EXCERCISES

CHAPTER 1

SEAN LI [1]

1. Reducted

Problem

(1.1) Simplify notation of the following terms

(a) $(\lambda x .(((x\ z)\ y)(x\ x)))$

(b) $((\lambda x .(\lambda y .(\lambda z .(z\ ((x\ y)\ z)))))(\lambda u\ .\ u))$

Solution.

(a) $\lambda x .(x\ z\ y)(x\ x)$

(b) $(\lambda x\ y\ z\ .\ x\ (x\ y\ z))(\lambda u\ .\ u)$

Problem

(1.2) Find the alpha equivalent terms to

$$\lambda x\ .\ x\ (\lambda x\ .\ x)$$

In

(a) $\lambda y\ .\ y\ (\lambda x\ .\ x)$

(b) $\lambda y\ .\ y\ (\lambda x\ .\ y)$

(c) $\lambda y\ .\ y\ (\lambda x\ .\ y)$

Solution. Only (a).

(1.3) Prove

$$\lambda x \,.\, x \,(\lambda z \,.\, y) \underset{\alpha}{=} \lambda z \,.\, z \,(\lambda z \,.\, y)$$

Solution.

Proof. By definition of alpha equivalence

$$M \underset{\alpha}{=} N \iff \exists \varphi, M^\varphi \underset{\alpha}{\rightarrow} N \wedge \text{FR } M = \text{FR } N$$

The witness of φ is substituting bound variable x with z, and z is not a free variable in the term, thus the two terms are alpha equivalent.

$$\lambda x \,.\, x \,(\lambda z \,.\, y)^{x \rightarrow z} \underset{\alpha}{\rightarrow} \lambda z \,.\, z \,(\lambda z \,.\, y)$$

∎

Problem

(1.4) Consider the following term:

$$U := (\lambda z \,.\, z \, x \, z)((\lambda y \,.\, x \, y) \, x)$$

1. Find Sub U
2. Draw tree rep of U
3. Find FV U
4. Find alpha equivalent terms to U from below and point out which of those follows the Barendregt convention:

$$\text{(a) } (\lambda y \,.\, y \, x \, y)((\lambda z \,.\, x \, z) \, x)$$
$$\text{(b) } (\lambda x \,.\, x \, y \, x)((\lambda z \,.\, y \, z) \, y)$$
$$\text{(c) } (\lambda y \,.\, y \, x \, y)((\lambda y \,.\, x \, y) \, x)$$
$$\text{(d) } (\lambda v \,.(v \, x) \, v)((\lambda u \,.\, u \, v) \, x)$$

1. Find Sub U.
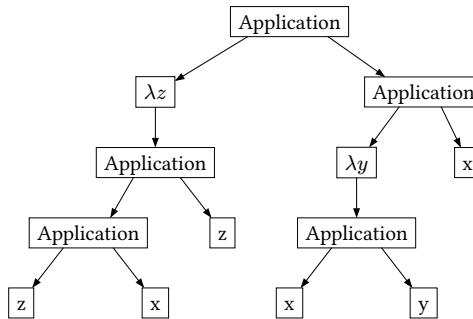
Solution.

Sub $U =$

$$\{(\lambda z \,.\, z\ x\ z)((\lambda y \,.\, x\ y)\ x), (\lambda z \,.\, z\ x\ z), ((\lambda y \,.\, x\ y)\ x)\} \cup$$
$$\{(\lambda y \,.\, x\ y), z\ x\ z, z\ x, x\ y\} \cup$$
$$\{z, x, y\}$$
$$= \{(\lambda z \,.\, z\ x\ z)((\lambda y \,.\, x\ y)\ x), (\lambda z \,.\, z\ x\ z), ((\lambda y \,.\, x\ y)\ x),$$
$$(\lambda y \,.\, x\ y), z\ x\ z, z\ x, x\ y, z, x, y\}$$

2. Draw a tree rep of U.

Solution.

3. Find FV U

Solution.

$$\text{FV } U = \text{FV } (\lambda y \,.\, y\ x\ y) \cup \text{FV } (\lambda z \,.\, x\ z)\ x$$
$$= (\text{FV } y\ x\ y) \setminus \{y\} \cup (\text{FV } \lambda z \,.\, x\ z) \cup \{x\}$$
$$= (\text{FV } y\ x) \setminus \{y\} \cup (\text{FV } x\ z) \setminus \{z\} \cup \{x\}$$
$$= \{x\}$$

4. Find an alpha-equivalent term.

Solution.

$$(a) \underset{\alpha}{=} (c) \underset{\alpha}{=} U$$

Only (a) follows the Barendregt convention.

Problem

(1.5) Give the results of the following substitutions

(a) $(\lambda x \,.\, y\ (\lambda y \,.\, x\ y))[y := \lambda z \,.\, z\ x]$

(b) $((x\ y\ z)[x := y])[y := z]$

(c) $((\lambda x \,.\, x\ y\ z)[x := y])[y := z]$

(d) $(\lambda y \,.\, y\ y\ x)[x := y\ z]$

Solution.

(a) $(\lambda v\,.(\lambda z\,.\,z\,x)(\lambda u\,.\,v\,u))$

(b) $(y\,y\,z)[y := z] = z\,z\,z$

(c) $(\lambda x\,.\,x\,y\,z)[y := z] = (\lambda x\,.\,x\,z\,z)$

(d) $(\lambda u\,.\,u\,u\,(y\,z))$

Problem

(1.6)

$$\neg\left(\forall\,M\,L\,N \in \Lambda,\, M\,[x := N, y := L] \underset{\alpha}{\equiv} M\,[x := N][y := L]\right)$$

Solution.

Proof. Because RHS $= M\,[x := N][y := L] = M\,[x := N\,[y := L]][y := L]$, if $y \in$ FV N, then what x gets substituted with will have y substituted for L, which is completely different with LHS. ∎
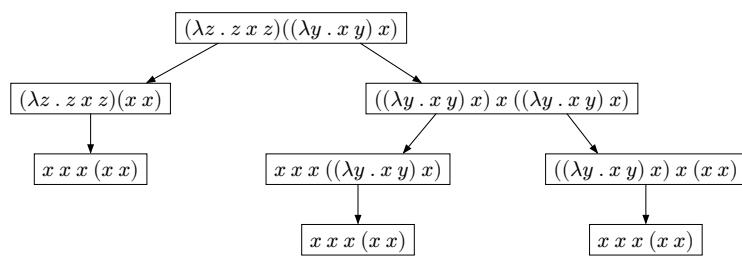
Problem

(1.7) Find all available redexes in

$$U := (\lambda z\,.\,z\,x\,z)((\lambda y\,.\,x\,y)\,x)$$

And all reduction pathes to the β-normal form.

Solution. The first redex is the term as an application itself; another the second term in the application.

(1.8) Show that

$$(\lambda x \;.\; x \; x) \; y \underset{\beta}{\neq} (\lambda x \; y \;.\; y \; x) \; x \; x$$

Solution. By Collorary 1.9.9, it suffices to prove the hypothesis with a proof of a common normal reduced form from LHS and RHS not existing.

Contradiction. By definition of $\underset{\beta}{=}$, there exists The set of all terms attainable from β -reduction on $(\lambda x \;.\; x \; x) \; y$ and $(\lambda x \; y \;.\; y \; x) \; x \; x$ do not intersect. Therefore,

$$\neg\left(\exists L \in \Lambda, (\lambda x \;.\; x \; x) \; y \underset{\beta}{\twoheadrightarrow} L \wedge (\lambda x \; y \;.\; y \; x) \; x \; x \underset{\beta}{\twoheadrightarrow} L\right) \implies \neg\left((\lambda x \;.\; x \; x) \; y \underset{\beta}{=} (\lambda x \; y \;.\; y \; x) \; x \; x\right)$$

∎

Problem

(1.9) Define the combinators

$$K := \lambda x \; y \;.\; x$$
$$S := \lambda x \; y \; z \;.\; x \; z \; (y \; z)$$

Prove that

$$\forall \; P \; Q \in \Lambda, K \; P \; Q \underset{\beta}{\twoheadrightarrow} P$$

$$\forall \; P \; Q \; R \in \Lambda, S \; P \; Q \; R \underset{\beta}{\twoheadrightarrow} P \; R \; (Q \; R)$$

Solution.

Proof.

$$K \; P \; Q = (\lambda x \; y \;.\; x) \; P \; Q \underset{\beta}{\rightarrow} (\lambda y \;.\; x)[x := P] \; Q \underset{\beta}{\rightarrow} P \; [y := Q] = P$$

$$S \; P \; Q \; R = (\lambda x \; y \; z \;.\; x \; z \; (y \; z)) \underset{\beta}{\twoheadrightarrow} (x \; z \; (y \; z))[x := P][y := Q][z := R] = P \; R \; (Q \; R)$$

∎

Problem

(1.10) We define the church numerals

$$\text{zero} := \lambda f\, x\,.\,x$$
$$\text{one} := \lambda f\, x\,.\,f\, x$$
$$\text{two} := \lambda f\, x\,.\,f\, f\, x$$
$$...$$
$$\text{num}_n := \lambda f\, x\,.\,f^n\, x$$

And operations

$$\text{add} := \lambda n\, m\, f\, x\,.\,m\, f\, (n\, f\, x)$$
$$\text{mul} := \lambda n\, m\, f\, x\,.\,m\, (n\, f)\, x$$

Show

(a) add one one $\twoheadrightarrow_\beta$ two

(b) add one one $\neq_\beta$ mul one zero

Solution.

(a) $\quad$ add one one $= (\lambda n\, m\, f\, x\,.\,m\, f\, (n\, f\, x))(\lambda f\, x\,.\,f\, x)(\lambda f\, x\,.\,f\, x)$

$\twoheadrightarrow_\beta (\lambda f\, x\,.(\lambda f\, x\,.\,f\, x)\, f\, ((\lambda f\, x\,.\,f\, x)\, f\, x))$

$\twoheadrightarrow_\beta (\lambda f\, x\,.(\lambda x\,.\,f\, x)\, f\, x)$

$\twoheadrightarrow_\beta (\lambda f\, x\,.\,f\, f\, x) = $ two

(b) $\quad$ mul one one $= (\lambda n\, m\, f\, x\,.\,m\, (n\, f)\, x)(\lambda f\, x\,.\,f\, x)(\lambda f\, x\,.\,f\, x)$

$\twoheadrightarrow_\beta \lambda f\, x\,.(\lambda f\, x\,.\,f\, x)((\lambda f\, x\,.\,f\, x)\, f)\, x$

$\twoheadrightarrow_\beta \lambda f\, x\,.\,f\, x = $ one

Because no intermediate form in the beta reduction process of the two terms are α-equivalent, by collorary 1.9.9 the two terms are not β-equivalent.

(1.11) We define

$$\text{succ} := \lambda m\, f\, x\,.\, f\, (m\, f\, x) \text{ s.t. } \forall \text{num}_n, \text{succ num}_n = \text{num}_{n+1}$$

Prove

$$\text{succ zero} \underset{\beta}{=} \text{one}$$

$$\text{succ one} \underset{\beta}{=} \text{two}$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$\text{succ zero} = (\lambda m\, f\, x\,.\, f\, (m\, f\, x))(\lambda f\, x\,.\, x)$$

$$\underset{\beta}{\rightarrow} (\lambda f\, x\,.\, f\, ((\lambda f\, x\,.\, x)\, f\, x))$$

$$\underset{\beta}{\twoheadrightarrow} (\lambda f\, x\,.\, f\, x) = \text{one}$$

The path succ zero $\underset{\beta}{\twoheadrightarrow}$ one derived above is the witness of a reduction chain from LHS to RHS.

$$\text{succ one} = (\lambda m\, f\, x\,.\, f\, (m\, f\, x))(\lambda f\, x\,.\, f\, x)$$

$$\underset{\beta}{\rightarrow} (\lambda f\, x\,.\, f\, ((\lambda f\, x\,.\, f\, x)\, f\, x))$$

$$\underset{\beta}{\twoheadrightarrow} (\lambda f\, x\,.\, f\, (f\, x)) = \text{two}$$

The path succ one $\underset{\beta}{\twoheadrightarrow}$ two derived above is the witness of a reduction chain from LHS to RHS. ∎

(1.12) We define the λ-terms $\top_\lambda$(true) and $\bot_\lambda$(false) and $\neg_\lambda$(not) by:

$$\top_\lambda := \lambda x\, y\,.\, x \quad \bot_\lambda := \lambda x\, y\,.\, y$$

$$\neg_\lambda := \lambda a\,.\, a\, \bot_\lambda \top_\lambda$$

Show that

$$\neg_\lambda(\neg_\lambda \top_\lambda) \underset{\beta}{=} \top_\lambda$$

$$\neg_\lambda(\neg_\lambda \bot_\lambda) \underset{\beta}{=} \bot_\lambda$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$\neg_\lambda(\neg_\lambda \top_\lambda) = \neg_\lambda((\lambda a \,.\, a \perp_\lambda \top_\lambda)(\lambda x\, y \,.\, x))$$
$$\underset{\beta}{\twoheadrightarrow} \neg_\lambda((\lambda x\, y \,.\, x)\perp_\lambda \top_\lambda)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda a \,.\, a \perp_\lambda \top_\lambda)\perp_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda x\, y \,.\, y)\perp_\lambda \top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} \top_\lambda$$

$$\neg_\lambda(\neg_\lambda \perp_\lambda) = \neg_\lambda((\lambda a \,.\, a \perp_\lambda \top_\lambda)(\lambda x\, y \,.\, y))$$
$$\underset{\beta}{\twoheadrightarrow} \neg_\lambda((\lambda x\, y \,.\, y)\perp_\lambda \top_\lambda)$$
$$\underset{\beta}{\twoheadrightarrow} \neg_\lambda \top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda a \,.\, a \perp_\lambda \top_\lambda)(\lambda x\, y \,.\, x)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda x\, y \,.\, x)\perp_\lambda \top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} \perp_\lambda$$

■

Problem

(1.13) Define

$$\text{iszero} := \lambda m \,.\, m\, (\lambda x \,.\perp_\lambda)\top_\lambda$$

Prove

$$\text{iszero zero} \underset{\beta}{\twoheadrightarrow} \top_\lambda$$
$$\forall\, n \in \mathbb{N}^+, \text{iszero num}_n \underset{\beta}{\twoheadrightarrow} \perp_\lambda$$

Solution.

$$\text{iszero zero} = (\lambda m \,.\, m\, (\lambda x \,.\perp_\lambda)\top_\lambda)(\lambda f\, x \,.\, x)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda f\, x \,.\, x)(\lambda x \,.\perp_\lambda)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} \top_\lambda$$

$$\text{iszero num}_n = (\lambda m \,.\, m \,(\lambda x \,.\bot_\lambda)\top_\lambda)(\lambda f \, x \,.\, f^n \, x)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda f \, x \,.\, f^n \, x)(\lambda x \,.\bot_\lambda)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda x \,.\bot_\lambda)\big((\lambda x \,.\bot_\lambda)^{n-1}\top_\lambda\big) \underset{\beta}{\twoheadrightarrow} \bot_\lambda$$

Problem

(1.14) If-else can be modeled as

$$\text{ifelse} = \lambda x \, t \, f \,.\, x \, t \, f$$

Where when x, then t, else f. Prove correctness by applying $\top_\lambda$ and $\bot_\lambda$ on ifelse.

Solution.

$$\text{ifelse } \top_\lambda = (\lambda x \, t \, f \,.\, x \, t \, f)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda t \, f \,.(\lambda x \, y \,.\, x) \, t \, f) \underset{\beta}{\twoheadrightarrow} (\lambda t \, f \,.\, t)$$

$$\text{ifelse } \top_\lambda = (\lambda x \, t \, f \,.\, x \, t \, f)\bot_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda t \, f \,.(\lambda x \, y \,.\, y) \, t \, f) \underset{\beta}{\twoheadrightarrow} (\lambda t \, f \,.\, f)$$

By applying the results to any two values, the correct corresponding value returns, ex, for ifelse $\top_\lambda$, t is always returned.

Problem

(1.15) Prove that $\Omega := (\lambda x \,.\, x \, x)(\lambda x \,.\, x \, x)$ does not have a β-nf.

Solution. Firstly let's prove Ω.

Proof. Induction on Ω's only reduction path proves that every $\Omega \underset{\beta}{\twoheadrightarrow} \Omega_i = \Omega$. For the base case because Ω has one and only one redux, it could only reduce to Ω_1 which is equivalent to itself. For the inductive step, $\Omega_i = \Omega$, therefore $\Omega_i \underset{\beta}{\rightarrow} \Omega_{i+1}$ is still Ω.

By definition, a term having a β-nf requires the existence of a form in β-nf such that the term can reduce to. By induction, Ω only reduces to Ω, and Ω is not in β-nf because it contains β-redex. Therefore, Ω can never reduce to a β-nf, thus it does not have a β-nf. ∎

Problem

(1.16) Let M be a λ-term with the following properties:
- M has a β-nf.
- There exists an infinite reduction path $M \equiv M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \dots$ on M.

Prove that every M_i has a β-nf, and give an example of M.

Solution. An example would be $(\lambda x\, y\, .\, y)\Omega$. Reduction can go on infinitely by reducing on Ω, but the β-nf of the term is $\lambda y\, .\, y$

Proof. Denote β-nf of M as M'. For any form in the reduction path, $M \underset{\beta}{\twoheadrightarrow} M_i$. In conjunction with $M \underset{\beta}{\twoheadrightarrow} M'$, by the Church-Rosser theorem, there exists L such that $M_i \underset{\beta}{\twoheadrightarrow} L$ and $M' \underset{\beta}{\twoheadrightarrow} L$. Because M' is in β-nf, L can only be M', thus $M_i \underset{\beta}{\twoheadrightarrow} M'$, so M_i is capable of reducing to M', a β-nf. Therefore, any form in the reduction path has a β-nf. ∎

Problem

(1.17) If $M\,N$ is strongly normalizing, then both M and N are strongly normalizing.

Solution.

Proof. If M is not strongly normalizing, then there exists a reduction path $M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \dots$ Therefore, $M\,N$ would have had a reduction path $M\,N \underset{\beta}{\to} M_1\,N \underset{\beta}{\to} \dots$ that is infinite, which contradicts with $M\,N$ being strongly normalizing. Vice versa for N. ∎

Problem

(1.18) Let $L, M, N \in \Lambda$ such that $L \underset{\beta}{=} M$ and $L \underset{\beta}{\twoheadrightarrow} N$. Moreover, N is in β-nf. Prove that $M \underset{\beta}{\twoheadrightarrow} N$.

Solution. Collorary 1.9.9.

(1.19) Define

$$U := \lambda z\, x\, .\, x\, (z\, z\, x) \quad \text{and} \quad Z := U\, U$$

Prove Z is a fixed point combinator.

Solution. Proving $\forall\, L \in \Lambda,\, L\,(Z\,L) \twoheadrightarrow_{\beta} Z\,L$.

Proof.

$$Z\,L = (\lambda z\, x\, .\, x\,(z\,z\,x))(\lambda z\, x\, .\, x\,(z\,z\,x))\,L$$
$$\twoheadrightarrow_{\beta} L\,((\lambda z\, x\, .\, x\,(z\,z\,x))(\lambda z\, x\, .\, x\,(z\,z\,x))\,L) \twoheadrightarrow_{\beta} L\,(Z\,L)$$

∎

(1.20) Solve for $M \in \Lambda$ in each equation:

$$M =_{\beta} \lambda x\, y\, .\, x\, M\, y$$
$$M\, x\, y\, z =_{\beta} x\, y\, z\, M$$

Solution. By the property of the Y combinator:

$$f\,(Y\,f) = Y\,f$$

The first equation can be remodeled as

$$M =_{\beta} L\,M \text{ where } L = \lambda m\, x\, y\, .\, x\, m\, y$$

Solving for fixed point of L :

$$M \equiv Y\,L \equiv L\,(Y\,L)$$
$$= (\lambda x\, .\, L\,(x\,x))(\lambda x\, .\, L\,(x\,x))$$
$$= (\lambda x\, .(\lambda m\, u\, v\, .\, u\, m\, v)(x\,x))(\lambda x\, .(\lambda m\, u\, v\, .\, u\, m\, v)(x\,x))$$
$$\twoheadrightarrow_{\beta} (\lambda x\, .(\lambda u\, v\, .\, u\,(x\,x)\,v))(\lambda x\, .(\lambda u\, v\, .\, u\,(x\,x)\,v))$$

The second equation can be η-reduced on both sides:

$$M\, x\, y\, z =_{\beta} x\, y\, z\, M$$
$$M =_{\beta} \lambda x\, y\, z\, .\, x\, y\, z\, M$$

Remodeling equation:

$$M = N\ M \text{ where } M = \lambda m\ x\ y\ z\ .\ x\ y\ z\ m$$

then

$$
\begin{aligned}
M &\equiv Y\ N \equiv N\ (Y\ N) \\
&= (\lambda x\ .\ N\ (x\ x))(\lambda x\ .\ N\ (x\ x)) \\
&= (\lambda x\ .(\lambda m\ u\ y\ z\ .\ u\ y\ z\ m)(x\ x))(\lambda x\ .(\lambda m\ u\ y\ z\ .\ u\ y\ z\ m)(x\ x)) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda x\ .(\lambda u\ y\ z\ .\ u\ y\ z\ (x\ x)))(\lambda x\ .(\lambda u\ y\ z\ .\ u\ y\ z\ (x\ x)))
\end{aligned}
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