Excercises

Chapter 5

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1. Reducted

Definition Some rules for reference.

λP Calculus Rules

$$\frac{}{\emptyset \vdash * : \square}\,\text{Sort} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}\,\text{Var} \qquad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}\,\text{Weak}$$

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi\, x : A \,.\, B : s}\,\text{Form} \qquad \frac{\Gamma \vdash M : \Pi\, x : A \,.\, B \quad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B\,[x := N]}\,\text{App}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi\, x : A \,.\, B : s}{\Gamma \vdash \lambda x : A \,.\, M : \Pi\, x : A \,.\, B}\,\text{Abst}$$

$$\frac{\Gamma \vdash A : B \quad B \overset{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'}\,\text{Conv}$$

Predicate Logic

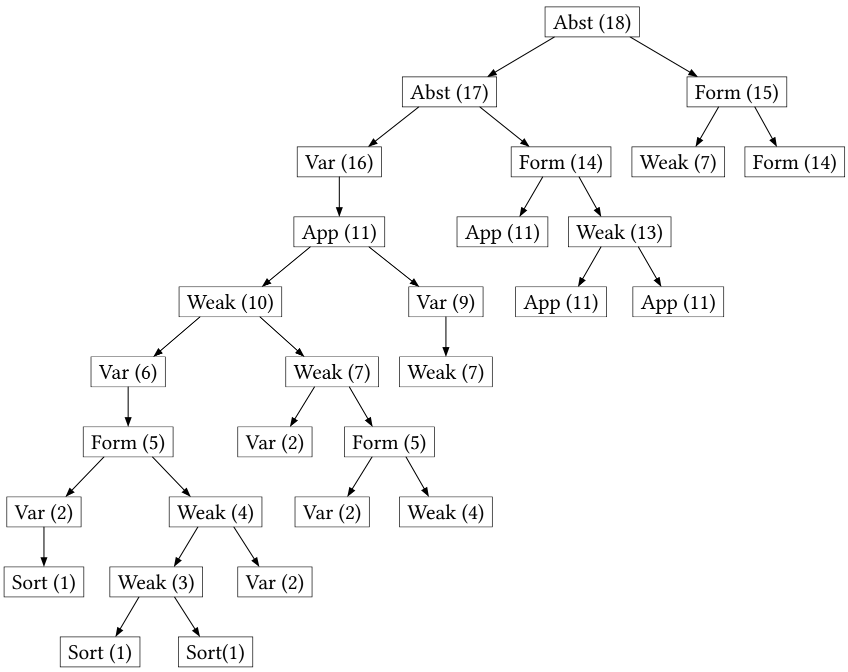
$$\frac{\begin{array}{ll} 1. & \text{Assume } A \\ 2. & \quad \dots \\ 3. & \quad B \end{array}}{A \Rightarrow B}\,{\Rightarrow}\text{I} \qquad \frac{A \Rightarrow B \quad A}{B}\,{\Rightarrow}\text{E} \qquad \frac{\begin{array}{ll} 1. & \text{Let } a \in S \\ 2. & \quad \dots \\ 3. & \quad P\,(a) \end{array}}{\forall\, a \in S, P\,(a)}\,\forall\text{I}$$

$$\frac{\forall\, a \in S \quad N \in S}{P\,(N)}\,\forall\text{E}$$

Problem

(5.1) Give a diagram of the tree corresponding to the complete tree derivation of line 18 of Section 5.3 (P 107)

Solution.



Problem

(5.2) Give a complete λP derivation of

$$S : * \vdash S \to S \to * : \square$$

In tree format and flag format.

Solution.

Tree Derivation.

$$(7)\,\dfrac{(3)\; S : * \vdash S : * \qquad (6)\,\dfrac{(4)\,\dfrac{\vdash * : \square \qquad \vdash * : \square}{S : * \vdash * : \square}\,\text{Weak} \qquad (3)\; S : * \vdash S : *}{S : *, x : S \vdash * : \square}\,\text{Weak}}{S : * \vdash S \to * : \square}$$

$$
\dfrac{(3)\ \dfrac{\vdash * : \square}{S : * \vdash S : *}\ \text{Var} \qquad (9)\ \dfrac{(7)\ S : * \vdash S \to * : \square \qquad (3)\ S : * \vdash S : *}{S : *, x : S \vdash S \to * : \square}\ \text{Weak}}{S : * \vdash S \to S \to * : \square}\ \text{Form}
$$

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Flag Derivation.

1.	$* : \square$	**Sort**
2.	$S : *$	
3.	$\quad S : *$	**1 Var**
4.	$\quad * : \square$	**1,1 Weak**
5.	$\quad x : S$	
6.	$\quad\quad * : \square$	**4,3 Weak**
7.	$\quad S \to * : \square$	**3,6 Form**
8.	$\quad x : S$	
9.	$\quad\quad S \to * : \square$	**7,3 Weak**
10.	$\quad S \to S \to * : \square$	**3,9 Form**

■

Problem

(5.3) Derive

$$
S : *, Q : S \to S \to * \vdash \Pi\, x : S\,.\Pi\, y : S\,.\,Q\, x\, y : *
$$

Solution.

1.	$* : \square$	**Sort**
2.	$S : *$	
3.	$\quad S : *$	**1 Var**
4.	$\quad * : \square$	**1,1 Weak**
5.	$\quad x : S$	
6.	$\quad\quad * : \square$	**4,3 Weak**
7.	$\quad S \to * : \square$	**3,6 Form**
8.	$\quad x : S$	
9.	$\quad\quad S \to * : \square$	**7,3 Weak**
10.	$\quad S \to S \to * : \square$	**3,9 Form**
11.	$\quad Q : S \to S \to *$	

12.	$Q : S \to S \to *$	**10 Var**
13.	$S : *$	**3,10 Weak**
14.	$* : \square$	**4,10 Weak**
15.	$x : S$	
16.	$* : \square$	**14,13 Weak**
17.	$S : *$	**13,13 Weak**
18.	$x : S$	**13 Var**
19.	$Q : S \to S \to *$	**12,13 Weak**
20.	$y : S$	
21.	$y : S$	**17 Var**
22.	$Q : S \to S \to *$	**19,17 Weak**
23.	$x : S$	**18,17 Weak**
24.	$Q\, x : S \to *$	**22,23 App**
25.	$Q\, x\, y : *$	**24,21 App**
26.	$\Pi\, y : S \,.\, Q\, x\, y : *$	**17,25 Form**
27.	$\Pi\, x : S \,.\Pi\, y : S \,.\, Q\, x\, y : *$	**13,26 Form**

Problem

(5.4) Prove that $*$ is the only valid kind in λP.

Solution.

Proof. The only possible way to construct a new kind is through the Form rule and the Sort axiom. Because we are trying to construct a kind, s here stands for $\square$.

$$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : \square}{\Gamma \vdash \Pi\, x : A \,.\, B : \square}\ \text{Form}$$

One could only construct new kinds with kinds, which requires $A : \square$ and $B : \square$. This contradicts with $A : *$. ∎

Problem

(5.5) Prove that $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ is a tautology by given a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash A \to (A \to B) \to B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$x : A$	
4.	$y : A \to B$	
5.	$y\,x : B$	**4,3 App**
6.	$\lambda y : A \to B \,.\, y\,x : (A \to B) \to B$	**5 Abst**
7.	$\lambda x : A \,.\lambda y : A \to B \,.\, y\,x : A \to (A \to B) \to B$	**5 Abst**

∎

Problem

(5.6 a) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ a tautology using natural deduction.

Solution.

Proof.

1.	Assume $A \Rightarrow (A \Rightarrow B)$	
2.	$A \Rightarrow (A \Rightarrow B)$	
3.	Assume A	
4.	A	
5.	$A \Rightarrow B$	**2,4 ⇒E**
6.	B	**5,4 ⇒E**
7.	$A \Rightarrow B$	**3,6 ⇒I**
8.	$(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$	**1,7 ⇒I**

∎

Problem

(5.6 b) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ using a shorthand λP derivation

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash (A \to A \to B) \to A \to B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$x : A \to A \to B$	
4.	$y : A$	
5.	$x\, y : A \to B$	**3,4 App**
6.	$x\, y\, y : B$	**5,4 App**
7.	$\lambda y : A \,.\, x\, y\, y : A \to B$	**6 Abst**
8.	$\lambda x : A \to A \to B \,.\lambda y : A \,.\, x\, y\, y : (A \to A \to B) \to A \to B$	**7 Abst**

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Problem

(5.7 a) Proof $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \to B) \to (B \to C) \to A \to C$$

And finding an inhabitant in a context only with definition of A, B, and C is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$C : *$	
4.	$x : A \to B$	
5.	$y : B \to C$	
6.	$a : A$	
7.	$x\, a : B$	**4,6 App**
8.	$y\,(x\, a) : C$	**5,7 App**

9.	$\lambda a : A \,.\, y\,(x\,z) : A \to C$	**8 Abst**
10.	$\lambda y : B \to C \,.\lambda a : A \,.\, y\,(x\,z) : (B \to C) \to A \to C$	**9 Abst**
11.	$\lambda x : A \to B \,.\lambda y : B \to C \,.\lambda a : A \,.\, y\,(x\,z)$ $: (A \to B) \to (B \to C) \to A \to C$	**10 Abst**

∎

Problem

(5.7 b) Proof $((A \Rightarrow B) \Rightarrow A) \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash ((A \to B) \to A) \to (A \to B) \to B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$x : (A \to B) \to A$	
4.	$y : A \to B$	
5.	$x\,y : A$	**3,4 App**
6.	$y\,(x\,y) : B$	**4,5 App**
7.	$\lambda y : A \to B \,.\, y\,(x\,y) : (A \to B) \to B$	**6 Abst**
8.	$\lambda x : (A \to B) \to A \,.\lambda y : A \to B \,.\, y\,(x\,y)$ $: ((A \to B) \to A) \to (A \to B) \to B$	**7 Abst**

∎

Problem

(5.7 c) Proof $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \to B \to C) \to (A \to B) \to A \to C$$

And finding an inhabitant in a context only with definition of A, B, and C is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$C : *$	
4.	$x : A \to B \to C$	
5.	$y : A \to B$	
6.	$a : A$	
7.	$x\, a : B \to C$	**4,6 App**
8.	$y\, a : B$	**5,6 App**
9.	$x\, a\, (y\, a) : C$	**7,8 App**
10.	$\lambda a : A\, .\, x\, a\, (y\, a) : A \to C$	**9 Abst**
11.	$\lambda y : A \to B\, .\lambda a : A\, .\, x\, a\, (y\, a) : (A \to B) \to A \to C$	**10 Abst**
12.	$\lambda x : A \to B \to C\, .\lambda y : A \to B\, .\lambda a : A\, .\, x\, a\, (y\, a)$ $: (A \to B \to C) \to (A \to B) \to A \to C$	**10 Abst**

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Problem

(5.8 a) Let $\Gamma \equiv S : *, P : S \to *, Q : S \to *$, find an inhabitant of

$$\Pi\, x : S\, .\, P\, x \to Q\, x \to P\, x$$

with respect to Γ and give a shorthand λP derivation

Solution.

1.	$S : *$	
2.	$P : S \to *$	
3.	$Q : S \to *$	
4.	$x : S$	
5.	$a : P\, x$	
6.	$b : Q\, x$	
7.	$a : P\, x$	**2,4 App**

8.	$\lambda b : Q\,x\,.\,S : Q\,x \to P\,x$	**7 Abst**
9.	$\lambda a : P\,x\,.\lambda b : Q\,x\,.\,a : P\,x \to Q\,x \to P\,x$	**8 Abst**
10.	$\lambda x : S\,.\lambda a : P\,x\,.\lambda b : Q\,x\,.\,a$ $: \Pi\,x : S\,.\,P\,x \to Q\,x \to P\,x$	**9 Abst**

Problem

(5.8 b) Let $\Gamma \equiv S : *, P : S \to *, Q : S \to *$, find an inhabitant of

$$\Pi\,x : S\,.\,P\,x \to Q\,x \to P\,x$$

By proving the corresponding proposition in natural deduction.

Solution. The corresponding proposition and premises are

$$\frac{S \in \text{Set} \qquad P : S \to \text{Prop} \qquad Q : S \to \text{Prop}}{\forall\, a \in S, P\,(a) \Rightarrow (Q\,(a) \Rightarrow P\,(a))}$$

Proof.

1.	Let $a \in S$	
2.	Assume $P\,(a)$	
3.	Assume $Q\,(a)$	
4.	$P\,(a)$	
5.	$Q\,(a) \Rightarrow P\,(a)$	**3,4 $\Rightarrow$I**
6.	$P\,(a) \Rightarrow (Q\,(a) \Rightarrow P\,(a))$	**2,5 $\Rightarrow$I**
7.	$\forall\, a \in S, P\,(a) \Rightarrow (Q\,(a) \Rightarrow P\,(a))$	**1,6 $\forall$I**

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