# Exercises

## Chapter 7

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1. Reducted

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**Reference - Propositional Logic in $\lambda C$**

$$\frac{A \quad \neg A}{\bot}\,\bot\text{I or }\neg\text{E} \qquad \frac{\bot}{A}\,\bot\text{E} \qquad \frac{A \quad B}{A \wedge B}\,\wedge\text{I} \qquad \frac{A \wedge B}{A}\,\wedge\text{EL}$$

$$\frac{A \wedge B}{B}\,\wedge\text{ER} \qquad \frac{a}{a \vee b}\,\vee\text{IL} \qquad \frac{b}{a \vee b}\,\vee\text{IR} \qquad \frac{A \Rightarrow B \quad A}{B}\,\Rightarrow\text{E}$$

$$\frac{A \vee B \quad A \to C \quad B \to C}{C}\,\vee\text{E} \qquad \frac{a \in S \quad P(a)}{\exists a \in S, P(a)}\,\exists\text{I}$$

$$\frac{\begin{array}{ll} 1. & A \\ 2. & \quad \dots \\ 3. & \quad \bot \end{array}}{\neg A}\,\neg\text{I} \qquad \frac{\begin{array}{ll} 1. & A \\ 2. & \quad \dots \\ 3. & \quad B \end{array}}{A \Rightarrow B}\,\Rightarrow\text{I} \qquad \frac{\begin{array}{ll} 1. & a \in S \\ 2. & \quad \dots \\ 3. & \quad P(a) \end{array}}{\forall a \in S, P(a)}\,\forall I$$

$$\frac{\exists x \in S, P(x) \quad \forall x \in S, (P(x) \Rightarrow A)}{A}\,\exists\text{E} \qquad \frac{a \in S \quad \forall x \in S, P(x)}{P(a)}\,\forall\text{E}$$

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**Reference - 2nd Encoding for Propositional Logic**

| **Proposition** | **Minimal Propositional Logic** |
|---|---|
| $\bot$ | $\forall A, A$ |
| $A \Rightarrow B$ | $A \Rightarrow B$ |
| $\neg A$ | $A \Rightarrow \bot$ |
| $A \wedge B$ | $\forall C, (A \Rightarrow B \Rightarrow C) \Rightarrow C$ |
| $A \vee B$ | $\forall C, (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$ |
| $\forall a \in S, P(a)$ | $\forall a \in S \, . \, P(a)$ |
| $\exists a \in S, P(a)$ | $\forall \alpha, (\forall a \in S, (P(a) \Rightarrow \alpha)) \Rightarrow \alpha$ |

**Problem**

*(7.1 a)* Prove in natural deduction and $\lambda C$ the tautology

$$B \Rightarrow (A \Rightarrow B)$$

*Solution.*

*Natural Deduction.*

| | | |
|---|---|---|
| 1. | $B$ | |
| 2. | $\quad A$ | |
| 3. | $\quad\quad B$ | |
| 4. | $\quad A \Rightarrow B$ | $\Rightarrow$**I** |
| 5. | $B \Rightarrow (A \Rightarrow B)$ | $\Rightarrow$**I** |

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*$\lambda C$.* Assuming context $\Gamma \equiv A : *, B : *$. By the PAT paradigm the proof is equivalent to an inhabitant of $B \to A \to B$.

| | | |
|---|---|---|
| 1. | $A : *, B : *$ | |
| 2. | $\quad x : B$ | |
| 3. | $\quad\quad y : A$ | |
| 4. | $\quad\quad\quad x : B$ | **Weak** |
| 5. | $\quad\quad \lambda y : A \, . \, x : A \to B$ | **4 Abst** |
| 6. | $\quad \lambda x : B \, . \lambda y : A \, . \, x : B \to A \to B$ | **5 Abst** |

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**Problem**

*(7.1 b)* Prove in natural deduction and $\lambda C$ the tautology

$$\neg A \Rightarrow (A \Rightarrow B)$$

*Solution.*

*Natural Deduction.*

| | | |
|---|---|---|
| 1. | $\neg A$ | |
| 2. | $\quad A$ | |
| 3. | $\quad\quad \neg A$ | |
| 4. | $\quad\quad A$ | |
| 5. | $\quad\quad \bot$ | $\bot$**I** |
| 6. | $\quad\quad B$ | $\bot$**E** |
| 7. | $\quad A \Rightarrow B$ | $\Rightarrow$**I** |
| 8. | $\neg A \Rightarrow (A \Rightarrow B)$ | $\Rightarrow$**I** |

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$\lambda C$. Assuming context $\Gamma \equiv A : *, B : *$. By the PAT paradigm the proof is equivalent to an inhabitant of $(A \to \bot) \to A \to B$.

| | | |
|---|---|---|
| 1. | $A : *, B : *$ | |
| 2. | $\quad x : \neg A$ | |
| 3. | $\quad\quad y : A$ | |
| 4. | $\quad\quad\quad x\, y : \Pi\alpha : * \,.\, \alpha$ | **2,3 App (Neg Elim)** |
| 5. | $\quad\quad\quad x\, y\, B : B$ | **4,1 App (Ex Falso)** |
| 6. | $\quad\quad \lambda y : A \,.\, x\, y\, B : A \to B$ | **5 Abst** |
| 7. | $\quad \lambda x : \neg A \,.\lambda y : A \,.\, x\, y\, B : \neg A \to A \to B$ | **6 Abst** |

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**Problem**

*(7.1 c)* Prove in natural deduction and $\lambda C$ the tautology

$$(A \Rightarrow \neg B) \Rightarrow ((A \Rightarrow B) \Rightarrow \neg A)$$

*Solution.*

*Natural Deduction.*

| | | |
|---|---|---|
| 1. | $A \Rightarrow \neg B$ | |
| 2. | $\quad A \Rightarrow B$ | |
| 3. | $\quad\quad A$ | |
| 4. | $\quad\quad\quad \neg B$ | **1,3 ⇒E** |
| 5. | $\quad\quad\quad B$ | **2,3 ⇒E** |
| 6. | $\quad\quad\quad \perp$ | **5,4 ⊥I** |
| 7. | $\quad\quad \neg A$ | **3,6 ¬I** |
| 8. | $\quad (A \Rightarrow B) \Rightarrow \neg A$ | **2,7 ⇒I** |
| 9. | $(A \Rightarrow \neg B) \Rightarrow ((A \Rightarrow B) \Rightarrow \neg A)$ | **1,8 ⇒I** |

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$\lambda C$. Assuming context $\Gamma \equiv A : *, B : *$. The proof should be equivalent to an inhabitant of $(A \to B \to \perp) \to (A \to B) \to A \to \perp$.

| | | |
|---|---|---|
| 1. | $A : *, B : *$ | |
| 2. | $\quad h : A \to \neg B$ | |
| 3. | $\quad\quad q : A \to B$ | |
| 4. | $\quad\quad\quad a : A$ | |
| 5. | $\quad\quad\quad\quad q\, a : B$ | **3,4 App** |
| 6. | $\quad\quad\quad\quad h\, a : B \to \perp$ | **2,4 App** |
| 7. | $\quad\quad\quad\quad h\, a\, (q\, a) : \perp$ | **6,5 App (Neg Elim)** |
| 8. | $\quad\quad\quad \lambda a : A\, .\, h\, a\, (q\, a) : \neg A$ | **7 Abst (Neg Intro)** |
| 9. | $\quad\quad \lambda q : A \to B\, .\lambda a : A\, .\, h\, a\, (q\, a) : A \to B \to \neg A$ | **8 Abst** |
| 10. | $\quad \lambda h : A \to \neg B\, .\lambda q : A \to B\, .\lambda a : A\, .\, h\, a\, (q\, a)$ $: (A \to \neg B) \to A \to B \to \neg A$ | **9 Abst** |

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**Problem**

*(7.1 d)* Prove in natural deduction and $\lambda C$ the tautology

$$\neg(A \Rightarrow B) \Rightarrow \neg B$$

*Solution.*

*Natural Deduction.*

1. $\neg(A \Rightarrow B)$
2. $B$
3. $A$
4. $B$
5. $A \Rightarrow B$ **3,4 $\Rightarrow$I**
6. $\bot$ **5,1 $\bot$I**
7. $\neg\, B$ **6 $\bot$E**
8. $\neg(A \Rightarrow B) \Rightarrow \neg\, B$ **1,7 $\Rightarrow$I**

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$\lambda C$. Assuming context $\Gamma \equiv A : *, B : *$. The proof should be equivalent to an inhabitant of $((A \to B) \to \bot) \to B \to \bot$.

1. $n : \neg(A \to B)$
2. $b : B$
3. $a : A$
4. $b : A$ **Weak**
5. $\lambda a : A \,.\, b : A \to B$ **4 Abst**
6. $n\ (\lambda a : A \,.\, b) : \bot$ **1,5 App (Neg Elim)**
7. $\lambda b : B \,.\, n\ (\lambda a : A \,.\, b) : \neg\, B$ **6 Abst (Neg Intro)**
8.

$\lambda n : \neg(A \to B).\lambda b : B \,.\, n\ (\lambda a : A \,.\, b)$
$\quad : \neg(A \to B) \to \neg\, B$ **7 Abst**

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