Excercises

Chapter 1

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1. Reducted

Problem

(1.1) Simplify notation of the following terms

(a) $(\lambda x.(((xz)y)(xx)))$

(b) $((\lambda x.(\lambda y.(\lambda z.(z((xy)z)))))(\lambda u.u))$

Solution.

(a) $\lambda x.(xzy)(xx)$

(b) $(\lambda xyz.x(xyz))(\lambda u.u)$

Problem

(1.2) Find the alpha equivalent terms to

$$\lambda x.x(\lambda x.x)$$

In

(a) $\lambda y.y(\lambda x.x)$

(b) $\lambda y.y(\lambda x.y)$

(c) $\lambda y.y(\lambda x.y)$

Solution. Only (a).

(1.3) Prove

$$\lambda x.x(\lambda z.y) =_{\alpha} \lambda z.z(\lambda z.y)$$

Solution.

Proof. By definition of alpha equivalence

$$M =_{\alpha} N \iff \exists \varphi, M^{\varphi} \to_{\alpha} N \wedge \text{FR } M = \text{FR } N$$

The witness of φ is substituting bound variable x with z, and z is not a free variable in the term, thus the two terms are alpha equivalent.

$$\lambda x.x(\lambda z.y)^{x \to z} \to_{\alpha} \lambda z.z(\lambda z.y)$$

∎

(1.4) Consider the following term:

$$U := (\lambda z.zxz)((\lambda y.xy)x)$$

1. Find Sub U
2. Draw tree rep of U
3. Find FV U
4. Find alpha equivalent terms to U from below and point out which of those follows the Barendregt convention:

$$\text{(a) } (\lambda y.yxy)((\lambda z.xz)x)$$
$$\text{(b) } (\lambda x.xyx)((\lambda z.yz)y)$$
$$\text{(c) } (\lambda y.yxy)((\lambda y.xy)x)$$
$$\text{(d) } (\lambda v.(vx)v)((\lambda u.uv)x)$$

1. Find Sub U.
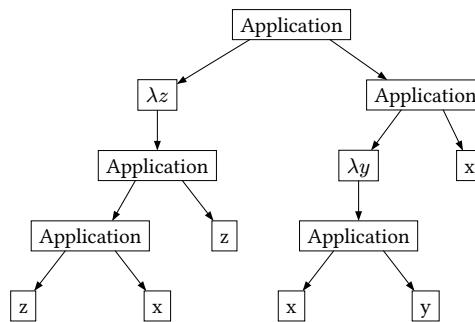
Solution.

Sub $U =$

$$\{(\lambda z.zxz)((\lambda y.xy)x), (\lambda z.zxz), ((\lambda y.xy)x)\} \cup$$
$$\{(\lambda y.xy), (y), (\lambda z.xz), (x)\} \cup$$
$$\{(\lambda y.x), (y)\} \cup \{(\lambda z.x), (z)\} \cup \{(y), (x)\}$$
$$= \{(\lambda z.zxz)(\lambda y.xy)x, (\lambda z.zxz), (\lambda y.xy)x,$$
$$(\lambda y.xy), (\lambda z.xz), (\lambda y.x), (\lambda z.x), y, x\}$$

2. Draw a tree rep of U.

Solution.

3. Find FV U

Solution.

$$\text{FV } U = \text{FV } (\lambda y.yxy) \cup \text{FV } (\lambda z.xz)x$$
$$= (\text{FV } yxy) \setminus \{y\} \cup (\text{FV } \lambda z.xz) \cup \{x\}$$
$$= (\text{FV } yx) \setminus \{y\} \cup (\text{FV } xz) \setminus \{z\} \cup \{x\}$$
$$= \{x\}$$

4. Find an alpha-equivalent term.

Solution.

$$(a) \underset{\alpha}{=} (c) \underset{\alpha}{=} U$$

Only (a) follows the Barendregt convention.

Problem

(1.5) Give the results of the following substitutions

(a) $(\lambda x.y(\lambda y.xy))[y := \lambda z.zx]$

(b) $((xyz)[x := y])[y := z]$

(c) $((\lambda x.xyz)[x := y])[y := z]$

(d) $(\lambda y.yyx)[x := yz]$

Solution.

(a) $(\lambda v.(\lambda z.zx)(\lambda u.vu))$

(b) $(yyz)[y := z] = zzz$

(c) $(\lambda x.xyz)[y := z] = (\lambda x.xzz)$

(d) $(\lambda u.uu(yz))$

Problem

(1.6)

$$\neg\Big(\forall MLN \in \Lambda, M[x := N, y := L] \underset{\alpha}{\equiv} M[x := N][y := L]\Big)$$

Solution.

Proof. Because RHS $= M[x := N][y := L] = M[x := N[y := L]][y := L]$, if $y \in$ FV N, then what x gets substituted with will have y substituted for L, which is completely different with LHS. ∎
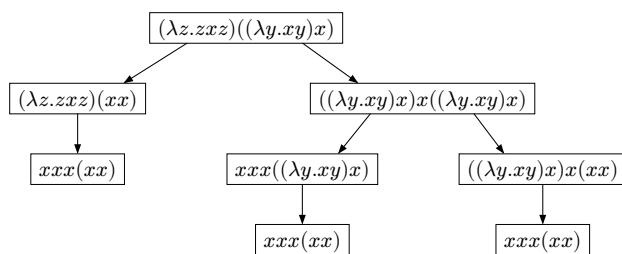
Problem

(1.7) Find all available redexes in

$$U := (\lambda z.zxz)((\lambda y.xy)x)$$

And all reduction pathes to the β-normal form.

Solution. The first redex is the term as an application itself; another the second term in the application.

> **Problem**
>
> *(1.8)* Show that
> $$(\lambda x.xx)y \neq_{\beta} (\lambda xy.yx)xx$$

Solution. By Collorary 1.9.9, it suffices to prove the hypothesis with a proof of a common normal reduced form from LHS and RHS not existing.

Contradiction. By definition of $=_{\beta}$, there exists The set of all terms attainable from β -reduction on $(\lambda x.xx)y$ and $(\lambda xy.yx)xx$ do not intersect. Therefore,

$$\neg\left(\exists L \in \Lambda, (\lambda x.xx)y \twoheadrightarrow_{\beta} L \wedge (\lambda xy.yx)xx \twoheadrightarrow_{\beta} L\right) \implies \neg\left((\lambda x.xx)y =_{\beta} (\lambda xy.yx)xx\right)$$

∎

> **Problem**
>
> *(1.9)* Define the combinators
> $$K := \lambda xy.x$$
> $$S := \lambda xyz.xz(yz)$$
>
> Prove that
> $$\forall PQ \in \Lambda, KPQ \twoheadrightarrow_{\beta} P$$
>
> $$\forall PQR \in \Lambda, SPQR \twoheadrightarrow_{\beta} PR(QR)$$

Solution.

Proof.

$$KPQ = (\lambda xy.x)PQ \rightarrow_{\beta} (\lambda y.x)[x := P]Q \rightarrow_{\beta} P[y := Q] = P$$

$$SPQR = (\lambda xyz.xz(yz)) \twoheadrightarrow_{\beta} (xz(yz))[x := P][y := Q][z := R] = PR(QR)$$

∎

(1.10) We define the church numerals

$$\text{zero} := \lambda fx.x$$
$$\text{one} := \lambda fx.fx$$
$$\text{two} := \lambda fx.ffx$$
$$...$$
$$\text{num}_n := \lambda fx.f^n x$$

And operations

$$\text{add} := \lambda nmfx.mf(nfx)$$
$$\text{mul} := \lambda nmfx.m(nf)x$$

Show

(a) add one one $\twoheadrightarrow_\beta$ two

(b) add one one $\neq_\beta$ mul one zero

Solution.

(a)   add one one $= (\lambda nmfx.mf(nfx))(\lambda fx.fx)(\lambda fx.fx)$

$\quad\quad \twoheadrightarrow_\beta (\lambda fx.(\lambda fx.fx)f((\lambda fx.fx)fx))$

$\quad\quad \twoheadrightarrow_\beta (\lambda fx.(\lambda x.fx)fx)$

$\quad\quad \twoheadrightarrow_\beta (\lambda fx.ffx) = $ two

(b)   mul one one $= (\lambda nmfx.m(nf)x)(\lambda fx.fx)(\lambda fx.fx)$

$\quad\quad \twoheadrightarrow_\beta \lambda fx.(\lambda fx.fx)((\lambda fx.fx)f)x$

$\quad\quad \twoheadrightarrow_\beta \lambda fx.fx = $ one

Because no intermediate form in the beta reduction process of the two terms are α-equivalent, by collorary 1.9.9 the two terms are not β-equivalent.

(1.11) We define

$$\text{succ} := \lambda mfx.f(mfx) \text{ s.t. } \forall \text{num}_n, \text{succ num}_n = \text{num}_{n+1}$$

Prove

$$\text{succ zero} \underset{\beta}{=} \text{one}$$

$$\text{succ one} \underset{\beta}{=} \text{two}$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$\text{succ zero} = (\lambda mfx.f(mfx))(\lambda fx.x)$$
$$\underset{\beta}{\to} (\lambda fx.f((\lambda fx.x)fx))$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda fx.fx) = \text{one}$$

The path succ zero $\underset{\beta}{\twoheadrightarrow}$ one derived above is the witness of a reduction chain from LHS to RHS.

$$\text{succ one} = (\lambda mfx.f(mfx))(\lambda fx.fx)$$
$$\underset{\beta}{\to} (\lambda fx.f((\lambda fx.fx)fx))$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda fx.f(fx)) = \text{two}$$

The path succ one $\underset{\beta}{\twoheadrightarrow}$ two derived above is the witness of a reduction chain from LHS to RHS. ∎

(1.12) We define the λ-terms $\top_\lambda$ (true) and $\bot_\lambda$ (false) and $\neg_\lambda$ (not) by:

$$\top_\lambda := \lambda xy.x \quad \bot_\lambda := \lambda xy.y$$
$$\neg_\lambda := \lambda a.a \bot_\lambda \top_\lambda$$

Show that

$$\neg_\lambda(\neg_\lambda \top_\lambda) \underset{\beta}{=} \top_\lambda$$

$$\neg_\lambda(\neg_\lambda \bot_\lambda) \underset{\beta}{=} \bot_\lambda$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$\neg_\lambda(\neg_\lambda \top_\lambda) = \neg_\lambda((\lambda a.a \bot_\lambda \top_\lambda)(\lambda xy.x))$$
$$\underset{\beta}{\twoheadrightarrow} \neg_\lambda((\lambda xy.x)\bot_\lambda \top_\lambda)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda a.a \bot_\lambda \top_\lambda)\bot_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda xy.y)\bot_\lambda \top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} \top_\lambda$$

$$\neg_\lambda(\neg_\lambda \bot_\lambda) = \neg_\lambda((\lambda a.a \bot_\lambda \top_\lambda)(\lambda xy.y))$$
$$\underset{\beta}{\twoheadrightarrow} \neg_\lambda((\lambda xy.x)\bot_\lambda \top_\lambda)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda a.a \bot_\lambda \top_\lambda)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda xy.x)\bot_\lambda \top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} \bot_\lambda$$

∎

Problem

(1.13) Define

$$\text{iszero} := \lambda m.m(\lambda x.\bot_\lambda)\top_\lambda$$

Prove

$$\text{iszero zero} \underset{\beta}{\twoheadrightarrow} \top_\lambda$$
$$\forall n \in \mathbb{N}^+, \text{iszero num}_n \underset{\beta}{\twoheadrightarrow} \bot_\lambda$$

Solution.

$$\text{iszero zero} = (\lambda m.m(\lambda x.\bot_\lambda)\top_\lambda)(\lambda fx.x)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda fx.x)(\lambda x.\bot_\lambda)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} \top_\lambda$$

$$\text{iszero num}_n = (\lambda m.m(\lambda x.\bot_\lambda)\top_\lambda)(\lambda fx.f^n x)$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda fx.f^n x)(\lambda x.\bot_\lambda)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda x.\bot_\lambda)\big((\lambda x.\bot_\lambda)^{n-1}\top_\lambda\big) \underset{\beta}{\twoheadrightarrow} \bot_\lambda$$

Problem

(1.14) If-else can be modeled as

$$\text{ifelse} = \lambda xtf.xtf$$

Where when x, then t, else f. Prove correctness by applying $\top_\lambda$ and $\bot_\lambda$ on ifelse.

Solution.

$$\text{ifelse } \top_\lambda = (\lambda xtf.xtf)\top_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda tf.(\lambda xy.x)tf) \underset{\beta}{\twoheadrightarrow} (\lambda tf.t)$$

$$\text{ifelse } \top_\lambda = (\lambda xtf.xtf)\bot_\lambda$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda tf.(\lambda xy.y)tf) \underset{\beta}{\twoheadrightarrow} (\lambda tf.f)$$

By applying the results to any two values, the correct corresponding value returns, ex, for ifelse $\top_\lambda$, t is always returned.

Problem

(1.15) Prove that $\Omega := (\lambda x.xx)(\lambda x.xx)$ does not have a β-nf.

Solution. Firstly let's prove Ω.

Proof. Induction on Ω's only reduction path proves that every $\Omega \underset{\beta}{\twoheadrightarrow} \Omega_i = \Omega$. For the base case because Ω has one and only one redux, it could only reduce to Ω_1 which is equivalent to itself. For the inductive step, $\Omega_i = \Omega$, therefore $\Omega_i \underset{\beta}{\rightarrow} \Omega_{i+1}$ is still Ω.

By definition, a term having a β-nf requires the existence of a form in β-nf such that the term can reduce to. By induction, Ω only reduces to Ω, and Ω is not in β-nf because it contains β-redex. Therefore, Ω can never reduce to a β-nf, thus it does not have a β-nf. $\blacksquare$

Problem

(1.16) Let M be a λ-term with the following properties:
- M has a β-nf.
- There exists an infinite reduction path $M \equiv M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \dots$ on M.

Prove that every M_i has a β-nf, and give an example of M.

Solution. An example would be $(\lambda xy.y)\Omega$. Reduction can go on infinitely by reducing on Ω, but the β-nf of the term is $\lambda y.y$

Proof. Denote β-nf of M as M'. For any form in the reduction path, $M \underset{\beta}{\twoheadrightarrow} M_i$. In conjunction with $M \underset{\beta}{\twoheadrightarrow} M'$, by the Church-Rosser theorem, there exists L such that $M_i \underset{\beta}{\twoheadrightarrow} L$ and $M' \underset{\beta}{\twoheadrightarrow} L$. Because M' is in β-nf, L can only be M', thus $M_i \underset{\beta}{\twoheadrightarrow} M'$, so M_i is capable of reducing to M', a β-nf. Therefore, any form in the reduction path has a β-nf. ∎

Problem

(1.17) If MN is strongly normalizing, then both M and N are strongly normalizing.

Solution.

Proof. If M is not strongly normalizing, then there exists a reduction path $M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \dots$ Therefore, MN would have had a reduction path $MN \underset{\beta}{\to} M_1 N \underset{\beta}{\to} \dots$ that is infinite, which contradicts with MN being strongly normalizing. Vice versa for N. ∎

Problem

(1.18) Let $L, M, N \in \Lambda$ such that $L \underset{\beta}{=} M$ and $L \underset{\beta}{\twoheadrightarrow} N$. Moreover, N is in β-nf. Prove that $M \underset{\beta}{\twoheadrightarrow} N$.

Solution. Collorary 1.9.9.

(1.19) Define

$$U := \lambda zx.x(zzx) \quad \text{and} \quad Z := UU$$

Prove Z is a fixed point combinator.

Solution. Proving $\forall L \in \Lambda, L(ZL) \underset{\beta}{\twoheadrightarrow} ZL$.

Proof.

$$ZL = (\lambda zx.x(zzx))(\lambda zx.x(zzx))L$$
$$\underset{\beta}{\twoheadrightarrow} L((\lambda zx.x(zzx))(\lambda zx.x(zzx))L) \underset{\beta}{\twoheadrightarrow} L(ZL)$$

∎

(1.20) Solve for $M \in \Lambda$ in each equation:

$$M \underset{\beta}{=} \lambda xy.xMy$$
$$Mxyz \underset{\beta}{=} xyzM$$

Solution. By the property of the Y combinator:

$$f(Yf) = Yf$$

The first equation can be remodeled as

$$M \underset{\beta}{=} LM \text{ where } L = \lambda mxy.xmy$$

Solving for fixed point of L :

$$M \equiv YL \equiv L(YL)$$
$$= (\lambda x.L(xx))(\lambda x.L(xx))$$
$$= (\lambda x.(\lambda muv.umv)(xx))(\lambda x.(\lambda muv.umv)(xx))$$
$$\underset{\beta}{\twoheadrightarrow} (\lambda x.(\lambda uv.u(xx)v))(\lambda x.(\lambda uv.u(xx)v))$$

The second equation can be η-reduced on both sides:

$$Mxyz \underset{\beta}{=} xyzM$$
$$M \underset{\beta}{=} \lambda xyz.xyzM$$

Remodeling equation:

$$M = NM \text{ where } M = \lambda mxyz.xyzm$$

then

$$
\begin{aligned}
M &\equiv YN \equiv N(YN) \\
&= (\lambda x.N(xx))(\lambda x.N(xx)) \\
&= (\lambda x.(\lambda muyz.uyzm)(xx))(\lambda x.(\lambda muyz.uyzm)(xx)) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda x.(\lambda uyz.uyz(xx)))(\lambda x.(\lambda uyz.uyz(xx)))
\end{aligned}
$$