Exercises

Chapter 7

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1. Reducted

Reference - Propositional Logic in λC

$$\frac{A \quad \neg\, A}{\bot}\ \bot\text{I or } \neg\text{E} \qquad \frac{\bot}{A}\ \bot\text{E} \qquad \frac{A \quad B}{A \wedge B}\ \wedge\text{I} \qquad \frac{A \wedge B}{A}\ \wedge\text{EL}$$

$$\frac{A \wedge B}{B}\ \wedge\text{ER} \qquad \frac{a}{a \vee b}\ \vee\text{IL} \qquad \frac{b}{a \vee b}\ \vee\text{IR} \qquad \frac{A \Rightarrow B \quad A}{B}\ \Rightarrow\text{E}$$

$$\frac{A \vee B \quad A \to C \quad B \to C}{C}\ \vee\text{E} \qquad \frac{a \in S \quad P\,(a)}{\exists a \in S, P\,(a)}\ \exists\text{I}$$

$$
\begin{array}{ll}
1. & A \\
2. & \ \vert\ \ldots \\
3. & \ \vert\ \bot \\
\hline
& \neg\, A
\end{array}\ \neg\text{I}
\qquad
\begin{array}{ll}
1. & A \\
2. & \ \vert\ \ldots \\
3. & \ \vert\ B \\
\hline
& A \Rightarrow B
\end{array}\ \Rightarrow\text{I}
\qquad
\begin{array}{ll}
1. & a \in S \\
2. & \ \vert\ \ldots \\
3. & \ \vert\ P\,(a) \\
\hline
& \forall a \in S, P\,(a)
\end{array}\ \forall I
$$

$$\frac{\exists x \in S, P\,(x) \quad \forall x \in S, (P\,(x) \Rightarrow A)}{A}\ \exists\text{E} \qquad \frac{a \in S \quad \forall x \in S, P\,(x)}{P\,(a)}\ \forall\text{E}$$

$$\frac{}{\neg\neg\, A \Rightarrow A}\ \text{DN (Classical)} \qquad \frac{}{A \vee \neg\, A}\ \text{ET (Classical)}$$

Problem

(7.1 a) Prove in natural deduction and λC the tautology

$$B \Rightarrow (A \Rightarrow B)$$

Solution.

Natural Deduction.

1. B
2. $\quad A$
3. $\quad\quad B$
4. $\quad A \Rightarrow B \qquad \Rightarrow$**I**
5. $B \Rightarrow (A \Rightarrow B) \quad \Rightarrow$**I**

■

λC. Assuming context $\Gamma \equiv A : *, B : *$. By the PAT paradigm the proof is equivalent to an inhabitant of $B \to A \to B$.

1. $A : *, B : *$
2. $\quad x : B$
3. $\quad\quad y : A$
4. $\quad\quad\quad x : B$ \qquad\qquad\qquad **Weak**
5. $\quad\quad \lambda y : A . x : A \to B$ \qquad **4 Abst**
6. $\quad \lambda x : B . \lambda y : A . x : B \to A \to B$ \quad **5 Abst**

■

(7.1 b) Prove in natural deduction and λC the tautology

$$\neg\, A \Rightarrow (A \Rightarrow B)$$

Solution.

Natural Deduction.

1. $\neg\, A$
2. A
3. $\neg\, A$
4. A
5. $\bot$ $\bot\mathbf{I}$
6. B $\bot\mathbf{E}$
7. $A \Rightarrow B$ $\Rightarrow\mathbf{I}$
8. $\neg\, A \Rightarrow (A \Rightarrow B)$ $\Rightarrow\mathbf{I}$

∎

λC. Assuming context $\Gamma \equiv A : *, B : *$. By the PAT paradigm the proof is equivalent to an inhabitant of $(A \to \bot) \to A \to B$.

1. $A : *, B : *$
2. $x : \neg\, A$
3. $y : A$
4. $x\, y : \Pi\alpha : * .\alpha$ **2,3 App (Neg Elim)**
5. $x\, y\, B : B$ **4,1 App (Ex Falso)**
6. $\lambda y : A\,.\, x\, y\, B : A \to B$ **5 Abst**
7. $\lambda x : \neg\, A\,.\lambda y : A\,.\, x\, y\, B : \neg\, A \to A \to B$ **6 Abst**

∎

(7.1 c) Prove in natural deduction and λC the tautology

$$(A \Rightarrow \neg\, B) \Rightarrow ((A \Rightarrow B) \Rightarrow \neg\, A)$$

Solution.

Natural Deduction.

1. $A \Rightarrow \neg B$
2. $\quad$ | $A \Rightarrow B$
3. $\quad$ | | A
4. $\quad$ | | | $\neg B$ $\qquad\qquad\qquad$ **1,3 $\Rightarrow$E**
5. $\quad$ | | | B $\qquad\qquad\qquad$ **2,3 $\Rightarrow$E**
6. $\quad$ | | | $\bot$ $\qquad\qquad\qquad$ **5,4 $\bot$I**
7. $\quad$ | | $\neg A$ $\qquad\qquad\qquad$ **3,6 $\neg$I**
8. $\quad$ | $(A \Rightarrow B) \Rightarrow \neg A$ $\qquad$ **2,7 $\Rightarrow$I**
9. $(A \Rightarrow \neg B) \Rightarrow ((A \Rightarrow B) \Rightarrow \neg A)$ $\quad$ **1,8 $\Rightarrow$I**

$\blacksquare$

λC. Assuming context $\Gamma \equiv A : *, B : *$. The proof should be equivalent to an inhabitant of $(A \to B \to \bot) \to (A \to B) \to A \to \bot$.

1. $A : *, B : *$
2. $\quad$ | $h : A \to \neg B$
3. $\quad$ | | $q : A \to B$
4. $\quad$ | | | $a : A$
5. $\quad$ | | | | $q\,a : B$ $\qquad\qquad\qquad$ **3,4 App**
6. $\quad$ | | | | $h\,a : B \to \bot$ $\qquad\qquad$ **2,4 App**
7. $\quad$ | | | | $h\,a\,(q\,a) : \bot$ $\qquad\qquad$ **6,5 App (Neg Elim)**
8. $\quad$ | | | $\lambda a : A \,.\, h\,a\,(q\,a) : \neg A$ $\qquad$ **7 Abst (Neg Intro)**
9. $\quad$ | | $\lambda q : A \to B \,.\lambda a : A \,.\, h\,a\,(q\,a) : A \to B \to \neg A$ $\quad$ **8 Abst**
$\quad$ | $\lambda h : A \to \neg B \,.\lambda q : A \to B \,.\lambda a : A \,.\, h\,a\,(q\,a)$
10. $\quad$ | $\quad : (A \to \neg B) \to A \to B \to \neg A$ $\qquad$ **9 Abst**

$\blacksquare$

Problem

(7.1 d) Prove in natural deduction and λC the tautology

$$\neg(A \Rightarrow B) \Rightarrow \neg B$$

Solution.

Natural Deduction.

1. $\neg(A \Rightarrow B)$

2. B

3. A

4. B

5. $A \Rightarrow B$ **3,4 $\Rightarrow$I**

6. $\bot$ **5,1 $\bot$I**

7. $\neg B$ **6 $\neg$I**

8. $\neg(A \Rightarrow B) \Rightarrow \neg B$ **1,7 $\Rightarrow$I**

■

λC.   Assuming context $\Gamma \equiv A : *, B : *$. The proof should be equivalent to an inhabitant of $((A \to B) \to \bot) \to B \to \bot$.

1. $n : \neg(A \to B)$

2. $b : B$

3. $a : A$

4. $b : B$ **Weak**

5. $\lambda a : A . b : A \to B$ **4 Abst**

6. $n \, (\lambda a : A . b) : \bot$ **1,5 App (Neg Elim)**

7. $\lambda b : B . n \, (\lambda a : A . b) : \neg B$ **6 Abst (Neg Intro)**

8.

$\lambda n : \neg(A \to B).\lambda b : B . n \, (\lambda a : A . b)$

  $: \neg(A \to B) \to \neg B$ **7 Abst**

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Problem

(7.2) Formulate the double negation law as an axiom in λC, and prove the following tautology in λC with DN.

$$(\neg A \Rightarrow A) \Rightarrow A$$

Solution. The rule

$$\frac{}{\neg\neg A \Rightarrow A} \text{ DN-E}$$

Could be translated into lambda calculus as

$$\Pi A : * .((A \rightarrow \bot) \rightarrow \bot) \rightarrow A$$

Proof. Assume context $\Gamma \equiv A : * .$

1. $A : *$
2. $h : \neg A \rightarrow A$
3. $x : \neg A$
4. $h\,x : A$ **2,3 App**
5. $x\,(h\,x) : \bot$ **3,4 App (Contradiction)**
6. $\lambda x : \neg A . x\,(h\,x) : \neg\neg A$ **5 Abst (Neg Intro)**
7. $\text{DN}\,A : \neg\neg A \rightarrow A$ **1,1 App**
8. $\text{DN}\,A\,(\lambda x : \neg A . x\,(h\,x)) : A$ **App (Axiom DN)**

      $\lambda h : \neg A \rightarrow A . \text{DN}\,A\,(\lambda x : \neg A . x\,(h\,x))$

9. $: (\neg A \rightarrow A) \rightarrow A$ **8 Abst**

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