

EXERCISES

CHAPTER 4

SEAN LI ¹

1. Reducted

Definition Some rules for reference.

$$\begin{array}{c} \frac{}{\emptyset \vdash * : \square} \text{Sort} \qquad \frac{\Gamma \vdash A : s \quad x \notin \text{dom } \Gamma}{\Gamma, x : A \vdash x : A} \text{Var} \\[10pt] \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s \quad x \notin \text{dom } \Gamma}{\Gamma, x : C \vdash A : B} \text{Weak} \qquad \frac{\Gamma \vdash A : s \quad \Gamma \vdash B : s}{\Gamma \vdash A \rightarrow B : s} \text{Form} \\[10pt] \frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B} \text{App} \\[10pt] \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash A \rightarrow B : s}{\Gamma \vdash \lambda x : A . M : A \rightarrow B} \text{Abst} \\[10pt] \frac{\Gamma \vdash A : B \quad \Gamma \vdash B' : s \quad B \stackrel{\beta}{=} B'}{\Gamma \vdash A : B'} \text{Conv} \end{array}$$

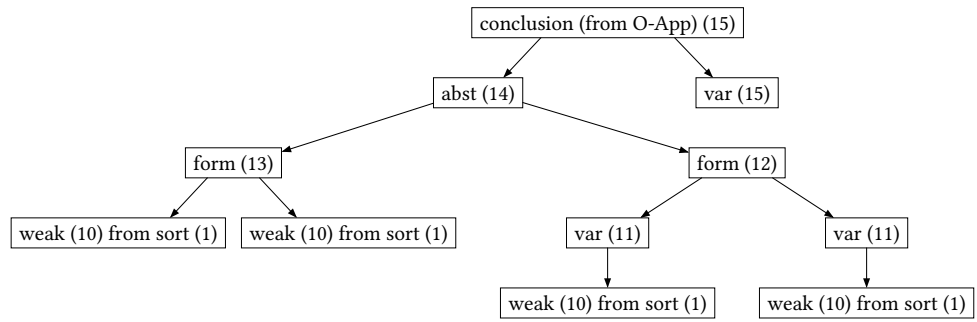
Previously an alternative version of the flag derivation was used, only putting up a flag for a local premise (abstraction unwrapping) to save horizontal space.

Currently, the standard flag derivation format will be used since now single lines will not be as long.

Problem

(4.1) Give a complete tree diagram of the derivation in section 4.5 (95)

Solution.



Problem

(4.2 a) Give a complete $\lambda\omega$ derivation in flag format of

$$\emptyset \vdash (* \rightarrow *) \rightarrow * : \square$$

Solution.

- | | |
|--|-----------------|
| 1. $* : \square$ | Sort |
| 2. $* \rightarrow * : \square$ | 1,1 Form |
| 3. $(* \rightarrow *) \rightarrow * : \square$ | 2,1 Form |

Problem

(4.2 b) Give a complete $\lambda\omega$ derivation in flag format of

$$\alpha : *, \beta : * \vdash (\alpha \rightarrow \beta) \rightarrow \alpha : *$$

Solution.

1.	$\emptyset \vdash * : \square$	Sort
2.	$\alpha : *$	
3.	$\alpha : *$	1 Var
4.	$* : \square$	1,1 Weak
5.	$\beta : *$	
6.	$\alpha : *$	3,4 Weak
7.	$\beta : *$	4 Var
8.	$\alpha \rightarrow \beta : *$	6,7 Form
9.	$(\alpha \rightarrow \beta) \rightarrow \alpha : *$	8,6 Form

Problem

(4.3 a) Give a complete $\lambda\omega$ derivation in flag format of

$$\alpha, \beta : *, x : \alpha, y : \alpha \rightarrow \beta \vdash y x : \beta$$

Solution.

1.	$* : \square$	Sort
2.	$\alpha : *$	
3.	$\alpha : *$	1 Var
4.	$* : \square$	1,1 Weak
5.	$\beta : *$	
6.	$\beta : *$	4 Var
7.	$\alpha : *$	3,4 Weak
8.	$* : \square$	4,4 Weak
9.	$x : \alpha$	
10.	$x : \alpha$	7 Var
11.	$\alpha : *$	7,7 Weak
12.	$\beta : *$	6,7 Weak
13.	$\alpha \rightarrow \beta : *$	11,12 Form
14.	$y : \alpha \rightarrow \beta$	
15.	$y : \alpha \rightarrow \beta$	13 Var
16.	$x : \alpha$	10,13 Weak
17.	$y x : \beta$	15,16 App

Problem

(4.3 b) Give a shortened λ_{ω} derivation in flag format of

$$\alpha, \beta : *, x : \alpha, y : \alpha \rightarrow \beta, z : \beta \rightarrow \alpha \vdash z (y x) : \alpha$$

(4.3 b) Give a shortened λ_{ω} derivation in flag format of

$$\alpha, \beta : *, x : \alpha, y : \alpha \rightarrow \beta, z : \beta \rightarrow \alpha \vdash z (y x) : \alpha$$

Solution.

1.	$\alpha : *$				
2.	$\beta : *$				
3.	$x : \alpha$				
4.	$y : \alpha \rightarrow \beta$				
5.	$x : \alpha$			3 Weak	
6.	$z : \beta \rightarrow \alpha$				
7.	$x : \alpha$			5 Weak	
8.	$y : \alpha \rightarrow \beta$			4 Weak	
9.	$y \ x : \beta$			8,7 App	
10.	$z \ (y \ x) : \alpha$			6,9 App	

Problem

(4.4 a) Give a shortened λ_{ω} derivation in flag format of

$$\alpha : *, \beta : * \rightarrow * \vdash \beta(\beta\alpha) : *$$

(4.4 a) Give a shortened λ_{ω} derivation in flag format of

$$\alpha : *, \beta : * \rightarrow * \vdash \beta(\beta\alpha) : *$$

Solution.

$$\begin{array}{ll}
1. & \alpha : * \\
2. & \left| \beta : * \rightarrow * \right. \\
3. & \left| \left| \beta\alpha : * \right. \right. \quad \mathbf{2,1 \text{ App}} \\
4. & \left| \left| \beta(\beta\alpha) : * \right. \right. \quad \mathbf{2,4 \text{ App}}
\end{array}$$

Problem

(4.4 b) Give a shortened $\lambda\underline{\omega}$ derivation in flag format of

$$\alpha : *, \beta : * \rightarrow *, x : \beta(\beta\alpha) \vdash \lambda y : \alpha. x : \alpha \rightarrow \beta(\beta\alpha)$$

(4.4 b) Give a shortened $\lambda\underline{\omega}$ derivation in flag format of

$$\alpha : *, \beta : * \rightarrow *, x : \beta(\beta\alpha) \vdash \lambda y : \alpha. x : \alpha \rightarrow \beta(\beta\alpha)$$

Solution.

$$\begin{array}{ll}
1. & \alpha : * \\
2. & \left| \beta : * \rightarrow * \right. \\
3. & \left| \left| x : \beta(\beta\alpha) \right. \right. \\
4. & \left| \left| \left| y : \alpha \right. \right. \right. \\
5. & \left| \left| \left| \left| x : \beta(\beta\alpha) \right. \right. \right. \quad \mathbf{3 \text{ Var}} \\
6. & \left| \left| \left| \left| \lambda y : \alpha. x : \alpha \rightarrow \beta(\beta\alpha) \right. \right. \right. \quad \mathbf{5 \text{ Abst}}
\end{array}$$

Problem

(4.4 c) Give a shortened $\lambda\omega$ derivation in flag format of

$$\emptyset \vdash \lambda\alpha : * . \lambda\beta : * \rightarrow * . \beta(\beta\alpha) : * \rightarrow (* \rightarrow *) \rightarrow *$$

Solution.

$$\begin{array}{ll}
1. & \alpha : * \\
2. & \left| \beta : * \rightarrow * \right. \\
3. & \left| \left| \beta\alpha : * \right. \right. \quad \mathbf{2,1 \text{ App}} \\
4. & \left| \left| \left| \beta(\beta\alpha) : * \right. \right. \right. \quad \mathbf{2,3 \text{ App}} \\
5. & \left| \left| \left| \lambda\beta : * \rightarrow * . \beta(\beta\alpha) : (* \rightarrow *) \rightarrow * \right. \right. \right. \quad \mathbf{4 \text{ Abst}} \\
6. & \lambda\alpha : * . \lambda\beta : * \rightarrow * . \beta(\beta\alpha) : * \rightarrow (* \rightarrow *) \rightarrow * \quad \mathbf{5 \text{ Abst}}
\end{array}$$

Problem

(4.4 d) Give a shortened $\lambda\omega$ derivation in flag format of

$$\mathbf{nat} : * \vdash (\lambda\alpha : * . \lambda\beta : * \rightarrow * . \beta(\beta\alpha)) \mathbf{nat} (\lambda\gamma : * . \gamma) : *$$

Solution.

$$\begin{array}{ll}
1. & \mathbf{nat} : * \\
2. & \left| \alpha : * \right. \\
3. & \left| \left| \beta : * \rightarrow * \right. \right. \\
4. & \left| \left| \left| \beta\alpha : * \right. \right. \right. \quad \mathbf{3,2 \text{ App}} \\
5. & \left| \left| \left| \left| \beta(\beta\alpha) : * \right. \right. \right. \quad \mathbf{3,4 \text{ App}} \\
6. & \left| \left| \left| \lambda\beta : * \rightarrow * . \beta(\beta\alpha) : (* \rightarrow *) \rightarrow * \right. \right. \right. \quad \mathbf{5 \text{ Abst}}
\end{array}$$

7.	$\lambda\alpha : * . \lambda\beta : * \rightarrow * . \beta(\beta\alpha) : * \rightarrow (* \rightarrow *) \rightarrow *$	6 Abst
8.	$(\lambda\alpha : * . \lambda\beta : * \rightarrow * . \beta(\beta\alpha)) \text{ nat} : (* \rightarrow *) \rightarrow *$	7,1 App
9.	$\gamma : *$	
10.	$\boxed{\gamma : *}$	9 Var
11.	$\lambda\gamma : * . \gamma : * \rightarrow *$	10 Abst
12.	$(\lambda\alpha : * . \lambda\beta : * \rightarrow * . \beta(\beta\alpha)) \text{ nat} (\lambda\gamma : * . \gamma) : *$	8,11 App

Problem

(4.5) Give a shortened $\lambda\omega$ derivation in flag format of

$$\alpha : * . x : \alpha \vdash \lambda y : \alpha . x : (\lambda\beta : * . \beta \rightarrow \beta)\alpha$$

Solution.

1.	$\alpha : *$	
2.	$x : \alpha$	
3.	$y : \alpha$	
4.	$\boxed{x : \alpha}$	2 Weak
5.	$\lambda y : \alpha . x : \alpha \rightarrow \alpha$	4 Abst
6.	$\beta : *$	
7.	$\boxed{\beta \rightarrow \beta : *}$	6,6 Form
8.	$\lambda\beta : * . \beta \rightarrow \beta : * \rightarrow *$	7 Abst
9.	$(\lambda\beta : * . \beta \rightarrow \beta)\alpha : *$	8,1 App
10.	$\boxed{\lambda y : \alpha . x : (\lambda\beta : * . \beta \rightarrow \beta)\alpha}$	5,9 Conv