

# EXERCISES

## CHAPTER 5

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1. Reducted

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**Definition** Some rules for reference.

$$\begin{array}{c} \frac{}{\emptyset \vdash * : \square} \text{Sort} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \text{Var} \quad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \text{Weak} \\[10pt] \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A . B : s} \text{Form} \quad \frac{\Gamma \vdash M : \Pi x : A . B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B [x := N]} \text{App} \\[10pt] \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A . B : s}{\Gamma \vdash \lambda x : A . M : \Pi x : A . B} \text{Abst} \\[10pt] \frac{\Gamma \vdash A : B \quad B \stackrel{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \text{Conv} \end{array}$$

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## Problem

(5.1) Give a diagram of the tree corresponding to the complete tree derivation of line 18 of Section 5.3 (P 107)

*Solution.*



## Problem

(5.2) Give a complete  $\lambda P$  derivation of

$$S : * \vdash S \rightarrow S \rightarrow * : \square$$

In tree format and flag format.

*Solution.*

*Tree Derivation.*

$$(7) \frac{(3) S : * \vdash S : * \quad (4) \frac{\vdash * : \square \quad \vdash * : \square}{(6) \frac{S : * \vdash * : \square}{S : *, x : S \vdash * : \square} \text{ Weak}} \text{ Weak}}{S : * \vdash S \rightarrow * : \square}$$

$$(3) \frac{\vdash * : \square}{S : * \vdash S : *} \text{Var} \quad (9) \frac{(7) S : * \vdash S \rightarrow * : \square \quad (3) S : * \vdash S : *}{S : *, x : S \vdash S \rightarrow * : \square} \text{Weak} \\ \frac{}{S : * \vdash S \rightarrow S \rightarrow * : \square} \text{Form}$$

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*Flag Derivation.*

1. $* : \square$	<b>Sort</b>
2. $S : *$	
3. $\boxed{S : *}$	<b>1 Var</b>
4. $\boxed{* : \square}$	<b>1,1 Weak</b>
5. $x : S$	
6. $\boxed{x : S}$	<b>4,3 Weak</b>
7. $\boxed{S \rightarrow * : \square}$	<b>3,6 Form</b>
8. $x : S$	
9. $\boxed{x : S}$	<b>7,3 Weak</b>
10. $\boxed{S \rightarrow S \rightarrow * : \square}$	<b>3,9 Form</b>

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### Problem

(5.3) Derive

$$S : *, Q : S \rightarrow S \rightarrow * \vdash \Pi x : S . \Pi y : S . Q x y : *$$

*Solution.*

1. $* : \square$	<b>Sort</b>
2. $S : *$	
3. $\boxed{S : *}$	<b>1 Var</b>
4. $\boxed{* : \square}$	<b>1,1 Weak</b>
5. $x : S$	
6. $\boxed{x : S}$	<b>4,3 Weak</b>
7. $\boxed{S \rightarrow * : \square}$	<b>3,6 Form</b>
8. $x : S$	
9. $\boxed{x : S}$	<b>7,3 Weak</b>
10. $\boxed{S \rightarrow S \rightarrow * : \square}$	<b>3,9 Form</b>
11. $\boxed{Q : S \rightarrow S \rightarrow *}$	

12.	$Q : S \rightarrow S \rightarrow *$	<b>10 Var</b>
13.	$S : *$	<b>3,10 Weak</b>
14.	$* : \square$	<b>4,10 Weak</b>
15.	$x : S$	
16.	$* : \square$	<b>14,13 Weak</b>
17.	$S : *$	<b>13,13 Weak</b>
18.	$x : S$	<b>13 Var</b>
19.	$Q : S \rightarrow S \rightarrow *$	<b>12,13 Weak</b>
20.	$y : S$	
21.	$y : S$	<b>17 Var</b>
22.	$Q : S \rightarrow S \rightarrow *$	<b>19,17 Weak</b>
23.	$x : S$	<b>18,17 Weak</b>
24.	$Q x : S \rightarrow *$	<b>22,23 App</b>
25.	$\underline{Q x y : *}$	<b>24,21 App</b>
26.	$\Pi y : S . Q x y : *$	<b>17,25 Form</b>
27.	$\Pi x : S . \Pi y : S . Q x y : *$	<b>13,26 Form</b>

### Problem

(5.4) Prove that  $*$  is the only valid kind in  $\lambda P$ .

*Solution.*

*Proof.* The only possible way to construct a new kind is through the Form rule and the Sort axiom. Because we are trying to construct a kind,  $s$  here stands for  $\square$ .

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : \square}{\Gamma \vdash \Pi x : A . B : \square} \text{Form}$$

One could only construct new kinds with kinds, which requires  $A : \square$  and  $B : \square$ . This contradicts with  $A : *$ .  $\blacksquare$

### Problem

(5.5) Prove that  $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$  is a tautology by given a shorthand  $\lambda P$  derivation.

*Solution.* By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash A \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of  $A$  and  $B$  is equivalent to a proof of tautologousness.

*Proof.*

1.	$A : *$	
2.	$B : *$	
3.	$x : A$	
4.	$y : A \rightarrow B$	
5.	$y x : B$	4,3 App
6.	$\lambda y : A \rightarrow B . y x : (A \rightarrow B) \rightarrow B$	5 Abst
7.	$\lambda x : A . \lambda y : A \rightarrow B . y x : A \rightarrow (A \rightarrow B) \rightarrow B$	5 Abst

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### Problem

(5.6 a) Prove  $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$  a tautology using natural deduction.

*Solution.*

*Proof.*

1.	Assume $A \Rightarrow (A \Rightarrow B)$	
2.	$A \Rightarrow (A \Rightarrow B)$	
3.	Assume $A$	
4.	$A$	
5.	$A \Rightarrow B$	2,4 $\Rightarrow E$
6.	$B$	5,4 $\Rightarrow E$
7.	$A \Rightarrow B$	3,6 $\Rightarrow I$
8.	$(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$	1,7 $\Rightarrow I$

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### Problem

(5.6 b) Prove  $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$  using a shorthand  $\lambda P$  derivation

*Solution.* By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$$

And finding an inhabitant in a context only with definition of  $A$  and  $B$  is equivalent to a proof of tautologousness.

*Proof.*

1.	$A : *$	
2.	$B : *$	
3.	$x : A \rightarrow A \rightarrow B$	
4.	$y : A$	
5.	$x y : A \rightarrow B$	3,4 App
6.	$x y y : B$	5,4 App
7.	$\lambda y : A . x y y : A \rightarrow B$	6 Abst
8.	$\lambda x : A \rightarrow A \rightarrow B . \lambda y : A . x y y : (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$	7 Abst

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### Problem

(5.7 a) Proof  $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$  using a shorthand  $\lambda P$  derivation.

*Solution.* By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of  $A$ ,  $B$ , and  $C$  is equivalent to a proof of tautologousness.

*Proof.*

1.	$A : *$	
2.	$B : *$	
3.	$C : *$	
4.	$x : A \rightarrow B$	
5.	$y : B \rightarrow C$	
6.	$a : A$	
7.	$x a : B$	4,6 App
8.	$y (x a) : C$	5,7 App

9.	$\lambda a : A . y (x z) : A \rightarrow C$	<b>8 Abst</b>
10.	$\lambda y : B \rightarrow C . \lambda a : A . y (x z) : (B \rightarrow C) \rightarrow A \rightarrow C$	<b>9 Abst</b>
11.	$\lambda x : A \rightarrow B . \lambda y : B \rightarrow C . \lambda a : A . y (x z)$ $: (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C$	<b>10 Abst</b>

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### Problem

(5.7 b) Proof  $((A \Rightarrow B) \Rightarrow A) \Rightarrow ((A \Rightarrow B) \Rightarrow B)$  using a shorthand  $\lambda P$  derivation.

*Solution.* By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of  $A$  and  $B$  is equivalent to a proof of tautologousness.

*Proof.*

1.	$A : *$	
2.	$B : *$	
3.	$x : (A \rightarrow B) \rightarrow A$	
4.	$y : A \rightarrow B$	
5.	$x y : A$	<b>3,4 App</b>
6.	$y (x y) : B$	<b>4,5 App</b>
7.	$\lambda y : A \rightarrow B . y (x y) : (A \rightarrow B) \rightarrow B$	<b>6 Abst</b>
8.	$\lambda x : (A \rightarrow B) \rightarrow A . \lambda y : A \rightarrow B . y (x y)$ $: ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B$	<b>7 Abst</b>

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### Problem

(5.7 c) Proof  $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$  using a shorthand  $\lambda P$  derivation.

*Solution.* By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of  $A$ ,  $B$ , and  $C$  is equivalent to a proof of tautologousness.

*Proof.*

1.	$A : *$	
2.	$B : *$	
3.	$C : *$	
4.	$x : A \rightarrow B \rightarrow C$	
5.	$y : A \rightarrow B$	
6.	$a : A$	
7.	$x a : B \rightarrow C$	4,6 App
8.	$y a : B$	5,6 App
9.	$x a (y a) : C$	7,8 App
10.	$\lambda a : A . x a (y a) : A \rightarrow C$	9 Abst
11.	$\lambda y : A \rightarrow B . \lambda a : A . x a (y a) : (A \rightarrow B) \rightarrow A \rightarrow C$	10 Abst
12.	$\lambda x : A \rightarrow B \rightarrow C . \lambda y : A \rightarrow B . \lambda a : A . x a (y a)$	
	$: (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$	10 Abst

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### Problem

(5.8 a) Let  $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *$ , find an inhabitant of

$$\Pi x : S . P x \rightarrow Q x \rightarrow P x$$

with respect to  $\Gamma$  and give a shorthand  $\lambda P$  derivation

*Solution.*

1.	$S : *$	
2.	$P : S \rightarrow *$	
3.	$Q : S \rightarrow *$	
4.	$x : S$	
5.	$a : P x$	
6.	$b : Q x$	
7.	$a : P x$	2,4 App

8.	$\boxed{\lambda b : Q x . S : Q x \rightarrow P x}$	<b>7 Abst</b>
9.	$\boxed{\lambda a : P x . \lambda b : Q x . a : P x \rightarrow Q x \rightarrow P x}$	<b>8 Abst</b>
10.	$\boxed{\lambda x : S . \lambda a : P x . \lambda b : Q x . a : P x \rightarrow Q x \rightarrow P x}$	<b>9 Abst</b>

### Problem

(5.8 b) Let  $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *$ , find an inhabitant of

$$\Pi x : S . P x \rightarrow Q x \rightarrow P x$$

By proving the corresponding proposition in natural deduction.

*Solution.* The corresponding proposition and premises are

$$\frac{S \in \text{Set} \quad P : S \rightarrow \text{Prop} \quad Q : S \rightarrow \text{Prop}}{\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a))}$$

*Proof.*

1. Let  $a \in S$
2. Assume  $P(a)$
3. Assume  $Q(a)$
4.  $\boxed{P(a)}$
5.  $\boxed{Q(a) \Rightarrow P(a)} \quad 3,4 \Rightarrow I$
6.  $\boxed{P(a) \Rightarrow (Q(a) \Rightarrow P(a))} \quad 2,5 \Rightarrow I$
7.  $\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a)) \quad 1,6 \forall I$

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