Excercises

Chapter 5

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1. Reducted

Reference - Calculus of Constructions

$$\frac{}{\emptyset \vdash * : \square}\,\text{Sort} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}\,\text{Var} \qquad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}\,\text{Weak}$$

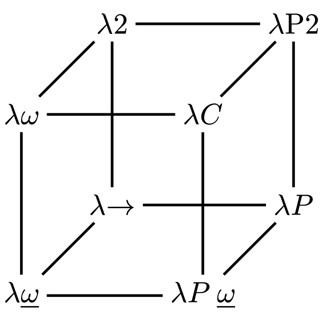
$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi\, x : A \,.\, B : s_2}\,\text{Form}$$

$$\frac{\Gamma \vdash M : \Pi\, x : A \,.\, B \quad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B\,[x := N]}\,\text{App}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi\, x : A \,.\, B : s}{\Gamma \vdash \lambda x : A \,.\, M : \Pi\, x : A \,.\, B}\,\text{Abst}$$

$$\frac{\Gamma \vdash A : B \quad B \overset{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'}\,\text{Conv}$$

The $\boldsymbol{\lambda}$-Cube



$\boldsymbol{\lambda\to}$	$(*, *)$			
$\boldsymbol{\lambda 2}$	$(*, *)$	$(\square, *)$		
$\boldsymbol{\lambda\underline{\omega}}$	$(*, *)$		$(\square, \square)$	
$\boldsymbol{\lambda P}$	$(*, *)$			$(*, \square)$
$\boldsymbol{\lambda\omega}$	$(*, *)$	$(\square, *)$	$(\square, \square)$	
$\boldsymbol{\lambda}$**2P**	$(*, *)$	$(\square, *)$		$(*, \square)$
$\boldsymbol{\lambda}$**P** $\underline{\boldsymbol{\omega}}$	$(*, *)$		$(\square, \square)$	$(*, \square)$
$\boldsymbol{\lambda C}$	$(*, *)$	$(\square, *)$	$(\square, \square)$	$(*, \square)$

Problem

(6.1 a) Give a complete derivation in tree format showing that

$$\bot \equiv \Pi\alpha : * \,.\alpha$$

is legal in λC.

Solution. Here we will show that there exists $s \in$ sort and Γ such that $\Gamma \vdash \bot : s$.

Proof.

$$\dfrac{\vdash * : \square \qquad \dfrac{\vdash * : \square}{\alpha : * \vdash \alpha : *}\,\text{Var}}{\vdash \Pi\alpha : * \,.\alpha : *}\,\text{Form}$$

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Problem

(6.1 a) Give a complete derivation in tree format showing that $\bot \to \bot$ is legal in λC where

$$\bot \equiv \Pi\alpha : * \,.\alpha$$

Solution. Here we will show that there exists $s \in$ sort and Γ such that $\Gamma \vdash \bot \to \bot : s$.

Proof.

$$\dfrac{\text{(6.1 a)}\,\dfrac{}{\vdash \bot : *} \qquad \dfrac{\text{(6.1 a)}\,\dfrac{}{\vdash \bot : *} \qquad \text{(6.1 a)}\,\dfrac{}{\vdash \bot : *}}{x : \bot \vdash \bot : *}\,\text{Weak}}{\vdash \Pi\, x : \bot . \bot : *}\,\text{Form}$$

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Problem

(6.1 c) To which systems of the λ-cube does $\bot$ belong? And $\bot \to \bot$?

Solution. The set of (s_1, s_2) pairs in formation rules of the derivation of $\bot$ is $\{(\square, *)\}$. The minimal system corresponding is $\lambda 2$. The same for $\bot \to \bot$. Therefore $\bot$ and $\bot \to \bot$ belongs to $\lambda 2$, $\lambda\omega$, λP and λC.

Problem

(6.2) Given context $\Gamma \equiv S : *, P : S \to *, A : *$. Prove by means of a flag derivation that the following expression is inhabited in λC with respect to Γ:

$$(\Pi\, x : S\, .(A \to P\, x)) \to A \to \Pi\, y : S\, .\, P\, y$$

Solution. The inhabitant is

$$M \equiv \lambda u : (\Pi\, x : S\, .(A \to P\, x)).\lambda v : A\, .\lambda y : S\, .\, u\, y\, v$$

Proof.

1.	$S : *, P : S \to *, A : *$	
2.	$\quad u : \Pi\, x : S\, .(A \to P\, x)$	
3.	$\quad\quad v : A$	
4.	$\quad\quad\quad y : S$	
5.	$\quad\quad\quad\quad u\, y : A \to P\, y$	**2,4 App**
6.	$\quad\quad\quad\quad u\, y\, v : P\, y$	**5,3 App**
7.	$\quad\quad\quad \lambda y : S\, .\, u\, y\, v : \Pi\, y : S\, .\, P\, y$	**6 Abst**
8.	$\quad\quad \lambda v : A\, .\lambda y : S\, .\, u\, y\, v : A \to \Pi\, y : S\, .\, P\, y$	**7 Abst**
9.	$\quad \lambda u : \Pi\, x : S\, .(A \to P\, x).\lambda v : A\, .\lambda y : S\, .\, u\, y\, v$ $: \Pi\, x : S\, .(A \to P\, x) \to A \to \Pi\, y : S\, .\, P\, y$	**8 Abst**

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Problem

(6.3 a) Let $\mathcal{J}$ be a judgement

$$\mathcal{J} \equiv S : *, P : S \to * \vdash \lambda x : S\, .(P\, x \to \bot) : S \to *$$

Derive $\mathcal{J}$ in λC with shorthand flag notation.

Solution.

1.	$S : *, P : S \to *$	
2.	$\quad x : S$	
3.	$\quad\quad P\, x : *$	**1,2 App**
4.	$\quad\quad \bot : *$	**Weak from 6.1 a**
5.	$\quad\quad P\, x \to \bot : *$	**3,4 Form**

6. $\left| \lambda x : S \,.\, P\, x \to \bot : S \to * \right.$ **5 Abst**

Problem

(6.3 b) Deternmine the (s_1, s_2) pairs corresponding to all Π abstractions occuring in $\mathcal{J}$.

Solution.

Abstraction	**Line Number**	(s_1, s_2)
$P : S \to *$	1	$(*, \square)$
$\bot \equiv \Pi\alpha : * \,.\alpha$	4	$(\square, *)$
$P\, x \to \bot$	5	$(\square, *)$
$\lambda x : S \,.\, P\, x \to \bot : S \to *$	6	$(*, \square)$

Problem

(6.3 c) What is the 'smallest' system in the λ-cube to which $\mathcal{J}$ belongs?

Solution. There are $(*, *)$ – $\lambda\!\to$ pairs, $(*, \square)$ – λP pairs, and $(\square, *)$ – $\lambda 2$. Therefore the minimal system $\mathcal{J}$ belongs to is λP2.

Problem

(6.4 a) Let $\Gamma \equiv S : *, Q : S \to S \to *$ and

$$M \equiv (\Pi\, x, y : S \,.(Q\, x\, y \to Q\, y\, x \to \bot)) \to \Pi\, z : S \,.(Q\, z\, z \to \bot)$$

Derive $\Gamma \vdash M : *$ and determine the smallest subsystemm to which this judgement belongs.

Solution.

1.	$S : *, Q : S \to S \to *$	
2.	$x : S$	
3.	$y : S$	
4.	$Q\,x : S \to *$	**1,2 App**
5.	$Q\,x\,y : *$	**4,3 App**
6.	$z : Q\,x\,y$	
7.	$Q\,y : S \to *$	**1,3 App**
8.	$Q\,y\,x : *$	**7,2 App**
9.	$t : Q\,y\,x$	
10.	$\bot : *$	**Weak from 6.1 a**
11.	$Q\,y\,x \to \bot : *$	**8,10 Form**
12.	$Q\,x\,y \to Q\,y\,x \to \bot : *$	**5,11 Form**
13.	$\Pi\,y : S\,.\,Q\,x\,y \to Q\,y\,x \to \bot : *$	**1,12 Form**
14.	$\Pi\,x, y : S\,.\,Q\,x\,y \to Q\,y\,x \to \bot : *$	**1,13 Form**
15.	$a : (\Pi\,x, y : S\,.(Q\,x\,y \to Q\,y\,x \to \bot))$	
16.	$z : S$	
17.	$Q\,z : S \to *$	**1,16 App**
18.	$Q\,z\,z : *$	**17,16 App**
19.	$b : Q\,z\,z$	
20.	$\bot : *$	**Weak from 6.1 a**
21.	$Q\,z\,z \to \bot : *$	**18,20 Form**
22.	$\Pi\,z : S\,.\,Q\,z\,z \to \bot : *$	**1,21 Form**
23.	$\Pi\,x, y : S\,.(Q\,x\,y \to Q\,y\,x \to \bot)$ $\to \Pi\,z : S\,.\,Q\,z\,z \to \bot : *$	**14,22 Form**

Here's a table of all Πs that appeared

Abstraction	**Line Number**	(s_1, s_2)
$S \to *$	1 / 4 / 7 / 17	$(*, \square)$
$S \to S \to *$	1	$(*, \square)$
$\bot$	10 / 11 / 12 / 13 / 14 / 15 / 20 / 21 / 22 / 23	$(\square, *)$
$Q\,y\,x \to \bot$	11 / 12 / 13 / 14 / 15 / 23	$(*, *)$
$Q\,x\,y \to Q\,y\,x \to \bot$	12 / 13 / 14 / 15 / 23	$(*, *)$
$\Pi\,y : S\,.\,Q\,x\,y \to Q\,y\,x \to \bot$	13 / 14 / 23	$(*, *)$

$\Pi\, x, y : S\,.\, Q\; x\; y \to Q\; y\; x \to \bot$	14 / 23	$(*, *)$
$Q\; z\; z \to \bot$	21 / 22 / 23	$(*, *)$
$\Pi\, z : S\,.\, Q\; z\; z \to \bot$	22 / 23	$(*, *)$
$\Pi\, x, y : S\,.\, Q\; x\; y \to Q\; y\; x \to \bot \to$ $\Pi\, z : S\,.\, Q\; z\; z \to \bot$	23	$(*, *)$

There are $(*, *)$ – $\lambda\!\to$ pairs, $(*, \square)$ – λP pairs, and $(\square, *)$ – $\lambda 2$ pairs. Therefore the mimimal system available is λP2.

Problem

(6.4 b) Prove in λC that M is inhabited in context Γ.

Solution. A shorthand derivation is given below:

Proof.

1.	$S : *, Q : S \to S \to *$	
2.	$h : \Pi\, x, y : S\,.(Q\; x\; y \to Q\; y\; x \to \bot)$	
3.	$z : S$	
4.	$a : Q\; z\; z$	
5.	$\alpha : *$	
6.	$h\; z : \Pi\, y : S\,.(Q\; z\; y \to Q\; y\; z \to \bot)$	**2,3 App**
7.	$h\; z\; z : Q\; z\; z \to Q\; z\; z \to \bot$	**6,3 App**
8.	$h\; z\; z\; a : Q\; z\; z \to \bot$	**7,4 App**
9.	$h\; z\; z\; a\; a : \Pi\alpha : *\,.\alpha$	**8,4 App**
10.	$h\; z\; z\; a\; a\; \alpha : \alpha$	**9,5 App**
11.	$\lambda\alpha : *\,.\, h\; z\; z\; a\; a\; \alpha : \Pi\alpha : *\,.\alpha$	**10 Abst**
12.	$\lambda a : Q\; z\; z\; \lambda\alpha : *\,.\, h\; z\; z\; a\; a\; \alpha : Q\; z\; z \to \bot$	**11 Abst**
13.	$\lambda z : S\,.\lambda a : Q\; z\; z\; \lambda\alpha : *\,.\, h\; z\; z\; a\; a\; \alpha : \Pi\, z : S\,.\, Q\; z\; z \to \bot$	**12 Abst**
14.	$\lambda h : \Pi\, x, y : S\,.(Q\; x\; y \to Q\; y\; x \to \bot)$ $\lambda z : S\,.\lambda a : Q\; z\; z\; \lambda\alpha : *\,.\, h\; z\; z\; a\; a\; \alpha$ $: \Pi\, x, y : S\,.(Q\; x\; y \to Q\; y\; x \to \bot) \to \Pi\, z : S\,.\, Q\; z\; z \to \bot$	**13 Abst**

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Problem

(6.4 c) We may consier Q to be a relation on set S. Moreover by PAT we may see $A \to \perp$ as the negation $\neg A$ of prop A. How can M then be interpreted by the PAT paradigm?

Solution. By a direct type-to-proposition translation we have

$$M \equiv \forall x, y \in S, (Q\,(x, y) \Rightarrow \neg\, Q\,(y, x)) \Rightarrow \forall z \in S, (\neg\, Q\,(z, z))$$

It expresses the fact if Q is asymmetric then it is irreflective.

Problem

(5.6 a) Let

$$\mathcal{J} \equiv S : * \vdash \lambda Q : S \to S \to * \,.\lambda x : S \,.\, Q\, x\, x : (S \to S \to *) \to S \to *$$

Give a shorthand derivation of $\mathcal{J}$ and determine the smallest subsystem to which $\mathcal{J}$ belongs.

Solution.

1.	$S : *$	
2.	$Q : S \to S \to *$	
3.	$x : S$	
4.	$Q\, x : S \to *$	**2,3 App**
5.	$Q\, x\, x : *$	**4,3 App**
6.	$\lambda x : S \,.\, Q\, x\, x : S \to *$	**5 Abst**
7.	$\lambda Q : S \to S \to * \,.\lambda x : S \,.\, Q\, x\, x : (S \to S \to *) \to S \to *$	**6 Abst**

Abstraction	**Line Number**	(s_1, s_2)
$S \to *$	2 / 4 / 6 / 7	$(*, \square)$
$S \to S \to *$	2 / 7	$(*, \square)$
$(S \to S \to *) \to (S \to *)$	7	$(\square, \square)$

The judgement contains $(*, \square)$ – λP pairs and $(\square, \square)$ – $\lambda \underline{\omega}$ pairs. Therefore the minimal system $\mathcal{J}$ belongs to is λP $\underline{\omega}$.

Problem

(6.5 b) In $\mathcal{J}$ of 6.5 a, we may consider the variable Q as expressing a relation on set S. How could you describe the subexpression $\lambda x : S \,.\, Q\ x\ x$ in this settings? And what is then the interpretation of the judgement $\mathcal{J}$?

Solution. By a informal translation, the term meant "Given a relation Q over set S and an arbitrary element of S, return whether if $Q\ (x, x)$ holds".