Excercises

Chapter 5

Sean Li [1]

1. Reducted

Definition Some rules for reference.

λP Calculus Rules

$$\frac{}{\emptyset \vdash * : \square}\,\text{Sort} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}\,\text{Var} \qquad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}\,\text{Weak}$$

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi\, x : A\, .\, B : s}\,\text{Form} \qquad \frac{\Gamma \vdash M : \Pi\, x : A\, .\, B \quad \Gamma \vdash N : A}{\Gamma \vdash M\, N : B\, [x := N]}\,\text{App}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi\, x : A\, .\, B : s}{\Gamma \vdash \lambda x : A\, .\, M : \Pi\, x : A\, .\, B}\,\text{Abst}$$

$$\frac{\Gamma \vdash A : B \quad B \overset{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'}\,\text{Conv}$$

Predicate Logic

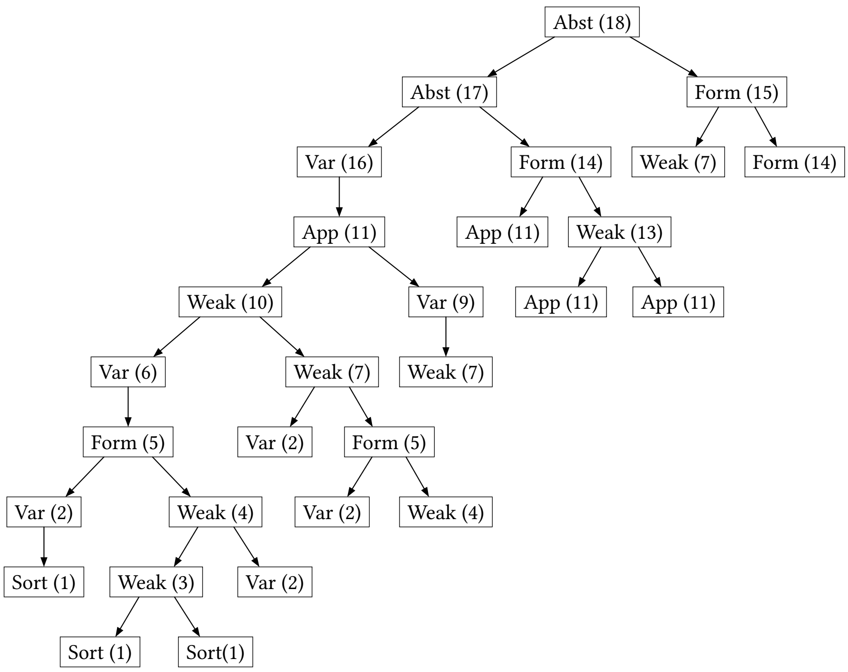
$$\frac{\begin{array}{ll} 1. & \text{Assume } A \\ 2. & \quad \dots \\ 3. & \quad B \end{array}}{A \Rightarrow B}\,{\Rightarrow}\text{I} \qquad \frac{A \Rightarrow B \quad A}{B}\,{\Rightarrow}\text{E} \qquad \frac{\begin{array}{ll} 1. & \text{Let } a \in S \\ 2. & \quad \dots \\ 3. & \quad P\,(a) \end{array}}{\forall a \in S, P\,(a)}\,\forall\text{I}$$

$$\frac{\forall a \in S \quad N \in S}{P\,(N)}\,\forall\text{E}$$

Problem

(5.1) Give a diagram of the tree corresponding to the complete tree derivation of line 18 of Section 5.3 (P 107)

Solution.



Problem

(5.2) Give a complete λP derivation of

$$S : * \vdash S \to S \to * : \square$$

In tree format and flag format.

Solution.

Tree Derivation.

$$(7)\ \dfrac{(3)\ S : * \vdash S : * \qquad (6)\ \dfrac{(4)\ \dfrac{\vdash * : \square \qquad \vdash * : \square}{S : * \vdash * : \square}\,\text{Weak} \qquad (3)\ S : * \vdash S : *}{S : *, x : S \vdash * : \square}\,\text{Weak}}{S : * \vdash S \to * : \square}$$

$$\dfrac{(3)\ \dfrac{\vdash * : \square}{S : * \vdash S : *}\,\text{Var} \qquad (9)\ \dfrac{(7)\ S : * \vdash S \to * : \square \qquad (3)\ S : * \vdash S : *}{S : *, x : S \vdash S \to * : \square}\,\text{Weak}}{S : * \vdash S \to S \to * : \square}\,\text{Form}$$

■

Flag Derivation.

1.	$* : \square$	**Sort**
2.	$S : *$	
3.	$S : *$	**1 Var**
4.	$* : \square$	**1,1 Weak**
5.	$x : S$	
6.	$* : \square$	**4,3 Weak**
7.	$S \to * : \square$	**3,6 Form**
8.	$x : S$	
9.	$S \to * : \square$	**7,3 Weak**
10.	$S \to S \to * : \square$	**3,9 Form**

■

Problem

(5.3) Derive

$$S : *, Q : S \to S \to * \vdash \Pi\, x : S\, .\Pi\, y : S\, .\, Q\, x\, y : *$$

Solution.

1.	$* : \square$	**Sort**
2.	$S : *$	
3.	$S : *$	**1 Var**
4.	$* : \square$	**1,1 Weak**
5.	$x : S$	
6.	$* : \square$	**4,3 Weak**
7.	$S \to * : \square$	**3,6 Form**
8.	$x : S$	
9.	$S \to * : \square$	**7,3 Weak**
10.	$S \to S \to * : \square$	**3,9 Form**
11.	$Q : S \to S \to *$	

12.	$Q : S \to S \to *$	**10 Var**
13.	$S : *$	**3,10 Weak**
14.	$* : \square$	**4,10 Weak**
15.	$x : S$	
16.	$* : \square$	**14,13 Weak**
17.	$S : *$	**13,13 Weak**
18.	$x : S$	**13 Var**
19.	$Q : S \to S \to *$	**12,13 Weak**
20.	$y : S$	
21.	$y : S$	**17 Var**
22.	$Q : S \to S \to *$	**19,17 Weak**
23.	$x : S$	**18,17 Weak**
24.	$Q\, x : S \to *$	**22,23 App**
25.	$Q\, x\, y : *$	**24,21 App**
26.	$\Pi\, y : S \,.\, Q\, x\, y : *$	**17,25 Form**
27.	$\Pi\, x : S \,.\Pi\, y : S \,.\, Q\, x\, y : *$	**13,26 Form**

Problem

(5.4) Prove that $*$ is the only valid kind in λP.

Solution.

Proof. The only possible way to construct a new kind is through the Form rule and the Sort axiom. Because we are trying to construct a kind, s here stands for $\square$.

$$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : \square}{\Gamma \vdash \Pi\, x : A \,.\, B : \square}\ \text{Form}$$

One could only construct new kinds with kinds, which requires $A : \square$ and $B : \square$. This contradicts with $A : *$. ∎

Problem

(5.5) Prove that $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ is a tautology by given a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash A \to (A \to B) \to B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$x : A$	
4.	$y : A \to B$	
5.	$y\, x : B$	**4,3 App**
6.	$\lambda y : A \to B \,.\, y\, x : (A \to B) \to B$	**5 Abst**
7.	$\lambda x : A \,.\lambda y : A \to B \,.\, y\, x : A \to (A \to B) \to B$	**5 Abst**

∎

Problem

(5.6 a) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ a tautology using natural deduction.

Solution.

Proof.

1.	Assume $A \Rightarrow (A \Rightarrow B)$	
2.	$A \Rightarrow (A \Rightarrow B)$	
3.	Assume A	
4.	A	
5.	$A \Rightarrow B$	**2,4 ⇒E**
6.	B	**5,4 ⇒E**
7.	$A \Rightarrow B$	**3,6 ⇒I**
8.	$(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$	**1,7 ⇒I**

∎

Problem

(5.6 b) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ using a shorthand λP derivation

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash (A \to A \to B) \to A \to B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1. $A : *$
2. $B : *$
3. $x : A \to A \to B$
4. $y : A$
5. $x\,y : A \to B$ **3,4 App**
6. $x\,y\,y : B$ **5,4 App**
7. $\lambda y : A \,.\, x\,y\,y : A \to B$ **6 Abst**
8. $\lambda x : A \to A \to B \,.\lambda y : A \,.\, x\,y\,y : (A \to A \to B) \to A \to B$ **7 Abst**

■

Problem

(5.7 a) Proof $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \to B) \to (B \to C) \to A \to C$$

And finding an inhabitant in a context only with definition of A, B, and C is equivalent to a proof of tautologousness.

Proof.

1. $A : *$
2. $B : *$
3. $C : *$
4. $x : A \to B$
5. $y : B \to C$
6. $a : A$
7. $x\,a : B$ **4,6 App**
8. $y\,(x\,a) : C$ **5,7 App**

9. $\lambda a : A \,.\, y\,(x\,a) : A \to C$ **8 Abst**

10. $\lambda y : B \to C \,.\lambda a : A \,.\, y\,(x\,z) : (B \to C) \to A \to C$ **9 Abst**

11. $\lambda x : A \to B \,.\lambda y : B \to C \,.\lambda a : A \,.\, y\,(x\,z)$
    $: (A \to B) \to (B \to C) \to A \to C$ **10 Abst**

■

Problem

(5.7 b) Proof $((A \Rightarrow B) \Rightarrow A) \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash ((A \to B) \to A) \to (A \to B) \to B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1. $A : *$
2. $B : *$
3. $x : (A \to B) \to A$
4. $y : A \to B$
5. $x\,y : A$ **3,4 App**
6. $y\,(x\,y) : B$ **4,5 App**
7. $\lambda y : A \to B \,.\, y\,(x\,y) : (A \to B) \to B$ **6 Abst**
8. $\lambda x : (A \to B) \to A \,.\lambda y : A \to B \,.\, y\,(x\,y)$
   $: ((A \to B) \to A) \to (A \to B) \to B$ **7 Abst**

■

Problem

(5.7 c) Proof $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \to B \to C) \to (A \to B) \to A \to C$$

And finding an inhabitant in a context only with definition of A, B, and C is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$C : *$	
4.	$x : A \to B \to C$	
5.	$y : A \to B$	
6.	$a : A$	
7.	$x\, a : B \to C$	**4,6 App**
8.	$y\, a : B$	**5,6 App**
9.	$x\, a\, (y\, a) : C$	**7,8 App**
10.	$\lambda a : A \,.\, x\, a\, (y\, a) : A \to C$	**9 Abst**
11.	$\lambda y : A \to B \,.\lambda a : A \,.\, x\, a\, (y\, a) : (A \to B) \to A \to C$	**10 Abst**
12.	$\lambda x : A \to B \to C \,.\lambda y : A \to B \,.\lambda a : A \,.\, x\, a\, (y\, a)$ $: (A \to B \to C) \to (A \to B) \to A \to C$	**10 Abst**

∎

Problem

(5.8 a) Let $\Gamma \equiv S : *, P : S \to *, Q : S \to *$, find an inhabitant of

$$\Pi\, x : S \,.\, P\, x \to Q\, x \to P\, x$$

with respect to Γ and give a shorthand λP derivation

Solution.

1.	$S : *$	
2.	$P : S \to *$	
3.	$Q : S \to *$	
4.	$x : S$	
5.	$a : P\, x$	
6.	$b : Q\, x$	
7.	$a : P\, x$	**2,4 App**

8.	$\lambda b : Q\ x\ .\ S : Q\ x \to P\ x$	**7 Abst**
9.	$\lambda a : P\ x\ .\lambda b : Q\ x\ .\ a : P\ x \to Q\ x \to P\ x$	**8 Abst**
10.	$\lambda x : S\ .\lambda a : P\ x\ .\lambda b : Q\ x\ .\ a$ $: \Pi\ x : S\ .\ P\ x \to Q\ x \to P\ x$	**9 Abst**

Problem

(5.8 b) Let $\Gamma \equiv S : *, P : S \to *, Q : S \to *$, find an inhabitant of

$$\Pi\ x : S\ .\ P\ x \to Q\ x \to P\ x$$

By proving the corresponding proposition in natural deduction.

Solution. The corresponding proposition and premises are

$$\frac{S \in \text{Set} \qquad P : S \to \text{Prop} \qquad Q : S \to \text{Prop}}{\forall a \in S, P\ (a) \Rightarrow (Q\ (a) \Rightarrow P\ (a))}$$

Proof.

1.	Let $a \in S$	
2.	Assume $P\ (a)$	
3.	Assume $Q\ (a)$	
4.	$P\ (a)$	
5.	$Q\ (a) \Rightarrow P\ (a)$	**3,4 $\Rightarrow$I**
6.	$P\ (a) \Rightarrow (Q\ (a) \Rightarrow P\ (a))$	**2,5 $\Rightarrow$I**
7.	$\forall a \in S, P\ (a) \Rightarrow (Q\ (a) \Rightarrow P\ (a))$	**1,6 $\forall$I**

■

Problem

(5.9 a) Give proof for

$$(\forall x \in S, Q\ (x)) \Rightarrow (\forall y \in S, P\ (y) \Rightarrow Q\ (y))$$

by natural deduction and a λP derivation.

Solution.

Natural Deduction.

1. Assume $\forall x \in S, Q(x)$
2. Let $y \in S$
3. Assume $P(y)$
4. $Q(y)$ **1,2 $\forall$E**
5. $P(y) \Rightarrow Q(y)$ **3,4 $\Rightarrow$I**
6. $\forall y \in S, P(y) \Rightarrow Q(y)$ **2,5 $\forall$I**
7. $(\forall x \in S, Q(x) \Rightarrow (\forall y \in S, P(y) \Rightarrow Q(y))$ **1,6 $\Rightarrow$I**

$\blacksquare$

λP Derivation. Corresponding type is

$$S : *, P : S \to *, Q : S \to * \vdash (\Pi\, x : S \,.\, Q\, x) \to (\Pi\, y : S \,.\, P\, y \to Q\, y)$$

1. $S : *$
2. $P : S \to *$
3. $Q : S \to *$
4. $a : \Pi\, x : S \,.\, Q\, x$
5. $y : S$
6. $z : P\, y$
7. $a\, y : Q\, y$ **4,5 App**
8. $\lambda z : P\, y \,.\, a\, y : P\, y \to Q\, y$ **7 Abst**
9. $\lambda y : S \,.\lambda z : P\, y \,.\, a\, y : \Pi\, y : S \,.\, P\, y \to Q\, y$ **7 Abst**
10. $\lambda a : \Pi\, x : S \,.\, Q\, x \,.\lambda y : S \,.\lambda z : P\, y \,.\, a\, y$ $: (\Pi\, x : S \,.\, Q\, x) \to (\Pi\, y : S \,.\, P\, y \to Q\, y)$ **7 Abst**

$\blacksquare$

Problem

(5.9 b) Give proof for

$$\forall x \in S, (P(x) \Rightarrow Q(x)) \Rightarrow ((\forall y \in S, P(y)) \Rightarrow (\forall z \in S, Q(z)))$$

by natural deduction and a λP derivation

Solution.

Natural Deduction.

1. Assume $\forall x \in S, (P(x) \Rightarrow Q(x))$

2. Assume $\forall y \in S, P\,(y)$
3. Let $z \in S$
4. $P\,(z)$ **2,3 $\forall$E**
5. $P\,(z) \Rightarrow Q\,(z)$ **1,3 $\forall$E**
6. $Q\,(z)$ **5,4 $\Rightarrow$E**
7. $\forall z \in S, Q\,(z)$ **3,6 $\forall$I**
8. $\forall y \in S, P\,(y) \Rightarrow (\forall z \in S, Q\,(z))$ **2,7 $\forall$I**
9. $\forall x \in S, (P\,(x) \Rightarrow Q\,(x)) \Rightarrow ((\forall y \in S, P\,(y)) \Rightarrow (\forall z \in S, Q\,(z)))$ **1,8 $\forall$I**

■

λP Derivation. Corresponding type is

$$S : *, P : S \to *, Q : S \to *$$
$$\vdash (\Pi\, x : S\, .\, P\, x \to Q\, x) \to (\Pi\, y : S\, .\, P\, y) \to (\Pi\, z : S\, .\, Q\, z)$$

1. $S : *$
2. $P : S \to *$
3. $Q : S \to *$
4. $a : \Pi\, x : S\, .\, P\, x \to Q\, x$
5. $b : \Pi\, y : S\, .\, P\, y$
6. $z : S$
7. $b\, z : P\, z$ **5,6 App**
8. $a\, z : P\, z \to Q\, z$ **4,6 App**
9. $a\, z\, (b\, z) : Q\, z$ **8,7 App**
10. $\lambda z : S\, .\, a\, z\, (b\, z) : \Pi\, z : S\, .\, Q\, z$ **9 Abst**
11. $\lambda b : (\Pi\, y : S\, .\, P\, y).\lambda z : S\, .\, a\, z\, (b\, z)$ $: (\Pi\, y : S\, .\, P\, y) \to \Pi\, z : S\, .\, Q\, z$ **10 Abst**
12. $\lambda a : (\Pi\, x : S\, .\, P\, x \to Q\, x).\lambda b : (\Pi\, y : S\, .\, P\, y).$ $\lambda z : S\, .\, a\, z\, (b\, z)$ $: (\Pi\, x : S\, .\, P\, x \to Q\, x) \to$ $(\Pi\, y : S\, .\, P\, y) \to \Pi\, z : S\, .\, Q\, z$ **10 Abst**

■

Problem

(5.10) Given a context

$$\Gamma \equiv S : *, P : S \to *, f : S \to S, g : S \to S,$$
$$u : \Pi\, x : S\, .(P\,(f\,x) \to P\,(g\,x)),$$
$$v : \Pi\, x, y : S\, .((P\,x \to P\,y) \to P\,(f\,x))$$

Let

$$M \equiv \lambda x : S\, .\, v\,(f\,x)(g\,x)(u\,x)$$

Type M under Γ.

Solution.

1.	$x : S$	
2.	$\quad f : S \to S$	
3.	$\quad f\,x : S$	**2,1 App**
4.	$\quad g : S \to S$	
5.	$\quad g\,x : S$	**4,1 App**
6.	$\quad u : \Pi\, x : S\, .(P\,(f\,x) \to P\,(g\,x))$	
7.	$\quad u\,x : P\,(f\,x) \to P\,(g\,x)$	**6,1 App**
8.	$\quad v : \Pi\, x, y : S\, .((P\,x \to P\,y) \to P\,(f\,x))$	
9.	$\quad v\,(f\,x) : \Pi\, y : S\, .((P\,(f\,x) \to P\,y) \to P\,(f\,(f\,x)))$	**8,3 App**
10.	$\quad v\,(f\,x)(g\,x) : (P\,(f\,x) \to P\,(g\,x)) \to P\,(f\,(f\,x))$	**9,5 App**
11.	$\quad v\,(f\,x)(g\,x)(u\,x) : P\,(f\,(f\,x))$	**10,7 App**
12.	$\lambda x : S\, .\, v\,(f\,x)(g\,x)(u\,x) : S \to P\,(f\,(f\,x))$	**11 Abst**

Problem

(5.11) Let S be a set, with Q and R relations on $S \times S$, and let f and g be functions from S to S. Assume

$$\forall x, y \in S\ (Q\ (x, f\ (y)) \Rightarrow Q\ (g\ (x), y))$$
$$\forall x, y \in S\ (Q\ (x, f\ (y)) \Rightarrow R\ (x, y))$$
$$\forall x \in S\ (Q\ (x, f\ (f\ (x))))$$

Prove that

$$\forall x \in S, R\ (g\ (g\ (x)), g\ (x))$$

By giving a context Γ and finding a term M such that

$$\Gamma \vdash M : \Pi\, x : S\ .\ R\ (g\ (g\ x))(g\ x)$$

Solution. Context Γ is as follows:

$$\begin{aligned}\Gamma \equiv\ & S : *, f : S \to S, g : S \to S\\ & Q : S \to S \to *, R : S \to S \to *\\ & A : \Pi\, x, y : S\ .(Q\ x\ (f\ y) \to Q\ (g\ x)\ y),\\ & B : \Pi\, x, y : S\ .(Q\ x\ (f\ y) \to R\ x\ y)\\ & C : \Pi\, x : S\ .\ Q\ x\ (f\ (f\ x))\end{aligned}$$

Derivation.

1.	$S : *, f : S \to S, g : S \to S$	
2.	$Q : S \to S \to *, R : S \to S \to *$	
3.	$A : \Pi\, x, y : S\ .(Q\ x\ (f\ y) \to Q\ (g\ x)\ y)$	
4.	$B : \Pi\, x, y : S\ .(Q\ x\ (f\ y) \to R\ x\ y)$	
5.	$C : \Pi\, x : S\ .\ Q\ x\ (f\ (f\ x))$	
6.	$x : S$	
7.	$g\ x : S$	**1,6 App**
8.	$C\ (g\ x) : Q\ (g\ x)(f\ (f\ (g\ x)))$	**5,7 App**
9.	$f\ (g\ x) : S$	**1,7 App**
10.	$A\ (g\ x) : \Pi\, y : S\ .(Q\ (g\ x)(f\ y)) \to (Q\ (g\ (g\ x))\ y)$	**3,7 App**
11.	$A\ (g\ x)(f\ (g\ x))$ $: (Q\ (g\ x)(f\ (f\ (g\ x))))$ $\to (Q\ (g\ (g\ x))(f\ (g\ x)))$	**10,9 App**

12. $A\,(g\,x)(f\,(g\,x))(C\,(g\,x))$
$\quad : (Q\,(g\,(g\,x))(f\,(g\,x)))$ **11,8 App**

13. $g\,(g\,x) : S$ **1,7 App**

14. $B\,(g\,(g\,x))$
$\quad : \Pi\,y : S\,.(Q\,(g\,(g\,x))(f\,y)$
$\qquad \to R\,(g\,(g\,x))\,y)$ **4,13 App**

15. $B\,(g\,(g\,x))(g\,x)$
$\quad : (Q\,(g\,(g\,x))(f\,(g\,x))) \to (R\,(g\,(g\,x))(g\,x))$ **14,7 App**

16. $B\,(g\,(g\,x))(g\,x)(A\,(g\,x)(f\,(g\,x))(C\,(g\,x)))$
$\quad : (R\,(g\,(g\,x))(g\,x))$ **15,12 App**

17.

$\lambda x : S\,.\,B\,(g\,(g\,x))(g\,x)(A\,(g\,x)(f\,(g\,x))(C\,(g\,x)))$
$\quad : \Pi\,x : S\,.(R\,(g\,(g\,x))(g\,x))$ **17 Abst**

■

Problem

(5.12 a) In λP, consider the context

$$\begin{aligned}\Gamma \equiv\ & S : *, R : S \to S \to *,\\ & u : \Pi\,x, y : S\,.\,R\,x\,y \to R\,y\,x\\ & v : \Pi\,x, y, z : S\,.\,R\,x\,y \to R\,x\,z \to R\,y\,z\end{aligned}$$

Show that R is reflexive over $S \times S$. That is, construct M such that

$$\Gamma \vdash M : \Pi\,x, y : S\,.\,R\,x\,y \to R\,x\,x$$

Solution.

Proof.

1. $S : *, R : S \to S \to *$
2. $A : \Pi\,u, v : S\,.\,R\,u\,v \to R\,v\,u$
3. $B : \Pi\,u, v, w : S\,.\,R\,u\,v \to R\,u\,w \to R\,v\,w$
4. $x : S$
5. $y : S$
6. $h : R\,x\,y$

7. $B\,x : \Pi\, v, w : S\,.\, R\,x\,v \to R\,v\,w \to R\,x\,w$ **3,4 App**

8. $B\,x\,y : \Pi\, w : S\,.\, R\,x\,y \to R\,y\,w \to R\,x\,w$ **7,5 App**

9. $B\,x\,y\,x : R\,x\,y \to R\,y\,x \to R\,x\,x$ **7,5 App**

10. $A\,x : \Pi\, v : R\,x\,v \to R\,v\,x$ **2,4 App**

11. $A\,x\,y : R\,x\,y \to R\,y\,x$ **10,5 App**

12. $A\,x\,y\,h : R\,y\,x$ **11,6 App**

13. $B\,x\,y\,x\,h : R\,y\,x \to R\,x\,x$ **9,6 App**

14. $B\,x\,y\,x\,h\,(A\,x\,y\,h) : R\,x\,x$ **13,12 App**

15. $\lambda h : R\,x\,y\,.\, B\,x\,y\,x\,h\,(A\,x\,y\,h) : R\,x\,y \to R\,x\,x$ **14 Abst**

16. $\lambda y : S\,.\lambda h : R\,x\,y\,.\, B\,x\,y\,x\,h\,(A\,x\,y\,h)$
$: \Pi\, y : S\,.\, R\,x\,y \to R\,x\,x$ **15 Abst**

17. $\lambda x, y : S\,.\lambda h : R\,x\,y\,.\, B\,x\,y\,x\,h\,(A\,x\,y\,h)$
$: \Pi\, x, y : S\,.\, R\,x\,y \to R\,x\,x$ **16 Abst**

■

Problem

(5.12 b) Given the context Γ in 5.12 a, prove transitivity of R by constructing M such that

$$\Gamma \vdash M : \Pi\, x, y, z : S\,.\, R\,x\,y \to R\,y\,z \to R\,x\,z$$

Solution.

Proof.

1. $S : *, R : S \to S \to *$
2. $A : \Pi\, u, v : S\,.\, R\,u\,v \to R\,v\,u$
3. $B : \Pi\, u, v, w : S\,.\, R\,u\,v \to R\,u\,w \to R\,v\,w$
4. $x : S$
5. $y : S$
6. $z : S$
7. $h : R\,x\,y$
8. $r : R\,y\,z$
9. $A\,x : \Pi\, v : S\,.\, R\,x\,v \to R\,v\,x$ **2,4 App**
10. $A\,x\,y : R\,x\,y \to R\,y\,x$ **9,5 App**
11. $A\,x\,y\,h : R\,y\,x$ **10,7 App**

12. $B\,y : \Pi\, v, w : S\,.\, R\,y\,v \to R\,y\,w \to R\,v\,w$ **3,5 App**

13. $B\,y\,x : \Pi\, w : S\,.\, R\,y\,x \to R\,y\,w \to R\,x\,w$ **12,4 App**

14. $B\,y\,x\,z : R\,y\,x \to R\,y\,z \to R\,x\,z$ **12,4 App**

15. $B\,y\,x\,z\,(A\,x\,y\,h) : R\,y\,z \to R\,x\,z$ **14,11 App**

16. $B\,y\,x\,z\,(A\,x\,y\,h)\,r : R\,x\,z$ **15,8 App**

17. $\lambda r : R\,y\,z\,.\, B\,y\,x\,z\,(A\,x\,y\,h)\,r : R\,y\,z \to R\,x\,z$ **16 Abst**

18. $\lambda h : R\,x\,y\,.\lambda r : R\,y\,z\,.\, B\,y\,x\,z\,(A\,x\,y\,h)\,r$
$\quad : R\,x\,y \to R\,y\,z \to R\,x\,z$ **17 Abst**

19. $\lambda z : S\,.\lambda h : R\,x\,y\,.\lambda r : R\,y\,z\,.\, B\,y\,x\,z\,(A\,x\,y\,h)\,r$
$\quad : \Pi\, z : S\,.\, R\,x\,y \to R\,y\,z \to R\,x\,z$ **18 Abst**

20. $\lambda y, z : S\,.\lambda h : R\,x\,y\,.\lambda r : R\,y\,z\,.\, B\,y\,x\,z\,(A\,x\,y\,h)\,r$
$\quad : \Pi\, y, z : S\,.\, R\,x\,y \to R\,y\,z \to R\,x\,z$ **19 Abst**

21. $\lambda x, y, z : S\,.\lambda h : R\,x\,y\,.\lambda r : R\,y\,z\,.\, B\,y\,x\,z\,(A\,x\,y\,h)\,r$
$\quad : \Pi\, x, y, z : S\,.\, R\,x\,y \to R\,y\,z \to R\,x\,z$ **20 Abst**

■

—

Completed Dec 22 6:51 pm.