Excercises

Chapter 5

Sean Li [1]

1. Reducted

Reference - Calculus of Constructions

$$\frac{}{\emptyset \vdash * : \square}\,\text{Sort} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}\,\text{Var} \qquad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}\,\text{Weak}$$

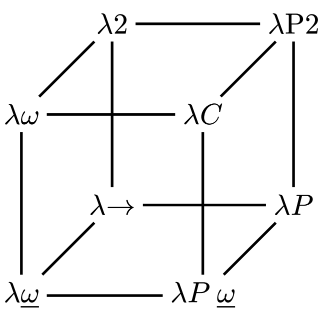
$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi\, x : A \,.\, B : s_2}\,\text{Form}$$

$$\frac{\Gamma \vdash M : \Pi\, x : A \,.\, B \quad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B\,[x := N]}\,\text{App}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi\, x : A \,.\, B : s}{\Gamma \vdash \lambda x : A \,.\, M : \Pi\, x : A \,.\, B}\,\text{Abst}$$

$$\frac{\Gamma \vdash A : B \quad B \overset{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'}\,\text{Conv}$$

The $\boldsymbol{\lambda}$-Cube



$\boldsymbol{\lambda\to}$	$(*,*)$			
$\boldsymbol{\lambda 2}$	$(*,*)$	$(\square,*)$		
$\boldsymbol{\lambda\underline{\omega}}$	$(*,*)$		$(\square,\square)$	
$\boldsymbol{\lambda P}$	$(*,*)$			$(*,\square)$
$\boldsymbol{\lambda\omega}$	$(*,*)$	$(\square,*)$	$(\square,\square)$	
$\boldsymbol{\lambda 2}$	$(*,*)$	$(\square,*)$		$(*,\square)$
$\boldsymbol{\lambda\underline{\omega}}$	$(*,*)$		$(\square,\square)$	$(*,\square)$
$\boldsymbol{\lambda P}$	$(*,*)$	$(\square,*)$	$(\square,\square)$	$(*,\square)$

Problem

(6.1 a) Give a complete derivation in tree format showing that

$$\bot \equiv \Pi\alpha : * .\alpha$$

is legal in λC.

Solution. Here we will show that there exists $s \in$ sort and Γ such that $\Gamma \vdash \bot : s$.

Proof.

$$\dfrac{\vdash * : \square \qquad \dfrac{\vdash * : \square}{\alpha : * \vdash \alpha : *}\,\text{Var}}{\vdash \Pi\alpha : * .\alpha : *}\,\text{Form}$$

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Problem

(6.1 a) Give a complete derivation in tree format showing that $\bot \to \bot$ is legal in λC where

$$\bot \equiv \Pi\alpha : * .\alpha$$

Solution. Here we will show that there exists $s \in$ sort and Γ such that $\Gamma \vdash \bot \to \bot : s$.

Proof.

$$\dfrac{(6.1\text{ a})\dfrac{}{\vdash \bot : *} \qquad \dfrac{(6.1\text{ a})\dfrac{}{\vdash \bot : *} \qquad (6.1\text{ a})\dfrac{}{\vdash \bot : *}}{x : \bot \vdash \bot : *}\,\text{Weak}}{\vdash \Pi\, x : \bot.\bot : *}\,\text{Form}$$

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Problem

(6.1 c) To which systems of the λ-cube does $\bot$ belong? And $\bot \to \bot$?

Solution. The set of (s_1, s_2) pairs in formation rules of the derivation of $\bot$ is $\{(\square, *)\}$. The minimal system corresponding is $\lambda 2$. The same for $\bot \to \bot$. Therefore $\bot$ and $\bot \to \bot$ belongs to $\lambda 2$, $\lambda\omega$, λP and λC.

Problem

(6.2) Given context $\Gamma \equiv S : *, P : S \to *, A : *$. Prove by means of a flag derivation that the following expression is inhabited in λC with respect to Γ:

$$(\Pi\, x : S\,.(A \to P\, x)) \to A \to \Pi\, y : S\,.\,P\, y$$

Solution. The inhabitant is

$$M \equiv \lambda u : (\Pi\, x : S\,.(A \to P\, x)).\lambda v : A\,.\lambda y : S\,.\,u\,y\,v$$

Proof.

1.	$S : *, P : S \to *, A : *$	
2.	$\quad u : \Pi\, x : S\,.(A \to P\, x)$	
3.	$\quad\quad v : A$	
4.	$\quad\quad\quad y : S$	
5.	$\quad\quad\quad\quad u\, y : A \to P\, y$	**2,4 App**
6.	$\quad\quad\quad\quad u\, y\, v : P\, y$	**5,3 App**
7.	$\quad\quad\quad \lambda y : S\,.\,u\,y\,v : \Pi\, y : S\,.\,P\, y$	**6 Abst**
8.	$\quad\quad \lambda v : A\,.\lambda y : S\,.\,u\,y\,v : A \to \Pi\, y : S\,.\,P\, y$	**7 Abst**
9.	$\quad \lambda u : \Pi\, x : S\,.(A \to P\, x).\lambda v : A\,.\lambda y : S\,.\,u\,y\,v$ $: \Pi\, x : S\,.(A \to P\, x) \to A \to \Pi\, y : S\,.\,P\, y$	**8 Abst**

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Problem

(6.3 a) Let $\mathcal{J}$ be a judgement

$$\mathcal{J} \equiv S : *, P : S \to * \vdash \lambda x : S\,.(P\, x \to \bot) : S \to *$$

Derive $\mathcal{J}$ in λC with shorthand flag notation.

Solution.

1.	$S : *, P : S \to *$	
2.	$\quad x : S$	
3.	$\quad\quad P\, x : *$	**1,2 App**
4.	$\quad\quad \bot : *$	**Weak from 6.1 a**
5.	$\quad\quad P\, x \to \bot : *$	**3,4 Form**

6. $\lambda x : S \,.\, P\,x \to \bot : S \to *$ **5 Abst**

Problem

(6.3 b) Deternmine the (s_1, s_2) pairs corresponding to all Π abstractions occuring in $\mathcal{J}$.

Solution.

Abstraction	**Line Number**	(s_1, s_2)
$P : S \to *$	1	$(*, \square)$
$\bot \equiv \Pi\alpha : * \,.\alpha$	4	$(\square, *)$
$P\,x \to \bot$	5	$(\square, *)$
$\lambda x : S \,.\, P\,x \to \bot : S \to *$	6	$(*, \square)$

Problem

(6.3 c) What is the 'smallest' system in the λ-cube to which $\mathcal{J}$ belongs?

Solution. There are $(*, *)$ – $\lambda\!\to$ pairs, $(*, \square)$ – λP pairs, and $(\square, *)$ – $\lambda 2$. Therefore the minimal system $\mathcal{J}$ belongs to is λP2.