Excercises

Chapter 1

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1. Reducted

Problem

(1.1) Simplify notation of the following terms

$$\text{(a)}\ (\lambda x\ .(((x\ z)\ y)(x\ x)))$$
$$\text{(b)}\ ((\lambda x\ .(\lambda y\ .(\lambda z\ .(z\ ((x\ y)\ z)))))(\lambda u\ .\ u))$$

Solution.

$$\text{(a)}\ \lambda x\ .(x\ z\ y)(x\ x)$$
$$\text{(b)}\ (\lambda x\ y\ z\ .\ x\ (x\ y\ z))(\lambda u\ .\ u)$$

Problem

(1.2) Find the alpha equivalent terms to

$$\lambda x\ .\ x\ (\lambda x\ .\ x)$$

In

$$\text{(a)}\ \lambda y\ .\ y\ (\lambda x\ .\ x)$$
$$\text{(b)}\ \lambda y\ .\ y\ (\lambda x\ .\ y)$$
$$\text{(c)}\ \lambda y\ .\ y\ (\lambda x\ .\ y)$$

Solution. Only (a).

Problem

(1.3) Prove

$$\lambda x \,.\, x\,(\lambda z \,.\, y) \underset{\alpha}{=} \lambda z \,.\, z\,(\lambda z \,.\, y)$$

Solution.

Proof. By definition of alpha equivalence

$$M \underset{\alpha}{=} N \iff \exists \varphi, M^{\varphi} \underset{\alpha}{\to} N \wedge \mathrm{FR}\, M = \mathrm{FR}\, N$$

The witness of φ is substituting bound variable x with z, and z is not a free variable in the term, thus the two terms are alpha equivalent.

$$\lambda x \,.\, x\,(\lambda z \,.\, y)^{x \to z} \underset{\alpha}{\to} \lambda z \,.\, z\,(\lambda z \,.\, y)$$

■

Problem

(1.4) Consider the following term:

$$U := (\lambda z \,.\, z\,x\,z)((\lambda y \,.\, x\,y)\,x)$$

1. Find Sub U
2. Draw tree rep of U
3. Find FV U
4. Find alpha equivalent terms to U from below and point out which of those follows the Barendregt convention:

   (a) $(\lambda y \,.\, y\,x\,y)((\lambda z \,.\, x\,z)\,x)$

   (b) $(\lambda x \,.\, x\,y\,x)((\lambda z \,.\, y\,z)\,y)$

   (c) $(\lambda y \,.\, y\,x\,y)((\lambda y \,.\, x\,y)\,x)$

   (d) $(\lambda v \,.(v\,x)\,v)((\lambda u \,.\, u\,v)\,x)$

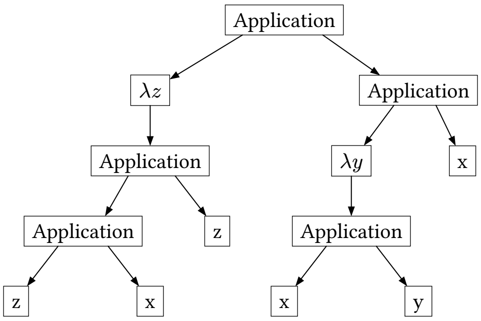
1. Find Sub U.

Solution.

$$
\begin{aligned}
\text{Sub}\, U =& \\
&\{(\lambda z\,.\,z\;x\;z)((\lambda y\,.\,x\;y)\;x), (\lambda z\,.\,z\;x\;z), ((\lambda y\,.\,x\;y)\;x)\} \cup \\
&\{(\lambda y\,.\,x\;y), (y), (\lambda z\,.\,x\;z), (x)\} \cup \\
&\{(\lambda y\,.\,x), (y)\} \cup \{(\lambda z\,.\,x), (z)\} \cup \{(y), (x)\} \\
=& \{(\lambda z\,.\,z\;x\;z)(\lambda y\,.\,x\;y)\;x, (\lambda z\,.\,z\;x\;z), (\lambda y\,.\,x\;y)\;x, \\
&(\lambda y\,.\,x\;y), (\lambda z\,.\,x\;z), (\lambda y\,.\,x), (\lambda z\,.\,x), y, x\}
\end{aligned}
$$

2. Draw a tree rep of U.

Solution.



3. Find FV U

Solution.

$$
\begin{aligned}
\text{FV}\, U &= \text{FV}\;(\lambda y\,.\,y\;x\;y) \cup \text{FV}\;(\lambda z\,.\,x\;z)\;x \\
&= (\text{FV}\;y\;x\;y) \setminus \{y\} \cup (\text{FV}\;\lambda z\,.\,x\;z) \cup \{x\} \\
&= (\text{FV}\;y\;x) \setminus \{y\} \cup (\text{FV}\;x\;z) \setminus \{z\} \cup \{x\} \\
&= \{x\}
\end{aligned}
$$

4. Find an alpha-equivalent term.

Solution.

$$(a) \underset{\alpha}{=} (c) \underset{\alpha}{=} U$$

Only (a) follows the Barendregt convention.

Problem

(1.5) Give the results of the following substitutions

$$
\begin{aligned}
&\text{(a)}\;\; (\lambda x\,.\,y\;(\lambda y\,.\,x\;y))[y := \lambda z\,.\,z\;x] \\
&\text{(b)}\;\; ((x\;y\;z)[x := y])[y := z] \\
&\text{(c)}\;\; ((\lambda x\,.\,x\;y\;z)[x := y])[y := z] \\
&\text{(d)}\;\; (\lambda y\,.\,y\;y\;x)[x := y\;z]
\end{aligned}
$$

Solution.

$$\begin{aligned}
&\text{(a)}\ (\lambda v\,.(\lambda z\,.\,z\,x)(\lambda u\,.\,v\,u))\\
&\text{(b)}\ (y\,y\,z)[y := z] = z\,z\,z\\
&\text{(c)}\ (\lambda x\,.\,x\,y\,z)[y := z] = (\lambda x\,.\,x\,z\,z)\\
&\text{(d)}\ (\lambda u\,.\,u\,u\,(y\,z))
\end{aligned}$$

Problem

(1.6)

$$\neg\Big(\forall\ M\ L\ N \in \Lambda, M\ [x := N, y := L] \underset{\alpha}{\equiv} M\ [x := N][y := L]\Big)$$

Solution.

Proof. Because RHS $= M\ [x := N][y := L] = M\ [x := N\ [y := L]][y := L]$, if $y \in \mathrm{FV}\ N$, then what x gets substituted with will have y substituted for L, which is completely different with LHS. ■

Problem

(1.7) Find all available redexes in

$$U := (\lambda z\,.\,z\,x\,z)((\lambda y\,.\,x\,y)\,x)$$

And all reduction pathes to the β-normal form.

Solution. The first redex is the term as an application itself; another the second term in the application.

$$\begin{tikzcd}
& \boxed{(\lambda z\,.\,z\,x\,z)((\lambda y\,.\,x\,y)\,x)} \arrow[dl] \arrow[dr] & & \\
\boxed{(\lambda z\,.\,z\,x\,z)(x\,x)} \arrow[d] & & \boxed{((\lambda y\,.\,x\,y)\,x)\,x\,((\lambda y\,.\,x\,y)\,x)} \arrow[dl] \arrow[dr] & \\
\boxed{x\,x\,x\,(x\,x)} & \boxed{x\,x\,x\,((\lambda y\,.\,x\,y)\,x)} \arrow[d] & & \boxed{((\lambda y\,.\,x\,y)\,x)\,x\,(x\,x)} \arrow[d] \\
& \boxed{x\,x\,x\,(x\,x)} & & \boxed{x\,x\,x\,(x\,x)}
\end{tikzcd}$$

Problem

(1.8) Show that

$$(\lambda x\ .\ x\ x)\ y \underset{\beta}{\neq} (\lambda x\ y\ .\ y\ x)\ x\ x$$

Solution. By Collorary 1.9.9, it suffices to prove the hypothesis with a proof of a common normal reducted form from LHS and RHS not existing.

Contradiction. By definition of $\underset{\beta}{=}$, there exists The set of all terms attainable from β -reduction on $(\lambda x\ .\ x\ x)\ y$ and $(\lambda x\ y\ .\ y\ x)\ x\ x$ do not intersect. Therefore,

$$\neg\left(\exists L \in \Lambda, (\lambda x\ .\ x\ x)\ y \underset{\beta}{\twoheadrightarrow} L \land (\lambda x\ y\ .\ y\ x)\ x\ x \underset{\beta}{\twoheadrightarrow} L\right) \implies \neg\left((\lambda x\ .\ x\ x)\ y \underset{\beta}{=} (\lambda x\ y\ .\ y\ x)\ x\ x\right)$$

■

Problem

(1.9) Define the combinators

$$K := \lambda x\ y\ .\ x$$
$$S := \lambda x\ y\ z\ .\ x\ z\ (y\ z)$$

Prove that

$$\forall\ P\ Q \in \Lambda, K\ P\ Q \underset{\beta}{\twoheadrightarrow} P$$

$$\forall\ P\ Q\ R \in \Lambda, S\ P\ Q\ R \underset{\beta}{\twoheadrightarrow} P\ R\ (Q\ R)$$

Solution.

Proof.

$$K\ P\ Q = (\lambda x\ y\ .\ x)\ P\ Q \underset{\beta}{\rightarrow} (\lambda y\ .\ x)[x := P]\ Q \underset{\beta}{\rightarrow} P\ [y := Q] = P$$

$$S\ P\ Q\ R = (\lambda x\ y\ z\ .\ x\ z\ (y\ z)) \underset{\beta}{\twoheadrightarrow} (x\ z\ (y\ z))[x := P][y := Q][z := R] = P\ R\ (Q\ R)$$

■

Problem

(1.10) We define the church numerals

$$\begin{aligned} \text{zero} &:= \lambda f\ x\ .\ x \\ \text{one} &:= \lambda f\ x\ .\ f\ x \\ \text{two} &:= \lambda f\ x\ .\ f\ f\ x \\ &\cdots \\ \text{num}_n &:= \lambda f\ x\ .\ f^n\ x \end{aligned}$$

And operations

$$\begin{aligned} \text{add} &:= \lambda n\ m\ f\ x\ .\ m\ f\ (n\ f\ x) \\ \text{mul} &:= \lambda n\ m\ f\ x\ .\ m\ (n\ f)\ x \end{aligned}$$

Show

$$\text{(a) add one one} \twoheadrightarrow_{\beta} \text{two}$$

$$\text{(b) add one one} \neq_{\beta} \text{mul one zero}$$

Solution.

$$\begin{aligned} \text{(a)} \quad \text{add one one} &= (\lambda n\ m\ f\ x\ .\ m\ f\ (n\ f\ x))(\lambda f\ x\ .\ f\ x)(\lambda f\ x\ .\ f\ x) \\ &\twoheadrightarrow_{\beta} (\lambda f\ x\ .(\lambda f\ x\ .\ f\ x)\ f\ ((\lambda f\ x\ .\ f\ x)\ f\ x)) \\ &\twoheadrightarrow_{\beta} (\lambda f\ x\ .(\lambda x\ .\ f\ x)\ f\ x) \\ &\twoheadrightarrow_{\beta} (\lambda f\ x\ .\ f\ f\ x) = \text{two} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \text{mul one one} &= (\lambda n\ m\ f\ x\ .\ m\ (n\ f)\ x)(\lambda f\ x\ .\ f\ x)(\lambda f\ x\ .\ f\ x) \\ &\twoheadrightarrow_{\beta} \lambda f\ x\ .(\lambda f\ x\ .\ f\ x)((\lambda f\ x\ .\ f\ x)\ f)\ x \\ &\twoheadrightarrow_{\beta} \lambda f\ x\ .\ f\ x = \text{one} \end{aligned}$$

Because no intermediate form in the beta reduction process of the two terms are α-equivalent, by collorary 1.9.9 the two terms are not β-equivalent.

Problem

(1.11) We define

$$\text{succ} := \lambda m\ f\ x\ .\ f\ (m\ f\ x)\ \text{ s.t. } \forall \text{num}_n, \text{succ num}_n = \text{num}_{n+1}$$

Prove

$$\text{succ zero} \underset{\beta}{=} \text{one}$$

$$\text{succ one} \underset{\beta}{=} \text{two}$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$\begin{aligned}\text{succ zero} &= (\lambda m\ f\ x\ .\ f\ (m\ f\ x))(\lambda f\ x\ .\ x)\\ &\underset{\beta}{\to} (\lambda f\ x\ .\ f\ ((\lambda f\ x\ .\ x)\ f\ x))\\ &\underset{\beta}{\twoheadrightarrow} (\lambda f\ x\ .\ f\ x) = \text{one}\end{aligned}$$

The path $\text{succ zero} \underset{\beta}{\twoheadrightarrow} \text{one}$ derived above is the witness of a reduction chain from LHS to RHS.

$$\begin{aligned}\text{succ one} &= (\lambda m\ f\ x\ .\ f\ (m\ f\ x))(\lambda f\ x\ .\ f\ x)\\ &\underset{\beta}{\to} (\lambda f\ x\ .\ f\ ((\lambda f\ x\ .\ f\ x)\ f\ x))\\ &\underset{\beta}{\twoheadrightarrow} (\lambda f\ x\ .\ f\ (f\ x)) = \text{two}\end{aligned}$$

The path $\text{succ one} \underset{\beta}{\twoheadrightarrow} \text{two}$ derived above is the witness of a reduction chain from LHS to RHS. ■

Problem

(1.12) We define the λ-terms $\top_\lambda$(true) and $\bot_\lambda$(false) and $\neg_\lambda$(not) by:

$$\top_\lambda := \lambda x\ y\ .\ x \quad \bot_\lambda := \lambda x\ y\ .\ y$$
$$\neg_\lambda := \lambda a\ .\ a\ \bot_\lambda \top_\lambda$$

Show that

$$\neg_\lambda(\neg_\lambda \top_\lambda) \underset{\beta}{=} \top_\lambda$$

$$\neg_\lambda(\neg_\lambda \bot_\lambda) \underset{\beta}{=} \bot_\lambda$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$\begin{aligned}
\neg_\lambda(\neg_\lambda \top_\lambda) &= \neg_\lambda((\lambda a \,.\, a \perp_\lambda \top_\lambda)(\lambda x\, y \,.\, x)) \\
&\twoheadrightarrow_\beta \neg_\lambda((\lambda x\, y \,.\, x)\perp_\lambda \top_\lambda) \\
&\twoheadrightarrow_\beta (\lambda a \,.\, a \perp_\lambda \top_\lambda)\perp_\lambda \\
&\twoheadrightarrow_\beta (\lambda x\, y \,.\, y)\perp_\lambda \top_\lambda \\
&\twoheadrightarrow_\beta \top_\lambda
\end{aligned}$$

$$\begin{aligned}
\neg_\lambda(\neg_\lambda \perp_\lambda) &= \neg_\lambda((\lambda a \,.\, a \perp_\lambda \top_\lambda)(\lambda x\, y \,.\, y)) \\
&\twoheadrightarrow_\beta \neg_\lambda((\lambda x\, y \,.\, x)\perp_\lambda \top_\lambda) \\
&\twoheadrightarrow_\beta (\lambda a \,.\, a \perp_\lambda \top_\lambda)\top_\lambda \\
&\twoheadrightarrow_\beta (\lambda x\, y \,.\, x)\perp_\lambda \top_\lambda \\
&\twoheadrightarrow_\beta \perp_\lambda
\end{aligned}$$

■

Problem

(1.13) Define

$$\text{iszero} \coloneqq \lambda m \,.\, m\ (\lambda x \,.\perp_\lambda)\top_\lambda$$

Prove

$$\text{iszero zero} \twoheadrightarrow_\beta \top_\lambda$$

$$\forall\, n \in \mathbb{N}^+, \text{iszero}\ \text{num}_n \twoheadrightarrow_\beta \perp_\lambda$$

Solution.

$$\begin{aligned}
\text{iszero zero} &= (\lambda m \,.\, m\ (\lambda x \,.\perp_\lambda)\top_\lambda)(\lambda f\, x \,.\, x) \\
&\twoheadrightarrow_\beta (\lambda f\, x \,.\, x)(\lambda x \,.\perp_\lambda)\top_\lambda \\
&\twoheadrightarrow_\beta \top_\lambda
\end{aligned}$$

$$
\begin{aligned}
\text{iszero num}_n &= (\lambda m\,.\,m\,(\lambda x\,.\bot_\lambda)\top_\lambda)(\lambda f\,x\,.\,f^n\,x) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda f\,x\,.\,f^n\,x)(\lambda x\,.\bot_\lambda)\top_\lambda \\
&\underset{\beta}{\twoheadrightarrow} (\lambda x\,.\bot_\lambda)\big((\lambda x\,.\bot_\lambda)^{n-1}\top_\lambda\big) \underset{\beta}{\twoheadrightarrow} \bot_\lambda
\end{aligned}
$$

Problem

(1.14) If-else can be modeled as

$$\text{ifelse} = \lambda x\,t\,f\,.\,x\,t\,f$$

Where when x, then t, else f. Prove correctness by applying $\top_\lambda$ and $\bot_\lambda$ on ifelse.

Solution.

$$
\begin{aligned}
\text{ifelse } \top_\lambda &= (\lambda x\,t\,f\,.\,x\,t\,f)\top_\lambda \\
&\underset{\beta}{\twoheadrightarrow} (\lambda t\,f\,.(\lambda x\,y\,.\,x)\,t\,f) \underset{\beta}{\twoheadrightarrow} (\lambda t\,f\,.\,t)
\end{aligned}
$$

$$
\begin{aligned}
\text{ifelse } \top_\lambda &= (\lambda x\,t\,f\,.\,x\,t\,f)\bot_\lambda \\
&\underset{\beta}{\twoheadrightarrow} (\lambda t\,f\,.(\lambda x\,y\,.\,y)\,t\,f) \underset{\beta}{\twoheadrightarrow} (\lambda t\,f\,.\,f)
\end{aligned}
$$

By applying the results to any two values, the correct corresponding value returns, ex, for ifelse $\top_\lambda$, t is always returned.

Problem

(1.15) Prove that $\Omega \coloneqq (\lambda x\,.\,x\,x)(\lambda x\,.\,x\,x)$ does not have a β-nf.

Solution. Firstly let's prove Ω.

Proof. Induction on Ω's only reduction path proves that every $\Omega \underset{\beta}{\twoheadrightarrow} \Omega_i = \Omega$. For the base case because Ω has one and only one redux, it could only reduce to Ω_1 which is equivalent to itself. For the inductive step, $\Omega_i = \Omega$, therefore $\Omega_i \underset{\beta}{\rightarrow} \Omega_{i+1}$ is still Ω.

By definition, a term having a β-nf requires the existence of a form in β-nf such that the term can reduce to. By induction, Ω only reduces to Ω, and Ω is not in β-nf because it contains β-redex. Therefore, Ω can never reduce to a β-nf, thus it does not have a β-nf. ∎

Problem

(1.16) Let M be a λ-term with the following properties:

- M has a β-nf.
- There exists an infinite reduction path $M \equiv M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \ldots$ on M.

Prove that every M_i has a β-nf, and give an example of M.

Solution. An example would be $(\lambda x\ y\ .\ y)\Omega$. Reduction can go on infinitely by reducing on Ω, but the β-nf of the term is $\lambda y\ .\ y$

Proof. Denote β-nf of M as M'. For any form in the reduction path, $M \underset{\beta}{\twoheadrightarrow} M_i$. In conjunction with $M \underset{\beta}{\twoheadrightarrow} M'$, by the Church-Rosser theorem, there exists L such that $M_i \underset{\beta}{\twoheadrightarrow} L$ and $M' \underset{\beta}{\twoheadrightarrow} L$. Because M' is in β-nf, L can only be M', thus $M_i \underset{\beta}{\twoheadrightarrow} M'$, so M_i is capable of reducing to M', a β-nf. Therefore, any form in the reduction path has a β-nf. ■

Problem

(1.17) If $M\ N$ is strongly normalizing, then both M and N are strongly normalizing.

Solution.

Proof. If M is not strongly normalizing, then there exists a reduction path $M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \ldots$ Therefore, $M\ N$ would have had a reduction path $M\ N \underset{\beta}{\to} M_1\ N \underset{\beta}{\to} \ldots$ that is infinite, which contradicts with $M\ N$ being strongly normalizing. Vice versa for N. ■

Problem

(1.18) Let $L, M, N \in \Lambda$ such that $L \underset{\beta}{=} M$ and $L \underset{\beta}{\twoheadrightarrow} N$. Moreover, N is in β-nf. Prove that $M \underset{\beta}{\twoheadrightarrow} N$.

Solution. Collorary 1.9.9.

Problem

(1.19) Define

$$U := \lambda z\, x\, .\, x\, (z\, z\, x) \quad \text{and} \quad Z := U\, U$$

Prove Z is a fixed point combinator.

Solution. Proving $\forall\, L \in \Lambda, L\, (Z\, L) \underset{\beta}{\twoheadrightarrow} Z\, L$.

Proof.

$$\begin{aligned} Z\, L &= (\lambda z\, x\, .\, x\, (z\, z\, x))(\lambda z\, x\, .\, x\, (z\, z\, x))\, L \\ &\underset{\beta}{\twoheadrightarrow} L\, ((\lambda z\, x\, .\, x\, (z\, z\, x))(\lambda z\, x\, .\, x\, (z\, z\, x))\, L) \underset{\beta}{\twoheadrightarrow} L\, (Z\, L) \end{aligned}$$

■

Problem

(1.20) Solve for $M \in \Lambda$ in each equation:

$$M \underset{\beta}{=} \lambda x\, y\, .\, x\, M\, y$$

$$M\, x\, y\, z \underset{\beta}{=} x\, y\, z\, M$$

Solution. By the property of the Y combinator:

$$f\, (Y\, f) = Y\, f$$

The first equation can be remodeled as

$$M \underset{\beta}{=} L\, M \text{ where } L = \lambda m\, x\, y\, .\, x\, m\, y$$

Solving for fixed point of L :

$$\begin{aligned} M &\equiv Y\, L \equiv L\, (Y\, L) \\ &= (\lambda x\, .\, L\, (x\, x))(\lambda x\, .\, L\, (x\, x)) \\ &= (\lambda x\, .(\lambda m\, u\, v\, .\, u\, m\, v)(x\, x))(\lambda x\, .(\lambda m\, u\, v\, .\, u\, m\, v)(x\, x)) \\ &\underset{\beta}{\twoheadrightarrow} (\lambda x\, .(\lambda u\, v\, .\, u\, (x\, x)\, v))(\lambda x\, .(\lambda u\, v\, .\, u\, (x\, x)\, v)) \end{aligned}$$

The second equation can be η-reduced on both sides:

$$\begin{aligned} M\, x\, y\, z &\underset{\beta}{=} x\, y\, z\, M \\ M &\underset{\beta}{=} \lambda x\, y\, z\, .\, x\, y\, z\, M \end{aligned}$$

Remodeling equation:

$$M = N\ M \text{ where } M = \lambda m\ x\ y\ z\ .\ x\ y\ z\ m$$

then

$$\begin{aligned} M &\equiv Y\ N \equiv N\ (Y\ N) \\ &= (\lambda x\ .\ N\ (x\ x))(\lambda x\ .\ N\ (x\ x)) \\ &= (\lambda x\ .(\lambda m\ u\ y\ z\ .\ u\ y\ z\ m)(x\ x))(\lambda x\ .(\lambda m\ u\ y\ z\ .\ u\ y\ z\ m)(x\ x)) \\ &\underset{\beta}{\twoheadrightarrow} (\lambda x\ .(\lambda u\ y\ z\ .\ u\ y\ z\ (x\ x)))(\lambda x\ .(\lambda u\ y\ z\ .\ u\ y\ z\ (x\ x))) \end{aligned}$$