

EXERCISES

CHAPTER 5

SEAN LI ¹

1. Reducted

Definition Some rules for reference.

λP Calculus Rules

$$\begin{array}{c}
 \frac{}{\emptyset \vdash * : \square} \text{Sort} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \text{Var} \quad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \text{Weak} \\
 \\
 \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A . B : s} \text{Form} \quad \frac{\Gamma \vdash M : \Pi x : A . B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B [x := N]} \text{App} \\
 \\
 \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A . B : s}{\Gamma \vdash \lambda x : A . M : \Pi x : A . B} \text{Abst} \\
 \\
 \frac{\Gamma \vdash A : B \quad B \stackrel{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \text{Conv}
 \end{array}$$

Predicate Logic

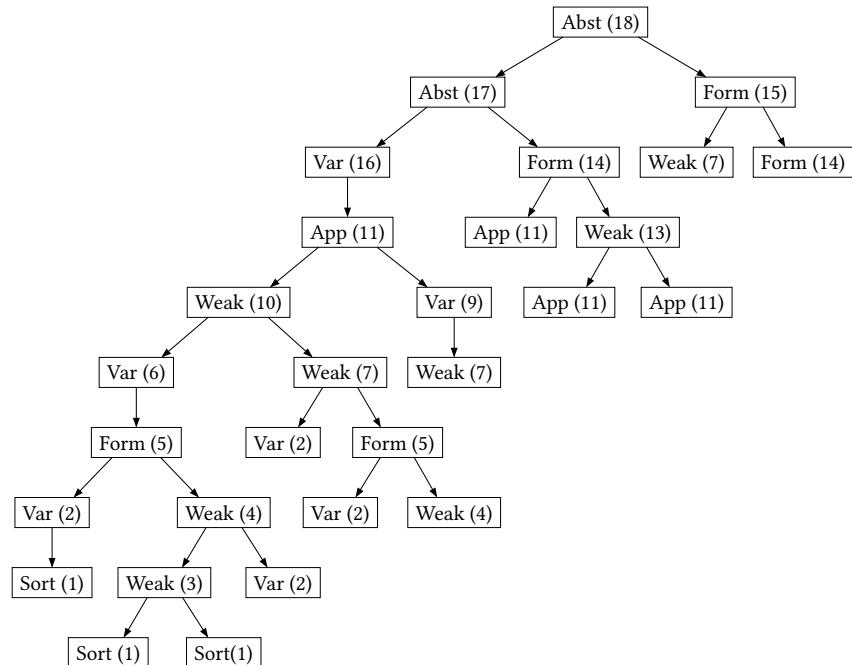
$$\begin{array}{ccc}
 \begin{array}{l}
 1. \text{ Assume } A \\
 2. \quad \boxed{\dots} \\
 3. \quad \boxed{B}
 \end{array}
 &
 \frac{}{A \Rightarrow B} \Rightarrow I \quad \frac{A \Rightarrow B \quad A}{B} \Rightarrow E &
 \begin{array}{l}
 1. \text{ Let } a \in S \\
 2. \quad \boxed{\dots} \\
 3. \quad \boxed{P(a)}
 \end{array}
 \\
 &
 &
 \frac{}{\forall a \in S, P(a)} \forall I
 \end{array}$$

$$\frac{\forall a \in S \quad N \in S}{P(N)} \forall E$$

Problem

(5.1) Give a diagram of the tree corresponding to the complete tree derivation of line 18 of Section 5.3 (P 107)

Solution.



Problem

(5.2) Give a complete λP derivation of

$$S : * \vdash S \rightarrow S \rightarrow * : \square$$

In tree format and flag format.

Solution.

Tree Derivation.

$$(7) \frac{(3) S : * \vdash S : * \quad (4) \frac{\vdash * : \square \quad \vdash * : \square}{(6) \frac{S : * \vdash * : \square}{S : *, x : S \vdash * : \square} \text{ Weak}} \text{ Weak}}{S : * \vdash S \rightarrow * : \square}$$

$$(3) \frac{\vdash * : \square}{S : * \vdash S : *} \text{Var} \quad (9) \frac{(7) S : * \vdash S \rightarrow * : \square \quad (3) S : * \vdash S : *}{S : *, x : S \vdash S \rightarrow * : \square} \text{Weak} \\ \frac{}{S : * \vdash S \rightarrow S \rightarrow * : \square} \text{Form}$$

■

Flag Derivation.

1. $* : \square$	Sort
2. $S : *$	
3. $\boxed{S : *}$	1 Var
4. $\boxed{* : \square}$	1,1 Weak
5. $x : S$	
6. $\boxed{x : S}$	4,3 Weak
7. $\boxed{S \rightarrow * : \square}$	3,6 Form
8. $x : S$	
9. $\boxed{x : S}$	7,3 Weak
10. $\boxed{S \rightarrow S \rightarrow * : \square}$	3,9 Form

■

Problem

(5.3) Derive

$$S : *, Q : S \rightarrow S \rightarrow * \vdash \Pi x : S . \Pi y : S . Q x y : *$$

Solution.

1. $* : \square$	Sort
2. $S : *$	
3. $\boxed{S : *}$	1 Var
4. $\boxed{* : \square}$	1,1 Weak
5. $x : S$	
6. $\boxed{x : S}$	4,3 Weak
7. $\boxed{S \rightarrow * : \square}$	3,6 Form
8. $x : S$	
9. $\boxed{x : S}$	7,3 Weak
10. $\boxed{S \rightarrow S \rightarrow * : \square}$	3,9 Form
11. $\boxed{Q : S \rightarrow S \rightarrow *}$	

12.	$Q : S \rightarrow S \rightarrow *$	10 Var
13.	$S : *$	3,10 Weak
14.	$* : \square$	4,10 Weak
15.	$x : S$	
16.	$* : \square$	14,13 Weak
17.	$S : *$	13,13 Weak
18.	$x : S$	13 Var
19.	$Q : S \rightarrow S \rightarrow *$	12,13 Weak
20.	$y : S$	
21.	$y : S$	17 Var
22.	$Q : S \rightarrow S \rightarrow *$	19,17 Weak
23.	$x : S$	18,17 Weak
24.	$Q x : S \rightarrow *$	22,23 App
25.	$\underline{Q x y : *}$	24,21 App
26.	$\Pi y : S . Q x y : *$	17,25 Form
27.	$\Pi x : S . \Pi y : S . Q x y : *$	13,26 Form

Problem

(5.4) Prove that $*$ is the only valid kind in λP .

Solution.

Proof. The only possible way to construct a new kind is through the Form rule and the Sort axiom. Because we are trying to construct a kind, s here stands for \square .

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : \square}{\Gamma \vdash \Pi x : A . B : \square} \text{Form}$$

One could only construct new kinds with kinds, which requires $A : \square$ and $B : \square$. This contradicts with $A : *$. \blacksquare

Problem

(5.5) Prove that $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ is a tautology by given a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash A \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$x : A$	
4.	$y : A \rightarrow B$	
5.	$\boxed{y x : B}$	4,3 App
6.	$\lambda y : A \rightarrow B . y x : (A \rightarrow B) \rightarrow B$	5 Abst
7.	$\lambda x : A . \lambda y : A \rightarrow B . y x : A \rightarrow (A \rightarrow B) \rightarrow B$	5 Abst

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Problem

(5.6 a) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ a tautology using natural deduction.

Solution.

Proof.

1.	Assume $A \Rightarrow (A \Rightarrow B)$	
2.	$A \Rightarrow (A \Rightarrow B)$	
3.	Assume A	
4.	A	
5.	$A \Rightarrow B$	2,4 $\Rightarrow E$
6.	B	5,4 $\Rightarrow E$
7.	$A \Rightarrow B$	3,6 $\Rightarrow I$
8.	$(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$	1,7 $\Rightarrow I$

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Problem

(5.6 b) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ using a shorthand λP derivation

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$x : A \rightarrow A \rightarrow B$	
4.	$y : A$	
5.	$x y : A \rightarrow B$	3,4 App
6.	$x y y : B$	5,4 App
7.	$\lambda y : A . x y y : A \rightarrow B$	6 Abst
8.	$\lambda x : A \rightarrow A \rightarrow B . \lambda y : A . x y y : (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$	7 Abst

■

Problem

(5.7 a) Proof $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of A , B , and C is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$C : *$	
4.	$x : A \rightarrow B$	
5.	$y : B \rightarrow C$	
6.	$a : A$	
7.	$x a : B$	4,6 App
8.	$y (x a) : C$	5,7 App

9.	$\lambda a : A . y (x z) : A \rightarrow C$	8 Abst
10.	$\lambda y : B \rightarrow C . \lambda a : A . y (x z) : (B \rightarrow C) \rightarrow A \rightarrow C$	9 Abst
11.	$\lambda x : A \rightarrow B . \lambda y : B \rightarrow C . \lambda a : A . y (x z)$ $: (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C$	10 Abst

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Problem

(5.7 b) Proof $((A \Rightarrow B) \Rightarrow A) \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$x : (A \rightarrow B) \rightarrow A$	
4.	$y : A \rightarrow B$	
5.	$x y : A$	3,4 App
6.	$y (x y) : B$	4,5 App
7.	$\lambda y : A \rightarrow B . y (x y) : (A \rightarrow B) \rightarrow B$	6 Abst
8.	$\lambda x : (A \rightarrow B) \rightarrow A . \lambda y : A \rightarrow B . y (x y)$ $: ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B$	7 Abst

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Problem

(5.7 c) Proof $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of A , B , and C is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$C : *$	
4.	$x : A \rightarrow B \rightarrow C$	
5.	$y : A \rightarrow B$	
6.	$a : A$	
7.	$x a : B \rightarrow C$	4,6 App
8.	$y a : B$	5,6 App
9.	$x a (y a) : C$	7,8 App
10.	$\lambda a : A . x a (y a) : A \rightarrow C$	9 Abst
11.	$\lambda y : A \rightarrow B . \lambda a : A . x a (y a) : (A \rightarrow B) \rightarrow A \rightarrow C$	10 Abst
12.	$\lambda x : A \rightarrow B \rightarrow C . \lambda y : A \rightarrow B . \lambda a : A . x a (y a)$	
	$: (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$	10 Abst

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Problem

(5.8 a) Let $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *$, find an inhabitant of

$$\Pi x : S . P x \rightarrow Q x \rightarrow P x$$

with respect to Γ and give a shorthand λP derivation

Solution.

1.	$S : *$	
2.	$P : S \rightarrow *$	
3.	$Q : S \rightarrow *$	
4.	$x : S$	
5.	$a : P x$	
6.	$b : Q x$	
7.	$a : P x$	2,4 App

8.	$\boxed{\lambda b : Q x . S : Q x \rightarrow P x}$	7 Abst
9.	$\boxed{\lambda a : P x . \lambda b : Q x . a : P x \rightarrow Q x \rightarrow P x}$	8 Abst
10.	$\boxed{\lambda x : S . \lambda a : P x . \lambda b : Q x . a : P x \rightarrow Q x \rightarrow P x}$	9 Abst

Problem

(5.8 b) Let $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *$, find an inhabitant of

$$\Pi x : S . P x \rightarrow Q x \rightarrow P x$$

By proving the corresponding proposition in natural deduction.

Solution. The corresponding proposition and premises are

$$\frac{S \in \text{Set} \quad P : S \rightarrow \text{Prop} \quad Q : S \rightarrow \text{Prop}}{\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a))}$$

Proof.

1. Let $a \in S$
2. Assume $P(a)$
3. Assume $Q(a)$
4. $\boxed{P(a)}$
5. $\boxed{Q(a) \Rightarrow P(a)} \quad 3,4 \Rightarrow I$
6. $\boxed{P(a) \Rightarrow (Q(a) \Rightarrow P(a))} \quad 2,5 \Rightarrow I$
7. $\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a)) \quad 1,6 \forall I$

■

Problem

(5.9 a) Give proof for

$$(\forall x \in S, Q(x)) \Rightarrow (\forall y \in S, P(y) \Rightarrow Q(y))$$

by natural deduction and a λP derivation.

Solution.

Natural Deduction.

1. Assume $\forall x \in S, Q(x)$
2. Let $y \in S$
3. Assume $P(y)$
4. $\boxed{Q(y)}$
5. $\boxed{P(y) \Rightarrow Q(y)}$
6. $\boxed{\forall y \in S, P(y) \Rightarrow Q(y)}$
7. $(\forall x \in S, Q(x)) \Rightarrow (\forall y \in S, P(y) \Rightarrow Q(y))$ **1,6 $\Rightarrow I$**

■

λP Derivation. Corresponding type is

$$S : *, P : S \rightarrow *, Q : S \rightarrow * \vdash (\Pi x : S . Q x) \rightarrow (\Pi y : S . P y \rightarrow Q y)$$

1. $S : *$
2. $P : S \rightarrow *$
3. $Q : S \rightarrow *$
4. $a : \Pi x : S . Q x$
5. $y : S$
6. $z : P y$
7. $\boxed{a y : Q y}$
8. $\boxed{\lambda z : P y . a y : P y \rightarrow Q y}$
9. $\boxed{\lambda y : S . \lambda z : P y . a y : \Pi y : S . P y \rightarrow Q y}$
10. $\boxed{\lambda a : \Pi x : S . Q x . \lambda y : S . \lambda z : P y . a y : (\Pi x : S . Q x) \rightarrow (\Pi y : S . P y \rightarrow Q y)}$ **7 Abst**

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Problem

(5.9 b) Give proof for

$$\forall x \in S, (P(x) \Rightarrow Q(x)) \Rightarrow ((\forall y \in S, P(y)) \Rightarrow (\forall z \in S, Q(z)))$$

by natural deduction and a λP derivation

Solution.

Natural Deduction.

1. Assume $\forall x \in S, (P(x) \Rightarrow Q(x))$

2.	Assume $\forall y \in S, P(y)$	
3.	Let $z \in S$	
4.	$P(z)$	2,3 $\forall E$
5.	$P(z) \Rightarrow Q(z)$	1,3 $\forall E$
6.	$Q(z)$	5,4 $\Rightarrow E$
7.	$\forall z \in S, Q(z)$	3,6 $\forall I$
8.	$\forall y \in S, P(y) \Rightarrow (\forall z \in S, Q(z))$	2,7 $\forall I$
9.	$\forall x \in S, (P(x) \Rightarrow Q(x)) \Rightarrow ((\forall y \in S, P(y)) \Rightarrow (\forall z \in S, Q(z)))$	1,8 $\forall I$

■

λP Derivation. Corresponding type is

$$\begin{aligned} & S : *, P : S \rightarrow *, Q : S \rightarrow * \\ & \vdash (\Pi x : S . P x \rightarrow Q x) \rightarrow (\Pi y : S . P y) \rightarrow (\Pi z : S . Q z) \end{aligned}$$

1.	$S : *$	
2.	$P : S \rightarrow *$	
3.	$Q : S \rightarrow *$	
4.	$a : \Pi x : S . P x \rightarrow Q x$	
5.	$b : \Pi y : S . P y$	
6.	$z : S$	
7.	$b z : P z$	5,6 App
8.	$a z : P z \rightarrow Q z$	4,6 App
9.	$a z(b z) : Q z$	8,7 App
10.	$\lambda z : S . a z(b z) : \Pi z : S . Q z$	9 Abst
11.	$\lambda b : (\Pi y : S . P y) . \lambda z : S . a z(b z)$ $: (\Pi y : S . P y) \rightarrow \Pi z : S . Q z$	10 Abst
12.	$\lambda a : (\Pi x : S . P x \rightarrow Q x) . \lambda b : (\Pi y : S . P y) .$ $\lambda z : S . a z(b z)$ $: (\Pi x : S . P x \rightarrow Q x) \rightarrow$ $(\Pi y : S . P y) \rightarrow \Pi z : S . Q z$	10 Abst

■

Problem

(5.10) Given a context

$$\begin{aligned}\Gamma &\equiv S : *, P : S \rightarrow *, f : S \rightarrow S, g : S \rightarrow S, \\ u &: \Pi x : S . (P (f x) \rightarrow P (g x)), \\ v &: \Pi x, y : S . ((P x \rightarrow P y) \rightarrow P (f x))\end{aligned}$$

Let

$$M \equiv \lambda x : S . v (f x)(g x)(u x)$$

Type M under Γ .

Solution.

- | | | |
|-----|---|-----------------|
| 1. | $x : S$ | |
| 2. | $f : S \rightarrow S$ | |
| 3. | $f x : S$ | 2,1 App |
| 4. | $g : S \rightarrow S$ | |
| 5. | $g x : S$ | 4,1 App |
| 6. | $u : \Pi x : S . (P (f x) \rightarrow P (g x))$ | |
| 7. | $u x : P (f x) \rightarrow P (g x)$ | 6,1 App |
| 8. | $v : \Pi x, y : S . ((P x \rightarrow P y) \rightarrow P (f x))$ | |
| 9. | $v (f x) : \Pi y : S . ((P (f x) \rightarrow P y) \rightarrow P (f (f x)))$ | 8,3 App |
| 10. | $v (f x)(g x) : (P (f x) \rightarrow P (g x)) \rightarrow P (f (f x))$ | 9,5 App |
| 11. | $v (f x)(g x)(u x) : P (f (f x))$ | 10,7 App |
| 12. | $\lambda x : S . v (f x)(g x)(u x) : S \rightarrow P (f (f x))$ | 11 Abst |

Problem

(5.11) Let S be a set, with Q and R relations on $S \times S$, and let f and g be functions from S to S . Assume

$$\begin{aligned}\forall x, y \in S (Q(x, f(y)) \Rightarrow Q(g(x), y)) \\ \forall x, y \in S (Q(x, f(y)) \Rightarrow R(x, y)) \\ \forall x \in S (Q(x, f(f(x))))\end{aligned}$$

Prove that

$$\forall x \in S, R(g(g(x)), g(x))$$

By giving a context Γ and finding a term M such that

$$\Gamma \vdash M : \Pi x : S . R(g(gx))(gx)$$

Solution. Context Γ is as follows:

$$\begin{aligned}\Gamma \equiv S : *, f : S \rightarrow S, g : S \rightarrow S \\ Q : S \rightarrow S \rightarrow *, R : S \rightarrow S \rightarrow * \\ A : \Pi x, y : S . (Q x (f y) \rightarrow Q (g x) y), \\ B : \Pi x, y : S . (Q x (f y) \rightarrow R x y) \\ C : \Pi x : S . Q x (f (f x))\end{aligned}$$

Derivation.

1.	$S : *, f : S \rightarrow S, g : S \rightarrow S$	
2.	$Q : S \rightarrow S \rightarrow *, R : S \rightarrow S \rightarrow *$	
3.	$A : \Pi x, y : S . (Q x (f y) \rightarrow Q (g x) y)$	
4.	$B : \Pi x, y : S . (Q x (f y) \rightarrow R x y)$	
5.	$C : \Pi x : S . Q x (f (f x))$	
6.	$x : S$	
7.	$gx : S$	1,6 App
8.	$C(gx) : Q(gx)(f(f(gx)))$	5,7 App
9.	$f(gx) : S$	1,7 App
10.	$A(gx) : \Pi y : S . (Q(gx)(fy) \rightarrow (Q(g(gx))y))$	3,7 App
	$A(gx)(f(gx))$	
	$: (Q(gx)(f(f(gx))))$	
11.	$\rightarrow (Q(g(gx))(f(gx)))$	
	10,9 App	

$$12. \quad \boxed{A(gx)(fx)(Cx) \\ : (Q(g(gx))(fx))}$$

11,8 App

$$13. \quad \boxed{g(gx) : S \\ B(g(gx)) \\ : \Pi y : S . (Q(g(gx))(fy) \\ \rightarrow R(g(gx))y)}$$

1,7 App

4,13 App

$$15. \quad \boxed{B(g(gx))(gx) \\ : (Q(g(gx))(fx)) \rightarrow (R(g(gx))(gx))}$$

14,7 App

$$16. \quad \boxed{B(g(gx))(gx)(Ax)(fx)(Cx) \\ : (R(g(gx))(gx))}$$

15,12 App

$$17. \quad \boxed{\lambda x : S . B(g(gx))(gx)(Ax)(fx)(Cx) \\ : \Pi x : S . (R(g(gx))(gx))}$$

17 Abst

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