

# EXERCISES

## CHAPTER 5

SEAN LI <sup>1</sup>

1. Reducted

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**Definition** Some rules for reference.

### $\lambda C$ Calculus Rules

$$\begin{array}{c} \frac{}{\emptyset \vdash * : \square} \text{Sort} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \text{Var} \quad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \text{Weak} \\ \\ \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A . B : s_2} \text{Form} \\ \\ \frac{\Gamma \vdash M : \Pi x : A . B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B [x := N]} \text{App} \\ \\ \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A . B : s}{\Gamma \vdash \lambda x : A . M : \Pi x : A . B} \text{Abst} \\ \\ \frac{\Gamma \vdash A : B \quad B \stackrel{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \text{Conv} \end{array}$$

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### Problem

(6.1 a) Give a complete derivation in tree format showing that

$$\perp \equiv \Pi \alpha : * . \alpha$$

is legal in  $\lambda C$ .

*Solution.* Here we will show that there exists  $s \in \text{sort}$  and  $\Gamma$  such that  $\Gamma \vdash \perp : s$ .

*Proof.*

$$\frac{\vdash * : \square \quad \frac{\vdash * : \square \quad \frac{\alpha : * \vdash \alpha : *}{\vdash \Pi \alpha : * . \alpha : *}}{\alpha : * \vdash \alpha : *} \text{Var}}{\vdash \Pi \alpha : * . \alpha : *} \text{Form}$$

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### Problem

(6.1 a) Give a complete derivation in tree format showing that  $\perp \rightarrow \perp$  is legal in  $\lambda C$  where

$$\perp \equiv \Pi \alpha : * . \alpha$$

*Solution.* Here we will show that there exists  $s \in \text{sort}$  and  $\Gamma$  such that  $\Gamma \vdash \perp \rightarrow \perp : s$ .

*Proof.*

$$\frac{(6.1 \text{ a}) \frac{}{\vdash \perp : *} \quad (6.1 \text{ a}) \frac{}{\vdash \perp : *} \quad (6.1 \text{ a}) \frac{}{\vdash \perp : *}}{\vdash \perp : *} \frac{\vdash \perp : * \quad \vdash \perp : *}{x : \perp \vdash \perp : *} \text{Weak}$$

$$\frac{x : \perp \vdash \perp : *}{\vdash \Pi x : \perp . \perp : *} \text{Form}$$

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### Problem

(6.1 c) To which systems of the  $\lambda$ -cube does  $\perp$  belong? And  $\perp \rightarrow \perp$ ?

*Solution.* The set of  $(s_1, s_2)$  pairs in formation rules of the derivation of  $\perp$  is  $\{(\square, *)\}$ . The minimal system corresponding is  $\lambda 2$ . The same for  $\perp \rightarrow \perp$ . Therefore  $\perp$  and  $\perp \rightarrow \perp$  belongs to  $\lambda 2, \lambda \omega, \lambda P$  and  $\lambda C$ .