

# EXERCISES

## CHAPTER 2

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### 1. Reducted

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**Definition** Some rules for reference.

$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash x : \sigma} \text{(T-Var)} \qquad \frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau} \text{(T-App)}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma. M : \sigma \rightarrow \tau} \text{(T-Abst)}$$

In this document, convention is that all type judgements in a proof tree, unless stated otherwise, is derived from a single context per tree.

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#### Problem

(2.1) Type the following terms

$$xxy \quad xyx \quad xyx \quad x(xy) \quad x(yx)$$

*Solution.* The first term cannot be typed.

*Proof.*  $xxy = (xx)y$ . Therefore,  $x$  is a function type, denote it as  $\tau \rightarrow \sigma$ . By the application rule, a subterm applied to  $x$  must be of  $\tau$ , which means that the application  $xx$  is not legally typed. ■

The second one is typable where  $x : \tau \rightarrow \tau \rightarrow \sigma$  and  $y : \tau$ .

1.  $x : \tau \rightarrow \tau \rightarrow \sigma \quad \neg \Gamma$

2.  $y : \tau \quad \neg \Gamma$

3.  $xy : \tau \rightarrow \sigma$       **1,2 T-App**
4.  $xyy : \sigma$       **3,2 T-App**

The third term is not typable.

*Proof.* Assume  $xyx = (xy)x$  is typable. Therefore,  $x : \tau$  where  $\tau \equiv \sigma \rightarrow \tau \rightarrow \alpha$  and  $y : \sigma$ . One can construct an infinite chain of function type by substituting  $\tau$ :  
 $\tau \equiv \sigma \rightarrow (\sigma \rightarrow (\sigma \rightarrow \dots \rightarrow \alpha) \rightarrow \alpha) \rightarrow \alpha$ . By induction, it can be proven that only lambda abstractions can construct function types, meaning that the term is of form

$$(\lambda n : \tau. \lambda m : \tau. \dots (\lambda a : \sigma. \lambda b : \sigma. \dots))y(\lambda n : \tau. \lambda m : \tau. \dots (\lambda a : \sigma. \lambda b : \sigma. \dots))$$

meaning that an infinite reduction path is needed. This is impossible in STLC. ■

The fourth type is typable where  $x : (\tau \rightarrow \tau)$  and  $y : \tau$ .

1.  $x : \tau \rightarrow \tau$        $\vdash \Gamma$
2.  $y : \tau$        $\vdash \Gamma$
3.  $xy : \tau$       **1,2 T-App**
4.  $x(xy) : \tau$       **1,3 T-App**

The fifth term is typable where  $x : (\tau \rightarrow \sigma)$  and  $y : (\tau \rightarrow \sigma) \rightarrow \tau$ :

1.  $x : \tau \rightarrow \sigma$        $\vdash \Gamma$
2.  $y : (\tau \rightarrow \sigma) \rightarrow \tau$        $\vdash \Gamma$
3.  $yx : \tau$       **2,1 T-App**
4.  $x(yx) : \sigma$       **1,3 T-App**

### Problem

(2.2) Find types for zero, one, and two

*Solution.* Term for zero is

$$\text{zero} := \lambda f x. x$$

Here  $x$  is only used as a

$$\text{zero} := \lambda f : \alpha. \lambda x : \beta. x$$

Type derivation shown as below:

1.  $f : \alpha$       **Bound**
2.  $\mid x : \beta$       **Bound**

3.  $\frac{}{x : \beta}$  **T-Var**
4.  $\frac{}{\lambda x.x : \beta \rightarrow \beta}$  **3 T-Abst**
5.  $\lambda f : \alpha.x : \beta.x : \alpha \rightarrow \beta \rightarrow \beta$  **4 T-Abst**

Term for one is

$$\text{one} := \lambda f x.f x$$

Let  $f$  be an arbitrary function type that consumes  $x$

$$\text{one} := \lambda f : \alpha \rightarrow \beta.x : \alpha.f x$$

Type derivation shown as below

1.  $f : \alpha \rightarrow \beta$  **Bound**
2.  $x : \alpha$  **Bound**
3.  $f : \alpha \rightarrow \beta$  **T-Var**
4.  $x : \alpha$  **T-Var**
5.  $f x : \beta$  **3,4 T-App**
6.  $\lambda x.f x : \alpha \rightarrow \beta$  **5 T-Abst**
7.  $\lambda f : \alpha \rightarrow \beta.x : \alpha.f x : (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$  **6 T-Abst**

Same type signatures can be given to two

$$\text{two} := \lambda f : \alpha \rightarrow \beta.\lambda x : \alpha.f f x$$

Type derivation shown as below

1.  $f : \alpha \rightarrow \beta$  **Bound**
2.  $x : \alpha$  **Bound**
3.  $f : \alpha \rightarrow \beta$  **T-Var**
4.  $x : \alpha$  **T-Var**
5.  $f x : \beta$  **3,4 T-App**
6.  $f f x : \beta$  **3,5 T-App**
7.  $\lambda x.f f x : \alpha \rightarrow \beta$  **6 T-Abst**
8.  $\lambda f : \alpha \rightarrow \beta.x : \alpha.f f x : (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$  **7 T-Abst**

## Problem

(2.3) Find types for

$$K := \lambda xy.x$$

$$S := \lambda xyz.xz(yz)$$

*Solution.* There are no occurrences of application in  $K$ 's subterms. Therefore all its binding variables could be given a simple base type.

$$K := \lambda x : \alpha. \lambda y : \beta. x$$

Type derivation shown as below

1.	$x : \alpha$	<b>Bound</b>
2.	$y : \beta$	<b>Bound</b>
3.	$x : \alpha$	<b>T-Var</b>
4.	$\lambda y : \beta. x : \beta \rightarrow \alpha$	<b>3 T-Abst</b>
5.	$\lambda x : \alpha. \lambda y : \beta. x : \alpha \rightarrow \beta \rightarrow \alpha$	<b>4 T-Abst</b>

For the  $S$  combinator, no term was applied to  $z$ . Therefore it can be given a simple base type  $\alpha$ . As  $z$  was applied to  $y$ , it implies that  $y : \alpha \rightarrow \beta$  for some output type  $\beta$ . As  $x$  takes  $z$  and  $(yz)$ , it must be of type  $\alpha \rightarrow \beta \rightarrow \delta$ .

$$S := \lambda x : \alpha \rightarrow \beta \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z. \alpha. xz(yz)$$

Complete type derivation shown as below:

1.	$x : \alpha \rightarrow \beta \rightarrow \delta$	<b>Bound</b>
2.	$y : \alpha \rightarrow \beta$	<b>Bound</b>
3.	$z : \alpha$	<b>Bound</b>
4.	$y : \alpha \rightarrow \beta$	<b>T-Var</b>
5.	$z : \alpha$	<b>T-Var</b>
6.	$yz : \beta$	<b>4,5 T-App</b>
7.	$x : \alpha \rightarrow \beta \rightarrow \delta$	<b>T-Var</b>
8.	$xz : \beta \rightarrow \delta$	<b>7,5 T-App</b>
9.	$xz(yz) : \delta$	<b>8,6 T-App</b>
10.	$\lambda z : \alpha. xz(yz) : \alpha \rightarrow \delta$	<b>9 T-Abstr</b>
11.	$\lambda y : \alpha \rightarrow \beta. \lambda z. \alpha. xz(yz) : (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \delta$	<b>10 T-Abstr</b>

12.

$$\lambda x : \alpha \rightarrow \beta \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z : \alpha. xz(yz) : (\alpha \rightarrow \beta \rightarrow \delta) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \delta \quad \mathbf{11\ T-Abstr}$$

### Problem

(2.4) Type the bound variables

$$\lambda xyz. x(yz)$$

$$\lambda xyz. y(xz)z$$

*Solution.* For the first term,  $z$  had nothing applied to it. Therefore it could be given a simple base type  $\alpha$ .  $z$  was applied to  $y$ , therefore  $y : \alpha \rightarrow \beta$  to satisfy the application rule. Because the application yielded a type of  $\beta$ , by the application rule  $x : \beta \rightarrow \delta$  for some type  $\delta$ .

$$\lambda x : \beta \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z : \alpha. x(yz)$$

Complete type derivation shown below

1.	$x : \beta \rightarrow \delta$	<b>Bound</b>
2.	$y : \alpha \rightarrow \beta$	<b>Bound</b>
3.	$z : \alpha$	<b>Bound</b>
4.	$y : \alpha \rightarrow \beta$	<b>T-Var</b>
5.	$z : \alpha$	<b>T-Var</b>
6.	$yz : \beta$	<b>4,5 T-App</b>
7.	$x : \beta \rightarrow \delta$	<b>T-Var</b>
8.	$x(yz) : \delta$	<b>7,6 T-App</b>
9.	$\lambda z : \alpha. x(yz) : \alpha \rightarrow \delta$	<b>8 T-Abst</b>
10.	$\lambda y : \alpha \rightarrow \beta. \lambda z : \alpha. x(yz) : (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \delta$	<b>9 T-Abst</b>
11.		

$$\lambda x : \beta \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z : \alpha. x(yz)$$

$$: (\beta \rightarrow \delta) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \delta \quad \mathbf{10\ T-Abstr}$$

In the second term  $z$  could still be given a simple base type  $z : \alpha$ . Therefore  $x : \alpha \rightarrow \beta$  for some type  $\beta$ .  $y$  takes  $xz : \beta$  and  $z : \alpha$ , therefore it is of type  $y : \beta \rightarrow \alpha \rightarrow \delta$  for some  $\delta$ .

$$\lambda x : \alpha \rightarrow \beta. \lambda y : \beta \rightarrow \alpha \rightarrow \delta. \lambda z : \alpha. y(xz)z$$

. Complete type derivation shown below

1.	$x : \alpha \rightarrow \beta$	<b>Bound</b>
2.	$y : \beta \rightarrow \alpha \rightarrow \delta$	<b>Bound</b>
3.	$z : \alpha$	<b>Bound</b>
4.	$x : \alpha \rightarrow \beta$	<b>T-Var</b>
5.	$z : \alpha$	<b>T-Var</b>
6.	$xz : \beta$	<b>4,5 T-App</b>
7.	$y : \beta \rightarrow \alpha \rightarrow \delta$	<b>T-Var</b>
8.	$y(xz) : \alpha \rightarrow \delta$	<b>7,6 T-App</b>
9.	$y(xz)z : \delta$	<b>8,5 T-App</b>
10.	$\lambda z : \alpha. y(xz)z : \alpha \rightarrow \delta$	<b>9 T-Abst</b>
11.	$\lambda y : \beta \rightarrow \alpha \rightarrow \delta. \lambda z. y(xz)z : (\beta \rightarrow \alpha \rightarrow \delta) \rightarrow \alpha \rightarrow \delta$	<b>10 T-Abst</b>
12.		

$\lambda x : \alpha \rightarrow \beta. \lambda y : \beta \rightarrow \alpha \rightarrow \delta. \lambda z. y(xz)z :$

$(\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \alpha \rightarrow \delta) \rightarrow \alpha \rightarrow \delta$

**11 T-Abst**

### Problem

(2.5) Try to type the following terms, and prove if not typable.

$\lambda xy. x(\lambda z. y)y$

$\lambda xy. x(\lambda z. x)y.$

*Solution.* The first term is trivially typable.

1.	$x : (\delta \rightarrow \alpha) \rightarrow \alpha \rightarrow \beta$	<b>Bound</b>
2.	$y : \alpha$	<b>Bound</b>
3.	$x : (\delta \rightarrow \alpha) \rightarrow \alpha \rightarrow \beta$	<b>T-Var</b>
4.	$z : \delta$	<b>Bound</b>
5.	$y : \alpha$	<b>T-Var</b>
6.	$\lambda z : \delta. y : \delta \rightarrow \alpha$	<b>5 T-Abst</b>
7.	$x(\lambda z : \delta. y) : \alpha \rightarrow \beta$	<b>3,6 T-App</b>
8.	$y : \alpha$	<b>T-Var</b>
9.	$x(\lambda z : \delta. y)y : \beta$	<b>7,8 T-App</b>
10.	$\lambda y : \alpha. x(\lambda z : \delta. y)y : \alpha \rightarrow \beta$	<b>9 T-Abst</b>

11.

$$\begin{aligned} \lambda x : ((\delta \rightarrow \alpha) \rightarrow \alpha \rightarrow \beta) \lambda y : \alpha. x(\lambda z : \delta. y) y \\ : ((\delta \rightarrow \alpha) \rightarrow \alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta \end{aligned} \quad \mathbf{10\ T-Abst}$$

The second term is not typable in STLC.

*Proof.* By induction on the type inference rule that constructed the type judgement for subterm  $x(\lambda z. x)$ . Because the term is an application, the only rule that applies is the application rule.

We denote the context inside the abstraction as  $\Gamma'$ . Suppose  $\mathcal{J} \equiv \Gamma' \vdash x(\lambda z. x) : \tau$ . By the inference rule of application,  $x$  must be a function type that accepts the type of  $(\lambda z. x)$ . Let  $\Gamma' \vdash z : \alpha$ , and type of  $x$  as  $\tau_x$ . Therefore,  $\Gamma' \vdash \lambda z : \alpha. x : \alpha \rightarrow \tau_x$ . Therefore,  $\tau_x = (\alpha \rightarrow \tau_x) \rightarrow \tau$ . This is a recursive type, which is not constructable as it requires infinitely nested lambda abstractions that requires infinite reduction paths to reach a normal form. ■

### Problem

(2.6) Prove the pretyped term below is legal.

$$\lambda x : ((\alpha \rightarrow \beta) \rightarrow \alpha). x(\lambda z : \alpha. y)$$

Using the tree format and the flag format.

*Solution.* We suppose a context  $\Gamma \vdash y : \beta$  that obviously exists.

*Proof.*

$$\frac{\frac{}{x : (\alpha \rightarrow \beta) \rightarrow \alpha} \text{(Bound)} \quad \frac{\Gamma, z : \alpha \vdash y : \beta}{\Gamma \vdash (\lambda z : \alpha. y) : \alpha \rightarrow \beta} \text{(T-Abst)}}{\Gamma \vdash (\lambda z : \alpha. y) : \alpha \rightarrow \beta} \text{(T-App)} \quad \frac{\Gamma, x : (\alpha \rightarrow \beta) \rightarrow \alpha \vdash (x(\lambda z : \alpha. y)) : \alpha}{\Gamma \vdash \lambda x : ((\alpha \rightarrow \beta) \rightarrow \alpha). x(\lambda z : \alpha. y) : ((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow \alpha} \text{(T-Abst)}$$

A valid type could be given to the term. Therefore, the term is typable under an existing context. ■

The flag derivation is given below:

1.	$x : (\alpha \rightarrow \beta) \rightarrow \alpha$	<b>Bound</b>
2.	$z : \alpha$	<b>Bound</b>
3.	$y : \beta$	$\vdash \Gamma$
4.	$(\lambda z : \alpha. y) : \alpha \rightarrow \beta$	<b>3 T-Abst</b>
5.	$x : (\alpha \rightarrow \beta) \rightarrow \alpha$	<b>T-Var</b>

6.  $\frac{x(\lambda z : \alpha.y) : \beta}{\text{5,4 T-App}}$
- 7.
- $\lambda x : ((\alpha \rightarrow \beta) \rightarrow \beta).x(\lambda z : \alpha.y)$
- $: (\alpha \rightarrow \beta) \rightarrow \beta \rightarrow \alpha$  **6 T-Abst**

### Problem

(2.7 a) Derive

$$f : A \rightarrow B \wedge g : B \rightarrow C \Rightarrow g \circ f : A \rightarrow C$$

Using the rules

$$\frac{f : A \rightarrow B, x \in A}{f(x) \in B} \text{(F-App)} \quad \frac{\forall x \in A, f(x) \in B}{f : A \rightarrow B} \text{(F-Abst)}$$

*Solution.*

*Proof.*

- |     |  |                      |
|-----|--|----------------------|
| 1.  | $f : A \rightarrow B \wedge g : B \rightarrow C$                                   | <b>Assumption</b>    |
| 2.  | $f : A \rightarrow B$  | $1 \wedge E$         |
| 3.  | $g : B \rightarrow C$  | $1 \wedge E$         |
| 4.  | $a \in A$  |                      |
| 5.  | $f(a) \in B$   | <b>3, 4 F-App</b>    |
| 6.  | $g(f(a)) \in C$  | <b>5, 4 F-App</b>    |
| 7.  | $(g \circ f)(a) \in C$   | <b>6 Compose Def</b> |
| 8.  | $\forall x \in A, (g \circ f)(x) \in C$  | $7 \forall E$        |
| 9.  | $g \circ f : A \rightarrow C$  | <b>8 F-Abst</b>      |
| 10. | $f : A \rightarrow B, g : B \rightarrow C \Rightarrow g \circ f : A \rightarrow C$ | $9 \Rightarrow I$    |

■



### Problem

(2.7 b) Give a derivation in natural deduction of the following:

$$(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$$

Using the rules

$$\frac{A \Rightarrow B \quad A}{B} (\Rightarrow E) \qquad \frac{\begin{array}{l} 1. \quad A \quad \text{Premise} \\ 2. \quad \vdots \\ 3. \quad B \end{array}}{A \Rightarrow B} (\Rightarrow I)$$

*Solution.*

*Proof.*

$$\begin{array}{ll} 1. & A \Rightarrow B \quad \text{Premise} \\ 2. & \begin{array}{|l} B \Rightarrow C \end{array} \quad \text{Premise} \\ 3. & \begin{array}{|l|} A \end{array} \quad \text{Premise} \\ 4. & \begin{array}{|l|} B \end{array} \quad 1, 3 \Rightarrow E \\ 5. & \begin{array}{|l|} C \end{array} \quad 2, 4 \Rightarrow E \\ 6. & \begin{array}{|l|} A \Rightarrow C \end{array} \quad 3-5 \Rightarrow I \\ 7. & \begin{array}{|l|} (B \Rightarrow C) \Rightarrow (A \Rightarrow C) \end{array} \quad 2-6 \Rightarrow I \\ 8. & (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)) \quad 1-7 \Rightarrow I \end{array}$$

■

### Problem

(2.7 c) Prove the following pre-typed term is legal using flag notation

$$\lambda z : \alpha. y(xz)$$

*Solution.*

*Proof.* Let  $\Gamma \vdash x : \alpha \rightarrow \beta, y : \beta \rightarrow \delta$  for some type  $\beta$  and  $\delta$ .

$$\begin{array}{ll} 1. & z : \alpha \quad \text{Bound} \\ 2. & \begin{array}{|l} x : \alpha \rightarrow \beta \end{array} \quad \neg \Gamma \\ 3. & \begin{array}{|l|} z : \alpha \end{array} \quad \text{T-Var} \end{array}$$

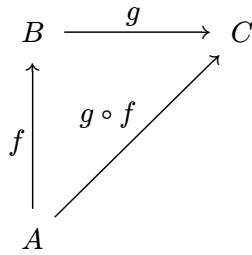
- |    |   |                  |
|----|---|------------------|
| 4. | $xz : \beta$  | <b>2,3 T-App</b> |
| 5. | $y : \beta \rightarrow \delta$                          | $\vdash \Gamma$  |
| 6. | $y(xz) : \delta$  | <b>5,4 T-App</b> |
| 7. | $\lambda z : \alpha. y(xz) : \alpha \rightarrow \delta$ | <b>6 T-Abst</b>  |

■

### Problem

(2.7 d) State the similarity between Q. 2.7 (a), (b), and (c).

*Solution.* All of these examples requires proving something about composing two maps together as like this:



### Problem

(2.8 a) Pre-type the bounding variables for the following term

$$\lambda xy. y(\lambda z. yx) : (\gamma \rightarrow \beta) \rightarrow ((\gamma \rightarrow \beta) \rightarrow \beta) \rightarrow \beta$$

*Solution.*

$$\lambda x : (\gamma \rightarrow \beta). y : ((\gamma \rightarrow \beta) \rightarrow \beta). y(\lambda z : \gamma. yx)$$

### Problem

(2.8 b) Give a derivation in tree format

*Solution.*

$$\begin{array}{c}
\text{(i)} \frac{}{x : (\gamma \rightarrow \beta)} \quad \text{(ii)} \frac{}{y : (\gamma \rightarrow \beta) \rightarrow \beta} \\
\text{(iii)} \frac{}{x : (\gamma \rightarrow \beta), y : ((\gamma \rightarrow \beta) \rightarrow \beta), z : \gamma \vdash yx : \beta} \text{T-App} \\
\text{(v)} \frac{}{y : ((\gamma \rightarrow \beta) \rightarrow \beta)} \quad \text{(iv)} \frac{}{x : (\gamma \rightarrow \beta), y : ((\gamma \rightarrow \beta) \rightarrow \beta) \vdash \lambda z : \gamma. yx : \gamma \rightarrow \beta} \text{T-Abst} \\
\text{(vi)} \frac{}{x : (\gamma \rightarrow \beta), y : ((\gamma \rightarrow \beta) \rightarrow \beta) \vdash y(\lambda z : \gamma. yx) : \beta} \text{T-App} \\
\text{(vii)} \frac{}{x : (\gamma \rightarrow \beta) \vdash \lambda y : ((\gamma \rightarrow \beta) \rightarrow \beta). y(\lambda z : \gamma. yx) : ((\gamma \rightarrow \beta) \rightarrow \beta) \rightarrow \beta} \text{T-Abst} \\
\text{(viii)} \frac{}{(\lambda x : (\gamma \rightarrow \beta). y : ((\gamma \rightarrow \beta) \rightarrow \beta). y(\lambda z : \gamma. yx)) : (\gamma \rightarrow \beta) \rightarrow ((\gamma \rightarrow \beta) \rightarrow \beta) \rightarrow \beta} \text{T-Abst}
\end{array}$$

### Problem

(2.8 c) Sketch a diagram of tree structure of derivation

*Solution.* Trivial.

### Problem

(2.8 d) Transform the derivation into flag notation

*Solution.*

	1.	$x : \gamma \rightarrow \beta$	<b>Bound</b>
	2.	$y : (\gamma \rightarrow \beta) \rightarrow \beta$	<b>Bound</b>
	3.	$z : \gamma$	<b>Bound</b>
(ii)	4.	$y : (\gamma \rightarrow \beta) \rightarrow \beta$	<b>T-Var</b>
(i)	5.	$x : \gamma \rightarrow \beta$	<b>T-Var</b>
(iii)	6.	$yx : \beta$	<b>4,5 T-App</b>
(iv)	7.	$\lambda z : \gamma. yx : \gamma \rightarrow \beta$	<b>6 T-Abst</b>
(v)	8.	$y : (\gamma \rightarrow \beta) \rightarrow \beta$	<b>T-Var</b>
(vi)	9.	$y(\lambda z : \gamma. yx) : \beta$	<b>8,7 T-App</b>
		$\lambda y : (\gamma \rightarrow \beta) \rightarrow \beta. y(\lambda z : \gamma. yx)$	
(vii)	10.	$: ((\gamma \rightarrow \beta) \rightarrow \beta) \rightarrow \beta$	<b>9 T-Abst</b>
(viii)	11.		
		$\lambda x : (\gamma \rightarrow \beta). \lambda y : (\gamma \rightarrow \beta) \rightarrow \beta. y(\lambda z : \gamma. yx)$	
		$: (\gamma \rightarrow \beta) \rightarrow ((\gamma \rightarrow \beta) \rightarrow \beta) \rightarrow \beta$	<b>10 T-Abst</b>

### Problem

(2.9 a) Give derivations of the following judgement

$$x : \delta \rightarrow \delta \rightarrow \alpha, y : \gamma \rightarrow \alpha, z : \alpha \rightarrow \beta \vdash \\ \lambda u : \delta. \lambda v : \gamma. z(yv) : \delta \rightarrow \gamma \rightarrow \beta$$

*Solution.*

1.	$u : \delta$	<b>Bound</b>
2.	$v : \gamma$	<b>Bound</b>
3.	$y : \gamma \rightarrow \alpha$	<b>T-Var</b>
4.	$v : \gamma$	<b>T-Var</b>
5.	$yv : \alpha$	<b>3,4 T-App</b>
6.	$z : \alpha \rightarrow \beta$	<b>T-Var</b>
7.	$z(yv) : \beta$	<b>6,5 T-App</b>
8.	$\lambda v : \gamma. z(yv) : \gamma \rightarrow \beta$	<b>7 T-Abst</b>
9.	$\lambda u : \delta. \lambda v : \gamma. z(yv) : \delta \rightarrow \gamma \rightarrow \beta$	<b>8 T-Abst</b>

### Problem

(2.9 b) Give derivations of the following judgement

$$x : \delta \rightarrow \delta \rightarrow \alpha, y : \gamma \rightarrow \alpha, z : \alpha \rightarrow \beta \vdash \\ \lambda u : \delta. \lambda v : \gamma. z(xuu) : \delta \rightarrow \gamma \rightarrow \beta$$

*Solution.*

1.	$u : \delta$	<b>Bound</b>
2.	$v : \gamma$	<b>Bound</b>
3.	$x : \delta \rightarrow \delta \rightarrow \alpha$	<b>T-Var</b>
4.	$u : \delta$	<b>T-Var</b>
5.	$xu : \delta \rightarrow \alpha$	<b>3,4 T-App</b>
6.	$xuu : \alpha$	<b>5,4 T-App</b>
7.	$z : \alpha \rightarrow \beta$	<b>T-Var</b>
8.	$z(xuu) : \beta$	<b>7,6 T-App</b>
9.	$\lambda v : \gamma. z(xuu) : \gamma \rightarrow \beta$	<b>8 T-Abst</b>
10.	$\lambda u : \delta. \lambda v : \gamma. z(xuu) : \delta \rightarrow \gamma \rightarrow \beta$	<b>9 T-Abst</b>

### Problem

(2.10 a) Give derivation for

$$xz(yz)$$

*Solution.* Assume an context

$$\Gamma \vdash x : \alpha \rightarrow \beta \rightarrow \gamma$$

$$\Gamma \vdash y : \alpha \rightarrow \beta$$

$$\Gamma \vdash z : \alpha$$

1.  $x : \alpha \rightarrow \beta \rightarrow \gamma$     **T-Var**
2.  $y : \alpha \rightarrow \beta$     **T-Var**
3.  $z : \alpha$     **T-Var**
4.  $xz : \beta \rightarrow \gamma$     **1,3 T-App**
5.  $yz : \beta$     **2,3 T-App**
6.  $xz(yz) : \gamma$     **4,5 T-App**

### Problem

(2.10 b) Give derivation for

$$\lambda x : (\alpha \rightarrow \beta) \rightarrow \beta.x(yz)$$

*Solution.* Assume an context

$$\Gamma \vdash y : \gamma \rightarrow (\alpha \rightarrow \beta)$$

$$\Gamma \vdash z : \gamma$$

1.  $x : (\alpha \rightarrow \beta) \rightarrow \beta$     **Bound**
2.  $x : (\alpha \rightarrow \beta) \rightarrow \beta$     **T-Var**
3.  $y : \gamma \rightarrow \alpha \rightarrow \beta$     **T-Var**
4.  $z : \gamma$     **T-Var**
5.  $yz : \alpha \rightarrow \beta$     **3,4 T-App**
6.  $x(yz) : \beta$     **2,5 T-App**
7.  $\lambda x : (\alpha \rightarrow \beta) \rightarrow \beta.x(yz) : ((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow \beta$     **6 T-Abst**

### Problem

(2.10 c) Give derivation for

$$\lambda y : \alpha. \lambda z : \beta \rightarrow \gamma. z(xyy)$$

*Solution.* Assume a context

$$\Gamma \vdash x : \alpha \rightarrow \alpha \rightarrow \beta$$

1.	$y : \alpha$	<b>Bound</b>
2.	$z : \beta \rightarrow \gamma$	<b>Bound</b>
3.	$z : \beta \rightarrow \gamma$	<b>T-Var</b>
4.	$x : \alpha \rightarrow \alpha \rightarrow \beta$	<b>T-Var</b>
5.	$y : \alpha$	<b>T-Var</b>
6.	$xy : \alpha \rightarrow \beta$	<b>4,5 T-App</b>
7.	$xyy : \beta$	<b>6,5 T-App</b>
8.	$z(xyy) : \gamma$	<b>3,6 T-App</b>
9.	$\lambda z : \beta \rightarrow \gamma. z(xyy) : (\beta \rightarrow \gamma) \rightarrow \gamma$	<b>8 T-Abst</b>
10.	$\lambda y : \alpha. \lambda z : \beta \rightarrow \gamma. z(xyy) : \alpha \rightarrow (\beta \rightarrow \gamma) \rightarrow \gamma$	<b>8 T-Abst</b>

### Problem

(2.10 d) Give derivation for

$$\lambda x : \alpha \rightarrow \beta. y(xz)z$$

*Solution.* Consider a context

$$\begin{aligned} \Gamma \vdash z : \alpha \\ \Gamma \vdash y : \beta \rightarrow \alpha \rightarrow \gamma \end{aligned}$$

1.	$x : \alpha \rightarrow \beta$	<b>Bound</b>
2.	$x : \alpha \rightarrow \beta$	<b>T-Var</b>
3.	$z : \alpha$	<b>T-Var</b>
4.	$xz : \beta$	<b>2,3 T-App</b>
5.	$y : \beta \rightarrow \alpha \rightarrow \gamma$	<b>T-Var</b>
6.	$y(xz) : \alpha \rightarrow \gamma$	<b>5,4 T-App</b>
7.	$y(xz)z : \gamma$	<b>3,5 T-App</b>

8.  $\lambda x : \alpha \rightarrow \beta. y(xz)z : (\alpha \rightarrow \beta) \rightarrow \gamma$  **7 T-Abst**

### Problem

(2.11 a) Find an inhabitant of type and prove through derivation

$$(\alpha \rightarrow \alpha \rightarrow \gamma) \rightarrow \alpha \rightarrow \beta \rightarrow \gamma$$

*Solution.*

$$\lambda x : (\alpha \rightarrow \alpha \rightarrow \gamma). \lambda y : (\alpha). \lambda z : (\beta). xyy$$

*Proof.*

1.	$x : \alpha \rightarrow \alpha \rightarrow \gamma$	<b>Bound</b>
2.	$y : \alpha$	<b>Bound</b>
3.	$z : \beta$	<b>Bound</b>
4.	$x : \alpha \rightarrow \alpha \rightarrow \gamma$	<b>T-Var</b>
5.	$y : \alpha$	<b>Bound</b>
6.	$xy : \alpha \rightarrow \gamma$	<b>4,5 T-App</b>
7.	$xyy : \gamma$	<b>6,5 T-App</b>
8.	$\lambda z : \beta. xyy : \beta \rightarrow \gamma$	<b>7 T-Abst</b>
9.	$\lambda y : \alpha. \lambda z : \beta. xyy : \alpha \rightarrow \beta \rightarrow \gamma$	<b>8 T-Abst</b>
10.		

$$\lambda x : \alpha \rightarrow \alpha \rightarrow \gamma. \lambda y : \alpha. \lambda z : \beta. xyy$$

$$: (\alpha \rightarrow \alpha \rightarrow \gamma) \rightarrow \alpha \rightarrow \beta \rightarrow \gamma \quad \mathbf{8\ T-Abst}$$

■

### Problem

(2.11 b) Find an inhabitant of type and prove through derivation

$$((\alpha \rightarrow \gamma) \rightarrow \alpha) \rightarrow (\alpha \rightarrow \gamma) \rightarrow \beta \rightarrow \gamma$$

*Solution.*

$$\lambda x : (\alpha \rightarrow \gamma) \rightarrow \alpha. \lambda y : (\alpha \rightarrow \gamma). \lambda z : \beta. y(xy)$$

*Proof.*

1.	$x : (\alpha \rightarrow \gamma) \rightarrow \alpha$	<b>Bound</b>
2.	$y : \alpha \rightarrow \gamma$	<b>Bound</b>
3.	$z : \beta$	<b>Bound</b>
4.	$x : (\alpha \rightarrow \gamma) \rightarrow \alpha$	<b>T-Var</b>
5.	$y : \alpha \rightarrow \gamma$	<b>T-Var</b>
6.	$xy : \alpha$	<b>4,5 T-App</b>
7.	$y(xy) : \gamma$	<b>5,6 T-App</b>
8.	$\lambda z : \beta. x(xy) : \beta \rightarrow \gamma$	<b>7 T-Abst</b>
9.	$\lambda y : \alpha \rightarrow \gamma. \lambda z : \beta. x(xy) : (\alpha \rightarrow \gamma) \rightarrow \beta \rightarrow \gamma$	<b>7 T-Abst</b>
10.		
	$\lambda x : (\alpha \rightarrow \gamma) \rightarrow \alpha. \lambda y : \alpha \rightarrow \gamma. \lambda z : \beta. x(xy)$	
	$: ((\alpha \rightarrow \gamma) \rightarrow \alpha) \rightarrow (\alpha \rightarrow \gamma) \rightarrow \beta \rightarrow \gamma$	<b>7 T-Abst</b>

■

### Problem

(2.12 a) Construct a term of type

$$((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha \rightarrow \beta) \rightarrow \alpha$$

*Solution.*

$$\lambda x : (\alpha \rightarrow \beta) \rightarrow \alpha. \lambda y : \alpha \rightarrow \alpha \rightarrow \beta. x(\lambda z : \alpha. yzz)$$

*Proof.*

1.	$x : (\alpha \rightarrow \beta) \rightarrow \alpha$	<b>Bound</b>
2.	$y : \alpha \rightarrow \alpha \rightarrow \beta$	<b>Bound</b>
3.	$x : (\alpha \rightarrow \beta) \rightarrow \alpha$	<b>T-Var</b>
4.	$z : \alpha$	<b>Bound</b>
5.	$y : \alpha \rightarrow \alpha \rightarrow \beta$	<b>T-Var</b>
6.	$z : \alpha$	<b>Bound</b>
7.	$yz : \alpha \rightarrow \beta$	<b>5,6 T-App</b>
8.	$yzz : \beta$	<b>7,6 T-App</b>
9.	$\lambda z : \alpha. yzz : \alpha \rightarrow \beta$	<b>8 T-Abst</b>
10.	$x(\lambda z : \alpha. yzz) : \alpha$	<b>3,9 T-App</b>
11.	$\lambda y : \alpha \rightarrow \alpha \rightarrow \beta. x(\lambda z : \alpha. yzz) : (\alpha \rightarrow \alpha \rightarrow \beta) \rightarrow \alpha$	<b>3,9 T-App</b>



12.

$$\lambda x : (\alpha \rightarrow \beta) \rightarrow \alpha. \lambda y : \alpha \rightarrow \alpha \rightarrow \beta. x(\lambda z : \alpha. yzz)$$

$$: ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha \rightarrow \beta) \rightarrow \alpha$$

**3,9 T-App**

■

### Problem

(2.12 b) Construct a term of type

$$((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha \rightarrow \beta) \rightarrow \beta$$

*Solution.*

$$\lambda x : (\alpha \rightarrow \beta) \rightarrow \alpha. \lambda y : \alpha \rightarrow \alpha \rightarrow \beta. (\lambda z : \alpha. yzz)(x(\lambda z : \alpha. yzz))$$

*Proof.*

1.	$x : (\alpha \rightarrow \beta) \rightarrow \alpha$	<b>Bound</b>
2.	$y : \alpha \rightarrow \alpha \rightarrow \beta$	<b>Bound</b>
3.	$x : (\alpha \rightarrow \beta) \rightarrow \alpha$	<b>T-Var</b>
4.	$z : \alpha$	<b>Bound</b>
5.	$y : \alpha \rightarrow \alpha \rightarrow \beta$	<b>T-Var</b>
6.	$z : \alpha$	<b>Bound</b>
7.	$yz : \alpha \rightarrow \beta$	<b>5,6 T-App</b>
8.	$yz z : \beta$	<b>7,6 T-App</b>
9.	have $f := \lambda z : \alpha. yzz : \alpha \rightarrow \beta$	<b>8 T-Abst</b>
10.	$xf : \alpha$	<b>3,9 T-App</b>
11.	$f(xf) : \beta$	<b>9,10 T-App</b>
12.	$\lambda y : \alpha \rightarrow \alpha \rightarrow \beta. f(xf) : (\alpha \rightarrow \alpha \rightarrow \beta) \rightarrow \beta$	<b>9,10 T-App</b>
	$\lambda x : (\alpha \rightarrow \beta) \rightarrow \alpha. \lambda y : \alpha \rightarrow \alpha \rightarrow \beta. f(xf)$	
13.	$: ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha \rightarrow \beta) \rightarrow \beta$	<b>9,10 T-App</b>

■

### Problem

(2.13 a) Find a term of type

$$(\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$$

in context  $\Gamma$

$$x : \alpha \rightarrow \beta \rightarrow \gamma \in \Gamma$$

*Solution.*

$$\lambda u : \alpha \rightarrow \beta. \lambda v : \alpha. x v (u v)$$

*Proof.*

1.	$u : \alpha \rightarrow \beta$	<b>Bound</b>
2.	$v : \alpha$	<b>Bound</b>
3.	$x : \alpha \rightarrow \beta \rightarrow \gamma$	<b>T-Var</b>
4.	$v : \alpha$	<b>T-Var</b>
5.	$x v : \beta \rightarrow \gamma$	<b>3,4 T-App</b>
6.	$u : \alpha \rightarrow \beta$	<b>T-Var</b>
7.	$u v : \beta$	<b>6,4 T-App</b>
8.	$x v (u v) : \gamma$	<b>5,7 T-App</b>
9.	$\lambda v : \alpha. x v (u v) : \alpha \rightarrow \gamma$	<b>8 T-Abst</b>
10.	$\lambda u : \alpha \rightarrow \beta. \lambda v : \alpha. x v (u v) : (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$	<b>9 T-Abst</b>

■

### Problem

(2.13 b) Find a term of type

$$\alpha \rightarrow (\alpha \rightarrow \beta) \rightarrow \gamma$$

in context  $\Gamma$

$$x : \alpha \rightarrow \beta \rightarrow \alpha \rightarrow \gamma \in \Gamma$$

*Solution.*

$$\lambda u : \alpha. \lambda v : \alpha \rightarrow \beta. x u (v u) u$$

*Proof.*

1.	$u : \alpha$	<b>Bound</b>
2.	$v : \alpha \rightarrow \beta$	<b>Bound</b>
3.	$x : \alpha \rightarrow \beta \rightarrow \alpha \rightarrow \gamma$	<b>T-Var</b>
4.	$u : \alpha$	<b>Bound</b>
5.	$v : \alpha \rightarrow \beta$	<b>Bound</b>
6.	$vu : \beta$	<b>5,4 T-App</b>
7.	$xu : \beta \rightarrow \alpha \rightarrow \gamma$	<b>3,4 T-App</b>
8.	$xu(vu) : \alpha \rightarrow \gamma$	<b>7,6 T-App</b>
9.	$xu(vu)u : \gamma$	<b>8,4 T-App</b>
10.	$\lambda v : \alpha \rightarrow \beta. xu(vu)u : (\alpha \rightarrow \beta)\gamma$	<b>9 T-Abst</b>
11.	$\lambda u : \alpha. \lambda v : \alpha \rightarrow \beta. xu(vu)u : \alpha \rightarrow (\alpha \rightarrow \beta)\gamma$	<b>10 T-Abst</b>

■

### Problem

(2.13 c) Find a term of type

$$(\alpha \rightarrow \gamma) \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma$$

in context  $\Gamma$

$$x : (\beta \rightarrow \gamma) \rightarrow \gamma \in \Gamma$$

*Solution.*

$$\lambda u : \alpha \rightarrow \gamma. \lambda v : \beta \rightarrow \alpha. x(\lambda y : \beta. u(vy))$$

*Proof.*

1.	$u : \alpha \rightarrow \gamma$	<b>Bound</b>
2.	$v : \beta \rightarrow \alpha$	<b>Bound</b>
3.	$x : (\beta \rightarrow \gamma) \rightarrow \gamma$	<b>T-Var</b>
4.	$y : \beta$	<b>Bound</b>
5.	$v : \beta \rightarrow \alpha$	<b>T-Var</b>
6.	$y : \beta$	<b>T-Var</b>
7.	$vy : \alpha$	<b>5,6 T-App</b>
8.	$u : \alpha \rightarrow \gamma$	<b>T-Var</b>
9.	$u(vy) : \gamma$	<b>8,7 T-App</b>
10.	$\lambda y : \beta. u(vy) : \beta \rightarrow \gamma$	<b>9 T-Abst</b>

11.	$\frac{x(\lambda y : \beta. u(vy)) : \gamma}{\lambda v : \beta \rightarrow \alpha. x(\lambda y : \beta. u(vy)) : (\beta \rightarrow \alpha) \rightarrow \gamma}$	<b>10,3 T-App</b>
12.		<b>11 T-Abst</b>
13.		
	$\lambda u : \alpha \rightarrow \gamma. \lambda v : \beta \rightarrow \alpha. x(\lambda y : \beta. u(vy))$	
	$: (\alpha \rightarrow \gamma) \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma$	<b>12 T-Abst</b>

■

### Problem

(2.14) Find an inhabitant of type  $\alpha \rightarrow \beta \rightarrow \gamma$  in the context  $\Gamma$

$$x : (\gamma \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma \in \Gamma$$

*Solution.*

$$\lambda u : \alpha. \lambda v : \beta. x(\lambda z : \gamma. v)u$$

*Proof.*

1.	$u : \alpha$	<b>Bound</b>
2.	$v : \beta$	<b>Bound</b>
3.	$x : (\gamma \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$	<b>T-Var</b>
4.	$z : \gamma$	<b>Bound</b>
5.	$v : \beta$	<b>T-Var</b>
6.	$\lambda z : \gamma. v : \gamma \rightarrow \beta$	<b>5 T-Abst</b>
7.	$x(\lambda z : \gamma. v) : \alpha \rightarrow \gamma$	<b>3,6 T-App</b>
8.	$u : \alpha$	<b>T-Var</b>
9.	$x(\lambda z : \gamma. v)u : \gamma$	<b>7,8 T-App</b>
10.	$\lambda v : \beta. x(\lambda z : \gamma. v)u : \beta \rightarrow \gamma$	<b>9 T-Abst</b>
11.	$\lambda u : \alpha. \lambda v : \beta. x(\lambda z : \gamma. v)u : \alpha \rightarrow \beta \rightarrow \gamma$	<b>10 T-Abst</b>

■

### Problem

(2.15) Prove the thinning lemma via induction.

*Solution.*

---

*Lemma 1.*     **Thinning Lemma.** Given two contexts  $\Gamma' \subseteq \Gamma$ ,

$$\Gamma' \vdash M : \sigma \Rightarrow \Gamma \vdash M : \sigma$$


---

We can do structural induction over the structure of  $M$  by the finite syntactic constructors of  $\lambda$  terms.

*Case 1 : Variable.* By the generation lemma  $M : \sigma \in \Gamma'$ . By the definition of subcontextes,  $M : \sigma \in \Gamma$ . It follows from the T-Var rule that  $\Gamma \vdash M : \sigma$ . ■

*Case 2 : Application.* Therefore the judgement must be of form

$$\Gamma' \vdash AB : \sigma$$

Where  $M = AB$ . By the generation lemma, there exists a type  $\tau$  such that  $\Gamma' \vdash A : \tau \rightarrow \sigma$  and  $\Gamma' \vdash B : \tau$ . By the principle of induction the thinning lemma holds for the terms  $A$  and  $B$ . By plugging in  $\Gamma \vdash A : \tau \rightarrow \sigma, B : \tau$ , the T-App rule proves  $\Gamma \vdash M : \sigma$ . ■

*Case 3 : Abstraction.* Therefore  $M$  is of a function type. We denote  $M$  as a abstraction term

$$\lambda x : \alpha. N : \alpha \rightarrow \beta$$

where  $x \notin \text{FR } N$ . By the generation lemma,  $\Gamma', x : \alpha \vdash N : \beta$ .

By the principle of induction, the thinning lemma already holds for  $N : \beta$ . Because  $\Gamma', x : \alpha \subseteq \Gamma, x : \alpha$ , the thinning lemma proves  $\Gamma, x : \alpha \vdash N : \beta$ . By the T-Abst rule,  $\Gamma \vdash \lambda x : \alpha. N : \alpha \rightarrow \beta$ . ■

### Problem

(2.16) Prove the subterm lemma.

*Solution.*

---

*Lemma 2.*     **Subterm Lemma.** If  $M \in \Lambda_{\mathbb{T}}$  is legal, then all subterms of  $M$  are.

---

Let's formalize the lemma. The definition of a legal term is a term  $M$  with the existence of a context  $\Gamma$  and a type  $\tau$  such that  $\Gamma \vdash M : \tau$ , in other words,  $M$  is typable. Because  $M$  is typable iff there's an applicable inference rule and each term appearing in the premise of each inference rule is an immediate subterm of  $M$  that

could construct  $M$ , structural induction could be done down the typing tree as it is also an induction down the term's structure, inducting over each subterm.

*Case 1 : Variable.*  $M$  is the only subterm of  $M$  thus is trivially typable and legal. ■

*Case 2 : Application.* Therefore the derivation must be of form

$$\frac{\Gamma \vdash A : \tau \rightarrow \sigma \quad \Gamma \vdash B : \tau}{\Gamma \vdash AB : \sigma} \text{T-App}$$

Where  $M = AB$ . From the generation lemma, those  $A$  and  $B$  must exist typable, thus legal. By the principle of induction, the subterm lemma holds for  $A$  and  $B$ . Denote the lemma as  $P(\Lambda_{\mathbb{T}})$ :

$$\begin{aligned} & P(AB) \wedge P(A) \wedge P(B) \wedge (\forall a \in \text{Sub } A, P(a)) \wedge (\forall b \in \text{Sub } B, P(b)) \\ &= \forall m \in \{AB, A, B\} \cup \text{Sub } A \cup \text{Sub } B, P(m) \\ &= \forall m \in \{AB\} \cup \text{Sub } A \cup \text{Sub } B, P(m) \\ &= \forall m \in \text{Sub } AB, P(m) = P(AB) = P(M) \end{aligned}$$

■

*Case 3 : Abstraction.* Therefore the derivation must be of form

$$\frac{\Gamma, x : \alpha \vdash N : \beta}{\Gamma \vdash \lambda x : \alpha. N : \alpha \rightarrow \beta} \text{T-Abst}$$

Where  $M = \lambda x : \alpha. N$  and  $x \notin \text{FR } N$ . By the generation lemma,  $N$  is typable under the context  $\Gamma, x : \alpha$ , thus valid. By the inductive hypothesis, the subterm lemma holds for  $N$ . Denote the lemma as  $P(\Lambda_{\mathbb{T}})$

$$\begin{aligned} & P(\lambda x : \alpha. N) \wedge (P(N) \wedge \forall n \in \text{Sub } N, P(n)) \\ &= \forall m \in \{\lambda x : \alpha. N, N\} \cup \text{Sub } N, P(m) \\ &= \forall m \in \{\lambda x : \alpha. N\} \cup \text{Sub } N, P(m) = P(\lambda x : \alpha. N) = P(M) \end{aligned}$$

■