Excercises

Chapter 1

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1. Redacted

Problem

(1.1) Simplify notation of the following terms

$$\text{(a)}\ (\lambda x\,.(((x\ z)\ y)(x\ x)))$$
$$\text{(b)}\ ((\lambda x\,.(\lambda y\,.(\lambda z\,.(z\ ((x\ y)\ z)))))(\lambda u\,.\,u))$$

Solution.

$$\text{(a)}\ \lambda x\,.(x\ z\ y)(x\ x)$$
$$\text{(b)}\ (\lambda x\ y\ z\,.\,x\ (x\ y\ z))(\lambda u\,.\,u)$$

Problem

(1.2) Find the alpha equivalent terms to

$$\lambda x\,.\,x\ (\lambda x\,.\,x)$$

In

$$\text{(a)}\ \lambda y\,.\,y\ (\lambda x\,.\,x)$$
$$\text{(b)}\ \lambda y\,.\,y\ (\lambda x\,.\,y)$$
$$\text{(c)}\ \lambda y\,.\,y\ (\lambda x\,.\,y)$$

Solution. Only (a).

Problem

(1.3) Prove

$$\lambda x \,.\, x\ (\lambda z \,.\, y) \underset{\alpha}{=} \lambda z \,.\, z\ (\lambda z \,.\, y)$$

Solution.

Proof. By definition of alpha equivalence

$$M \underset{\alpha}{=} N \iff \exists \varphi, M^{\varphi} \underset{\alpha}{\to} N \land \mathrm{FR}\ M = \mathrm{FR}\ N$$

The witness of φ is substituting bound variable x with z, and z is not a free variable in the term, thus the two terms are alpha equivalent.

$$\lambda x \,.\, x\ (\lambda z \,.\, y)^{x \to z} \underset{\alpha}{\to} \lambda z \,.\, z\ (\lambda z \,.\, y)$$

■

Problem

(1.4) Consider the following term:

$$U \coloneqq (\lambda z \,.\, z\ x\ z)((\lambda y \,.\, x\ y)\ x)$$

1. Find Sub U
2. Draw tree rep of U
3. Find FV U
4. Find alpha equivalent terms to U from below and point out which of those follows the Barendregt convention:

(a) $(\lambda y \,.\, y\ x\ y)((\lambda z \,.\, x\ z)\ x)$

(b) $(\lambda x \,.\, x\ y\ x)((\lambda z \,.\, y\ z)\ y)$

(c) $(\lambda y \,.\, y\ x\ y)((\lambda y \,.\, x\ y)\ x)$

(d) $(\lambda v \,.(v\ x)\ v)((\lambda u \,.\, u\ v)\ x)$

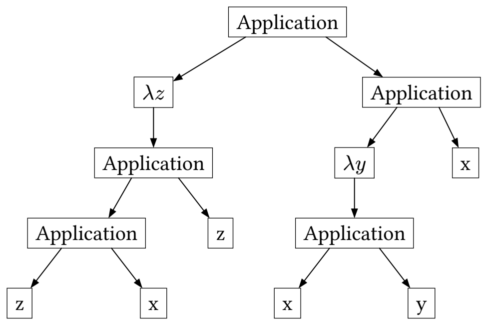
1. Find Sub U.

Solution.

$$
\begin{aligned}
\text{Sub}\,U = {} & \\
& \{(\lambda z\,.\,z\;x\;z)((\lambda y\,.\,x\;y)\;x), (\lambda z\,.\,z\;x\;z), ((\lambda y\,.\,x\;y)\;x)\} \cup \\
& \{(\lambda y\,.\,x\;y), z\;x\;z, z\;x, x\;y\} \cup \\
& \{z, x, y\} \\
= {} & \{(\lambda z\,.\,z\;x\;z)((\lambda y\,.\,x\;y)\;x), (\lambda z\,.\,z\;x\;z), ((\lambda y\,.\,x\;y)\;x), \\
& (\lambda y\,.\,x\;y), z\;x\;z, z\;x, x\;y, z, x, y\}
\end{aligned}
$$

2. Draw a tree rep of U.

Solution.



3. Find FV U

Solution.

$$
\begin{aligned}
\text{FV}\,U &= \text{FV}\;(\lambda y\,.\,y\;x\;y) \cup \text{FV}\;(\lambda z\,.\,x\;z)\;x \\
&= (\text{FV}\;y\;x\;y) \setminus \{y\} \cup (\text{FV}\;\;\lambda z\,.\,x\;z) \cup \{x\} \\
&= (\text{FV}\;y\;x) \setminus \{y\} \cup (\text{FV}\;x\;z) \setminus \{z\} \cup \{x\} \\
&= \{x\}
\end{aligned}
$$

4. Find an alpha-equivalent term.

Solution.

$$(a) \underset{\alpha}{=} (c) \underset{\alpha}{=} U$$

Only (a) follows the Barendregt convention.

Problem

(1.5) Give the results of the following substitutions

(a) $(\lambda x\,.\,y\;(\lambda y\,.\,x\;y))[y := \lambda z\,.\,z\;x]$

(b) $((x\;y\;z)[x := y])[y := z]$

(c) $((\lambda x\,.\,x\;y\;z)[x := y])[y := z]$

(d) $(\lambda y\,.\,y\;y\;x)[x := y\;z]$

Solution.

(a) $(\lambda v \,.(\lambda z \,.\, z\, x)(\lambda u \,.\, v\, u))$

(b) $(y\, y\, z)[y := z] = z\, z\, z$

(c) $(\lambda x \,.\, x\, y\, z)[y := z] = (\lambda x \,.\, x\, z\, z)$

(d) $(\lambda u \,.\, u\, u\, (y\, z))$

Problem

(1.6)

$$\neg\Big(\forall\ M\ L\ N \in \Lambda, M\ [x := N, y := L] \underset{\alpha}{\equiv} M\ [x := N][y := L]\Big)$$

Solution.

Proof. Because RHS $= M\ [x := N][y := L] = M\ [x := N\ [y := L]][y := L]$, if $y \in \mathrm{FV}\ N$, then what x gets substituted with will have y substituted for L, which is completely different with LHS. ■

Problem

(1.7) Find all available redexes in

$$U := (\lambda z \,.\, z\, x\, z)((\lambda y \,.\, x\, y)\, x)$$

And all reduction pathes to the β-normal form.

Solution. The first redex is the term as an application itself; another the second term in the application.

$$\begin{tikzcd}
& (\lambda z \,.\, z\, x\, z)((\lambda y \,.\, x\, y)\, x) \arrow[dl] \arrow[drr] & & & \\
(\lambda z \,.\, z\, x\, z)(x\, x) \arrow[d] & & & ((\lambda y \,.\, x\, y)\, x)\, x\, ((\lambda y \,.\, x\, y)\, x) \arrow[dl] \arrow[dr] & \\
x\, x\, x\, (x\, x) & & x\, x\, x\, ((\lambda y \,.\, x\, y)\, x) \arrow[d] & & ((\lambda y \,.\, x\, y)\, x)\, x\, (x\, x) \arrow[d] \\
& & x\, x\, x\, (x\, x) & & x\, x\, x\, (x\, x)
\end{tikzcd}$$

Problem

(1.8) Show that

$$(\lambda x\,.\,x\ x)\ y \underset{\beta}{\neq} (\lambda x\ y\,.\,y\ x)\ x\ x$$

Solution. By Collorary 1.9.9, it suffices to prove the hypothesis with a proof of a common normal reducted form from LHS and RHS not existing.

Contradiction. By definition of $\underset{\beta}{=}$, there exists The set of all terms attainable from β-reduction on $(\lambda x\,.\,x\ x)\ y$ and $(\lambda x\ y\,.\,y\ x)\ x\ x$ do not intersect. Therefore,

$$\neg\left(\exists L \in \Lambda, (\lambda x\,.\,x\ x)\ y \underset{\beta}{\twoheadrightarrow} L \wedge (\lambda x\ y\,.\,y\ x)\ x\ x \underset{\beta}{\twoheadrightarrow} L\right) \implies \neg\left((\lambda x\,.\,x\ x)\ y \underset{\beta}{=} (\lambda x\ y\,.\,y\ x)\ x\ x\right)$$

■

Problem

(1.9) Define the combinators

$$K := \lambda x\ y\,.\,x$$
$$S := \lambda x\ y\ z\,.\,x\ z\ (y\ z)$$

Prove that

$$\forall\ P\ Q \in \Lambda, K\ P\ Q \underset{\beta}{\twoheadrightarrow} P$$

$$\forall\ P\ Q\ R \in \Lambda, S\ P\ Q\ R \underset{\beta}{\twoheadrightarrow} P\ R\ (Q\ R)$$

Solution.

Proof.

$$K\ P\ Q = (\lambda x\ y\,.\,x)\ P\ Q \underset{\beta}{\rightarrow} (\lambda y\,.\,x)[x := P]\ Q \underset{\beta}{\rightarrow} P\ [y := Q] = P$$

$$S\ P\ Q\ R = (\lambda x\ y\ z\,.\,x\ z\ (y\ z)) \underset{\beta}{\twoheadrightarrow} (x\ z\ (y\ z))[x := P][y := Q][z := R] = P\ R\ (Q\ R)$$

■

Problem

(1.10) We define the church numerals

$$\begin{aligned} \text{zero} &:= \lambda f\ x\ .\ x \\ \text{one} &:= \lambda f\ x\ .\ f\ x \\ \text{two} &:= \lambda f\ x\ .\ f\ f\ x \\ &\dots \\ \text{num}_n &:= \lambda f\ x\ .\ f^n\ x \end{aligned}$$

And operations

$$\begin{aligned} \text{add} &:= \lambda n\ m\ f\ x\ .\ m\ f\ (n\ f\ x) \\ \text{mul} &:= \lambda n\ m\ f\ x\ .\ m\ (n\ f)\ x \end{aligned}$$

Show

$$\text{(a) add one one} \twoheadrightarrow_\beta \text{two}$$

$$\text{(b) add one one} \neq_\beta \text{mul one zero}$$

Solution.

$$\begin{aligned} \text{(a)} \quad \text{add one one} &= (\lambda n\ m\ f\ x\ .\ m\ f\ (n\ f\ x))(\lambda f\ x\ .\ f\ x)(\lambda f\ x\ .\ f\ x) \\ &\twoheadrightarrow_\beta (\lambda f\ x\ .(\lambda f\ x\ .\ f\ x)\ f\ ((\lambda f\ x\ .\ f\ x)\ f\ x)) \\ &\twoheadrightarrow_\beta (\lambda f\ x\ .(\lambda x\ .\ f\ x)\ f\ x) \\ &\twoheadrightarrow_\beta (\lambda f\ x\ .\ f\ f\ x) = \text{two} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \text{mul one one} &= (\lambda n\ m\ f\ x\ .\ m\ (n\ f)\ x)(\lambda f\ x\ .\ f\ x)(\lambda f\ x\ .\ f\ x) \\ &\twoheadrightarrow_\beta \lambda f\ x\ .(\lambda f\ x\ .\ f\ x)((\lambda f\ x\ .\ f\ x)\ f)\ x \\ &\twoheadrightarrow_\beta \lambda f\ x\ .\ f\ x = \text{one} \end{aligned}$$

Because no intermediate form in the beta reduction process of the two terms are α-equivalent, by collorary 1.9.9 the two terms are not β-equivalent.

Problem

(1.11) We define

$$\text{succ} := \lambda m\ f\ x\ .\ f\ (m\ f\ x)\ \text{ s.t. } \forall \text{num}_n, \text{succ num}_n = \text{num}_{n+1}$$

Prove

$$\text{succ zero} \underset{\beta}{=} \text{one}$$

$$\text{succ one} \underset{\beta}{=} \text{two}$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$\begin{aligned} \text{succ zero} &= (\lambda m\ f\ x\ .\ f\ (m\ f\ x))(\lambda f\ x\ .\ x) \\ &\underset{\beta}{\to} (\lambda f\ x\ .\ f\ ((\lambda f\ x\ .\ x)\ f\ x)) \\ &\underset{\beta}{\twoheadrightarrow} (\lambda f\ x\ .\ f\ x) = \text{one} \end{aligned}$$

The path succ zero $\underset{\beta}{\twoheadrightarrow}$ one derived above is the witness of a reduction chain from LHS to RHS.

$$\begin{aligned} \text{succ one} &= (\lambda m\ f\ x\ .\ f\ (m\ f\ x))(\lambda f\ x\ .\ f\ x) \\ &\underset{\beta}{\to} (\lambda f\ x\ .\ f\ ((\lambda f\ x\ .\ f\ x)\ f\ x)) \\ &\underset{\beta}{\twoheadrightarrow} (\lambda f\ x\ .\ f\ (f\ x)) = \text{two} \end{aligned}$$

The path succ one $\underset{\beta}{\twoheadrightarrow}$ two derived above is the witness of a reduction chain from LHS to RHS. ∎

Problem

(1.12) We define the λ-terms $\top_\lambda$(true) and $\bot_\lambda$(false) and $\neg_\lambda$(not) by:

$$\top_\lambda := \lambda x\ y\ .\ x \quad \bot_\lambda := \lambda x\ y\ .\ y$$
$$\neg_\lambda := \lambda a\ .\ a\ \bot_\lambda \top_\lambda$$

Show that

$$\neg_\lambda(\neg_\lambda \top_\lambda) \underset{\beta}{\equiv} \top_\lambda$$

$$\neg_\lambda(\neg_\lambda \bot_\lambda) \underset{\beta}{\equiv} \bot_\lambda$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$\begin{aligned}
\neg_\lambda(\neg_\lambda \top_\lambda) &= \neg_\lambda((\lambda a \,.\, a \perp_\lambda \top_\lambda)(\lambda x\, y \,.\, x)) \\
&\twoheadrightarrow_\beta \neg_\lambda((\lambda x\, y \,.\, x)\perp_\lambda \top_\lambda) \\
&\twoheadrightarrow_\beta (\lambda a \,.\, a \perp_\lambda \top_\lambda)\perp_\lambda \\
&\twoheadrightarrow_\beta (\lambda x\, y \,.\, y)\perp_\lambda \top_\lambda \\
&\twoheadrightarrow_\beta \top_\lambda
\end{aligned}$$

$$\begin{aligned}
\neg_\lambda(\neg_\lambda \perp_\lambda) &= \neg_\lambda((\lambda a \,.\, a \perp_\lambda \top_\lambda)(\lambda x\, y \,.\, y)) \\
&\twoheadrightarrow_\beta \neg_\lambda((\lambda x\, y \,.\, y)\perp_\lambda \top_\lambda) \\
&\twoheadrightarrow_\beta \neg_\lambda \top_\lambda \\
&\twoheadrightarrow_\beta (\lambda a \,.\, a \perp_\lambda \top_\lambda)(\lambda x\, y \,.\, x) \\
&\twoheadrightarrow_\beta (\lambda x\, y \,.\, x)\perp_\lambda \top_\lambda \\
&\twoheadrightarrow_\beta \perp_\lambda
\end{aligned}$$

■

Problem

(1.13) Define

$$\text{iszero} \coloneqq \lambda m \,.\, m\, (\lambda x \,.\perp_\lambda)\top_\lambda$$

Prove

$$\text{iszero zero} \twoheadrightarrow_\beta \top_\lambda$$

$$\forall\, n \in \mathbb{N}^+, \text{iszero num}_n \twoheadrightarrow_\beta \perp_\lambda$$

Solution.

$$\begin{aligned}
\text{iszero zero} &= (\lambda m \,.\, m\, (\lambda x \,.\perp_\lambda)\top_\lambda)(\lambda f\, x \,.\, x) \\
&\twoheadrightarrow_\beta (\lambda f\, x \,.\, x)(\lambda x \,.\perp_\lambda)\top_\lambda \\
&\twoheadrightarrow_\beta \top_\lambda
\end{aligned}$$

$$
\begin{aligned}
\text{iszero num}_n &= (\lambda m\,.\,m\,(\lambda x\,.\bot_\lambda)\top_\lambda)(\lambda f\,x\,.\,f^n\,x)\\
&\underset{\beta}{\twoheadrightarrow} (\lambda f\,x\,.\,f^n\,x)(\lambda x\,.\bot_\lambda)\top_\lambda\\
&\underset{\beta}{\twoheadrightarrow} (\lambda x\,.\bot_\lambda)\big((\lambda x\,.\bot_\lambda)^{n-1}\top_\lambda\big) \underset{\beta}{\twoheadrightarrow} \bot_\lambda
\end{aligned}
$$

Problem

(1.14) If-else can be modeled as

$$\text{ifelse} = \lambda x\;t\;f\;.\;x\;t\;f$$

Where when x, then t, else f. Prove correctness by applying $\top_\lambda$ and $\bot_\lambda$ on ifelse.

Solution.

$$
\begin{aligned}
\text{ifelse}\ \top_\lambda &= (\lambda x\;t\;f\;.\;x\;t\;f)\top_\lambda\\
&\underset{\beta}{\twoheadrightarrow} (\lambda t\;f\;.(\lambda x\;y\;.\;x)\;t\;f) \underset{\beta}{\twoheadrightarrow} (\lambda t\;f\;.\;t)
\end{aligned}
$$

$$
\begin{aligned}
\text{ifelse}\ \top_\lambda &= (\lambda x\;t\;f\;.\;x\;t\;f)\bot_\lambda\\
&\underset{\beta}{\twoheadrightarrow} (\lambda t\;f\;.(\lambda x\;y\;.\;y)\;t\;f) \underset{\beta}{\twoheadrightarrow} (\lambda t\;f\;.\;f)
\end{aligned}
$$

By applying the results to any two values, the correct corresponding value returns, ex, for ifelse $\top_\lambda$, t is always returned.

Problem

(1.15) Prove that $\Omega \coloneqq (\lambda x\;.\;x\;x)(\lambda x\;.\;x\;x)$ does not have a β-nf.

Solution. Firstly let's prove Ω.

Proof. Induction on Ω's only reduction path proves that every $\Omega \underset{\beta}{\twoheadrightarrow} \Omega_i = \Omega$. For the base case because Ω has one and only one redux, it could only reduce to Ω_1 which is equivalent to itself. For the inductive step, $\Omega_i = \Omega$, therefore $\Omega_i \underset{\beta}{\rightarrow} \Omega_{i+1}$ is still Ω.

By definition, a term having a β-nf requires the existence of a form in β-nf such that the term can reduce to. By induction, Ω only reduces to Ω, and Ω is not in β-nf because it contains β-redex. Therefore, Ω can never reduce to a β-nf, thus it does not have a β-nf. ■

Problem

(1.16) Let M be a λ-term with the following properties:

- M has a β-nf.
- There exists an infinite reduction path $M \equiv M_0 \underset{\beta}{\rightarrow} M_1 \underset{\beta}{\rightarrow} \dots$ on M.

Prove that every M_i has a β-nf, and give an example of M.

Solution. An example would be $(\lambda x\ y\ .\ y)\Omega$. Reduction can go on infinitely by reducing on Ω, but the β-nf of the term is $\lambda y\ .\ y$

Proof. Denote β-nf of M as M'. For any form in the reduction path, $M \underset{\beta}{\twoheadrightarrow} M_i$. In conjunction with $M \underset{\beta}{\twoheadrightarrow} M'$, by the Church-Rosser theorem, there exists L such that $M_i \underset{\beta}{\twoheadrightarrow} L$ and $M' \underset{\beta}{\twoheadrightarrow} L$. Because M' is in β-nf, L can only be M', thus $M_i \underset{\beta}{\twoheadrightarrow} M'$, so M_i is capable of reducing to M', a β-nf. Therefore, any form in the reduction path has a β-nf. ■

Problem

(1.17) If $M\ N$ is strongly normalizing, then both M and N are strongly normalizing.

Solution.

Proof. If M is not strongly normalizing, then there exists a reduction path $M_0 \underset{\beta}{\rightarrow} M_1 \underset{\beta}{\rightarrow} \dots$ Therefore, $M\ N$ would have had a reduction path $M\ N \underset{\beta}{\rightarrow} M_1\ N \underset{\beta}{\rightarrow} \dots$ that is infinite, which contradicts with $M\ N$ being strongly normalizing. Vice versa for N. ■

Problem

(1.18) Let $L, M, N \in \Lambda$ such that $L \underset{\beta}{=} M$ and $L \underset{\beta}{\twoheadrightarrow} N$. Moreover, N is in β-nf. Prove that $M \underset{\beta}{\twoheadrightarrow} N$.

Solution. Collorary 1.9.9.

Problem

(1.19) Define

$$U := \lambda z\ x\ .\ x\ (z\ z\ x) \quad \text{and} \quad Z := U\ U$$

Prove Z is a fixed point combinator.

Solution. Proving $\forall\, L \in \Lambda, L\,(Z\,L) \twoheadrightarrow_{\beta} Z\,L$.

Proof.

$$\begin{aligned} Z\,L &= (\lambda z\,x\,.\,x\,(z\,z\,x))(\lambda z\,x\,.\,x\,(z\,z\,x))\,L \\ &\twoheadrightarrow_{\beta} L\,((\lambda z\,x\,.\,x\,(z\,z\,x))(\lambda z\,x\,.\,x\,(z\,z\,x))\,L) \twoheadrightarrow_{\beta} L\,(Z\,L) \end{aligned}$$

■

Problem

(1.20) Solve for $M \in \Lambda$ in each equation:

$$M =_{\beta} \lambda x\,y\,.\,x\,M\,y$$

$$M\,x\,y\,z =_{\beta} x\,y\,z\,M$$

Solution. By the property of the Y combinator:

$$f\,(Y\,f) = Y\,f$$

The first equation can be remodeled as

$$M =_{\beta} L\,M \text{ where } L = \lambda m\,x\,y\,.\,x\,m\,y$$

Solving for fixed point of L :

$$\begin{aligned} M &\equiv Y\,L \equiv L\,(Y\,L) \\ &= (\lambda x\,.\,L\,(x\,x))(\lambda x\,.\,L\,(x\,x)) \\ &= (\lambda x\,.(\lambda m\,u\,v\,.\,u\,m\,v)(x\,x))(\lambda x\,.(\lambda m\,u\,v\,.\,u\,m\,v)(x\,x)) \\ &\twoheadrightarrow_{\beta} (\lambda x\,.(\lambda u\,v\,.\,u\,(x\,x)\,v))(\lambda x\,.(\lambda u\,v\,.\,u\,(x\,x)\,v)) \end{aligned}$$

The second equation can be η-reduced on both sides:

$$M\,x\,y\,z =_{\beta} x\,y\,z\,M$$

$$M =_{\beta} \lambda x\,y\,z\,.\,x\,y\,z\,M$$

Remodeling equation:

$$M = N\,M \text{ where } M = \lambda m\,x\,y\,z\,.\,x\,y\,z\,m$$

then

$$
\begin{aligned}
M &\equiv Y\ N \equiv N\ (Y\ N) \\
&= (\lambda x\ .\ N\ (x\ x))(\lambda x\ .\ N\ (x\ x)) \\
&= (\lambda x\ .(\lambda m\ u\ y\ z\ .\ u\ y\ z\ m)(x\ x))(\lambda x\ .(\lambda m\ u\ y\ z\ .\ u\ y\ z\ m)(x\ x)) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda x\ .(\lambda u\ y\ z\ .\ u\ y\ z\ (x\ x)))(\lambda x\ .(\lambda u\ y\ z\ .\ u\ y\ z\ (x\ x)))
\end{aligned}
$$