

EXERCISES

CHAPTER 5

SEAN LI ¹

1. Redacted

Definition Some rules for reference.

λP Calculus Rules

$$\begin{array}{c} \frac{}{\emptyset \vdash * : \square} \text{Sort} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \text{Var} \quad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \text{Weak} \\[10pt] \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A . B : s} \text{Form} \quad \frac{\Gamma \vdash M : \Pi x : A . B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B [x := N]} \text{App} \\[10pt] \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A . B : s}{\Gamma \vdash \lambda x : A . M : \Pi x : A . B} \text{Abst} \\[10pt] \frac{\Gamma \vdash A : B \quad B \stackrel{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \text{Conv} \end{array}$$

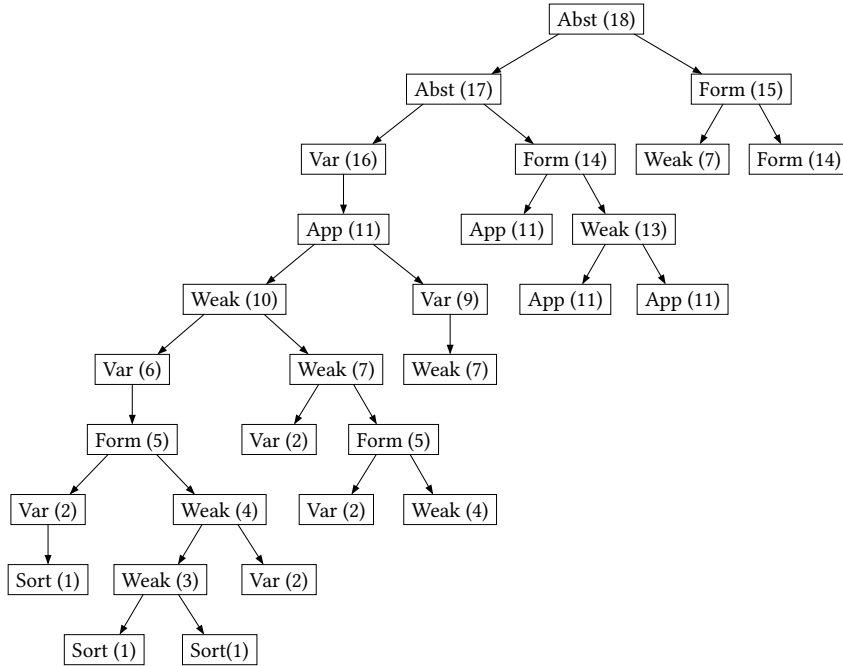
Predicate Logic

$$\begin{array}{c} \begin{array}{l} 1. \text{ Assume } A \\ 2. \quad \left| \begin{array}{l} \dots \\ 3. \quad B \end{array} \right. \\ \hline A \Rightarrow B \end{array} \Rightarrow I \quad \frac{A \Rightarrow B \quad A}{B} \Rightarrow E \quad \begin{array}{l} 1. \text{ Let } a \in S \\ 2. \quad \left| \begin{array}{l} \dots \\ 3. \quad P(a) \end{array} \right. \\ \hline \forall a \in S, P(a) \end{array} \forall I \\[10pt] \frac{\forall a \in S \quad N \in S}{P(N)} \forall E \end{array}$$

Problem

(5.1) Give a diagram of the tree corresponding to the complete tree derivation of line 18 of Section 5.3 (P 107)

Solution.



Problem

(5.2) Give a complete λP derivation of

$$S : * \vdash S \rightarrow S \rightarrow * : \square$$

In tree format and flag format.

Solution.

Tree Derivation.

$$\begin{array}{c}
 (4) \frac{\vdash * : \square \quad \vdash * : \square}{S : * \vdash * : \square} \text{Weak} \quad (3) S : * \vdash S : * \\
 (6) \frac{S : * \vdash * : \square}{S : *, x : S \vdash * : \square} \text{Weak} \\
 (7) \frac{(3) S : * \vdash S : *}{S : * \vdash S \rightarrow * : \square}
 \end{array}$$

$$\begin{array}{c}
(3) \frac{\vdash * : \Box}{S : * \vdash S : *} \text{Var} \quad (9) \frac{(7) S : * \vdash S \rightarrow * : \Box \quad (3) S : * \vdash S : *}{S : *, x : S \vdash S \rightarrow * : \Box} \text{Weak} \\
\hline
S : * \vdash S \rightarrow S \rightarrow * : \Box \text{Form}
\end{array}$$

■

Flag Derivation.

| | | |
|-----|--|-----------------|
| 1. | $* : \Box$ | Sort |
| 2. | $S : *$ | |
| 3. | $S : *$ | 1 Var |
| 4. | $* : \Box$ | 1,1 Weak |
| 5. | $x : S$ | |
| 6. | $\boxed{* : \Box}$ | 4,3 Weak |
| 7. | $S \rightarrow * : \Box$ | 3,6 Form |
| 8. | $x : S$ | |
| 9. | $\boxed{S \rightarrow * : \Box}$ | 7,3 Weak |
| 10. | $S \rightarrow S \rightarrow * : \Box$ | 3,9 Form |

■

Problem

(5.3) Derive

$$S : *, Q : S \rightarrow S \rightarrow * \vdash \Pi x : S . \Pi y : S . Q x y : *$$

Solution.

| | | |
|-----|--|-----------------|
| 1. | $* : \Box$ | Sort |
| 2. | $S : *$ | |
| 3. | $S : *$ | 1 Var |
| 4. | $* : \Box$ | 1,1 Weak |
| 5. | $x : S$ | |
| 6. | $\boxed{* : \Box}$ | 4,3 Weak |
| 7. | $S \rightarrow * : \Box$ | 3,6 Form |
| 8. | $x : S$ | |
| 9. | $\boxed{S \rightarrow * : \Box}$ | 7,3 Weak |
| 10. | $S \rightarrow S \rightarrow * : \Box$ | 3,9 Form |
| 11. | $Q : S \rightarrow S \rightarrow *$ | |

| | | |
|-----|-------------------------------------|-------------------|
| 12. | $Q : S \rightarrow S \rightarrow *$ | 10 Var |
| 13. | $S : *$ | 3,10 Weak |
| 14. | $* : \square$ | 4,10 Weak |
| 15. | $x : S$ | |
| 16. | $* : \square$ | 14,13 Weak |
| 17. | $S : *$ | 13,13 Weak |
| 18. | $x : S$ | 13 Var |
| 19. | $Q : S \rightarrow S \rightarrow *$ | 12,13 Weak |
| 20. | $y : S$ | |
| 21. | $y : S$ | 17 Var |
| 22. | $Q : S \rightarrow S \rightarrow *$ | 19,17 Weak |
| 23. | $x : S$ | 18,17 Weak |
| 24. | $Q x : S \rightarrow *$ | 22,23 App |
| 25. | $Q x y : *$ | 24,21 App |
| 26. | $\Pi y : S . Q x y : *$ | 17,25 Form |
| 27. | $\Pi x : S . \Pi y : S . Q x y : *$ | 13,26 Form |

Problem

(5.4) Prove that $*$ is the only valid kind in λP .

Solution.

Proof. The only possible way to construct a new kind is through the Form rule and the Sort axiom. Because we are trying to construct a kind, s here stands for \square .

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : \square}{\Gamma \vdash \Pi x : A . B : \square} \text{Form}$$

One could only construct new kinds with kinds, which requires $A : \square$ and $B : \square$. This contradicts with $A : *$. ■

Problem

(5.5) Prove that $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ is a tautology by given a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash A \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

| | | |
|----|---|----------------|
| 1. | $A : *$ | |
| 2. | $B : *$ | |
| 3. | $x : A$ | |
| 4. | $y : A \rightarrow B$ | |
| 5. | $y x : B$ | 4,3 App |
| 6. | $\lambda y : A \rightarrow B . y x : (A \rightarrow B) \rightarrow B$ | 5 Abst |
| 7. | $\lambda x : A . \lambda y : A \rightarrow B . y x : A \rightarrow (A \rightarrow B) \rightarrow B$ | 5 Abst |

■

Problem

(5.6 a) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ a tautology using natural deduction.

Solution.

Proof.

| | | |
|----|---|--------------------------------------|
| 1. | Assume $A \Rightarrow (A \Rightarrow B)$ | |
| 2. | $A \Rightarrow (A \Rightarrow B)$ | |
| 3. | Assume A | |
| 4. | A | |
| 5. | $A \Rightarrow B$ | 2,4 \RightarrowE |
| 6. | B | 5,4 \RightarrowE |
| 7. | $A \Rightarrow B$ | 3,6 \RightarrowI |
| 8. | $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ | 1,7 \RightarrowI |

■

Problem

(5.6 b) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ using a shorthand λP derivation

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

| | | |
|----|---|---------|
| 1. | $A : *$ | |
| 2. | $B : *$ | |
| 3. | $x : A \rightarrow A \rightarrow B$ | |
| 4. | $y : A$ | |
| 5. | $x y : A \rightarrow B$ | 3,4 App |
| 6. | $x y y : B$ | 5,4 App |
| 7. | $\lambda y : A . x y y : A \rightarrow B$ | 6 Abst |
| 8. | $\lambda x : A \rightarrow A \rightarrow B . \lambda y : A . x y y : (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$ | 7 Abst |

■

Problem

(5.7 a) Proof $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of A , B , and C is equivalent to a proof of tautologousness.

Proof.

| | | |
|-----|---|---------|
| 1. | $A : *$ | |
| 2. | $B : *$ | |
| 3. | $C : *$ | |
| 4. | $x : A \rightarrow B$ | |
| 5. | $y : B \rightarrow C$ | |
| 6. | $a : A$ | |
| 7. | $x a : B$ | 4,6 App |
| 8. | $y (x a) : C$ | 5,7 App |
| 9. | $\lambda a : A . y (x a) : A \rightarrow C$ | 8 Abst |
| 10. | $\lambda y : B \rightarrow C . \lambda a : A . y (x a) : (B \rightarrow C) \rightarrow A \rightarrow C$ | 9 Abst |

$$\begin{array}{c}
11. \quad \left| \begin{array}{l} \lambda x : A \rightarrow B . \lambda y : B \rightarrow C . \lambda a : A . y (x z) \\ : (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C \end{array} \right. \quad \mathbf{10 \text{ Abst}}
\end{array}$$

■

Problem

(5.7 b) Proof $((A \Rightarrow B) \Rightarrow A) \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

$$\begin{array}{c}
1. \quad A : * \\
2. \quad \left| B : * \right. \\
3. \quad \left| \left| x : (A \rightarrow B) \rightarrow A \right. \right. \\
4. \quad \left| \left| \left| y : A \rightarrow B \right. \right. \right. \\
5. \quad \left| \left| \left| \left| x y : A \right. \right. \right. \quad \mathbf{3,4 \text{ App}} \\
6. \quad \left| \left| \left| \left| y (x y) : B \right. \right. \right. \quad \mathbf{4,5 \text{ App}} \\
7. \quad \left| \left| \left| \lambda y : A \rightarrow B . y (x y) : (A \rightarrow B) \rightarrow B \right. \right. \quad \mathbf{6 \text{ Abst}} \\
8. \quad \left| \left| \lambda x : (A \rightarrow B) \rightarrow A . \lambda y : A \rightarrow B . y (x y) \right. \right. \\
\quad \left| : ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B \quad \mathbf{7 \text{ Abst}} \right.
\end{array}$$

■

Problem

(5.7 c) Proof $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of A , B , and C is equivalent to a proof of tautologousness.

Proof.

| | | | |
|-----|---|---------|--|
| 1. | $A : *$ | | |
| 2. | $B : *$ | | |
| 3. | $C : *$ | | |
| 4. | $x : A \rightarrow B \rightarrow C$ | | |
| 5. | $y : A \rightarrow B$ | | |
| 6. | $a : A$ | | |
| 7. | $x a : B \rightarrow C$ | 4,6 App | |
| 8. | $y a : B$ | 5,6 App | |
| 9. | $x a (y a) : C$ | 7,8 App | |
| 10. | $\lambda a : A . x a (y a) : A \rightarrow C$ | 9 Abst | |
| 11. | $\lambda y : A \rightarrow B . \lambda a : A . x a (y a) : (A \rightarrow B) \rightarrow A \rightarrow C$ | 10 Abst | |
| 12. | $\lambda x : A \rightarrow B \rightarrow C . \lambda y : A \rightarrow B . \lambda a : A . x a (y a) : (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$ | 10 Abst | |

■

Problem

(5.8 a) Let $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *$, find an inhabitant of

$$\Pi x : S . P x \rightarrow Q x \rightarrow P x$$

with respect to Γ and give a shorthand λP derivation

Solution.

| | | | |
|----|---|---------|--|
| 1. | $S : *$ | | |
| 2. | $P : S \rightarrow *$ | | |
| 3. | $Q : S \rightarrow *$ | | |
| 4. | $x : S$ | | |
| 5. | $a : P x$ | | |
| 6. | $b : Q x$ | | |
| 7. | $a : P x$ | 2,4 App | |
| 8. | $\lambda b : Q x . S : Q x \rightarrow P x$ | 7 Abst | |
| 9. | $\lambda a : P x . \lambda b : Q x . a : P x \rightarrow Q x \rightarrow P x$ | 8 Abst | |

$$10. \quad \left| \left| \left| \begin{array}{l} \lambda x : S . \lambda a : P \ x . \lambda b : Q \ x . a \\ : \Pi x : S . P \ x \rightarrow Q \ x \rightarrow P \ x \end{array} \right. \right. \right. \quad \mathbf{9 \text{ Abst}}$$

Problem

(5.8 b) Let $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *$, find an inhabitant of

$$\Pi x : S . P \ x \rightarrow Q \ x \rightarrow P \ x$$

By proving the corresponding proposition in natural deduction.

Solution. The corresponding proposition and premises are

$$\frac{S \in \text{Set} \quad P : S \rightarrow \text{Prop} \quad Q : S \rightarrow \text{Prop}}{\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a))}$$

Proof.

1. Let $a \in S$
2. | Assume $P(a)$
3. | | Assume $Q(a)$
4. | | | $P(a)$
5. | | | $Q(a) \Rightarrow P(a)$ **3,4 \Rightarrow I**
6. | | $P(a) \Rightarrow (Q(a) \Rightarrow P(a))$ **2,5 \Rightarrow I**
7. $\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a))$ **1,6 \forall I**

■

Problem

(5.9 a) Give proof for

$$(\forall x \in S, Q(x)) \Rightarrow (\forall y \in S, P(y) \Rightarrow Q(y))$$

by natural deduction and a λP derivation.

Solution.

Natural Deduction.

1. Assume $\forall x \in S, Q(x)$
2. | Let $y \in S$

| | | | |
|----|--|--|---------------------------------------|
| 3. | | Assume $P(y)$ | |
| 4. | | $Q(y)$ | 1,2 $\forall E$ |
| 5. | | $P(y) \Rightarrow Q(y)$ | 3,4 $\Rightarrow I$ |
| 6. | | $\forall y \in S, P(y) \Rightarrow Q(y)$ | 2,5 $\forall I$ |
| 7. | | $(\forall x \in S, Q(x) \Rightarrow (\forall y \in S, P(y) \Rightarrow Q(y)))$ | 1,6 $\Rightarrow I$ |

■

λP Derivation. Corresponding type is

$$S : *, P : S \rightarrow *, Q : S \rightarrow * \vdash (\Pi x : S . Q x) \rightarrow (\Pi y : S . P y \rightarrow Q y)$$

| | | | |
|-----|--|---------|--|
| 1. | $S : *$ | | |
| 2. | $P : S \rightarrow *$ | | |
| 3. | $Q : S \rightarrow *$ | | |
| 4. | $a : \Pi x : S . Q x$ | | |
| 5. | $y : S$ | | |
| 6. | $z : P y$ | | |
| 7. | $a y : Q y$ | 4,5 App | |
| 8. | $\lambda z : P y . a y : P y \rightarrow Q y$ | 7 Abst | |
| 9. | $\lambda y : S . \lambda z : P y . a y : \Pi y : S . P y \rightarrow Q y$ | 7 Abst | |
| 10. | $\lambda a : \Pi x : S . Q x . \lambda y : S . \lambda z : P y . a y$ $: (\Pi x : S . Q x) \rightarrow (\Pi y : S . P y \rightarrow Q y)$ | 7 Abst | |

■

Problem

(5.9 b) Give proof for

$$\forall x \in S, (P(x) \Rightarrow Q(x)) \Rightarrow ((\forall y \in S, P(y)) \Rightarrow (\forall z \in S, Q(z)))$$

by natural deduction and a λP derivation

Solution.

Natural Deduction.

| | |
|----|---|
| 1. | Assume $\forall x \in S, (P(x) \Rightarrow Q(x))$ |
| 2. | Assume $\forall y \in S, P(y)$ |
| 3. | Let $z \in S$ |

| | | | | |
|----|--|--|--|---------------------------------------|
| 4. | | | $P(z)$ | 2,3 $\forall E$ |
| 5. | | | $P(z) \Rightarrow Q(z)$ | 1,3 $\forall E$ |
| 6. | | | $Q(z)$ | 5,4 $\Rightarrow E$ |
| 7. | | | $\forall z \in S, Q(z)$ | 3,6 $\forall I$ |
| 8. | | | $\forall y \in S, P(y) \Rightarrow (\forall z \in S, Q(z))$ | 2,7 $\forall I$ |
| 9. | | | $\forall x \in S, (P(x) \Rightarrow Q(x)) \Rightarrow ((\forall y \in S, P(y)) \Rightarrow (\forall z \in S, Q(z)))$ | 1,8 $\forall I$ |

■

λP Derivation. Corresponding type is

$$S : *, P : S \rightarrow *, Q : S \rightarrow *$$

$$\vdash (\Pi x : S. P x \rightarrow Q x) \rightarrow (\Pi y : S. P y) \rightarrow (\Pi z : S. Q z)$$

| | | | |
|-----|---|---------|--|
| 1. | $S : *$ | | |
| 2. | $P : S \rightarrow *$ | | |
| 3. | $Q : S \rightarrow *$ | | |
| 4. | $a : \Pi x : S . P x \rightarrow Q x$ | | |
| 5. | $b : \Pi y : S . P y$ | | |
| 6. | $z : S$ | | |
| 7. | $b z : P z$ | 5,6 App | |
| 8. | $a z : P z \rightarrow Q z$ | 4,6 App | |
| 9. | $a z (b z) : Q z$ | 8,7 App | |
| 10. | $\lambda z : S . a z (b z) : \Pi z : S . Q z$ | 9 Abst | |
| 11. | $\lambda b : (\Pi y : S . P y) . \lambda z : S . a z (b z)$ | | |
| | $: (\Pi y : S . P y) \rightarrow \Pi z : S . Q z$ | 10 Abst | |
| | $\lambda a : (\Pi x : S . P x \rightarrow Q x) . \lambda b : (\Pi y : S . P y) .$ | | |
| | $\lambda z : S . a z (b z)$ | | |
| | $: (\Pi x : S . P x \rightarrow Q x) \rightarrow$ | | |
| 12. | $(\Pi y : S . P y) \rightarrow \Pi z : S . Q z$ | 10 Abst | |

■

Problem

(5.10) Given a context

$$\begin{aligned}\Gamma &\equiv S : *, P : S \rightarrow *, f : S \rightarrow S, g : S \rightarrow S, \\ u &: \Pi x : S . (P (f x) \rightarrow P (g x)), \\ v &: \Pi x, y : S . ((P x \rightarrow P y) \rightarrow P (f x))\end{aligned}$$

Let

$$M \equiv \lambda x : S . v (f x)(g x)(u x)$$

Type M under Γ .

Solution.

| | | |
|-----|---|-----------------|
| 1. | $x : S$ | |
| 2. | $f : S \rightarrow S$ | |
| 3. | $f x : S$ | 2,1 App |
| 4. | $g : S \rightarrow S$ | |
| 5. | $g x : S$ | 4,1 App |
| 6. | $u : \Pi x : S . (P (f x) \rightarrow P (g x))$ | |
| 7. | $u x : P (f x) \rightarrow P (g x)$ | 6,1 App |
| 8. | $v : \Pi x, y : S . ((P x \rightarrow P y) \rightarrow P (f x))$ | |
| 9. | $v (f x) : \Pi y : S . ((P (f x) \rightarrow P y) \rightarrow P (f (f x)))$ | 8,3 App |
| 10. | $v (f x)(g x) : (P (f x) \rightarrow P (g x)) \rightarrow P (f (f x))$ | 9,5 App |
| 11. | $v (f x)(g x)(u x) : P (f (f x))$ | 10,7 App |
| 12. | $\lambda x : S . v (f x)(g x)(u x) : S \rightarrow P (f (f x))$ | 11 Abst |

Problem

(5.11) Let S be a set, with Q and R relations on $S \times S$, and let f and g be functions from S to S . Assume

$$\forall x, y \in S (Q(x, f(y)) \Rightarrow Q(g(x), y))$$

$$\forall x, y \in S (Q(x, f(y)) \Rightarrow R(x, y))$$

$$\forall x \in S (Q(x, f(f(x))))$$

Prove that

$$\forall x \in S, R(g(g(x)), g(x))$$

By giving a context Γ and finding a term M such that

$$\Gamma \vdash M : \Pi x : S . R(g(g(x)))(g(x))$$

Solution. Context Γ is as follows:

$$\Gamma \equiv S : *, f : S \rightarrow S, g : S \rightarrow S$$

$$Q : S \rightarrow S \rightarrow *, R : S \rightarrow S \rightarrow *$$

$$A : \Pi x, y : S . (Q(x, f(y)) \rightarrow Q(g(x), y)),$$

$$B : \Pi x, y : S . (Q(x, f(y)) \rightarrow R(x, y))$$

$$C : \Pi x : S . Q(x, f(f(x)))$$

Derivation.

- | | | |
|-----|--|-----------------|
| 1. | $S : *, f : S \rightarrow S, g : S \rightarrow S$ | |
| 2. | $Q : S \rightarrow S \rightarrow *, R : S \rightarrow S \rightarrow *$ | |
| 3. | $A : \Pi x, y : S . (Q(x, f(y)) \rightarrow Q(g(x), y))$ | |
| 4. | $B : \Pi x, y : S . (Q(x, f(y)) \rightarrow R(x, y))$ | |
| 5. | $C : \Pi x : S . Q(x, f(f(x)))$ | |
| 6. | $x : S$ | |
| 7. | $g(x) : S$ | 1,6 App |
| 8. | $C(g(x)) : Q(g(x))(f(f(g(x))))$ | 5,7 App |
| 9. | $f(g(x)) : S$ | 1,7 App |
| 10. | $A(g(x)) : \Pi y : S . (Q(g(x), f(y)) \rightarrow Q(g(g(x)), y))$ | 3,7 App |
| | $A(g(x))(f(g(x)))$ | |
| | $: (Q(g(x))(f(f(g(x))))$ | |
| 11. | $\rightarrow (Q(g(g(x)))(f(g(x))))$ | 10,9 App |

| | | |
|-----|---|------------------|
| 12. | $A(g\ x)(f(g\ x))(C(g\ x))$ $: (Q(g(g\ x))(f(g\ x)))$ | 11,8 App |
| 13. | $g(g\ x) : S$ $B(g(g\ x))$ $: \Pi y : S . (Q(g(g\ x))(f\ y))$ | 1,7 App |
| 14. | $\rightarrow R(g(g\ x))\ y)$ | 4,13 App |
| 15. | $B(g(g\ x))(g\ x)$ $: (Q(g(g\ x))(f(g\ x))) \rightarrow (R(g(g\ x))(g\ x))$ | 14,7 App |
| 16. | $B(g(g\ x))(g\ x)(A(g\ x)(f(g\ x))(C(g\ x)))$ $: (R(g(g\ x))(g\ x))$ | 15,12 App |
| 17. | $\lambda x : S . B(g(g\ x))(g\ x)(A(g\ x)(f(g\ x))(C(g\ x)))$ $: \Pi x : S . (R(g(g\ x))(g\ x))$ | 17 Abst |

■

Problem

(5.12 a) In λP , consider the context

$$\begin{aligned}\Gamma &\equiv S : *, R : S \rightarrow S \rightarrow *, \\ u &: \Pi x, y : S . R\ x\ y \rightarrow R\ y\ x \\ v &: \Pi x, y, z : S . R\ x\ y \rightarrow R\ x\ z \rightarrow R\ y\ z\end{aligned}$$

Show that R is reflexive over $S \times S$. That is, construct M such that

$$\Gamma \vdash M : \Pi x, y : S . R\ x\ y \rightarrow R\ x\ x$$

Solution.

Proof.

1. $S : *, R : S \rightarrow S \rightarrow *$
2. $A : \Pi u, v : S . R\ u\ v \rightarrow R\ v\ u$
3. $B : \Pi u, v, w : S . R\ u\ v \rightarrow R\ u\ w \rightarrow R\ v\ w$
4. $\left| \begin{array}{l} x : S \\ y : S \\ h : R\ x\ y \end{array} \right|$
- 5.
- 6.

| | | |
|-----|--|-----------|
| 7. | $B x : \Pi v, w : S . R x v \rightarrow R v w \rightarrow R x w$ | 3,4 App |
| 8. | $B x y : \Pi w : S . R x y \rightarrow R y w \rightarrow R x w$ | 7,5 App |
| 9. | $B x y x : R x y \rightarrow R y x \rightarrow R x x$ | 7,5 App |
| 10. | $A x : \Pi v : R x v \rightarrow R v x$ | 2,4 App |
| 11. | $A x y : R x y \rightarrow R y x$ | 10,5 App |
| 12. | $A x y h : R y x$ | 11,6 App |
| 13. | $B x y x h : R y x \rightarrow R x x$ | 9,6 App |
| 14. | $B x y x h (A x y h) : R x x$ | 13,12 App |
| 15. | $\lambda h : R x y . B x y x h (A x y h) : R x y \rightarrow R x x$ | 14 Abst |
| 16. | $\lambda y : S . \lambda h : R x y . B x y x h (A x y h)$ $: \Pi y : S . R x y \rightarrow R x x$ | 15 Abst |
| 17. | $\lambda x, y : S . \lambda h : R x y . B x y x h (A x y h)$ $: \Pi x, y : S . R x y \rightarrow R x x$ | 16 Abst |

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Problem

(5.12 b) Given the context Γ in 5.12 a, prove transitivity of R by constructing M such that

$$\Gamma \vdash M : \Pi x, y, z : S . R x y \rightarrow R y z \rightarrow R x z$$

Solution.

Proof.

| | | |
|-----|---|----------|
| 1. | $S : *, R : S \rightarrow S \rightarrow *$ | |
| 2. | $A : \Pi u, v : S . R u v \rightarrow R v u$ | |
| 3. | $B : \Pi u, v, w : S . R u v \rightarrow R u w \rightarrow R v w$ | |
| 4. | $x : S$ | |
| 5. | $y : S$ | |
| 6. | $z : S$ | |
| 7. | $h : R x y$ | |
| 8. | $r : R y z$ | |
| 9. | $A x : \Pi v : S . R x v \rightarrow R v x$ | 2,4 App |
| 10. | $A x y : R x y \rightarrow R y x$ | 9,5 App |
| 11. | $A x y h : R y x$ | 10,7 App |

| | | |
|-----|--|------------------|
| 12. | $B y : \Pi v, w : S . R y v \rightarrow R y w \rightarrow R v w$ | 3,5 App |
| 13. | $B y x : \Pi w : S . R y x \rightarrow R y w \rightarrow R x w$ | 12,4 App |
| 14. | $B y x z : R y x \rightarrow R y z \rightarrow R x z$ | 12,4 App |
| 15. | $B y x z (A x y h) : R y z \rightarrow R x z$ | 14,11 App |
| 16. | $B y x z (A x y h) r : R x z$ | 15,8 App |
| 17. | $\lambda r : R y z . B y x z (A x y h) r : R y z \rightarrow R x z$ | 16 Abst |
| 18. | $\lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$ $: R x y \rightarrow R y z \rightarrow R x z$ | 17 Abst |
| 19. | $\lambda z : S . \lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$ $: \Pi z : S . R x y \rightarrow R y z \rightarrow R x z$ | 18 Abst |
| 20. | $\lambda y, z : S . \lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$ $: \Pi y, z : S . R x y \rightarrow R y z \rightarrow R x z$ | 19 Abst |
| 21. | $\lambda x, y, z : S . \lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$ $: \Pi x, y, z : S . R x y \rightarrow R y z \rightarrow R x z$ | 20 Abst |

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Completed Dec 22 6:51 pm.