Exercises

Chapter 7

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1. Redacted

Reference - Propositional Logic in λC

$$\frac{A \quad \neg A}{\bot}\ \bot\text{I or } \neg\text{E} \qquad \frac{\bot}{A}\ \bot\text{E} \qquad \frac{A \quad B}{A \wedge B}\ \wedge\text{I} \qquad \frac{A \wedge B}{A}\ \wedge\text{EL}$$

$$\frac{A \wedge B}{B}\ \wedge\text{ER} \qquad \frac{a}{a \vee b}\ \vee\text{IL} \qquad \frac{b}{a \vee b}\ \vee\text{IR} \qquad \frac{A \Rightarrow B \quad A}{B}\ \Rightarrow\text{E}$$

$$\frac{A \vee B \quad A \to C \quad B \to C}{C}\ \vee\text{E} \qquad \frac{a \in S \quad P\,(a)}{\exists a \in S, P\,(a)}\ \exists\text{I}$$

$$\frac{\begin{array}{ll}1. & A\\ 2. & \vdots\\ 3. & \bot\end{array}}{\neg A}\ \neg\text{I} \qquad \frac{\begin{array}{ll}1. & A\\ 2. & \vdots\\ 3. & B\end{array}}{A \Rightarrow B}\ \Rightarrow\text{I} \qquad \frac{\begin{array}{ll}1. & a \in S\\ 2. & \vdots\\ 3. & P\,(a)\end{array}}{\forall a \in S, P\,(a)}\ \forall I$$

$$\frac{\exists x \in S, P\,(x) \quad \forall x \in S, (P\,(x) \Rightarrow A)}{A}\ \exists\text{E} \qquad \frac{a \in S \quad \forall x \in S, P\,(x)}{P\,(a)}\ \forall\text{E}$$

$$\frac{}{\neg\neg A \Rightarrow A}\ \text{DN (Classical)} \qquad \frac{}{A \vee \neg A}\ \text{ET (Classical)}$$

Reference - 2nd Encoding for Propositional Logic

Proposition	**Minimal Propositional Logic**
$\bot$	$\forall A, A$
$A \Rightarrow B$	$A \Rightarrow B$
$\neg A$	$A \Rightarrow \bot$
$A \wedge B$	$\forall C, (A \Rightarrow B \Rightarrow C) \Rightarrow C$
$A \vee B$	$\forall C, (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$
$\forall a \in S, P\,(a)$	$\forall a \in S\,.\,P\,(a)$
$\exists a \in S, P\,(a)$	$\forall \alpha, (\forall a \in S, (P\,(a) \Rightarrow \alpha)) \Rightarrow \alpha$

Problem

(7.1 a) Prove in natural deduction and λC the tautology

$$B \Rightarrow (A \Rightarrow B)$$

Solution.

Natural Deduction.

1. B
2. $\quad A$
3. $\quad\quad B$
4. $\quad A \Rightarrow B$ $\quad \Rightarrow\mathbf{I}$
5. $B \Rightarrow (A \Rightarrow B)$ $\quad \Rightarrow\mathbf{I}$

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λC. Assuming context $\Gamma \equiv A : *, B : *$. By the PAT paradigm the proof is equivalent to an inhabitant of $B \to A \to B$.

1. $A : *, B : *$
2. $\quad x : B$
3. $\quad\quad y : A$
4. $\quad\quad\quad x : B$ $\quad$ **Weak**
5. $\quad\quad \lambda y : A\,.\,x : A \to B$ $\quad$ **4 Abst**
6. $\quad \lambda x : B\,.\lambda y : A\,.\,x : B \to A \to B$ $\quad$ **5 Abst**

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Problem

(7.1 b) Prove in natural deduction and λC the tautology

$$\neg A \Rightarrow (A \Rightarrow B)$$

Solution.

Natural Deduction.

1.	$\neg A$	
2.	$\quad A$	
3.	$\quad\quad \neg A$	
4.	$\quad\quad A$	
5.	$\quad\quad \bot$	$\bot$**I**
6.	$\quad\quad B$	$\bot$**E**
7.	$\quad A \Rightarrow B$	$\Rightarrow$**I**
8.	$\neg A \Rightarrow (A \Rightarrow B)$	$\Rightarrow$**I**

■

λC. Assuming context $\Gamma \equiv A : *, B : *$. By the PAT paradigm the proof is equivalent to an inhabitant of $(A \to \bot) \to A \to B$.

1.	$A : *, B : *$	
2.	$\quad x : \neg A$	
3.	$\quad\quad y : A$	
4.	$\quad\quad\quad x\ y : \Pi\alpha : * .\alpha$	**2,3 App (Neg Elim)**
5.	$\quad\quad\quad x\ y\ B : B$	**4,1 App (Ex Falso)**
6.	$\quad\quad \lambda y : A\ .\ x\ y\ B : A \to B$	**5 Abst**
7.	$\quad \lambda x : \neg A\ .\lambda y : A\ .\ x\ y\ B : \neg A \to A \to B$	**6 Abst**

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Problem

(7.1 c) Prove in natural deduction and λC the tautology

$$(A \Rightarrow \neg B) \Rightarrow ((A \Rightarrow B) \Rightarrow \neg A)$$

Solution.

Natural Deduction.

1.	$A \Rightarrow \neg B$	
2.	$\quad A \Rightarrow B$	
3.	$\quad\quad A$	
4.	$\quad\quad\quad \neg B$	**1,3 ⇒E**
5.	$\quad\quad\quad B$	**2,3 ⇒E**
6.	$\quad\quad\quad \bot$	**5,4 ⊥I**
7.	$\quad\quad \neg A$	**3,6 ¬I**
8.	$\quad (A \Rightarrow B) \Rightarrow \neg A$	**2,7 ⇒I**
9.	$(A \Rightarrow \neg B) \Rightarrow ((A \Rightarrow B) \Rightarrow \neg A)$	**1,8 ⇒I**

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λC. Assuming context $\Gamma \equiv A : *, B : *$. The proof should be equivalent to an inhabitant of $(A \to B \to \bot) \to (A \to B) \to A \to \bot$.

1.	$A : *, B : *$	
2.	$\quad h : A \to \neg B$	
3.	$\quad\quad q : A \to B$	
4.	$\quad\quad\quad a : A$	
5.	$\quad\quad\quad\quad q\ a : B$	**3,4 App**
6.	$\quad\quad\quad\quad h\ a : B \to \bot$	**2,4 App**
7.	$\quad\quad\quad\quad h\ a\ (q\ a) : \bot$	**6,5 App (Neg Elim)**
8.	$\quad\quad\quad \lambda a : A\ .\ h\ a\ (q\ a) : \neg A$	**7 Abst (Neg Intro)**
9.	$\quad\quad \lambda q : A \to B\ .\lambda a : A\ .\ h\ a\ (q\ a) : A \to B \to \neg A$	**8 Abst**
10.	$\quad \lambda h : A \to \neg B\ .\lambda q : A \to B\ .\lambda a : A\ .\ h\ a\ (q\ a)$ $\quad : (A \to \neg B) \to A \to B \to \neg A$	**9 Abst**

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Problem

(7.1 d) Prove in natural deduction and λC the tautology

$$\neg(A \Rightarrow B) \Rightarrow \neg B$$

Solution.

Natural Deduction.

1.	$\neg(A \Rightarrow B)$	
2.	$\quad B$	
3.	$\quad\quad A$	
4.	$\quad\quad\quad B$	
5.	$\quad\quad A \Rightarrow B$	**3,4 ⇒I**
6.	$\quad\quad \bot$	**5,1 ⊥I**
7.	$\quad \neg B$	**6 ¬I**
8.	$\neg(A \Rightarrow B) \Rightarrow \neg B$	**1,7 ⇒I**

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λC. Assuming context $\Gamma \equiv A : *, B : *$. The proof should be equivalent to an inhabitant of $((A \to B) \to \bot) \to B \to \bot$.

1.	$n : \neg(A \to B)$	
2.	$\quad b : B$	
3.	$\quad\quad a : A$	
4.	$\quad\quad\quad b : B$	**Weak**
5.	$\quad\quad \lambda a : A \,.\, b : A \to B$	**4 Abst**
6.	$\quad\quad n\ (\lambda a : A \,.\, b) : \bot$	**1,5 App (Neg Elim)**
7.	$\quad \lambda b : B \,.\, n\ (\lambda a : A \,.\, b) : \neg B$	**6 Abst (Neg Intro)**
8.		

$\lambda n : \neg(A \to B).\lambda b : B \,.\, n\ (\lambda a : A \,.\, b)$
$\quad : \neg(A \to B) \to \neg B$ **7 Abst**

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Problem

(7.2) Formulate the double negation law as an axiom in λC, and prove the following tautology in λC with DN.

$$(\neg A \Rightarrow A) \Rightarrow A$$

Solution. The rule

$$\frac{}{\neg\neg A \Rightarrow A}\ \text{DN-E}$$

Could be translated into lambda calculus as

$$\Pi A : * . ((A \to \bot) \to \bot) \to A$$

Proof. Assume context $\Gamma \equiv A : *$.

1.	$A : *$	
2.	$h : \neg A \to A$	
3.	$x : \neg A$	
4.	$h\ x : A$	**2,3 App**
5.	$x\ (h\ x) : \bot$	**3,4 App (Contradiction)**
6.	$\lambda x : \neg A\ .\ x\ (h\ x) : \neg\neg A$	**5 Abst (Neg Intro)**
7.	$\mathrm{DN}\ A : \neg\neg A \to A$	**1,1 App**
8.	$\mathrm{DN}\ A\ (\lambda x : \neg A\ .\ x\ (h\ x)) : A$	**App (Axiom DN)**
	$\lambda h : \neg A \to A\ .\ \mathrm{DN}\ A\ (\lambda x : \neg A\ .\ x\ (h\ x))$	
9.	$: (\neg A \to A) \to A$	**8 Abst**

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