Excercises

Chapter 2

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1. Reducted

Definition Some rules for reference.

$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash x : \sigma}\text{(T-Var)} \qquad \frac{\Gamma \vdash M : \sigma \to \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}\text{(T-App)}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma.M : \sigma \to \tau}\text{(T-Abst)}$$

In this document, convention is that all type judgements in a proof tree, unless stated otherwise, is derived from a single context per tree.

> **Problem**
>
> *(2.1)* Type the following terms
>
> $$xxy \quad xyy \quad xyx \quad x(xy) \quad x(yx)$$

Solution. The first term cannot be typed.

Proof. $xxy = (xx)y$. Therefore, x is a function type, denote it as $\tau \to \sigma$. By the application rule, a subterm applied to x must be of τ, which means that the application xx is not legally typed. ∎

The second one is typable where $x : \tau \to \tau \to \sigma$ and $y : \tau$.

1. $x : \tau \to \tau \to \sigma \quad \dashv \Gamma$
2. $y : \tau \qquad\qquad \dashv \Gamma$

3. $xy : \tau \to \sigma$ **1,2 T-App**

4. $xyy : \sigma$ **3,2 T-App**

The third term is not typable.

Proof. Assume $xyx = (xy)x$ is typable. Therefore, $x : \tau$ where $\tau = \sigma \to \tau \to \alpha$ and $y : \sigma$. One can construct an infinite chain of function type by substituting τ: $\tau = \sigma \to (\sigma \to (\sigma \to ... \to \alpha) \to \alpha) \to \alpha$. By induction, it can be proven that only lambda abstractions can construct function types, meaning that the term is of form

$$(\lambda n : \tau.\lambda m : \tau....(\lambda a : \sigma.\lambda b : \sigma....))y(\lambda n : \tau.\lambda m : \tau....(\lambda a : \sigma.\lambda b : \sigma....))$$

meaning that an infinite reduction path is needed. This is impossible in STLC.   ∎

The fourth type is typable where $x : (\tau \to \tau)$ and $y : \tau$.

1. $x : \tau \to \tau$ $\dashv \Gamma$

2. $y : \tau$ $\dashv \Gamma$

3. $xy : \tau$ **1,2 T-App**

4. $x(xy) : \tau$ **1,3 T-App**

The fifth term is typable where $x : (\tau \to \sigma)$ and $y : (\tau \to \sigma) \to \tau$:

1. $x : \tau \to \sigma$ $\dashv \Gamma$

2. $y : (\tau \to \sigma) \to \tau$ $\dashv \Gamma$

3. $yx : \tau$ **2,1 T-App**

4. $x(yx) : \sigma$ **1,3 T-App**

Problem

(2.2) Find types for zero, one, and two

Solution. Term for zero is

$$\text{zero} := \lambda fx.x$$

Here x is only used as a

$$\text{zero} := \lambda f : \alpha.\lambda x : \beta.x$$

Type derivation shown as below:

1. $f : \alpha$ **Bound**

2. $\big|\, x : \beta$ **Bound**

3. $x : \beta$ **T-Var**

4. $\lambda x.x : \beta \to \beta$ **3 T-Abst**

5. $\lambda f : \alpha.x : \beta.x : \alpha \to \beta \to \beta$ **4 T-Abst**

Term for one is

$$\text{one} := \lambda f x.f x$$

Let f be an arbitrary function type that consumes x

$$\text{one} := \lambda f : \alpha \to \beta.x : \alpha.f x$$

Type derivation shown as below

1. $f : \alpha \to \beta$ **Bound**

2. $x : \alpha$ **Bound**

3. $f : \alpha \to \beta$ **T-Var**

4. $x : \alpha$ **T-Var**

5. $f x : \beta$ **3,4 T-App**

6. $\lambda x.f x : \alpha \to \beta$ **5 T-Abst**

7. $\lambda f : \alpha \to \beta.x : \alpha.f x : (\alpha \to \beta) \to \alpha \to \beta$ **6 T-Abst**

Same type signatures can be given to two

$$\text{two} := \lambda f : \alpha \to \beta.\lambda x : \alpha.f f x$$

Type derivation shown as below

1. $f : \alpha \to \beta$ **Bound**

2. $x : \alpha$ **Bound**

3. $f : \alpha \to \beta$ **T-Var**

4. $x : \alpha$ **T-Var**

5. $f x : \beta$ **3,4 T-App**

6. $f f x : \beta$ **3,5 T-App**

7. $\lambda x.f f x : \alpha \to \beta$ **6 T-Abst**

8. $\lambda f : \alpha \to \beta.x : \alpha.f f x : (\alpha \to \beta) \to \alpha \to \beta$ **7 T-Abst**

(2.3) Find types for

$$K := \lambda xy.x$$
$$S := \lambda xyz.xz(yz)$$

Solution. There are no occurences of application in K's subterms. Therefore all its binding variables could be given a simple base type.

$$K := \lambda x : \alpha.\lambda y : \beta.x$$

Type derivation shown as below

1. $x : \alpha$ **Bound**
2. $y : \beta$ **Bound**
3. $x : \alpha$ **T-Var**
4. $\lambda y : \beta.x : \beta \to \alpha$ **3 T-Abst**
5. $\lambda x : \alpha.\lambda y : \beta.x : \alpha \to \beta \to \alpha$ **4 T-Abst**

For the S combinator, no term was applied to z. Therefore it can be given a simple base type α. As z was applied to y, it implies that $y : \alpha \to \beta$ for some output type β. As x takes z and (yz), it must be of type $\alpha \to \beta \to \delta$.

$$S := \lambda x : \alpha \to \beta \to \delta.\lambda y : \alpha \to \beta.\lambda z.\alpha.xz(yz)$$

Complete type derivation shown as below:

1. $x : \alpha \to \beta \to \delta$ **Bound**
2. $y : \alpha \to \beta$ **Bound**
3. $z : \alpha$ **Bound**
4. $y : \alpha \to \beta$ **T-Var**
5. $z : \alpha$ **T-Var**
6. $yz : \beta$ **4,5 T-App**
7. $x : \alpha \to \beta \to \delta$ **T-Var**
8. $xz : \beta \to \delta$ **7,5 T-App**
9. $xz(yz) : \delta$ **8,6 T-App**
10. $\lambda z : \alpha.xz(yz) : \alpha \to \delta$ **9 T-Abstr**
11. $\lambda y : \alpha \to \beta.\lambda z.\alpha.xz(yz) : (\alpha \to \beta) \to \alpha \to \delta$ **10 T-Abstr**

12.

$$\lambda x : \alpha \to \beta \to \delta.\lambda y : \alpha \to \beta.\lambda z$$
$$: \alpha.xz(yz) : (\alpha \to \beta \to \delta) \to (\alpha \to \beta) \to \alpha \to \delta \qquad \textbf{11 T-Abstr}$$

Problem

(2.4) Type the bound variables

$$\lambda xyz.x(yz)$$
$$\lambda xyz.y(xz)z$$

Solution. For the first term, z had nothing applied to it. Therefore it could be given a simple base type α. z was applied to y, therefore $y : \alpha \to \beta$ to satisfy the application rule. Because the application yielded a type of β, by the application rule $x : \beta \to \delta$ for some type δ.

$$\lambda x : \beta \to \delta.\lambda y : \alpha \to \beta.\lambda z : \alpha.x(yz)$$

Complete type derivation shown below

1. $\quad x : \beta \to \delta$		**Bound**
2. $\quad\quad y : \alpha \to \beta$		**Bound**
3. $\quad\quad\quad z : \alpha$		**Bound**
4. $\quad\quad\quad\quad y : \alpha \to \beta$		**T-Var**
5. $\quad\quad\quad\quad z : \alpha$		**T-Var**
6. $\quad\quad\quad\quad yz : \beta$		**4,5 T-App**
7. $\quad\quad\quad\quad x : \beta \to \delta$		**T-Var**
8. $\quad\quad\quad\quad x(yz) : \delta$		**7,6 T-App**
9. $\quad\quad\quad \lambda z : \alpha.x(yz) : \alpha \to \delta$		**8 T-Abst**
10. $\quad\quad \lambda y : \alpha \to \beta.\lambda z : \alpha.x(yz) : (\alpha \to \beta) \to \alpha \to \delta$		**9 T-Abst**
11.		

$$\lambda x : \beta \to \delta.\lambda y : \alpha \to \beta.\lambda z : \alpha.x(yz)$$
$$: (\beta \to \delta) \to (\alpha \to \beta) \to \alpha \to \delta \qquad \textbf{10 T-Abst}$$

In the second term z could still be given a simple base type $z : \alpha$. Therefore $x : \alpha \to \beta$ for some type β. y takes $xz : \beta$ and $z : \alpha$, therefore it is of type $y : \beta \to \alpha \to \delta$ for some δ.

$$\lambda x : \alpha \to \beta.\lambda y : \beta \to \alpha \to \delta.\lambda z : \alpha.y(xz)z$$

. Complete type derivation shown below

1.	$x : \alpha \to \beta$	**Bound**
2.	$y : \beta \to \alpha \to \delta$	**Bound**
3.	$z : \alpha$	**Bound**
4.	$x : \alpha \to \beta$	**T-Var**
5.	$z : \alpha$	**T-Var**
6.	$xz : \beta$	**4,5 T-App**
7.	$y : \beta \to \alpha \to \delta$	**T-Var**
8.	$y(xz) : \alpha \to \delta$	**7,6 T-App**
9.	$y(xz)z : \delta$	**8,5 T-App**
10.	$\lambda z : \alpha.y(xz)z : \alpha \to \delta$	**9 T-Abst**
11.	$\lambda y : \beta \to \alpha \to \delta.\lambda z.y(xz)z : (\beta \to \alpha \to \delta) \to \alpha \to \delta$	**10 T-Abst**
12.		

$$\lambda x : \alpha \to \beta.\lambda y : \beta \to \alpha \to \delta.\lambda z.y(xz)z :$$
$$(\alpha \to \beta) \to (\beta \to \alpha \to \delta) \to \alpha \to \delta$$

11 T-Abst

Problem

(2.5) Try to type the following terms, and prove if not typable.

$$\lambda xy.x(\lambda z.y)y$$
$$\lambda xy.x(\lambda z.x)y.$$

Solution. The first term is trivially typable.

1.	$x : (\delta \to \alpha) \to \alpha \to \beta$	**Bound**
2.	$y : \alpha$	**Bound**
3.	$x : (\delta \to \alpha) \to \alpha \to \beta$	**T-Var**
4.	$z : \delta$	**Bound**
5.	$y : \alpha$	**T-Var**
6.	$\lambda z : \delta.y : \delta \to \alpha$	**5 T-Abst**
7.	$x(\lambda z : \delta.y) : \alpha \to \beta$	**3,6 T-App**
8.	$y : \alpha$	**T-Var**
9.	$x(\lambda z : \delta.y)y : \beta$	**7,8 T-App**
10.	$\lambda y : \alpha.x(\lambda z : \delta.y)y : \alpha \to \beta$	**9 T-Abst**

6.

11.

$$\lambda x : ((\delta \rightarrow \alpha) \rightarrow \alpha \rightarrow \beta)\lambda y : \alpha.x(\lambda z : \delta.y)y$$
$$: ((\delta \rightarrow \alpha) \rightarrow \alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta \qquad \textbf{10 T-Abst}$$

The second term is not typable in STLC.

Proof. By induction on the type inference rule that constructed the type judgement for subterm $x(\lambda z.x)$. Because the term is an application, the only rule that applies is the application rule.

We denote the context inside the abstraction as Γ'. Suppose $\mathcal{J} \equiv \Gamma' \vdash x(\lambda z.x) : \tau$. By the inference rule of application, x must be a function type that accepts the type of $(\lambda z.x)$. Let $\Gamma' \vdash z : \alpha$, and type of x as τ_x. Therefore, $\Gamma' \vdash \lambda z : \alpha.x : \alpha \rightarrow \tau_x$. Therefore, $\tau_x = (\alpha \rightarrow \tau_x) \rightarrow \tau$. This is a recursive type, which is not constructable as it requires infinitely nested lambda abstractions that requires infinite reduction paths to reach a normal form. ∎

> ### Problem
>
> *(2.6)* Prove the pretyped term below is legal.
>
> $$\lambda x : ((\alpha \rightarrow \beta) \rightarrow \alpha).x(\lambda z : \alpha.y)$$
>
> Using the tree format and the flag format.

Solution. We suppose a context $\Gamma \vdash y : \beta$ that obviously exists.

Proof.

$$\cfrac{\cfrac{}{x : (\alpha \rightarrow \beta) \rightarrow \alpha}\text{(Bound)} \quad \cfrac{\cfrac{\Gamma, z : \alpha \vdash y : \beta}{\Gamma \vdash (\lambda z : \alpha.y) : \alpha \rightarrow \beta}\text{(T-Abst)}}{}}{\cfrac{\Gamma, x : (\alpha \rightarrow \beta) \rightarrow \alpha \vdash (x(\lambda z : \alpha.y)) : \alpha}{\Gamma \vdash \lambda x : ((\alpha \rightarrow \beta) \rightarrow \alpha).x(\lambda z : \alpha.y) : ((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow \alpha}\text{(T-Abst)}}\text{(T-App)}$$

A valid type could be given to the term. Therefore, the term is typable under an existing context. ∎

The flag derivation is given below:

1.	$x : (\alpha \rightarrow \beta) \rightarrow \alpha$	**Bound**
2.	$z : \alpha$	**Bound**
3.	$y : \beta$	$\dashv \Gamma$
4.	$(\lambda z : \alpha.y) : \alpha \rightarrow \beta$	**3 T-Abst**
5.	$x : (\alpha \rightarrow \beta) \rightarrow \alpha$	**T-Var**

6. $\quad$ | $x(\lambda z : \alpha.y) : \beta$ $\quad\quad\quad\quad$ **5,4 T-App**

7.

$\lambda x : ((\alpha \to \beta) \to \beta).x(\lambda z : \alpha.y)$

$\quad\quad : (\alpha \to \beta) \to \beta \to \alpha$ $\quad\quad$ **6 T-Abst**

Problem

(2.7 a) Derive

$$f : A \to B \wedge g : B \to C \Rightarrow g \circ f : A \to C$$

Using the rules

$$\frac{f : A \to B, x \in A}{f(x) \in B}\text{(F-App)} \qquad \frac{\forall x \in A, f(x) \in B}{f : A \to B}\text{(F-Abst)}$$

Solution.

Proof.

1. $f : A \to B \wedge g : B \to C$ $\quad\quad\quad\quad\quad$ **Assumption**
2. $\quad$ | $f : A \to B$ $\quad\quad\quad\quad\quad\quad\quad$ $1 \wedge E$
3. $\quad$ | $g : B \to C$ $\quad\quad\quad\quad\quad\quad\quad$ $1 \wedge E$
4. $\quad$ | $a \in A$
5. $\quad$ | | $f(a) \in B$ $\quad\quad\quad\quad\quad\quad$ **3, 4 F-App**
6. $\quad$ | | $g(f(a)) \in C$ $\quad\quad\quad\quad\quad$ **5, 4 F-App**
7. $\quad$ | | $(g \circ f)(a) \in C$ $\quad\quad\quad\quad$ **6 Compose Def**
8. $\quad$ | $\forall x \in A, (g \circ f)(x) \in C$ $\quad\quad$ $7 \forall E$
9. $\quad$ | $g \circ f : A \to C$ $\quad\quad\quad\quad\quad$ **8 F-Abst**
10. $f : A \to B, g : B \to C \Rightarrow g \circ f : A \to C$ $\quad$ $9 \Rightarrow I$

$\blacksquare$

(2.7 b) Give a derivation in natural deduction of the following:

$$(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$$

Using the rules

$$\frac{A \Rightarrow B \quad A}{B} \, (\Rightarrow E)$$

1.	A	**Premise**
2.	$\vdots$	...
3.	B	

$$\frac{}{A \Rightarrow B} \, (\Rightarrow I)$$

Solution.

Proof.

1.	$A \Rightarrow B$	**Premise**
2.	$B \Rightarrow C$	**Premise**
3.	A	**Premise**
4.	B	$1,3 \Rightarrow E$
5.	C	$2,4 \Rightarrow E$
6.	$A \Rightarrow C$	$3\text{-}5 \Rightarrow I$
7.	$(B \Rightarrow C) \Rightarrow (A \Rightarrow C)$	$2\text{-}6 \Rightarrow I$
8.	$(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$	$1\text{-}7 \Rightarrow I$

■

(2.7 c) Prove the following pre-typed term is legal using flag notation

$$\lambda z : \alpha . y(xz)$$

Solution.

Proof. Let $\Gamma \vdash x : \alpha \to \beta, y : \beta \to \delta$ for some type β and δ.

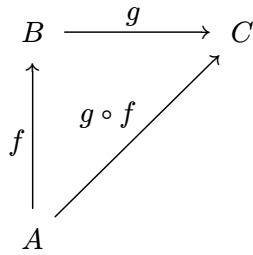| 1.	$z : \alpha$	**Bound**
2.	$x : \alpha \to \beta$	$\dashv \Gamma$
3.	$z : \alpha$	**T-Var**

4.	$xz : \beta$	**2,3 T-App**
5.	$y : \beta \to \delta$	$\dashv \Gamma$
6.	$y(xz) : \delta$	**5,4 T-App**
7.	$\lambda z : \alpha . y(xz) : \alpha \to \delta$	**6 T-Abst**

∎

Problem

(2.7 d) State the similarity between Q. 2.7 (a), (b), and (c).

Solution. All of these examples requires proving something about composing two maps together as like this:

Problem

(2.8 a) Pre-type the bounding variables for the following term

$$\lambda xy.y(\lambda z.yx) : (\gamma \to \beta) \to ((\gamma \to \beta) \to \beta) \to \beta$$

Solution.

$$\lambda x : (\gamma \to \beta).y : ((\gamma \to \beta) \to \beta).y(\lambda z : \gamma.yx)$$

Problem

(2.8 b) Give a derivation in tree format

Solution.

$$\cfrac{\text{(v)} \cfrac{}{}}{\text{(vi)} \cfrac{y : ((\gamma \to \beta) \to \beta)}{\text{(vii)} \cfrac{x : (\gamma \to \beta), y : ((\gamma \to \beta) \to \beta) \vdash y(\lambda z : \gamma.yx) : \beta}{\text{(viii)} \cfrac{x : (\gamma \to \beta) \vdash \lambda y : ((\gamma \to \beta) \to \beta).y(\lambda z : \gamma.yx) : ((\gamma \to \beta) \to \beta) \to \beta}{(\lambda x : (\gamma \to \beta).y : ((\gamma \to \beta) \to \beta).y(\lambda z : \gamma.yx)) : (\gamma \to \beta) \to ((\gamma \to \beta) \to \beta) \to \beta} \text{T-Abst}}}}$$

(i) $\cfrac{}{x : (\gamma \to \beta)}$ (ii) $\cfrac{}{y : (\gamma \to \beta) \to \beta}$ T-App

(iii) $\cfrac{x : (\gamma \to \beta), y : ((\gamma \to \beta) \to \beta), z : \gamma \vdash yx : \beta}{}$ T-Abst

(iv) $\cfrac{x : (\gamma \to \beta), y : ((\gamma \to \beta) \to \beta) \vdash \lambda z : \gamma.yx : \gamma \to \beta}{}$ T-App

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Problem

(2.8 c) Sketch a diagram of tree structure of derivation

Solution. Trivial.

Problem

(2.8 d) Transform the derivation into flag notation

Solution.

	1.	$x : \gamma \to \beta$	**Bound**
	2.	$y : (\gamma \to \beta) \to \beta$	**Bound**
	3.	$z : \gamma$	**Bound**
(ii)	4.	$y : (\gamma \to \beta) \to \beta$	**T-Var**
(i)	5.	$x : \gamma \to \beta$	**T-Var**
(iii)	6.	$yx : \beta$	**4,5 T-App**
(iv)	7.	$\lambda z : \gamma.yx : \gamma \to \beta$	**6 T-Abst**
(v)	8.	$y : (\gamma \to \beta) \to \beta$	**T-Var**
(vi)	9.	$y(\lambda z : \gamma.yx) : \beta$	**8,7 T-App**
(vii)	10.	$\lambda y : (\gamma \to \beta) \to \beta.y(\lambda z : \gamma.yx)$ $: ((\gamma \to \beta) \to \beta) \to \beta$	**9 T-Abst**
(viii)	11.	$\lambda x : (\gamma \to \beta).\lambda y : (\gamma \to \beta) \to \beta.y(\lambda z : \gamma.yx)$ $: (\gamma \to \beta) \to ((\gamma \to \beta) \to \beta) \to \beta$	**10 T-Abst**

Problem

(2.9 a) Give derivations of the following judgement

$$x : \delta \to \delta \to \alpha, y : \gamma \to \alpha, z : \alpha \to \beta \vdash$$
$$\lambda u : \delta.\lambda v : \gamma.z(yv) : \delta \to \gamma \to \beta$$

Solution.

1.	$u : \delta$	**Bound**
2.	$v : \gamma$	**Bound**
3.	$y : \gamma \to \alpha$	**T-Var**
4.	$v : \gamma$	**T-Var**
5.	$yv : \alpha$	**3,4 T-App**
6.	$z : \alpha \to \beta$	**T-Var**
7.	$z(yv) : \beta$	**6,5 T-App**
8.	$\lambda v : \gamma.z(yv) : \gamma \to \beta$	**7 T-Abst**
9.	$\lambda u : \delta.\lambda v : \gamma.z(yv) : \delta \to \gamma \to \beta$	**8 T-Abst**

Problem

(2.9 b) Give derivations of the following judgement

$$x : \delta \to \delta \to \alpha, y : \gamma \to \alpha, z : \alpha \to \beta \vdash$$
$$\lambda u : \delta.\lambda v : \gamma.z(xuu) : \delta \to \gamma \to \beta$$

Solution.

1.	$u : \delta$	**Bound**
2.	$v : \gamma$	**Bound**
3.	$x : \delta \to \delta \to \alpha$	**T-Var**
4.	$u : \delta$	**T-Var**
5.	$xu : \delta \to \alpha$	**3,4 T-App**
6.	$xuu : \alpha$	**5,4 T-App**
7.	$z : \alpha \to \beta$	**T-Var**
8.	$z(xuu) : \beta$	**7,6 T-App**
9.	$\lambda v : \gamma.z(xuu) : \gamma \to \beta$	**8 T-Abst**
10.	$\lambda u : \delta.\lambda v : \gamma.z(xuu) : \delta \to \gamma \to \beta$	**9 T-Abst**

Problem

(2.10 a) Give derivation for

$$xz(yz)$$

Solution. Assume an context

$$\Gamma \vdash x : \alpha \to \beta \to \gamma$$
$$\Gamma \vdash y : \alpha \to \beta$$
$$\Gamma \vdash z : \alpha$$

1. $x : \alpha \to \beta \to \gamma$ **T-Var**
2. $y : \alpha \to \beta$ **T-Var**
3. $z : \alpha$ **T-Var**
4. $xz : \beta \to \gamma$ **1,3 T-App**
5. $yz : \beta$ **2,3 T-App**
6. $xz(yz) : \gamma$ **4,5 T-App**

Problem

(2.10 b) Give derivation for

$$\lambda x : (\alpha \to \beta) \to \beta.x(yz)$$

Solution. Assume an context

$$\Gamma \vdash y : \gamma \to (\alpha \to \beta)$$
$$\Gamma \vdash z : \gamma$$

1. $x : (\alpha \to \beta) \to \beta$ **Bound**
2. $x : (\alpha \to \beta) \to \beta$ **T-Var**
3. $y : \gamma \to \alpha \to \beta$ **T-Var**
4. $z : \gamma$ **T-Var**
5. $yz : \alpha \to \beta$ **3,4 T-App**
6. $x(yz) : \beta$ **2,5 T-App**
7. $\lambda x : (\alpha \to \beta) \to \beta.x(yz) : ((\alpha \to \beta) \to \beta) \to \beta$ **6 T-Abst**

(2.10 c) Give derivation for

$$\lambda y : \alpha.\lambda z : \beta \to \gamma.z(xyy)$$

Solution. Assume a context

$$\Gamma \vdash x : \alpha \to \alpha \to \beta$$

1.	$y : \alpha$	**Bound**
2.	$z : \beta \to \gamma$	**Bound**
3.	$z : \beta \to \gamma$	**T-Var**
4.	$x : \alpha \to \alpha \to \beta$	**T-Var**
5.	$y : \alpha$	**T-Var**
6.	$xy : \alpha \to \beta$	**4,5 T-App**
7.	$xyy : \beta$	**6,5 T-App**
8.	$z(xyy) : \gamma$	**3,6 T-App**
9.	$\lambda z : \beta \to \gamma.z(xyy) : (\beta \to \gamma) \to \gamma$	**8 T-Abst**
10.	$\lambda y : \alpha.\lambda z : \beta \to \gamma.z(xyy) : \alpha \to (\beta \to \gamma) \to \gamma$	**8 T-Abst**

(2.10 d) Give derivation for

$$\lambda x : \alpha \to \beta.y(xz)z$$

Solution. Consider a context

$$\Gamma \vdash z : \alpha$$
$$\Gamma \vdash y : \beta \to \alpha \to \gamma$$

1.	$x : \alpha \to \beta$	**Bound**
2.	$x : \alpha \to \beta$	**T-Var**
3.	$z : \alpha$	**T-Var**
4.	$xz : \beta$	**2,3 T-App**
5.	$y : \beta \to \alpha \to \gamma$	**T-Var**
6.	$y(xz) : \alpha \to \gamma$	**5,4 T-App**
7.	$y(xz)z : \gamma$	**3,5 T-App**

8. $\lambda x : \alpha \to \beta.y(xz)z : (\alpha \to \beta) \to \gamma$ **7 T-Abst**

Problem

(2.11 a) Find an inhabitant of type and prove through derivation

$$(\alpha \to \alpha \to \gamma) \to \alpha \to \beta \to \gamma$$

Solution.

$$\lambda x : (\alpha \to \alpha \to \gamma).\lambda y : (\alpha).\lambda z : (\beta).xyy$$

Proof.

1.	$x : \alpha \to \alpha \to \gamma$	**Bound**
2.	$y : \alpha$	**Bound**
3.	$z : \beta$	**Bound**
4.	$x : \alpha \to \alpha \to \gamma$	**T-Var**
5.	$y : \alpha$	**Bound**
6.	$xy : \alpha \to \gamma$	**4,5 T-App**
7.	$xyy : \gamma$	**6,5 T-App**
8.	$\lambda z : \beta.xyy : \beta \to \gamma$	**7 T-Abst**
9.	$\lambda y : \alpha.\lambda z : \beta.xyy : \alpha \to \beta \to \gamma$	**8 T-Abst**
10.		

$\lambda x : \alpha \to \alpha \to \gamma.\lambda y : \alpha.\lambda z : \beta.xyy$
   $: (\alpha \to \alpha \to \gamma) \to \alpha \to \beta \to \gamma$ **8 T-Abst**

                                                   ■

Problem

(2.11 b) Find an inhabitant of type and prove through derivation

$$((\alpha \to \gamma) \to \alpha) \to (\alpha \to \gamma) \to \beta \to \gamma$$

Solution.

$$\lambda x : (\alpha \to \gamma) \to \alpha.\lambda y : (\alpha \to \gamma).\lambda z : \beta.y(xy)$$

Proof.

1.	$x : (\alpha \to \gamma) \to \alpha$	**Bound**
2.	$y : \alpha \to \gamma$	**Bound**
3.	$z : \beta$	**Bound**
4.	$x : (\alpha \to \gamma) \to \alpha$	**T-Var**
5.	$y : \alpha \to \gamma$	**T-Var**
6.	$xy : \alpha$	**4,5 T-App**
7.	$y(xy) : \gamma$	**5,6 T-App**
8.	$\lambda z : \beta.x(xy) : \beta \to \gamma$	**7 T-Abst**
9.	$\lambda y : \alpha \to \gamma.\lambda z : \beta.x(xy) : (\alpha \to \gamma) \to \beta \to \gamma$	**7 T-Abst**
10.		

$\lambda x : (\alpha \to \gamma) \to \alpha.\lambda y : \alpha \to \gamma.\lambda z : \beta.x(xy)$

$\quad : ((\alpha \to \gamma) \to \alpha) \to (\alpha \to \gamma) \to \beta \to \gamma$ **7 T-Abst**

∎

Problem

(2.12 a) Construct a term of type

$$((\alpha \to \beta) \to \alpha) \to (\alpha \to \alpha \to \beta) \to \alpha$$

Solution.

$$\lambda x : (\alpha \to \beta) \to \alpha.\lambda y : \alpha \to \alpha \to \beta.x(\lambda z : \alpha.yzz)$$

Proof.

1.	$x : (\alpha \to \beta) \to \alpha$	**Bound**
2.	$y : \alpha \to \alpha \to \beta$	**Bound**
3.	$x : (\alpha \to \beta) \to \alpha$	**T-Var**
4.	$z : \alpha$	**Bound**
5.	$y : \alpha \to \alpha \to \beta$	**T-Var**
6.	$z : \alpha$	**Bound**
7.	$yz : \alpha \to \beta$	**5,6 T-App**
8.	$yzz : \beta$	**7,6 T-App**
9.	$\lambda z : \alpha.yzz : \alpha \to \beta$	**8 T-Abst**
10.	$x(\lambda z : \alpha.yzz) : \alpha$	**3,9 T-App**
11.	$\lambda y : \alpha \to \alpha \to \beta.x(\lambda z : \alpha.yzz) : (\alpha \to \alpha \to \beta) \to \alpha$	**3,9 T-App**

12.

$$\lambda x : (\alpha \to \beta) \to \alpha.\lambda y : \alpha \to \alpha \to \beta.x(\lambda z : \alpha.yzz)$$
$$: ((\alpha \to \beta) \to \alpha) \to (\alpha \to \alpha \to \beta) \to \alpha \qquad \textbf{3,9 T-App}$$

∎

Problem

(2.12 b) Construct a term of type

$$((\alpha \to \beta) \to \alpha) \to (\alpha \to \alpha \to \beta) \to \beta$$

Solution.

$$\lambda x : (\alpha \to \beta) \to \alpha.\lambda y : \alpha \to \alpha \to \beta.(\lambda z : \alpha.yzz)(x(\lambda z : \alpha.yzz))$$

Proof.

1.	$x : (\alpha \to \beta) \to \alpha$	**Bound**
2.	$y : \alpha \to \alpha \to \beta$	**Bound**
3.	$x : (\alpha \to \beta) \to \alpha$	**T-Var**
4.	$z : \alpha$	**Bound**
5.	$y : \alpha \to \alpha \to \beta$	**T-Var**
6.	$z : \alpha$	**Bound**
7.	$yz : \alpha \to \beta$	**5,6 T-App**
8.	$yzz : \beta$	**7,6 T-App**
9.	have $f := \lambda z : \alpha.yzz : \alpha \to \beta$	**8 T-Abst**
10.	$xf : \alpha$	**3,9 T-App**
11.	$f(xf) : \beta$	**9,10 T-App**
12.	$\lambda y : \alpha \to \alpha \to \beta.f(xf) : (\alpha \to \alpha \to \beta) \to \beta$	**9,10 T-App**
	$\lambda x : (\alpha \to \beta) \to \alpha.\lambda y : \alpha \to \alpha \to \beta.f(xf)$	
13.	$: (\alpha \to \beta) \to \alpha \to (\alpha \to \alpha \to \beta) \to \beta$	**9,10 T-App**

∎

(2.13 a) Find a term of type

$$(\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$$

in context Γ

$$x : \alpha \rightarrow \beta \rightarrow \gamma \in \Gamma$$

Solution.

$$\lambda u : \alpha \rightarrow \beta.\lambda v : \alpha.xv(uv)$$

Proof.

1.	$u : \alpha \rightarrow \beta$	**Bound**
2.	$v : \alpha$	**Bound**
3.	$x : \alpha \rightarrow \beta \rightarrow \gamma$	**T-Var**
4.	$v : \alpha$	**T-Var**
5.	$xv : \beta \rightarrow \gamma$	**3,4 T-App**
6.	$u : \alpha \rightarrow \beta$	**T-Var**
7.	$uv : \beta$	**6,4 T-App**
8.	$xv(uv) : \gamma$	**5,7 T-App**
9.	$\lambda v : \alpha.xv(uv) : \alpha \rightarrow \gamma$	**8 T-Abst**
10.	$\lambda u : \alpha \rightarrow \beta.\lambda v : \alpha.xv(uv) : (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$	**9 T-Abst**

■

(2.13 b) Find a term of type

$$\alpha \rightarrow (\alpha \rightarrow \beta) \rightarrow \gamma$$

in context Γ

$$x : \alpha \rightarrow \beta \rightarrow \alpha \rightarrow \gamma \in \Gamma$$

Solution.

$$\lambda u : \alpha.\lambda v : \alpha \rightarrow \beta.xu(vu)u$$

Proof.

1.	$u : \alpha$	**Bound**
2.	$v : \alpha \to \beta$	**Bound**
3.	$x : \alpha \to \beta \to \alpha \to \gamma$	**T-Var**
4.	$u : \alpha$	**Bound**
5.	$v : \alpha \to \beta$	**Bound**
6.	$vu : \beta$	**5,4 T-App**
7.	$xu : \beta \to \alpha \to \gamma$	**3,4 T-App**
8.	$xu(vu) : \alpha \to \gamma$	**7,6 T-App**
9.	$xu(vu)u : \gamma$	**8,4 T-App**
10.	$\lambda v : \alpha \to \beta.xu(vu)u : (\alpha \to \beta)\gamma$	**9 T-Abst**
11.	$\lambda u : \alpha.\lambda v : \alpha \to \beta.xu(vu)u : \alpha \to (\alpha \to \beta)\gamma$	**10 T-Abst**

∎

Problem

(2.13 c) Find a term of type

$$(\alpha \to \gamma) \to (\beta \to \alpha) \to \gamma$$

in context Γ

$$x : (\beta \to \gamma) \to \gamma \in \Gamma$$

Solution.

$$\lambda u : \alpha \to \gamma.\lambda v : \beta \to \alpha.x(\lambda y : \beta.u(vy))$$

Proof.

1.	$u : \alpha \to \gamma$	**Bound**
2.	$v : \beta \to \alpha$	**Bound**
3.	$x : (\beta \to \gamma) \to \gamma$	**T-Var**
4.	$y : \beta$	**Bound**
5.	$v : \beta \to \alpha$	**T-Var**
6.	$y : \beta$	**T-Var**
7.	$vy : \alpha$	**5,6 T-App**
8.	$u : \alpha \to \gamma$	**T-Var**
9.	$u(vy) : \gamma$	**8,7 T-App**
10.	$\lambda y : \beta.u(vy) : \beta \to \gamma$	**9 T-Abst**

11. $\quad$ | $x(\lambda y : \beta.u(vy)) : \gamma$ $\qquad$ **10,3 T-App**

12. $\quad$ | $\lambda v : \beta \to \alpha.x(\lambda y : \beta.u(vy)) : (\beta \to \alpha) \to \gamma$ $\quad$ **11 T-Abst**

13.

$\lambda u : \alpha \to \gamma.\lambda v : \beta \to \alpha.x(\lambda y : \beta.u(vy))$

$\qquad : (\alpha \to \gamma) \to (\beta \to \alpha) \to \gamma$ $\qquad$ **12 T-Abst**

∎

<div style="background:#2a8a8a;color:white;padding:4px;">

Problem

</div>

(2.14) Find an inhabitant of type $\alpha \to \beta \to \gamma$ in the context Γ

$$x : (\gamma \to \beta) \to \alpha \to \gamma \in \Gamma$$

Solution.

$$\lambda u : \alpha.\lambda v : \beta.x(\lambda z : \gamma.v)u$$

Proof.

1. $\quad u : \alpha$ $\qquad$ **Bound**

2. $\quad$ | $v : \beta$ $\qquad$ **Bound**

3. $\quad$ | | $x : (\gamma \to \beta) \to \alpha \to \gamma$ $\qquad$ **T-Var**

4. $\quad$ | | $z : \gamma$ $\qquad$ **Bound**

5. $\quad$ | | | $v : \beta$ $\qquad$ **T-Var**

6. $\quad$ | | $\lambda z : \gamma.v : \gamma \to \beta$ $\qquad$ **5 T-Abst**

7. $\quad$ | | $x(\lambda z : \gamma.v) : \alpha \to \gamma$ $\qquad$ **3,6 T-App**

8. $\quad$ | | $u : \alpha$ $\qquad$ **T-Var**

9. $\quad$ | | $x(\lambda z : \gamma.v)u : \gamma$ $\qquad$ **7,8 T-App**

10. $\quad$ | $\lambda v : \beta.x(\lambda z : \gamma.v)u : \beta \to \gamma$ $\qquad$ **9 T-Abst**

11. $\lambda u : \alpha.\lambda v : \beta.x(\lambda z : \gamma.v)u : \alpha \to \beta \to \gamma$ $\quad$ **10 T-Abst**

∎