

EXERCISES

CHAPTER 7

SEAN LI ¹

1. Redacted

Reference - Propositional Logic in λC

$$\begin{array}{c} \frac{A \quad \neg A}{\perp} \perp\text{I or } \neg\text{E} \quad \frac{\perp}{A} \perp\text{E} \quad \frac{A \quad B}{A \wedge B} \wedge\text{I} \quad \frac{A \wedge B}{A} \wedge\text{EL} \quad \frac{A \wedge B}{B} \wedge\text{ER} \\ \\ \frac{a}{a \vee b} \vee\text{IL} \quad \frac{b}{a \vee b} \vee\text{IR} \quad \frac{A \Rightarrow B \quad A}{B} \Rightarrow\text{E} \\ \\ \frac{A \vee B \quad A \rightarrow C \quad B \rightarrow C}{C} \vee\text{E} \quad \frac{a \in S \quad P(a)}{\exists a \in S, P(a)} \exists\text{I} \quad \frac{\begin{array}{l} 1. \quad A \\ 2. \quad \left| \begin{array}{l} \dots \\ \perp \end{array} \right. \\ 3. \end{array}}{\neg A} \neg\text{I} \\ \\ \frac{\begin{array}{l} 1. \quad A \\ 2. \quad \left| \begin{array}{l} \dots \\ B \end{array} \right. \\ 3. \end{array}}{A \Rightarrow B} \Rightarrow\text{I} \quad \frac{\begin{array}{l} 1. \quad a \in S \\ 2. \quad \left| \begin{array}{l} \dots \\ P(a) \end{array} \right. \\ 3. \end{array}}{\forall a \in S, P(a)} \forall\text{I} \\ \\ \frac{\exists x \in S, P(x) \quad \forall x \in S, (P(x) \Rightarrow A)}{A} \exists\text{E} \quad \frac{a \in S \quad \forall x \in S, P(x)}{P(a)} \forall\text{E} \\ \\ \frac{}{\neg\neg A \Rightarrow A} \text{DN (Classical)} \quad \frac{}{A \vee \neg A} \text{ET (Classical)} \end{array}$$

Reference - 2nd Encoding for Propositional Logic

Proposition	Minimal Propositional Logic
\perp	$\forall A, A$
$A \Rightarrow B$	$A \Rightarrow B$
$\neg A$	$A \Rightarrow \perp$
$A \wedge B$	$\forall C, (A \Rightarrow B \Rightarrow C) \Rightarrow C$
$A \vee B$	$\forall C, (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$
$\forall a \in S, P(a)$	$\forall a \in S. P(a)$
$\exists a \in S, P(a)$	$\forall \alpha, (\forall a \in S, (P(a) \Rightarrow \alpha)) \Rightarrow \alpha$

Problem

(7.1 a) Prove in natural deduction and λC the tautology

$$B \Rightarrow (A \Rightarrow B)$$

Solution.

Natural Deduction.

1. B
2. $\left| \begin{array}{l} A \\ \left| \begin{array}{l} B \\ \hline A \Rightarrow B \end{array} \right. \end{array} \right. \Rightarrow \text{I}$
3. $\left| \begin{array}{l} A \\ \left| \begin{array}{l} B \\ \hline A \Rightarrow B \end{array} \right. \end{array} \right. \Rightarrow \text{I}$
4. $\left| \begin{array}{l} A \\ \left| \begin{array}{l} B \\ \hline A \Rightarrow B \end{array} \right. \end{array} \right. \Rightarrow \text{I}$
5. $B \Rightarrow (A \Rightarrow B) \Rightarrow \text{I}$

■

λC . Assuming context $\Gamma \equiv A : *, B : *$. By the PAT paradigm the proof is equivalent to an inhabitant of $B \rightarrow A \rightarrow B$.

1. $A : *, B : *$
2. $\left| \begin{array}{l} x : B \\ \left| \begin{array}{l} y : A \\ \left| \begin{array}{l} x : B \\ \hline \lambda y : A. x : A \rightarrow B \end{array} \right. \end{array} \right. \end{array} \right. \text{Weak}$
3. $\left| \begin{array}{l} x : B \\ \left| \begin{array}{l} y : A \\ \left| \begin{array}{l} x : B \\ \hline \lambda y : A. x : A \rightarrow B \end{array} \right. \end{array} \right. \end{array} \right. \text{4 Abst}$
4. $\left| \begin{array}{l} x : B \\ \left| \begin{array}{l} y : A \\ \left| \begin{array}{l} x : B \\ \hline \lambda y : A. x : A \rightarrow B \end{array} \right. \end{array} \right. \end{array} \right. \text{5 Abst}$
5. $\left| \begin{array}{l} x : B \\ \left| \begin{array}{l} y : A \\ \left| \begin{array}{l} x : B \\ \hline \lambda y : A. x : A \rightarrow B \end{array} \right. \end{array} \right. \end{array} \right. \text{5 Abst}$
6. $\left| \begin{array}{l} x : B \\ \left| \begin{array}{l} y : A \\ \left| \begin{array}{l} x : B \\ \hline \lambda y : A. x : A \rightarrow B \end{array} \right. \end{array} \right. \end{array} \right. \text{5 Abst}$

■

Problem

(7.1 b) Prove in natural deduction and λC the tautology

$$\neg A \Rightarrow (A \Rightarrow B)$$

Solution.

Natural Deduction.

1. $\neg A$
2. $\begin{array}{|l} A \end{array}$
3. $\begin{array}{|l} \neg A \end{array}$
4. $\begin{array}{|l} A \end{array}$
5. $\begin{array}{|l} \perp \end{array} \quad \perp \text{I}$
6. $\begin{array}{|l} B \end{array} \quad \perp \text{E}$
7. $\begin{array}{|l} A \Rightarrow B \end{array} \quad \Rightarrow \text{I}$
8. $\neg A \Rightarrow (A \Rightarrow B) \quad \Rightarrow \text{I}$

■

λC . Assuming context $\Gamma \equiv A : *, B : *$. By the PAT paradigm the proof is equivalent to an inhabitant of $(A \rightarrow \perp) \rightarrow A \rightarrow B$.

1. $A : *, B : *$
2. $\begin{array}{|l} x : \neg A \end{array}$
3. $\begin{array}{|l} y : A \end{array}$
4. $\begin{array}{|l} x y : \Pi \alpha : *. \alpha \end{array} \quad \mathbf{2,3 \text{ App (Neg Elim)}}$
5. $\begin{array}{|l} x y B : B \end{array} \quad \mathbf{4,1 \text{ App (Ex Falso)}}$
6. $\begin{array}{|l} \lambda y : A. x y B : A \rightarrow B \end{array} \quad \mathbf{5 \text{ Abst}}$
7. $\begin{array}{|l} \lambda x : \neg A. \lambda y : A. x y B : \neg A \rightarrow A \rightarrow B \end{array} \quad \mathbf{6 \text{ Abst}}$

■

Problem

(7.1 c) Prove in natural deduction and λC the tautology

$$(A \Rightarrow \neg B) \Rightarrow ((A \Rightarrow B) \Rightarrow \neg A)$$

Solution.

Natural Deduction.

1. $A \Rightarrow \neg B$
2. $\begin{array}{|l} A \Rightarrow B \end{array}$
3. $\begin{array}{|l} A \end{array}$
4. $\begin{array}{|l} \neg B \end{array}$ **1,3 \Rightarrow E**
5. $\begin{array}{|l} B \end{array}$ **2,3 \Rightarrow E**
6. $\begin{array}{|l} \perp \end{array}$ **5,4 \perp I**
7. $\begin{array}{|l} \neg A \end{array}$ **3,6 \neg I**
8. $\begin{array}{|l} (A \Rightarrow B) \Rightarrow \neg A \end{array}$ **2,7 \Rightarrow I**
9. $(A \Rightarrow \neg B) \Rightarrow ((A \Rightarrow B) \Rightarrow \neg A)$ **1,8 \Rightarrow I**

■

λC . Assuming context $\Gamma \equiv A : *, B : *$. The proof should be equivalent to an inhabitant of $(A \rightarrow B \rightarrow \perp) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow \perp$.

1. $A : *, B : *$
2. $\begin{array}{|l} h : A \rightarrow \neg B \end{array}$
3. $\begin{array}{|l} q : A \rightarrow B \end{array}$
4. $\begin{array}{|l} a : A \end{array}$
5. $\begin{array}{|l} q a : B \end{array}$ **3,4 App**
6. $\begin{array}{|l} h a : B \rightarrow \perp \end{array}$ **2,4 App**
7. $\begin{array}{|l} h a (q a) : \perp \end{array}$ **6,5 App (Neg Elim)**
8. $\begin{array}{|l} \lambda a : A . h a (q a) : \neg A \end{array}$ **7 Abst (Neg Intro)**
9. $\begin{array}{|l} \lambda q : A \rightarrow B . \lambda a : A . h a (q a) : A \rightarrow B \rightarrow \neg A \end{array}$ **8 Abst**
10. $\begin{array}{|l} \lambda h : A \rightarrow \neg B . \lambda q : A \rightarrow B . \lambda a : A . h a (q a) : (A \rightarrow \neg B) \rightarrow A \rightarrow B \rightarrow \neg A \end{array}$ **9 Abst**

■

Problem

(7.1 d) Prove in natural deduction and λC the tautology

$$\neg(A \Rightarrow B) \Rightarrow \neg B$$

Solution.

Natural Deduction.

$$\begin{array}{ll}
1. & \neg(A \Rightarrow B) \\
2. & \begin{array}{|l} B \\ \hline 3. \quad \begin{array}{|l} A \\ \hline 4. \quad \begin{array}{|l} B \\ \hline 5. \quad A \Rightarrow B \quad \quad 3,4 \Rightarrow I \\ \hline 6. \quad \perp \quad \quad 5,1 \perp I \\ \hline 7. \quad \neg B \quad \quad 6 \neg I \end{array} \end{array} \\ \hline 8. & \neg(A \Rightarrow B) \Rightarrow \neg B \quad 1,7 \Rightarrow I
\end{array}$$

■

λC . Assuming context $\Gamma \equiv A : *, B : *$. The proof should be equivalent to an inhabitant of $((A \rightarrow B) \rightarrow \perp) \rightarrow B \rightarrow \perp$.

$$\begin{array}{ll}
1. & n : \neg(A \rightarrow B) \\
2. & \begin{array}{|l} b : B \\ \hline 3. \quad \begin{array}{|l} a : A \\ \hline 4. \quad \begin{array}{|l} b : B \\ \hline 5. \quad \lambda a : A . b : A \rightarrow B \quad \text{Weak} \\ \hline 6. \quad n (\lambda a : A . b) : \perp \quad \text{4 Abst} \\ \hline 7. \quad \lambda b : B . n (\lambda a : A . b) : \neg B \quad \text{1,5 App (Neg Elim)} \end{array} \end{array} \\ \hline 8. & \lambda b : B . n (\lambda a : A . b) : \neg B \quad \text{6 Abst (Neg Intro)}
\end{array}$$

$$\begin{array}{l}
\lambda n : \neg(A \rightarrow B) . \lambda b : B . n (\lambda a : A . b) \\
: \neg(A \rightarrow B) \rightarrow \neg B \quad \text{7 Abst}
\end{array}$$

■

Problem

(7.2) Formulate the double negation law as an axiom in λC , and prove the following tautology in λC with DN.

$$(\neg A \Rightarrow A) \Rightarrow A$$

Solution. The rule

$$\frac{}{\neg\neg A \Rightarrow A} \text{DN-E}$$

Could be translated into lambda calculus as

$$\Pi A : * . ((A \rightarrow \perp) \rightarrow \perp) \rightarrow A$$

Proof. Assume context $\Gamma \equiv A : *$.

1.	$A : *$	
2.	$h : \neg A \rightarrow A$	
3.	$x : \neg A$	
4.	$h x : A$	2,3 App
5.	$x (h x) : \perp$	3,4 App (Contradiction)
6.	$\lambda x : \neg A . x (h x) : \neg \neg A$	5 Abst (Neg Intro)
7.	$\text{DN } A : \neg \neg A \rightarrow A$	1,1 App
8.	$\text{DN } A (\lambda x : \neg A . x (h x)) : A$	App (Axiom DN)
9.	$\lambda h : \neg A \rightarrow A . \text{DN } A (\lambda x : \neg A . x (h x)) : (\neg A \rightarrow A) \rightarrow A$	8 Abst

■

Problem

(7.3 a) Prove the following tautology in classical logic using λC

$$(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$$

Solution. Assume context $\Gamma \equiv A : *, B : *$. It suffices the proof to find an inhabitant of type

$$(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$$

Proof.

1.	$A : *, B : *$	
2.	$h : A \rightarrow B$	
3.	$b : \neg B$	
4.	$a : A$	
5.	$h a : B$	5,2 App
6.	$b (h a) : \perp$	6,3 App (Contradiction)
7.	$b (h a)(\neg A) : \neg A$	7,5 App (Ex Falso)
8.	$\lambda a : A . b (h a)(\neg A) : A \rightarrow \neg A$	7 Abst
9.	$a : \neg A$	
10.	$a : \neg A$	Var
11.	$\lambda a : \neg A . a : (\neg A \rightarrow \neg A)$	10 Abst

12.	ET $A : A \vee \neg A$	App (Axiom ET)
	ET $A (\neg A) : (A \rightarrow \neg A) \rightarrow$	
13.	$(\neg A \rightarrow \neg A) \rightarrow \neg A$	12 App
	ET $A (\neg A)(\lambda a : A . b (h a)(\neg A)) :$	
14.	$(\neg A \rightarrow \neg A) \rightarrow \neg A$	13,8 App
	ET $A (\neg A)$	
	$(\lambda a : A . b (h a)(\neg A))$	
15.	$(\lambda a : \neg A . a) : \neg A$	14,11 App
	$\lambda b : \neg B . \text{ET } A (\neg A)$	
	$(\lambda a : A . b (h a)(\neg A))$	
16.	$(\lambda a : \neg A . a) : \neg B \rightarrow \neg A$	15 Abst
	$\lambda h : A \rightarrow B . \lambda b : \neg B . \text{ET } A (\neg A)$	
	$(\lambda a : A . b (h a)(\neg A))$	
	$(\lambda a : \neg A . a) :$	
17.	$(A \rightarrow B) \rightarrow \neg B \rightarrow \neg A$	16 Abst

■

Problem

(7.3 b) Prove the following tautology in classical logic using λC

$$(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$$

Solution. Assume context $\Gamma \equiv A : *, B : *$. It suffices the proof to find an inhabitant of type

$$(\neg B \rightarrow \neg A) \rightarrow A \rightarrow B$$

Proof.

1.	$A : *, B : *$	
2.	$h : \neg B \rightarrow \neg A$	
3.	$a : A$	
4.	$b : B$	
5.	$b : B$	Weak
6.	$\lambda b : B . b : B \rightarrow B$	
7.	$b : \neg B$	

8.				$h\ b : \neg A$	2,7 App
9.				$h\ b\ a : \perp$	8,2 App (Neg Elim)
10.				$h\ b\ a\ B : B$	9 App (Ex Falso)
11.				$\lambda b : \neg B . h\ b\ a\ B : \neg B \rightarrow B$	10 Abst
12.				ET $B : B \vee \neg B$	1 App (Axiom ET)
13.				ET $B\ B : (B \rightarrow B) \rightarrow (\neg B \rightarrow B) \rightarrow B$	12,1 App
14.				ET $B\ B\ (\lambda b : B . b) : (\neg B \rightarrow B) \rightarrow B$	13,6 App
15.				ET $B\ B\ (\lambda b : B . b)(\lambda b : \neg B . h\ b\ a\ B) : B$	14,11 App
				$\lambda a : A . \text{ET } B\ B\ (\lambda b : B . b)$	
16.				$(\lambda b : \neg B . h\ b\ a\ B) : A \rightarrow B$	15 Abst
				$\lambda h : \neg B \rightarrow \neg A . \lambda a : A .$	
				ET $B\ B\ (\lambda b : B . b)(\lambda b : \neg B . h\ b\ a\ B)$	
17.				$: (\neg B \rightarrow \neg A) \rightarrow A \rightarrow B$	16 Abst

■

Problem

(7.4) Derive \wedge -EL and \wedge -ER in λC .

Solution. The derivation is the same as proving

$$M : \Pi C . (A \rightarrow B \rightarrow C) \rightarrow C \vdash N_1 : A$$

$$M : \Pi C . (A \rightarrow B \rightarrow C) \rightarrow C \vdash N_2 : B$$

Left Projection.

1.	$M : \Pi C . (A \rightarrow B \rightarrow C) \rightarrow C$	
2.		$M\ A : (A \rightarrow B \rightarrow A) \rightarrow A$
3.		$x : A$
4.		
5.		
6.		
7.		
8.		

Weak

5 Abst

6 Abst

2,7 App

■

Right Projection.

1.	$M : \Pi C . (A \rightarrow B \rightarrow C) \rightarrow C$	
2.	$M B : (A \rightarrow B \rightarrow B) \rightarrow B$	
3.	$x : A$	
4.	$b : B$	
5.	$b : B$	Weak
6.	$\lambda b : B . b : B \rightarrow B$	5 Abst
7.	$\lambda x : A . \lambda b : B . b : A \rightarrow B \rightarrow B$	6 Abst
8.	$N_2 \equiv M B (\lambda x : A . \lambda b : B . b) : B$	2,7 App

■

Problem

(7.5 a) Prove tautology under classical logic

$$\neg(A \Rightarrow B) \Rightarrow A$$

Solution. Assume context $\Gamma \equiv A : *, B : *$. Therefore the proof suffices to find an inhabitant of

$$((A \rightarrow B) \rightarrow \perp) \rightarrow A$$

Proof.

1.	$A : *, B : *$	
2.	$h : \neg(A \rightarrow B)$	
3.	$a : A$	
4.	$a : A$	Weak
5.	$\lambda a : A . a : A \rightarrow A$	4 Abst
6.	$a : \neg A$	
7.	$n : A$	
8.	$a n : \perp$	6,7 App (Contradiction)
9.	$a n B : B$	8 App (Ex Falso)
10.	$\lambda n : A . a n B : A \rightarrow B$	9 Abst
11.	$h (\lambda n : A . a n B) : \perp$	10,2 App (Contra.)
12.	$h (\lambda n : A . a n B) A : A$	11 App (Ex Falso)
	$\lambda a : \neg A . h (\lambda n : A . a n B) A$	
13.	$: \neg A \rightarrow A$	12 Abst
14.	$ET : \Pi S : * . S \vee \neg S$	Axiom ET

15.	ET $A : A \vee \neg A$	14,1 App
16.	ET $A A : (A \rightarrow A) \rightarrow (\neg A \rightarrow A) \rightarrow A$	15,1 App
	ET $A A (\lambda a : A . a)$	
17.	$: (\neg A \rightarrow A) \rightarrow A$	16,5 App
	ET $A A (\lambda a : A . a)$	
18.	$(\lambda a : \neg A . h (\lambda n : A . a \ n B) A) : A$	17,13 App
	$\lambda h : \neg(A \rightarrow B) . \text{ET } A A (\lambda a : A . a)$	
	$(\lambda a : \neg A . h (\lambda n : A . a \ n B) A)$	
19.	$: \neg(A \rightarrow B) \rightarrow A$	18 Abst

■

Problem

(7.5 b) Prove tautology under classical logic

$$\neg(A \Rightarrow B) \Rightarrow (A \wedge \neg B)$$

Solution. Assume context $\Gamma \equiv A : *, B : *$. Therefore the proof suffices to find an inhabitant of

$$((A \rightarrow B) \rightarrow \perp) \rightarrow \Pi C : * . ((A \rightarrow (B \rightarrow \perp) \rightarrow C) \rightarrow C)$$

Proof.

1.	$A : *, B : *$	
2.	$p : \neg(A \rightarrow B)$	
3.	$C : *$	
4.	$h : A \rightarrow \neg B \rightarrow C$	
5.	$hb : B$	
6.	$ha : A$	
7.	$hb : B$	Weak
8.	$\lambda ha : A . hb : A \rightarrow B$	7 Abst
9.	$p (\lambda ha : A . hb) : \perp$	2,8 App (Contra.)
10.	$\lambda hb : B . p (\lambda ha : A . hb) : \neg B$	9 Abst (Neg Intro)
11.	$hna : \neg A$	
12.	$ha' : A$	
13.	$hna ha' : \perp$	11,12 App (Contra.)

14.	$\boxed{hna\ ha' B : B}$	13,1 App (Ex Falso)
15.	$\lambda ha' : A . hna\ ha' B : A \rightarrow B$	14 Abst
16.	$p(\lambda ha' : A . hna\ ha' B) : \perp$	2,15 App (Contra.)
17.	$\boxed{p(\lambda ha' : A . hna\ ha' B) A : A}$	2,15 App (Contra.)
	$\lambda hna : \neg A . p$	
	$(\lambda ha' : A . hna\ ha' B) A$	
18.	$: \neg A \rightarrow A$	17 Abst
19.	$ha'' : A$	
20.	$\boxed{ha'' : A}$	Var
21.	$\lambda ha'' : A . ha'' : A \rightarrow A$	20 Abst
22.	$ET : \Pi S : * . S \vee \neg S$	Axiom ET
23.	$ET\ A : A \vee \neg A$	22,1 App
24.	$ET\ A\ A : (A \rightarrow A) \rightarrow (\neg A \rightarrow A) \rightarrow A$	23,1 App
	$ET\ A\ A$	
	$(\lambda ha'' : A . ha'')$	
25.	$: (\neg A \rightarrow A) \rightarrow A$	24,21 App
	$ET\ A\ A$	
	$(\lambda ha'' : A . ha'')$	
	$(\lambda hna : \neg A . p$	
26.	$(\lambda ha' : A . hna\ ha' B) A) : A$	25,18 App
	$h\ (ET\ A\ A$	
	$(\lambda ha'' : A . ha'')$	
	$(\lambda hna : \neg A . p$	
	$(\lambda ha' : A . hna\ ha' B) A))$	
27.	$: \neg B \rightarrow C$	4,26 App
	$h\ (ET\ A\ A$	
	$(\lambda ha'' : A . ha'')$	
	$(\lambda hna : \neg A . p$	
	$(\lambda ha' : A . hna\ ha' B) A))$	
28.	$(\lambda hb : B . p(\lambda ha : A . hb)) : C$	27,10 App

29.	$ \begin{array}{l} \lambda h : A \rightarrow \neg B \rightarrow C . h \text{ (ET } A \ A \\ (\lambda ha'' : A . ha'') \\ (\lambda hna : \neg A . p (\lambda ha' : A . hna \ ha' \ B) \ A)) \\ (\lambda hbb : B . p (\lambda ha : A . hb)) \\ : (A \rightarrow \neg B \rightarrow C) \rightarrow C \end{array} $	28 Abst
30.	$ \begin{array}{l} \lambda C : * . \lambda h : A \rightarrow \neg B \rightarrow C . \\ h \text{ (ET } A \ A \\ (\lambda ha'' : A . ha'') \\ (\lambda hna : \neg A . p (\lambda ha' : A . hna \ ha' \ B) \ A)) \\ (\lambda hbb : B . p (\lambda ha : A . hb)) : A \wedge \neg B \end{array} $	29 Abst
31.	$ \begin{array}{l} \lambda p : \neg(A \rightarrow B). \\ \lambda C : * . \lambda h : A \rightarrow \neg B \rightarrow C . \\ h \text{ (ET } A \ A \\ (\lambda ha'' : A . ha'') \\ (\lambda hna : \neg A . p (\lambda ha' : A . hna \ ha' \ B) \ A)) \\ (\lambda hbb : B . p (\lambda ha : A . hb)) \\ : \neg(A \rightarrow B) \rightarrow (A \wedge \neg B) \end{array} $	30 Abst

■