# Exercises

## Chapter 9

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1. Redacted

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**Definition** Extended Rules for $\lambda D_0$

$$\frac{\Delta;\Gamma \vdash K : L \qquad \Delta;\overline{a} : \overline{M} \vdash M : N}{\Delta, \left(\overline{x} : \overline{A} \rhd a\left(\overline{x}\right) := M : N\right);\Gamma \vdash K : L}\,\text{def}$$

$$\frac{\Delta, \mathcal{D};\Gamma \vdash * : \square \qquad \Delta, \mathcal{D};\Gamma \vdash \overline{U} : \overline{A\left[\overline{x} := \overline{U}\right]} \qquad \mathcal{D} \equiv \overline{x} : \overline{A} \rhd a\left(\overline{x}\right) := M : N}{\Delta, \mathcal{D};\Gamma \vdash a\left(\overline{U}\right) : N\left[\overline{x} := \overline{U}\right]}\,\text{inst}$$

$$\frac{\Delta;\Gamma \vdash x : A \qquad \Delta;\Gamma \vdash A : s \qquad A \overset{\Delta,\beta}{=} B}{\Delta;\Gamma \vdash x : B}\,\text{conv}$$

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*Lemma 1.* Given $\mathcal{D} \equiv \overline{x} : \overline{A} \rhd a\left(\overline{x}\right) := M : N$ and $a \notin \Delta$

$$\frac{\Delta;\overline{x} : \overline{A} \vdash M : N}{\Delta, \mathcal{D};\overline{x} : \overline{A} \vdash a\left(\overline{x}\right) : N}\,\text{par}$$

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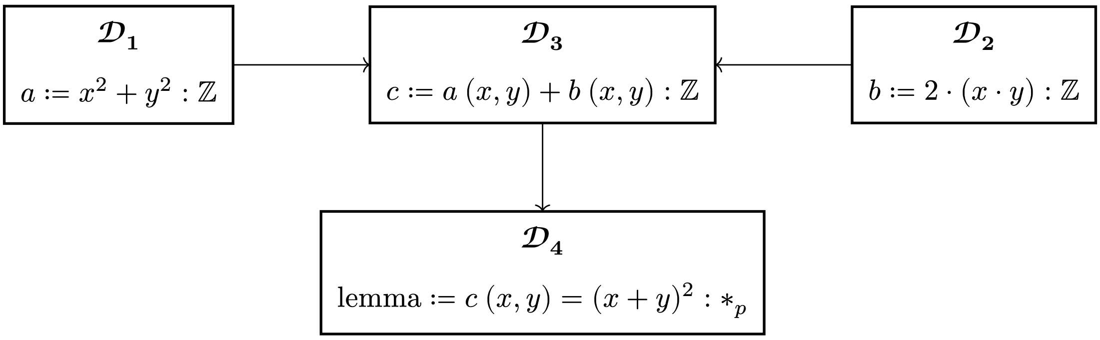
## Problem

*(9.1)* Given

$$
\begin{array}{llclll}
(\mathcal{D}_1) & x:\mathbb{Z}, y:\mathbb{Z} & \rhd & a\,(x,y) & := x^2+y^2 & :\mathbb{Z} \\
(\mathcal{D}_2) & x:\mathbb{Z}, y:\mathbb{Z} & \rhd & b\,(x,y) & := 2\cdot(x\cdot y) & :\mathbb{Z} \\
(\mathcal{D}_3) & x:\mathbb{Z}, y:\mathbb{Z} & \rhd & c\,(x,y) & := a\,(x,y)+b\,(x,y) & :\mathbb{Z} \\
(\mathcal{D}_4) & x:\mathbb{Z}, y:\mathbb{Z} & \rhd & \mathrm{lemma}(x,y) & := c\,(x,y)=(x+y)^2 & :*_p
\end{array}
$$

Consider $\Delta \equiv \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$. Describe the dependencies between the four definitions and give all possible linearizations of the corresponding partial order.

*Solution.* Hasse diagram given below




The only incomparable pair is $(\mathcal{D}_1, \mathcal{D}_2)$. Therefore there are two possible linearizations:

$$
\begin{array}{ll}
(1) & \mathcal{D}_1 \leq \mathcal{D}_2 \leq \mathcal{D}_3 \leq \mathcal{D}_4 \\
(2) & \mathcal{D}_2 \leq \mathcal{D}_1 \leq \mathcal{D}_3 \leq \mathcal{D}_4
\end{array}
$$

## Problem

*(9.2)* Consider

$$\mathcal{D}_i \equiv \overline{x} : \overline{A} \rhd a\,(\overline{x}) := K : L$$
$$\mathcal{D}_j \equiv \overline{y} : \overline{B} \rhd b\,(\overline{y}) := M : N$$

Let $\Delta; \Gamma \vdash U : V$ and assume $\mathcal{D}_i$ and $\mathcal{D}_j$ are elements of list $\Delta$, where $\mathcal{D}_i$ precedes $\mathcal{D}_j$. Describe precisely where constant $a$ may occur in $\mathcal{D}_i$ and $\mathcal{D}_j$ and where constant $b$ may occur in $\Delta$.

*Solution.* In order for $\mathcal{D}_i$ to be a valid definition, $\overline{x} : \overline{A} \vdash K : L$ must be legal. Therefore by the free variable lemma any free variables in $K$ and $L$ must be in $\overline{x} : \overline{A}$, which by the time, does not yet contain $a$'s definition. Therefore, $a$ could only appear in $\mathcal{D}_j$.

By similar reasoning $b$ could only have appeared in definitions after $\mathcal{D}_j$. Assuming the list sorted by the suffix, then $b$ could only have been in any $\mathcal{D}_k$ where $k > j$.

**Problem**

*(9.3)* Recall Q 8.2

1. $V : *_s$
2. $\quad u : V \subseteq \mathbb{R}$
3. $\quad\quad \text{bounded-from-above}(V, u) := \exists y : R\, . \forall x : \mathbb{R}, (x \in V \Rightarrow x \leq y) : *_p$
4. $\quad\quad s : \mathbb{R}$
5. $\quad\quad\quad \text{upper-bound}(V, u, s) := \forall x \in \mathbb{R}.(x \in V \Rightarrow x \leq s) : *_p$
6. $\quad\quad\quad \text{least-upper-bound}(V, u, s) := \text{upper-bound}(V, u, s) \wedge \forall x : \mathbb{R}.(x < s \Rightarrow \neg\ \text{upper-bound}(V, u, x)) : *_p$
7. $\quad\quad v : V \neq \emptyset$
8. $\quad\quad\quad w : \text{bounded-from-above}(V, u)$
9. $\quad\quad\quad\quad p_4\ (V, u, w\ v) := \texttt{sorry} : \exists^1\ s : \mathbb{R}.\ \text{least-upper-bound}(V, u, s)$
10. $S := \left\{x : \mathbb{R} \mid \exists n : \mathbb{R}.\left(n \in \mathbb{N} \wedge x = \frac{n}{n+1}\right)\right\}$
11. $p_6 := \texttt{sorry} : S \subseteq \mathbb{R}$
12. $p_7 := \texttt{sorry} : \text{bounded-from-above}(S, p_6)$
13. $p_8 := \texttt{sorry} : \text{least-upper-bound}(S, p_6, 1)$

Write $p_8$ out such that all definitions have been unfolded.

*Solution.*

$$
\begin{aligned}
p_8 &:= \text{least-upper-bound}(S, p_6, 1) \\
&\underset{\delta}{=} \text{upper-bound}(S, p_6, 1) \wedge \forall x : \mathbb{R}.(x < 1 \Rightarrow \neg\ \text{upper-bound}(S, p_6, 1)) \\
&\underset{\delta}{=} \forall x : \mathbb{R}.(x \in S \Rightarrow x \leq 1) \wedge \forall x : \mathbb{R}.(x < 1 \Rightarrow \neg(\forall y : \mathbb{R}.(y \in S \Rightarrow y \leq x))) \\
&\underset{\delta}{=} \forall x : \mathbb{R}.\left(x \in \left\{x : \mathbb{R} \mid \exists n : \mathbb{R}.\left(n \in \mathbb{N} \wedge x = \frac{n}{n+1}\right)\right\} \Rightarrow x \leq 1\right) \wedge \\
&\qquad \forall x : \mathbb{R}.\left(x < 1 \Rightarrow \neg\forall y : \mathbb{R}.\left(y \in \left\{k : \mathbb{R} \mid \exists n : \mathbb{R}.\left(n \in \mathbb{N} \wedge k = \frac{n}{n+1}\right)\right\} \Rightarrow y \leq x\right)\right)
\end{aligned}
$$

## Problem

*(9.4)* Recall $\Delta \equiv \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ from 9.1. Give a complete $\delta$-reduction diagram for

$$c\ (a\ (u, v), b\ (w, w))$$

*Solution.* Too long to contain. An algorithm for finding the graph is proposed as below:

**Let** $V := \emptyset$ : `Set` of type $\mathcal{E}_{\lambda D}$
**Let** $E := \emptyset$ : `Set` of type $(\mathcal{E}_{\lambda D} \times \mathcal{E}_{\lambda D})$
**Define** *procedure* reduce($t : \mathcal{E}_{\lambda D}, \Delta$ : `Env`) *do*
  **If** $t \in V$ **then** *terminate*
  **Else**
    **Set** $V := V \cup \{t\}$
    **Loop** for each redex $r$ of $t$ *do*
      **Let** $r' :=$ outermost one-step $\delta$-reduction of $r$
      **Let** $t' := t\ [r := r']$
      **Set** $E := E \cup \{(t, t')\}$
      **Execute** reduce($t', \Delta$)
    **End** *loop*
  **End** *if*
**End** *reduce*
**Main**
  **Define** $\Delta := \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$
  **Execute** reduce($c\ (a\ (u, v), b\ (w, w)), \Delta$) *and discard result*
  **Graph** $(V, E)$
  **Terminates**
**End** *main*

## Problem

*(9.5)* Check that all instantiations in the 8.2 proof is legal under *inst* rule.

*Solution.* It is trivial that the first (well-formed context and enviroment) and third (definition existence) holds for all instantiations.

The instantiation on line 6 ...upper-bound$(V, u, s)$... requires the following premises after expansion:

$$\begin{aligned}
&\Delta; \Gamma \vdash V : (*_s\ [V \coloneqq V, u \coloneqq u, s \coloneqq s] \text{ which is } *_s) && \checkmark \\
&\Delta; \Gamma \vdash u : (V \subseteq \mathbb{R}[V \coloneqq V, u \coloneqq u, s \coloneqq s] \text{ which is } V) && \checkmark \\
&\Delta; \Gamma \vdash s : (\mathbb{R}[V \coloneqq V, u \coloneqq u, s \coloneqq s] \text{ which is } \mathbb{R}) && \checkmark
\end{aligned}$$

The instantiation on line 6 …upper-bound$(V, u, x)$… requires the following premises after expansion:

$$\begin{aligned}
&\Delta; \Gamma \vdash V : (*_s\ [V \coloneqq V, u \coloneqq u, s \coloneqq x] \text{ which is } *_s) && \checkmark \\
&\Delta; \Gamma \vdash u : (V \subseteq \mathbb{R}[V \coloneqq V, u \coloneqq u, s \coloneqq x] \text{ which is } V) && \checkmark \\
&\Delta; \Gamma \vdash x : (\mathbb{R}[V \coloneqq V, u \coloneqq u, s \coloneqq x] \text{ which is } \mathbb{R}) && \checkmark
\end{aligned}$$

The instantiation on line 8 …bounded-from-above$(V, u)$… requires the following premises after expansion:

$$\begin{aligned}
&\Delta; \Gamma \vdash V : (*_s\ [V \coloneqq V, u \coloneqq u] \text{ which is } *_s) && \checkmark \\
&\Delta; \Gamma \vdash u : (V \subseteq \mathbb{R}[V \coloneqq V, u \coloneqq u] \text{ which is } V \subseteq \mathbb{R}) && \checkmark
\end{aligned}$$

The instantiation on line 9 …least-upper-bound$(V, u, s)$… requires the following premises after expansion:

$$\begin{aligned}
&\Delta; \Gamma \vdash V : (*_s\ [V \coloneqq V, u \coloneqq u, s \coloneqq s] \text{ which is } *_s) && \checkmark \\
&\Delta; \Gamma \vdash u : ((V \subseteq \mathbb{R})[V \coloneqq V, u \coloneqq u, s \coloneqq s] \text{ which is } V \subseteq \mathbb{R}) && \checkmark \\
&\Delta; \Gamma \vdash s : (\mathbb{R}[V \coloneqq V, u \coloneqq u, s \coloneqq s] \text{ which is } \mathbb{R}) && \checkmark
\end{aligned}$$

The instantiation on line 12 …bounded-from-above$(S, p_6)$… requires the following premises after expansion:

$$\begin{aligned}
&\Delta; \Gamma \vdash S : (*_s\ [V \coloneqq S, u \coloneqq p_6] \text{ which is } *_s) && \checkmark \\
&\Delta; \Gamma \vdash p_6 : ((V \subseteq \mathbb{R})[V \coloneqq S, u \coloneqq p_6] \text{ which is } S \subseteq \mathbb{R}) && \checkmark
\end{aligned}$$

The instantiation on line 13 …least-upper-bound$(S, p_6, 1)$… requires the following premises after expansion:

$$\begin{aligned}
&\Delta; \Gamma \vdash S : (*_s\ [V \coloneqq S, u \coloneqq p_6, s \coloneqq 1] \text{ which is } *_s) && \checkmark \\
&\Delta; \Gamma \vdash p_6 : ((V \subseteq \mathbb{R})[V \coloneqq S, u \coloneqq p_6, s \coloneqq 1] \text{ which is } S \subseteq \mathbb{R}) && \checkmark \\
&\Delta; \Gamma \vdash 1 : (\mathbb{R}[V \coloneqq S, u \coloneqq p_6; s \coloneqq 1] \text{ which is } \mathbb{R}) && \checkmark
\end{aligned}$$

Therefore all of the instantiations are valid.

**Problem**

*(9.6 a)* Consider the following formal enviroment $\Delta$ consisting of six definitions, in which we use, for the sake of convenience, some well-known formats such as the summation sigma and infix-notations:

$$\mathcal{D}_1 \equiv f : \mathbb{N} \to \mathbb{R}, n : \mathbb{N} \rhd a_1\ (f, n) := \sum_{i=0}^{n} (f\ i) : \mathbb{R}$$

$$\mathcal{D}_2 \equiv f : \mathbb{N} \to \mathbb{R}, d : \mathbb{R} \rhd a_2\ (f, d) := \forall_{n:\mathbb{N}} (f\ (n+1) - f\ n = d) : *_p$$

$$\mathcal{D}_3 \equiv f : \mathbb{N} \to \mathbb{R}, d : \mathbb{R}, u : a_2\ (f, d), n : \mathbb{N} \rhd$$
$$a_3\ (f, d, u, n) := \mathtt{sorry} : f\ n = f\ 0 + n \times d$$

$$\mathcal{D}_4 \equiv f : \mathbb{N} \to \mathbb{R}, d : \mathbb{R}, u : a_2\ (f, d), n : \mathbb{N} \rhd$$
$$a_4\ (f, d, u, n) := \mathtt{sorry} : a_1\ (f, n) = (n+1) \times (f\ 0 + f\ n) \div 2$$

$$\mathcal{D}_5 \equiv f : \mathbb{N} \to \mathbb{R}, d : \mathbb{R}, u : a_2\ (f, d), n : \mathbb{N} \rhd$$
$$a_5\ (f, d, u, )) := \mathtt{sorry} : a_1\ (f, n) = (f\ 0) \times (n+1) + n \times (n+1) \times d \div 2$$

$$\mathcal{D}_6 \equiv \emptyset \rhd a_6 := \mathtt{sorry} : \sum_{i=0}^{100} i = 5050$$

Rewrite this in flag notation.

*Solution.* All meta-level notations appearing below are assumed to be formally defined. That is we are intentionally not expanding common notational abbreviations into explicit flags. Moreover, all sets – unless explicitly defined in-text – and literals denoting elements of them are assumed to be defined earlier.

1. $f : \mathbb{N} \to \mathbb{R}$
2. $\quad n : \mathbb{N}$
3. $\quad\quad a_1\ (f, n) := \sum_{i=0}^{n} (f\ i) : \mathbb{R}$ $\qquad \mathcal{D}_1$
4. $\quad d : \mathbb{R}$
5. $\quad\quad a_2\ (f, d) := \forall n : \mathbb{N}.(f\ (n+1) - f\ n = d) : *_p$ $\qquad \mathcal{D}_2$
6. $\quad\quad u : a_2\ (f, d)$
7. $\quad\quad\quad n : \mathbb{N}$
8. $\quad\quad\quad\quad a_3\ (f, d, u, n) := \mathtt{sorry} : f\ n = f\ 0 + n \times d$ $\qquad \mathcal{D}_3$
9. $\quad\quad\quad\quad a_4\ (f, d, u, n) := \mathtt{sorry} : a_1\ (f, n) = \frac{1}{2}(n+1) \times (f\ 0 + f\ n)$ $\qquad \mathcal{D}_4$

10. $\left| \; \left| \; \left| \; \left| \; \begin{array}{l} a_5\ (f, d, u, n) := \texttt{sorry} : \\ \quad a_1\ (f, n) = (f\ 0) \times (n + 1) + \frac{1}{2}\, n \times (n + 1) \times d \end{array} \right. \right. \right. \right.$ $\mathcal{D}_5$

11.

$a_6 := \texttt{sorry} : \sum_{i=0}^{100} i = 5050$ $\mathcal{D}_6$

**Problem**

*(9.6 b)* What is the name of $a_2$ in standard literature?

*Solution.* It is obvious that $f$ is a sequence over $\mathbb{R}$. $a_2$ provides a proof that the difference between any two term in $f$ is a constant. Therefore $f$ is an **arithmetic progression**.

Thus $a_2$ is a **predicate over an sequence $f$ and real $d$ encoding the fact of $f$ being a arithmetic progression with common difference $d$**.

**Problem**

*(9.6 c)* Find the $\delta$-normal form with respect to $\Delta$ of $a_5\ ((\lambda x : \mathbb{N}.2\ x), 2, u, 100)$ where u is an inhabitant of $a_2\ ((\lambda x : N\ .2\ x), 2)$

*Solution.*

$$
\begin{aligned}
& a_5\ ((\lambda x : \mathbb{N}.2\ x), 2, u, 100) \\
\overset{\Delta}{\twoheadrightarrow}\ & \sum_{i=0}^{100} ((\lambda x : \mathbb{N}.2\ x)\ i) = ((\lambda x : \mathbb{N}.2\ x) 0) \times (100 + 1) + \frac{1}{2} 100 \times (100 + 1) \times 2 \\
\overset{\beta}{\twoheadrightarrow}\ & \sum_{i=0}^{100} (2 \times i) = 0 \times (100 + 1) + \frac{1}{2} 100 \times (100 + 1) \times 2 \\
\overset{\text{arith.}}{=}\ & \sum_{i=0}^{100} 2 \times i = 10100
\end{aligned}
$$

Actually line 2 is already a valid answer. Its just that line 4 looks nicer lol