

EXERCISES

CHAPTER 4

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1. Reducted

Definition Some rules for reference.

$$\begin{array}{c} \frac{}{\emptyset \vdash * : \square} \text{Sort} \qquad \frac{\Gamma \vdash A : s \quad x \notin \text{dom } \Gamma}{\Gamma, x : A \vdash x : A} \text{Var} \\[10pt] \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s \quad x \notin \text{dom } \Gamma}{\Gamma, x : C \vdash A : B} \text{Weak} \qquad \frac{\Gamma \vdash A : s \quad \Gamma \vdash B : s}{\Gamma \vdash A \rightarrow B : s} \text{Form} \\[10pt] \frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B} \text{App} \\[10pt] \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash A \rightarrow B : s}{\Gamma \vdash \lambda x : A . M : A \rightarrow B} \text{Abst} \\[10pt] \frac{\Gamma \vdash A : B \quad \Gamma \vdash B' : s \quad B \stackrel{\beta}{=} B'}{\Gamma \vdash A : B'} \text{Conv} \end{array}$$

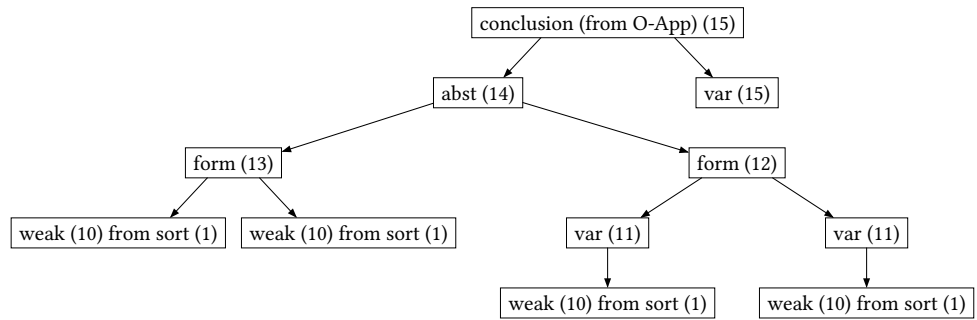
Previously an alternative version of the flag derivation was used, only putting up a flag for a local premise (abstraction unwrapping) to save horizontal space.

Currently, the standard flag derivation format will be used since now single lines will not be as long.

Problem

(4.1) Give a complete tree diagram of the derivation in section 4.5 (95)

Solution.



Problem

(4.2 a) Give a complete $\lambda\omega$ derivation in flag format of

$$\emptyset \vdash (* \rightarrow *) \rightarrow * : \square$$

Solution.

- | | |
|--|-----------------|
| 1. $* : \square$ | Sort |
| 2. $* \rightarrow * : \square$ | 1,1 Form |
| 3. $(* \rightarrow *) \rightarrow * : \square$ | 2,1 Form |

Problem

(4.2 b) Give a complete $\lambda\omega$ derivation in flag format of

$$\alpha : *, \beta : * \vdash (\alpha \rightarrow \beta) \rightarrow \alpha : *$$

Solution.

1.	$\emptyset \vdash * : \square$	Sort
2.	$\alpha : *$	
3.	$\alpha : *$	1 Var
4.	$* : \square$	1,1 Weak
5.	$\beta : *$	
6.	$\alpha : *$	3,4 Weak
7.	$\beta : *$	4 Var
8.	$\alpha \rightarrow \beta : *$	6,7 Form
9.	$(\alpha \rightarrow \beta) \rightarrow \alpha : *$	8,6 Form

Problem

(4.3 a) Give a complete $\lambda\omega$ derivation in flag format of

$$\alpha, \beta : *, x : \alpha, y : \alpha \rightarrow \beta \vdash y x : \beta$$

Solution.

1.	$* : \square$	Sort
2.	$\alpha : *$	
3.	$\alpha : *$	1 Var
4.	$* : \square$	1,1 Weak
5.	$\beta : *$	
6.	$\beta : *$	4 Var
7.	$\alpha : *$	3,4 Weak
8.	$* : \square$	4,4 Weak
9.	$x : \alpha$	
10.	$x : \alpha$	7 Var
11.	$\alpha : *$	7,7 Weak
12.	$\beta : *$	6,7 Weak
13.	$\alpha \rightarrow \beta : *$	11,12 Form
14.	$y : \alpha \rightarrow \beta$	
15.	$y : \alpha \rightarrow \beta$	13 Var
16.	$x : \alpha$	10,13 Weak
17.	$y x : \beta$	15,16 App

Problem

(4.3 b) Give a shortened λ_{ω} derivation in flag format of

$$\alpha, \beta : *, x : \alpha, y : \alpha \rightarrow \beta, z : \beta \rightarrow \alpha \vdash z (y x) : \alpha$$

(4.3 b) Give a shortened λ_{ω} derivation in flag format of

$$\alpha, \beta : *, x : \alpha, y : \alpha \rightarrow \beta, z : \beta \rightarrow \alpha \vdash z (y x) : \alpha$$

Solution.

1.	$\alpha : *$			
2.	$\beta : *$			
3.	$x : \alpha$			
4.	$y : \alpha \rightarrow \beta$			
5.	$x : \alpha$			3 Weak
6.	$z : \beta \rightarrow \alpha$			
7.	$x : \alpha$			5 Weak
8.	$y : \alpha \rightarrow \beta$			4 Weak
9.	$y\ x : \beta$			8,7 App
10.	$z\ (y\ x) : \alpha$			6,9 App

Problem

(4.4 a) Give a shortened $\lambda_{\underline{\omega}}$ derivation in flag format of

$$\alpha : *, \beta : * \rightarrow * \vdash \beta(\beta\alpha) : *$$

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$$\alpha : *, \beta : * \rightarrow * \vdash \beta(\beta\alpha) : *$$

Solution.

$$\begin{array}{lcl}
1. & \alpha : * & \\
2. & \left| \beta : * \rightarrow * \right. & \\
3. & \left| \left| \beta \alpha : * \right. \right. & \mathbf{2,1 \text{ App}} \\
4. & \left| \left| \beta(\beta \alpha) : * \right. \right. & \mathbf{2,4 \text{ App}}
\end{array}$$

Problem

(4.4 b) Give a shortened $\lambda\underline{\omega}$ derivation in flag format of

$$\alpha : *, \beta : * \rightarrow *, x : \beta(\beta\alpha) \vdash \lambda y : \alpha. x : \alpha \rightarrow \beta(\beta\alpha)$$

(4.4 b) Give a shortened $\lambda\underline{\omega}$ derivation in flag format of

$$\alpha : *, \beta : * \rightarrow *, x : \beta(\beta\alpha) \vdash \lambda y : \alpha. x : \alpha \rightarrow \beta(\beta\alpha)$$

Solution.

$$\begin{array}{lcl}
1. & \alpha : * & \\
2. & \left| \beta : * \rightarrow * \right. & \\
3. & \left| \left| x : \beta(\beta\alpha) \right. \right. & \\
4. & \left| \left| \left| y : \alpha \right. \right. \right. & \\
5. & \left| \left| \left| \left| x : \beta(\beta\alpha) \right. \right. \right. & \mathbf{3\ Var} \\
6. & \left| \left| \left| \left| \lambda y : \alpha. x : \alpha \rightarrow \beta(\beta\alpha) \right. \right. \right. & \mathbf{5\ Abst}
\end{array}$$

Problem

(4.4 c) Give a shortened $\lambda\omega$ derivation in flag format of

$$\emptyset \vdash \lambda\alpha : *. \lambda\beta : * \rightarrow *. \beta(\beta\alpha) : * \rightarrow (* \rightarrow *) \rightarrow *$$

Solution.

$$\begin{array}{lcl}
1. & \alpha : * & \\
2. & \left| \beta : * \rightarrow * \right. & \\
3. & \left| \left| \beta\alpha : * \right. \right. & \mathbf{2,1\ App} \\
4. & \left| \left| \left| \beta(\beta\alpha) : * \right. \right. \right. & \mathbf{2,3\ App} \\
5. & \left| \left| \left| \lambda\beta : * \rightarrow *. \beta(\beta\alpha) : (* \rightarrow *) \rightarrow * \right. \right. \right. & \mathbf{4\ Abst} \\
6. & \lambda\alpha : *. \lambda\beta : * \rightarrow *. \beta(\beta\alpha) : * \rightarrow (* \rightarrow *) \rightarrow * & \mathbf{5\ Abst}
\end{array}$$

Problem

(4.4 d) Give a shortened $\lambda\omega$ derivation in flag format of

$$\mathbf{nat} : * \vdash (\lambda\alpha : *. \lambda\beta : * \rightarrow *. \beta(\beta\alpha)) \mathbf{nat} (\lambda\gamma : *. \gamma) : *$$

Solution.

$$\begin{array}{lcl}
1. & \mathbf{nat} : * & \\
2. & \left| \alpha : * \right. & \\
3. & \left| \left| \beta : * \rightarrow * \right. \right. & \\
4. & \left| \left| \left| \beta\alpha : * \right. \right. \right. & \mathbf{3,2\ App} \\
5. & \left| \left| \left| \left| \beta(\beta\alpha) : * \right. \right. \right. & \mathbf{3,4\ App} \\
6. & \left| \left| \left| \lambda\beta : * \rightarrow *. \beta(\beta\alpha) : (* \rightarrow *) \rightarrow * \right. \right. \right. & \mathbf{5\ Abst}
\end{array}$$

7.	$\lambda\alpha : * . \lambda\beta : * \rightarrow * . \beta(\beta\alpha) : * \rightarrow (* \rightarrow *) \rightarrow *$	6 Abst
8.	$(\lambda\alpha : * . \lambda\beta : * \rightarrow * . \beta(\beta\alpha)) \text{ nat} : (* \rightarrow *) \rightarrow *$	7,1 App
9.	$\gamma : *$	
10.	$\boxed{\gamma : *}$	9 Var
11.	$\lambda\gamma : * . \gamma : * \rightarrow *$	10 Abst
12.	$(\lambda\alpha : * . \lambda\beta : * \rightarrow * . \beta(\beta\alpha)) \text{ nat} (\lambda\gamma : * . \gamma) : *$	8,11 App

Problem

(4.5) Give a shortened $\lambda\omega$ derivation in flag format of

$$\alpha : * . x : \alpha \vdash \lambda y : \alpha . x : (\lambda\beta : * . \beta \rightarrow \beta)\alpha$$

Solution.

1.	$\alpha : *$	
2.	$x : \alpha$	
3.	$y : \alpha$	
4.	$\boxed{x : \alpha}$	2 Weak
5.	$\lambda y : \alpha . x : \alpha \rightarrow \alpha$	4 Abst
6.	$\beta : *$	
7.	$\boxed{\beta \rightarrow \beta : *}$	6,6 Form
8.	$\lambda\beta : * . \beta \rightarrow \beta : * \rightarrow *$	7 Abst
9.	$(\lambda\beta : * . \beta \rightarrow \beta)\alpha : *$	8,1 App
10.	$\boxed{\lambda y : \alpha . x : (\lambda\beta : * . \beta \rightarrow \beta)\alpha}$	5,9 Conv

Problem

(4.6 a) Show that no such context Γ and term M in $\lambda\omega$ such that

$$\Gamma \vdash \square : M$$

is derivable.

Solution. Proof by induction on inference rules. Rules like Sort, Var, Form, Abst, App has syntactically or semantic different conclusions than $\square : M$.

Case 1 : Rule Weak. Let $\Gamma', C : s \equiv \Gamma$. Therefore this derivation requires a premise $\Gamma' \vdash \square : M$. By the inductive hypothesis this is impossible. ■

Case 2 : Rule Conv. This derivation requires a premise $\Gamma \vdash \square : M'$ such that $M \equiv_{\beta} M'$. By the inductive hypothesis this is impossible. ■

By the principle of induction this proves that there's no derivation that could give $\Gamma \vdash \square : M$.

Problem

(4.6 b) Prove there are no such context Γ and terms M and N in $\lambda\omega$ such that $\Gamma \vdash M \rightarrow \square : N$

Solution. Proof by induction on inference rules. Rules like Sort, Var, Abst, App has syntactically or semantically conclusions than $M \rightarrow \square : N$.

Case 1 : Rule Weak. Let $\Gamma', C : s \equiv \Gamma$. The derivation requires a premise $\Gamma' \vdash M \rightarrow \square : N$. By the inductive hypothesis this is impossible. ■

Case 2 : Rule Form. This requires a derivation with premise $\Gamma \vdash \square : N$, which by 4.6 a is not possible. ■

Case 3 : Rule Conv. This requires a premise $\Gamma \vdash M \rightarrow \square : N'$ such that $N \equiv N'$. By the inductive hypothesis this is impossible. ■

By the principle of induction this proves there's no derivation that could give $\Gamma \vdash M \rightarrow \square : N$.

Problem

(4.7 a) Give $\lambda\omega$ definition of the notion legal term, $\lambda\omega$ context and domain.

Solution.

Definition *Legal Term* are typable terms. That is, a term M is legal iff there exists a context Γ and a legal higher-sorted term α under Γ such that $\Gamma \vdash M : \alpha$.

Definition $\lambda\omega$ Context.

1. \emptyset is a $\lambda\omega$ context.
 2. When Γ is a valid $\lambda\omega$ context, α is valid under Γ , and type of x is α , and $x \notin \text{dom } \Gamma$, then the context $\Gamma, x : \alpha$ is valid in $\lambda\omega$.
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Definition *Domain*

1. $\text{dom } \emptyset = \{\}$
 2. $\text{dom } \Gamma, x : s = \text{dom } \Gamma \cup \{x\}$
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Problem

(4.7 b) Formulate the Free Variables Lemma, Thinning Lemma, and Substitution Lemma for $\lambda\omega$.

Solution.

Lemma 1. **FV Lemma ($\lambda\omega$).** For any legal term M under Γ , $\text{FV } M \subseteq \text{dom } \Gamma$.

More specifically,

$$\begin{aligned} \forall M, \alpha \in \Lambda_{\omega}, \Gamma \vdash \alpha : s, \Gamma \vdash M : \alpha &\implies \text{FV } M \subseteq \text{dom } \Gamma \\ M \equiv \square &\implies \text{FV } M \equiv \emptyset \subseteq \text{dom } \Gamma \end{aligned}$$

Lemma 2. **Thinning Lemma ($\lambda\omega$).** For any legal term M in Γ' and $\Gamma' \subseteq \Gamma$, M is legal under Γ .

Lemma 3. **Substitution Lemma ($\lambda\omega$).** Assume term $\kappa : s$ under context Γ' . Under another context Γ'' given a term $\Gamma'' \vdash N : \kappa$ and another context Γ such that $\Gamma, x : \kappa, \Gamma' \vdash M : A$ for some type $A : s$ under Γ . Then

$$\Gamma, \Gamma', \Gamma'' \vdash M [x := N] : A$$
