

# EXERCISES

## CHAPTER 3

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1. Redacted

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**Definition** Some rules for reference.

$$\frac{x : \sigma \in \Gamma \quad \Gamma \text{ is a } \lambda 2 \text{ context}}{\Gamma \vdash x : \sigma} \text{ (T-Var)} \quad \frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M N : \tau} \text{ (T-App)}$$
$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma. M : \sigma \rightarrow \tau} \text{ (T-Abst)} \quad \frac{\alpha \in \mathbb{T}_2 \quad \text{FV } \alpha \subseteq \text{dom } \Gamma}{\alpha : * \in \Gamma} \text{ (T-Form)}$$
$$\frac{\Gamma \vdash M : \Pi_{\alpha : *} A \quad \Gamma \vdash B : *}{\Gamma \vdash M B : A [\alpha := B]} \text{ (T2-App)} \quad \frac{\Gamma, \alpha : * \vdash M : A}{\Gamma \vdash \lambda \alpha : *. M : \Pi_{\alpha : *} A} \text{ (T2-Abst)}$$

In this document, convention is that all type judgements in a proof tree, unless stated otherwise, is derived from a single and unique  $\lambda 2$  context per tree. Multiple conclusions might be drawn on a single line from usage of the same inference rule for compactness. Eg:

*ex* 1.  $\alpha, \beta : *$  **T-Form**

Is shorthand for

*ex* 1.  $\Gamma \vdash \alpha : *$  **T-Form**

*ex* 2.  $\Gamma \vdash \beta : *$  **T-Form**

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## Problem

(3.1) How many  $\lambda 2$  contexts consisting of four and only four declarations

- (1)  $\Gamma \vdash \alpha : *$
- (2)  $\Gamma \vdash \beta : *$
- (3)  $\Gamma \vdash f : \alpha \rightarrow \beta$
- (4)  $\Gamma \vdash x : \alpha$

*Solution.* The last two declarations depend on the first two. Therefore this is an easy combinatorics problem:  $2! \times 2! = 4$  contexts:

$$\begin{array}{ll} 1 - 2 - 3 - 4 & 1 - 2 - 4 - 3 \\ 2 - 1 - 3 - 4 & 2 - 1 - 4 - 3 \end{array}$$

## Problem

(3.2) Give a full derivation in  $\lambda 2$  to show the following type term is legal:

$$M \equiv \lambda \alpha, \beta, \gamma : * . \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(f x)$$

*Solution.*

|     |                                                                                                                                                                                                                    |                    |
|-----|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------|
| 1.  | $\alpha : *$                                                                                                                                                                                                       | <b>Bound</b>       |
| 2.  | $\beta : *$                                                                                                                                                                                                        | <b>Bound</b>       |
| 3.  | $\gamma : *$                                                                                                                                                                                                       | <b>Bound</b>       |
| 4.  | $f : \alpha \rightarrow \beta$                                                                                                                                                                                     | <b>Bound</b>       |
| 5.  | $g : \beta \rightarrow \gamma$                                                                                                                                                                                     | <b>Bound</b>       |
| 6.  | $x : \alpha$                                                                                                                                                                                                       | <b>Bound</b>       |
| 7.  | $\alpha, \beta, \gamma : *$                                                                                                                                                                                        | <b>T-Form</b>      |
| 8.  | $\alpha \rightarrow \beta, \beta \rightarrow \gamma : *$                                                                                                                                                           | <b>T-Form</b>      |
| 9.  | $f : \alpha \rightarrow \beta, x : \alpha$                                                                                                                                                                         | <b>T-Var</b>       |
| 10. | $f x : \beta$                                                                                                                                                                                                      | <b>8,8 T-App</b>   |
| 11. | $g : \beta \rightarrow \gamma$                                                                                                                                                                                     | <b>T-Var</b>       |
| 12. | $g(f x) : \gamma$                                                                                                                                                                                                  | <b>11,10 T-App</b> |
| 13. | $\lambda x : \alpha. g(f x) : \alpha \rightarrow \gamma$                                                                                                                                                           | <b>12 T-Abst</b>   |
| 14. | $\lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(f x) : (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$                                                                              | <b>13 T-Abst</b>   |
| 15. | $\lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(f x) : (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$ | <b>14 T-Abst</b>   |

|     |                                                                                                                                                                                                                                                                                                                      |                   |
|-----|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------|
| 16. | $\begin{array}{l} \lambda\gamma : * . \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(f x) \\ : \Pi\gamma : * . (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma \end{array}$                               | <b>15 T2-Abst</b> |
| 17. | $\begin{array}{l} \lambda\beta, \gamma : * . \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(f x) \\ : \Pi\beta, \gamma : * . (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma \end{array}$                 | <b>16 T2-Abst</b> |
| 18. |                                                                                                                                                                                                                                                                                                                      |                   |
|     | $\begin{array}{l} \lambda\alpha, \beta, \gamma : * . \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(f x) \\ : \Pi\alpha, \beta, \gamma : * . (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma \end{array}$ | <b>17 T2-Abst</b> |

### Problem

(3.3 a) Given  $M$  in 3.2, and a context  $\Gamma$  such that

$$\begin{aligned} \Gamma &\vdash \mathbf{nat} : * \\ \Gamma &\vdash \mathbf{bool} : * \\ \Gamma &\vdash \mathbf{succ} : \mathbf{nat} \rightarrow \mathbf{nat} \\ \Gamma &\vdash \mathbf{even} : \mathbf{nat} \rightarrow \mathbf{bool} \end{aligned}$$

Prove  $M \mathbf{nat} \mathbf{nat} \mathbf{bool} \mathbf{succ} \mathbf{even}$  is legal.

*Solution.* Proof by deriving the term's type.

*Proof.*

1.  $M : \Pi\alpha, \beta, \gamma : * . (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$  **T-Var**
  2.  $\mathbf{nat}, \mathbf{bool} : *$  **T-Form**
  3.  $M \mathbf{nat} : \Pi\beta, \gamma : * . (\mathbf{nat} \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \mathbf{nat} \rightarrow \gamma$  **2,1 T2-App**
  4.  $M \mathbf{nat} \mathbf{nat} : \Pi\gamma : * . (\mathbf{nat} \rightarrow \mathbf{nat}) \rightarrow (\mathbf{nat} \rightarrow \gamma) \rightarrow \mathbf{nat} \rightarrow \gamma$  **2,3 T2-App**
  - 5.
- $M \mathbf{nat} \mathbf{nat} \mathbf{bool} : (\mathbf{nat} \rightarrow \mathbf{nat}) \rightarrow (\mathbf{nat} \rightarrow \mathbf{bool}) \rightarrow \mathbf{nat} \rightarrow \mathbf{bool}$  **2,3 T2-App**
6.  $\mathbf{succ} : \mathbf{nat} \rightarrow \mathbf{nat}, \mathbf{even} : \mathbf{nat} \rightarrow \mathbf{bool}$  **T-Var**
  7.  $M \mathbf{nat} \mathbf{nat} \mathbf{bool} \mathbf{succ} : (\mathbf{nat} \rightarrow \mathbf{bool}) \rightarrow \mathbf{nat} \rightarrow \mathbf{bool}$  **6,5 T-App**
  8.  $M \mathbf{nat} \mathbf{nat} \mathbf{bool} \mathbf{succ} \mathbf{even} : \mathbf{nat} \rightarrow \mathbf{bool}$  **6,7 T-App**

■

### Problem

(3.3 b.i) Prove  $\lambda x : \text{nat}. \text{even}(\text{succ } x)$  via 3.3 a.

*Solution.* The result of beta reduction on the term in 3.3 a is what we are proving.

*Proof.*

$$\begin{aligned} M &\equiv \text{nat nat bool succ even} \\ &\equiv (\lambda\alpha, \beta, \gamma, f, g. \lambda x : \alpha. g(f x)) \text{ nat nat bool succ even} \\ &\xrightarrow[\beta]{\beta} (\lambda f : \text{nat} \rightarrow \text{nat}. \lambda g : \text{nat} \rightarrow \text{bool}. \lambda x : \text{nat}. g(f(x))) \text{ succ even} \\ &\xrightarrow[\beta]{\beta} (\lambda x : \text{nat}. \text{even}(\text{succ } x)) \end{aligned}$$

By the subject reduction lemma,  $\lambda x : \text{nat}. \text{even}(\text{succ } x) : \text{nat} \rightarrow \text{bool}$ , thus is legal. ■

### Problem

(3.3 b.ii) Prove  $\lambda x : \text{nat}. \text{even}(\text{succ } x)$  via derivation in the context provided in 3.3 a.

*Solution.*

*Proof.*

|    |                                                                                            |                  |
|----|--------------------------------------------------------------------------------------------|------------------|
| 1. | $\text{nat}, \text{bool} : *$                                                              | <b>T-Form</b>    |
| 2. | $x : \text{nat}$                                                                           | <b>Bound</b>     |
| 3. | $\text{succ} : \text{nat} \rightarrow \text{nat}$                                          | <b>T-Var</b>     |
| 4. | $x : \text{nat}$                                                                           | <b>T-Var</b>     |
| 5. | $\text{succ } x : \text{nat}$                                                              | <b>3,4 T-App</b> |
| 6. | $\text{even} : \text{nat} \rightarrow \text{bool}$                                         | <b>T-Var</b>     |
| 7. | $\text{even}(\text{succ } x) : \text{bool}$                                                | <b>6,5 T-App</b> |
| 8. | $\lambda x : \text{nat}. \text{even}(\text{succ } x) : \text{nat} \rightarrow \text{bool}$ | <b>7 T-Abst</b>  |

■

### Problem

(3.4) Give a shorthanded (omit T-Var and T-Form) derivation in  $\lambda 2$  to show the following term is valid in  $\Gamma \equiv \text{nat} : *, \text{bool} : *$

$$(\lambda\alpha, \beta : *. \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(x))) \text{ nat bool}$$

*Solution.*

*Proof.*

|     |                                                                                                                                                             |                         |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------|
| 1.  | $\alpha, \beta : *$                                                                                                                                         | <b>Bound</b>            |
| 2.  | $f : \alpha \rightarrow \alpha$                                                                                                                             | <b>Bound</b>            |
| 3.  | $g : \alpha \rightarrow \beta$                                                                                                                              | <b>Bound</b>            |
| 4.  | $x : \alpha$                                                                                                                                                | <b>Bound</b>            |
| 5.  | $f x : \alpha$                                                                                                                                              | $^{*,*}$ <b>T-App</b>   |
| 6.  | $f(f x) : \alpha$                                                                                                                                           | $^{*,5}$ <b>T-App</b>   |
| 7.  | $g(f(f x)) : \beta$                                                                                                                                         | $^{*,6}$ <b>T-App</b>   |
| 8.  | $\lambda x : \alpha. g(f(f x)) : \alpha \rightarrow \beta$                                                                                                  | <b>7 T-Abst</b>         |
| 9.  | $\lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(f x)) : (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$                     | <b>8 T-Abst</b>         |
| 10. | $\lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(f x))$                                                | <b>9 T-Abst</b>         |
|     | $: (\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$                                                 |                         |
| 11. |                                                                                                                                                             |                         |
|     | $\lambda \alpha, \beta : * . \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(f x))$                    |                         |
|     | $: \Pi \alpha, \beta : * . (\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$                         | <b>10 T2-Abst</b>       |
| 12. |                                                                                                                                                             |                         |
|     | $(\lambda \alpha, \beta : * . \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(f x))) \text{ nat}$      |                         |
|     | $: \Pi \beta : * . (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta$                 | $^{*,11}$ <b>T2-App</b> |
| 13. |                                                                                                                                                             |                         |
|     | $(\lambda \alpha, \beta : * . \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(f x))) \text{ nat bool}$ |                         |
|     | $: \Pi \beta : * . (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \text{bool}) \rightarrow \text{nat} \rightarrow \text{bool}$     | $^{*,12}$ <b>T2-App</b> |

■

### Problem

(3.5 a) Let  $\perp \equiv \Pi \alpha : * . \alpha$ . Prove  $\perp$  is legal.

*Solution.* Here a notion called kind checking is introduced. This has not yet been discussed in this book (?)

*Proof.*

|    |              |               |
|----|--------------|---------------|
| 1. | $\alpha : *$ | <b>Bound</b>  |
| 2. | $\alpha : *$ | <b>T-Form</b> |

3.  $\Pi\alpha : * . \alpha : * \rightarrow *$  **T2-Abst**

■

### Problem

(3.5 b) Consider the context  $\Gamma \equiv \beta : *, x : \perp$ . Find an inhabitant of type  $\beta$  under  $\Gamma$ .

*Solution.*  $x \beta$  is. Because  $x$  is of second-order type, it must be parametric to a type, thus  $x$  is of form  $\lambda\alpha : * . M$  where  $\Gamma, \alpha : * \vdash M : \alpha$ .

*Proof.*

1.  $x : \Pi\alpha : * . \alpha$  **T-Var**
2.  $\beta : *$  **T-Form**
3.  $x \beta : \beta$  **1,2 T2-App**

■

### Problem

(3.5 c) Give three inhabitants of  $\beta \rightarrow \beta$  in  $\beta$ -nf under  $\Gamma$  in 3.5 b.

*Solution.*

1.  $\lambda y : \beta . y$ .

*Proof.*

1.  $y : \beta$  **Bound**
2.  $\boxed{y : \beta}$  **T-Var**
3.  $\lambda y : \beta . y : \beta \rightarrow \beta$  **2 T-Abst**

■

2.  $\lambda y : \beta . x \beta$ .

*Proof.*

1.  $y : \beta$  **Bound**
2.  $\boxed{x : \Pi\alpha : * . \alpha}$  **T-Var**
3.  $\boxed{\beta : *}$  **T-Form**
4.  $\boxed{x \beta : \beta}$  **2,3 T2-App**

5.  $\lambda y : \beta. x \beta : \beta \rightarrow \beta$  **4 T-Abst**

■

3.  $\lambda y : \beta. x (\beta \rightarrow \beta) y.$

*Proof.*

|    |                                                                              |                   |
|----|------------------------------------------------------------------------------|-------------------|
| 1. | $y : \beta$                                                                  | <b>Bound</b>      |
| 2. | $x : \Pi \alpha : * . \alpha$                                                | <b>T-Var</b>      |
| 3. | $\beta \rightarrow \beta : *$                                                | <b>T-Form</b>     |
| 4. | $x (\beta \rightarrow \beta) : \beta \rightarrow \beta$                      | <b>2,3 T2-App</b> |
| 5. | $y : \beta$                                                                  | <b>T-Var</b>      |
| 6. | $x (\beta \rightarrow \beta) y : \beta$                                      | <b>4,5 T-App</b>  |
| 7. | $\lambda y : \beta. x (\beta \rightarrow \beta) y : \beta \rightarrow \beta$ | <b>5 T-Abst</b>   |

■

### Problem

(3.5 d) Prove that the following terms inhabit the same type in  $\Gamma$ :

$$\lambda f : \beta \rightarrow \beta \rightarrow \beta. f (x \beta)(x \beta)$$

$$x ((\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta)$$

*Solution.* We simply derive the types.

*First Term.*

|    |                                                                                                                                               |                   |
|----|-----------------------------------------------------------------------------------------------------------------------------------------------|-------------------|
| 1. | $f : \beta \rightarrow \beta \rightarrow \beta$                                                                                               | <b>Bound</b>      |
| 2. | $f : \beta \rightarrow \beta \rightarrow \beta$                                                                                               | <b>T-Var</b>      |
| 3. | $x : \Pi \alpha : * . \alpha$                                                                                                                 | <b>T-Var</b>      |
| 4. | $\beta : *$                                                                                                                                   | <b>T-Form</b>     |
| 5. | $x \beta : \beta$                                                                                                                             | <b>3,4 T2-App</b> |
| 6. | $f (x \beta) : \beta \rightarrow \beta$                                                                                                       | <b>2,5 T-App</b>  |
| 7. | $f (x \beta)(x \beta) : \beta$                                                                                                                | <b>6,5 T-App</b>  |
| 8. | $\lambda f : \beta \rightarrow \beta \rightarrow \beta. f (x \beta)(x \beta) : (\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$ | <b>6 T-Abst</b>   |

■

*Second Term.*

1.  $(\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta : *$  **T-Form**
2.  $x : \Pi \alpha : * . \alpha$  **T-Var**
3.  $x ((\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta) : (\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$  **2,1 T2-App**

■

The two terms were shown to both inhabit  $(\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$ .

### Problem

(3.6 a) Find inhabitant of type

$$\Pi \alpha, \beta : * . (\text{nat} \rightarrow \alpha) \rightarrow (\alpha \rightarrow \text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta$$

In context  $\Gamma \equiv \text{nat} : *$ .

*Solution.*

$$\lambda \alpha, \beta : * . \lambda x : \text{nat} \rightarrow \alpha. \lambda y : (\alpha \rightarrow \text{nat} \rightarrow \beta). \lambda z : \text{nat}. y (x z) z$$

*Proof.*

|     |                                                                                                                                                                                                                                                                              |                    |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------|
| 1.  | $\alpha, \beta : *$                                                                                                                                                                                                                                                          | <b>Bound</b>       |
| 2.  | $\text{nat} \rightarrow \alpha : *$                                                                                                                                                                                                                                          | <b>T-Form</b>      |
| 3.  | $x : \text{nat} \rightarrow \alpha$                                                                                                                                                                                                                                          | <b>Bound</b>       |
| 4.  | $\alpha \rightarrow \text{nat} \rightarrow \beta : *$                                                                                                                                                                                                                        | <b>T-Form</b>      |
| 5.  | $y : \alpha \rightarrow \text{nat} \rightarrow \beta$                                                                                                                                                                                                                        | <b>Bound</b>       |
| 6.  | $\text{nat} : *$                                                                                                                                                                                                                                                             | <b>Bound</b>       |
| 7.  | $z : \text{nat}$                                                                                                                                                                                                                                                             | <b>Bound</b>       |
| 8.  | $y : \alpha \rightarrow \text{nat} \rightarrow \beta$                                                                                                                                                                                                                        | <b>T-Var</b>       |
| 9.  | $x : \text{nat} \rightarrow \alpha$                                                                                                                                                                                                                                          | <b>T-Var</b>       |
| 10. | $z : \text{nat}$                                                                                                                                                                                                                                                             | <b>T-Var</b>       |
| 11. | $x z : \alpha$                                                                                                                                                                                                                                                               | <b>9,10 T-App</b>  |
| 12. | $y (x z) : \text{nat} \rightarrow \beta$                                                                                                                                                                                                                                     | <b>8,11 T-App</b>  |
| 13. | $y (x z) z : \beta$                                                                                                                                                                                                                                                          | <b>12,10 T-App</b> |
| 14. | $\lambda z : \text{nat}. y (x z) z : \text{nat} \rightarrow \beta$                                                                                                                                                                                                           | <b>13 T-Abst</b>   |
| 15. | $\lambda y : \alpha \rightarrow \text{nat} \rightarrow \beta. \lambda z : \text{nat}. y (x z) z : (\alpha \rightarrow \text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta$                                                                                | <b>14 T-Abst</b>   |
| 16. | $\lambda x : \text{nat} \rightarrow \alpha. y : \alpha \rightarrow \text{nat} \rightarrow \beta. \lambda z : \text{nat}. y (x z) z : (\text{nat} \rightarrow \alpha) \rightarrow (\alpha \rightarrow \text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta$ | <b>15 T2-Abst</b>  |

17.

$$\begin{aligned} \lambda\alpha, \beta : * . & x : \text{nat} \rightarrow \alpha . y : \alpha \rightarrow \text{nat} \rightarrow \beta . \lambda z : \text{nat} . y(xz)z \\ & : \Pi\alpha, \beta : * . (\text{nat} \rightarrow \alpha) \rightarrow (\alpha \rightarrow \text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta \end{aligned}$$

**16 T2-Abst**

■

### Problem

(3.6 b) Find inhabitant of type

$$\Pi\delta : * . ((\alpha \rightarrow \gamma) \rightarrow \delta) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta$$

In context  $\Gamma \equiv \alpha : *, \beta : *, \gamma : *$

*Solution.*

$$\lambda\delta : * . \lambda x : (\alpha \rightarrow \gamma) \rightarrow \delta . \lambda y : \alpha \rightarrow \beta . \lambda z : \beta \rightarrow \gamma . x(\lambda u : \alpha . z(yu))$$

A derivation in shorthand will be given (omitting T-Form / T-Var)

*Proof.*

|                                                                                                                                                                                                                                                                                                                                               | Bound      |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|
| 1. $\delta : *$                                                                                                                                                                                                                                                                                                                               |            |
| 2. $x : (\alpha \rightarrow \gamma) \rightarrow \delta$                                                                                                                                                                                                                                                                                       | Bound      |
| 3. $y : \alpha \rightarrow \beta$                                                                                                                                                                                                                                                                                                             | Bound      |
| 4. $z : \beta \rightarrow \gamma$                                                                                                                                                                                                                                                                                                             | Bound      |
| 5. $u : \alpha$                                                                                                                                                                                                                                                                                                                               | Bound      |
| 6. $y u : \beta$                                                                                                                                                                                                                                                                                                                              | *,* T-App  |
| 7. $\underline{z(yu) : \gamma}$                                                                                                                                                                                                                                                                                                               | *,6 T-App  |
| 8. $\lambda u : \alpha . z(yu) : \alpha \rightarrow \gamma$                                                                                                                                                                                                                                                                                   | 7 T-Abst   |
| 9. $\underline{x(\lambda u : \alpha . z(yu)) : \delta}$                                                                                                                                                                                                                                                                                       | 8 T-Abst   |
| 10. $\lambda z : \beta \rightarrow \gamma . x(\lambda u : \alpha . z(yu)) : (\beta \rightarrow \gamma) \rightarrow \delta$                                                                                                                                                                                                                    | 9 T-Abst   |
| 11. $\underline{\lambda y : \alpha \rightarrow \beta . \lambda z : \beta \rightarrow \gamma . x(\lambda u : \alpha . z(yu)) : (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta}$                                                                                                                          | 10 T-Abst  |
| 12. $\underline{\lambda x : (\alpha \rightarrow \gamma) \rightarrow \delta . \lambda y : \alpha \rightarrow \beta . \lambda z : \beta \rightarrow \gamma . x(\lambda u : \alpha . z(yu)) : ((\alpha \rightarrow \gamma) \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta}$ | 11 T-Abst  |
| 13.                                                                                                                                                                                                                                                                                                                                           |            |
| $\lambda\delta : * . \lambda x : (\alpha \rightarrow \gamma) \rightarrow \delta . \lambda y : \alpha \rightarrow \beta . \lambda z : \beta \rightarrow \gamma . x(\lambda u : \alpha . z(yu))$                                                                                                                                                |            |
| $: \Pi\delta : * . ((\alpha \rightarrow \gamma) \rightarrow \delta) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta$                                                                                                                                                                         | 12 T2-Abst |

■

### Problem

(3.6 c) Find inhabitant of type

$$\Pi\alpha, \beta, \gamma : * . (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$$

In the empty context

*Solution.*

$$\lambda\alpha, \beta, \gamma : * . \lambda f : (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) . \lambda x : \alpha . f x (\lambda u : \beta . x)$$

*Proof.*

|     |                                                                                                                                   |                  |
|-----|-----------------------------------------------------------------------------------------------------------------------------------|------------------|
| 1.  | $\alpha, \beta, \gamma$                                                                                                           | <b>Bound</b>     |
| 2.  | $f : \alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma$                                                            | <b>Bound</b>     |
| 3.  | $x : \alpha$                                                                                                                      | <b>Bound</b>     |
| 4.  | $f x : (\beta \rightarrow \alpha) \rightarrow \gamma$                                                                             | <b>** T-App</b>  |
| 5.  | $u : \beta$                                                                                                                       | <b>Bound</b>     |
| 6.  | $x : \alpha$                                                                                                                      | <b>T-Var</b>     |
| 7.  | $\lambda u : \beta . x : \beta \rightarrow \alpha$                                                                                | <b>6 T-Abst</b>  |
| 8.  | $f x (\lambda u : \beta . x) : \gamma$                                                                                            | <b>4,7 T-App</b> |
| 9.  | $\lambda x : \alpha . f x (\lambda u : \beta . x) : \alpha \rightarrow \gamma$                                                    | <b>8 T-Abst</b>  |
| 10. | $\lambda f : \alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma . \lambda x : \alpha . f x (\lambda u : \beta . x)$ |                  |
| 11. | $: (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$                      | <b>9 T-Abst</b>  |

$$\lambda\alpha, \beta, \gamma : * . \lambda f : \alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma . \lambda x : \alpha . f x (\lambda u : \beta . x)$$

$$: \Pi\alpha, \beta, \gamma : * . (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$$

**10 T2-Abst**

■

### Problem

(3.7) Let  $\perp \equiv \Pi\alpha : * . \alpha$  and context  $\Gamma \equiv \alpha : *, \beta : *, x : \alpha \rightarrow \perp, f : (\alpha \rightarrow \alpha) \rightarrow \alpha$ . Give a derivation that successively calculate an inhabitant of  $\alpha$  and  $\beta$ , both in context  $\Gamma$ .

*Solution.* Have  $M : \alpha := f (\lambda n : \alpha . n)$ . Then  $\Gamma \vdash x M \beta : \beta$ .

*Typing M.*

1.  $f : (\alpha \rightarrow \alpha) \rightarrow \alpha$  **T-Var**
2.  $n : \alpha$  **Bound**
3.  $\boxed{n : \alpha}$  **T-Var**
4.  $\lambda n : \alpha. n : \alpha \rightarrow \alpha$  **3 T-Abst**
5.  $f(\lambda n : \alpha. n) : \alpha$  **1,4 T-App**

■

*Typing  $x M \beta$ .*

1.  $M : \alpha$  **T-Var**
2.  $x : \alpha \rightarrow \Pi \alpha : * . \alpha$  **T-Var**
3.  $M x : \Pi \alpha : * . \alpha$  **2,1 T-App**
4.  $M x \beta : \beta$  **3,\* T2-App**

■

### Problem

(3.8) Recall  $K \equiv \lambda x y . x \in \Lambda$  from untyped lambda calculus. Consider the following types

$$T_1 \equiv \Pi \alpha, \beta : * . \alpha \rightarrow \beta \rightarrow \alpha \quad T_2 \equiv \Pi \alpha : * . \alpha \rightarrow (\Pi \beta : * . \beta \rightarrow \alpha)$$

Find inhabitants of both type  $t_1 : T_1$  and  $t_2 : T_2$  under the empty context, which may be considered the closed  $\lambda 2$  form of  $K \in \Lambda_{T_2}$ .

*Solution.*

$$\begin{aligned} & \lambda \alpha, \beta : * . \lambda x : \alpha. \lambda y : \beta. x \\ & \lambda \alpha : * . \lambda x : \alpha. \lambda \beta : * . \lambda y : \beta. x \end{aligned}$$

*First Form.*

1.  $\alpha, \beta : *$  **Bound**
2.  $x : \alpha$  **Bound**
3.  $y : \beta$  **Bound**
4.  $x : \alpha$  **T-Var**
5.  $\boxed{\lambda y : \beta. x : \beta \rightarrow \alpha}$  **4 T-Abst**
6.  $\lambda x : \alpha. \lambda y : \beta. x : \alpha \rightarrow \beta \rightarrow \alpha$  **5 T-Abst**

7.  $\lambda\alpha, \beta : * . \lambda x : \alpha. \lambda y : \beta. x : \Pi\alpha, \beta : * . \alpha \rightarrow \beta \rightarrow \alpha$     **5 T2-Abst**

■

*Second Form.*

|    |                                                                                                                                                  |                  |
|----|--------------------------------------------------------------------------------------------------------------------------------------------------|------------------|
| 1. | $\alpha : *$                                                                                                                                     | <b>Bound</b>     |
| 2. | $x : \alpha$                                                                                                                                     | <b>Bound</b>     |
| 3. | $\beta : *$                                                                                                                                      | <b>Bound</b>     |
| 4. | $y : \beta$                                                                                                                                      | <b>Bound</b>     |
| 5. | $x : \alpha$                                                                                                                                     | <b>T-Var</b>     |
| 6. | $\lambda y : \beta. x : \beta \rightarrow \alpha$                                                                                                | <b>5 T-Abst</b>  |
| 7. | $\lambda\beta : * . \lambda y : \beta. x : \Pi\beta : * . \beta \rightarrow \alpha$                                                              | <b>6 T2-Abst</b> |
| 8. | $\lambda x : \alpha. \lambda\beta : * . \lambda y : \beta. x : \alpha \rightarrow (\Pi\beta : * . \beta \rightarrow \alpha)$                     | <b>7 T-Abst</b>  |
| 9. | $\lambda\alpha : * . \lambda x : \alpha. \lambda\beta : * . \lambda y : \beta. x : \alpha \rightarrow (\Pi\beta : * . \beta \rightarrow \alpha)$ | <b>8 T2-Abst</b> |

■

### Problem

(3.9) Pretype the combinator

$$S \equiv \lambda x y z . x z (y z)$$

In closed form (typable in an empty context) in  $\Lambda_{\text{T2}}$ .

*Solution.*

$$S \equiv \lambda\alpha, \beta, \gamma : * . \lambda x : \alpha \rightarrow \beta \rightarrow \gamma. \lambda y : \alpha \rightarrow \beta. \lambda z : \alpha. x z (y z)$$

*Proof.*

|    |                                                                                                                                          |                  |
|----|------------------------------------------------------------------------------------------------------------------------------------------|------------------|
| 1. | $\alpha, \beta, \gamma : *$                                                                                                              | <b>Bound</b>     |
| 2. | $x : \alpha \rightarrow \beta \rightarrow \gamma$                                                                                        | <b>Bound</b>     |
| 3. | $y : \alpha \rightarrow \beta$                                                                                                           | <b>Bound</b>     |
| 4. | $z : \alpha$                                                                                                                             | <b>Bound</b>     |
| 5. | $x z : \beta \rightarrow \gamma$                                                                                                         | <b>,* T-App</b>  |
| 6. | $y x : \beta$                                                                                                                            | <b>,* T-App</b>  |
| 7. | $x z (y x) : \gamma$                                                                                                                     | <b>5,6 T-App</b> |
| 8. | $\lambda z : \alpha. x z (y x) : \alpha \rightarrow \gamma$                                                                              | <b>7 T-Abst</b>  |
|    | $\lambda y : \alpha \rightarrow \beta. \lambda z : \alpha. x z (y x) : (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$ | <b>8 T-Abst</b>  |

$$10. \quad \boxed{\begin{array}{l} \lambda x : \alpha \rightarrow \beta \rightarrow \gamma. \lambda y : \alpha \rightarrow \beta. \lambda z : \alpha. x z (y x) \\ : (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma \end{array}} \quad \mathbf{9 \text{ T-Abst}}$$

11.

$$\lambda \alpha, \beta, \gamma : * . \lambda x : \alpha \rightarrow \beta \rightarrow \gamma. \lambda y : \alpha \rightarrow \beta. \lambda z : \alpha. x z (y x) \\ : \Pi \alpha, \beta, \gamma : * . (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma \quad \mathbf{10 \text{ T2-Abst}}$$

■

### Problem

(3.10 a) Consider the term

$$M \equiv \lambda x : \Pi \alpha : * . \alpha \rightarrow \alpha. x (\sigma \rightarrow \sigma)(x \sigma)$$

Prove that  $M$  is legal.

*Solution.* For a term to be legal there must exist a context so that the term could be typed. Here, a witness context is  $\Gamma \equiv \sigma : *$ .

*Proof.*

|                                                                                                                  |                       |
|------------------------------------------------------------------------------------------------------------------|-----------------------|
| 1. $x : \Pi \alpha : * . \alpha \rightarrow \alpha$                                                              | <b>Bound</b>          |
| 2. $\boxed{x (\sigma \rightarrow \sigma) : (\sigma \rightarrow \sigma) \rightarrow (\sigma \rightarrow \sigma)}$ | $*; * \text{ T2-App}$ |
| 3. $\boxed{x \sigma : \sigma \rightarrow \sigma}$                                                                | $*; * \text{ T2-App}$ |
| 4. $\boxed{x (\sigma \rightarrow \sigma)(x \sigma) : \sigma \rightarrow \sigma}$                                 | $2,3 \text{ T-App}$   |

5.

$$\lambda x : \Pi \alpha : * . \alpha \rightarrow \alpha. x (\sigma \rightarrow \sigma)(x \sigma) \\ : (\Pi \alpha : * . \alpha \rightarrow \alpha) \rightarrow \sigma \rightarrow \sigma \quad \mathbf{4 \text{ T-Abst}}$$

■

### Problem

(3.10 b) Find a term  $N$  such that  $M N$  is legal and may be considered to be a way to add type information to  $(\lambda x . x x)(\lambda y . y)$

*Solution.*

$$M \sigma N \equiv (\lambda x : \Pi \alpha : * . \alpha \rightarrow \alpha. x (\sigma \rightarrow \sigma)(x \sigma)) \sigma (\lambda y : \sigma. y)$$

Is the same as  $(\lambda x . x x)(\lambda y . y)$  modulo type annotations.

*Proof.*

1.  $M : (\Pi \alpha : * . \alpha \rightarrow \alpha) \rightarrow \sigma \rightarrow \sigma$  **T-Var**
2.  $M \sigma : (\sigma \rightarrow \sigma) \rightarrow \sigma \rightarrow \sigma$  **1,\* T2-App**
3.  $y : \sigma$  **Bound**
4.  $\underline{y : \sigma}$  **T-Var**
5. have  $N := \lambda y : \sigma. y : \sigma \rightarrow \sigma$  **4 T-Abst**
6.  $M \sigma N : \sigma \rightarrow \sigma$  **2,5 T-Abst**

■

### Problem

(3.11) Recall  $\perp \equiv \Pi \alpha : * . \alpha$  from 3.5. Type and prove the following term legal:

$$\lambda x : \perp. x (\perp \rightarrow \perp \rightarrow \perp)(x (\perp \rightarrow \perp) x)(x (\perp \rightarrow \perp \rightarrow \perp) x x)$$

*Solution.*

*Proof.* The type  $\perp \rightarrow \perp$  is closed and well formed. Therefore, the term is legal.

1.  $\perp : * \equiv \Pi \alpha : * . \alpha$  **T-Form**
  2.  $x : \perp$  **Bound**
  3.  $x (\perp \rightarrow \perp \rightarrow \perp) : \perp \rightarrow \perp \rightarrow \perp$  **\*,\* T2-App**
  4.  $x (\perp \rightarrow \perp) : \perp \rightarrow \perp$  **\*,\* T2-App**
  5.  $x (\perp \rightarrow \perp) x : \perp$  **4,\* T-App**
  6.  $x (\perp \rightarrow \perp \rightarrow \perp)(x (\perp \rightarrow \perp) x) : \perp \rightarrow \perp$  **3,5 T-App**
  7.  $x (\perp \rightarrow \perp \rightarrow \perp) x : \perp \rightarrow \perp$  **3,\* T-App**
  8.  $x (\perp \rightarrow \perp \rightarrow \perp) x x : \perp$  **7,\* T-App**
  9.  $x (\perp \rightarrow \perp \rightarrow \perp)(x (\perp \rightarrow \perp) x)(x (\perp \rightarrow \perp \rightarrow \perp) x x) : \perp$  **6,8 T-App**
  - 10.
- $\lambda x : \perp. x (\perp \rightarrow \perp \rightarrow \perp)(x (\perp \rightarrow \perp) x)$   
 $(x (\perp \rightarrow \perp \rightarrow \perp) x x) : \perp \rightarrow \perp$  **9 T-Abst**

■

## Problem

(3.12) Given the Polymorphic Church Numerals:

$$\begin{aligned}
 \text{nat} &\in \mathbb{T}_2 := \Pi\alpha : * . (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha \\
 \bar{0} &\equiv \lambda\alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . x : \text{nat} \\
 \bar{1} &\equiv \lambda\alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . f x : \text{nat} \\
 \bar{2} &\equiv \lambda\alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . f(f x) : \text{nat} \\
 \text{succ} &\equiv \lambda n : \text{nat} . \lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f(n\beta f x)
 \end{aligned}$$

Prove that

$$\begin{aligned}
 \text{succ } \bar{0} &\underset{\beta}{=} \bar{1} \\
 \text{succ } \bar{1} &\underset{\beta}{=} \bar{2}
 \end{aligned}$$

*Solution.*

$$\begin{aligned}
 &\text{succ } \bar{0} \\
 &\equiv (\lambda n : \text{nat} . \lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f(n\beta f x))(\lambda\alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . x) \\
 &\xrightarrow{\beta} (\lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f((\lambda\alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . x)\beta f x)) \\
 &\xrightarrow{\beta_{\mathbb{T}_2}} (\lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f((\lambda f : \beta \rightarrow \beta . \lambda x : \beta . x) f x)) \\
 &\xrightarrow{\beta} (\lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f x) \underset{\alpha_{\mathbb{T}_2}}{\equiv} \bar{1} \\
 \\
 &\text{succ } \bar{1} \\
 &\equiv (\lambda n : \text{nat} . \lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f(n\beta f x))(\lambda\alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . f x) \\
 &\xrightarrow{\beta} (\lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f((\lambda\alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . f x)\beta f x)) \\
 &\xrightarrow{\beta_{\mathbb{T}_2}} (\lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f((\lambda f : \beta \rightarrow \beta . \lambda x : \beta . f x) f x)) \\
 &\xrightarrow{\beta} (\lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f(f x)) \underset{\alpha_{\mathbb{T}_2}}{\equiv} \bar{2}
 \end{aligned}$$

## Problem

(3.13 a) We define addition in Polymorphic Church Numerals as

$$\text{add} \equiv \lambda m, n : \text{nat}. \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \text{nat}. m \alpha f (n \alpha f x)$$

Show that

$$\text{add } \bar{1} \quad \bar{1} \xrightarrow{\beta} \bar{2}$$

*Solution.*

$$\begin{aligned} & \text{add } \bar{1} \quad \bar{1} \\ & \equiv (\lambda m, n : \text{nat}. \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m \alpha f (n \alpha f x)) \bar{1} \quad \bar{1} \\ & \xrightarrow[\beta]{\gg} \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. \bar{1} \alpha f (\bar{1} \alpha f x) \\ & \equiv \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. \bar{1} \alpha f ((\lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f x) \alpha f x) \\ & \xrightarrow[\beta]{\gg} \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. \bar{1} \alpha f (f x) \\ & \equiv \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. (\lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f x) \alpha f (f x) \\ & \xrightarrow[\beta]{\gg} \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f (f x) \equiv \bar{2} \end{aligned}$$

## Problem

(3.13 b) Find a term mul simulates multiplication on **nat**.

*Solution.*

$$\text{mul} \equiv \lambda m, n : \text{nat}. \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m \alpha (n \alpha f) x$$

*Proof.* We derive the type first to prove a legal term.

| 1. $m, n : \text{nat}$                                                            | <b>Bound</b>           |
|-----------------------------------------------------------------------------------|------------------------|
| 2. $\alpha : *$                                                                   | <b>Bound</b>           |
| 3. $f : \alpha \rightarrow \alpha$                                                | <b>Bound</b>           |
| 4. $x : \alpha$                                                                   | <b>Bound</b>           |
| 5. $m \alpha : (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$ | $^{*,*} \text{T2-App}$ |
| 6. $n \alpha : (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$ | $^{*,*} \text{T2-App}$ |
| 7. $n \alpha f : \alpha \rightarrow \alpha$                                       | $6,^* \text{T-App}$    |
| 8. $m \alpha (n \alpha f) : \alpha \rightarrow \alpha$                            | $5,7 \text{T-App}$     |

|     |                                                                                                                                                         |                   |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------|
| 9.  | $m \alpha(n \alpha f) x : \alpha$                                                                                                                       | <b>8,* T-App</b>  |
| 10. | $\lambda x : \alpha. m \alpha(n \alpha f) x : \alpha \rightarrow \alpha$                                                                                | <b>9 T-Abst</b>   |
| 11. | $\lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m \alpha(n \alpha f) x : (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$ | <b>10 T-Abst</b>  |
| 12. | $\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m \alpha(n \alpha f) x : \text{nat}$                                    | <b>11 T2-Abst</b> |
| 13. |                                                                                                                                                         |                   |
|     | $\lambda m, n : \text{nat}. \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m \alpha(n \alpha f) x$                      |                   |
|     | $: \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$                                                                                            | <b>12 T-Abst</b>  |

This proves that the term does indeed produce a natural number from two. Next let's prove that

$$\forall \bar{n}, \bar{m} : \text{nat} \quad \text{mul } \bar{n} \cdot \bar{m} = \overline{\bar{n} \times \bar{m}}$$

It could be proven by induction that

$$\begin{aligned} \forall \bar{a} : \text{nat} \quad \bar{a} &\equiv \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f^a x \\ \text{mul } \bar{n} \cdot \bar{m} &\equiv (\lambda m, n : \text{nat}. \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m \alpha(n \alpha f) x)(\bar{n})(\bar{m}) \\ &\xrightarrow[\beta]{} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. \bar{n} \alpha(\bar{m} \alpha f) x \\ &\xrightarrow[\beta]{} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. \bar{n} \alpha(\lambda u : \alpha. f^m u) x \\ &\xrightarrow[\beta]{} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. (\lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f^n x)(\lambda u : \alpha. f^m u) x \\ &\xrightarrow[\beta]{} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. (\lambda u : \alpha. f^m u)^n x \end{aligned}$$

By induction this can be further beta-reduced to

$$\xrightarrow[\beta]{} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f^{mn} x \equiv \overline{m \bar{n}}$$

■

### Problem

(3.14) We present the Church-Encoded Boolean:

$$\begin{aligned} \text{bool} &\in \mathbb{T}_2 := \Pi \alpha : *. \alpha \rightarrow \alpha \rightarrow \alpha \\ \text{true} &\equiv \lambda \alpha : *. \lambda x, y : \alpha. x \\ \text{false} &\equiv \lambda \alpha : *. \lambda x, y : \alpha. y \end{aligned}$$

Construct a  $\lambda 2$  term neg such that  $\text{neg true} \underset{\beta}{=} \text{false}$  and  $\text{neg false} \underset{\beta}{=} \text{true}$ .

*Solution.*

$$\text{neg} \equiv \lambda b : \text{bool}. \lambda \alpha : *. b \alpha(\text{false } \alpha)(\text{true } \alpha)$$

*Neg True.*

$$\begin{aligned}
 \text{neg true} &\equiv (\lambda b : \text{bool}. \lambda \alpha : * . b \alpha(\text{false } \alpha)(\text{true } \alpha)) \text{ true} \\
 &\xrightarrow[\beta]{\Rightarrow} \lambda \alpha : * . (\lambda x, y : \alpha. x)(\text{false } \alpha)(\text{true } \alpha) \\
 &\xrightarrow[\beta]{\rightarrow} \lambda \alpha : * . \text{false } \alpha \xrightarrow[\eta]{\rightarrow} \text{false}
 \end{aligned}$$

■

*Neg False.*

$$\begin{aligned}
 \text{neg false} &\equiv (\lambda b : \text{bool}. \lambda \alpha : * . b \alpha(\text{false } \alpha)(\text{true } \alpha)) \text{ false} \\
 &\xrightarrow[\beta]{\Rightarrow} \lambda \alpha : * . (\lambda x, y : \alpha. y)(\text{false } \alpha)(\text{true } \alpha) \\
 &\xrightarrow[\beta]{\rightarrow} \lambda \alpha : * . \text{true } \alpha \xrightarrow[\eta]{\rightarrow} \text{true}
 \end{aligned}$$

■

### Problem

(3.15) Define

$$M \equiv \lambda u, v : \text{bool}. \lambda \beta : * . \lambda x, y : \beta. u \beta(v \beta x y)(v \beta y y)$$

And reduce  $M$  true true,  $M$  true false,  $M$  false true,  $M$  false false, and decide which logical operator is represented by  $M$ .

*Solution.*

$$\begin{aligned}
 M \text{ true true} &\equiv (\lambda u, v : \text{bool}. \lambda \beta : * . \lambda x, y : \beta. u \beta(v \beta x y)(v \beta y y)) \text{ true true} \\
 &\xrightarrow[\beta]{\Rightarrow} \lambda \beta : * . \lambda x, y : \beta. \text{true } \beta(\text{true } \beta x y)(\text{true } \beta y y) \\
 &\xrightarrow[\beta]{\Rightarrow} \lambda \beta : * . \lambda x, y : \beta. (\text{true } \beta x y) \\
 &\xrightarrow[\beta]{\Rightarrow} \lambda \beta : * . \lambda x, y : \beta. x \xrightarrow[\alpha]{\equiv} \text{true}
 \end{aligned}$$

$$\begin{aligned}
 M \text{ true false} &\equiv (\lambda u, v : \text{bool}. \lambda \beta : * . \lambda x, y : \beta. u \beta(v \beta x y)(v \beta y y)) \text{ true false} \\
 &\xrightarrow[\beta]{\Rightarrow} \lambda \beta : * . \lambda x, y : \beta. \text{true } \beta(\text{false } \beta x y)(\text{false } \beta y y) \\
 &\xrightarrow[\beta]{\Rightarrow} \lambda \beta : * . \lambda x, y : \beta. (\text{false } \beta x y) \\
 &\xrightarrow[\beta]{\Rightarrow} \lambda \beta : * . \lambda x, y : \beta. y \xrightarrow[\alpha]{\equiv} \text{false}
 \end{aligned}$$

$$M \text{ false true} \equiv (\lambda u, v : \text{bool}. \lambda \beta : * . \lambda x, y : \beta. u \beta(v \beta x y)(v \beta y y)) \text{ false true}$$

$$\xrightarrow[\beta]{\beta} \lambda \beta : * . \lambda x, y : \beta. \text{false } \beta(\text{true } \beta x y)(\text{true } \beta y y)$$

$$\xrightarrow[\beta]{\beta} \lambda \beta : * . \lambda x, y : \beta. (\text{true } \beta y y)$$

$$\xrightarrow[\beta]{\beta} \lambda \beta : * . \lambda x, y : \beta. y \equiv_{\alpha} \text{false}$$

$$M \text{ false false} \equiv (\lambda u, v : \text{bool}. \lambda \beta : * . \lambda x, y : \beta. u \beta(v \beta x y)(v \beta y y)) \text{ false false}$$

$$\xrightarrow[\beta]{\beta} \lambda \beta : * . \lambda x, y : \beta. \text{false } \beta(\text{false } \beta x y)(\text{false } \beta y y)$$

$$\xrightarrow[\beta]{\beta} \lambda \beta : * . \lambda x, y : \beta. (\text{false } \beta y y)$$

$$\xrightarrow[\beta]{\beta} \lambda \beta : * . \lambda x, y : \beta. y \equiv_{\alpha} \text{false}$$

Therefore  $M$  is equivalent to logical AND.

### Problem

(3.16) Find  $\lambda 2$  term representing the logical OR, XOR, IMP.

*Solution.*

$$\text{OR} \equiv \lambda u, v : \text{bool}. \lambda \beta : * . \lambda x, y : \beta. u \beta x (v \beta x y)$$

$$\text{XOR} \equiv \lambda u, v : \text{bool}. \lambda \beta : * . \lambda x, y : \beta. u \beta(v \beta y x)(v \beta x y)$$

$$\text{IMP} \equiv \lambda u, v : \text{bool}. \lambda \beta : * . \lambda x, y : \beta. u \beta(v \beta x y) x$$

All of them could be checked by finite enumeration over  $\text{bool} \times \text{bool}$ .

### Problem

(3.17) Find  $\text{isZero} : \text{nat} \rightarrow \text{bool}$  such that  $\forall n : \text{nat}, \text{isZero } n \equiv_{\beta} \text{false}$  except when  $n \equiv \bar{0}$ .

*Solution.*

$$\text{isZero} \equiv \lambda n : \text{nat}. n \text{ bool } (\lambda u : \text{bool}. \text{false}) \text{ true}$$

*Proof.*

$$\begin{aligned}
 \text{isZero } \bar{0} &\equiv (\lambda n : \text{nat}. n \text{ bool } (\lambda u : \text{bool}. \text{false}) \text{ true}) \bar{0} \\
 &\xrightarrow[\beta]{} (\lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. x) \text{ bool } (\lambda u : \text{bool}. \text{false}) \text{ true} \\
 &\xrightarrow[\beta]{} (\lambda f : \text{bool} \rightarrow \text{bool}. \lambda x : \text{bool}. x)(\lambda u : \text{bool}. \text{false}) \text{ true} \\
 &\xrightarrow[\beta]{} \text{true}
 \end{aligned}$$

■

By induction it could be proven that any other natural numbers must be applied  $\lambda u : \text{bool}$ .  $\text{false}$  to the body, making the result  $\text{false}$ , except for  $\bar{0}$ , where the function  $f : \alpha \rightarrow \alpha$  never got applied.

### Problem

(3.18 a) Define type

$$\text{tree} \equiv \Pi \alpha : *. (\text{bool} \rightarrow \alpha) \rightarrow (\text{bool} \rightarrow \alpha \rightarrow \alpha \rightarrow \alpha) \rightarrow \alpha$$

Then we construct an inhabitant

$$\lambda \alpha. * . \lambda u : \text{bool} \rightarrow \alpha. \lambda v : \text{bool} \rightarrow \alpha \rightarrow \alpha \rightarrow \alpha. M$$

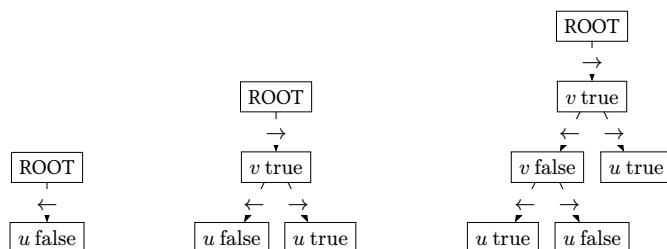
Is a node of a binary tree. Sketch graphs of trees where  $M$  is

$$\begin{aligned}
 &u \text{ false} \\
 &v \text{ true } (u \text{ false})(u \text{ true}) \\
 &v \text{ true } (u \text{ true})(v \text{ false } (u \text{ true})(u \text{ false}))
 \end{aligned}$$

*Solution.* A binary tree is usually defined as this:

```
inductive Tree (α : Type) where
| leaf (value : α) : Tree α
| node (left right : Tree α) : Tree α
```

With two constructors: a leaf or a node. Here  $\alpha$  is the type of the payload at each node. There are two constructors:  $u$  is the left constructor, taking a  $\text{bool}$  value for the direction of the node. The  $v$  term is the node constructor, taking a  $\text{bool}$  as the direction, two  $\alpha$ -typed terms as its children.



### Problem

(3.18 b) Give a  $\lambda 2$  term, which, given input a polymorphic boolean  $p$  and two trees  $s$  and  $t$ , delivers the combined tree with  $p$  on top, left subtree  $s$  and right subtree  $t$ .

*Solution.*

$$\begin{aligned} \text{leaf} &:= \lambda p : \text{bool}. \lambda s, t : \text{tree}. \\ &\quad \lambda \alpha. \lambda u : \text{bool} \rightarrow \alpha. \lambda v : \text{bool} \rightarrow \alpha \rightarrow \alpha \rightarrow \alpha. \\ &\quad v p (s \alpha u v)(t \alpha u v) \end{aligned}$$

*Proof.* We suppose

$$\begin{aligned} s &\equiv \lambda \beta : * . \lambda u_s : \text{bool} \rightarrow \beta. \lambda v_s : \text{bool} \rightarrow \beta \rightarrow \beta \rightarrow \beta. S \\ t &\equiv \lambda \gamma : * . \lambda u_t : \text{bool} \rightarrow \gamma. \lambda v_t : \text{bool} \rightarrow \gamma \rightarrow \gamma \rightarrow \gamma. T \end{aligned}$$

We want

$$\text{leaf } p \ s \ t =_{\beta} \lambda \alpha : * . \lambda u : \text{bool} \rightarrow \alpha. \lambda v : \text{bool} \rightarrow \alpha \rightarrow \alpha \rightarrow \alpha. v p \ S \ T$$

For compactness we denote

$$\begin{aligned} \alpha : * \vdash \tau_{\text{mkleaf}} &\equiv \text{bool} \rightarrow \alpha \\ \alpha : * \vdash \tau_{\text{mknnode}} &\equiv \text{bool} \rightarrow \alpha \rightarrow \alpha \rightarrow \alpha \end{aligned}$$

By beta reduction we have

$$\begin{aligned} \text{leaf } p \ s \ t &\equiv (\lambda p : \text{bool}. \lambda s, t : \text{tree}...). p \ s \ t \\ &\xrightarrow{\beta} (\lambda \alpha : * . \lambda u . \tau_{\text{mkleaf}}. \lambda v : \tau_{\text{mknnode}}. v p (s \alpha u v)(t \alpha u v)) \\ &\xrightarrow{\beta} (\lambda \alpha : * . \lambda u . \tau_{\text{mkleaf}}. \lambda v : \tau_{\text{mknnode}}. v p (s \alpha u v)(t \alpha u v)) \\ &\xrightarrow{\beta} \lambda \alpha : * . \lambda u . \tau_{\text{mkleaf}}. \lambda v : \tau_{\text{mknnode}}. v p \ S \ T \end{aligned}$$

■

### Problem

(3.19) If  $\Gamma \vdash L : \sigma$ , then  $\Gamma$  is a valid  $\lambda 2$  context.

*Solution.* The definition of “valid” here would be taken as relative to a judgement as the fact to be able to derive the judgement. Thus, it meant a complete inference path could be made using only statements and judgements derived from the context. Now proof by induction on inference rule that deducted  $L : \sigma$ .

*Case 1 : T-Var.* The premise is that  $\Gamma$  is a  $\lambda 2$  context. ■

*Case 2 : T-App.* Therefore  $L \equiv M N$  for some  $M, N \in \Lambda_{\mathbb{T}_2}$ . Therefore,

$$\Gamma \vdash M : \tau \rightarrow \sigma \quad \Gamma \vdash N : \tau$$

for some  $\tau \in \mathbb{T}_2$ . By the inductive hypothesis on any premise  $\Gamma$  is a valid  $\lambda 2$  context. ■

*Case 3 : T-Abst.* Therefore  $L \equiv \lambda x : \alpha. N$  such that

$$\Gamma, x : \alpha \vdash N : \beta$$

for some  $\alpha, \beta \in \mathbb{T}_2$  such that  $\sigma \equiv \alpha \rightarrow \beta$ . By the inductive hypothesis,  $\Gamma, x : \alpha$ . By the recursive definition of  $\lambda 2$  contexts, for some valid context  $\Delta$ ,  $\forall n \in \text{dom } \Delta$ ,  $n$  could not depend on statement declared after  $n$  in the context. Therefore, no statement in  $\Gamma$  could depend on  $x : \alpha$ . Therefore,  $\Gamma$  is a valid context. ■

*Case 4 : T-Form.* The premise is that  $\Gamma$  is a valid  $\lambda 2$  context. ■

*Case 5 : T2-App.* Therefore  $L \equiv N B$  for some  $N, B \in \mathbb{V}_2$  such that

$$\Gamma \vdash N : \Pi \alpha : * . \sigma \quad \Gamma \vdash B : *$$

By the inductive hypothesis on the any premise  $\Gamma$  is a valid  $\lambda 2$  context. ■

*Case 6 : T2-Abst.* Therefore  $L \equiv \lambda \alpha : * . M$  for some  $M \in \Lambda_{\mathbb{T}_2}$  such that

$$\Gamma, \alpha : * \vdash M : \beta$$

and  $\sigma \equiv \Pi \alpha : * . \beta$ . By reasoning in *Case 3* no statement in the context could depend on any statement before the latter's declaration. Therefore no statement in  $\Gamma$  could depend on  $\alpha : *$ , making it a valid context. ■

### Problem

(3.20) Prove the free variable lemma for  $\lambda 2$ .

$$\Gamma \vdash L : \sigma \Rightarrow \text{FV } L \subseteq \text{dom } \Gamma$$

*Solution.* Proof by induction on inference rules that deduced  $L : \sigma$ . The only rule not considered is T-Form since all terms apparent in  $\mathbb{T}_2$ .

*Case 1 : T-Var.* Therefore  $L$  is the only free variable in  $L$ . By the generation lemma  $L : \sigma \in \Gamma$ , so  $\{L\} \subseteq \text{dom } \Gamma$  ■

*Case 2 : T-App.* Therefore by the generation lemma  $L \equiv M N$  for some  $M, N \in \Lambda_{\mathbb{T}_2}$  such that

$$\Gamma \vdash M : \tau \rightarrow \sigma \quad \Gamma \vdash N : \tau$$

For some type  $\tau$ . By the inductive hypothesis  $\text{FV } M \subseteq \Gamma$  and  $\text{FV } N \subseteq \text{dom } \Gamma$ . Therefore  $\text{FV } L = (\text{FV } M) \cup (\text{FV } N) \subseteq \text{dom } \Gamma$ . ■

*Case 3 : T-Abst.* Therefore by the generation lemma  $L \equiv \lambda x : \alpha . M$  for some  $M \in \Lambda_{\mathbb{T}_2}$  such that

$$\Gamma, x : \alpha \vdash M : \beta$$

and  $\sigma \equiv \alpha \rightarrow \beta$ . By the inductive hypothesis  $\text{FV } M \subseteq \text{dom } \Gamma \cup \{x\}$ . Therefore

$$\text{FV } L = \text{FV } M \setminus \{x\} \subseteq (\text{dom } \Gamma \cup \{x\}) \setminus \{x\} = \text{dom } \Gamma$$

■

*Case 4 : T2-App.* Therefore  $L \equiv B$  for some  $N, B \in \mathbb{V}_2$  such that

$$\Gamma \vdash N : \Pi \alpha : * . \sigma \quad \Gamma \vdash B : *$$

By the inductive hypothesis  $\text{FV } N \subseteq \text{dom } \Gamma$ . Since  $B \in \mathbb{T}_2$  then  $\text{FV } B = \emptyset$ . Therefore  $\text{FV } L = \text{FV } N \subseteq \text{dom } \Gamma$ .

■

*Case 5 : T2-Abst.* Therefore  $L \equiv \lambda \alpha : * . M$  for some  $M \in \Lambda_{\mathbb{T}_2}$  such that

$$\Gamma, \alpha : * \vdash M : \beta$$

and  $\sigma \equiv \Pi \alpha : * . \beta$ . Because  $\alpha \in \mathbb{T}_2$  so  $\alpha \notin \text{FV } M$  and  $\text{FV } L = \text{FV } M$ , thus  $\text{FV } M = \text{FV } L \subseteq \Gamma$ .

■

### Problem

(3.21) Give a recursive definition for  $\text{FTV} : \mathbb{T}_2 \cup \Lambda_{\mathbb{T}_2} \rightarrow \mathbb{V}_2$

*Solution.* Here  $\alpha \in \mathbb{V}_2$ ,  $A, B \in \mathbb{T}_2$ ,  $x \in V$  and  $M \in \Lambda_{\mathbb{T}_2}$ .

| Form                                   | Value                                |
|----------------------------------------|--------------------------------------|
| $\text{FTV } \alpha$                   | $\{\alpha\}$                         |
| $\text{FTV } x$                        | $\emptyset$                          |
| $\text{FTV } (A \rightarrow B)$        | $\text{FTV}(A) \cup \text{FTV}(B)$   |
| $\text{FTV } (\Pi \alpha : * . A)$     | $\text{FTV}(A) \setminus \{\alpha\}$ |
| $\text{FTV } (M N)$                    | $\text{FTV}(M) \cup \text{FTV}(N)$   |
| $\text{FTV } (\lambda x : A . M)$      | $\text{FTV}(A) \cup \text{FTV}(M)$   |
| $\text{FTV } (M B)$                    | $\text{FV}(M) \cup \text{FTV}(B)$    |
| $\text{FTV } (\lambda \alpha : * . M)$ | $\text{FTV}(M) \setminus \{\alpha\}$ |

—

Completed Dec 16 10:11 pm.