

EXERCISES

CHAPTER 5

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1. Reducted

Definition Some rules for reference.

$$\begin{array}{c} \frac{}{\emptyset \vdash * : \square} \text{Sort} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \text{Var} \quad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \text{Weak} \\[10pt] \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A . B : s} \text{Form} \quad \frac{\Gamma \vdash M : \Pi x : A . B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B [x := N]} \text{App} \\[10pt] \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A . B : s}{\Gamma \vdash \lambda x : A . M : \Pi x : A . B} \text{Abst} \\[10pt] \frac{\Gamma \vdash A : B \quad B \stackrel{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \text{Conv} \end{array}$$

Problem

(5.1) Give a diagram of the tree corresponding to the complete tree derivation of line 18 of Section 5.3 (P 107)

Solution.



Problem

(5.2) Give a complete λP derivation of

$$S : * \vdash S \rightarrow S \rightarrow * : \square$$

In tree format and flag format.

Solution.

Tree Derivation.

$$(7) \frac{(3) S : * \vdash S : * \quad (4) \frac{\vdash * : \square \quad \vdash * : \square}{(6) \frac{S : * \vdash * : \square}{S : *, x : S \vdash * : \square} \text{ Weak}} \text{ Weak}}{S : * \vdash S \rightarrow * : \square}$$

$$(3) \frac{\vdash * : \square}{S : * \vdash S : *} \text{Var} \quad (9) \frac{(7) S : * \vdash S \rightarrow * : \square \quad (3) S : * \vdash S : *}{S : *, x : S \vdash S \rightarrow * : \square} \text{Weak} \\ \frac{}{S : * \vdash S \rightarrow S \rightarrow * : \square} \text{Form}$$

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Flag Derivation.

| | |
|---|-----------------|
| 1. $* : \square$ | Sort |
| 2. $S : *$ | |
| 3. $\boxed{S : *}$ | 1 Var |
| 4. $\boxed{* : \square}$ | 1,1 Weak |
| 5. $x : S$ | |
| 6. $\boxed{x : \boxed{* : \square}}$ | 4,3 Weak |
| 7. $\boxed{S \rightarrow * : \square}$ | 3,6 Form |
| 8. $x : S$ | |
| 9. $\boxed{x : \boxed{S \rightarrow * : \square}}$ | 7,3 Weak |
| 10. $\boxed{S \rightarrow S \rightarrow * : \square}$ | 3,9 Form |

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Problem

(5.3) Derive

$$S : *, Q : S \rightarrow S \rightarrow * \vdash \Pi x : S . \Pi y : S . Q x y : *$$

Solution.

| | |
|---|-----------------|
| 1. $* : \square$ | Sort |
| 2. $S : *$ | |
| 3. $\boxed{S : *}$ | 1 Var |
| 4. $\boxed{* : \square}$ | 1,1 Weak |
| 5. $x : S$ | |
| 6. $\boxed{x : \boxed{* : \square}}$ | 4,3 Weak |
| 7. $\boxed{S \rightarrow * : \square}$ | 3,6 Form |
| 8. $x : S$ | |
| 9. $\boxed{x : \boxed{S \rightarrow * : \square}}$ | 7,3 Weak |
| 10. $\boxed{S \rightarrow S \rightarrow * : \square}$ | 3,9 Form |
| 11. $\boxed{Q : S \rightarrow S \rightarrow *}$ | |

| | | |
|-----|-------------------------------------|-------------------|
| 12. | $Q : S \rightarrow S \rightarrow *$ | 10 Var |
| 13. | $S : *$ | 3,10 Weak |
| 14. | $* : \square$ | 4,10 Weak |
| 15. | $x : S$ | |
| 16. | $* : \square$ | 14,13 Weak |
| 17. | $S : *$ | 13,13 Weak |
| 18. | $x : S$ | 13 Var |
| 19. | $Q : S \rightarrow S \rightarrow *$ | 12,13 Weak |
| 20. | $y : S$ | |
| 21. | $y : S$ | 17 Var |
| 22. | $Q : S \rightarrow S \rightarrow *$ | 19,17 Weak |
| 23. | $x : S$ | 18,17 Weak |
| 24. | $Q x : S \rightarrow *$ | 22,23 App |
| 25. | $\underline{Q x y : *}$ | 24,21 App |
| 26. | $\Pi y : S . Q x y : *$ | 17,25 Form |
| 27. | $\Pi x : S . \Pi y : S . Q x y : *$ | 13,26 Form |

Problem

(5.4) Prove that $*$ is the only valid kind in λP .

Solution.

Proof. The only possible way to construct a new kind is through the Form rule and the Sort axiom. Because we are trying to construct a kind, s here stands for \square .

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : \square}{\Gamma \vdash \Pi x : A . B : \square} \text{Form}$$

One could only construct new kinds with kinds, which requires $A : \square$ and $B : \square$. This contradicts with $A : *$. ■