# EXCERCISES

## CHAPTER 1

SEAN LI [1]

1. Reducted

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**Problem**

Simplify notation of the following terms

$$\text{(a)} \ (\lambda x.(((xz)y)(xx)))$$
$$\text{(b)} \ ((\lambda x.(\lambda y.(\lambda z.(z((xy)z)))))(\lambda u.u))$$

*Solution.*

$$\text{(a)} \ \lambda x.(xzy)(xx)$$
$$\text{(b)} \ (\lambda xyz.x(xyz))(\lambda u.u)$$

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**Problem**

Find the alpha equivalent terms to

$$\lambda x.x(\lambda x.x)$$

In

$$\text{(a)} \ \lambda y.y(\lambda x.x)$$
$$\text{(b)} \ \lambda y.y(\lambda x.y)$$
$$\text{(c)} \ \lambda y.y(\lambda x.y)$$

*Solution.* Only (a).

Prove

$$\lambda x.x(\lambda z.y) =_\alpha \lambda z.z(\lambda z.y)$$

*Solution.*

*Proof.* By definition of alpha equivalence

$$M =_\alpha N \iff \exists \varphi, M^\varphi \to_\alpha N \wedge \text{FR } M = \text{FR } N$$

The witness of $\varphi$ is substituting bound variable $x$ with $z$, and $z$ is not a free variable in the term, thus the two terms are alpha equivalent.

$$\lambda x.x(\lambda z.y)^{x \to z} \to_\alpha \lambda z.z(\lambda z.y)$$

∎

**Problem**

Consider the following term:

$$U := (\lambda z.zxz)((\lambda y.xy)x)$$

1. Find Sub $U$
2. Draw tree rep of $U$
3. Find FV $U$
4. Find alpha equivalent terms to $U$ from below and point out which of those follows the Barendregt convention:

   (a) $(\lambda y.yxy)((\lambda z.xz)x)$

   (b) $(\lambda x.xyx)((\lambda z.yz)y)$

   (c) $(\lambda y.yxy)((\lambda y.xy)x)$
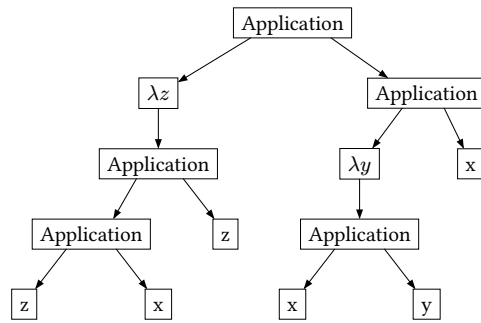
   (d) $(\lambda v.(vx)v)((\lambda u.uv)x)$

1. Find Sub $U$.

*Solution.*

$$\text{Sub } U = \{$$

$$(\lambda z.zxz)((\lambda y.xy)x), (\lambda z.zxz), ((\lambda y.xy)x),$$

$$(\lambda y.yx), (y), (\lambda z.xz), (x),$$

$$(\lambda y.y), (x), (\lambda z.x), (z), (y), (x)$$

$$\}$$

2. Draw a tree rep of $U$.

*Solution.*



3. Find FV $U$

*Solution.*

$$\begin{aligned}
\text{FV } U &= \text{FV } (\lambda y.yxy) \cup \text{FV } (\lambda z.xz)x \\
&= (\text{FV } yxy) \setminus \{y\} \cup (\text{FV } \lambda z.xz) \cup \{x\} \\
&= (\text{FV } yx) \setminus \{y\} \cup (\text{FV } xz) \setminus \{z\} \cup \{x\} \\
&= \{x\}
\end{aligned}$$

4. Find an alpha-equivalent term.

*Solution.*

$$(a) \underset{\alpha}{=} (c) \underset{\alpha}{=} U$$

(a), (b), and (c) all follow the Barendregt convention. In (d) the free variable $v$ was bound in the first abstraction.

**Problem**

Give the results of the following substitutions

$$(a) \; (\lambda x.y(\lambda y.xy))[y := \lambda z.zx]$$
$$(b) \; ((xyz)[x := y])[y := z]$$
$$(c) \; ((\lambda x.xyz)[x := y])[y := z]$$
$$(d) \; (\lambda y.yyx)[x := yz]$$

*Solution.*

(a) $(\lambda v.(\lambda z.zx)(\lambda u.vu))$

(b) $(yyz)[y := z] = zzz$

(c) $(\lambda x.xyz)[y := z] = (\lambda x.xzz)$

(d) $(\lambda u.uu(yz))$

## Problem

$$\neg\Big(\forall MLN \in \Lambda, M[x := N, y := L] \underset{\alpha}{\equiv} M[x := N][y := L]\Big)$$

*Solution.*

*Proof.* Because RHS $= M[x := N][y := L] = M[x := N[y := L]][y := L]$, if $y \in$ FV $N$, then what $x$ gets substituted with will have $y$ substituted for $L$, which is completely different with LHS. ∎

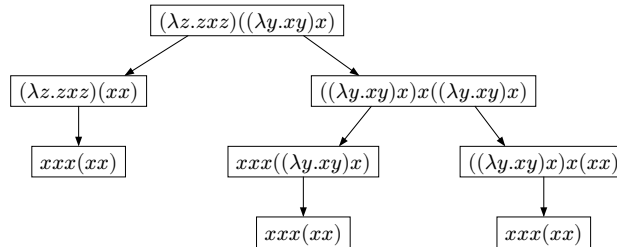## Problem

Find all available redexes in

$$U := (\lambda z.zxz)((\lambda y.xy)x)$$

And all reduction pathes to the $\beta$-normal form.

*Solution.* The first redex is the term as an application itself; another the second term in the application.



## Problem

Show that

$$(\lambda x.xx)y \underset{\beta}{\neq} (\lambda xy.yx)xx$$

*Solution.* By Collorary 1.9.9, it suffices to prove the hypothesis with a proof of a common normal reduced form from LHS and RHS not existing.

*Contradiction.* By definition of $\underset{\beta}{=}$, there exists The set of all terms attainable from $\beta$-reduction on $(\lambda x.xx)y$ and $(\lambda xy.yx)xx$ do not intersect. Therefore,

$$\neg\left(\exists L \in \Lambda, (\lambda x.xx)y \underset{\beta}{\twoheadrightarrow} L \wedge (\lambda xy.yx)xx \underset{\beta}{\twoheadrightarrow} L\right) \implies \neg\left((\lambda x.xx)y \underset{\beta}{=} (\lambda xy.yx)xx\right)$$

∎

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**Problem**

Define the combinators

$$K := \lambda xy.x$$
$$S := \lambda xyz.xz(yz)$$

Prove that

$$\forall PQ \in \Lambda, KPQ \underset{\beta}{\twoheadrightarrow} P$$

$$\forall PQR \in \Lambda, SPQR \underset{\beta}{\twoheadrightarrow} PR(QR)$$

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*Solution.*

*Proof.*

$$KPQ = (\lambda xy.x)PQ \underset{\beta}{\rightarrow} (\lambda y.x)[x := P]Q \underset{\beta}{\rightarrow} P[y := Q] = P$$

$$SPQR = (\lambda xyz.xz(yz)) \underset{\beta}{\twoheadrightarrow} (xz(yz))[x := P][y := Q][z := R] = PR(QR)$$

∎

**Problem**

We define the church numerals

$$\text{zero} := \lambda fx.x$$
$$\text{one} := \lambda fx.fx$$
$$\text{two} := \lambda fx.ffx$$
$$\ldots$$
$$\text{num}_n := \lambda fx.f^n x$$

And operations

$$\text{add} := \lambda nmfx.mf(nfx)$$
$$\text{mul} := \lambda nmfx.m(nf)x$$

Show

(a) add one one $\underset{\beta}{\twoheadrightarrow}$ two

(b) add one one $\underset{\beta}{\neq}$ mul one zero

*Solution.*

$$
\begin{aligned}
\text{(a)} \quad \text{add one one} &= (\lambda nmfx.mf(nfx))(\lambda fx.fx)(\lambda fx.fx) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda fx.(\lambda fx.fx)f((\lambda fx.fx)fx)) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda fx.(\lambda x.fx)fx) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda fx.ffx) = \text{two}
\end{aligned}
$$

$$
\begin{aligned}
\text{(b)} \quad \text{mul one one} &= (\lambda nmfx.m(nf)x)(\lambda fx.fx)(\lambda fx.fx) \\
&\underset{\beta}{\twoheadrightarrow} \lambda fx.(\lambda fx.fx)((\lambda fx.fx)f)x \\
&\underset{\beta}{\twoheadrightarrow} \lambda fx.fx = \text{one}
\end{aligned}
$$

Because no intermediate form in the beta reduction process of the two terms are $\alpha$-equivalent, by collorary 1.9.9 the two terms are not $\beta$-equivalent.

*Solution.* It suffices to provide a witness of a reduction chain from one side to the other to prove $\beta$-equivalence.

*Proof.*

$$\text{succ zero} = (\lambda mfx.f(mfx))(\lambda fx.x)$$
$$\underset{\beta}{\rightarrow} (\lambda fx.f((\lambda fx.x)fx))$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda fx.fx) = \text{one}$$

The path succ zero $\underset{\beta}{\twoheadrightarrow}$ one derived above is the witness of a reduction chain from LHS to RHS.

$$\text{succ one} = (\lambda mfx.f(mfx))(\lambda fx.fx)$$
$$\underset{\beta}{\rightarrow} (\lambda fx.f((\lambda fx.fx)fx))$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda fx.f(fx)) = \text{two}$$

The path succ one $\underset{\beta}{\twoheadrightarrow}$ two derived above is the witness of a reduction chain from LHS to RHS. ∎

*Solution.* It suffices to provide a witness of a reduction chain from one side to the other to prove $\beta$-equivalence.

*Proof.*

$$\neg_\lambda(\neg_\lambda\top_\lambda) = \neg_\lambda((\lambda a.a\bot_\lambda\top_\lambda)(\lambda xy.x))$$
$$\underset{\beta}{\twoheadrightarrow} \neg_\lambda((\lambda xy.x)\bot_\lambda\top_\lambda)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda a.a\bot_\lambda\top_\lambda)\bot_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda xy.y)\bot_\lambda\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} \top_\lambda$$

$$\neg_\lambda(\neg_\lambda\bot_\lambda) = \neg_\lambda((\lambda a.a\bot_\lambda\top_\lambda)(\lambda xy.y))$$
$$\underset{\beta}{\twoheadrightarrow} \neg_\lambda((\lambda xy.x)\bot_\lambda\top_\lambda)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda a.a\bot_\lambda\top_\lambda)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda xy.x)\bot_\lambda\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} \bot_\lambda$$

∎

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**Problem**

Define

$$\text{iszero} := \lambda m.m(\lambda x.\bot_\lambda)\top_\lambda$$

Prove

$$\text{iszero zero} \underset{\beta}{\twoheadrightarrow} \top_\lambda$$
$$\forall n \in \mathbb{N}^+, \text{iszero num}_n \underset{\beta}{\twoheadrightarrow} \bot_\lambda$$

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*Solution.*

$$\text{iszero zero} = (\lambda m.m(\lambda x.\bot_\lambda)\top_\lambda)(\lambda fx.x)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda fx.x)(\lambda x.\bot_\lambda)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} \top_\lambda$$

$$\text{iszero num}_n = (\lambda m.m(\lambda x.\perp_\lambda)\top_\lambda)(\lambda fx.f^n x)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda fx.f^n x)(\lambda x.\perp_\lambda)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda x.\perp_\lambda)\big((\lambda x.\perp_\lambda)^{n-1}\top_\lambda\big) \underset{\beta}{\twoheadrightarrow} \perp_\lambda$$

## Problem

If-else can be modeled as

$$\text{ifelse} = \lambda xtf.xtf$$

Where when $x$, then $t$, else $f$. Prove correctness by applying $\top_\lambda$ and $\perp_\lambda$ on ifelse.

*Solution.*

$$\text{ifelse } \top_\lambda = (\lambda xtf.xtf)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda tf.(\lambda xy.x)tf) \underset{\beta}{\twoheadrightarrow} (\lambda tf.t)$$

$$\text{ifelse } \top_\lambda = (\lambda xtf.xtf)\perp_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda tf.(\lambda xy.y)tf) \underset{\beta}{\twoheadrightarrow} (\lambda tf.f)$$

By applying the results to any two values, the correct corresponding value returns, ex, for ifelse $\top_\lambda$, $t$ is always returned.

## Problem

Prove that $\Omega := (\lambda x.xx)(\lambda x.xx)$.

*Solution.* Firstly let's prove $\Omega$.

*Proof.* Induction on $\Omega$'s only reduction path proves that every $\Omega \underset{\beta}{\twoheadrightarrow} \Omega_i = \Omega$. For the base case because $\Omega$ has one and only one redux, it could only reduce to $\Omega_1$ which is equivalent to itself. For the inductive step, $\Omega_i = \Omega$, therefore $\Omega_i \underset{\beta}{\to} \Omega_{i+1}$ is still $\Omega$.

By definition, a term having a $\beta$-nf requires the existence of a form in $\beta$-nf such that the term can reduce to. By induction, $\Omega$ only reduces to $\Omega$, and $\Omega$ is not in $\beta$-nf because it contains $\beta$-redex. Therefore, $\Omega$ can never reduce to a $\beta$-nf, thus it does not have a $\beta$-nf. ∎

**Problem**

Let $M$ be a $\lambda$-term with the following properties:
- $M$ has a $\beta$-nf.
- There exists an infinite reduction path $M \equiv M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \ldots$ on $M$.

Prove that every $M_i$ has a $\beta$-nf, and give an example of $M$.

*Solution.* An example would be $(\lambda xy.y)\Omega$. Reduction can go on infinitely by reducing on $\Omega$, but the $\beta$-nf of the term is $\lambda y.y$

*Proof.* Denote $\beta$-nf of $M$ as $M'$. For any form in the reduction path, $M \underset{\beta}{\twoheadrightarrow} M_i$. In conjunction with $M \underset{\beta}{\twoheadrightarrow} M'$, by the Church-Rosser theorem, there exists $L$ such that $M_i \underset{\beta}{\twoheadrightarrow} L$ and $M' \underset{\beta}{\twoheadrightarrow} L$. Because $M'$ is in $\beta$-nf, $L$ can only be $M'$, thus $M_i \underset{\beta}{\twoheadrightarrow} M'$, so $M_i$ is capable of reducing to $M'$, a $\beta$-nf. Therefore, any form in the reduction path has a $\beta$-nf. ∎

**Problem**

If $MN$ is strongly normalizing, then both $M$ and $N$ are strongly normalizing.

*Solution.*

*Proof.* If $M$ is not strongly normalizing, then there exists a reduction path $M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \ldots$ Therefore, $MN$ would have had a reduction path $MN \underset{\beta}{\to} M_1 N \underset{\beta}{\to} \ldots$ that is infinite, which contradicts with $MN$ being strongly normalizing. Vice versa for $N$. ∎

**Problem**

Let $L, M, N \in \Lambda$ such that $L \underset{\beta}{=} M$ and $L \underset{\beta}{\twoheadrightarrow} N$. Moreover, $N$ is in $\beta$-nf. Prove that $M \underset{\beta}{\twoheadrightarrow} N$.

*Solution.* Collorary 1.9.9.

Define

$$U := \lambda zx.x(zzx) \quad \text{and} \quad Z := UU$$

Prove $Z$ is a fixed point combinator.

*Solution.* Proving $\forall L \in \Lambda, L(ZL) \underset{\beta}{\twoheadrightarrow} ZL$.

*Proof.*

$$ZL = (\lambda zx.x(zzx))(\lambda zx.x(zzx))L$$
$$\underset{\beta}{\twoheadrightarrow} L((\lambda zx.x(zzx))(\lambda zx.x(zzx))L) \underset{\beta}{\twoheadrightarrow} L(ZL)$$

∎

**Problem**

Solve for $M \in \Lambda$ in each equation:

$$M \underset{\beta}{=} \lambda xy.xMy$$

$$Mxyz \underset{\beta}{=} xyzM$$

*Solution.* By the property of the $Y$ combinator:

$$f(Yf) = Yf$$

The first equation can be remodeled as

$$M \underset{\beta}{=} LM \text{ where } L = \lambda mxy.xmy$$

Solving for fixed point of $L$ :

$$M \equiv YL \equiv L(YL)$$
$$= (\lambda x.L(xx))(\lambda x.L(xx))$$
$$= (\lambda x.(\lambda muv.umv)(xx))(\lambda x.(\lambda muv.umv)(xx))$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda x.(\lambda uv.u(xx)v))(\lambda x.(\lambda uv.u(xx)v))$$

The second equation can be $\eta$-reduced on both sides:

$$Mxyz \underset{\beta}{=} xyzM$$

$$M \underset{\beta}{=} \lambda zyx.xyzM$$

Remodeling equation:

$$M = NM \text{ where } M = \lambda mzyx.xyzm$$

then

$$
\begin{aligned}
M &\equiv YN \equiv N(YN) \\
&= (\lambda x.N(xx))(\lambda x.N(xx)) \\
&= (\lambda x.(\lambda mzyu.uyzm)(xx))(\lambda x.(\lambda mzyu.uyzm)(xx)) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda x.(\lambda zyu.uyz(xx)))(\lambda x.(\lambda zyu.uyz(xx)))
\end{aligned}
$$