EXCERCISES

CHAPTER 1

SEAN LI [1]

1. Redacted

Problem

(1.1) Simplify notation of the following terms

(a) $(\lambda x .(((x\ z)\ y)(x\ x)))$

(b) $((\lambda x .(\lambda y .(\lambda z .(z\ ((x\ y)\ z)))))(\lambda u\ .\ u))$

Solution.

(a) $\lambda x .(x\ z\ y)(x\ x)$

(b) $(\lambda x\ y\ z\ .\ x\ (x\ y\ z))(\lambda u\ .\ u)$

Problem

(1.2) Find the alpha equivalent terms to

$$\lambda x\ .\ x\ (\lambda x\ .\ x)$$

In

(a) $\lambda y\ .\ y\ (\lambda x\ .\ x)$

(b) $\lambda y\ .\ y\ (\lambda x\ .\ y)$

(c) $\lambda y\ .\ y\ (\lambda x\ .\ y)$

Solution. Only (a).

Problem

(1.3) Prove

$$\lambda x \,.\, x \,(\lambda z \,.\, y) \underset{\alpha}{=} \lambda z \,.\, z \,(\lambda z \,.\, y)$$

Solution.

Proof. By definition of alpha equivalence

$$M \underset{\alpha}{=} N \iff \exists \varphi, M^\varphi \underset{\alpha}{\to} N \wedge \text{FR } M = \text{FR } N$$

The witness of φ is substituting bound variable x with z, and z is not a free variable in the term, thus the two terms are alpha equivalent.

$$\lambda x \,.\, x \,(\lambda z \,.\, y)^{x \to z} \underset{\alpha}{\to} \lambda z \,.\, z \,(\lambda z \,.\, y)$$

∎

Problem

(1.4) Consider the following term:

$$U := (\lambda z \,.\, z\,x\,z)((\lambda y \,.\, x\,y)\,x)$$

1. Find Sub U
2. Draw tree rep of U
3. Find FV U
4. Find alpha equivalent terms to U from below and point out which of those follows the Barendregt convention:

$$\text{(a) } (\lambda y \,.\, y\,x\,y)((\lambda z \,.\, x\,z)\,x)$$
$$\text{(b) } (\lambda x \,.\, x\,y\,x)((\lambda z \,.\, y\,z)\,y)$$
$$\text{(c) } (\lambda y \,.\, y\,x\,y)((\lambda y \,.\, x\,y)\,x)$$
$$\text{(d) } (\lambda v \,.(v\,x)\,v)((\lambda u \,.\, u\,v)\,x)$$

1. Find Sub U.
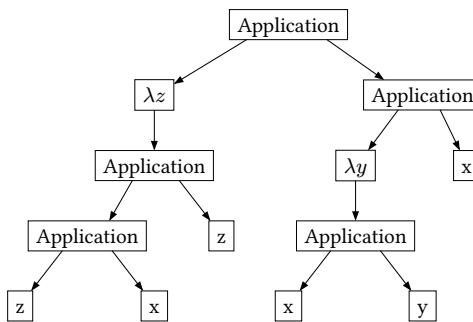
Solution.

Sub $U =$

$$\{(\lambda z \,.\, z\, x\, z)((\lambda y \,.\, x\, y)\, x), (\lambda z \,.\, z\, x\, z), ((\lambda y \,.\, x\, y)\, x)\} \cup$$
$$\{(\lambda y \,.\, x\, y), z\, x\, z, z\, x, x\, y\} \cup$$
$$\{z, x, y\}$$
$$= \{(\lambda z \,.\, z\, x\, z)((\lambda y \,.\, x\, y)\, x), (\lambda z \,.\, z\, x\, z), ((\lambda y \,.\, x\, y)\, x),$$
$$(\lambda y \,.\, x\, y), z\, x\, z, z\, x, x\, y, z, x, y\}$$

2. Draw a tree rep of U.

Solution.

3. Find FV U

Solution.

$$\text{FV } U = \text{FV }(\lambda y \,.\, y\, x\, y) \cup \text{FV }(\lambda z \,.\, x\, z)\, x$$
$$= (\text{FV } y\, x\, y) \setminus \{y\} \cup (\text{FV } \lambda z \,.\, x\, z) \cup \{x\}$$
$$= (\text{FV } y\, x) \setminus \{y\} \cup (\text{FV } x\, z) \setminus \{z\} \cup \{x\}$$
$$= \{x\}$$

4. Find an alpha-equivalent term.

Solution.

$$(a) \underset{\alpha}{=} (c) \underset{\alpha}{=} U$$

Only (a) follows the Barendregt convention.

Problem

(1.5) Give the results of the following substitutions

$$(a)\ (\lambda x \,.\, y\, (\lambda y \,.\, x\, y))[y := \lambda z \,.\, z\, x]$$
$$(b)\ ((x\, y\, z)[x := y])[y := z]$$
$$(c)\ ((\lambda x \,.\, x\, y\, z)[x := y])[y := z]$$
$$(d)\ (\lambda y \,.\, y\, y\, x)[x := y\, z]$$

Solution.

$$
\begin{aligned}
&\text{(a) } (\lambda v .(\lambda z . z\, x)(\lambda u . v\, u)) \\
&\text{(b) } (y\, y\, z)[y := z] = z\, z\, z \\
&\text{(c) } (\lambda x . x\, y\, z)[y := z] = (\lambda x . x\, z\, z) \\
&\text{(d) } (\lambda u . u\, u\, (y\, z))
\end{aligned}
$$

Problem

(1.6)

$$
\neg\left(\forall\, M\, L\, N \in \Lambda,\, M\, [x := N, y := L] \underset{\alpha}{\equiv} M\, [x := N][y := L]\right)
$$

Solution.

Proof. Because RHS $= M\, [x := N][y := L] = M\, [x := N\, [y := L]][y := L]$, if $y \in$ FV N, then what x gets substituted with will have y substituted for L, which is completely different with LHS. ∎
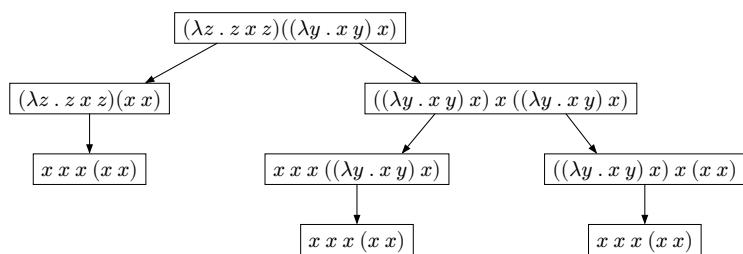
Problem

(1.7) Find all available redexes in

$$
U := (\lambda z . z\, x\, z)((\lambda y . x\, y)\, x)
$$

And all reduction pathes to the β-normal form.

Solution. The first redex is the term as an application itself; another the second term in the application.

(1.8) Show that

$$(\lambda x \; . \; x \; x) \; y \underset{\beta}{\neq} (\lambda x \; y \; . \; y \; x) \; x \; x$$

Solution. By Collorary 1.9.9, it suffices to prove the hypothesis with a proof of a common normal reduced form from LHS and RHS not existing.

Contradiction. By definition of $\underset{\beta}{=}$, there exists The set of all terms attainable from β-reduction on $(\lambda x \; . \; x \; x) \; y$ and $(\lambda x \; y \; . \; y \; x) \; x \; x$ do not intersect. Therefore,

$$\neg \Big(\exists L \in \Lambda, (\lambda x \; . \; x \; x) \; y \underset{\beta}{\twoheadrightarrow} L \wedge (\lambda x \; y \; . \; y \; x) \; x \; x \underset{\beta}{\twoheadrightarrow} L \Big) \implies \neg \Big((\lambda x \; . \; x \; x) \; y \underset{\beta}{=} (\lambda x \; y \; . \; y \; x) \; x \; x \Big)$$

∎

Problem

(1.9) Define the combinators

$$K := \lambda x \; y \; . \; x$$
$$S := \lambda x \; y \; z \; . \; x \; z \; (y \; z)$$

Prove that

$$\forall \; P \; Q \in \Lambda, K \; P \; Q \underset{\beta}{\twoheadrightarrow} P$$

$$\forall \; P \; Q \; R \in \Lambda, S \; P \; Q \; R \underset{\beta}{\twoheadrightarrow} P \; R \; (Q \; R)$$

Solution.

Proof.

$$K \; P \; Q = (\lambda x \; y \; . \; x) \; P \; Q \underset{\beta}{\rightarrow} (\lambda y \; . \; x)[x := P] \; Q \underset{\beta}{\rightarrow} P \; [y := Q] = P$$

$$S \; P \; Q \; R = (\lambda x \; y \; z \; . \; x \; z \; (y \; z)) \underset{\beta}{\twoheadrightarrow} (x \; z \; (y \; z))[x := P][y := Q][z := R] = P \; R \; (Q \; R)$$

∎

(1.10) We define the church numerals

$$\text{zero} := \lambda f\,x\,.\,x$$
$$\text{one} := \lambda f\,x\,.\,f\,x$$
$$\text{two} := \lambda f\,x\,.\,f\,f\,x$$
$$...$$
$$\text{num}_n := \lambda f\,x\,.\,f^n\,x$$

And operations

$$\text{add} := \lambda n\,m\,f\,x\,.\,m\,f\,(n\,f\,x)$$
$$\text{mul} := \lambda n\,m\,f\,x\,.\,m\,(n\,f)\,x$$

Show

(a) add one one $\twoheadrightarrow_\beta$ two

(b) add one one $\neq_\beta$ mul one zero

Solution.

(a)   add one one $= (\lambda n\,m\,f\,x\,.\,m\,f\,(n\,f\,x))(\lambda f\,x\,.\,f\,x)(\lambda f\,x\,.\,f\,x)$

$\twoheadrightarrow_\beta (\lambda f\,x\,.(\lambda f\,x\,.\,f\,x)\,f\,((\lambda f\,x\,.\,f\,x)\,f\,x))$

$\twoheadrightarrow_\beta (\lambda f\,x\,.(\lambda x\,.\,f\,x)\,f\,x)$

$\twoheadrightarrow_\beta (\lambda f\,x\,.\,f\,f\,x) = $ two

(b)   mul one one $= (\lambda n\,m\,f\,x\,.\,m\,(n\,f)\,x)(\lambda f\,x\,.\,f\,x)(\lambda f\,x\,.\,f\,x)$

$\twoheadrightarrow_\beta \lambda f\,x\,.(\lambda f\,x\,.\,f\,x)((\lambda f\,x\,.\,f\,x)\,f)\,x$

$\twoheadrightarrow_\beta \lambda f\,x\,.\,f\,x = $ one

Because no intermediate form in the beta reduction process of the two terms are α-equivalent, by collorary 1.9.9 the two terms are not β-equivalent.

(1.11) We define

$$\text{succ} := \lambda m\ f\ x\ .\ f\ (m\ f\ x)\ \text{s.t.}\ \forall \text{num}_n, \text{succ num}_n = \text{num}_{n+1}$$

Prove

$$\text{succ zero} \underset{\beta}{=} \text{one}$$

$$\text{succ one} \underset{\beta}{=} \text{two}$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$\text{succ zero} = (\lambda m\ f\ x\ .\ f\ (m\ f\ x))(\lambda f\ x\ .\ x)$$

$$\underset{\beta}{\rightarrow} (\lambda f\ x\ .\ f\ ((\lambda f\ x\ .\ x)\ f\ x))$$

$$\underset{\beta}{\twoheadrightarrow} (\lambda f\ x\ .\ f\ x) = \text{one}$$

The path succ zero $\underset{\beta}{\twoheadrightarrow}$ one derived above is the witness of a reduction chain from LHS to RHS.

$$\text{succ one} = (\lambda m\ f\ x\ .\ f\ (m\ f\ x))(\lambda f\ x\ .\ f\ x)$$

$$\underset{\beta}{\rightarrow} (\lambda f\ x\ .\ f\ ((\lambda f\ x\ .\ f\ x)\ f\ x))$$

$$\underset{\beta}{\twoheadrightarrow} (\lambda f\ x\ .\ f\ (f\ x)) = \text{two}$$

The path succ one $\underset{\beta}{\twoheadrightarrow}$ two derived above is the witness of a reduction chain from LHS to RHS. ∎

(1.12) We define the λ-terms $\top_\lambda$ (true) and $\bot_\lambda$ (false) and $\neg_\lambda$ (not) by:

$$\top_\lambda := \lambda x\ y\ .\ x \quad \bot_\lambda := \lambda x\ y\ .\ y$$

$$\neg_\lambda := \lambda a\ .\ a \bot_\lambda \top_\lambda$$

Show that

$$\neg_\lambda(\neg_\lambda \top_\lambda) \underset{\beta}{=} \top_\lambda$$

$$\neg_\lambda(\neg_\lambda \bot_\lambda) \underset{\beta}{=} \bot_\lambda$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$\neg_\lambda(\neg_\lambda\top_\lambda) = \neg_\lambda((\lambda a \,.\, a \perp_\lambda \top_\lambda)(\lambda x\, y \,.\, x))$$
$$\underset{\beta}{\twoheadrightarrow} \neg_\lambda((\lambda x\, y \,.\, x)\perp_\lambda \top_\lambda)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda a \,.\, a \perp_\lambda \top_\lambda)\perp_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda x\, y \,.\, y)\perp_\lambda \top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} \top_\lambda$$

$$\neg_\lambda(\neg_\lambda\perp_\lambda) = \neg_\lambda((\lambda a \,.\, a \perp_\lambda \top_\lambda)(\lambda x\, y \,.\, y))$$
$$\underset{\beta}{\twoheadrightarrow} \neg_\lambda((\lambda x\, y \,.\, y)\perp_\lambda \top_\lambda)$$
$$\underset{\beta}{\twoheadrightarrow} \neg_\lambda \top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda a \,.\, a \perp_\lambda \top_\lambda)(\lambda x\, y \,.\, x)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda x\, y \,.\, x)\perp_\lambda \top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} \perp_\lambda$$

∎

Problem

(1.13) Define

$$\text{iszero} := \lambda m \,.\, m \,(\lambda x \,.\perp_\lambda)\top_\lambda$$

Prove

$$\text{iszero zero} \underset{\beta}{\twoheadrightarrow} \top_\lambda$$
$$\forall\, n \in \mathbb{N}^+, \text{iszero num}_n \underset{\beta}{\twoheadrightarrow} \perp_\lambda$$

Solution.

$$\text{iszero zero} = (\lambda m \,.\, m \,(\lambda x \,.\perp_\lambda)\top_\lambda)(\lambda f\, x \,.\, x)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda f\, x \,.\, x)(\lambda x \,.\perp_\lambda)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} \top_\lambda$$

$$\text{iszero num}_n = (\lambda m \,.\, m \,(\lambda x \,.\perp_\lambda)\top_\lambda)(\lambda f\, x \,.\, f^n\, x)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda f\, x \,.\, f^n\, x)(\lambda x \,.\perp_\lambda)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda x \,.\perp_\lambda)\big((\lambda x \,.\perp_\lambda)^{n-1}\top_\lambda\big) \underset{\beta}{\twoheadrightarrow} \perp_\lambda$$

Problem

(1.14) If-else can be modeled as

$$\text{ifelse} = \lambda x\, t\, f \,.\, x\, t\, f$$

Where when x, then t, else f. Prove correctness by applying $\top_\lambda$ and $\perp_\lambda$ on ifelse.

Solution.

$$\text{ifelse } \top_\lambda = (\lambda x\, t\, f \,.\, x\, t\, f)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda t\, f \,.(\lambda x\, y \,.\, x)\, t\, f) \underset{\beta}{\twoheadrightarrow} (\lambda t\, f \,.\, t)$$

$$\text{ifelse } \top_\lambda = (\lambda x\, t\, f \,.\, x\, t\, f)\perp_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda t\, f \,.(\lambda x\, y \,.\, y)\, t\, f) \underset{\beta}{\twoheadrightarrow} (\lambda t\, f \,.\, f)$$

By applying the results to any two values, the correct corresponding value returns, ex, for ifelse $\top_\lambda$, t is always returned.

Problem

(1.15) Prove that $\Omega := (\lambda x \,.\, x\, x)(\lambda x \,.\, x\, x)$ does not have a β-nf.

Solution. Firstly let's prove Ω.

Proof. Induction on Ω's only reduction path proves that every $\Omega \underset{\beta}{\twoheadrightarrow} \Omega_i = \Omega$. For the base case because Ω has one and only one redux, it could only reduce to Ω_1 which is equivalent to itself. For the inductive step, $\Omega_i = \Omega$, therefore $\Omega_i \underset{\beta}{\rightarrow} \Omega_{i+1}$ is still Ω.

By definition, a term having a β-nf requires the existence of a form in β-nf such that the term can reduce to. By induction, Ω only reduces to Ω, and Ω is not in β-nf because it contains β-redex. Therefore, Ω can never reduce to a β-nf, thus it does not have a β-nf. ∎

(1.16) Let M be a λ-term with the following properties:
- M has a β-nf.
- There exists an infinite reduction path $M \equiv M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \dots$ on M.

Prove that every M_i has a β-nf, and give an example of M.

Solution. An example would be $(\lambda x\, y \,.\, y)\Omega$. Reduction can go on infinitely by reducing on Ω, but the β-nf of the term is $\lambda y \,.\, y$

Proof. Denote β-nf of M as M'. For any form in the reduction path, $M \underset{\beta}{\twoheadrightarrow} M_i$. In conjunction with $M \underset{\beta}{\twoheadrightarrow} M'$, by the Church-Rosser theorem, there exists L such that $M_i \underset{\beta}{\twoheadrightarrow} L$ and $M' \underset{\beta}{\twoheadrightarrow} L$. Because M' is in β-nf, L can only be M', thus $M_i \underset{\beta}{\twoheadrightarrow} M'$, so M_i is capable of reducing to M', a β-nf. Therefore, any form in the reduction path has a β-nf. ∎

(1.17) If $M\,N$ is strongly normalizing, then both M and N are strongly normalizing.

Solution.

Proof. If M is not strongly normalizing, then there exists a reduction path $M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \dots$ Therefore, $M\,N$ would have had a reduction path $M\,N \underset{\beta}{\to} M_1\,N \underset{\beta}{\to} \dots$ that is infinite, which contradicts with $M\,N$ being strongly normalizing. Vice versa for N. ∎

(1.18) Let $L, M, N \in \Lambda$ such that $L \underset{\beta}{=} M$ and $L \underset{\beta}{\twoheadrightarrow} N$. Moreover, N is in β-nf. Prove that $M \underset{\beta}{\twoheadrightarrow} N$.

Solution. Collorary 1.9.9.

(1.19) Define

$$U := \lambda z\, x \,.\, x\,(z\,z\,x) \quad \text{and} \quad Z := U\,U$$

Prove Z is a fixed point combinator.

Solution. Proving $\forall\, L \in \Lambda, L\ (Z\ L) \underset{\beta}{\twoheadrightarrow} Z\ L.$

Proof.

$$Z\ L = (\lambda z\ x\ .\ x\ (z\ z\ x))(\lambda z\ x\ .\ x\ (z\ z\ x))\ L$$
$$\underset{\beta}{\twoheadrightarrow} L\ ((\lambda z\ x\ .\ x\ (z\ z\ x))(\lambda z\ x\ .\ x\ (z\ z\ x))\ L) \underset{\beta}{\twoheadrightarrow} L\ (Z\ L)$$

∎

Problem

(1.20) Solve for $M \in \Lambda$ in each equation:

$$M \underset{\beta}{=} \lambda x\ y\ .\ x\ M\ y$$

$$M\ x\ y\ z \underset{\beta}{=} x\ y\ z\ M$$

Solution. By the property of the Y combinator:

$$f\ (Y\ f) = Y\ f$$

The first equation can be remodeled as

$$M \underset{\beta}{=} L\ M \text{ where } L = \lambda m\ x\ y\ .\ x\ m\ y$$

Solving for fixed point of L :

$$M \equiv Y\ L \equiv L\ (Y\ L)$$
$$= (\lambda x\ .\ L\ (x\ x))(\lambda x\ .\ L\ (x\ x))$$
$$= (\lambda x\ .(\lambda m\ u\ v\ .\ u\ m\ v)(x\ x))(\lambda x\ .(\lambda m\ u\ v\ .\ u\ m\ v)(x\ x))$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda x\ .(\lambda u\ v\ .\ u\ (x\ x)\ v))(\lambda x\ .(\lambda u\ v\ .\ u\ (x\ x)\ v))$$

The second equation can be η-reduced on both sides:

$$M\ x\ y\ z \underset{\beta}{=} x\ y\ z\ M$$

$$M \underset{\beta}{=} \lambda x\ y\ z\ .\ x\ y\ z\ M$$

Remodeling equation:

$$M = N\ M \text{ where } M = \lambda m\ x\ y\ z\ .\ x\ y\ z\ m$$

then

$$M \equiv Y\,N \equiv N\,(Y\,N)$$
$$= (\lambda x \,.\, N\,(x\,x))(\lambda x \,.\, N\,(x\,x))$$
$$= (\lambda x \,.(\lambda m\,u\,y\,z \,.\, u\,y\,z\,m)(x\,x))(\lambda x \,.(\lambda m\,u\,y\,z \,.\, u\,y\,z\,m)(x\,x))$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda x \,.(\lambda u\,y\,z \,.\, u\,y\,z\,(x\,x)))(\lambda x \,.(\lambda u\,y\,z \,.\, u\,y\,z\,(x\,x)))$$