Excercises

Chapter 3

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1. Reducted

Definition Some rules for reference.

$$\frac{x : \sigma \in \Gamma \qquad \Gamma \text{ is a } \lambda 2 \text{ context}}{\Gamma \vdash x : \sigma}\,(\text{T-Var}) \qquad \frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}\,(\text{T-App})$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma . M : \sigma \to \tau}\,(\text{T-Abst}) \qquad \frac{\alpha \in \mathbb{T}_2 \qquad \forall \tau \in \text{FV } \alpha, \Gamma \vdash \tau : *}{\alpha : * \in \Gamma}\,(\text{T-Form})$$

$$\frac{\Gamma \vdash M : \Pi_{\alpha:*}.A \qquad \Gamma \vdash B : *}{\Gamma \vdash MB : A[\alpha := B]}\,(\text{T2-App}) \qquad \frac{\Gamma, \alpha : * \vdash M : A}{\Gamma \vdash \lambda \alpha : * .M : \Pi_{\alpha:*} : A}\,(\text{T2-Abst})$$

In this document, convention is that all type judgements in a proof tree, unless stated otherwise, is derived from a single and unique $\lambda 2$ context per tree. Multiple conclusions might be drawn on a single line from usage of the same inference rule for compactness. Eg:

ex 1. $\alpha, \beta : *$ **T-Form**

Is shorthand for

ex 1. $\Gamma \vdash \alpha : *$ **T-Form**
ex 2. $\Gamma \vdash \beta : *$ **T-Form**

Problem

(3.1) How many $\lambda 2$ contexts consisting of four and only four declarations

$$\begin{array}{ll} (1)\ \Gamma \vdash \alpha : * & (2)\ \Gamma \vdash \beta : * \\ (3)\ \Gamma \vdash f : \alpha \to \beta & (4)\ \Gamma \vdash x : \alpha \end{array}$$

Solution. The last two declarations depende on the first two. Therefore this is an easy combinatorics problem: $2! \times 2! = 4$ contexts:

$$\begin{array}{ll} 1-2-3-4 & 1-2-4-3 \\ 2-1-3-4 & 2-1-4-3 \end{array}$$

Problem

(3.2) Give a full derivation in $\lambda 2$ to show the following type term is legal:

$$M \equiv \lambda \alpha, \beta, \gamma : * . \lambda f : \alpha \to \beta . \lambda g : \beta \to \gamma . \lambda x : \alpha . g(fx)$$

Solution.

1.	$\alpha : *$	**Bound**
2.	$\beta : *$	**Bound**
3.	$\gamma : *$	**Bound**
4.	$f : \alpha \to \beta$	**Bound**
5.	$g : \beta \to \gamma$	**Bound**
6.	$x : \alpha$	**Bound**
7.	$\alpha, \beta, \gamma : *$	**T-Form**
8.	$\alpha \to \beta, \beta \to \gamma : *$	**T-Form**
9.	$f : \alpha \to \beta, x : \alpha$	**T-Var**
10.	$fx : \beta$	**8,8 T-App**
11.	$g : \beta \to \gamma$	**T-Var**
12.	$g(fx) : \gamma$	**11,10 T-App**
13.	$\lambda x : \alpha . g(fx) : \alpha \to \gamma$	**12 T-Abst**
14.	$\lambda g : \beta \to \gamma . \lambda x : \alpha . g(fx) : (\beta \to \gamma) \to \alpha \to \gamma$	**13 T-Abst**
15.	$\lambda f : \alpha \to \beta . \lambda g : \beta \to \gamma . \lambda x : \alpha . g(fx)$ $: (\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$	**14 T-Abst**

16. $\left|\begin{array}{l}\lambda\gamma:*.\lambda f:\alpha\to\beta.\lambda g:\beta\to\gamma.\lambda x:\alpha.g(fx) \\ \quad :\Pi\gamma:*.(\alpha\to\beta)\to(\beta\to\gamma)\to\alpha\to\gamma\end{array}\right.$ **15 T2-Abst**

17. $\lambda\beta,\gamma:*.\lambda f:\alpha\to\beta.\lambda g:\beta\to\gamma.\lambda x:\alpha.g(fx)$
$\quad :\Pi\beta,\gamma:*.(\alpha\to\beta)\to(\beta\to\gamma)\to\alpha\to\gamma$ **16 T2-Abst**

18.

$\lambda\alpha,\beta,\gamma:*.\lambda f:\alpha\to\beta.\lambda g:\beta\to\gamma.\lambda x:\alpha.g(fx)$
$\quad :\Pi\alpha,\beta,\gamma:*.(\alpha\to\beta)\to(\beta\to\gamma)\to\alpha\to\gamma$ **17 T2-Abst**

Problem

(3.3 a) Given M in 3.2, and a context Γ such that

$$\Gamma\vdash\texttt{nat}:*$$

$$\Gamma\vdash\texttt{bool}:*$$

$$\Gamma\vdash\text{succ}:\texttt{nat}\to\texttt{nat}$$

$$\Gamma\vdash\text{even}:\texttt{nat}\to\texttt{bool}$$

Prove M $\texttt{nat}$ $\texttt{nat}$ $\texttt{bool}$ succ even is legal.

Solution. Proof by deriving the term's type.

Proof.

1. $M:\Pi\alpha,\beta,\gamma:*.(\alpha\to\beta)\to(\beta\to\gamma)\to\alpha\to\gamma$ **T-Var**
2. $\texttt{nat},\texttt{bool}:*$ **T-Form**
3. $M\ \texttt{nat}:\Pi\beta,\gamma:*.(\texttt{nat}\to\beta)\to(\beta\to\gamma)\to\texttt{nat}\to\gamma$ **2,1 T2-App**
4. $M\ \texttt{nat}\ \texttt{nat}:\Pi\gamma:*.(\texttt{nat}\to\texttt{nat})\to(\texttt{nat}\to\gamma)\to\texttt{nat}\to\gamma$ **2,3 T2-App**
5.

$M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}:(\texttt{nat}\to\texttt{nat})\to(\texttt{nat}\to\texttt{bool})\to\texttt{nat}\to\texttt{bool}$ **2,3 T2-App**

6. $\text{succ}:\texttt{nat}\to\texttt{nat},\text{even}:\texttt{nat}\to\texttt{bool}$ **T-Var**
7. $M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \text{succ}:(\texttt{nat}\to\texttt{bool})\to\texttt{nat}\to\texttt{bool}$ **6,5 T-App**
8. $M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \text{succ}\ \text{even}:\texttt{nat}\to\texttt{bool}$ **6,7 T-App**

■

Problem

(3.3 b.i) Prove $\lambda x : \mathtt{nat}.\ \text{even}\ (\text{succ}\ x)$ via 3.3 a.

Solution. The result of beta reduction on the term in 3.3 a is what we are proving.

Proof.

$$
\begin{aligned}
& M\ \mathtt{nat}\ \mathtt{nat}\ \mathtt{bool}\ \text{succ even} \\
\equiv\ & (\lambda \alpha, \beta, \gamma, f, g.\lambda x : \alpha.g(fx))\ \mathtt{nat}\ \mathtt{nat}\ \mathtt{bool}\ \text{succ even} \\
\underset{\beta}{\twoheadrightarrow}\ & (\lambda f : \mathtt{nat} \to \mathtt{nat}.\lambda g : \mathtt{nat} \to \mathtt{bool}.\lambda x : \mathtt{nat}.g(f(x)))\ \text{succ even} \\
\underset{\beta}{\to}\ & (\lambda x : \mathtt{nat}.\ \text{even}\ (\text{succ}\ x))
\end{aligned}
$$

By the subject reduction lemma, $\lambda x : \mathtt{nat}.\ \text{even}\ (\text{succ}\ x)) : \mathtt{nat} \to \mathtt{bool}$, thus is legal. ■

Problem

(3.3 b.ii) Prove $\lambda x : \mathtt{nat}.\ \text{even}\ (\text{succ}\ x)$ via derivation in the context provided in 3.3 a.

Solution.

Proof.

1.	$\mathtt{nat}, \mathtt{bool} : *$	**T-Form**
2.	$x : \mathtt{nat}$	**Bound**
3.	$\quad \text{succ} : \mathtt{nat} \to \mathtt{nat}$	**T-Var**
4.	$\quad x : \mathtt{nat}$	**T-Var**
5.	$\quad \text{succ}\ x : \mathtt{nat}$	**3,4 T-App**
6.	$\quad \text{even} : \mathtt{nat} \to \mathtt{bool}$	**T-Var**
7.	$\quad \text{even}\ (\text{succ}\ x) : \mathtt{bool}$	**6,5 T-App**
8.	$\lambda x : \mathtt{nat}.\ \text{even}\ (\text{succ}\ x) : \mathtt{nat} \to \mathtt{bool}$	**7 T-Abst**

■

Problem

(3.4) Give a shorthanded (omit T-Var and T-Form) derivation in $\lambda 2$ to show the following term is valid in $\Gamma \equiv \texttt{nat} : *, \texttt{bool} : *$

$$(\lambda \alpha, \beta : * . \lambda f : \alpha \to \alpha . \lambda g : \alpha \to \beta . \lambda x : \alpha . g(f(fx)))\ \texttt{nat}\ \texttt{bool}$$

Solution.

Proof.

1. $\alpha, \beta : *$ — **Bound**
2. $f : \alpha \to \alpha$ — **Bound**
3. $g : \alpha \to \beta$ — **Bound**
4. $x : \alpha$ — **Bound**
5. $fx : \alpha$ — ***,* T-App**
6. $f(fx) : \alpha$ — ***,5 T-App**
7. $g(f(fx)) : \beta$ — ***,6 T-App**
8. $\lambda x : \alpha . g(f(fx)) : \alpha \to \beta$ — **7 T-Abst**
9. $\lambda g : \alpha \to \beta . \lambda x : \alpha . g(f(fx)) : (\alpha \to \beta) \to \alpha \to \beta$ — **8 T-Abst**
10. $\lambda f : \alpha \to \alpha . \lambda g : \alpha \to \beta . \lambda x : \alpha . g(f(fx))$
    $: (\alpha \to \alpha) \to (\alpha \to \beta) \to \alpha \to \beta$ — **9 T-Abst**
11. $\lambda \alpha, \beta : * . \lambda f : \alpha \to \alpha . \lambda g : \alpha \to \beta . \lambda x : \alpha . g(f(fx))$
    $: \Pi \alpha, \beta : * . (\alpha \to \alpha) \to (\alpha \to \beta) \to \alpha \to \beta$ — **10 T2-Abst**
12. $(\lambda \alpha, \beta : * . \lambda f : \alpha \to \alpha . \lambda g : \alpha \to \beta . \lambda x : \alpha . g(f(fx)))\ \texttt{nat}$
    $: \Pi \beta : * . (\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \beta) \to \texttt{nat} \to \beta$ — ***,11 T2-App**
13. $(\lambda \alpha, \beta : * . \lambda f : \alpha \to \alpha . \lambda g : \alpha \to \beta . \lambda x : \alpha . g(f(fx)))\ \texttt{nat}\ \texttt{bool}$
    $: \Pi \beta : * . (\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \texttt{bool}) \to \texttt{nat} \to \texttt{bool}$ — ***,12 T2-App**

■

Problem

(3.5 a) Let $\bot \equiv \Pi \alpha : * . \alpha$. Prove $\bot$ is legal.

Solution. Here a notion called kind checking is introduced. This has not yet been discussed in this book (?

Proof.

1.	$\alpha : *$	**Bound**
2.	$\quad \alpha : *$	**T-Form**
3.	$\Pi\alpha : * .\alpha : * \to *$	**T-Form**

■

Problem

(3.5 b) Consider the context $\Gamma \equiv \beta : *, x : \bot$. Find an inhabitant of type β under Γ.

Solution. $x\beta$ is. Because x is of second-order type, it must be parametric to a type, thus x is of form $\lambda\alpha : * .M$ where $\Gamma, \alpha : * \vdash M : \alpha$.

Proof.

1.	$x : \Pi\alpha : * .\alpha$	**T-Var**
2.	$\beta : *$	**T-Form**
3.	$x\beta : \beta$	**1,2 T2-App**

■

Problem

(3.5 c) Give three inhabitants of $\beta \to \beta$ in β-nf under Γ in 3.5 b.

Solution.
1. $\lambda y : \beta.y$.

Proof.

1.	$y : \beta$	**Bound**
2.	$\quad y : \beta$	**T-Var**
3.	$\lambda y : \beta.y : \beta \to \beta$	**2 T-Abst**

■

2. $\lambda y : \beta.x\beta$.

Proof.

1.	$y : \beta$	**Bound**
2.	$x : \Pi\alpha : * .\alpha$	**T-Var**
3.	$\beta : *$	**T-Form**
4.	$x\beta : \beta$	**2,3 T2-App**
5.	$\lambda y : \beta.x\beta : \beta \to \beta$	**4 T-Abst**

■

3. $\lambda y : \beta.x(\beta \to \beta)y$.

Proof.

1.	$y : \beta$	**Bound**
2.	$x : \Pi\alpha : * .\alpha$	**T-Var**
3.	$\beta \to \beta : *$	**T-Form**
4.	$x(\beta \to \beta) : \beta \to \beta$	**2,3 T2-App**
5.	$y : \beta$	**T-Var**
6.	$x(\beta \to \beta)y : \beta$	**4,5 T-App**
7.	$\lambda y : \beta.x(\beta \to \beta)y : \beta \to \beta$	**5 T-Abst**

■

Problem

(3.5 d) Prove that the following terms inhabit the same type in Γ:

$$\lambda f : \beta \to \beta \to \beta.f(x\beta)(x\beta)$$

$$x((\beta \to \beta \to \beta) \to \beta)$$

Solution. We simply derive the types.

First Term.

1.	$f : \beta \to \beta \to \beta$	**Bound**
2.	$f : \beta \to \beta \to \beta$	**T-Var**
3.	$x : \Pi\alpha : * .\alpha$	**T-Var**
4.	$\beta : *$	**T-Form**
5.	$x\beta : \beta$	**3,4 T2-App**
6.	$f(x\beta) : \beta \to \beta$	**2,5 T-App**

7. $\quad \big| \; f(x\beta)(x\beta) : \beta$ **6,5 T-App**

8. $\lambda f : \beta \to \beta \to \beta . f(x\beta)(x\beta) : (\beta \to \beta \to \beta) \to \beta$ **6 T-Abst**

$\blacksquare$

Second Term.

1. $(\beta \to \beta \to \beta) \to \beta : *$ **T-Form**
2. $x : \Pi\alpha : * . \alpha$ **T-Var**
3. $x((\beta \to \beta \to \beta) \to \beta) : (\beta \to \beta \to \beta) \to \beta$ **2,1 T2-App**

$\blacksquare$

The two terms were shown to both inhabit $(\beta \to \beta \to \beta) \to \beta$.