EXCERCISES

CHAPTER 1

SEAN LI [1]

1. Redacted

Problem

(1.1) Simplify notation of the following terms

(a) $(\lambda x .(((x\ z)\ y)(x\ x)))$

(b) $((\lambda x .(\lambda y .(\lambda z .(z\ ((x\ y)\ z)))))(\lambda u\ .\ u))$

Solution.

(a) $\lambda x .(x\ z\ y)(x\ x)$

(b) $(\lambda x\ y\ z\ .\ x\ (x\ y\ z))(\lambda u\ .\ u)$

Problem

(1.2) Find the alpha equivalent terms to

$$\lambda x\ .\ x\ (\lambda x\ .\ x)$$

In

(a) $\lambda y\ .\ y\ (\lambda x\ .\ x)$

(b) $\lambda y\ .\ y\ (\lambda x\ .\ y)$

(c) $\lambda y\ .\ y\ (\lambda x\ .\ y)$

Solution. Only (a).

Problem

(1.3) Prove

$$\lambda x \, . \, x \, (\lambda z \, . \, y) \underset{\alpha}{=} \lambda z \, . \, z \, (\lambda z \, . \, y)$$

Solution.

Proof. By definition of alpha equivalence

$$M \underset{\alpha}{=} N \iff \exists \varphi, M^\varphi \underset{\alpha}{\to} N \wedge \mathrm{FR} \, M = \mathrm{FR} \, N$$

The witness of φ is substituting bound variable x with z, and z is not a free variable in the term, thus the two terms are alpha equivalent.

$$\lambda x \, . \, x \, (\lambda z \, . \, y)^{x \to z} \underset{\alpha}{\to} \lambda z \, . \, z \, (\lambda z \, . \, y)$$

■

Problem

(1.4) Consider the following term:

$$U := (\lambda z \, . \, z \, x \, z)((\lambda y \, . \, x \, y) \, x)$$

1. Find Sub U
2. Draw tree rep of U
3. Find FV U
4. Find alpha equivalent terms to U from below and point out which of those follows the Barendregt convention:

(a) $(\lambda y \, . \, y \, x \, y)((\lambda z \, . \, x \, z) \, x)$

(b) $(\lambda x \, . \, x \, y \, x)((\lambda z \, . \, y \, z) \, y)$

(c) $(\lambda y \, . \, y \, x \, y)((\lambda y \, . \, x \, y) \, x)$

(d) $(\lambda v \, . (v \, x) \, v)((\lambda u \, . \, u \, v) \, x)$

1. Find Sub U.
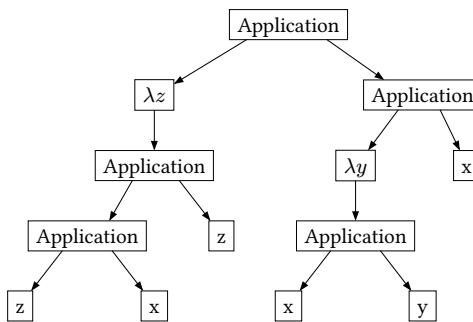
Solution.

Sub $U =$

$\quad \{(\lambda z \,.\, z\, x\, z)((\lambda y \,.\, x\, y)\, x), (\lambda z \,.\, z\, x\, z), ((\lambda y \,.\, x\, y)\, x)\} \cup$

$\quad \{(\lambda y \,.\, x\, y), z\, x\, z, z\, x, x\, y\} \cup$

$\quad \{z, x, y\}$

$= \{(\lambda z \,.\, z\, x\, z)((\lambda y \,.\, x\, y)\, x), (\lambda z \,.\, z\, x\, z), ((\lambda y \,.\, x\, y)\, x),$

$\quad (\lambda y \,.\, x\, y), z\, x\, z, z\, x, x\, y, z, x, y\}$

2. Draw a tree rep of U.

Solution.

3. Find FV U

Solution.

$$\begin{aligned}
\text{FV } U &= \text{FV } (\lambda y \,.\, y\, x\, y) \cup \text{FV } (\lambda z \,.\, x\, z)\, x \\
&= (\text{FV } y\, x\, y) \setminus \{y\} \cup (\text{FV } \lambda z \,.\, x\, z) \cup \{x\} \\
&= (\text{FV } y\, x) \setminus \{y\} \cup (\text{FV } x\, z) \setminus \{z\} \cup \{x\} \\
&= \{x\}
\end{aligned}$$

4. Find an alpha-equivalent term.

Solution.

$$(a) \underset{\alpha}{=} (c) \underset{\alpha}{=} U$$

Only (a) follows the Barendregt convention.

Problem

(1.5) Give the results of the following substitutions

$\quad$ (a) $(\lambda x \,.\, y\, (\lambda y \,.\, x\, y))[y := \lambda z \,.\, z\, x]$

$\quad$ (b) $((x\, y\, z)[x := y])[y := z]$

$\quad$ (c) $((\lambda x \,.\, x\, y\, z)[x := y])[y := z]$

$\quad$ (d) $(\lambda y \,.\, y\, y\, x)[x := y\, z]$

Solution.

$$\text{(a) } (\lambda v \,.(\lambda z \,.\, z\, x)(\lambda u \,.\, v\, u))$$
$$\text{(b) } (y\, y\, z)[y := z] = z\, z\, z$$
$$\text{(c) } (\lambda x \,.\, x\, y\, z)[y := z] = (\lambda x \,.\, x\, z\, z)$$
$$\text{(d) } (\lambda u \,.\, u\, u\, (y\, z))$$

Problem

(1.6)

$$\neg\Big(\forall\, M\, L\, N \in \Lambda,\, M\,[x := N, y := L] \underset{\alpha}{\equiv} M\,[x := N][y := L] \Big)$$

Solution.

Proof. Because RHS $= M\,[x := N][y := L] = M\,[x := N\,[y := L]][y := L]$, if $y \in \mathrm{FV}\, N$, then what x gets substituted with will have y substituted for L, which is completely different with LHS. ∎
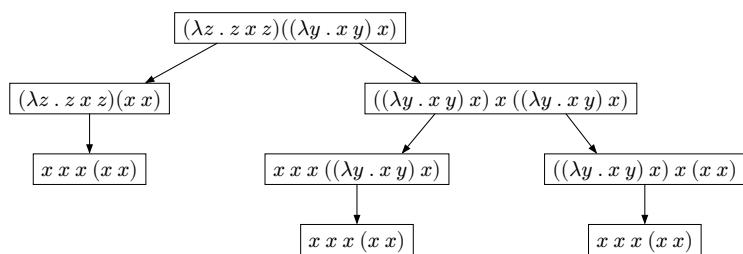

Problem

(1.7) Find all available redexes in

$$U := (\lambda z \,.\, z\, x\, z)((\lambda y \,.\, x\, y)\, x)$$

And all reduction pathes to the β-normal form.

Solution. The first redex is the term as an application itself; another the second term in the application.

(1.8) Show that

$$(\lambda x \,.\, x\,x)\,y \underset{\beta}{\neq} (\lambda x\,y \,.\, y\,x)\,x\,x$$

Solution. By Collorary 1.9.9, it suffices to prove the hypothesis with a proof of a common normal reduced form from LHS and RHS not existing.

Contradiction. By definition of $\underset{\beta}{=}$, there exists The set of all terms attainable from β-reduction on $(\lambda x \,.\, x\,x)\,y$ and $(\lambda x\,y \,.\, y\,x)\,x\,x$ do not intersect. Therefore,

$$\neg\Big(\exists L \in \Lambda, (\lambda x \,.\, x\,x)\,y \underset{\beta}{\twoheadrightarrow} L \wedge (\lambda x\,y \,.\, y\,x)\,x\,x \underset{\beta}{\twoheadrightarrow} L\Big) \implies \neg\Big((\lambda x \,.\, x\,x)\,y \underset{\beta}{=} (\lambda x\,y \,.\, y\,x)\,x\,x\Big)$$

∎

(1.9) Define the combinators

$$K := \lambda x\,y \,.\, x$$
$$S := \lambda x\,y\,z \,.\, x\,z\,(y\,z)$$

Prove that

$$\forall\,P\,Q \in \Lambda, K\,P\,Q \underset{\beta}{\twoheadrightarrow} P$$

$$\forall\,P\,Q\,R \in \Lambda, S\,P\,Q\,R \underset{\beta}{\twoheadrightarrow} P\,R\,(Q\,R)$$

Solution.

Proof.

$$K\,P\,Q = (\lambda x\,y \,.\, x)\,P\,Q \underset{\beta}{\to} (\lambda y \,.\, x)[x := P]\,Q \underset{\beta}{\to} P\,[y := Q] = P$$

$$S\,P\,Q\,R = (\lambda x\,y\,z \,.\, x\,z\,(y\,z)) \underset{\beta}{\twoheadrightarrow} (x\,z\,(y\,z))[x := P][y := Q][z := R] = P\,R\,(Q\,R)$$

∎

(1.10) We define the church numerals

$$\text{zero} := \lambda f\, x\, .\, x$$
$$\text{one} := \lambda f\, x\, .\, f\, x$$
$$\text{two} := \lambda f\, x\, .\, f\, f\, x$$
$$...$$
$$\text{num}_n := \lambda f\, x\, .\, f^n\, x$$

And operations

$$\text{add} := \lambda n\, m\, f\, x\, .\, m\, f\, (n\, f\, x)$$
$$\text{mul} := \lambda n\, m\, f\, x\, .\, m\, (n\, f)\, x$$

Show

(a) add one one $\twoheadrightarrow_{\beta}$ two

(b) add one one $\neq_{\beta}$ mul one zero

Solution.

(a) $\text{add one one} = (\lambda n\, m\, f\, x\, .\, m\, f\, (n\, f\, x))(\lambda f\, x\, .\, f\, x)(\lambda f\, x\, .\, f\, x)$

$\twoheadrightarrow_{\beta} (\lambda f\, x\, .(\lambda f\, x\, .\, f\, x)\, f\, ((\lambda f\, x\, .\, f\, x)\, f\, x))$

$\twoheadrightarrow_{\beta} (\lambda f\, x\, .(\lambda x\, .\, f\, x)\, f\, x)$

$\twoheadrightarrow_{\beta} (\lambda f\, x\, .\, f\, f\, x) = \text{two}$

(b) $\text{mul one one} = (\lambda n\, m\, f\, x\, .\, m\, (n\, f)\, x)(\lambda f\, x\, .\, f\, x)(\lambda f\, x\, .\, f\, x)$

$\twoheadrightarrow_{\beta} \lambda f\, x\, .(\lambda f\, x\, .\, f\, x)((\lambda f\, x\, .\, f\, x)\, f)\, x$

$\twoheadrightarrow_{\beta} \lambda f\, x\, .\, f\, x = \text{one}$

Because no intermediate form in the beta reduction process of the two terms are α-equivalent, by collorary 1.9.9 the two terms are not β-equivalent.

(1.11) We define

$$\text{succ} := \lambda m\, f\, x\,.\, f\,(m\, f\, x) \text{ s.t. } \forall \text{num}_n, \text{succ num}_n = \text{num}_{n+1}$$

Prove

$$\text{succ zero} \underset{\beta}{=} \text{one}$$

$$\text{succ one} \underset{\beta}{=} \text{two}$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$\text{succ zero} = (\lambda m\, f\, x\,.\, f\,(m\, f\, x))(\lambda f\, x\,.\, x)$$

$$\underset{\beta}{\to} (\lambda f\, x\,.\, f\,((\lambda f\, x\,.\, x)\, f\, x))$$

$$\underset{\beta}{\twoheadrightarrow} (\lambda f\, x\,.\, f\, x) = \text{one}$$

The path succ zero $\underset{\beta}{\twoheadrightarrow}$ one derived above is the witness of a reduction chain from LHS to RHS.

$$\text{succ one} = (\lambda m\, f\, x\,.\, f\,(m\, f\, x))(\lambda f\, x\,.\, f\, x)$$

$$\underset{\beta}{\to} (\lambda f\, x\,.\, f\,((\lambda f\, x\,.\, f\, x)\, f\, x))$$

$$\underset{\beta}{\twoheadrightarrow} (\lambda f\, x\,.\, f\,(f\, x)) = \text{two}$$

The path succ one $\underset{\beta}{\twoheadrightarrow}$ two derived above is the witness of a reduction chain from LHS to RHS. ∎

(1.12) We define the λ-terms $\top_\lambda$(true) and $\bot_\lambda$(false) and $\neg_\lambda$(not) by:

$$\top_\lambda := \lambda x\, y\,.\, x \quad \bot_\lambda := \lambda x\, y\,.\, y$$

$$\neg_\lambda := \lambda a\,.\, a\, \bot_\lambda \top_\lambda$$

Show that

$$\neg_\lambda(\neg_\lambda \top_\lambda) \underset{\beta}{=} \top_\lambda$$

$$\neg_\lambda(\neg_\lambda \bot_\lambda) \underset{\beta}{=} \bot_\lambda$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$
\begin{aligned}
\neg_\lambda(\neg_\lambda \top_\lambda) = \neg_\lambda((\lambda a \,.\, a \perp_\lambda \top_\lambda)(\lambda x\, y \,.\, x)) \\
\underset{\beta}{\twoheadrightarrow} \neg_\lambda((\lambda x\, y \,.\, x)\perp_\lambda \top_\lambda) \\
\underset{\beta}{\twoheadrightarrow} (\lambda a \,.\, a \perp_\lambda \top_\lambda)\perp_\lambda \\
\underset{\beta}{\twoheadrightarrow} (\lambda x\, y \,.\, y)\perp_\lambda \top_\lambda \\
\underset{\beta}{\twoheadrightarrow} \top_\lambda
\end{aligned}
$$

$$
\begin{aligned}
\neg_\lambda(\neg_\lambda \perp_\lambda) = \neg_\lambda((\lambda a \,.\, a \perp_\lambda \top_\lambda)(\lambda x\, y \,.\, y)) \\
\underset{\beta}{\twoheadrightarrow} \neg_\lambda((\lambda x\, y \,.\, y)\perp_\lambda \top_\lambda) \\
\underset{\beta}{\twoheadrightarrow} \neg_\lambda \top_\lambda \\
\underset{\beta}{\twoheadrightarrow} (\lambda a \,.\, a \perp_\lambda \top_\lambda)(\lambda x\, y \,.\, x) \\
\underset{\beta}{\twoheadrightarrow} (\lambda x\, y \,.\, x)\perp_\lambda \top_\lambda \\
\underset{\beta}{\twoheadrightarrow} \perp_\lambda
\end{aligned}
$$

∎

Problem

(1.13) Define

$$\text{iszero} := \lambda m \,.\, m \,(\lambda x \,.\perp_\lambda)\top_\lambda$$

Prove

$$\text{iszero zero} \underset{\beta}{\twoheadrightarrow} \top_\lambda$$

$$\forall\, n \in \mathbb{N}^+, \text{iszero num}_n \underset{\beta}{\twoheadrightarrow} \perp_\lambda$$

Solution.

$$
\begin{aligned}
\text{iszero zero} = (\lambda m \,.\, m \,(\lambda x \,.\perp_\lambda)\top_\lambda)(\lambda f\, x \,.\, x) \\
\underset{\beta}{\twoheadrightarrow} (\lambda f\, x \,.\, x)(\lambda x \,.\perp_\lambda)\top_\lambda \\
\underset{\beta}{\twoheadrightarrow} \top_\lambda
\end{aligned}
$$

8.

$$\text{iszero num}_n = (\lambda m \, . \, m \, (\lambda x \, .\bot_\lambda) \top_\lambda)(\lambda f \, x \, . \, f^n \, x)$$
$$\twoheadrightarrow_\beta (\lambda f \, x \, . \, f^n \, x)(\lambda x \, .\bot_\lambda)\top_\lambda$$
$$\twoheadrightarrow_\beta (\lambda x \, .\bot_\lambda)\big((\lambda x \, .\bot_\lambda)^{n-1}\top_\lambda\big) \twoheadrightarrow_\beta \bot_\lambda$$

Problem

(1.14) If-else can be modeled as

$$\text{ifelse} = \lambda x \, t \, f \, . \, x \, t \, f$$

Where when x, then t, else f. Prove correctness by applying $\top_\lambda$ and $\bot_\lambda$ on ifelse.

Solution.

$$\text{ifelse } \top_\lambda = (\lambda x \, t \, f \, . \, x \, t \, f)\top_\lambda$$
$$\twoheadrightarrow_\beta (\lambda t \, f \, .(\lambda x \, y \, . \, x) \, t \, f) \twoheadrightarrow_\beta (\lambda t \, f \, . \, t)$$

$$\text{ifelse } \top_\lambda = (\lambda x \, t \, f \, . \, x \, t \, f)\bot_\lambda$$
$$\twoheadrightarrow_\beta (\lambda t \, f \, .(\lambda x \, y \, . \, y) \, t \, f) \twoheadrightarrow_\beta (\lambda t \, f \, . \, f)$$

By applying the results to any two values, the correct corresponding value returns, ex, for ifelse $\top_\lambda$, t is always returned.

Problem

(1.15) Prove that $\Omega := (\lambda x \, . \, x \, x)(\lambda x \, . \, x \, x)$ does not have a β-nf.

Solution. Firstly let's prove Ω.

Proof. Induction on Ω's only reduction path proves that every $\Omega \twoheadrightarrow_\beta \Omega_i = \Omega$. For the base case because Ω has one and only one redux, it could only reduce to Ω_1 which is equivalent to itself. For the inductive step, $\Omega_i = \Omega$, therefore $\Omega_i \to_\beta \Omega_{i+1}$ is still Ω.

By definition, a term having a β-nf requires the existence of a form in β-nf such that the term can reduce to. By induction, Ω only reduces to Ω, and Ω is not in β-nf because it contains β-redex. Therefore, Ω can never reduce to a β-nf, thus it does not have a β-nf. ∎

(1.16) Let M be a λ-term with the following properties:
- M has a β-nf.
- There exists an infinite reduction path $M \equiv M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \ldots$ on M.

Prove that every M_i has a β-nf, and give an example of M.

Solution. An example would be $(\lambda x\, y \,.\, y)\Omega$. Reduction can go on infinitely by reducing on Ω, but the β-nf of the term is $\lambda y\,.\,y$

Proof. Denote β-nf of M as M'. For any form in the reduction path, $M \underset{\beta}{\twoheadrightarrow} M_i$. In conjunction with $M \underset{\beta}{\twoheadrightarrow} M'$, by the Church-Rosser theorem, there exists L such that $M_i \underset{\beta}{\twoheadrightarrow} L$ and $M' \underset{\beta}{\twoheadrightarrow} L$. Because M' is in β-nf, L can only be M', thus $M_i \underset{\beta}{\twoheadrightarrow} M'$, so M_i is capable of reducing to M', a β-nf. Therefore, any form in the reduction path has a β-nf. ∎

(1.17) If $M\,N$ is strongly normalizing, then both M and N are strongly normalizing.

Solution.

Proof. If M is not strongly normalizing, then there exists a reduction path $M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \ldots$ Therefore, $M\,N$ would have had a reduction path $M\,N \underset{\beta}{\to} M_1\,N \underset{\beta}{\to} \ldots$ that is infinite, which contradicts with $M\,N$ being strongly normalizing. Vice versa for N. ∎

(1.18) Let $L, M, N \in \Lambda$ such that $L \underset{\beta}{=} M$ and $L \underset{\beta}{\twoheadrightarrow} N$. Moreover, N is in β-nf. Prove that $M \underset{\beta}{\twoheadrightarrow} N$.

Solution. Collorary 1.9.9.

(1.19) Define

$$U := \lambda z\, x \,.\, x\,(z\,z\,x) \quad \text{and} \quad Z := U\,U$$

Prove Z is a fixed point combinator.

Solution. Proving $\forall L \in \Lambda, L\ (Z\ L) \twoheadrightarrow_\beta Z\ L$.

Proof.

$$Z\ L = (\lambda z\ x\ .\ x\ (z\ z\ x))(\lambda z\ x\ .\ x\ (z\ z\ x))\ L$$
$$\twoheadrightarrow_\beta L\ ((\lambda z\ x\ .\ x\ (z\ z\ x))(\lambda z\ x\ .\ x\ (z\ z\ x))\ L) \twoheadrightarrow_\beta L\ (Z\ L)$$

∎

Problem

(1.20) Solve for $M \in \Lambda$ in each equation:

$$M =_\beta \lambda x\ y\ .\ x\ M\ y$$
$$M\ x\ y\ z =_\beta x\ y\ z\ M$$

Solution. By the property of the Y combinator:

$$f\ (Y\ f) = Y\ f$$

The first equation can be remodeled as

$$M =_\beta L\ M \text{ where } L = \lambda m\ x\ y\ .\ x\ m\ y$$

Solving for fixed point of L :

$$M \equiv Y\ L \equiv L\ (Y\ L)$$
$$= (\lambda x\ .\ L\ (x\ x))(\lambda x\ .\ L\ (x\ x))$$
$$= (\lambda x\ .(\lambda m\ u\ v\ .\ u\ m\ v)(x\ x))(\lambda x\ .(\lambda m\ u\ v\ .\ u\ m\ v)(x\ x))$$
$$\twoheadrightarrow_\beta (\lambda x\ .(\lambda u\ v\ .\ u\ (x\ x)\ v))(\lambda x\ .(\lambda u\ v\ .\ u\ (x\ x)\ v))$$

The second equation can be η-reduced on both sides:

$$M\ x\ y\ z =_\beta x\ y\ z\ M$$
$$M =_\beta \lambda x\ y\ z\ .\ x\ y\ z\ M$$

Remodeling equation:

$$M = N\ M \text{ where } M = \lambda m\ x\ y\ z\ .\ x\ y\ z\ m$$

then

$$M \equiv Y\,N \equiv N\,(Y\,N)$$
$$= (\lambda x \,.\, N\,(x\,x))(\lambda x \,.\, N\,(x\,x))$$
$$= (\lambda x \,.(\lambda m\,u\,y\,z\,.\,u\,y\,z\,m)(x\,x))(\lambda x \,.(\lambda m\,u\,y\,z\,.\,u\,y\,z\,m)(x\,x))$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda x \,.(\lambda u\,y\,z\,.\,u\,y\,z\,(x\,x)))(\lambda x \,.(\lambda u\,y\,z\,.\,u\,y\,z\,(x\,x)))$$