Exercises

Chapter 6

Sean Li [1]

1. Redacted

Reference - Calculus of Constructions

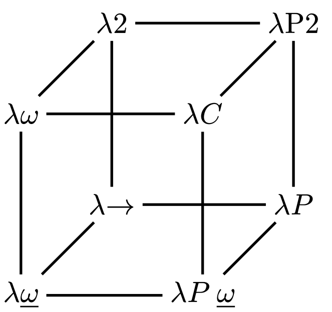
$$\frac{}{\emptyset \vdash * : \square}\,\text{Sort} \qquad \frac{\Gamma \vdash A : s \qquad x \notin \operatorname{dom}\Gamma}{\Gamma, x : A \vdash x : A}\,\text{Var} \qquad \frac{\Gamma \vdash A : B \qquad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}\,\text{Weak}$$

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A \,.\, B : s_2}\,\text{Form} \qquad \frac{\Gamma \vdash M : \Pi x : A \,.\, B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\, N : B\,[x := N]}\,\text{App}$$

$$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash \Pi x : A \,.\, B : s}{\Gamma \vdash \lambda x : A \,.\, M : \Pi x : A \,.\, B}\,\text{Abst}$$

$$\frac{\Gamma \vdash A : B \qquad B \stackrel{\beta}{=} B' \qquad \Gamma \vdash B' : s}{\Gamma \vdash A : B'}\,\text{Conv}$$

The $\boldsymbol{\lambda}$-Cube



$\boldsymbol{\lambda\to}$	$(*, *)$			
$\boldsymbol{\lambda 2}$	$(*, *)$	$(\square, *)$		
$\boldsymbol{\lambda \underline{\omega}}$	$(*, *)$		$(\square, \square)$	
$\boldsymbol{\lambda P}$	$(*, *)$			$(*, \square)$
$\boldsymbol{\lambda \omega}$	$(*, *)$	$(\square, *)$	$(\square, \square)$	
$\boldsymbol{\lambda 2P}$	$(*, *)$	$(\square, *)$		$(*, \square)$
$\boldsymbol{\lambda P\ \underline{\omega}}$	$(*, *)$		$(\square, \square)$	$(*, \square)$
$\boldsymbol{\lambda C}$	$(*, *)$	$(\square, *)$	$(\square, \square)$	$(*, \square)$

Problem

(6.1 a) Give a complete derivation in tree format showing that

$$\bot \equiv \Pi\alpha : * .\alpha$$

is legal in λC.

Solution. Here we will show that there exists $s \in$ sort and Γ such that $\Gamma \vdash \bot : s$.

Proof.

$$\dfrac{\vdash * : \square \qquad \dfrac{\vdash * : \square}{\alpha : * \vdash \alpha : *}\,\text{Var}}{\vdash \Pi\alpha : * .\alpha : *}\,\text{Form}$$

■

Problem

(6.1 b) Give a complete derivation in tree format showing that $\bot \to \bot$ is legal in λC where

$$\bot \equiv \Pi\alpha : * .\alpha$$

Solution. Here we will show that there exists $s \in$ sort and Γ such that $\Gamma \vdash \bot \to \bot : s$.

Proof.

$$\dfrac{\text{(6.1 a)}\,\dfrac{}{\vdash \bot : *} \qquad \dfrac{\text{(6.1 a)}\,\dfrac{}{\vdash \bot : *} \qquad \text{(6.1 a)}\,\dfrac{}{\vdash \bot : *}}{x : \bot \vdash \bot : *}\,\text{Weak}}{\vdash \Pi x : \bot.\bot : *}\,\text{Form}$$

■

Problem

(6.1 c) To which systems of the λ-cube does $\bot$ belong? And $\bot \to \bot$?

Solution. The set of (s_1, s_2) pairs in formation rules of the derivation of $\bot$ is $\{(\square, *)\}$. The minimal system corresponding is $\lambda 2$. The same for $\bot \to \bot$. Therefore $\bot$ and $\bot \to \bot$ belongs to $\lambda 2$, $\lambda\omega$, λP and λC.

Problem

(6.2) Given context $\Gamma \equiv S : *, P : S \to *, A : *$. Prove by means of a flag derivation that the following expression is inhabited in λC with respect to Γ:

$$(\Pi x : S . (A \to P\,x)) \to A \to \Pi y : S . P\,y$$

Solution. The inhabitant is

$$M \equiv \lambda u : (\Pi x : S . (A \to P\,x)) . \lambda v : A . \lambda y : S . u\,y\,v$$

Proof.

1.	$S : *, P : S \to *, A : *$	
2.	$\quad u : \Pi x : S . (A \to P\,x)$	
3.	$\quad\quad v : A$	
4.	$\quad\quad\quad y : S$	
5.	$\quad\quad\quad\quad u\,y : A \to P\,y$	**2,4 App**
6.	$\quad\quad\quad\quad u\,y\,v : P\,y$	**5,3 App**
7.	$\quad\quad\quad \lambda y : S . u\,y\,v : \Pi y : S . P\,y$	**6 Abst**
8.	$\quad\quad \lambda v : A . \lambda y : S . u\,y\,v : A \to \Pi y : S . P\,y$	**7 Abst**
9.	$\quad \lambda u : \Pi x : S . (A \to P\,x) . \lambda v : A . \lambda y : S . u\,y\,v$ $: \Pi x : S . (A \to P\,x) \to A \to \Pi y : S . P\,y$	**8 Abst**

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Problem

(6.3 a) Let $\mathcal{J}$ be a judgement

$$\mathcal{J} \equiv S : *, P : S \to * \vdash \lambda x : S . (P\,x \to \bot) : S \to *$$

Derive $\mathcal{J}$ in λC with shorthand flag notation.

Solution.

1.	$S : *, P : S \to *$	
2.	$\quad x : S$	
3.	$\quad\quad P\,x : *$	**1,2 App**
4.	$\quad\quad \bot : *$	**Weak from 6.1 a**
5.	$\quad\quad P\,x \to \bot : *$	**3,4 Form**

6. $\lambda x : S \,.\, P\, x \to \bot : S \to *$ **5 Abst**

Problem

(6.3 b) Determine the (s_1, s_2) pairs corresponding to all Π abstractions occurring in $\mathcal{J}$.

Solution.

Abstraction	**Line Number**	(s_1, s_2)
$P : S \to *$	1	$(*, \square)$
$\bot \equiv \Pi\alpha : * \,.\alpha$	4	$(\square, *)$
$P\, x \to \bot$	5	$(\square, *)$
$\lambda x : S \,.\, P\, x \to \bot : S \to *$	6	$(*, \square)$

Problem

(6.3 c) What is the 'smallest' system in the λ-cube to which $\mathcal{J}$ belongs?

Solution. There are $(*, *)$ – $\lambda\!\to$ pairs, $(*, \square)$ – λP pairs, and $(\square, *)$ – $\lambda 2$. Therefore the minimal system $\mathcal{J}$ belongs to is λP2.

Problem

(6.4 a) Let $\Gamma \equiv S : *, Q : S \to S \to *$ and

$$M \equiv (\Pi x, y : S \,.(Q\, x\, y \to Q\, y\, x \to \bot)) \to \Pi z : S \,.(Q\, z\, z \to \bot)$$

Derive $\Gamma \vdash M : *$ and determine the smallest subsystem to which this judgement belongs.

Solution.

1.	$S : *, Q : S \to S \to *$	
2.	$x : S$	
3.	$y : S$	
4.	$Q\,x : S \to *$	**1,2 App**
5.	$Q\,x\,y : *$	**4,3 App**
6.	$z : Q\,x\,y$	
7.	$Q\,y : S \to *$	**1,3 App**
8.	$Q\,y\,x : *$	**7,2 App**
9.	$t : Q\,y\,x$	
10.	$\bot : *$	**Weak from 6.1 a**
11.	$Q\,y\,x \to \bot : *$	**8,10 Form**
12.	$Q\,x\,y \to Q\,y\,x \to \bot : *$	**5,11 Form**
13.	$\Pi y : S \,.\, Q\,x\,y \to Q\,y\,x \to \bot : *$	**1,12 Form**
14.	$\Pi x, y : S \,.\, Q\,x\,y \to Q\,y\,x \to \bot : *$	**1,13 Form**
15.	$a : (\Pi x, y : S \,.(Q\,x\,y \to Q\,y\,x \to \bot))$	
16.	$z : S$	
17.	$Q\,z : S \to *$	**1,16 App**
18.	$Q\,z\,z : *$	**17,16 App**
19.	$b : Q\,z\,z$	
20.	$\bot : *$	**Weak from 6.1 a**
21.	$Q\,z\,z \to \bot : *$	**18,20 Form**
22.	$\Pi z : S \,.\, Q\,z\,z \to \bot : *$	**1,21 Form**
23.	$\Pi x, y : S \,.(Q\,x\,y \to Q\,y\,x \to \bot) \to \Pi z : S \,.\, Q\,z\,z \to \bot : *$	**14,22 Form**

Here's a table of all Πs that appeared

Abstraction	**Line Number**	(s_1, s_2)
$S \to *$	1 / 4 / 7 / 17	$(*, \square)$
$S \to S \to *$	1	$(*, \square)$
$\bot$	10 / 11 / 12 / 13 / 14 / 15 / 20 / 21 / 22 / 23	$(\square, *)$
$Q\,y\,x \to \bot$	11 / 12 / 13 / 14 / 15 / 23	$(*, *)$
$Q\,x\,y \to Q\,y\,x \to \bot$	12 / 13 / 14 / 15 / 23	$(*, *)$
$\Pi y : S \,.\, Q\,x\,y \to Q\,y\,x \to \bot$	13 / 14 / 23	$(*, *)$

$\Pi x, y : S \,.\, Q\ x\ y \to Q\ y\ x \to \bot$	14 / 23	$(*, *)$
$Q\ z\ z \to \bot$	21 / 22 / 23	$(*, *)$
$\Pi z : S \,.\, Q\ z\ z \to \bot$	22 / 23	$(*, *)$
$\Pi x, y : S \,.\, Q\ x\ y \to Q\ y\ x \to \bot \to$ $\Pi z : S \,.\, Q\ z\ z \to \bot$	23	$(*, *)$

There are $(*, *)$ – $\lambda\!\to$ pairs, $(*, \square)$ – λP pairs, and $(\square, *)$ – $\lambda 2$ pairs. Therefore the minimal system available is λP2.

Problem

(6.4 b) Prove in λC that M is inhabited in context Γ.

Solution. A shorthand derivation is given below:

Proof.

1.	$S : *, Q : S \to S \to *$	
2.	$h : \Pi x, y : S \,.(Q\ x\ y \to Q\ y\ x \to \bot)$	
3.	$z : S$	
4.	$a : Q\ z\ z$	
5.	$\alpha : *$	
6.	$h\ z : \Pi y : S \,.(Q\ z\ y \to Q\ y\ z \to \bot)$	**2,3 App**
7.	$h\ z\ z : Q\ z\ z \to Q\ z\ z \to \bot$	**6,3 App**
8.	$h\ z\ z\ a : Q\ z\ z \to \bot$	**7,4 App**
9.	$h\ z\ z\ a\ a : \Pi \alpha : * \,.\alpha$	**8,4 App**
10.	$h\ z\ z\ a\ a\ \alpha : \alpha$	**9,5 App**
11.	$\lambda \alpha : * \,.\, h\ z\ z\ a\ a\ \alpha : \Pi \alpha : * \,.\alpha$	**10 Abst**
12.	$\lambda a : Q\ z\ z\ \lambda \alpha : * \,.\, h\ z\ z\ a\ a\ \alpha : Q\ z\ z \to \bot$	**11 Abst**
13.	$\lambda z : S \,.\lambda a : Q\ z\ z\ \lambda \alpha : * \,.\, h\ z\ z\ a\ a\ \alpha : \Pi z : S \,.\, Q\ z\ z \to \bot$	**12 Abst**
14.	$\lambda h : \Pi x, y : S \,.(Q\ x\ y \to Q\ y\ x \to \bot)$ $\lambda z : S \,.\lambda a : Q\ z\ z\ \lambda \alpha : * \,.\, h\ z\ z\ a\ a\ \alpha$ $: \Pi x, y : S \,.(Q\ x\ y \to Q\ y\ x \to \bot) \to \Pi z : S \,.\, Q\ z\ z \to \bot$	**13 Abst**

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Problem

(6.4 c) We may consier Q to be a relation on set S. Moreover by PAT we may see $A \to \bot$ as the negation $\neg A$ of prop A. How can M then be interpreted by the PAT paradigm?

Solution. By a direct type-to-proposition translation we have

$$M \equiv \forall x, y \in S, (Q\ (x, y) \Rightarrow \neg\, Q\ (y, x)) \Rightarrow \forall z \in S, (\neg\, Q\ (z, z))$$

It expresses the fact if Q is asymmetric then it is irreflective.

Problem

(6.5 a) Let

$$\mathcal{J} \equiv S : * \vdash \lambda Q : S \to S \to * \,.\, \lambda x : S \,.\, Q\ x\ x : (S \to S \to *) \to S \to *$$

Give a shorthand derivation of $\mathcal{J}$ and determine the smallest subsystem to which $\mathcal{J}$ belongs.

Solution.

1.	$S : *$	
2.	$\quad Q : S \to S \to *$	
3.	$\quad\quad x : S$	
4.	$\quad\quad\quad Q\ x : S \to *$	**2,3 App**
5.	$\quad\quad\quad Q\ x\ x : *$	**4,3 App**
6.	$\quad\quad \lambda x : S \,.\, Q\ x\ x : S \to *$	**5 Abst**
7.	$\quad \lambda Q : S \to S \to * \,.\, \lambda x : S \,.\, Q\ x\ x : (S \to S \to *) \to S \to *$	**6 Abst**

Abstraction	**Line Number**	(s_1, s_2)
$S \to *$	2 / 4 / 6 / 7	$(*, \square)$
$S \to S \to *$	2 / 7	$(*, \square)$
$(S \to S \to *) \to (S \to *)$	7	$(\square, \square)$

The judgement contains $(*, \square)$ – λP pairs and $(\square, \square)$ – $\lambda\underline{\omega}$ pairs. Therefore the minimal system $\mathcal{J}$ belongs to is λP $\underline{\omega}$.

Problem

(6.5 b) In $\mathcal{J}$ of 6.5 a, we may consider the variable Q as expressing a relation on set S. How could you describe the subexpression $\lambda x : S \,.\, Q\;x\;x$ in this settings? And what is then the interpretation of the judgement $\mathcal{J}$?

Solution. By a informal translation, the term meant "Given a relation Q over set S and an arbitrary element of S, return whether if $Q\,(x, x)$ holds".

Problem

(6.6 a & b) Let

$$M \equiv \lambda S : * \,.\lambda P : S \to * \,.\lambda x : S \,.(P\;x \to \bot)$$

Prove M legal and determine its type. What is the smallest system in which the λ-cube in which M may occur?

Solution.

1. $* : \square$	**Sort**
2. $S : *$	
3. $a : S$	
4. $* : \square$	
5. $S \to * : \square$	**2,3 Form**
6. $P : S \to *$	
7. $x : S$	
8. $P\;x : *$	**6,7 App**
9. $a : P\;x$	
10. $\bot : *$	**Weak from 6.1 a**
11. $P\;x \to \bot : *$	**8,10 Form**
12. $\lambda x : S \,.\, P\;x \to \bot : S \to *$	**11,5 Abst**
13. $S \to * : \square$	**5,5 Weak**
14. $(S \to *) \to S \to * : \square$	**5,13 Form**
15. $\lambda P : S \to * \,.\lambda x : S \,.\, P\;x \to \bot : (S \to *) \to S \to *$	**12,5 Abst**
16. $\Pi S : * \,.(S \to *) \to S \to * : \square$	**1,14 Form**

17.

$$\lambda S : * . \lambda P : S \to * . \lambda x : S \, . \, P \, x \to \bot$$
$$: \Pi S : * . (S \to *) \to S \to *$$

15,16 Abst

Abstraction	**Line Number**	(s_1, s_2)
$S \to *$	5 / 12 / 13 / 14 / 15	$(*, \square)$
$\bot \equiv \Pi\alpha : * . \alpha$	10 / 11	$(\square, *)$
$P \, x \to \bot$	11 / 12	$(*, *)$
$(S \to *) \to S \to *$	14 / 15	$(\square, *)$
$\Pi S : * . (S \to *) \to S \to *$	16 / 17	$(\square, *)$

The derivation contains $(*, *)$ – $\lambda\!\to$ pairs, $(*, \square)$ – λP pairs, and $(\square, *)$ – $\lambda 2$ pairs. Therefore the minimal system in which M is legal is λP2.

Problem

(6.6 c) How could you interpret the constructor M from 6.6 a, if $A \to \bot$ encodes $\neg A$?

Solution. Converting into mathematical function notation,

$$M \; (S, P, x) = \neg \; P \; (x) \text{ where } S \in \text{set}, P \subseteq S, x \in S$$

M constructs the negation of a predicate P over a set S applied to x, an element of S. An inhabitant of M would prove the negation.

Problem

(6.7 a) Given

$$\Gamma \equiv S : *, Q : S \to S \to *$$

We define under Γ terms

$$\begin{aligned} M_1 &\equiv \lambda x, y : S \, . \Pi R : S \to S \to * . ((\Pi z : S \, . \, R \, z \, z) \to R \, x \, y) \\ M_2 &\equiv \lambda x, y : S \, . \Pi R : S \to S \to * \, . \\ &\qquad ((\Pi u, v : S \, . (Q \, u \, v \to R \, u \, v)) \to R \, x \, y) \end{aligned}$$

Give an inhabitant of $\Pi a : S \, . \, M_1 \; a \; a$ and a shorthand derivation proving your answer.

Solution. By β-reduction

$$
\begin{aligned}
&\Pi a : S \,.\, M_1\, a\, a \\
&\equiv \Pi a : S \,.(\lambda x, y : S \,.\Pi R : S \to S \to * \,.((\Pi z : S \,.\, R\, z\, z) \to R\, x\, y))\, a\, a \\
&\twoheadrightarrow_{\beta} \Pi a : S \,.\Pi R : S \to S \to * \,.((\Pi z : S \,.\, R\, z\, z) \to R\, a\, a)
\end{aligned}
$$

One such term

$$M \equiv \lambda a : S \,.\lambda R : S \to S \to * \,.(\lambda h : (\Pi z : S \,.\, R\, z\, z).\, h\, a)$$

Is an inhabitant.

Proof.

1.	$S : *, Q : S \to S \to *$	
2.	$a : S$	
3.	$b : S$	
4.	$c : S$	
5.	$* : \square$	**Sort**
6.	$S \to * : \square$	**1,5 Form**
7.	$S \to S \to * : \square$	**1,6 Form**
8.	$R : S \to S \to *$	
9.	$z : S$	
10.	$R\, z : S \to *$	**8,9 App**
11.	$R\, z\, z : *$	**10,9 App**
12.	$\Pi z : S \,.\, R\, z\, z : *$	**1,11 Form**
13.	$h : \Pi z : S \,.\, R\, z\, z$	
14.	$R\, a : S \to *$	**8,2 App**
15.	$R\, a\, a : *$	**14,2 App**
16.	$h\, a : R\, a\, a$	**13,2 App**
17.	$\Pi z : S \,.\, R\, z\, z \to R\, a\, a : *$	**12,15 Form**
18.	$\lambda h : \Pi z : S \,.\, R\, z\, z \,.\, h\, a : (\Pi z : S \,.\, R\, z\, z) \to R\, a\, a$	**16,17 Abst**
19.	$\Pi R : S \to S \to * \,.(\Pi z : S \,.\, R\, z\, z) \to R\, a\, a : *$	**7,17 Form**
20.	$\lambda R : S \to S \to * \,.\lambda h : \Pi z : S \,.\, R\, z\, z \,.\, h\, a$ $: \Pi R : S \to S \to * \,.(\Pi z : S \,.\, R\, z\, z) \to R\, a\, a$	**18,19 Abst**
21.	$\Pi a : S \,.\Pi R : S \to S \to * \,.(\Pi z : S \,.\, R\, z\, z) \to R\, a\, a : *$	**1,19 Form**
22.	$\lambda a : S \,.\lambda R : S \to S \to * \,.\lambda h : \Pi z : S \,.\, R\, z\, z \,.\, h\, a$ $: \Pi a : S \,.\Pi R : S \to S \to * \,.(\Pi z : S \,.\, R\, z\, z) \to R\, a\, a$	**20,21 Abst**

■

Problem

(6.7 b) Under Γ of 6.7 a, give an inhabitant of $\Pi a, b : S \,.(Q\ a\ b \to M_2\ a\ b)$ and a shorthand derivation proving your answer.

Solution. By β-reduction

$$
\begin{aligned}
&\Pi a, b : S \,.(Q\ a\ b \to M_2\ a\ b) \\
\equiv\ &\Pi a, b : S \,.\, Q\ a\ b \to (\lambda x, y : S \,.\Pi R : S \to S \to * \,. \\
&\quad (\Pi u, v : S \,.(Q\ u\ v \to R\ u\ v)) \to R\ x\ y)\ a\ b \\
\twoheadrightarrow_\beta\ &\Pi a, b : S \,.\, Q\ a\ b \to (\Pi R : S \to S \to * \,.\Pi u, v : S \,.(Q\ u\ v \to R\ u\ v) \to R\ a\ b)
\end{aligned}
$$

One such term

$$
\begin{aligned}
M \equiv\ &\lambda a, b : S \,.\lambda h : Q\ a\ b \,.\lambda R : S \to S \to * \,.\lambda r : (\Pi u, v : S \,.\, Q\ u\ v \to R\ u\ v). \\
&\quad r\ a\ b\ h
\end{aligned}
$$

is an inhabitant.

Note: this proof would be too long with all formation rules included. For now, legality of abstraction types is assumed. Since we are omitting many lines, line labels are also removed from rule labels.

Proof.

1. $S : *, Q : S \to S \to *$
2. $\quad a : S$
3. $\quad\quad b : S$
4. $\quad\quad\quad h : Q\ a\ b$
5. $\quad\quad\quad\quad R : S \to S \to *$
6. $\quad\quad\quad\quad\quad r : \Pi u, v : S \,.\, Q\ u\ v \to R\ u\ v$
7. $\quad\quad\quad\quad\quad\quad r\ a : \Pi v : S \,.\, Q\ a\ v \to R\ a\ v$ **App**
8. $\quad\quad\quad\quad\quad\quad r\ a\ b : Q\ a\ b \to R\ a\ b$ **App**
9. $\quad\quad\quad\quad\quad\quad r\ a\ b\ h : R\ a\ b$ **App**
10. $\quad\quad\quad\quad\quad \lambda r : \Pi u, v : S \,.\, Q\ u\ v \to R\ u\ v \,.\, r\ a\ b\ h : (\Pi u, v : S \,.\, Q\ u\ v \to R\ u\ v) \to R\ a\ b$ **Abst**
11. $\quad\quad\quad\quad \lambda R : S \to S \to * \,.\, \lambda r : \Pi u, v : S \,.\, Q\ u\ v \to R\ u\ v \,.\, r\ a\ b\ h : \Pi R : S \to S \to * \,.\, (\Pi u, v : S \,.\, Q\ u\ v \to R\ u\ v) \to R\ a\ b$ **Abst**

$$
\begin{array}{lll}
 & \quad\quad \lambda h : Q\,a\,b\,.\lambda R : S \to S \to *\,. & \\
 & \quad\quad\quad \lambda r : \Pi u, v : S\,.\,Q\,u\,v \to R\,u\,v\,.\,r\,a\,b\,h & \\
 & \quad\quad\quad\quad : Q\,a\,b \to \Pi R : S \to S \to *\,. & \\
12. & \quad\quad\quad\quad\quad (\Pi u, v : S\,.\,Q\,u\,v \to R\,u\,v) \to R\,a\,b & \textbf{Abst} \\
 & \quad \lambda b : S\,.\lambda h : Q\,a\,b\,.\lambda R : S \to S \to *\,. & \\
 & \quad\quad \lambda r : \Pi u, v : S\,.\,Q\,u\,v \to R\,u\,v\,.\,r\,a\,b\,h & \\
 & \quad\quad\quad : \Pi b : S\,.\,Q\,a\,b \to \Pi R : S \to S \to *\,. & \\
13. & \quad\quad\quad\quad (\Pi u, v : S\,.\,Q\,u\,v \to R\,u\,v) \to R\,a\,b & \textbf{Abst} \\
 & \lambda a, b : S\,.\lambda h : Q\,a\,b\,.\lambda R : S \to S \to *\,. & \\
 & \quad \lambda r : \Pi u, v : S\,.\,Q\,u\,v \to R\,u\,v\,.\,r\,a\,b\,h & \\
 & \quad\quad : \Pi a, b : S\,.\,Q\,a\,b \to \Pi R : S \to S \to *\,. & \\
14. & \quad\quad\quad (\Pi u, v : S\,.\,Q\,u\,v \to R\,u\,v) \to R\,a\,b & \textbf{Abst}
\end{array}
$$

■

Problem

(6.8 a) Let $\Gamma \equiv S : *, P : S \to *$. Find an inhabitant of

$$
\begin{aligned}
N \equiv [\Pi \alpha : *\,.((\Pi x : S\,.(P\,x \to \alpha)) \to \alpha)] \to \\
[\Pi x : S\,.(P\,x \to \bot)] \to \bot
\end{aligned}
$$

Under Γ by means of a shortened derivation.

Solution. One such term

$$
\begin{aligned}
M \equiv \lambda a : \Pi \alpha : *\,.((\Pi x : S\,.(P\,x \to \alpha)) \to \alpha) \\
\lambda b : \Pi x : S\,.(P\,x \to \bot). \\
a\,\bot\,b
\end{aligned}
$$

is an inhabitant.

Proof.

$$
\begin{array}{lll}
1. & S : *, P : S \to * & \\
2. & \quad a : \Pi \alpha : *\,.((\Pi x : S\,.(P\,x \to \alpha)) \to \alpha) & \\
3. & \quad\quad b : \Pi x : S\,.(P\,x \to \bot) & \\
4. & \quad\quad\quad a\,\bot : (\Pi x : S\,.\,P\,x \to \bot) \to \bot & \textbf{App} \\
5. & \quad\quad\quad a\,\bot\,b : \bot & \textbf{App} \\
6. & \quad\quad \lambda b : \Pi x : S\,.(P\,x \to \bot).\,a\,\bot\,b : [\Pi x : S\,.(P\,x \to \bot)] \to \bot & \textbf{Abst}
\end{array}
$$

7. $$\begin{array}{|l} \lambda a : \Pi\alpha : * .((\Pi x : S .(P\,x \to \alpha)) \to \alpha) \\ \quad \lambda b : \Pi x : S .(P\,x \to \bot).\, a \perp b : \\ \quad\quad [\Pi\alpha : * .((\Pi x : S .(P\,x \to \alpha)) \to \alpha)] \to \\ \quad\quad\quad [\Pi x : S .(P\,x \to \bot)] \to \bot \\ \hline \end{array}$$ **Abst**

■

Problem

(6.8 b) What is the smallest system in the λ-cube in which the derivation in 6.8 a may be executed?

Solution.

Abstraction	(s_1, s_2)
$S \to *$	$(*, \Box)$
$P\,x \to \alpha$	$(*, *)$
$\Pi x : S .(P\,x \to \alpha)$	$(*, *)$
$(\Pi x : S .(P\,x \to \alpha)) \to \alpha$	$(*, *)$
$\Pi\alpha : * .((\Pi x : S .(P\,x \to \alpha)) \to \alpha)$	$(\Box, *)$
$P\,x \to \bot$	$(*, *)$
$\Pi x : S .(P\,x \to \bot)$	$(*, *)$
$[\Pi x : S .(P\,x \to \bot)] \to \bot$	$(*, *)$
N	$(*, *)$
$\bot \equiv \Pi\alpha : * .\alpha$	$(\Box, *)$

The derivation contains $(*, *)$ – $\lambda\!\to$ pairs, $(*, \Box)$ – λP pairs, and $(\Box, *)$ – $\lambda 2$ pairs. Therefore the minimal system in which the derivation may be executed is λP2.

Problem

(6.8 c) The expression $\Pi\alpha : * .(\Pi x : S .(P\,x \to \alpha)) \to \alpha$ maybe consider as an encoding of $\exists x \in S, P\,(x)$ under the PAT paradigm. With $A \to \bot \equiv \neg\, A$ in mind, how can we interpret the content of the expression N?

Solution.

$$N \equiv (\exists x \in S, P\,(x)) \Rightarrow \neg(\forall x \in S, \neg\, P\,(x))$$

Problem

(6.9) Given $S : *, P : S \to *$ and $f : S \to S$, we define in λC the expression

$$M \equiv \lambda x : S \,.\Pi Q : S \to * \,.(\Pi z : S \,.(Q\ z \to Q\ (f\ z))) \to Q\ x$$

Give a term of type $\Pi a : S \,.(M\ a \to M\ (f\ a))$ and a (shortened) derivation proving this.

Solution. By β-reduction

$$\begin{aligned}
& \Pi a : S \,.(M\ a \to M\ (f\ a)) \\
\equiv\ & \Pi a : S \,.((\lambda x : S \,.\Pi Q : S \to * \,.(\Pi z : S \,.(Q\ z \to Q\ (f\ z))) \to Q\ x)\ a \to M\ (f\ a)) \\
\underset{\beta}{\to}\ & \Pi a : S \,.(\Pi Q : S \to * \,.(\Pi z : S \,.(Q\ z \to Q\ (f\ z))) \to Q\ a \to M\ (f\ a)) \\
\equiv\ & \Pi a : S \,.(\Pi Q : S \to * \,.(\Pi z : S \,.(Q\ z \to Q\ (f\ z))) \to Q\ a \\
& \quad \to (\lambda x : S \,.\Pi Q : S \to * \,.(\Pi z : S \,.(Q\ z \to Q\ (f\ z))) \to Q\ x)(f\ a)) \\
\underset{\beta}{\to}\ & \Pi a : S \,.(\Pi Q : S \to * \,.(\Pi z : S \,.(Q\ z \to Q\ (f\ z))) \to Q\ a) \\
& \quad \to \Pi Q : S \to * \,.(\Pi z : S \,.(Q\ z \to Q\ (f\ z))) \to Q\ (f\ a)
\end{aligned}$$

One such term

$$\begin{aligned}
M \equiv\ & \lambda a : S \,.\lambda h : \Pi Q : S \to * \,.(\Pi z : S \,.(Q\ z \to Q\ (f\ z))) \to Q\ a \,. \\
& \quad \lambda Q : S \to * \,.\lambda b : (\Pi z : S \,.(Q\ z \to Q\ (f\ z))). \\
& \quad\quad b\ a\ (h\ Q\ b)
\end{aligned}$$

Is an inhabitant of the type above.

Proof.

1.	$S : *, P : S \to *, f : S \to S$	
2.	$\mid\ a : S$	
3.	$\mid\ \mid\ h : \Pi Q : S \to * \,.(\Pi z : S \,.(Q\ z \to Q\ (f\ z))) \to Q\ a$	
4.	$\mid\ \mid\ \mid\ Q : S \to *$	
5.	$\mid\ \mid\ \mid\ \mid\ b : \Pi z : S \,.(Q\ z \to Q\ (f\ z))$	
6.	$\mid\ \mid\ \mid\ \mid\ \mid\ h\ Q : (\Pi z : S \,.(Q\ z \to Q\ (f\ z))) \to Q\ a$	**App**
7.	$\mid\ \mid\ \mid\ \mid\ \mid\ h\ Q\ b : Q\ a$	**App**
8.	$\mid\ \mid\ \mid\ \mid\ \mid\ b\ a : Q\ a \to Q\ (f\ a)$	**App**
9.	$\mid\ \mid\ \mid\ \mid\ \mid\ b\ a\ (h\ Q\ b) : Q\ (f\ a)$	**App**
10.	$\mid\ \mid\ \mid\ \mid\ \lambda b : \Pi z : S \,.(Q\ z \to Q\ (f\ z)).\, b\ a\ (h\ Q\ b)$ $: (\Pi z : S \,.(Q\ z \to Q\ (f\ z))) \to Q\ (f\ a)$	**Abst**

$$\begin{array}{llr}
 & \lambda Q : S \to * .\lambda b : \Pi z : S .(Q\,z \to Q\,(f\,z)).\,b\,a\,(h\,Q\,b) & \\
11. & \quad : \Pi Q : S \to * .(\Pi z : S .(Q\,z \to Q\,(f\,z))) \to Q\,(f\,a) & \textbf{Abst} \\
 & \lambda h : \Pi Q : S \to * .(\Pi z : S .(Q\,z \to Q\,(f\,z))) \to Q\,a & \\
 & \quad \lambda Q : S \to * .\lambda b : \Pi z : S .(Q\,z \to Q\,(f\,z)).\,b\,a\,(h\,Q\,b) & \\
 & \qquad : (\Pi Q : S \to * .(\Pi z : S .(Q\,z \to Q\,(f\,z))) \to Q\,a) \to & \\
12. & \qquad\quad \Pi Q : S \to * .(\Pi z : S .(Q\,z \to Q\,(f\,z))) \to Q\,(f\,a) & \textbf{Abst} \\
 & \lambda a : S .\lambda h : \Pi Q : S \to * .(\Pi z : S .(Q\,z \to Q\,(f\,z))) \to Q\,a & \\
 & \quad \lambda Q : S \to * .\lambda b : \Pi z : S .(Q\,z \to Q\,(f\,z)).\,b\,a\,(h\,Q\,b) & \\
 & \qquad : \Pi a : S .(\Pi Q : S \to * .(\Pi z : S .(Q\,z \to Q\,(f\,z))) \to Q\,a) & \\
13. & \qquad\quad \to \Pi Q : S \to * .(\Pi z : S .(Q\,z \to Q\,(f\,z))) \to Q\,(f\,a) & \textbf{Abst}
\end{array}$$

∎

Problem

(6.10 a) Given $S : *$ and $P_1, P_2 : S \to *$, we define in λC the expression

$$R \equiv \lambda x : S .\Pi Q : S \to * .(\Pi y : S .(P_1\,y \to P_2\,y \to Q\,y)) \to Q\,x$$

We claim that R codes the intersection of P_1 and P_2. In order to show this, give inhabitants of

$$\begin{array}{ll}
(1) & \Pi x : S .(P_1\,x \to P_2\,x \to R\,x) \\
(2) & \Pi x : S .\,R\,x \to P_1\,x \\
(3) & \Pi x : S .\,R\,x \to P_2\,x
\end{array}$$

Why do (a), (b), and (c) entail that R is this intersection?

Solution. By β-reduction

$$\begin{aligned}
(1) &\equiv \Pi x : S .(P_1\,x \to P_2\,x \to R\,x) \\
&\equiv \Pi x : S\,(P_1\,x \to P_2\,x \to (\lambda x : S .\Pi Q : S \to * . \\
&\qquad (\Pi y : S .(P_1\,y \to P_2\,y \to Q\,y)) \to Q\,x)\,x) \\
&\xrightarrow{\beta} \Pi x : S\,(P_1\,x \to P_2\,x \to (\Pi Q : S \to * . \\
&\qquad (\Pi y : S .(P_1\,y \to P_2\,y \to Q\,y)) \to Q\,x))
\end{aligned}$$

$$
\begin{aligned}
(2) &\equiv \Pi x : S \,.\, R\, x \to P_1\, x \\
&\equiv \Pi x : S \,.(\lambda x : S \,.\Pi Q : S \to * \,. \\
&\qquad (\Pi y : S \,.(P_1\, y \to P_2\, y \to Q\, y)) \to Q\, x)\, x \to P_1\, x \\
&\underset{\beta}{\to} \Pi x : S \,.(\Pi Q : S \to * \,.(\Pi y : S \,.(P_1\, y \to P_2\, y \to Q\, y)) \to Q\, x) \to P_1\, x
\end{aligned}
$$

$$
\begin{aligned}
(3) &\equiv \Pi x : S \,.\, R\, x \to P_2\, x \\
&\equiv \Pi x : S \,.(\lambda x : S \,.\Pi Q : S \to * \,. \\
&\qquad (\Pi y : S \,.(P_1\, y \to P_2\, y \to Q\, y)) \to Q\, x)\, x \to P_2\, x \\
&\underset{\beta}{\to} \Pi x : S \,.(\Pi Q : S \to * \,.(\Pi y : S \,.(P_1\, y \to P_2\, y \to Q\, y)) \to Q\, x) \to P_2\, x
\end{aligned}
$$

Terms

$$
\begin{aligned}
M_I &\equiv \lambda x : S \,.\lambda a : P_1\, x \,.\lambda b : P_2\, x \,.\lambda Q : S \to * \,. \\
&\qquad \lambda h : \Pi y : S \,.(P_1\, y \to P_2\, y \to Q\, y). \\
&\qquad\quad h\, x\, a\, b \\
\pi_1 &\equiv \lambda x : S \,.\lambda h : \Pi Q : S \to * \,.(\Pi y : S \,.(P_1\, y \to P_2\, y \to Q\, y)) \to Q\, x \,. \\
&\qquad h\, P_1\, (\lambda y : S \,.\lambda a : P_1\, y \,.\lambda b : P_2\, y \,.\, a) \\
\pi_2 &\equiv \lambda x : S \,.\lambda h : \Pi Q : S \to * \,.(\Pi y : S \,.(P_1\, y \to P_2\, y \to Q\, y)) \to Q\, x \,. \\
&\qquad h\, P_2\, (\lambda y : S \,.\lambda a : P_1\, y \,.\lambda b : P_2\, y \,.\, b)
\end{aligned}
$$

inhabits (1), (2) and (3) respectively.

Proof 1.

1. $S : *, P_1, P_2 : S \to *$
2. $\quad x : S$
3. $\quad\quad a : P_1\, x$
4. $\quad\quad\quad b : P_2\, x$
5. $\quad\quad\quad\quad Q : S \to *$
6. $\quad\quad\quad\quad\quad h : \Pi y : S \,.(P_1\, y \to P_2\, y \to Q\, y)$
7. $\quad\quad\quad\quad\quad\quad h\, x : P_1\, x \to P_2\, x \to Q\, x$
8. $\quad\quad\quad\quad\quad\quad h\, x\, a : P_2\, x \to Q\, x$
9. $\quad\quad\quad\quad\quad\quad h\, x\, a\, b : Q\, x$
10. $\quad\quad\quad\quad\quad \lambda h : \Pi y : S \,.(P_1\, y \to P_2\, y \to Q\, y).\, h\, x\, a\, b$ $\quad : (\Pi y : S \,.(P_1\, y \to P_2\, y \to Q\, y)) \to Q\, x$ **Abst**
11. $\quad\quad\quad\quad \lambda Q : S \to * \,.\lambda h : \Pi y : S \,.(P_1\, y \to P_2\, y \to Q\, y).\, h\, x\, a\, b$ $\quad : \Pi Q : S \to * \,.(\Pi y : S \,.(P_1\, y \to P_2\, y \to Q\, y)) \to Q\, x$ **Abst**

$\lambda b : P_2\, x\, .\lambda Q : S \to * .$
$\quad \lambda h : \Pi y : S\, .(P_1\, y \to P_2\, y \to Q\, y).\ h\, x\, a\, b$
$\quad\quad : P_2\, x \to \Pi Q : S \to * .$

12. $\quad\quad\quad (\Pi y : S\, .(P_1\, y \to P_2\, y \to Q\, y)) \to Q\, x$ **Abst**

$\lambda a : P_1\, x\, .\lambda b : P_2\, x\, .\lambda Q : S \to * .$
$\quad \lambda h : \Pi y : S\, .(P_1\, y \to P_2\, y \to Q\, y).\ h\, x\, a\, b$
$\quad\quad : P_1\, x \to P_2\, x \to \Pi Q : S \to * .$

13. $\quad\quad\quad (\Pi y : S\, .(P_1\, y \to P_2\, y \to Q\, y)) \to Q\, x$ **Abst**

$\lambda x : S\, .\lambda a : P_1\, x\, .\lambda b : P_2\, x\, .\lambda Q : S \to * .$
$\quad \lambda h : \Pi y : S\, .(P_1\, y \to P_2\, y \to Q\, y).\ h\, x\, a\, b$
$\quad\quad : \Pi x : S\, .\ P_1\, x \to P_2\, x \to \Pi Q : S \to * .$

14. $\quad\quad\quad (\Pi y : S\, .(P_1\, y \to P_2\, y \to Q\, y)) \to Q\, x$ **Abst**

■

Proof 2..

1. $S : *, P_1, P_2 : S \to *$
2. $x : S$
3. $h : \Pi Q : S \to * .(\Pi y : S\, .(P_1\, y \to P_2\, y \to Q\, y)) \to Q\, x$
4. $y : S$
5. $a : P_1\, y$
6. $b : P_2\, y$
7. $a : P_1\, y$ **Weak**
8. $\lambda b : P_2\, y\, .\ a : P_2\, y \to P_1\, y$ **Abst**
9. $\lambda a : P_1\, y\, .\lambda b : P_2\, y\, .\ a : P_1\, y \to P_2\, y \to P_1\, y$ **Abst**
10. $\lambda y : S\, .\lambda a : P_1\, y\, .\lambda b : P_2\, y\, .\ a : \Pi y : S\, .\ P_1\, y \to P_2\, y \to P_1\, y$ **Abst**
11. $h\, P_1 : (\Pi y : S\, .\ P_1\, y \to P_2\, y \to P_1\, y) \to P_1\, x$ **App**
12. $h\, P_1\, (\lambda y : S\, .\lambda a : P_1\, y\, .\lambda b : P_2\, y\, .\ a) : P_1\, x$ **App**

$\lambda h : \Pi Q : S \to * .(\Pi y : S\, .(P_1\, y \to P_2\, y \to Q\, y)) \to Q\, x$
$\quad h\, P_1\, (\lambda y : S\, .\lambda a : P_1\, y\, .\lambda b : P_2\, y\, .\ a) :$
$\quad\quad (\Pi Q : S \to * .(\Pi y : S\, .$

13. $\quad\quad\quad (P_1\, y \to P_2\, y \to Q\, y)) \to Q\, x) \to P_1\, x$ **Abst**

$\lambda x : S\,.\lambda h : \Pi Q : S \to *\,.(\Pi y : S\,.(P_1\,y \to P_2\,y \to Q\,y)) \to Q\,x$
$h\,P_1\,(\lambda y : S\,.\lambda a : P_1\,y\,.\lambda b : P_2\,y\,.\,a) :$
$\Pi x : S\,.(\Pi Q : S \to *\,.(\Pi y : S\,.$
14. $(P_1\,y \to P_2\,y \to Q\,y)) \to Q\,x) \to P_1\,x$ **Abst**

■

Proof 3..

1. $S : *, P_1, P_2 : S \to *$
2. $x : S$
3. $h : \Pi Q : S \to *\,.(\Pi y : S\,.(P_1\,y \to P_2\,y \to Q\,y)) \to Q\,x$
4. $y : S$
5. $a : P_1\,y$
6. $b : P_2\,y$
7. $b : P_2\,y$ **Weak**
8. $\lambda b : P_2\,y\,.\,b : P_2\,y \to P_2\,y$ **Abst**
9. $\lambda a : P_1\,y\,.\lambda b : P_2\,y\,.\,b : P_1\,y \to P_2\,y \to P_2\,y$ **Abst**
10. $\lambda y : S\,.\lambda a : P_1\,y\,.\lambda b : P_2\,y\,.\,b : \Pi y : S\,.\,P_1\,y \to P_2\,y \to P_2\,y$ **Abst**
11. $h\,P_2 : (\Pi y : S\,.\,P_1\,y \to P_2\,y \to P_2\,y) \to P_2\,x$ **App**
12. $h\,P_2\,(\lambda y : S\,.\lambda a : P_1\,y\,.\lambda b : P_2\,y\,.\,b) : P_2\,x$ **App**

$\lambda h : \Pi Q : S \to *\,.(\Pi y : S\,.(P_1\,y \to P_2\,y \to Q\,y)) \to Q\,x$
$h\,P_2\,(\lambda y : S\,.\lambda a : P_1\,y\,.\lambda b : P_2\,y\,.\,b) :$
$(\Pi Q : S \to *\,.(\Pi y : S\,.$
13. $(P_1\,y \to P_2\,y \to Q\,y)) \to Q\,x) \to P_2\,x$ **Abst**

$\lambda x : S\,.\lambda h : \Pi Q : S \to *\,.(\Pi y : S\,.(P_1\,y \to P_2\,y \to Q\,y)) \to Q\,x$
$h\,P_2\,(\lambda y : S\,.\lambda a : P_1\,y\,.\lambda b : P_2\,y\,.\,b) :$
$\Pi x : S\,.(\Pi Q : S \to *\,.(\Pi y : S\,.$
14. $(P_1\,y \to P_2\,y \to Q\,y)) \to Q\,x) \to P_2\,x$ **Abst**

■

The three terms correspond to the three rules of conjunction in predicate logic.

$$M_I \equiv \forall x \in S, P_1\,(x) \Rightarrow P_2\,(x) \Rightarrow P_1\,(x) \wedge P_2\,(x)$$
$$\pi_1 \equiv \forall x \in S, P_1\,(x) \wedge P_2\,(x) \Rightarrow P_1\,(x)$$
$$\pi_2 \equiv \forall x \in S, P_1\,(x) \wedge P_2\,(x) \Rightarrow P_2\,(x)$$

Problem

(6.11 a) Let $\Gamma \equiv x_1 : A_1, \ldots x_n : A_n$ be a well-formed context in λC. Prove $x_1, \ldots, x_n$ distinct.

Solution. We prove by induction over n.

Base Case. When $n = 1$, only one variable x_1 exists, which is trivially distinct. ■

Inductive Step. Assume $x_1, \ldots, x_n$ are distinct. We show x_{n+1} is distinct from all of them. By the Var rule:

$$\frac{\dfrac{\ldots}{x_1 : A_1, \ldots, x_n : A_n \vdash A_{n+1} : s} \qquad x_{n+1} \notin \{x_1, x_2, \ldots, x_n\}}{x_1 : A_1, \ldots, x_n : A_n, x_{n+1} : A_{n+1} \vdash x_{n+1} : A_{n+1}} \text{Var}$$

The side condition of the Var rule requires $x_{n+1} \notin \operatorname{dom}(\Gamma)$, i.e., $x_{n+1} \notin \{x_1, \ldots, x_n\}$. Thus $x_1, \ldots, x_{n+1}$ are distinct. ■

By the principle of mathematical induction, $x_1, \ldots, x_n$ are distinct for any n.

Problem

(6.11 b) Prove the Free Variables Lemma for λC.

Solution.

Lemma 1. *Free Variables Lemma.* If $\Gamma \vdash A : B$, then $\mathrm{FV}\ A \cup \mathrm{FV}\ B \subseteq \operatorname{dom} \Gamma$.

We prove by structural induction over the inference rule that derived $\Gamma \vdash A : B$.

Base Case - Sort Axiom. $\mathrm{FV} * \cup \mathrm{FV}\ \square = \emptyset \subseteq \operatorname{dom} \Gamma$. ■

Var Rule. Then there exists Γ' and variable x s.t. $\Gamma', x : B \equiv \Gamma$. Therefore $\operatorname{dom} \Gamma' \subseteq \operatorname{dom} \Gamma$. The inference rule is

$$\frac{\Gamma' \vdash B : s \qquad x \notin \operatorname{dom} \Gamma'}{\Gamma', x : B \vdash x : B}$$

By the inductive hypothesis on the first premise, $\mathrm{FV}\ B \subseteq \operatorname{dom} \Gamma'$, thus $\mathrm{FV}\ B \subseteq \operatorname{dom} \Gamma$ since $\operatorname{dom} \Gamma' \subseteq \operatorname{dom} \Gamma$. Since $A \equiv x$ and $x \in \operatorname{dom} \Gamma$ by construction, we have $\mathrm{FV}\ A = \{x\} \subseteq \operatorname{dom} \Gamma$. Therefore $\mathrm{FV}\ A \cup \mathrm{FV}\ B \subseteq \operatorname{dom} \Gamma$. ■

Weak Rule. Then there exists Γ' s.t. $\Gamma', x : C \equiv \Gamma$ for some C legal under Γ'. The rule is

$$\frac{\Gamma' \vdash A : B \qquad \Gamma' \vdash C : s}{\Gamma', x : C \vdash A : B}$$

By the inductive hypothesis, $\mathrm{FV}\ A \cup \mathrm{FV}\ B \subseteq \mathrm{dom}\ \Gamma'$. Since $\mathrm{dom}\ \Gamma' \subseteq \mathrm{dom}\ \Gamma$, we have $\mathrm{FV}\ A \cup \mathrm{FV}\ B \subseteq \mathrm{dom}\ \Gamma$. ■

Form Rule. Then $A \equiv \Pi x : C\ .\ D$ and $B \equiv s_2$ for some sort s_2. The rule is

$$\frac{\Gamma \vdash C : s_1 \qquad \Gamma, x : C \vdash D : s_2}{\Gamma \vdash \Pi x : C\ .\ D : s_2}$$

Since B is a sort, $\mathrm{FV}\ B = \emptyset$. By the definition of FV, $\mathrm{FV}\ A = \mathrm{FV}\ C \cup (\mathrm{FV}\ D \setminus \{x\})$.

By the inductive hypothesis on the first premise, $\mathrm{FV}\ C \subseteq \mathrm{dom}\ \Gamma$. By the inductive hypothesis on the second premise, $\mathrm{FV}\ D \subseteq \mathrm{dom}\ (\Gamma, x : C) = \mathrm{dom}\ \Gamma \cup \{x\}$. Thus $\mathrm{FV}\ D \setminus \{x\} \subseteq \mathrm{dom}\ \Gamma$.

Therefore $\mathrm{FV}\ A = \mathrm{FV}\ C \cup (\mathrm{FV}\ D \setminus \{x\}) \subseteq \mathrm{dom}\ \Gamma$, and $\mathrm{FV}\ A \cup \mathrm{FV}\ B \subseteq \mathrm{dom}\ \Gamma$. ■

App Rule. Then $A \equiv M\ N$ and $B \equiv D\,[x := N]$. The rule is

$$\frac{\Gamma \vdash M : \Pi x : C\ .\ D \qquad \Gamma \vdash N : C}{\Gamma \vdash M\ N : D\,[x := N]}$$

By definition, $\mathrm{FV}\ A = \mathrm{FV}\ M \cup \mathrm{FV}\ N$ and $\mathrm{FV}\ B = \mathrm{FV}\ D\,[x := N] \subseteq (\mathrm{FV}\ D \setminus \{x\}) \cup \mathrm{FV}\ N$.

By the inductive hypothesis on the first premise, $\mathrm{FV}\ M \cup \mathrm{FV}\ (\Pi x : C\ .\ D) \subseteq \mathrm{dom}\ \Gamma$. Since $\mathrm{FV}\ (\Pi x : C\ .\ D) = \mathrm{FV}\ C \cup (\mathrm{FV}\ D \setminus \{x\})$, we have $\mathrm{FV}\ D \setminus \{x\} \subseteq \mathrm{dom}\ \Gamma$.

By the inductive hypothesis on the second premise, $\mathrm{FV}\ N \subseteq \mathrm{dom}\ \Gamma$.

Therefore $\mathrm{FV}\ A \cup \mathrm{FV}\ B \subseteq \mathrm{FV}\ M \cup \mathrm{FV}\ N \cup (\mathrm{FV}\ D \setminus \{x\}) \subseteq \mathrm{dom}\ \Gamma$. ■

Abst Rule. Then $A \equiv \lambda x : C\ .\ M$ and $B \equiv \Pi x : C\ .\ D$. The rule is

$$\frac{\Gamma, x : C \vdash M : D \qquad \Gamma \vdash \Pi x : C\ .\ D : s}{\Gamma \vdash \lambda x : C\ .\ M : \Pi x : C\ .\ D}$$

By definition, $\mathrm{FV}\ A = \mathrm{FV}\ C \cup (\mathrm{FV}\ M \setminus \{x\})$ and $\mathrm{FV}\ B = \mathrm{FV}\ C \cup (\mathrm{FV}\ D \setminus \{x\})$.

By the inductive hypothesis on the first premise, $\mathrm{FV}\ M \cup \mathrm{FV}\ D \subseteq \mathrm{dom}\ (\Gamma, x : C) = \mathrm{dom}\ \Gamma \cup \{x\}$. Thus $\mathrm{FV}\ M \setminus \{x\} \subseteq \mathrm{dom}\ \Gamma$ and $\mathrm{FV}\ D \setminus \{x\} \subseteq \mathrm{dom}\ \Gamma$.

By the inductive hypothesis on the second premise, $\mathrm{FV}\ (\Pi x : C\ .\ D) \subseteq \mathrm{dom}\ \Gamma$, so $\mathrm{FV}\ C \subseteq \mathrm{dom}\ \Gamma$.

Therefore $\mathrm{FV}\ A \cup \mathrm{FV}\ B = \mathrm{FV}\ C \cup (\mathrm{FV}\ M \setminus \{x\}) \cup (\mathrm{FV}\ D \setminus \{x\}) \subseteq \mathrm{dom}\ \Gamma$. ■

Conv Rule. Then $B =_\beta B'$ for some B'. The rule is

$$\frac{\Gamma \vdash A : B' \qquad \Gamma \vdash B : s \qquad B =_\beta B'}{\Gamma \vdash A : B}$$

By the inductive hypothesis on the first premise, $\text{FV } A \subseteq \text{dom } \Gamma$. By the inductive hypothesis on the second premise, $\text{FV } B \subseteq \text{dom } \Gamma$. Therefore $\text{FV } A \cup \text{FV } B \subseteq \text{dom } \Gamma$. ■

By the principle of structural induction, for any derivation of $\Gamma \vdash A : B$, we have $\text{FV } A \cup \text{FV } B \subseteq \text{dom } \Gamma$.

Problem

(6.11 c) Take $\Gamma \equiv x_1 : A_1, ..., x_n : A_n$ from 6.11 a. Prove

$$\forall i < n, \text{FV } A_i \subseteq \{x_j : 1 \leq j \leq n\}$$

Solution. We first need a lemma

$$\text{dom } \Gamma = \{x_i : 1 \leq i \leq n\}$$

Proof by induction on n.

Base Case. Only declaration in Γ, trivial. ■

Inductive Step. We assume a $\Gamma' \equiv \Gamma, x_{n+1} : A_{n+1}$. Then

$$\begin{aligned} \text{dom } \Gamma' &\equiv \text{dom } \Gamma \cup \{x_{n+1}\} \\ &\equiv \{x_1, ..., x_n\} \cup \{x_{n+1}\} \equiv \{x_i : 1 \leq i \leq n+1\} \end{aligned}$$

■

Main Proof. Let $\Gamma' \equiv x_1 : A_1, ..., x_{i-1} : A_{i-1}$ be the prefix of Γ before the i-th declaration. By the Var rule, for $x_i : A_i$ to extend Γ', we require $\Gamma' \vdash A_i : s$ for some sort s.

By the Free Variables Lemma (6.11 b), $\text{FV } A_i \subseteq \text{dom } \Gamma'$. By the lemma above, $\text{dom } \Gamma' = \{x_1, ..., x_{i-1}\}$.

Therefore $\text{FV } A_i \subseteq \{x_1, ..., x_{i-1}\} \subseteq \{x_j : 1 \leq j \leq n\}$. ■

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Completed Dec 24 1:36 am.