Excercises

Chapter 5

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1. Reducted

Definition Some rules for reference.

$$\frac{}{\emptyset \vdash * : \square}\,\text{Sort} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}\,\text{Var} \qquad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}\,\text{Weak}$$

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi\, x : A \,.\, B : s}\,\text{Form} \qquad \frac{\Gamma \vdash M : \Pi\, x : A \,.\, B \quad \Gamma \vdash N : A}{\Gamma \vdash M\; N : B\,[x := N]}\,\text{App}$$

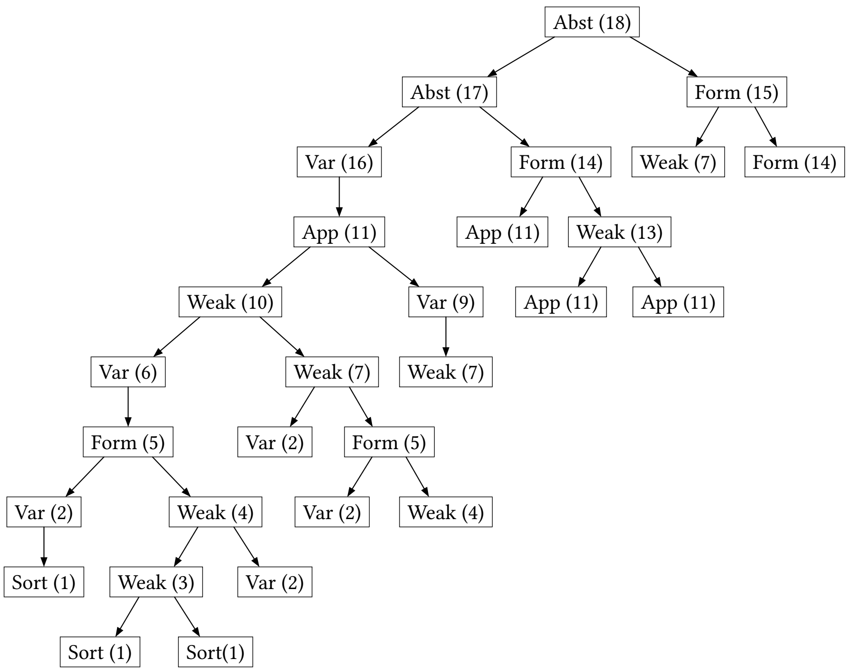
$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi\, x : A \,.\, B : s}{\Gamma \vdash \lambda x : A \,.\, M : \Pi\, x : A \,.\, B}\,\text{Abst}$$

$$\frac{\Gamma \vdash A : B \quad B \overset{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'}\,\text{Conv}$$

Problem

(5.1) Give a diagram of the tree corresponding to the complete tree derivation of line 18 of Section 5.3 (P 107)

Solution.



Problem

(5.2) Give a complete λP derivation of

$$S : * \vdash S \to S \to * : \square$$

In tree format and flag format.

Solution.

Tree Derivation.

$$(7)\ \dfrac{(3)\ S:*\vdash S:* \qquad (6)\ \dfrac{(4)\ \dfrac{\vdash *:\square \qquad \vdash *:\square}{S:*\vdash *:\square}\,\text{Weak} \qquad (3)\ S:*\vdash S:*}{S:*,x:S\vdash *:\square}\,\text{Weak}}{S:*\vdash S\to *:\square}$$

$$\dfrac{(3)\ \dfrac{\vdash * : \square}{S : * \vdash S : *}\,\text{Var} \qquad (9)\ \dfrac{(7)\ S : * \vdash S \to * : \square \qquad (3)\ S : * \vdash S : *}{S : *, x : S \vdash S \to * : \square}\,\text{Weak}}{S : * \vdash S \to S \to * : \square}\,\text{Form}$$

■

Flag Derivation.

1.	$* : \square$	**Sort**
2.	$S : *$	
3.	$\quad S : *$	**1 Var**
4.	$\quad * : \square$	**1,1 Weak**
5.	$\quad x : S$	
6.	$\quad\quad * : \square$	**4,3 Weak**
7.	$\quad S \to * : \square$	**3,6 Form**
8.	$\quad x : S$	
9.	$\quad\quad S \to * : \square$	**7,3 Weak**
10.	$\quad S \to S \to * : \square$	**3,9 Form**

■

Problem

(5.3) Derive

$$S : *, Q : S \to S \to * \vdash \Pi\, x : S\, .\Pi\, y : S\, .\, Q\, x\, y : *$$

Solution.

1.	$* : \square$	**Sort**
2.	$S : *$	
3.	$\quad S : *$	**1 Var**
4.	$\quad * : \square$	**1,1 Weak**
5.	$\quad x : S$	
6.	$\quad\quad * : \square$	**4,3 Weak**
7.	$\quad S \to * : \square$	**3,6 Form**
8.	$\quad x : S$	
9.	$\quad\quad S \to * : \square$	**7,3 Weak**
10.	$\quad S \to S \to * : \square$	**3,9 Form**
11.	$\quad Q : S \to S \to *$	

12.	$Q : S \to S \to *$	**10 Var**
13.	$S : *$	**3,10 Weak**
14.	$* : \square$	**4,10 Weak**
15.	$x : S$	
16.	$\quad * : \square$	**14,13 Weak**
17.	$\quad S : *$	**13,13 Weak**
18.	$\quad x : S$	**13 Var**
19.	$\quad Q : S \to S \to *$	**12,13 Weak**
20.	$\quad y : S$	
21.	$\quad\quad y : S$	**17 Var**
22.	$\quad\quad Q : S \to S \to *$	**19,17 Weak**
23.	$\quad\quad x : S$	**18,17 Weak**
24.	$\quad\quad Q\, x : S \to *$	**22,23 App**
25.	$\quad\quad Q\, x\, y : *$	**24,21 App**
26.	$\quad \Pi\, y : S \,.\, Q\, x\, y : *$	**17,25 Form**
27.	$\Pi\, x : S \,.\Pi\, y : S \,.\, Q\, x\, y : *$	**13,26 Form**

Problem

(5.4) Prove that $*$ is the only valid kind in λP.

Solution.

Proof. The only possible way to construct a new kind is through the Form rule and the Sort axiom. Because we are trying to construct a kind, s here stands for $\square$.

$$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : \square}{\Gamma \vdash \Pi\, x : A \,.\, B : \square}\ \text{Form}$$

One could only construct new kinds with kinds, which requires $A : \square$ and $B : \square$. This contradicts with $A : *$. ■

Problem

(5.5) Prove that $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ is a tautology by given a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash A \to (A \to B) \to B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1. $A : *$
2. $\quad B : *$
3. $\quad\quad x : A$
4. $\quad\quad\quad y : A \to B$
5. $\quad\quad\quad\quad y\, x : B$ — **4,3 App**
6. $\quad\quad\quad \lambda y : A \to B \,.\, y\, x : (A \to B) \to B$ — **5 Abst**
7. $\quad\quad \lambda x : A \,.\lambda y : A \to B \,.\, y\, x : A \to (A \to B) \to B$ — **5 Abst**

■

Problem

(5.6 a) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ a tautology using natural deduction.

Solution.

Proof.

1. Assume $A \Rightarrow (A \Rightarrow B)$
2. $\quad A \Rightarrow (A \Rightarrow B)$
3. $\quad$ Assume A
4. $\quad\quad A$
5. $\quad\quad A \Rightarrow B$ — **2,4 ⇒E**
6. $\quad\quad B$ — **5,4 ⇒E**
7. $\quad A \Rightarrow B$ — **3,6 ⇒I**
8. $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ — **1,7 ⇒I**

■

Problem

(5.6 b) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ using a shorthand λP derivation

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash (A \to A \to B) \to A \to B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1. $A : *$
2. $B : *$
3. $x : A \to A \to B$
4. $y : A$
5. $x\, y : A \to B$ **3,4 App**
6. $x\, y\, y : B$ **5,4 App**
7. $\lambda y : A \,.\, x\, y\, y : A \to B$ **6 Abst**
8. $\lambda x : A \to A \to B \,.\lambda y : A \,.\, x\, y\, y : (A \to A \to B) \to A \to B$ **7 Abst**

■

Problem

(5.7 a) Proof $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \to B) \to (B \to C) \to A \to C$$

And finding an inhabitant in a context only with definition of A, B, and C is equivalent to a proof of tautologousness.

Proof.

1. $A : *$
2. $B : *$
3. $C : *$
4. $x : A \to B$
5. $y : B \to C$
6. $a : A$
7. $x\, a : B$ **4,6 App**
8. $y\,(x\, a) : C$ **5,7 App**

9. $\lambda a : A \,.\, y\,(x\,z) : A \to C$ **8 Abst**

10. $\lambda y : B \to C \,.\lambda a : A \,.\, y\,(x\,z) : (B \to C) \to A \to C$ **9 Abst**

11. $\lambda x : A \to B \,.\lambda y : B \to C \,.\lambda a : A \,.\, y\,(x\,z)$
$: (A \to B) \to (B \to C) \to A \to C$ **10 Abst**

■

Problem

(5.7 b) Proof $((A \Rightarrow B) \Rightarrow A) \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash ((A \to B) \to A) \to (A \to B) \to B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1. $A : *$
2. $B : *$
3. $x : (A \to B) \to A$
4. $y : A \to B$
5. $x\,y : A$ **3,4 App**
6. $y\,(x\,y) : B$ **4,5 App**
7. $\lambda y : A \to B \,.\, y\,(x\,y) : (A \to B) \to B$ **6 Abst**
8. $\lambda x : (A \to B) \to A \,.\lambda y : A \to B \,.\, y\,(x\,y)$
$: ((A \to B) \to A) \to (A \to B) \to B$ **7 Abst**

■

Problem

(5.7 c) Proof $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \to B \to C) \to (A \to B) \to A \to C$$

And finding an inhabitant in a context only with definition of A, B, and C is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$C : *$	
4.	$x : A \to B \to C$	
5.	$y : A \to B$	
6.	$a : A$	
7.	$x\, a : B \to C$	**4,6 App**
8.	$y\, a : B$	**5,6 App**
9.	$x\, a\, (y\, a) : C$	**7,8 App**
10.	$\lambda a : A \,.\, x\, a\, (y\, a) : A \to C$	**9 Abst**
11.	$\lambda y : A \to B \,.\lambda a : A \,.\, x\, a\, (y\, a) : (A \to B) \to A \to C$	**10 Abst**
12.	$\lambda x : A \to B \to C \,.\lambda y : A \to B \,.\lambda a : A \,.\, x\, a\, (y\, a)$ $: (A \to B \to C) \to (A \to B) \to A \to C$	**10 Abst**

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