

EXERCISES

CHAPTER 5

SEAN LI ¹

1. Redacted

Definition Some rules for reference.

λP Calculus Rules

$$\begin{array}{c} \frac{}{\emptyset \vdash * : \square} \text{Sort} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \text{Var} \quad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \text{Weak} \\ \\ \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A . B : s} \text{Form} \quad \frac{\Gamma \vdash M : \Pi x : A . B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B [x := N]} \text{App} \\ \\ \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A . B : s}{\Gamma \vdash \lambda x : A . M : \Pi x : A . B} \text{Abst} \\ \\ \frac{\Gamma \vdash A : B \quad B \stackrel{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \text{Conv} \end{array}$$

Predicate Logic

$$\begin{array}{ccc} \begin{array}{c} 1. \text{ Assume } A \\ 2. \quad \boxed{\dots} \\ 3. \quad \boxed{B} \end{array} & \frac{A \Rightarrow B \quad A}{B} \Rightarrow E & \begin{array}{c} 1. \text{ Let } a \in S \\ 2. \quad \boxed{\dots} \\ 3. \quad \boxed{P(a)} \end{array} \\ \hline \frac{}{A \Rightarrow B} \Rightarrow I & & \frac{}{\forall a \in S, P(a)} \forall I \end{array}$$

$$\frac{\forall a \in S \quad N \in S}{P(N)} \forall E$$

Problem

(5.1) Give a diagram of the tree corresponding to the complete tree derivation of line 18 of Section 5.3 (P 107)

Solution.



Problem

(5.2) Give a complete λP derivation of

$$S : * \vdash S \rightarrow S \rightarrow * : \square$$

In tree format and flag format.

Solution.

Tree Derivation.

$$\begin{array}{c}
 (4) \frac{\vdash * : \square \quad \vdash * : \square}{S : * \vdash * : \square} \text{ Weak} \\
 (6) \frac{(3) S : * \vdash S : * \quad (3) S : * \vdash S : *}{S : *, x : S \vdash * : \square} \text{ Weak} \\
 (7) \frac{(3) S : * \vdash S : *}{S : * \vdash S \rightarrow * : \square}
 \end{array}$$

$$(3) \frac{\vdash * : \square}{S : * \vdash S : *} \text{Var} \quad (9) \frac{(7) S : * \vdash S \rightarrow * : \square \quad (3) S : * \vdash S : *}{S : *, x : S \vdash S \rightarrow * : \square} \text{Weak}$$

$$\frac{}{S : * \vdash S \rightarrow S \rightarrow * : \square} \text{Form}$$

■

Flag Derivation.

| | | |
|-----|---|-----------------|
| 1. | $* : \square$ | Sort |
| 2. | $S : *$ | |
| 3. | $S : *$ | 1 Var |
| 4. | $* : \square$ | 1,1 Weak |
| 5. | $x : S$ | |
| 6. | $\boxed{* : \square}$ | 4,3 Weak |
| 7. | $S \rightarrow * : \square$ | 3,6 Form |
| 8. | $x : S$ | |
| 9. | $\boxed{S \rightarrow * : \square}$ | 7,3 Weak |
| 10. | $S \rightarrow S \rightarrow * : \square$ | 3,9 Form |

■

Problem

(5.3) Derive

$$S : *, Q : S \rightarrow S \rightarrow * \vdash \Pi x : S . \Pi y : S . Q x y : *$$

Solution.

| | | |
|-----|---|-----------------|
| 1. | $* : \square$ | Sort |
| 2. | $S : *$ | |
| 3. | $S : *$ | 1 Var |
| 4. | $* : \square$ | 1,1 Weak |
| 5. | $x : S$ | |
| 6. | $\boxed{* : \square}$ | 4,3 Weak |
| 7. | $S \rightarrow * : \square$ | 3,6 Form |
| 8. | $x : S$ | |
| 9. | $\boxed{S \rightarrow * : \square}$ | 7,3 Weak |
| 10. | $S \rightarrow S \rightarrow * : \square$ | 3,9 Form |
| 11. | $Q : S \rightarrow S \rightarrow *$ | |

| | | |
|-----|-------------------------------------|-------------------|
| 12. | $Q : S \rightarrow S \rightarrow *$ | 10 Var |
| 13. | $S : *$ | 3,10 Weak |
| 14. | $* : \square$ | 4,10 Weak |
| 15. | $x : S$ | |
| 16. | $* : \square$ | 14,13 Weak |
| 17. | $S : *$ | 13,13 Weak |
| 18. | $x : S$ | 13 Var |
| 19. | $Q : S \rightarrow S \rightarrow *$ | 12,13 Weak |
| 20. | $y : S$ | |
| 21. | $y : S$ | 17 Var |
| 22. | $Q : S \rightarrow S \rightarrow *$ | 19,17 Weak |
| 23. | $x : S$ | 18,17 Weak |
| 24. | $Q x : S \rightarrow *$ | 22,23 App |
| 25. | $Q x y : *$ | 24,21 App |
| 26. | $\Pi y : S . Q x y : *$ | 17,25 Form |
| 27. | $\Pi x : S . \Pi y : S . Q x y : *$ | 13,26 Form |

Problem

(5.4) Prove that $*$ is the only valid kind in λP .

Solution.

Proof. The only possible way to construct a new kind is through the Form rule and the Sort axiom. Because we are trying to construct a kind, s here stands for \square .

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : \square}{\Gamma \vdash \Pi x : A . B : \square} \text{Form}$$

One could only construct new kinds with kinds, which requires $A : \square$ and $B : \square$. This contradicts with $A : *$. ■

Problem

(5.5) Prove that $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ is a tautology by given a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash A \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

| | | |
|----|---|---------|
| 1. | $A : *$ | |
| 2. | $B : *$ | |
| 3. | $x : A$ | |
| 4. | $y : A \rightarrow B$ | |
| 5. | $\boxed{y x : B}$ | 4,3 App |
| 6. | $\lambda y : A \rightarrow B . y x : (A \rightarrow B) \rightarrow B$ | 5 Abst |
| 7. | $\boxed{\lambda x : A . \lambda y : A \rightarrow B . y x : A \rightarrow (A \rightarrow B) \rightarrow B}$ | 5 Abst |

■

Problem

(5.6 a) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ a tautology using natural deduction.

Solution.

Proof.

| | | |
|----|---|---------------------|
| 1. | Assume $A \Rightarrow (A \Rightarrow B)$ | |
| 2. | $A \Rightarrow (A \Rightarrow B)$ | |
| 3. | Assume A | |
| 4. | A | |
| 5. | $A \Rightarrow B$ | 2,4 $\Rightarrow E$ |
| 6. | B | 5,4 $\Rightarrow E$ |
| 7. | $\boxed{A \Rightarrow B}$ | 3,6 $\Rightarrow I$ |
| 8. | $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ | 1,7 $\Rightarrow I$ |

■

Problem

(5.6 b) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ using a shorthand λP derivation

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

| | | |
|----|---|---------|
| 1. | $A : *$ | |
| 2. | $B : *$ | |
| 3. | $x : A \rightarrow A \rightarrow B$ | |
| 4. | $y : A$ | |
| 5. | $x y : A \rightarrow B$ | 3,4 App |
| 6. | $x y y : B$ | 5,4 App |
| 7. | $\lambda y : A . x y y : A \rightarrow B$ | 6 Abst |
| 8. | $\lambda x : A \rightarrow A \rightarrow B . \lambda y : A . x y y : (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$ | 7 Abst |

■

Problem

(5.7 a) Proof $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of A , B , and C is equivalent to a proof of tautologousness.

Proof.

| | | |
|-----|---|---------|
| 1. | $A : *$ | |
| 2. | $B : *$ | |
| 3. | $C : *$ | |
| 4. | $x : A \rightarrow B$ | |
| 5. | $y : B \rightarrow C$ | |
| 6. | $a : A$ | |
| 7. | $x a : B$ | 4,6 App |
| 8. | $y (x a) : C$ | 5,7 App |
| 9. | $\lambda a : A . y (x a) : A \rightarrow C$ | 8 Abst |
| 10. | $\lambda y : B \rightarrow C . \lambda a : A . y (x a) : (B \rightarrow C) \rightarrow A \rightarrow C$ | 9 Abst |

$$11. \quad \boxed{\begin{array}{c} \lambda x : A \rightarrow B . \lambda y : B \rightarrow C . \lambda a : A . y(x z) \\ : (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C \end{array}} \quad \mathbf{10 \ Abst}$$

■

Problem

(5.7 b) Proof $((A \Rightarrow B) \Rightarrow A) \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

$$\begin{array}{ll} 1. & A : * \\ 2. & \boxed{B : *} \\ 3. & \boxed{x : (A \rightarrow B) \rightarrow A} \\ 4. & \boxed{y : A \rightarrow B} \\ 5. & \boxed{\begin{array}{l} x y : A \\ y(x y) : B \end{array}} \quad \mathbf{3,4 \ App} \\ 6. & \boxed{y(x y) : B} \quad \mathbf{4,5 \ App} \\ 7. & \boxed{\begin{array}{l} \lambda y : A \rightarrow B . y(x y) : (A \rightarrow B) \rightarrow B \\ \lambda x : (A \rightarrow B) \rightarrow A . \lambda y : A \rightarrow B . y(x y) \end{array}} \quad \mathbf{6 \ Abst} \\ 8. & \boxed{\begin{array}{l} \lambda x : (A \rightarrow B) \rightarrow A . \lambda y : A \rightarrow B . y(x y) \\ : ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B \end{array}} \quad \mathbf{7 \ Abst} \end{array}$$

■

Problem

(5.7 c) Proof $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of A , B , and C is equivalent to a proof of tautologousness.

Proof.

| | | |
|-----|---|----------------|
| 1. | $A : *$ | |
| 2. | $B : *$ | |
| 3. | $C : *$ | |
| 4. | $x : A \rightarrow B \rightarrow C$ | |
| 5. | $y : A \rightarrow B$ | |
| 6. | $a : A$ | |
| 7. | $x a : B \rightarrow C$ | 4,6 App |
| 8. | $y a : B$ | 5,6 App |
| 9. | $x a (y a) : C$ | 7,8 App |
| 10. | $\lambda a : A . x a (y a) : A \rightarrow C$ | 9 Abst |
| 11. | $\lambda y : A \rightarrow B . \lambda a : A . x a (y a) : (A \rightarrow B) \rightarrow A \rightarrow C$ | 10 Abst |
| | $\lambda x : A \rightarrow B \rightarrow C . \lambda y : A \rightarrow B . \lambda a : A . x a (y a)$ | |
| 12. | $: (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$ | 10 Abst |

■

Problem

(5.8 a) Let $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *,$ find an inhabitant of

$$\Pi x : S . P x \rightarrow Q x \rightarrow P x$$

with respect to Γ and give a shorthand λP derivation

Solution.

| | | |
|----|---|----------------|
| 1. | $S : *$ | |
| 2. | $P : S \rightarrow *$ | |
| 3. | $Q : S \rightarrow *$ | |
| 4. | $x : S$ | |
| 5. | $a : P x$ | |
| 6. | $b : Q x$ | |
| 7. | $a : P x$ | 2,4 App |
| 8. | $\lambda b : Q x . S : Q x \rightarrow P x$ | 7 Abst |
| 9. | $\lambda a : P x . \lambda b : Q x . a : P x \rightarrow Q x \rightarrow P x$ | 8 Abst |

$$10. \quad \boxed{\begin{array}{l} \lambda x : S . \lambda a : P x . \lambda b : Q x . a \\ : \Pi x : S . P x \rightarrow Q x \rightarrow P x \end{array}} \quad \mathbf{9 \ Abst}$$

Problem

(5.8 b) Let $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *$, find an inhabitant of
 $\Pi x : S . P x \rightarrow Q x \rightarrow P x$

By proving the corresponding proposition in natural deduction.

Solution. The corresponding proposition and premises are

$$\frac{S \in \text{Set} \quad P : S \rightarrow \text{Prop} \quad Q : S \rightarrow \text{Prop}}{\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a))}$$

Proof.

1. Let $a \in S$
2. Assume $P(a)$
3. Assume $Q(a)$
4. $P(a)$
5. $\boxed{Q(a) \Rightarrow P(a)} \quad 3,4 \Rightarrow I$
6. $P(a) \Rightarrow (Q(a) \Rightarrow P(a)) \quad 2,5 \Rightarrow I$
7. $\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a)) \quad 1,6 \forall I$

■

Problem

(5.9 a) Give proof for

$$(\forall x \in S, Q(x)) \Rightarrow (\forall y \in S, P(y) \Rightarrow Q(y))$$

by natural deduction and a λP derivation.

Solution.

Natural Deduction.

1. Assume $\forall x \in S, Q(x)$
2. Let $y \in S$

| | | |
|----|--|---------------------|
| 3. | Assume $P(y)$ | |
| 4. | $\boxed{Q(y)}$ | 1,2 $\forall E$ |
| 5. | $\boxed{P(y) \Rightarrow Q(y)}$ | 3,4 $\Rightarrow I$ |
| 6. | $\forall y \in S, P(y) \Rightarrow Q(y)$ | 2,5 $\forall I$ |
| 7. | $(\forall x \in S, Q(x)) \Rightarrow (\forall y \in S, P(y) \Rightarrow Q(y))$ | 1,6 $\Rightarrow I$ |

■

λP Derivation. Corresponding type is

$$S : *, P : S \rightarrow *, Q : S \rightarrow * \vdash (\Pi x : S . Q x) \rightarrow (\Pi y : S . P y \rightarrow Q y)$$

| | | |
|-----|---|---------|
| 1. | $S : *$ | |
| 2. | $P : S \rightarrow *$ | |
| 3. | $Q : S \rightarrow *$ | |
| 4. | $a : \Pi x : S . Q x$ | |
| 5. | $y : S$ | |
| 6. | $z : P y$ | |
| 7. | $\boxed{a y : Q y}$ | 4,5 App |
| 8. | $\boxed{\lambda z : P y . a y : P y \rightarrow Q y}$ | 7 Abst |
| 9. | $\boxed{\lambda y : S . \lambda z : P y . a y : \Pi y : S . P y \rightarrow Q y}$ | 7 Abst |
| 10. | $\boxed{\lambda a : \Pi x : S . Q x . \lambda y : S . \lambda z : P y . a y : (\Pi x : S . Q x) \rightarrow (\Pi y : S . P y \rightarrow Q y)}$ | 7 Abst |

■

Problem

(5.9 b) Give proof for

$$\forall x \in S, (P(x) \Rightarrow Q(x)) \Rightarrow ((\forall y \in S, P(y)) \Rightarrow (\forall z \in S, Q(z)))$$

by natural deduction and a λP derivation

Solution.

Natural Deduction.

| | | |
|----|---|--|
| 1. | Assume $\forall x \in S, (P(x) \Rightarrow Q(x))$ | |
| 2. | Assume $\forall y \in S, P(y)$ | |
| 3. | Let $z \in S$ | |

| | | |
|----|--|---------------------|
| 4. | $P(z)$ | 2,3 $\forall E$ |
| 5. | $P(z) \Rightarrow Q(z)$ | 1,3 $\forall E$ |
| 6. | $\boxed{Q(z)}$ | 5,4 $\Rightarrow E$ |
| 7. | $\boxed{\forall z \in S, Q(z)}$ | 3,6 $\forall I$ |
| 8. | $\boxed{\forall y \in S, P(y) \Rightarrow (\forall z \in S, Q(z))}$ | 2,7 $\forall I$ |
| 9. | $\forall x \in S, (P(x) \Rightarrow Q(x)) \Rightarrow ((\forall y \in S, P(y)) \Rightarrow (\forall z \in S, Q(z)))$ | 1,8 $\forall I$ |

■

λP Derivation. Corresponding type is

$$\begin{aligned} & S : *, P : S \rightarrow *, Q : S \rightarrow * \\ & \vdash (\Pi x : S . P x \rightarrow Q x) \rightarrow (\Pi y : S . P y) \rightarrow (\Pi z : S . Q z) \end{aligned}$$

| | | |
|-----|---|---------|
| 1. | $S : *$ | |
| 2. | $P : S \rightarrow *$ | |
| 3. | $Q : S \rightarrow *$ | |
| 4. | $a : \Pi x : S . P x \rightarrow Q x$ | |
| 5. | $b : \Pi y : S . P y$ | |
| 6. | $z : S$ | |
| 7. | $b z : P z$ | 5,6 App |
| 8. | $a z : P z \rightarrow Q z$ | 4,6 App |
| 9. | $\boxed{a z (b z) : Q z}$ | 8,7 App |
| 10. | $\boxed{\lambda z : S . a z (b z) : \Pi z : S . Q z}$ | 9 Abst |
| | $\lambda b : (\Pi y : S . P y) . \lambda z : S . a z (b z)$ | |
| 11. | $\vdash : (\Pi y : S . P y) \rightarrow \Pi z : S . Q z$ | 10 Abst |
| | $\lambda a : (\Pi x : S . P x \rightarrow Q x) . \lambda b : (\Pi y : S . P y) .$ | |
| | $\lambda z : S . a z (b z)$ | |
| | $\vdash : (\Pi x : S . P x \rightarrow Q x) \rightarrow$ | |
| 12. | $\vdash (\Pi y : S . P y) \rightarrow \Pi z : S . Q z$ | 10 Abst |

■

Problem

(5.10) Given a context

$$\begin{aligned}\Gamma \equiv & S : *, P : S \rightarrow *, f : S \rightarrow S, g : S \rightarrow S, \\ & u : \Pi x : S . (P(f x) \rightarrow P(g x)), \\ & v : \Pi x, y : S . ((P x \rightarrow P y) \rightarrow P(f x))\end{aligned}$$

Let

$$M \equiv \lambda x : S . v(f x)(g x)(u x)$$

Type M under Γ .

Solution.

- | | | |
|-----|---|-----------------|
| 1. | $x : S$ | |
| 2. | $f : S \rightarrow S$ | |
| 3. | $f x : S$ | 2,1 App |
| 4. | $g : S \rightarrow S$ | |
| 5. | $g x : S$ | 4,1 App |
| 6. | $u : \Pi x : S . (P(f x) \rightarrow P(g x))$ | |
| 7. | $u x : P(f x) \rightarrow P(g x)$ | 6,1 App |
| 8. | $v : \Pi x, y : S . ((P x \rightarrow P y) \rightarrow P(f x))$ | |
| 9. | $v(f x) : \Pi y : S . ((P(f x) \rightarrow P y) \rightarrow P(f(f x)))$ | 8,3 App |
| 10. | $v(f x)(g x) : (P(f x) \rightarrow P(g x)) \rightarrow P(f(f x))$ | 9,5 App |
| 11. | $v(f x)(g x)(u x) : P(f(f x))$ | 10,7 App |
| 12. | $\lambda x : S . v(f x)(g x)(u x) : S \rightarrow P(f(f x))$ | 11 Abst |

Problem

(5.11) Let S be a set, with Q and R relations on $S \times S$, and let f and g be functions from S to S . Assume

$$\begin{aligned}\forall x, y \in S (Q(x, f(y)) \Rightarrow Q(g(x), y)) \\ \forall x, y \in S (Q(x, f(y)) \Rightarrow R(x, y)) \\ \forall x \in S (Q(x, f(f(x))))\end{aligned}$$

Prove that

$$\forall x \in S, R(g(g(x)), g(x))$$

By giving a context Γ and finding a term M such that

$$\Gamma \vdash M : \Pi x : S . R(g(g x))(g x)$$

Solution. Context Γ is as follows:

$$\begin{aligned}\Gamma \equiv & S : *, f : S \rightarrow S, g : S \rightarrow S \\ & Q : S \rightarrow S \rightarrow *, R : S \rightarrow S \rightarrow * \\ & A : \Pi x, y : S . (Q x (f y) \rightarrow Q (g x) y), \\ & B : \Pi x, y : S . (Q x (f y) \rightarrow R x y) \\ & C : \Pi x : S . Q x (f (f x))\end{aligned}$$

Derivation.

1. $S : *, f : S \rightarrow S, g : S \rightarrow S$
2. $Q : S \rightarrow S \rightarrow *, R : S \rightarrow S \rightarrow *$
3. $A : \Pi x, y : S . (Q x (f y) \rightarrow Q (g x) y)$
4. $B : \Pi x, y : S . (Q x (f y) \rightarrow R x y)$
5. $C : \Pi x : S . Q x (f (f x))$
6. $x : S$
7. $g x : S$ **1,6 App**
8. $C(g x) : Q(g x)(f(f(g x)))$ **5,7 App**
9. $f(g x) : S$ **1,7 App**
10. $A(g x) : \Pi y : S . (Q(g x)(f y)) \rightarrow (Q(g(g x)) y)$ **3,7 App**
11.
$$\begin{aligned}A(g x)(f(g x)) \\ : (Q(g x)(f(f(g x)))) \\ \rightarrow (Q(g(g x))(f(g x)))\end{aligned}$$
 10,9 App

| | | |
|-----|---|------------------|
| | $A(gx)(fx)(Cx)$ | |
| 12. | $: (Q(g(gx))(fx))$ | 11,8 App |
| 13. | $g(gx) : S$ | 1,7 App |
| | $B(g(gx))$ | |
| | $: \Pi y : S . (Q(g(gx))(fy))$ | |
| 14. | $\rightarrow R(g(gx))y$ | 4,13 App |
| | $B(g(gx))(gx)$ | |
| 15. | $: (Q(g(gx))(fx)) \rightarrow (R(g(gx))(gx))$ | 14,7 App |
| | $B(g(gx))(gx)(Ax)(fx)(Cx)$ | |
| 16. | $: (R(g(gx))(gx))$ | 15,12 App |
| 17. | $\lambda x : S . B(g(gx))(gx)(Ax)(fx)(Cx)$ | |
| | $: \Pi x : S . (R(g(gx))(gx))$ | 17 Abst |

■

Problem

(5.12 a) In λP , consider the context

$$\begin{aligned}\Gamma &\equiv S : *, R : S \rightarrow S \rightarrow *, \\ u &: \Pi x, y : S . R x y \rightarrow R y x \\ v &: \Pi x, y, z : S . R x y \rightarrow R x z \rightarrow R y z\end{aligned}$$

Show that R is reflexive over $S \times S$. That is, construct M such that

$$\Gamma \vdash M : \Pi x, y : S . R x y \rightarrow R x x$$

Solution.

Proof.

1. $S : *, R : S \rightarrow S \rightarrow *$
2. $A : \Pi u, v : S . R u v \rightarrow R v u$
3. $B : \Pi u, v, w : S . R u v \rightarrow R u w \rightarrow R v w$
4. $x : S$
5. $y : S$
6. $h : R x y$

| | | |
|-----|---|-----------|
| 7. | $B x : \Pi v, w : S . R x v \rightarrow R v w \rightarrow R x w$ | 3,4 App |
| 8. | $B x y : \Pi w : S . R x y \rightarrow R y w \rightarrow R x w$ | 7,5 App |
| 9. | $B x y x : R x y \rightarrow R y x \rightarrow R x x$ | 7,5 App |
| 10. | $A x : \Pi v : R x v \rightarrow R v x$ | 2,4 App |
| 11. | $A x y : R x y \rightarrow R y x$ | 10,5 App |
| 12. | $A x y h : R y x$ | 11,6 App |
| 13. | $B x y x h : R y x \rightarrow R x x$ | 9,6 App |
| 14. | $B x y x h (A x y h) : R x x$ | 13,12 App |
| 15. | $\lambda h : R x y . B x y x h (A x y h) : R x y \rightarrow R x x$ | 14 Abst |
| | $\lambda y : S . \lambda h : R x y . B x y x h (A x y h)$ | |
| 16. | $: \Pi y : S . R x y \rightarrow R x x$ | 15 Abst |
| | $\lambda x, y : S . \lambda h : R x y . B x y x h (A x y h)$ | |
| 17. | $: \Pi x, y : S . R x y \rightarrow R x x$ | 16 Abst |

■

Problem

(5.12 b) Given the context Γ in 5.12 a, prove transitivity of R by constructing M such that

$$\Gamma \vdash M : \Pi x, y, z : S . R x y \rightarrow R y z \rightarrow R x z$$

Solution.

Proof.

| | | |
|-----|---|----------|
| 1. | $S : *, R : S \rightarrow S \rightarrow *$ | |
| 2. | $A : \Pi u, v : S . R u v \rightarrow R v u$ | |
| 3. | $B : \Pi u, v, w : S . R u v \rightarrow R u w \rightarrow R v w$ | |
| 4. | $x : S$ | |
| 5. | $y : S$ | |
| 6. | $z : S$ | |
| 7. | $h : R x y$ | |
| 8. | $r : R y z$ | |
| 9. | $A x : \Pi v : S . R x v \rightarrow R v x$ | 2,4 App |
| 10. | $A x y : R x y \rightarrow R y x$ | 9,5 App |
| 11. | $A x y h : R y x$ | 10,7 App |

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|-----|--|--|-----------|
| 12. | | $B y : \Pi v, w : S . R y v \rightarrow R y w \rightarrow R v w$ | 3,5 App |
| 13. | | $B y x : \Pi w : S . R y x \rightarrow R y w \rightarrow R x w$ | 12,4 App |
| 14. | | $B y x z : R y x \rightarrow R y z \rightarrow R x z$ | 12,4 App |
| 15. | | $B y x z (A x y h) : R y z \rightarrow R x z$ | 14,11 App |
| 16. | | $B y x z (A x y h) r : R x z$ | 15,8 App |
| 17. | | $\lambda r : R y z . B y x z (A x y h) r : R y z \rightarrow R x z$ | 16 Abst |
| 18. | | $\lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$: $R x y \rightarrow R y z \rightarrow R x z$ | 17 Abst |
| 19. | | $\lambda z : S . \lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$: $\Pi z : S . R x y \rightarrow R y z \rightarrow R x z$ | 18 Abst |
| 20. | | $\lambda y, z : S . \lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$: $\Pi y, z : S . R x y \rightarrow R y z \rightarrow R x z$ | 19 Abst |
| 21. | | $\lambda x, y, z : S . \lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$: $\Pi x, y, z : S . R x y \rightarrow R y z \rightarrow R x z$ | 20 Abst |

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Completed Dec 22 6:51 pm.