

EXERCISES

CHAPTER 5

SEAN LI ¹

1. Reducted

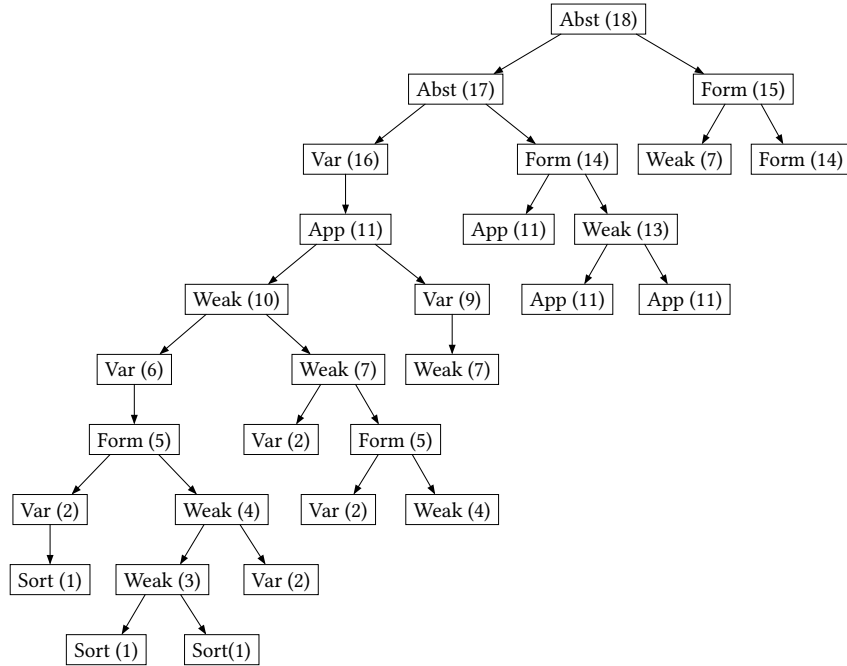
Definition Some rules for reference.

$$\begin{array}{c} \frac{}{\emptyset \vdash * : \square} \text{Sort} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \text{Var} \quad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \text{Weak} \\[10pt] \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A . B : s} \text{Form} \quad \frac{\Gamma \vdash M : \Pi x : A . B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B[x := N]} \text{App} \\[10pt] \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A . B : s}{\Gamma \vdash \lambda x : A . M : \Pi x : A . B} \text{Abst} \\[10pt] \frac{\Gamma \vdash A : B \quad B \stackrel{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \text{Conv} \end{array}$$

Problem

(5.1) Give a diagram of the tree corresponding to the complete tree derivation of line 18 of Section 5.3 (P 107)

Solution.



Problem

(5.2) Give a complete λP derivation of

$$S : * \vdash S \rightarrow S \rightarrow * : \square$$

In tree format and flag format.

Solution.

Tree Derivation.

$$\begin{array}{c}
 (4) \frac{\vdash * : \square \quad \vdash * : \square}{S : * \vdash * : \square} \text{Weak} \quad (3) S : * \vdash S : * \\
 (6) \frac{S : * \vdash * : \square \quad (3) S : * \vdash S : *}{S : *, x : S \vdash * : \square} \text{Weak} \\
 (7) \frac{(3) S : * \vdash S : * \quad S : *, x : S \vdash * : \square}{S : * \vdash S \rightarrow * : \square}
 \end{array}$$

$$\begin{array}{c}
(3) \frac{\vdash * : \square}{S : * \vdash S : *} \text{Var} \quad (9) \frac{(7) S : * \vdash S \rightarrow * : \square \quad (3) S : * \vdash S : *}{S : *, x : S \vdash S \rightarrow * : \square} \text{Weak} \\
\hline
S : * \vdash S \rightarrow S \rightarrow * : \square \text{Form}
\end{array}$$

■

Flag Derivation.

1.	$* : \square$	Sort
2.	$S : *$	
3.	$S : *$	1 Var
4.	$* : \square$	1,1 Weak
5.	$x : S$	
6.	$\boxed{* : \square}$	4,3 Weak
7.	$S \rightarrow * : \square$	3,6 Form
8.	$x : S$	
9.	$\boxed{S \rightarrow * : \square}$	7,3 Weak
10.	$\boxed{S \rightarrow S \rightarrow * : \square}$	3,9 Form

■

Problem

(5.3) Derive

$$S : *, Q : S \rightarrow S \rightarrow * \vdash \Pi x : S . \Pi y : S . Q x y : *$$

Solution.

1.	$* : \square$	Sort
2.	$S : *$	
3.	$S : *$	1 Var
4.	$* : \square$	1,1 Weak
5.	$x : S$	
6.	$\boxed{* : \square}$	4,3 Weak
7.	$S \rightarrow * : \square$	3,6 Form
8.	$x : S$	
9.	$\boxed{S \rightarrow * : \square}$	7,3 Weak
10.	$S \rightarrow S \rightarrow * : \square$	3,9 Form
11.	$Q : S \rightarrow S \rightarrow *$	

12.	$Q : S \rightarrow S \rightarrow *$	10 Var
13.	$S : *$	3,10 Weak
14.	$* : \square$	4,10 Weak
15.	$x : S$	
16.	$* : \square$	14,13 Weak
17.	$S : *$	13,13 Weak
18.	$x : S$	13 Var
19.	$Q : S \rightarrow S \rightarrow *$	12,13 Weak
20.	$y : S$	
21.	$y : S$	17 Var
22.	$Q : S \rightarrow S \rightarrow *$	19,17 Weak
23.	$x : S$	18,17 Weak
24.	$Q x : S \rightarrow *$	22,23 App
25.	$Q x y : *$	24,21 App
26.	$\Pi y : S . Q x y : *$	17,25 Form
27.	$\Pi x : S . \Pi y : S . Q x y : *$	13,26 Form

Problem

(5.4) Prove that $*$ is the only valid kind in λP .

Solution.

Proof. The only possible way to construct a new kind is through the Form rule and the Sort axiom. Because we are trying to construct a kind, s here stands for \square .

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : \square}{\Gamma \vdash \Pi x : A . B : \square} \text{Form}$$

One could only construct new kinds with kinds, which requires $A : \square$ and $B : \square$. This contradicts with $A : *$. ■

Problem

(5.5) Prove that $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ is a tautology by given a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash A \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$x : A$	
4.	$y : A \rightarrow B$	
5.	$y x : B$	4,3 App
6.	$\lambda y : A \rightarrow B . y x : (A \rightarrow B) \rightarrow B$	5 Abst
7.	$\lambda x : A . \lambda y : A \rightarrow B . y x : A \rightarrow (A \rightarrow B) \rightarrow B$	5 Abst

■

Problem

(5.6 a) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ a tautology using natural deduction.

Solution.

Proof.

1.	Assume $A \Rightarrow (A \Rightarrow B)$	
2.	$A \Rightarrow (A \Rightarrow B)$	
3.	Assume A	
4.	A	
5.	$A \Rightarrow B$	2,4 \Rightarrow E
6.	B	5,4 \Rightarrow E
7.	$A \Rightarrow B$	3,6 \Rightarrow I
8.	$(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$	1,7 \Rightarrow I

■

Problem

(5.6 b) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ using a shorthand λP derivation

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$x : A \rightarrow A \rightarrow B$	
4.	$y : A$	
5.	$x y : A \rightarrow B$	3,4 App
6.	$x y y : B$	5,4 App
7.	$\lambda y : A . x y y : A \rightarrow B$	6 Abst
8.	$\lambda x : A \rightarrow A \rightarrow B . \lambda y : A . x y y : (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$	7 Abst

■

Problem

(5.7 a) Proof $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of A , B , and C is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$C : *$	
4.	$x : A \rightarrow B$	
5.	$y : B \rightarrow C$	
6.	$a : A$	
7.	$x a : B$	4,6 App
8.	$y (x a) : C$	5,7 App

9.				$\lambda a : A . y (x z) : A \rightarrow C$	8 Abst
10.				$\lambda y : B \rightarrow C . \lambda a : A . y (x z) : (B \rightarrow C) \rightarrow A \rightarrow C$	9 Abst
11.				$\lambda x : A \rightarrow B . \lambda y : B \rightarrow C . \lambda a : A . y (x z)$ $: (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C$	10 Abst

■

Problem

(5.7 b) Proof $((A \Rightarrow B) \Rightarrow A) \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$x : (A \rightarrow B) \rightarrow A$	
4.	$y : A \rightarrow B$	
5.	$xy : A$	3,4 App
6.	$y(xy) : B$	4,5 App
7.	$\lambda y : A \rightarrow B . y(xy) : (A \rightarrow B) \rightarrow B$	6 Abst
8.	$\lambda x : (A \rightarrow B) \rightarrow A . \lambda y : A \rightarrow B . y(xy)$ $: ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B$	7 Abst

■

Problem

(5.7 c) Proof $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of A , B , and C is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$C : *$	
4.	$x : A \rightarrow B \rightarrow C$	
5.	$y : A \rightarrow B$	
6.	$a : A$	
7.	$x a : B \rightarrow C$	4,6 App
8.	$y a : B$	5,6 App
9.	$x a (y a) : C$	7,8 App
10.	$\lambda a : A . x a (y a) : A \rightarrow C$	9 Abst
11.	$\lambda y : A \rightarrow B . \lambda a : A . x a (y a) : (A \rightarrow B) \rightarrow A \rightarrow C$	10 Abst
12.	$\lambda x : A \rightarrow B \rightarrow C . \lambda y : A \rightarrow B . \lambda a : A . x a (y a)$ $: (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$	10 Abst

■

Problem

(5.8 a) Let $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *$, find an inhabitant of

$$\Pi x : S . P x \rightarrow Q x \rightarrow P x$$

with respect to Γ and give a shorthand λP derivation

Solution.

1.	$S : *$	
2.	$P : S \rightarrow *$	
3.	$Q : S \rightarrow *$	
4.	$x : S$	
5.	$a : P x$	
6.	$b : Q x$	
7.	$a : P x$	2,4 App

8.				$\lambda b : Q\ x . S : Q\ x \rightarrow P\ x$	7 Abst
9.				$\lambda a : P\ x . \lambda b : Q\ x . a : P\ x \rightarrow Q\ x \rightarrow P\ x$	8 Abst
10.				$\lambda x : S . \lambda a : P\ x . \lambda b : Q\ x . a$ $: \Pi x : S . P\ x \rightarrow Q\ x \rightarrow P\ x$	9 Abst

Problem

(5.8 b) Let $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *$, find an inhabitant of

$$\Pi x : S . P\ x \rightarrow Q\ x \rightarrow P\ x$$

By proving the corresponding proposition in natural deduction.

Solution. The corresponding proposition and premises are

$$\frac{S \in \text{Set} \quad P : S \rightarrow \text{Prop} \quad Q : S \rightarrow \text{Prop}}{\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a))}$$

Proof.

1. Let $a \in S$
2. Assume $P(a)$
3. Assume $Q(a)$
4. Assume $P(a)$
5. $Q(a) \Rightarrow P(a)$ **3,4 \Rightarrow I**
6. $P(a) \Rightarrow (Q(a) \Rightarrow P(a))$ **2,5 \Rightarrow I**
7. $\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a))$ **1,6 \forall I**

■