

EXERCISES

CHAPTER 3

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1. Reducted

Definition Some rules for reference.

$$\frac{x : \sigma \in \Gamma \quad \Gamma \text{ is a } \lambda 2 \text{ context}}{\Gamma \vdash x : \sigma} \text{ (T-Var)} \quad \frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau} \text{ (T-App)}$$
$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma. M : \sigma \rightarrow \tau} \text{ (T-Abst)} \quad \frac{\alpha \in \mathbb{T}_2 \quad \forall \tau \in \text{FV } \alpha, \Gamma \vdash \tau : *}{\alpha : * \in \Gamma} \text{ (T-Form)}$$
$$\frac{\Gamma \vdash M : \Pi_{\alpha : *} A \quad \Gamma \vdash B : *}{\Gamma \vdash MB : A[\alpha := B]} \text{ (T2-App)} \quad \frac{\Gamma, \alpha : * \vdash M : A}{\Gamma \vdash \lambda \alpha : *. M : \Pi_{\alpha : *} A} \text{ (T2-Abst)}$$

In this document, convention is that all type judgements in a proof tree, unless stated otherwise, is derived from a single and unique $\lambda 2$ context per tree. Multiple conclusions might be drawn on a single line from usage of the same inference rule for compactness. Eg:

ex 1. $\alpha, \beta : *$ **T-Form**

Is shorthand for

ex 1. $\Gamma \vdash \alpha : *$ **T-Form**

ex 2. $\Gamma \vdash \beta : *$ **T-Form**

Problem

(3.1) How many $\lambda 2$ contexts consisting of four and only four declarations

- (1) $\Gamma \vdash \alpha : *$
- (2) $\Gamma \vdash \beta : *$
- (3) $\Gamma \vdash f : \alpha \rightarrow \beta$
- (4) $\Gamma \vdash x : \alpha$

Solution. The last two declarations depende on the first two. Therefore this is an easy combinatorics problem: $2! \times 2! = 4$ contexts:

$$\begin{array}{ll} 1 - 2 - 3 - 4 & 1 - 2 - 4 - 3 \\ 2 - 1 - 3 - 4 & 2 - 1 - 4 - 3 \end{array}$$

Problem

(3.2) Give a full derivation in $\lambda 2$ to show the following type term is legal:

$$M \equiv \lambda \alpha, \beta, \gamma : * . \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx)$$

Solution.

		Bound
1.	$\alpha : *$	
2.	$\beta : *$	Bound
3.	$\gamma : *$	Bound
4.	$f : \alpha \rightarrow \beta$	Bound
5.	$g : \beta \rightarrow \gamma$	Bound
6.	$x : \alpha$	Bound
7.	$\alpha, \beta, \gamma : *$	T-Form
8.	$\alpha \rightarrow \beta, \beta \rightarrow \gamma : *$	T-Form
9.	$f : \alpha \rightarrow \beta, x : \alpha$	T-Var
10.	$fx : \beta$	8,8 T-App
11.	$g : \beta \rightarrow \gamma$	T-Var
12.	$g(fx) : \gamma$	11,10 T-App
13.	$\lambda x : \alpha. g(fx) : \alpha \rightarrow \gamma$	12 T-Abst
14.	$\lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) : (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	13 T-Abst
15.	$\lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) : (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	14 T-Abst

16.	$\begin{array}{l} \lambda\gamma : * . \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) \\ : \Pi\gamma : * . (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma \end{array}$	15 T2-Abst
17.	$\begin{array}{l} \lambda\beta, \gamma : * . \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) \\ : \Pi\beta, \gamma : * . (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma \end{array}$	16 T2-Abst
18.	$\begin{array}{l} \lambda\alpha, \beta, \gamma : * . \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) \\ : \Pi\alpha, \beta, \gamma : * . (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma \end{array}$	17 T2-Abst

Problem

(3.3 a) Given M in 3.2, and a context Γ such that

$$\Gamma \vdash \text{nat} : *$$

$$\Gamma \vdash \text{bool} : *$$

$$\Gamma \vdash \text{succ} : \text{nat} \rightarrow \text{nat}$$

$$\Gamma \vdash \text{even} : \text{nat} \rightarrow \text{bool}$$

Prove $M \text{ nat } \text{nat } \text{bool } \text{succ } \text{even}$ is legal.

Solution. Proof by deriving the term's type.

Proof.

1.	$M : \Pi\alpha, \beta, \gamma : * . (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	T-Var
2.	$\text{nat}, \text{bool} : *$	T-Form
3.	$M \text{ nat} : \Pi\beta, \gamma : * . (\text{nat} \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \text{nat} \rightarrow \gamma$	2,1 T2-App
4.	$M \text{ nat } \text{nat} : \Pi\gamma : * . (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \gamma) \rightarrow \text{nat} \rightarrow \gamma$	2,3 T2-App
5.	$M \text{ nat } \text{nat } \text{bool} : (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \text{bool}) \rightarrow \text{nat} \rightarrow \text{bool}$	2,3 T2-App
6.	$\text{succ} : \text{nat} \rightarrow \text{nat}, \text{even} : \text{nat} \rightarrow \text{bool}$	T-Var
7.	$M \text{ nat } \text{nat } \text{bool } \text{succ} : (\text{nat} \rightarrow \text{bool}) \rightarrow \text{nat} \rightarrow \text{bool}$	6,5 T-App
8.	$M \text{ nat } \text{nat } \text{bool } \text{succ } \text{even} : \text{nat} \rightarrow \text{bool}$	6,7 T-App

■

Problem

(3.3 b.i) Prove $\lambda x : \text{nat}. \text{even}(\text{succ } x)$ via 3.3 a.

Solution. The result of beta reduction on the term in 3.3 a is what we are proving.

Proof.

$$\begin{aligned} M & \text{ nat nat bool succ even} \\ & \equiv (\lambda \alpha, \beta, \gamma, f, g. \lambda x : \alpha. g(fx)) \text{ nat nat bool succ even} \\ & \xrightarrow[\beta]{\beta} (\lambda f : \text{nat} \rightarrow \text{nat}. \lambda g : \text{nat} \rightarrow \text{bool}. \lambda x : \text{nat}. g(f(x))) \text{ succ even} \\ & \xrightarrow[\beta]{} (\lambda x : \text{nat}. \text{even}(\text{succ } x)) \end{aligned}$$

By the subject reduction lemma, $\lambda x : \text{nat}. \text{even}(\text{succ } x) : \text{nat} \rightarrow \text{bool}$, thus is legal. ■

Problem

(3.3 b.ii) Prove $\lambda x : \text{nat}. \text{even}(\text{succ } x)$ via derivation in the context provided in 3.3 a.

Solution.

Proof.

1.	$\text{nat}, \text{bool} : *$	T-Form
2.	$x : \text{nat}$	Bound
3.	$\text{succ} : \text{nat} \rightarrow \text{nat}$	T-Var
4.	$x : \text{nat}$	T-Var
5.	$\text{succ } x : \text{nat}$	3,4 T-App
6.	$\text{even} : \text{nat} \rightarrow \text{bool}$	T-Var
7.	$\text{even}(\text{succ } x) : \text{bool}$	6,5 T-App
8.	$\lambda x : \text{nat}. \text{even}(\text{succ } x) : \text{nat} \rightarrow \text{bool}$	7 T-Abst

Problem

(3.4) Give a shorthanded (omit T-Var and T-Form) derivation in $\lambda 2$ to show the following term is valid in $\Gamma \equiv \text{nat} : *, \text{bool} : *$

$$(\lambda\alpha, \beta : * . \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx))) \text{ nat bool}$$

Solution.

Proof.

1.	$\alpha, \beta : *$	Bound
2.	$f : \alpha \rightarrow \alpha$	Bound
3.	$g : \alpha \rightarrow \beta$	Bound
4.	$x : \alpha$	Bound
5.	$fx : \alpha$	$^{*,*} \text{T-App}$
6.	$f(fx) : \alpha$	$^{*,5} \text{T-App}$
7.	$g(f(fx)) : \beta$	$^{*,6} \text{T-App}$
8.	$\lambda x : \alpha. g(f(fx)) : \alpha \rightarrow \beta$	7 T-Abst
9.	$\lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx)) : (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$	8 T-Abst
10.	$\lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx))$	
	$: (\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$	9 T-Abst
11.		
12.	$\lambda\alpha, \beta : * . \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx))$	
	$: \Pi\alpha, \beta : * . (\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$	10 T2-Abst
13.	$(\lambda\alpha, \beta : * . \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx))) \text{ nat}$	
	$: \Pi\beta : * . (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta$	$^{*,11} \text{T2-App}$
	$(\lambda\alpha, \beta : * . \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx))) \text{ nat bool}$	
	$: \Pi\beta : * . (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \text{bool}) \rightarrow \text{nat} \rightarrow \text{bool}$	$^{*,12} \text{T2-App}$

■

Problem

(3.5 a) Let $\perp \equiv \Pi\alpha : * . \alpha$. Prove \perp is legal.

Solution. Here a notion called kind checking is introduced. This has not yet been discussed in this book (?)

Proof.

- | | |
|--|---------------|
| 1. $\alpha : *$ | Bound |
| 2. $\boxed{\alpha : *}$ | T-Form |
| 3. $\Pi\alpha : *. \alpha : * \rightarrow *$ | T-Form |

■

Problem

(3.5 b) Consider the context $\Gamma \equiv \beta : *, x : \perp$. Find an inhabitant of type β under Γ .

Solution. $x\beta$ is. Because x is of second-order type, it must be parametric to a type, thus x is of form $\lambda\alpha : *. M$ where $\Gamma, \alpha : * \vdash M : \alpha$.

Proof.

- | | |
|--------------------------------|-------------------|
| 1. $x : \Pi\alpha : *. \alpha$ | T-Var |
| 2. $\beta : *$ | T-Form |
| 3. $x\beta : \beta$ | 1,2 T2-App |

■

Problem

(3.5 c) Give three inhabitants of $\beta \rightarrow \beta$ in β -nf under Γ in 3.5 b.

Solution.

1. $\lambda y : \beta. y$.

Proof.

- | | |
|---|-----------------|
| 1. $y : \beta$ | Bound |
| 2. $\boxed{y : \beta}$ | T-Var |
| 3. $\lambda y : \beta. y : \beta \rightarrow \beta$ | 2 T-Abst |

■

2. $\lambda y : \beta. x\beta$.

Proof.

1.	$y : \beta$	Bound
2.	$x : \Pi\alpha : * . \alpha$	T-Var
3.	$\beta : *$	T-Form
4.	$x\beta : \beta$	2,3 T2-App
5.	$\lambda y : \beta. x\beta : \beta \rightarrow \beta$	4 T-Abst

■

$$3. \lambda y : \beta. x(\beta \rightarrow \beta)y.$$

Proof.

1.	$y : \beta$	Bound
2.	$x : \Pi\alpha : * . \alpha$	T-Var
3.	$\beta \rightarrow \beta : *$	T-Form
4.	$x(\beta \rightarrow \beta) : \beta \rightarrow \beta$	2,3 T2-App
5.	$y : \beta$	T-Var
6.	$x(\beta \rightarrow \beta)y : \beta$	4,5 T-App
7.	$\lambda y : \beta. x(\beta \rightarrow \beta)y : \beta \rightarrow \beta$	5 T-Abst

■

Problem

(3.5 d) Prove that the following terms inhabit the same type in Γ :

$$\lambda f : \beta \rightarrow \beta \rightarrow \beta. f(x\beta)(x\beta)$$

$$x((\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta)$$

Solution. We simply derive the types.

First Term.

1.	$f : \beta \rightarrow \beta \rightarrow \beta$	Bound
2.	$f : \beta \rightarrow \beta \rightarrow \beta$	T-Var
3.	$x : \Pi\alpha : * . \alpha$	T-Var
4.	$\beta : *$	T-Form
5.	$x\beta : \beta$	3,4 T2-App
6.	$f(x\beta) : \beta \rightarrow \beta$	2,5 T-App

7. $f(x\beta)(x\beta) : \beta$ **6,5 T-App**
 8. $\lambda f : \beta \rightarrow \beta \rightarrow \beta. f(x\beta)(x\beta) : (\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$ **6 T-Abst**

■

Second Term.

1. $(\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta : *$ **T-Form**
 2. $x : \Pi \alpha : *. \alpha$ **T-Var**
 3. $x((\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta) : (\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$ **2,1 T2-App**

■

The two terms were shown to both inhabit $(\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$.**Problem**

(3.6 a) Find inhabitant of type

$$\Pi \alpha, \beta : *. (\text{nat} \rightarrow \alpha) \rightarrow (\alpha \rightarrow \text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta$$

In context $\Gamma \equiv \text{nat} : *$.*Solution.*

$$\lambda \alpha, \beta : *. \lambda x : \text{nat} \rightarrow \alpha. \lambda y : (\alpha \rightarrow \text{nat} \rightarrow \beta). \lambda z : \text{nat}. y(xz)z$$

Proof.

- | | |
|--|--------------------|
| 1. $\alpha, \beta : *$ | Bound |
| 2. $\text{nat} \rightarrow \alpha : *$ | T-Form |
| 3. $x : \text{nat} \rightarrow \alpha$ | Bound |
| 4. $\alpha \rightarrow \text{nat} \rightarrow \beta : *$ | T-Form |
| 5. $y : \alpha \rightarrow \text{nat} \rightarrow \beta$ | Bound |
| 6. $\text{nat} : *$ | Bound |
| 7. $z : \text{nat}$ | Bound |
| 8. $y : \alpha \rightarrow \text{nat} \rightarrow \beta$ | T-Var |
| 9. $x : \text{nat} \rightarrow \alpha$ | T-Var |
| 10. $z : \text{nat}$ | T-Var |
| 11. $xz : \alpha$ | 9,10 T-App |
| 12. $y(xz) : \text{nat} \rightarrow \beta$ | 8,11 T-App |
| 13. $y(xz)z : \beta$ | 12,10 T-App |

14.	$\boxed{\lambda z : \text{nat}.y(xz)z : \text{nat} \rightarrow \beta}$	13 T-Abst
15.	$\boxed{\lambda y : \alpha \rightarrow \text{nat} \rightarrow \beta. \lambda z : \text{nat}.y(xz)z}$	14 T-Abst
16.	$\boxed{\lambda x : \text{nat} \rightarrow \alpha. y : \alpha \rightarrow \text{nat} \rightarrow \beta. \lambda z : \text{nat}.y(xz)z}$	15 T2-Abst
17.	$\boxed{\lambda x : \text{nat} \rightarrow \alpha. y : \alpha \rightarrow \text{nat} \rightarrow \beta. \lambda z : \text{nat}.y(xz)z}$	
	$\boxed{\lambda x : \text{nat} \rightarrow \alpha. y : \alpha \rightarrow \text{nat} \rightarrow \beta. \lambda z : \text{nat}.y(xz)z}$	16 T2-Abst

■

Problem

(3.6 b) Find inhabitant of type

$$\Pi\delta : * . ((\alpha \rightarrow \gamma) \rightarrow \delta) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta$$

In context $\Gamma \equiv \alpha : *, \beta : *, \gamma : *$

Solution.

$$\lambda\delta : * . \lambda x : (\alpha \rightarrow \gamma) \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x(\lambda u : \alpha. z(yu))$$

A derivation in shorthand will be given (omitting T-Form / T-Var)

Proof.

1.	$\delta : *$	Bound
2.	$x : (\alpha \rightarrow \gamma) \rightarrow \delta$	Bound
3.	$y : \alpha \rightarrow \beta$	Bound
4.	$z : \beta \rightarrow \gamma$	Bound
5.	$u : \alpha$	Bound
6.	$yu : \beta$	*, T-App
7.	$z(yu) : \gamma$	*,6 T-App
8.	$\lambda u : \alpha. z(yu) : \alpha \rightarrow \gamma$	7 T-Abst
9.	$x(\lambda u : \alpha. z(yu)) : \delta$	8 T-Abst
10.	$\lambda z : \beta \rightarrow \gamma. x(\lambda u : \alpha. z(yu)) : (\beta \rightarrow \gamma) \rightarrow \delta$	9 T-Abst
11.	$\lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x(\lambda u : \alpha. z(yu))$	
	$: (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta$	10 T-Abst

12. $\boxed{\begin{array}{l} \lambda x : (\alpha \rightarrow \gamma) \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x(\lambda u : \alpha. z(yu)) \\ : ((\alpha \rightarrow \gamma) \rightarrow \delta) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta \end{array}}$	11 T-Abst
13. $\lambda \delta : * . \lambda x : (\alpha \rightarrow \gamma) \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x(\lambda u : \alpha. z(yu))$ $: \Pi \delta : * . ((\alpha \rightarrow \gamma) \rightarrow \delta) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta$	12 T2-Abst

■

Problem

(3.6 c) Find inhabitant of type

$$\Pi \alpha, \beta, \gamma : * . (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$$

In the empty context

Solution.

$$\lambda \alpha, \beta, \gamma : * . \lambda f : (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) . \lambda x : \alpha. f x (\lambda u : \beta. x)$$

Proof.

1. α, β, γ	Bound
2. $f : \alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma$	Bound
3. $x : \alpha$	Bound
4. $fx : (\beta \rightarrow \alpha) \rightarrow \gamma$	** T-App
5. $u : \beta$	Bound
6. $x : \alpha$	T-Var
7. $\lambda u : \beta. x : \beta \rightarrow \alpha$	6 T-Abst
8. $fx(\lambda u : \beta. x) : \gamma$	4,7 T-App
9. $\lambda x : \alpha. fx(\lambda u : \beta. x) : \alpha \rightarrow \gamma$	8 T-Abst
10. $\lambda f : \alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma. \lambda x : \alpha. fx(\lambda u : \beta. x)$ $: (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	9 T-Abst
11. $\lambda \alpha, \beta, \gamma : * . \lambda f : \alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma. \lambda x : \alpha. fx(\lambda u : \beta. x)$ $: \Pi \alpha, \beta, \gamma : * . (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	10 T2-Abst

■

Problem

(3.7) Let $\perp \equiv \Pi\alpha : * . \alpha$ and context $\Gamma \equiv \alpha : *, \beta : *, x : \alpha \rightarrow \perp, f : (\alpha \rightarrow \alpha) \rightarrow \alpha$. Give a derivation that successively calculate an inhabitant of α and β , both in context Γ .

Solution. Have $M : \alpha := f(\lambda n : \alpha.n)$. Then $\Gamma \vdash xM\beta : \beta$.

Typing M.

1. $f : (\alpha \rightarrow \alpha) \rightarrow \alpha$ **T-Var**
2. $n : \alpha$ **Bound**
3. $\boxed{n : \alpha}$ **T-Var**
4. $\lambda n : \alpha. n : \alpha \rightarrow \alpha$ **3 T-Abst**
5. $f(\lambda n : \alpha. n) : \alpha$ **1,4 T-App**

Typing $xM\beta$.

1. $M : \alpha$ **T-Var**
2. $x : \alpha \rightarrow \Pi\alpha : * . \alpha$ **T-Var**
3. $Mx : \Pi\alpha : * . \alpha$ **2,1 T-App**
4. $Mx\beta : \beta$ **3,* T2-App**

Problem

(3.8) Recall $K \equiv \lambda xy.x \in \Lambda$ from untyped lambda calculus. Consider the following types

$$T_1 \equiv \Pi\alpha, \beta : * . \alpha \rightarrow \beta \rightarrow \alpha \quad T_2 \equiv \Pi\alpha : * . \alpha \rightarrow (\Pi\beta : * . \beta \rightarrow \alpha)$$

Find inhabitants of both type $t_1 : T_1$ and $t_2 : T_2$ under the empty context, which may be considered the closed $\lambda 2$ form of $K \in \Lambda_{T_2}$.

Solution.

$$\lambda\alpha, \beta : * . \lambda x : \alpha. \lambda y : \beta. x$$

$$\lambda\alpha : * . \lambda x : \alpha. \lambda\beta : * . \lambda y : \beta. x$$

First Form.

1.	$\alpha, \beta : *$	Bound
2.	$x : \alpha$	Bound
3.	$y : \beta$	Bound
4.	$x : \alpha$	T-Var
5.	$\lambda y : \beta. x : \beta \rightarrow \alpha$	4 T-Abst
6.	$\lambda x : \alpha. \lambda y : \beta. x : \alpha \rightarrow \beta \rightarrow \alpha$	5 T-Abst
7.	$\lambda \alpha, \beta : * . \lambda x : \alpha. \lambda y : \beta. x : \Pi \alpha, \beta : * . \alpha \rightarrow \beta \rightarrow \alpha$	5 T2-Abst

■

Second Form.

1.	$\alpha : *$	Bound
2.	$x : \alpha$	Bound
3.	$\beta : *$	Bound
4.	$y : \beta$	Bound
5.	$x : \alpha$	T-Var
6.	$\lambda y : \beta. x : \beta \rightarrow \alpha$	5 T-Abst
7.	$\lambda \beta : * . \lambda y : \beta. x : \Pi \beta : * . \beta \rightarrow \alpha$	6 T2-Abst
8.	$\lambda x : \alpha. \lambda \beta : * . \lambda y : \beta. x : \alpha \rightarrow (\Pi \beta : * . \beta \rightarrow \alpha)$	7 T-Abst
9.	$\lambda \alpha : * . \lambda x : \alpha. \lambda \beta : * . \lambda y : \beta. x : \Pi \alpha : * . \alpha \rightarrow (\Pi \beta : * . \beta \rightarrow \alpha)$	8 T2-Abst

■

Problem

(3.9) Pretype the combinator

$$S \equiv \lambda xyz. xz(yz)$$

In closed form (typable in an empty context) in Λ_{T2} .*Solution.*

$$S \equiv \lambda \alpha, \beta, \gamma : * . \lambda x : \alpha \rightarrow \beta \rightarrow \gamma. \lambda y : \alpha \rightarrow \beta. \lambda z : \alpha. xz(yz)$$

Proof.

1.	$\alpha, \beta, \gamma : *$	Bound
2.	$x : \alpha \rightarrow \beta \rightarrow \gamma$	Bound
3.	$y : \alpha \rightarrow \beta$	Bound

4.	$z : \alpha$	Bound
5.	$xz : \beta \rightarrow \gamma$	*,* T-App
6.	$yx : \beta$	*,* T-App
7.	$xz(yx) : \gamma$	5,6 T-App
8.	$\lambda z : \alpha.xz(yx) : \alpha \rightarrow \gamma$	7 T-Abst
9.	$\lambda y : \alpha \rightarrow \beta.\lambda z : \alpha.xz(yx) : (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$	8 T-Abst
10.	$\lambda x : \alpha \rightarrow \beta \rightarrow \gamma.\lambda y : \alpha \rightarrow \beta.\lambda z : \alpha.xz(yx)$	9 T-Abst
11.	$\lambda x : \alpha \rightarrow \beta \rightarrow \gamma.\lambda y : \alpha \rightarrow \beta.\lambda z : \alpha.xz(yx) : (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$	10 T2-Abst

■

Problem

(3.10 a) Consider the term

$$M \equiv \lambda x : \Pi \alpha : * . \alpha \rightarrow \alpha. x(\sigma \rightarrow \sigma)(x\sigma)$$

Prove that M is legal.

Solution. For a term to be legal there must exist a context so that the term could be typed. Here, a witness context is $\Gamma \equiv \sigma : *$.

Proof.

1.	$x : \Pi \alpha : * . \alpha \rightarrow \alpha$	Bound
2.	$x(\sigma \rightarrow \sigma) : (\sigma \rightarrow \sigma) \rightarrow (\sigma \rightarrow \sigma)$	*,* T2-App
3.	$x\sigma : \sigma \rightarrow \sigma$	*,* T2-App
4.	$x(\sigma \rightarrow \sigma)(x\sigma) : \sigma \rightarrow \sigma$	2,3 T-App
5.	$\lambda x : \Pi \alpha : * . \alpha \rightarrow \alpha. x(\sigma \rightarrow \sigma)(x\sigma) : (\Pi \alpha : * . \alpha \rightarrow \alpha) \rightarrow \sigma \rightarrow \sigma$	4 T-Abst

■

Problem

(3.10 b) Find a term N such that MN is legal and may be considered to be a way to add type information to $(\lambda x.xx)(\lambda y.y)$

Solution.

$$M\sigma N \equiv (\lambda x : \Pi\alpha : * . \alpha \rightarrow \alpha. x(\sigma \rightarrow \sigma)(x\sigma))\sigma(\lambda y : \sigma. y)$$

Is the same as $(\lambda x. xx)(\lambda y. y)$ modulo type annotations.

Proof.

- | | |
|--|-------------------|
| 1. $M : (\Pi\alpha : * . \alpha \rightarrow \alpha) \rightarrow \sigma \rightarrow \sigma$ | T-Var |
| 2. $M\sigma : (\sigma \rightarrow \sigma) \rightarrow \sigma \rightarrow \sigma$ | 1,* T2-App |
| 3. $y : \sigma$ | Bound |
| 4. $\boxed{y : \sigma}$ | T-Var |
| 5. have $N := \lambda y : \sigma. y : \sigma \rightarrow \sigma$ | 4 T-Abst |
| 6. $M\sigma N : \sigma \rightarrow \sigma$ | 2,5 T-Abst |

■

Problem

(3.11) Recall $\perp \equiv \Pi\alpha : * . \alpha$ from 3.5. Type and prove the following term legal:

$$\lambda x : \perp. x(\perp \rightarrow \perp \rightarrow \perp)(x(\perp \rightarrow \perp)x)(x(\perp \rightarrow \perp \rightarrow \perp)xx)$$

Solution.

Proof. The type $\perp \rightarrow \perp$ is closed and well formed. Therefore, the term is legal.

- | | |
|--|-------------------|
| 1. $\perp : * \equiv \Pi\alpha : * . \alpha$ | T-Form |
| 2. $x : \perp$ | Bound |
| 3. $x(\perp \rightarrow \perp \rightarrow \perp) : \perp \rightarrow \perp \rightarrow \perp$ | *,* T2-App |
| 4. $x(\perp \rightarrow \perp) : \perp \rightarrow \perp$ | *,* T2-App |
| 5. $x(\perp \rightarrow \perp)x : \perp$ | 4,* T-App |
| 6. $x(\perp \rightarrow \perp \rightarrow \perp)(x(\perp \rightarrow \perp)x) : \perp \rightarrow \perp$ | 3,5 T-App |
| 7. $x(\perp \rightarrow \perp \rightarrow \perp)x : \perp \rightarrow \perp$ | 3,* T-App |
| 8. $x(\perp \rightarrow \perp \rightarrow \perp)xx : \perp$ | 7,* T-App |
| 9. $x(\perp \rightarrow \perp \rightarrow \perp)(x(\perp \rightarrow \perp)x)(x(\perp \rightarrow \perp \rightarrow \perp)xx) : \perp$ | 6,8 T-App |
| 10. | |

$$\lambda x : \perp. x(\perp \rightarrow \perp \rightarrow \perp)(x(\perp \rightarrow \perp)x)$$

$$(x(\perp \rightarrow \perp \rightarrow \perp)xx) : \perp \rightarrow \perp$$

9 T-Abst

■

Problem

(3.12) Given the Polymorphic Church Numerals:

$$\begin{aligned}
 \text{nat} &\in \mathbb{T}_2 := \Pi\alpha : * . (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha \\
 \bar{0} &\equiv \lambda\alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . x : \text{nat} \\
 \bar{1} &\equiv \lambda\alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . f x : \text{nat} \\
 \bar{2} &\equiv \lambda\alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . f(fx) : \text{nat} \\
 \text{succ} &\equiv \lambda n : \text{nat} . \lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f(n\beta fx)
 \end{aligned}$$

Prove that

$$\begin{aligned}
 \text{succ } \bar{0} &\underset{\beta}{=} \bar{1} \\
 \text{succ } \bar{1} &\underset{\beta}{=} \bar{2}
 \end{aligned}$$

Solution.

$$\begin{aligned}
 &\text{succ } \bar{0} \\
 &\equiv (\lambda n : \text{nat} . \lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f(n\beta fx))(\lambda\alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . x) \\
 &\xrightarrow[\beta]{} (\lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f((\lambda\alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . x)\beta fx)) \\
 &\xrightarrow[\beta_{\mathbb{T}2}]{} (\lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f((\lambda f : \beta \rightarrow \beta . \lambda x : \beta . x)fx)) \\
 &\twoheadrightarrow[\beta]{} (\lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . fx) \underset{\alpha_{\mathbb{T}2}}{\equiv} \bar{1} \\
 \\
 &\text{succ } \bar{1} \\
 &\equiv (\lambda n : \text{nat} . \lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f(n\beta fx))(\lambda\alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . fx) \\
 &\xrightarrow[\beta]{} (\lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f((\lambda\alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . fx)\beta fx)) \\
 &\xrightarrow[\beta_{\mathbb{T}2}]{} (\lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f((\lambda f : \beta \rightarrow \beta . \lambda x : \beta . fx)fx)) \\
 &\twoheadrightarrow[\beta]{} (\lambda\beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f(fx)) \underset{\alpha_{\mathbb{T}2}}{\equiv} \bar{2}
 \end{aligned}$$

Problem

(3.13 a) We define addition in Polymorphic Church Numerals as

$$\text{add} \equiv \lambda m, n : \text{nat}. \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \text{nat}. m\alpha f(n\alpha fx)$$

Show that

$$\text{add } \bar{1} \quad \bar{1} \underset{\beta}{=} \bar{2}$$

Solution.

$$\begin{aligned} & \text{add } \bar{1} \quad \bar{1} \\ & \equiv (\lambda m, n : \text{nat}. \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m\alpha f(n\alpha fx)) \bar{1} \quad \bar{1} \\ & \xrightarrow{\beta} \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. \bar{1}\alpha f(\bar{1}\alpha fx) \\ & \equiv \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. \bar{1}\alpha f((\lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. fx)\alpha fx) \\ & \xrightarrow{\beta} \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. \bar{1}\alpha f(fx) \\ & \equiv \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. (\lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. fx)\alpha f(fx) \\ & \xrightarrow{\beta} \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f(fx) \underset{\alpha}{=} \bar{2} \end{aligned}$$

Problem

(3.13 b) Find a term mul simulates multiplication on **nat**.

Solution.

$$\text{mul} \equiv \lambda m, n : \text{nat}. \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m\alpha(n\alpha f)x$$

Proof. We derive the type first to prove a legal term.

1. $m, n : \text{nat}$	Bound
2. $\alpha : *$	Bound
3. $f : \alpha \rightarrow \alpha$	Bound
4. $x : \alpha$	Bound
5. $m\alpha : (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$	*, T2-App
6. $n\alpha : (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$	*, T2-App
7. $n\alpha f : \alpha \rightarrow \alpha$	6, T-App
8. $m\alpha(n\alpha f) : \alpha \rightarrow \alpha$	5,7 T-App

9.	$m\alpha(n\alpha f)x : \alpha$	8,* T-App
10.	$\lambda x : \alpha. m\alpha(n\alpha f)x : \alpha \rightarrow \alpha$	9 T-Abst
11.	$\lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m\alpha(n\alpha f)x : (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$	10 T-Abst
12.	$\lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m\alpha(n\alpha f)x : \text{nat}$	11 T2-Abst
13.	$\lambda m, n : \text{nat}. \lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m\alpha(n\alpha f)x : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$	12 T-Abst

This proves that the term does indeed produce a natural number from two. Next let's prove that

$$\forall \bar{n}, \bar{m} : \text{nat} \quad \text{mul } \bar{n} \cdot \bar{m} = \bar{n} \times \bar{m}$$

It could be proven by induction that

$$\begin{aligned} \forall \bar{a} : \text{nat} \quad \bar{a} &\equiv \lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f^a x \\ \text{mul } \bar{n} \cdot \bar{m} &\equiv (\lambda m, n : \text{nat}. \lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m\alpha(n\alpha f)x)(\bar{n})(\bar{m}) \\ &\xrightarrow[\beta]{\Rightarrow} \lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. \bar{n}\alpha(\bar{m}\alpha f)x \\ &\xrightarrow[\beta]{\Rightarrow} \lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. \bar{n}\alpha(\lambda u : \alpha. f^m u)x \\ &\xrightarrow[\beta]{\Rightarrow} \lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. (\lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f^n x)(\lambda u : \alpha. f^m u)x \\ &\xrightarrow[\beta]{\Rightarrow} \lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. (\lambda u : \alpha. f^m u)^n x \end{aligned}$$

By induction this can be further beta-reduced to

$$\xrightarrow[\beta]{\Rightarrow} \lambda\alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f^{mn} x \equiv \bar{m}\bar{n}$$

■