Exercises

Chapter 9

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1. Redacted

Definition Extended Rules for λD_0

$$\frac{\Delta; \Gamma \vdash K : L \qquad \Delta; \overline{a} : \overline{M} \vdash M : N}{\Delta, \left(\overline{x} : \overline{A} \rhd a\left(\overline{x}\right) := M : N\right); \Gamma \vdash K : L}\text{def}$$

$$\frac{\Delta, \mathcal{D}; \Gamma \vdash * : \square \qquad \Delta, \mathcal{D}; \Gamma \vdash \overline{U} : \overline{A\left[\overline{x} := \overline{U}\right]} \qquad \mathcal{D} \equiv \overline{x} : \overline{A} \rhd a\left(\overline{x}\right) := M : N}{\Delta, \mathcal{D}; \Gamma \vdash a\left(\overline{U}\right) : N\left[\overline{x} := \overline{U}\right]}\text{inst}$$

$$\frac{\Delta; \Gamma \vdash x : A \qquad \Delta; \Gamma \vdash A : s \qquad A \overset{\Delta,\beta}{=} B}{\Delta; \Gamma \vdash x : B}\text{conv}$$

Lemma 1. Given $\mathcal{D} \equiv \overline{x} : \overline{A} \rhd a\left(\overline{x}\right) := M : N$ and $a \notin \Delta$

$$\frac{\Delta; \overline{x} : \overline{A} \vdash M : N}{\Delta, \mathcal{D}; \overline{x} : \overline{A} \vdash a\left(\overline{x}\right) : N}\text{par}$$

Problem

(9.1) Given

$$\begin{array}{llllll} (\mathcal{D}_1) & x:\mathbb{Z}, y:\mathbb{Z} & \rhd & a\,(x,y) & := x^2+y^2 & :\mathbb{Z} \\ (\mathcal{D}_2) & x:\mathbb{Z}, y:\mathbb{Z} & \rhd & b\,(x,y) & := 2\cdot(x\cdot y) & :\mathbb{Z} \\ (\mathcal{D}_3) & x:\mathbb{Z}, y:\mathbb{Z} & \rhd & c\,(x,y) & := a\,(x,y)+b\,(x,y) & :\mathbb{Z} \\ (\mathcal{D}_4) & x:\mathbb{Z}, y:\mathbb{Z} & \rhd & \mathrm{lemma}(x,y) & := c\,(x,y)=(x+y)^2 & :*_p \end{array}$$

Consider $\Delta \equiv \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$. Describe the dependencies between the four definitions and give all possible linearizations of the corresponding partial order.

Solution. Hasse diagram given below

$$\begin{tikzcd} \boxed{\begin{array}{c}\boldsymbol{\mathcal{D}_1}\\ a := x^2+y^2 : \mathbb{Z}\end{array}} \arrow[r] & \boxed{\begin{array}{c}\boldsymbol{\mathcal{D}_3}\\ c := a\,(x,y)+b\,(x,y) : \mathbb{Z}\end{array}} \arrow[d] & \boxed{\begin{array}{c}\boldsymbol{\mathcal{D}_2}\\ b := 2\cdot(x\cdot y) : \mathbb{Z}\end{array}} \arrow[l] \\ & \boxed{\begin{array}{c}\boldsymbol{\mathcal{D}_4}\\ \mathrm{lemma} := c\,(x,y)=(x+y)^2 : *_p\end{array}} & \end{tikzcd}$$

The only incomparable pair is $(\mathcal{D}_1, \mathcal{D}_2)$. Therefore there are two possible linearizations:

$$\begin{array}{ll} (1) & \mathcal{D}_1 \le \mathcal{D}_2 \le \mathcal{D}_3 \le \mathcal{D}_4 \\ (2) & \mathcal{D}_2 \le \mathcal{D}_1 \le \mathcal{D}_3 \le \mathcal{D}_4 \end{array}$$

Problem

(9.2) Consider

$$\mathcal{D}_i \equiv \overline{x} : \overline{A} \rhd a\,(\overline{x}) := K : L$$
$$\mathcal{D}_j \equiv \overline{y} : \overline{B} \rhd b\,(\overline{y}) := M : N$$

Let $\Delta; \Gamma \vdash U : V$ and assume $\mathcal{D}_i$ and $\mathcal{D}_j$ are elements of list Δ, where $\mathcal{D}_i$ precedes $\mathcal{D}_j$. Describe precisely where constant a may occur in $\mathcal{D}_i$ and $\mathcal{D}_j$ and where constant b may occur in Δ.

Solution. In order for $\mathcal{D}_i$ to be a valid definition, $\overline{x} : \overline{A} \vdash K : L$ must be legal. Therefore by the free variable lemma any free variables in K and L must be in $\overline{x} : \overline{A}$, which by the time, does not yet contain a's definition. Therefore, a could only appear in $\mathcal{D}_j$.

By similar reasoning b could only have appeared in definitions after $\mathcal{D}_j$. Assuming the list sorted by the suffix, then b could only have been in any $\mathcal{D}_k$ where $k > j$.

Problem

(9.3) Recall Q 8.2

1. $V : *_s$
2. $\quad u : V \subseteq \mathbb{R}$
3. $\quad\quad \text{bounded-from-above}(V, u) := \exists y : R\, .\forall x : \mathbb{R}, (x \in V \Rightarrow x \leq y) : *_p$
4. $\quad\quad s : \mathbb{R}$
5. $\quad\quad\quad \text{upper-bound}(V, u, s) := \forall x \in \mathbb{R}.(x \in V \Rightarrow x \leq s) : *_p$
6. $\quad\quad\quad \text{least-upper-bound}(V, u, s) := \text{upper-bound}(V, u, s) \wedge \forall x : \mathbb{R}.(x < s \Rightarrow \neg \text{ upper-bound}(V, u, x)) : *_p$
7. $\quad\quad v : V \neq \emptyset$
8. $\quad\quad\quad w : \text{bounded-from-above}(V, u)$
9. $\quad\quad\quad\quad p_4\ (V, u, w\ v) := \texttt{sorry} : \exists^1\ s : \mathbb{R}.\ \text{least-upper-bound}(V, u, s)$
10. $S := \left\{x : \mathbb{R} \mid \exists n : \mathbb{R}.\left(n \in \mathbb{N} \wedge x = \frac{n}{n+1}\right)\right\}$
11. $p_6 := \texttt{sorry} : S \subseteq \mathbb{R}$
12. $p_7 := \texttt{sorry} : \text{bounded-from-above}(S, p_6)$
13. $p_8 := \texttt{sorry} : \text{least-upper-bound}(S, p_6, 1)$

Write p_8 out such that all definitions have been unfolded.

Solution.

$$\begin{aligned}
p_8 &:= \text{least-upper-bound}(S, p_6, 1) \\
&\underset{\delta}{=} \text{upper-bound}(S, p_6, 1) \wedge \forall x : \mathbb{R}.(x < 1 \Rightarrow \neg \text{ upper-bound}(S, p_6, 1)) \\
&\underset{\delta}{=} \forall x : \mathbb{R}.(x \in S \Rightarrow x \leq 1) \wedge \forall x : \mathbb{R}.(x < 1 \Rightarrow \neg(\forall y : \mathbb{R}.(y \in S \Rightarrow y \leq x))) \\
&\underset{\delta}{=} \forall x : \mathbb{R}.\left(x \in \left\{x : \mathbb{R} \mid \exists n : \mathbb{R}.\left(n \in \mathbb{N} \wedge x = \frac{n}{n+1}\right)\right\} \Rightarrow x \leq 1\right) \wedge \\
&\qquad \forall x : \mathbb{R}.\left(x < 1 \Rightarrow \neg\forall y : \mathbb{R}.\left(y \in \left\{k : \mathbb{R} \mid \exists n : \mathbb{R}.\left(n \in \mathbb{N} \wedge k = \frac{n}{n+1}\right)\right\} \Rightarrow y \leq x\right)\right)
\end{aligned}$$

Problem

(9.4) Recall $\Delta \equiv \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ from 9.1. Give a complete δ-reduction diagram for

$$c\ (a\ (u, v), b\ (w, w))$$

Solution. Too long to contain. An algorithm for finding the graph is proposed as below:

Let $V := \emptyset : \texttt{Set}$ of type $\mathcal{E}_{\lambda D}$
Let $E := \emptyset : \texttt{Set}$ of type $(\mathcal{E}_{\lambda D} \times \mathcal{E}_{\lambda D})$
Define *procedure* reduce$(t : \mathcal{E}_{\lambda D}, \Delta : \texttt{Env})$ *do*
    If $t \in V$ **then** *terminate*
    Else
        Set $V := V \cup \{t\}$
        Loop for each redex r of t *do*
            Let $r' :=$ outermost one-step δ-reduction of r
            Let $t' := t\ [r := r']$
            Set $E := E \cup \{(t, t')\}$
            Execute reduce(t', Δ)
        End *loop*
    End *if*
End *reduce*
Main
    Define $\Delta := \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$
    Execute reduce$(c\ (a\ (u, v), b\ (w, w)), \Delta)$ *and discard result*
    Graph (V, E)
    Terminates
End *main*

Problem

(9.5) Check that all instantiations in the 8.2 proof is legal under *inst* rule.

Solution. It is trivial that the first (well-formed context and enviroment) and third (definition existence) holds for all instantiations.

The instantiation on line 6 …upper-bound(V, u, s)… requires the following premises after expansion:

$$\begin{aligned}
&\Delta;\Gamma \vdash V : (*_s\ [V := V, u := u, s := s] \text{ which is } *_s) && \checkmark \\
&\Delta;\Gamma \vdash u : (V \subseteq \mathbb{R}[V := V, u := u, s := s] \text{ which is } V) && \checkmark \\
&\Delta;\Gamma \vdash s : (\mathbb{R}[V := V, u := u, s := s] \text{ which is } \mathbb{R}) && \checkmark
\end{aligned}$$

The instantiation on line 6 ...upper-bound(V, u, x)... requires the following premises after expansion:

$$\begin{aligned}
&\Delta;\Gamma \vdash V : (*_s\ [V := V, u := u, s := x] \text{ which is } *_s) && \checkmark \\
&\Delta;\Gamma \vdash u : (V \subseteq \mathbb{R}[V := V, u := u, s := x] \text{ which is } V) && \checkmark \\
&\Delta;\Gamma \vdash x : (\mathbb{R}[V := V, u := u, s := x] \text{ which is } \mathbb{R}) && \checkmark
\end{aligned}$$

The instantiation on line 8 ...bounded-from-above(V, u)... requires the following premises after expansion:

$$\begin{aligned}
&\Delta;\Gamma \vdash V : (*_s\ [V := V, u := u] \text{ which is } *_s) && \checkmark \\
&\Delta;\Gamma \vdash u : (V \subseteq \mathbb{R}[V := V, u := u] \text{ which is } V \subseteq \mathbb{R}) && \checkmark
\end{aligned}$$

The instantiation on line 9 ...least-upper-bound(V, u, s)... requires the following premises after expansion:

$$\begin{aligned}
&\Delta;\Gamma \vdash V : (*_s\ [V := V, u := u, s := s] \text{ which is } *_s) && \checkmark \\
&\Delta;\Gamma \vdash u : ((V \subseteq \mathbb{R})[V := V, u := u, s := s] \text{ which is } V \subseteq \mathbb{R}) && \checkmark \\
&\Delta;\Gamma \vdash s : (\mathbb{R}[V := V, u := u, s := s] \text{ which is } \mathbb{R}) && \checkmark
\end{aligned}$$

The instantiation on line 12 ...bounded-from-above(S, p_6)... requires the following premises after expansion:

$$\begin{aligned}
&\Delta;\Gamma \vdash S : (*_s\ [V := S, u := p_6] \text{ which is } *_s) && \checkmark \\
&\Delta;\Gamma \vdash p_6 : ((V \subseteq \mathbb{R})[V := S, u := p_6] \text{ which is } S \subseteq \mathbb{R}) && \checkmark
\end{aligned}$$

The instantiation on line 13 ...least-upper-bound$(S, p_6, 1)$... requires the following premises after expansion:

$$\begin{aligned}
&\Delta;\Gamma \vdash S : (*_s\ [V := S, u := p_6, s := 1] \text{ which is } *_s) && \checkmark \\
&\Delta;\Gamma \vdash p_6 : ((V \subseteq \mathbb{R})[V := S, u := p_6, s := 1] \text{ which is } S \subseteq \mathbb{R}) && \checkmark \\
&\Delta;\Gamma \vdash 1 : (\mathbb{R}[V := S, u := p_6; s := 1] \text{ which is } \mathbb{R}) && \checkmark
\end{aligned}$$

Therefore all of the instantiations are valid.