

EXERCISES

CHAPTER 7

SEAN LI ¹

1. Reduced

Reference - Propositional Logic in λC

$$\frac{A \quad \neg A}{\perp} \perp\text{I or } \neg\text{E} \quad \frac{\perp}{A} \perp\text{E} \quad \frac{A \quad B}{A \wedge B} \wedge\text{I} \quad \frac{A \wedge B}{A} \wedge\text{EL}$$

$$\frac{A \wedge B}{B} \wedge\text{ER} \quad \frac{a}{a \vee b} \vee\text{IL} \quad \frac{b}{a \vee b} \vee\text{IR} \quad \frac{A \Rightarrow B \quad A}{B} \Rightarrow\text{E}$$

$$\frac{A \vee B \quad A \rightarrow C \quad B \rightarrow C}{C} \vee\text{E} \quad \frac{a \in S \quad P(a)}{\exists a \in S, P(a)} \exists\text{I}$$

$$\begin{array}{c} 1. \quad A \\ 2. \quad \left| \begin{array}{l} \dots \\ 3. \quad \perp \end{array} \right. \\ \hline \neg A \end{array} \neg\text{I} \quad \begin{array}{c} 1. \quad A \\ 2. \quad \left| \begin{array}{l} \dots \\ 3. \quad B \end{array} \right. \\ \hline A \Rightarrow B \end{array} \Rightarrow\text{I} \quad \begin{array}{c} 1. \quad a \in S \\ 2. \quad \left| \begin{array}{l} \dots \\ 3. \quad P(a) \end{array} \right. \\ \hline \forall a \in S, P(a) \end{array} \forall\text{I}$$

$$\frac{\exists x \in S, P(x) \quad \forall x \in S, (P(x) \Rightarrow A)}{A} \exists\text{E} \quad \frac{a \in S \quad \forall x \in S, P(x)}{P(a)} \forall\text{E}$$

$$\frac{}{\neg\neg A \Rightarrow A} \text{DN (Classical)} \quad \frac{}{A \vee \neg A} \text{ET (Classical)}$$

Reference - 2nd Encoding for Propositional Logic

Proposition	Minimal Propositional Logic
\perp	$\forall A, A$
$A \Rightarrow B$	$A \Rightarrow B$
$\neg A$	$A \Rightarrow \perp$
$A \wedge B$	$\forall C, (A \Rightarrow B \Rightarrow C) \Rightarrow C$
$A \vee B$	$\forall C, (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$
$\forall a \in S, P(a)$	$\forall a \in S. P(a)$
$\exists a \in S, P(a)$	$\forall \alpha, (\forall a \in S, (P(a) \Rightarrow \alpha)) \Rightarrow \alpha$

Problem

(7.1 a) Prove in natural deduction and λC the tautology

$$B \Rightarrow (A \Rightarrow B)$$

Solution.

Natural Deduction.

1. B
2. $\left| \begin{array}{l} A \\ \left| \begin{array}{l} B \\ \hline A \Rightarrow B \end{array} \right. \end{array} \right. \Rightarrow \mathbf{I}$
3. $\left| \begin{array}{l} A \\ \left| \begin{array}{l} B \\ \hline A \Rightarrow B \end{array} \right. \end{array} \right. \Rightarrow \mathbf{I}$
4. $\left| \begin{array}{l} A \\ \left| \begin{array}{l} B \\ \hline A \Rightarrow B \end{array} \right. \end{array} \right. \Rightarrow \mathbf{I}$
5. $B \Rightarrow (A \Rightarrow B) \Rightarrow \mathbf{I}$

■

λC . Assuming context $\Gamma \equiv A : *, B : *$. By the PAT paradigm the proof is equivalent to an inhabitant of $B \rightarrow A \rightarrow B$.

1. $A : *, B : *$
2. $\left| \begin{array}{l} x : B \\ \left| \begin{array}{l} y : A \\ \left| \begin{array}{l} x : B \\ \hline \lambda y : A. x : A \rightarrow B \end{array} \right. \end{array} \right. \end{array} \right. \text{Weak}$
3. $\left| \begin{array}{l} x : B \\ \left| \begin{array}{l} y : A \\ \left| \begin{array}{l} x : B \\ \hline \lambda y : A. x : A \rightarrow B \end{array} \right. \end{array} \right. \end{array} \right. \text{4 Abst}$
4. $\left| \begin{array}{l} x : B \\ \left| \begin{array}{l} y : A \\ \left| \begin{array}{l} x : B \\ \hline \lambda y : A. x : A \rightarrow B \end{array} \right. \end{array} \right. \end{array} \right. \text{5 Abst}$
5. $\left| \begin{array}{l} x : B \\ \left| \begin{array}{l} y : A \\ \left| \begin{array}{l} x : B \\ \hline \lambda y : A. x : A \rightarrow B \end{array} \right. \end{array} \right. \end{array} \right. \text{5 Abst}$
6. $\left| \begin{array}{l} x : B \\ \left| \begin{array}{l} y : A \\ \left| \begin{array}{l} x : B \\ \hline \lambda y : A. x : A \rightarrow B \end{array} \right. \end{array} \right. \end{array} \right. \text{5 Abst}$

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Problem

(7.1 b) Prove in natural deduction and λC the tautology

$$\neg A \Rightarrow (A \Rightarrow B)$$

Solution.

Natural Deduction.

1. $\neg A$
2. $\begin{array}{|l} A \end{array}$
3. $\begin{array}{|l} \neg A \end{array}$
4. $\begin{array}{|l} A \end{array}$
5. $\begin{array}{|l} \perp \end{array} \quad \perp \text{I}$
6. $\begin{array}{|l} B \end{array} \quad \perp \text{E}$
7. $\begin{array}{|l} A \Rightarrow B \end{array} \quad \Rightarrow \text{I}$
8. $\neg A \Rightarrow (A \Rightarrow B) \quad \Rightarrow \text{I}$

■

λC . Assuming context $\Gamma \equiv A : *, B : *$. By the PAT paradigm the proof is equivalent to an inhabitant of $(A \rightarrow \perp) \rightarrow A \rightarrow B$.

1. $A : *, B : *$
2. $\begin{array}{|l} x : \neg A \end{array}$
3. $\begin{array}{|l} y : A \end{array}$
4. $\begin{array}{|l} x y : \Pi \alpha : * . \alpha \end{array} \quad \mathbf{2,3 \text{ App (Neg Elim)}}$
5. $\begin{array}{|l} x y B : B \end{array} \quad \mathbf{4,1 \text{ App (Ex Falso)}}$
6. $\begin{array}{|l} \lambda y : A . x y B : A \rightarrow B \end{array} \quad \mathbf{5 \text{ Abst}}$
7. $\begin{array}{|l} \lambda x : \neg A . \lambda y : A . x y B : \neg A \rightarrow A \rightarrow B \end{array} \quad \mathbf{6 \text{ Abst}}$

■

Problem

(7.1 c) Prove in natural deduction and λC the tautology

$$(A \Rightarrow \neg B) \Rightarrow ((A \Rightarrow B) \Rightarrow \neg A)$$

Solution.

Natural Deduction.

1.	$A \Rightarrow \neg B$	
2.	$A \Rightarrow B$	
3.	A	
4.	$\neg B$	1,3 $\Rightarrow E$
5.	B	2,3 $\Rightarrow E$
6.	\perp	5,4 $\perp I$
7.	$\neg A$	3,6 $\neg I$
8.	$(A \Rightarrow B) \Rightarrow \neg A$	2,7 $\Rightarrow I$
9.	$(A \Rightarrow \neg B) \Rightarrow ((A \Rightarrow B) \Rightarrow \neg A)$	1,8 $\Rightarrow I$

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λC . Assuming context $\Gamma \equiv A : *, B : *$. The proof should be equivalent to an inhabitant of $(A \rightarrow B \rightarrow \perp) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow \perp$.

1.	$A : *, B : *$	
2.	$h : A \rightarrow \neg B$	
3.	$q : A \rightarrow B$	
4.	$a : A$	
5.	$q a : B$	3,4 App
6.	$h a : B \rightarrow \perp$	2,4 App
7.	$h a (q a) : \perp$	6,5 App (Neg Elim)
8.	$\lambda a : A . h a (q a) : \neg A$	7 Abst (Neg Intro)
9.	$\lambda q : A \rightarrow B . \lambda a : A . h a (q a) : A \rightarrow B \rightarrow \neg A$	8 Abst
10.	$\lambda h : A \rightarrow \neg B . \lambda q : A \rightarrow B . \lambda a : A . h a (q a) : (A \rightarrow \neg B) \rightarrow A \rightarrow B \rightarrow \neg A$	9 Abst

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Problem

(7.1 d) Prove in natural deduction and λC the tautology

$$\neg(A \Rightarrow B) \Rightarrow \neg B$$

Solution.

Natural Deduction.

$$\begin{array}{ll}
1. & \neg(A \Rightarrow B) \\
2. & \begin{array}{|l} B \\ \hline \end{array} \\
3. & \begin{array}{|l} \begin{array}{|l} A \\ \hline \end{array} \\ \hline \end{array} \\
4. & \begin{array}{|l} \begin{array}{|l} B \\ \hline \end{array} \\ \hline \end{array} \\
5. & \begin{array}{|l} A \Rightarrow B \\ \hline \end{array} \quad \mathbf{3,4 \Rightarrow I} \\
6. & \begin{array}{|l} \perp \\ \hline \end{array} \quad \mathbf{5,1 \perp I} \\
7. & \begin{array}{|l} \neg B \\ \hline \end{array} \quad \mathbf{6 \neg I} \\
8. & \neg(A \Rightarrow B) \Rightarrow \neg B \quad \mathbf{1,7 \Rightarrow I}
\end{array}$$

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λC . Assuming context $\Gamma \equiv A : *, B : *$. The proof should be equivalent to an inhabitant of $((A \rightarrow B) \rightarrow \perp) \rightarrow B \rightarrow \perp$.

$$\begin{array}{ll}
1. & n : \neg(A \rightarrow B) \\
2. & \begin{array}{|l} b : B \\ \hline \end{array} \\
3. & \begin{array}{|l} \begin{array}{|l} a : A \\ \hline \end{array} \\ \hline \end{array} \\
4. & \begin{array}{|l} \begin{array}{|l} b : B \\ \hline \end{array} \\ \hline \end{array} \quad \mathbf{Weak} \\
5. & \begin{array}{|l} \lambda a : A . b : A \rightarrow B \\ \hline \end{array} \quad \mathbf{4 Abst} \\
6. & \begin{array}{|l} n (\lambda a : A . b) : \perp \\ \hline \end{array} \quad \mathbf{1,5 App (Neg Elim)} \\
7. & \begin{array}{|l} \lambda b : B . n (\lambda a : A . b) : \neg B \\ \hline \end{array} \quad \mathbf{6 Abst (Neg Intro)} \\
8. &
\end{array}$$

$$\begin{array}{ll}
\lambda n : \neg(A \rightarrow B) . \lambda b : B . n (\lambda a : A . b) & \\
: \neg(A \rightarrow B) \rightarrow \neg B & \mathbf{7 Abst}
\end{array}$$

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Problem

(7.2) Formulate the double negation law as an axiom in λC , and prove the following tautology in λC with DN.

$$(\neg A \Rightarrow A) \Rightarrow A$$

Solution. The rule

$$\frac{}{\neg\neg A \Rightarrow A} \text{DN-E}$$

Could be translated into lambda calculus as

$$\Pi A : * . ((A \rightarrow \perp) \rightarrow \perp) \rightarrow A$$

Proof. Assume context $\Gamma \equiv A : *$.

1.	$A : *$	
2.	$h : \neg A \rightarrow A$	
3.	$x : \neg A$	
4.	$h x : A$	2,3 App
5.	$x (h x) : \perp$	3,4 App (Contradiction)
6.	$\lambda x : \neg A . x (h x) : \neg \neg A$	5 Abst (Neg Intro)
7.	$\text{DN } A : \neg \neg A \rightarrow A$	1,1 App
8.	$\text{DN } A (\lambda x : \neg A . x (h x)) : A$	App (Axiom DN)
	$\lambda h : \neg A \rightarrow A . \text{DN } A (\lambda x : \neg A . x (h x))$	
9.	$: (\neg A \rightarrow A) \rightarrow A$	8 Abst

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