

EXERCISES

CHAPTER 5

SEAN LI ¹

1. Redacted

Definition Some rules for reference.

λP Calculus Rules

$$\begin{array}{c} \frac{}{\emptyset \vdash * : \square} \text{Sort} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \text{Var} \quad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \text{Weak} \\[10pt] \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A . B : s} \text{Form} \quad \frac{\Gamma \vdash M : \Pi x : A . B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B [x := N]} \text{App} \\[10pt] \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A . B : s}{\Gamma \vdash \lambda x : A . M : \Pi x : A . B} \text{Abst} \\[10pt] \frac{\Gamma \vdash A : B \quad B \stackrel{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \text{Conv} \end{array}$$

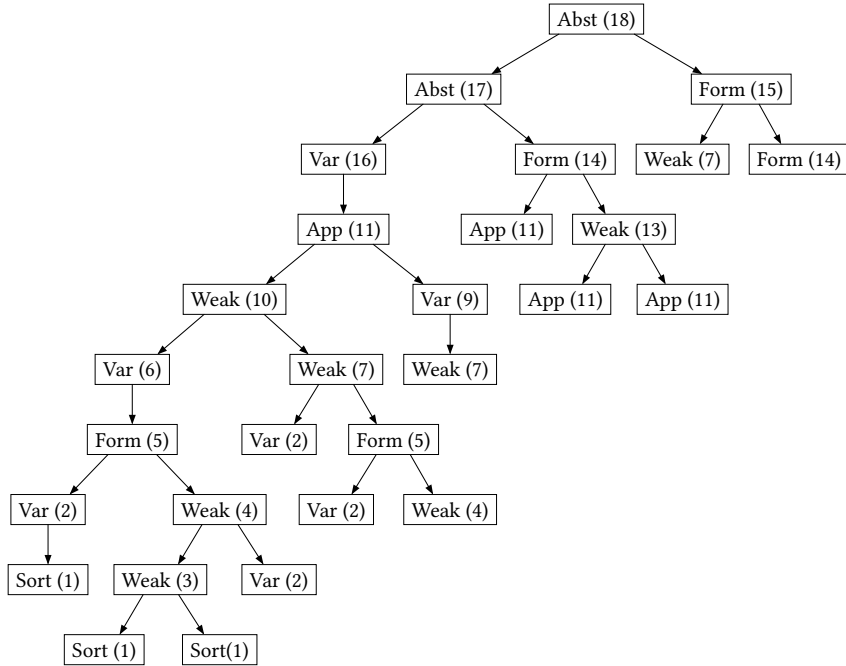
Predicate Logic

$$\begin{array}{c} \begin{array}{l} 1. \text{ Assume } A \\ 2. \quad \left| \begin{array}{l} \dots \\ 3. \quad B \end{array} \right. \\ \hline A \Rightarrow B \end{array} \Rightarrow I \quad \frac{A \Rightarrow B \quad A}{B} \Rightarrow E \quad \begin{array}{l} 1. \text{ Let } a \in S \\ 2. \quad \left| \begin{array}{l} \dots \\ 3. \quad P(a) \end{array} \right. \\ \hline \forall a \in S, P(a) \end{array} \forall I \\[10pt] \frac{\forall a \in S \quad N \in S}{P(N)} \forall E \end{array}$$

Problem

(5.1) Give a diagram of the tree corresponding to the complete tree derivation of line 18 of Section 5.3 (P 107)

Solution.



Problem

(5.2) Give a complete λP derivation of

$$S : * \vdash S \rightarrow S \rightarrow * : \square$$

In tree format and flag format.

Solution.

Tree Derivation.

$$\begin{array}{c}
 (4) \frac{\vdash * : \square \quad \vdash * : \square}{S : * \vdash * : \square} \text{Weak} \\
 (6) \frac{(3) S : * \vdash S : * \quad (4) S : * \vdash * : \square}{S : *, x : S \vdash * : \square} \text{Weak} \\
 (7) \frac{(3) S : * \vdash S : * \quad (6) S : *, x : S \vdash * : \square}{S : * \vdash S \rightarrow * : \square}
 \end{array}$$

$$\begin{array}{c}
(3) \frac{\vdash * : \square}{S : * \vdash S : *} \text{Var} \quad (9) \frac{(7) S : * \vdash S \rightarrow * : \square \quad (3) S : * \vdash S : *}{S : *, x : S \vdash S \rightarrow * : \square} \text{Weak} \\
\hline
S : * \vdash S \rightarrow S \rightarrow * : \square \text{Form}
\end{array}$$

■

Flag Derivation.

1.	$* : \square$	Sort
2.	$S : *$	
3.	$S : *$	1 Var
4.	$* : \square$	1,1 Weak
5.	$x : S$	
6.	$\boxed{* : \square}$	4,3 Weak
7.	$S \rightarrow * : \square$	3,6 Form
8.	$x : S$	
9.	$\boxed{S \rightarrow * : \square}$	7,3 Weak
10.	$S \rightarrow S \rightarrow * : \square$	3,9 Form

■

Problem

(5.3) Derive

$$S : *, Q : S \rightarrow S \rightarrow * \vdash \Pi x : S . \Pi y : S . Q x y : *$$

Solution.

1.	$* : \square$	Sort
2.	$S : *$	
3.	$S : *$	1 Var
4.	$* : \square$	1,1 Weak
5.	$x : S$	
6.	$\boxed{* : \square}$	4,3 Weak
7.	$S \rightarrow * : \square$	3,6 Form
8.	$x : S$	
9.	$\boxed{S \rightarrow * : \square}$	7,3 Weak
10.	$S \rightarrow S \rightarrow * : \square$	3,9 Form
11.	$Q : S \rightarrow S \rightarrow *$	

12.	$Q : S \rightarrow S \rightarrow *$	10 Var
13.	$S : *$	3,10 Weak
14.	$* : \square$	4,10 Weak
15.	$x : S$	
16.	$* : \square$	14,13 Weak
17.	$S : *$	13,13 Weak
18.	$x : S$	13 Var
19.	$Q : S \rightarrow S \rightarrow *$	12,13 Weak
20.	$y : S$	
21.	$y : S$	17 Var
22.	$Q : S \rightarrow S \rightarrow *$	19,17 Weak
23.	$x : S$	18,17 Weak
24.	$Q x : S \rightarrow *$	22,23 App
25.	$Q x y : *$	24,21 App
26.	$\Pi y : S . Q x y : *$	17,25 Form
27.	$\Pi x : S . \Pi y : S . Q x y : *$	13,26 Form

Problem

(5.4) Prove that $*$ is the only valid kind in λP .

Solution.

Proof. The only possible way to construct a new kind is through the Form rule and the Sort axiom. Because we are trying to construct a kind, s here stands for \square .

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : \square}{\Gamma \vdash \Pi x : A . B : \square} \text{Form}$$

One could only construct new kinds with kinds, which requires $A : \square$ and $B : \square$. This contradicts with $A : *$. ■

Problem

(5.5) Prove that $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ is a tautology by given a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash A \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$x : A$	
4.	$y : A \rightarrow B$	
5.	$y x : B$	4,3 App
6.	$\lambda y : A \rightarrow B . y x : (A \rightarrow B) \rightarrow B$	5 Abst
7.	$\lambda x : A . \lambda y : A \rightarrow B . y x : A \rightarrow (A \rightarrow B) \rightarrow B$	5 Abst

■

Problem

(5.6 a) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ a tautology using natural deduction.

Solution.

Proof.

1.	Assume $A \Rightarrow (A \Rightarrow B)$	
2.	$A \Rightarrow (A \Rightarrow B)$	
3.	Assume A	
4.	A	
5.	$A \Rightarrow B$	2,4 \RightarrowE
6.	B	5,4 \RightarrowE
7.	$A \Rightarrow B$	3,6 \RightarrowI
8.	$(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$	1,7 \RightarrowI

■

Problem

(5.6 b) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ using a shorthand λP derivation

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$x : A \rightarrow A \rightarrow B$	
4.	$y : A$	
5.	$x y : A \rightarrow B$	3,4 App
6.	$x y y : B$	5,4 App
7.	$\lambda y : A . x y y : A \rightarrow B$	6 Abst
8.	$\lambda x : A \rightarrow A \rightarrow B . \lambda y : A . x y y : (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$	7 Abst

■

Problem

(5.7 a) Proof $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of A , B , and C is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$C : *$	
4.	$x : A \rightarrow B$	
5.	$y : B \rightarrow C$	
6.	$a : A$	
7.	$x a : B$	4,6 App
8.	$y (x a) : C$	5,7 App
9.	$\lambda a : A . y (x a) : A \rightarrow C$	8 Abst
10.	$\lambda y : B \rightarrow C . \lambda a : A . y (x a) : (B \rightarrow C) \rightarrow A \rightarrow C$	9 Abst

$$\begin{array}{c}
11. \quad \left| \begin{array}{l} \lambda x : A \rightarrow B . \lambda y : B \rightarrow C . \lambda a : A . y (x z) \\ : (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C \end{array} \right. \quad \mathbf{10 \text{ Abst}}
\end{array}$$

■

Problem

(5.7 b) Proof $((A \Rightarrow B) \Rightarrow A) \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

$$\begin{array}{c}
1. \quad A : * \\
2. \quad \left| B : * \right. \\
3. \quad \left| \left| x : (A \rightarrow B) \rightarrow A \right. \right. \\
4. \quad \left| \left| \left| y : A \rightarrow B \right. \right. \right. \\
5. \quad \left| \left| \left| \left| x y : A \right. \right. \right. \quad \mathbf{3,4 \text{ App}} \\
6. \quad \left| \left| \left| \left| y (x y) : B \right. \right. \right. \quad \mathbf{4,5 \text{ App}} \\
7. \quad \left| \left| \left| \lambda y : A \rightarrow B . y (x y) : (A \rightarrow B) \rightarrow B \right. \right. \quad \mathbf{6 \text{ Abst}} \\
8. \quad \left| \left| \lambda x : (A \rightarrow B) \rightarrow A . \lambda y : A \rightarrow B . y (x y) \right. \right. \\
\quad \left| : ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B \quad \mathbf{7 \text{ Abst}} \right.
\end{array}$$

■

Problem

(5.7 c) Proof $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of A , B , and C is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$		
2.	$B : *$		
3.	$C : *$		
4.	$x : A \rightarrow B \rightarrow C$		
5.	$y : A \rightarrow B$		
6.	$a : A$		
7.	$x a : B \rightarrow C$	4,6 App	
8.	$y a : B$	5,6 App	
9.	$x a (y a) : C$	7,8 App	
10.	$\lambda a : A . x a (y a) : A \rightarrow C$	9 Abst	
11.	$\lambda y : A \rightarrow B . \lambda a : A . x a (y a) : (A \rightarrow B) \rightarrow A \rightarrow C$	10 Abst	
12.	$\lambda x : A \rightarrow B \rightarrow C . \lambda y : A \rightarrow B . \lambda a : A . x a (y a) : (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$	10 Abst	

■

Problem

(5.8 a) Let $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *$, find an inhabitant of

$$\Pi x : S . P x \rightarrow Q x \rightarrow P x$$

with respect to Γ and give a shorthand λP derivation

Solution.

1.	$S : *$		
2.	$P : S \rightarrow *$		
3.	$Q : S \rightarrow *$		
4.	$x : S$		
5.	$a : P x$		
6.	$b : Q x$		
7.	$a : P x$	2,4 App	
8.	$\lambda b : Q x . S : Q x \rightarrow P x$	7 Abst	
9.	$\lambda a : P x . \lambda b : Q x . a : P x \rightarrow Q x \rightarrow P x$	8 Abst	

$$10. \quad \left| \left| \left| \begin{array}{l} \lambda x : S . \lambda a : P \ x . \lambda b : Q \ x . a \\ : \Pi x : S . P \ x \rightarrow Q \ x \rightarrow P \ x \end{array} \right. \right. \right. \quad \mathbf{9 \text{ Abst}}$$

Problem

(5.8 b) Let $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *$, find an inhabitant of

$$\Pi x : S . P \ x \rightarrow Q \ x \rightarrow P \ x$$

By proving the corresponding proposition in natural deduction.

Solution. The corresponding proposition and premises are

$$\frac{S \in \text{Set} \quad P : S \rightarrow \text{Prop} \quad Q : S \rightarrow \text{Prop}}{\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a))}$$

Proof.

1. Let $a \in S$
2. $\left| \text{Assume } P(a) \right.$
3. $\left| \left| \text{Assume } Q(a) \right. \right.$
4. $\left| \left| \left| P(a) \right. \right. \right.$
5. $\left| \left| Q(a) \Rightarrow P(a) \right. \right. \quad \mathbf{3,4 \Rightarrow I}$
6. $\left| P(a) \Rightarrow (Q(a) \Rightarrow P(a)) \right. \quad \mathbf{2,5 \Rightarrow I}$
7. $\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a)) \quad \mathbf{1,6 \forall I}$

■

Problem

(5.9 a) Give proof for

$$(\forall x \in S, Q(x)) \Rightarrow (\forall y \in S, P(y) \Rightarrow Q(y))$$

by natural deduction and a λP derivation.

Solution.

Natural Deduction.

1. Assume $\forall x \in S, Q(x)$
2. $\left| \text{Let } y \in S \right.$

3.		Assume $P(y)$	
4.		$Q(y)$	1,2 $\forall E$
5.		$P(y) \Rightarrow Q(y)$	3,4 $\Rightarrow I$
6.		$\forall y \in S, P(y) \Rightarrow Q(y)$	2,5 $\forall I$
7.		$(\forall x \in S, Q(x) \Rightarrow (\forall y \in S, P(y) \Rightarrow Q(y)))$	1,6 $\Rightarrow I$

■

λP Derivation. Corresponding type is

$$S : *, P : S \rightarrow *, Q : S \rightarrow * \vdash (\Pi x : S . Q x) \rightarrow (\Pi y : S . P y \rightarrow Q y)$$

1.	$S : *$		
2.	$P : S \rightarrow *$		
3.	$Q : S \rightarrow *$		
4.	$a : \Pi x : S . Q x$		
5.	$y : S$		
6.	$z : P y$		
7.	$a y : Q y$	4,5 App	
8.	$\lambda z : P y . a y : P y \rightarrow Q y$	7 Abst	
9.	$\lambda y : S . \lambda z : P y . a y : \Pi y : S . P y \rightarrow Q y$	7 Abst	
10.	$\lambda a : \Pi x : S . Q x . \lambda y : S . \lambda z : P y . a y$ $: (\Pi x : S . Q x) \rightarrow (\Pi y : S . P y \rightarrow Q y)$	7 Abst	

■

Problem

(5.9 b) Give proof for

$$\forall x \in S, (P(x) \Rightarrow Q(x)) \Rightarrow ((\forall y \in S, P(y)) \Rightarrow (\forall z \in S, Q(z)))$$

by natural deduction and a λP derivation

Solution.

Natural Deduction.

1.	Assume $\forall x \in S, (P(x) \Rightarrow Q(x))$
2.	Assume $\forall y \in S, P(y)$
3.	Let $z \in S$

4.	$P(z)$	2,3 $\forall E$
5.	$P(z) \Rightarrow Q(z)$	1,3 $\forall E$
6.	$Q(z)$	5,4 $\Rightarrow E$
7.	$\forall z \in S, Q(z)$	3,6 $\forall I$
8.	$\forall y \in S, P(y) \Rightarrow (\forall z \in S, Q(z))$	2,7 $\forall I$
9.	$\forall x \in S, (P(x) \Rightarrow Q(x)) \Rightarrow ((\forall y \in S, P(y)) \Rightarrow (\forall z \in S, Q(z)))$	1,8 $\forall I$

■

λP Derivation. Corresponding type is

$$S : *, P : S \rightarrow *, Q : S \rightarrow *$$

$$\vdash (\Pi x : S. P x \rightarrow Q x) \rightarrow (\Pi y : S. P y) \rightarrow (\Pi z : S. Q z)$$

1.	$S : *$	
2.	$P : S \rightarrow *$	
3.	$Q : S \rightarrow *$	
4.	$a : \Pi x : S. P x \rightarrow Q x$	
5.	$b : \Pi y : S. P y$	
6.	$z : S$	
7.	$b z : P z$	5,6 App
8.	$a z : P z \rightarrow Q z$	4,6 App
9.	$a z (b z) : Q z$	8,7 App
10.	$\lambda z : S. a z (b z) : \Pi z : S. Q z$	9 Abst
11.	$\lambda b : (\Pi y : S. P y). \lambda z : S. a z (b z)$ $: (\Pi y : S. P y) \rightarrow \Pi z : S. Q z$	10 Abst
12.	$\lambda a : (\Pi x : S. P x \rightarrow Q x). \lambda b : (\Pi y : S. P y).$ $\lambda z : S. a z (b z)$ $: (\Pi x : S. P x \rightarrow Q x) \rightarrow$ $(\Pi y : S. P y) \rightarrow \Pi z : S. Q z$	10 Abst

■

Problem

(5.10) Given a context

$$\begin{aligned}\Gamma &\equiv S : *, P : S \rightarrow *, f : S \rightarrow S, g : S \rightarrow S, \\ u &: \Pi x : S . (P (f x) \rightarrow P (g x)), \\ v &: \Pi x, y : S . ((P x \rightarrow P y) \rightarrow P (f x))\end{aligned}$$

Let

$$M \equiv \lambda x : S . v (f x)(g x)(u x)$$

Type M under Γ .

Solution.

1.	$x : S$	
2.	$f : S \rightarrow S$	
3.	$f x : S$	2,1 App
4.	$g : S \rightarrow S$	
5.	$g x : S$	4,1 App
6.	$u : \Pi x : S . (P (f x) \rightarrow P (g x))$	
7.	$u x : P (f x) \rightarrow P (g x)$	6,1 App
8.	$v : \Pi x, y : S . ((P x \rightarrow P y) \rightarrow P (f x))$	
9.	$v (f x) : \Pi y : S . ((P (f x) \rightarrow P y) \rightarrow P (f (f x)))$	8,3 App
10.	$v (f x)(g x) : (P (f x) \rightarrow P (g x)) \rightarrow P (f (f x))$	9,5 App
11.	$v (f x)(g x)(u x) : P (f (f x))$	10,7 App
12.	$\lambda x : S . v (f x)(g x)(u x) : S \rightarrow P (f (f x))$	11 Abst

Problem

(5.11) Let S be a set, with Q and R relations on $S \times S$, and let f and g be functions from S to S . Assume

$$\forall x, y \in S (Q(x, f(y)) \Rightarrow Q(g(x), y))$$

$$\forall x, y \in S (Q(x, f(y)) \Rightarrow R(x, y))$$

$$\forall x \in S (Q(x, f(f(x))))$$

Prove that

$$\forall x \in S, R(g(g(x)), g(x))$$

By giving a context Γ and finding a term M such that

$$\Gamma \vdash M : \Pi x : S. R(g(g(x)))(g(x))$$

Solution. Context Γ is as follows:

$$\Gamma \equiv S : *, f : S \rightarrow S, g : S \rightarrow S$$

$$Q : S \rightarrow S \rightarrow *, R : S \rightarrow S \rightarrow *$$

$$A : \Pi x, y : S. (Q(x, f(y)) \rightarrow Q(g(x), y)),$$

$$B : \Pi x, y : S. (Q(x, f(y)) \rightarrow R(x, y))$$

$$C : \Pi x : S. Q(x, f(f(x)))$$

Derivation.

- | | | |
|-----|------------------------------------------------------------------------|-----------------|
| 1. | $S : *, f : S \rightarrow S, g : S \rightarrow S$ | |
| 2. | $Q : S \rightarrow S \rightarrow *, R : S \rightarrow S \rightarrow *$ | |
| 3. | $A : \Pi x, y : S. (Q(x, f(y)) \rightarrow Q(g(x), y))$ | |
| 4. | $B : \Pi x, y : S. (Q(x, f(y)) \rightarrow R(x, y))$ | |
| 5. | $C : \Pi x : S. Q(x, f(f(x)))$ | |
| 6. | $x : S$ | |
| 7. | $g(x) : S$ | 1,6 App |
| 8. | $C(g(x)) : Q(g(x))(f(f(g(x))))$ | 5,7 App |
| 9. | $f(g(x)) : S$ | 1,7 App |
| 10. | $A(g(x)) : \Pi y : S. (Q(g(x), f(y)) \rightarrow Q(g(g(x)), y))$ | 3,7 App |
| | $A(g(x))(f(g(x)))$ | |
| | $: (Q(g(x))(f(f(g(x))))$ | |
| 11. | $\rightarrow (Q(g(g(x)))(f(g(x))))$ | 10,9 App |

12.	$A(g\ x)(f(g\ x))(C(g\ x))$ $: (Q(g(g\ x))(f(g\ x)))$	11,8 App
13.	$g(g\ x) : S$ $B(g(g\ x))$ $: \Pi y : S . (Q(g(g\ x))(f\ y))$	1,7 App
14.	$\rightarrow R(g(g\ x))\ y)$ $B(g(g\ x))(g\ x)$ $: (Q(g(g\ x))(f(g\ x))) \rightarrow (R(g(g\ x))(g\ x))$	4,13 App
15.	$B(g(g\ x))(g\ x)(A(g\ x)(f(g\ x))(C(g\ x)))$ $: (R(g(g\ x))(g\ x))$	14,7 App
16.		15,12 App
17.	$\lambda x : S . B(g(g\ x))(g\ x)(A(g\ x)(f(g\ x))(C(g\ x)))$ $: \Pi x : S . (R(g(g\ x))(g\ x))$	17 Abst

■

Problem

(5.12 a) In λP , consider the context

$$\begin{aligned}\Gamma &\equiv S : *, R : S \rightarrow S \rightarrow *, \\ u &: \Pi x, y : S . R\ x\ y \rightarrow R\ y\ x \\ v &: \Pi x, y, z : S . R\ x\ y \rightarrow R\ x\ z \rightarrow R\ y\ z\end{aligned}$$

Show that R is reflexive over $S \times S$. That is, construct M such that

$$\Gamma \vdash M : \Pi x, y : S . R\ x\ y \rightarrow R\ x\ x$$

Solution.

Proof.

1. $S : *, R : S \rightarrow S \rightarrow *$
2. $A : \Pi u, v : S . R\ u\ v \rightarrow R\ v\ u$
3. $B : \Pi u, v, w : S . R\ u\ v \rightarrow R\ u\ w \rightarrow R\ v\ w$
4. $\left| \begin{array}{l} x : S \\ y : S \\ h : R\ x\ y \end{array} \right|$
- 5.
- 6.

7.	$B x : \Pi v, w : S . R x v \rightarrow R v w \rightarrow R x w$	3,4 App
8.	$B x y : \Pi w : S . R x y \rightarrow R y w \rightarrow R x w$	7,5 App
9.	$B x y x : R x y \rightarrow R y x \rightarrow R x x$	7,5 App
10.	$A x : \Pi v : R x v \rightarrow R v x$	2,4 App
11.	$A x y : R x y \rightarrow R y x$	10,5 App
12.	$A x y h : R y x$	11,6 App
13.	$B x y x h : R y x \rightarrow R x x$	9,6 App
14.	$B x y x h (A x y h) : R x x$	13,12 App
15.	$\lambda h : R x y . B x y x h (A x y h) : R x y \rightarrow R x x$	14 Abst
16.	$\lambda y : S . \lambda h : R x y . B x y x h (A x y h)$ $: \Pi y : S . R x y \rightarrow R x x$	15 Abst
17.	$\lambda x, y : S . \lambda h : R x y . B x y x h (A x y h)$ $: \Pi x, y : S . R x y \rightarrow R x x$	16 Abst

■

Problem

(5.12 b) Given the context Γ in 5.12 a, prove transitivity of R by constructing M such that

$$\Gamma \vdash M : \Pi x, y, z : S . R x y \rightarrow R y z \rightarrow R x z$$

Solution.

Proof.

1.	$S : *, R : S \rightarrow S \rightarrow *$	
2.	$A : \Pi u, v : S . R u v \rightarrow R v u$	
3.	$B : \Pi u, v, w : S . R u v \rightarrow R u w \rightarrow R v w$	
4.	$x : S$	
5.	$y : S$	
6.	$z : S$	
7.	$h : R x y$	
8.	$r : R y z$	
9.	$A x : \Pi v : S . R x v \rightarrow R v x$	2,4 App
10.	$A x y : R x y \rightarrow R y x$	9,5 App
11.	$A x y h : R y x$	10,7 App

12.	$B y : \Pi v, w : S . R y v \rightarrow R y w \rightarrow R v w$	3,5 App
13.	$B y x : \Pi w : S . R y x \rightarrow R y w \rightarrow R x w$	12,4 App
14.	$B y x z : R y x \rightarrow R y z \rightarrow R x z$	12,4 App
15.	$B y x z (A x y h) : R y z \rightarrow R x z$	14,11 App
16.	$B y x z (A x y h) r : R x z$	15,8 App
17.	$\lambda r : R y z . B y x z (A x y h) r : R y z \rightarrow R x z$	16 Abst
18.	$\lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$ $: R x y \rightarrow R y z \rightarrow R x z$	17 Abst
19.	$\lambda z : S . \lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$ $: \Pi z : S . R x y \rightarrow R y z \rightarrow R x z$	18 Abst
20.	$\lambda y, z : S . \lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$ $: \Pi y, z : S . R x y \rightarrow R y z \rightarrow R x z$	19 Abst
21.	$\lambda x, y, z : S . \lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$ $: \Pi x, y, z : S . R x y \rightarrow R y z \rightarrow R x z$	20 Abst

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Completed Dec 22 6:51 pm.