\end{array}$$

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