# Excercises

## Chapter 5

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1. Reducted

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**Definition** Some rules for reference.

**$\lambda P$ Calculus Rules**

$$\frac{}{\emptyset \vdash * : \square}\,\text{Sort} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}\,\text{Var} \qquad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}\,\text{Weak}$$

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi\, x : A\, .\, B : s}\,\text{Form} \qquad \frac{\Gamma \vdash M : \Pi\, x : A\, .\, B \quad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B\,[x := N]}\,\text{App}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi\, x : A\, .\, B : s}{\Gamma \vdash \lambda x : A\, .\, M : \Pi\, x : A\, .\, B}\,\text{Abst}$$

$$\frac{\Gamma \vdash A : B \quad B \overset{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'}\,\text{Conv}$$

**Predicate Logic**

$$\frac{\begin{array}{ll} 1. & \text{Assume } A \\ 2. & \quad \dots \\ 3. & \quad B \end{array}}{A \Rightarrow B}\,{\Rightarrow}\text{I} \qquad \frac{A \Rightarrow B \quad A}{B}\,{\Rightarrow}\text{E} \qquad \frac{\begin{array}{ll} 1. & \text{Let } a \in S \\ 2. & \quad \dots \\ 3. & \quad P\,(a) \end{array}}{\forall a \in S, P\,(a)}\,\forall\text{I}$$

$$\frac{\forall a \in S \quad N \in S}{P\,(N)}\,\forall\text{E}$$

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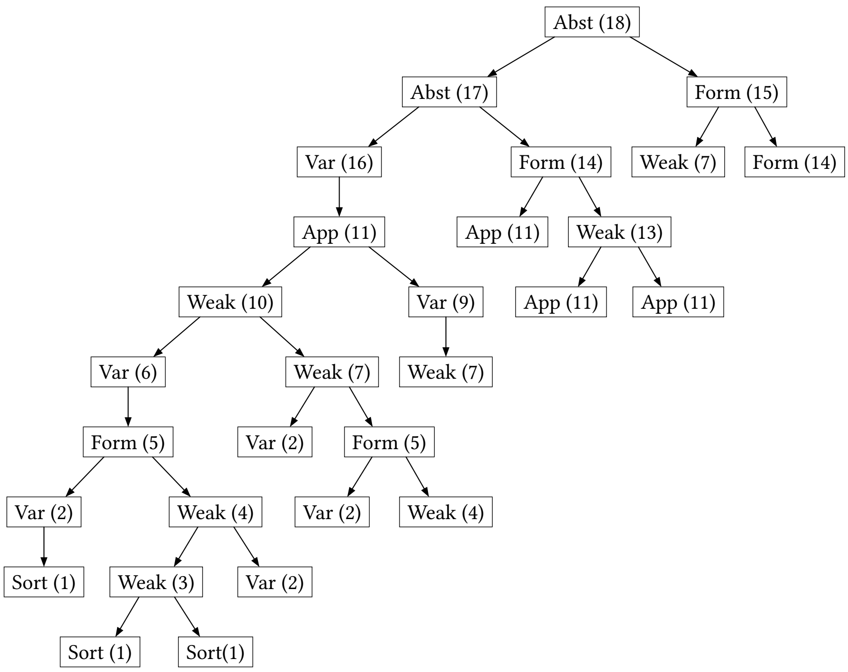
## Problem

*(5.1)* Give a diagram of the tree corresponding to the complete tree derivation of line 18 of Section 5.3 (P 107)

*Solution.*




## Problem

*(5.2)* Give a complete $\lambda P$ derivation of

$$S : * \vdash S \to S \to * : \square$$

In tree format and flag format.

*Solution.*

*Tree Derivation.*

$$(7)\ \dfrac{(3) S : * \vdash S : * \qquad (6)\ \dfrac{(4)\ \dfrac{\vdash * : \square \qquad \vdash * : \square}{S : * \vdash * : \square}\ \text{Weak} \qquad (3) S : * \vdash S : *}{S : *, x : S \vdash * : \square}\ \text{Weak}}{S : * \vdash S \to * : \square}$$

$$\dfrac{(3)\,\dfrac{\vdash * : \Box}{S : * \vdash S : *}\,\text{Var} \qquad (9)\,\dfrac{(7) S : * \vdash S \to * : \Box \qquad (3) S : * \vdash S : *}{S : *, x : S \vdash S \to * : \Box}\,\text{Weak}}{S : * \vdash S \to S \to * : \Box}\,\text{Form}$$

■

*Flag Derivation.*

| | | |
|---|---|---|
| 1. | $* : \Box$ | **Sort** |
| 2. | $S : *$ | |
| 3. | $S : *$ | **1 Var** |
| 4. | $* : \Box$ | **1,1 Weak** |
| 5. | $x : S$ | |
| 6. | $* : \Box$ | **4,3 Weak** |
| 7. | $S \to * : \Box$ | **3,6 Form** |
| 8. | $x : S$ | |
| 9. | $S \to * : \Box$ | **7,3 Weak** |
| 10. | $S \to S \to * : \Box$ | **3,9 Form** |

■

**Problem**

*(5.3)* Derive

$$S : *, Q : S \to S \to * \vdash \Pi\, x : S\, .\Pi\, y : S\, .\, Q\, x\, y : *$$

*Solution.*

| | | |
|---|---|---|
| 1. | $* : \Box$ | **Sort** |
| 2. | $S : *$ | |
| 3. | $S : *$ | **1 Var** |
| 4. | $* : \Box$ | **1,1 Weak** |
| 5. | $x : S$ | |
| 6. | $* : \Box$ | **4,3 Weak** |
| 7. | $S \to * : \Box$ | **3,6 Form** |
| 8. | $x : S$ | |
| 9. | $S \to * : \Box$ | **7,3 Weak** |
| 10. | $S \to S \to * : \Box$ | **3,9 Form** |
| 11. | $Q : S \to S \to *$ | |

| | | |
|---|---|---|
| 12. | $Q : S \to S \to *$ | **10 Var** |
| 13. | $S : *$ | **3,10 Weak** |
| 14. | $* : \square$ | **4,10 Weak** |
| 15. | $x : S$ | |
| 16. | $\quad * : \square$ | **14,13 Weak** |
| 17. | $\quad S : *$ | **13,13 Weak** |
| 18. | $\quad x : S$ | **13 Var** |
| 19. | $\quad Q : S \to S \to *$ | **12,13 Weak** |
| 20. | $\quad y : S$ | |
| 21. | $\quad\quad y : S$ | **17 Var** |
| 22. | $\quad\quad Q : S \to S \to *$ | **19,17 Weak** |
| 23. | $\quad\quad x : S$ | **18,17 Weak** |
| 24. | $\quad\quad Q\, x : S \to *$ | **22,23 App** |
| 25. | $\quad\quad Q\, x\, y : *$ | **24,21 App** |
| 26. | $\quad \Pi\, y : S\, .\, Q\, x\, y : *$ | **17,25 Form** |
| 27. | $\Pi\, x : S\, .\Pi\, y : S\, .\, Q\, x\, y : *$ | **13,26 Form** |

**Problem**

*(5.4)* Prove that $*$ is the only valid kind in $\lambda P$.

*Solution.*

*Proof.* The only possible way to construct a new kind is through the Form rule and the Sort axiom. Because we are trying to construct a kind, $s$ here stands for $\square$.

$$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : \square}{\Gamma \vdash \Pi\, x : A\, .\, B : \square}\ \text{Form}$$

One could only construct new kinds with kinds, which requires $A : \square$ and $B : \square$. This contradicts with $A : *$. ∎

**Problem**

*(5.5)* Prove that $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ is a tautology by given a shorthand $\lambda P$ derivation.

*Solution.* By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash A \to (A \to B) \to B$$

And finding an inhabitant in a context only with definition of $A$ and $B$ is equivalent to a proof of tautologousness.

*Proof.*

1. $A : *$
2. $\quad B : *$
3. $\quad\quad x : A$
4. $\quad\quad\quad y : A \to B$
5. $\quad\quad\quad\quad y\, x : B$ — **4,3 App**
6. $\quad\quad\quad \lambda y : A \to B \,.\, y\, x : (A \to B) \to B$ — **5 Abst**
7. $\quad\quad \lambda x : A \,.\lambda y : A \to B \,.\, y\, x : A \to (A \to B) \to B$ — **5 Abst**

∎

**Problem**

*(5.6 a)* Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ a tautology using natural deduction.

*Solution.*

*Proof.*

1. Assume $A \Rightarrow (A \Rightarrow B)$
2. $\quad A \Rightarrow (A \Rightarrow B)$
3. $\quad$ Assume $A$
4. $\quad\quad A$
5. $\quad\quad A \Rightarrow B$ — **2,4 ⇒E**
6. $\quad\quad B$ — **5,4 ⇒E**
7. $\quad A \Rightarrow B$ — **3,6 ⇒I**
8. $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ — **1,7 ⇒I**

∎

**Problem**

*(5.6 b)* Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ using a shorthand $\lambda P$ derivation

*Solution.* By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash (A \to A \to B) \to A \to B$$

And finding an inhabitant in a context only with definition of $A$ and $B$ is equivalent to a proof of tautologousness.

*Proof.*

| | | |
|---|---|---|
| 1. | $A : *$ | |
| 2. | $B : *$ | |
| 3. | $x : A \to A \to B$ | |
| 4. | $y : A$ | |
| 5. | $x\, y : A \to B$ | **3,4 App** |
| 6. | $x\, y\, y : B$ | **5,4 App** |
| 7. | $\lambda y : A \,.\, x\, y\, y : A \to B$ | **6 Abst** |
| 8. | $\lambda x : A \to A \to B \,.\lambda y : A \,.\, x\, y\, y : (A \to A \to B) \to A \to B$ | **7 Abst** |

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**Problem**

*(5.7 a)* Proof $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ using a shorthand $\lambda P$ derivation.

*Solution.* By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \to B) \to (B \to C) \to A \to C$$

And finding an inhabitant in a context only with definition of $A$, $B$, and $C$ is equivalent to a proof of tautologousness.

*Proof.*

| | | |
|---|---|---|
| 1. | $A : *$ | |
| 2. | $B : *$ | |
| 3. | $C : *$ | |
| 4. | $x : A \to B$ | |
| 5. | $y : B \to C$ | |
| 6. | $a : A$ | |
| 7. | $x\, a : B$ | **4,6 App** |
| 8. | $y\,(x\, a) : C$ | **5,7 App** |

9. $\lambda a : A \,.\, y\,(x\,z) : A \to C$ **8 Abst**

10. $\lambda y : B \to C \,.\lambda a : A \,.\, y\,(x\,z) : (B \to C) \to A \to C$ **9 Abst**

$\lambda x : A \to B \,.\lambda y : B \to C \,.\lambda a : A \,.\, y\,(x\,z)$

11. $: (A \to B) \to (B \to C) \to A \to C$ **10 Abst**

■

**Problem**

*(5.7 b)* Proof $((A \Rightarrow B) \Rightarrow A) \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ using a shorthand $\lambda P$ derivation.

*Solution.* By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash ((A \to B) \to A) \to (A \to B) \to B$$

And finding an inhabitant in a context only with definition of $A$ and $B$ is equivalent to a proof of tautologousness.

*Proof.*

1. $A : *$
2. $B : *$
3. $x : (A \to B) \to A$
4. $y : A \to B$
5. $x\,y : A$ **3,4 App**
6. $y\,(x\,y) : B$ **4,5 App**
7. $\lambda y : A \to B \,.\, y\,(x\,y) : (A \to B) \to B$ **6 Abst**

$\lambda x : (A \to B) \to A \,.\lambda y : A \to B \,.\, y\,(x\,y)$

8. $: ((A \to B) \to A) \to (A \to B) \to B$ **7 Abst**

■

**Problem**

*(5.7 c)* Proof $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ using a shorthand $\lambda P$ derivation.

*Solution.* By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \to B \to C) \to (A \to B) \to A \to C$$

And finding an inhabitant in a context only with definition of $A$, $B$, and $C$ is equivalent to a proof of tautologousness.

*Proof.*

1. $A : *$
2. $B : *$
3. $C : *$
4. $x : A \to B \to C$
5. $y : A \to B$
6. $a : A$
7. $x\, a : B \to C$ — **4,6 App**
8. $y\, a : B$ — **5,6 App**
9. $x\, a\, (y\, a) : C$ — **7,8 App**
10. $\lambda a : A\, .\, x\, a\, (y\, a) : A \to C$ — **9 Abst**
11. $\lambda y : A \to B\, .\lambda a : A\, .\, x\, a\, (y\, a) : (A \to B) \to A \to C$ — **10 Abst**
12. $\lambda x : A \to B \to C\, .\lambda y : A \to B\, .\lambda a : A\, .\, x\, a\, (y\, a)$
    $: (A \to B \to C) \to (A \to B) \to A \to C$ — **10 Abst**

∎

**Problem**

*(5.8 a)* Let $\Gamma \equiv S : *, P : S \to *, Q : S \to *$, find an inhabitant of

$$\Pi\, x : S\, .\, P\, x \to Q\, x \to P\, x$$

with respect to $\Gamma$ and give a shorthand $\lambda P$ derivation

*Solution.*

1. $S : *$
2. $P : S \to *$
3. $Q : S \to *$
4. $x : S$
5. $a : P\, x$
6. $b : Q\, x$
7. $a : P\, x$ — **2,4 App**

| | | |
|---|---|---|
| 8. | $\lambda b : Q\,x\,.\,S : Q\,x \to P\,x$ | **7 Abst** |
| 9. | $\lambda a : P\,x\,.\lambda b : Q\,x\,.\,a : P\,x \to Q\,x \to P\,x$ | **8 Abst** |
| 10. | $\lambda x : S\,.\lambda a : P\,x\,.\lambda b : Q\,x\,.\,a$ $: \Pi\,x : S\,.\,P\,x \to Q\,x \to P\,x$ | **9 Abst** |

**Problem**

*(5.8 b)* Let $\Gamma \equiv S : *, P : S \to *, Q : S \to *$, find an inhabitant of

$$\Pi\,x : S\,.\,P\,x \to Q\,x \to P\,x$$

By proving the corresponding proposition in natural deduction.

*Solution.* The corresponding proposition and premises are

$$\frac{S \in \text{Set} \qquad P : S \to \text{Prop} \qquad Q : S \to \text{Prop}}{\forall a \in S, P\,(a) \Rightarrow (Q\,(a) \Rightarrow P\,(a))}$$

*Proof.*

| | | |
|---|---|---|
| 1. | Let $a \in S$ | |
| 2. | Assume $P\,(a)$ | |
| 3. | Assume $Q\,(a)$ | |
| 4. | $P\,(a)$ | |
| 5. | $Q\,(a) \Rightarrow P\,(a)$ | **3,4 $\Rightarrow$I** |
| 6. | $P\,(a) \Rightarrow (Q\,(a) \Rightarrow P\,(a))$ | **2,5 $\Rightarrow$I** |
| 7. | $\forall a \in S, P\,(a) \Rightarrow (Q\,(a) \Rightarrow P\,(a))$ | **1,6 $\forall$I** |

∎

**Problem**

*(5.9 a)* Give proof for

$$(\forall x \in S, Q\,(x)) \Rightarrow (\forall y \in S, P\,(y) \Rightarrow Q\,(y))$$

by natural deduction and a $\lambda P$ derivation.

*Solution.*

*Natural Deduction.*

1. Assume $\forall x \in S, Q\ (x)$
2. Let $y \in S$
3. Assume $P\ (y)$
4. $Q\ (y)$ **1,2 $\forall$E**
5. $P\ (y) \Rightarrow Q\ (y)$ **3,4 $\Rightarrow$I**
6. $\forall y \in S, P\ (y) \Rightarrow Q\ (y)$ **2,5 $\forall$I**
7. $(\forall x \in S, Q\ (x) \Rightarrow (\forall y \in S, P\ (y) \Rightarrow Q\ (y))$ **1,6 $\Rightarrow$I**

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*$\lambda P$ Derivation.* Corresponding type is

$$S : *, P : S \to *, Q : S \to * \vdash (\Pi\, x : S\ .\ Q\ x) \to (\Pi\, y : S\ .\ P\ y \to Q\ y)$$

1. $S : *$
2. $P : S \to *$
3. $Q : S \to *$
4. $a : \Pi\, x : S\ .\ Q\ x$
5. $y : S$
6. $z : P\ y$
7. $a\ y : Q\ y$ **4,5 App**
8. $\lambda z : P\ y\ .\ a\ y : P\ y \to Q\ y$ **7 Abst**
9. $\lambda y : S\ .\lambda z : P\ y\ .\ a\ y : \Pi\, y : S\ .\ P\ y \to Q\ y$ **7 Abst**
10. $\lambda a : \Pi\, x : S\ .\ Q\ x\ .\lambda y : S\ .\lambda z : P\ y\ .\ a\ y$ $: (\Pi\, x : S\ .\ Q\ x) \to (\Pi\, y : S\ .\ P\ y \to Q\ y)$ **7 Abst**

■

**Problem**

*(5.9 b)* Give proof for

$$\forall x \in S, (P\ (x) \Rightarrow Q\ (x)) \Rightarrow ((\forall y \in S, P\ (y)) \Rightarrow (\forall z \in S, Q\ (z)))$$

by natural deduction and a $\lambda P$ derivation

*Solution.*

*Natural Deduction.*

1. Assume $\forall x \in S, (P\ (x) \Rightarrow Q\ (x))$

| | | |
|---|---|---|
| 2. | Assume $\forall y \in S, P\ (y)$ | |
| 3. | Let $z \in S$ | |
| 4. | $P\ (z)$ | **2,3 $\forall$E** |
| 5. | $P\ (z) \Rightarrow Q\ (z)$ | **1,3 $\forall$E** |
| 6. | $Q\ (z)$ | **5,4 $\Rightarrow$E** |
| 7. | $\forall z \in S, Q\ (z)$ | **3,6 $\forall$I** |
| 8. | $\forall y \in S, P\ (y) \Rightarrow (\forall z \in S, Q\ (z))$ | **2,7 $\forall$I** |
| 9. | $\forall x \in S, (P\ (x) \Rightarrow Q\ (x)) \Rightarrow ((\forall y \in S, P\ (y)) \Rightarrow (\forall z \in S, Q\ (z)))$ | **1,8 $\forall$I** |

■

*$\lambda P$ Derivation.* Corresponding type is

$$S : *, P : S \to *, Q : S \to *$$
$$\vdash (\Pi\, x : S\, .\, P\ x \to Q\ x) \to (\Pi\, y : S\, .\, P\ y) \to (\Pi\, z : S\, .\, Q\ z)$$

| | | |
|---|---|---|
| 1. | $S : *$ | |
| 2. | $P : S \to *$ | |
| 3. | $Q : S \to *$ | |
| 4. | $a : \Pi\, x : S\, .\, P\ x \to Q\ x$ | |
| 5. | $b : \Pi\, y : S\, .\, P\ y$ | |
| 6. | $z : S$ | |
| 7. | $b\ z : P\ z$ | **5,6 App** |
| 8. | $a\ z : P\ z \to Q\ z$ | **4,6 App** |
| 9. | $a\ z\ (b\ z) : Q\ z$ | **8,7 App** |
| 10. | $\lambda z : S\, .\, a\ z\ (b\ z) : \Pi\, z : S\, .\, Q\ z$ | **9 Abst** |
| 11. | $\lambda b : (\Pi\, y : S\, .\, P\ y). \lambda z : S\, .\, a\ z\ (b\ z)$ $: (\Pi\, y : S\, .\, P\ y) \to \Pi\, z : S\, .\, Q\ z$ | **10 Abst** |
| 12. | $\lambda a : (\Pi\, x : S\, .\, P\ x \to Q\ x). \lambda b : (\Pi\, y : S\, .\, P\ y).$ $\lambda z : S\, .\, a\ z\ (b\ z)$ $: (\Pi\, x : S\, .\, P\ x \to Q\ x) \to$ $(\Pi\, y : S\, .\, P\ y) \to \Pi\, z : S\, .\, Q\ z$ | **10 Abst** |

■

## Problem

*(5.10)* Given a context

$$\begin{aligned}\Gamma \equiv S : *, P : S \to *, f : S \to S, g : S \to S,\\ u : \Pi\, x : S\,.(P\,(f\,x) \to P\,(g\,x)),\\ v : \Pi\, x, y : S\,.((P\,x \to P\,y) \to P\,(f\,x))\end{aligned}$$

Let

$$M \equiv \lambda x : S\,.\,v\,(f\,x)(g\,x)(u\,x)$$

Type $M$ under $\Gamma$.

*Solution.*

| | | |
|---|---|---|
| 1. | $x : S$ | |
| 2. | $\quad f : S \to S$ | |
| 3. | $\quad f\,x : S$ | **2,1 App** |
| 4. | $\quad g : S \to S$ | |
| 5. | $\quad g\,x : S$ | **4,1 App** |
| 6. | $\quad u : \Pi\, x : S\,.(P\,(f\,x) \to P\,(g\,x))$ | |
| 7. | $\quad u\,x : P\,(f\,x) \to P\,(g\,x)$ | **6,1 App** |
| 8. | $\quad v : \Pi\, x, y : S\,.((P\,x \to P\,y) \to P\,(f\,x))$ | |
| 9. | $\quad v\,(f\,x) : \Pi\, y : S\,.((P\,(f\,x) \to P\,y) \to P\,(f\,(f\,x)))$ | **8,3 App** |
| 10. | $\quad v\,(f\,x)(g\,x) : (P\,(f\,x) \to P\,(g\,x)) \to P\,(f\,(f\,x))$ | **9,5 App** |
| 11. | $\quad v\,(f\,x)(g\,x)(u\,x) : P\,(f\,(f\,x))$ | **10,7 App** |
| 12. | $\lambda x : S\,.\,v\,(f\,x)(g\,x)(u\,x) : S \to P\,(f\,(f\,x))$ | **11 Abst** |