

EXERCISES

CHAPTER 5

SEAN LI ¹

1. Reduced

Definition Some rules for reference.

λP Calculus Rules

$$\begin{array}{c} \frac{}{\emptyset \vdash * : \square} \text{Sort} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \text{Var} \quad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \text{Weak} \\[10pt] \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A . B : s} \text{Form} \quad \frac{\Gamma \vdash M : \Pi x : A . B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B [x := N]} \text{App} \\[10pt] \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A . B : s}{\Gamma \vdash \lambda x : A . M : \Pi x : A . B} \text{Abst} \\[10pt] \frac{\Gamma \vdash A : B \quad B \stackrel{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \text{Conv} \end{array}$$

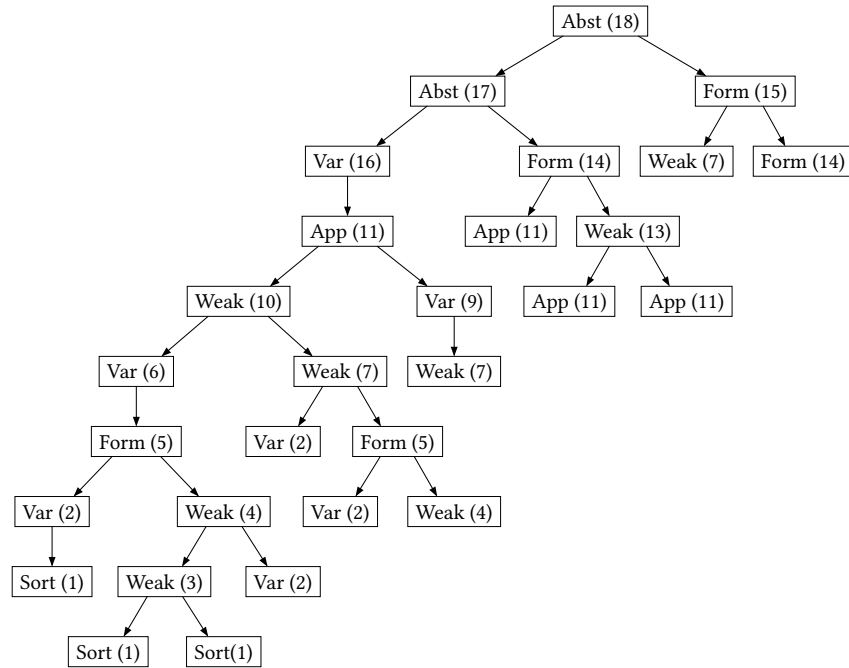
Predicate Logic

$$\begin{array}{c} \begin{array}{l} 1. \text{ Assume } A \\ 2. \quad \left| \begin{array}{l} \dots \\ 3. \quad B \end{array} \right. \\ \hline A \Rightarrow B \end{array} \Rightarrow I \quad \frac{A \Rightarrow B \quad A}{B} \Rightarrow E \quad \begin{array}{l} 1. \text{ Let } a \in S \\ 2. \quad \left| \begin{array}{l} \dots \\ 3. \quad P(a) \end{array} \right. \\ \hline \forall a \in S, P(a) \end{array} \forall I \\[10pt] \frac{\forall a \in S \quad N \in S}{P(N)} \forall E \end{array}$$

Problem

(5.1) Give a diagram of the tree corresponding to the complete tree derivation of line 18 of Section 5.3 (P 107)

Solution.



Problem

(5.2) Give a complete λP derivation of

$$S : * \vdash S \rightarrow S \rightarrow * : \square$$

In tree format and flag format.

Solution.

Tree Derivation.

$$\begin{array}{c}
 (4) \frac{\vdash * : \square \quad \vdash * : \square}{S : * \vdash * : \square} \text{Weak} \quad (3) S : * \vdash S : * \\
 (6) \frac{S : * \vdash * : \square}{S : *, x : S \vdash * : \square} \text{Weak} \\
 (7) \frac{(3) S : * \vdash S : *}{S : * \vdash S \rightarrow * : \square}
 \end{array}$$

$$\begin{array}{c}
(3) \frac{\vdash * : \square}{S : * \vdash S : *} \text{Var} \quad (9) \frac{(7) S : * \vdash S \rightarrow * : \square \quad (3) S : * \vdash S : *}{S : *, x : S \vdash S \rightarrow * : \square} \text{Weak} \\
\hline
S : * \vdash S \rightarrow S \rightarrow * : \square \text{Form}
\end{array}$$

■

Flag Derivation.

1.	$* : \square$	Sort
2.	$S : *$	
3.	$S : *$	1 Var
4.	$* : \square$	1,1 Weak
5.	$x : S$	
6.	$\boxed{* : \square}$	4,3 Weak
7.	$S \rightarrow * : \square$	3,6 Form
8.	$x : S$	
9.	$\boxed{S \rightarrow * : \square}$	7,3 Weak
10.	$S \rightarrow S \rightarrow * : \square$	3,9 Form

■

Problem

(5.3) Derive

$$S : *, Q : S \rightarrow S \rightarrow * \vdash \Pi x : S . \Pi y : S . Q x y : *$$

Solution.

1.	$* : \square$	Sort
2.	$S : *$	
3.	$S : *$	1 Var
4.	$* : \square$	1,1 Weak
5.	$x : S$	
6.	$\boxed{* : \square}$	4,3 Weak
7.	$S \rightarrow * : \square$	3,6 Form
8.	$x : S$	
9.	$\boxed{S \rightarrow * : \square}$	7,3 Weak
10.	$S \rightarrow S \rightarrow * : \square$	3,9 Form
11.	$Q : S \rightarrow S \rightarrow *$	

12.	$Q : S \rightarrow S \rightarrow *$	10 Var
13.	$S : *$	3,10 Weak
14.	$* : \square$	4,10 Weak
15.	$x : S$	
16.	$* : \square$	14,13 Weak
17.	$S : *$	13,13 Weak
18.	$x : S$	13 Var
19.	$Q : S \rightarrow S \rightarrow *$	12,13 Weak
20.	$y : S$	
21.	$y : S$	17 Var
22.	$Q : S \rightarrow S \rightarrow *$	19,17 Weak
23.	$x : S$	18,17 Weak
24.	$Q x : S \rightarrow *$	22,23 App
25.	$Q x y : *$	24,21 App
26.	$\Pi y : S . Q x y : *$	17,25 Form
27.	$\Pi x : S . \Pi y : S . Q x y : *$	13,26 Form

Problem

(5.4) Prove that $*$ is the only valid kind in λP .

Solution.

Proof. The only possible way to construct a new kind is through the Form rule and the Sort axiom. Because we are trying to construct a kind, s here stands for \square .

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : \square}{\Gamma \vdash \Pi x : A . B : \square} \text{Form}$$

One could only construct new kinds with kinds, which requires $A : \square$ and $B : \square$. This contradicts with $A : *$. ■

Problem

(5.5) Prove that $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ is a tautology by given a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash A \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$x : A$	
4.	$y : A \rightarrow B$	
5.	$y x : B$	4,3 App
6.	$\lambda y : A \rightarrow B . y x : (A \rightarrow B) \rightarrow B$	5 Abst
7.	$\lambda x : A . \lambda y : A \rightarrow B . y x : A \rightarrow (A \rightarrow B) \rightarrow B$	5 Abst

■

Problem

(5.6 a) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ a tautology using natural deduction.

Solution.

Proof.

1.	Assume $A \Rightarrow (A \Rightarrow B)$	
2.	$A \Rightarrow (A \Rightarrow B)$	
3.	Assume A	
4.	A	
5.	$A \Rightarrow B$	2,4 \Rightarrow E
6.	B	5,4 \Rightarrow E
7.	$A \Rightarrow B$	3,6 \Rightarrow I
8.	$(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$	1,7 \Rightarrow I

■

Problem

(5.6 b) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ using a shorthand λP derivation

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$x : A \rightarrow A \rightarrow B$	
4.	$y : A$	
5.	$x y : A \rightarrow B$	3,4 App
6.	$x y y : B$	5,4 App
7.	$\lambda y : A . x y y : A \rightarrow B$	6 Abst
8.	$\lambda x : A \rightarrow A \rightarrow B . \lambda y : A . x y y : (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$	7 Abst

■

Problem

(5.7 a) Proof $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of A , B , and C is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$C : *$	
4.	$x : A \rightarrow B$	
5.	$y : B \rightarrow C$	
6.	$a : A$	
7.	$x a : B$	4,6 App
8.	$y (x a) : C$	5,7 App

9.				$\lambda a : A . y (x z) : A \rightarrow C$	8 Abst
10.				$\lambda y : B \rightarrow C . \lambda a : A . y (x z) : (B \rightarrow C) \rightarrow A \rightarrow C$	9 Abst
11.				$\lambda x : A \rightarrow B . \lambda y : B \rightarrow C . \lambda a : A . y (x z)$ $: (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C$	10 Abst

■

Problem

(5.7 b) Proof $((A \Rightarrow B) \Rightarrow A) \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$			
2.		$B : *$		
3.		$x : (A \rightarrow B) \rightarrow A$		
4.			$y : A \rightarrow B$	
5.				$x y : A$ 3,4 App
6.				$y (x y) : B$ 4,5 App
7.			$\lambda y : A \rightarrow B . y (x y) : (A \rightarrow B) \rightarrow B$	6 Abst
8.		$\lambda x : (A \rightarrow B) \rightarrow A . \lambda y : A \rightarrow B . y (x y)$ $: ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B$ 7 Abst		

■

Problem

(5.7 c) Proof $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of A , B , and C is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$C : *$	
4.	$x : A \rightarrow B \rightarrow C$	
5.	$y : A \rightarrow B$	
6.	$a : A$	
7.	$x a : B \rightarrow C$	4,6 App
8.	$y a : B$	5,6 App
9.	$x a (y a) : C$	7,8 App
10.	$\lambda a : A . x a (y a) : A \rightarrow C$	9 Abst
11.	$\lambda y : A \rightarrow B . \lambda a : A . x a (y a) : (A \rightarrow B) \rightarrow A \rightarrow C$	10 Abst
12.	$\lambda x : A \rightarrow B \rightarrow C . \lambda y : A \rightarrow B . \lambda a : A . x a (y a) : (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$	10 Abst

■

Problem

(5.8 a) Let $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *$, find an inhabitant of

$$\Pi x : S . P x \rightarrow Q x \rightarrow P x$$

with respect to Γ and give a shorthand λP derivation

Solution.

1.	$S : *$	
2.	$P : S \rightarrow *$	
3.	$Q : S \rightarrow *$	
4.	$x : S$	
5.	$a : P x$	
6.	$b : Q x$	
7.	$a : P x$	2,4 App

8.				$\lambda b : Q\ x . S : Q\ x \rightarrow P\ x$	7 Abst
9.				$\lambda a : P\ x . \lambda b : Q\ x . a : P\ x \rightarrow Q\ x \rightarrow P\ x$	8 Abst
10.				$\lambda x : S . \lambda a : P\ x . \lambda b : Q\ x . a$	
				$: \Pi x : S . P\ x \rightarrow Q\ x \rightarrow P\ x$	9 Abst

Problem

(5.8 b) Let $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *$, find an inhabitant of

$$\Pi x : S . P\ x \rightarrow Q\ x \rightarrow P\ x$$

By proving the corresponding proposition in natural deduction.

Solution. The corresponding proposition and premises are

$$\frac{S \in \text{Set} \quad P : S \rightarrow \text{Prop} \quad Q : S \rightarrow \text{Prop}}{\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a))}$$

Proof.

1. Let $a \in S$
2. Assume $P(a)$
3. Assume $Q(a)$
4. Assume $P(a)$
5. $Q(a) \Rightarrow P(a)$ **3,4 \Rightarrow I**
6. $P(a) \Rightarrow (Q(a) \Rightarrow P(a))$ **2,5 \Rightarrow I**
7. $\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a))$ **1,6 \forall I**

■

Problem

(5.9 a) Give proof for

$$(\forall x \in S, Q(x)) \Rightarrow (\forall y \in S, P(y) \Rightarrow Q(y))$$

by natural deduction and a λP derivation.

Solution.

Natural Deduction.

1. Assume $\forall x \in S, Q(x)$
2. Let $y \in S$
3. Assume $P(y)$
4. | $Q(y)$ **1,2 $\forall E$**
5. | $P(y) \Rightarrow Q(y)$ **3,4 $\Rightarrow I$**
6. $\forall y \in S, P(y) \Rightarrow Q(y)$ **2,5 $\forall I$**
7. $(\forall x \in S, Q(x) \Rightarrow (\forall y \in S, P(y) \Rightarrow Q(y)))$ **1,6 $\Rightarrow I$**

■

λP Derivation. Corresponding type is

$$S : *, P : S \rightarrow *, Q : S \rightarrow * \vdash (\Pi x : S . Q x) \rightarrow (\Pi y : S . P y \rightarrow Q y)$$

1. $S : *$
2. $P : S \rightarrow *$
3. $Q : S \rightarrow *$
4. | $a : \Pi x : S . Q x$
5. | $y : S$
6. | | $z : P y$
7. | | | $a y : Q y$ **4,5 App**
8. | | $\lambda z : P y . a y : P y \rightarrow Q y$ **7 Abst**
9. | $\lambda y : S . \lambda z : P y . a y : \Pi y : S . P y \rightarrow Q y$ **7 Abst**
10. | $\lambda a : \Pi x : S . Q x . \lambda y : S . \lambda z : P y . a y$
10. | $: (\Pi x : S . Q x) \rightarrow (\Pi y : S . P y \rightarrow Q y)$ **7 Abst**

■

Problem

(5.9 b) Give proof for

$$\forall x \in S, (P(x) \Rightarrow Q(x)) \Rightarrow ((\forall y \in S, P(y)) \Rightarrow (\forall z \in S, Q(z)))$$

by natural deduction and a λP derivation

Solution.

Natural Deduction.

1. Assume $\forall x \in S, (P(x) \Rightarrow Q(x))$

2.	Assume $\forall y \in S, P(y)$	
3.	Let $z \in S$	
4.	$P(z)$	2,3 $\forall E$
5.	$P(z) \Rightarrow Q(z)$	1,3 $\forall E$
6.	$Q(z)$	5,4 $\Rightarrow E$
7.	$\forall z \in S, Q(z)$	3,6 $\forall I$
8.	$\forall y \in S, P(y) \Rightarrow (\forall z \in S, Q(z))$	2,7 $\forall I$
9.	$\forall x \in S, (P(x) \Rightarrow Q(x)) \Rightarrow ((\forall y \in S, P(y)) \Rightarrow (\forall z \in S, Q(z)))$	1,8 $\forall I$

■

λP Derivation. Corresponding type is

$$S : *, P : S \rightarrow *, Q : S \rightarrow *$$

$$\vdash (\Pi x : S. P x \rightarrow Q x) \rightarrow (\Pi y : S. P y) \rightarrow (\Pi z : S. Q z)$$

1.	$S : *$	
2.	$P : S \rightarrow *$	
3.	$Q : S \rightarrow *$	
4.	$a : \Pi x : S. P x \rightarrow Q x$	
5.	$b : \Pi y : S. P y$	
6.	$z : S$	
7.	$b z : P z$	5,6 App
8.	$a z : P z \rightarrow Q z$	4,6 App
9.	$a z (b z) : Q z$	8,7 App
10.	$\lambda z : S. a z (b z) : \Pi z : S. Q z$	9 Abst
11.	$\lambda b : (\Pi y : S. P y). \lambda z : S. a z (b z)$ $: (\Pi y : S. P y) \rightarrow \Pi z : S. Q z$	10 Abst
12.	$\lambda a : (\Pi x : S. P x \rightarrow Q x). \lambda b : (\Pi y : S. P y).$ $\lambda z : S. a z (b z)$ $: (\Pi x : S. P x \rightarrow Q x) \rightarrow$ $(\Pi y : S. P y) \rightarrow \Pi z : S. Q z$	10 Abst

■

Problem

(5.10) Given a context

$$\begin{aligned}\Gamma &\equiv S : *, P : S \rightarrow *, f : S \rightarrow S, g : S \rightarrow S, \\ u &: \Pi x : S . (P (f x) \rightarrow P (g x)), \\ v &: \Pi x, y : S . ((P x \rightarrow P y) \rightarrow P (f x))\end{aligned}$$

Let

$$M \equiv \lambda x : S . v (f x)(g x)(u x)$$

Type M under Γ .

Solution.

1.	$x : S$	
2.	$f : S \rightarrow S$	
3.	$f x : S$	2,1 App
4.	$g : S \rightarrow S$	
5.	$g x : S$	4,1 App
6.	$u : \Pi x : S . (P (f x) \rightarrow P (g x))$	
7.	$u x : P (f x) \rightarrow P (g x)$	6,1 App
8.	$v : \Pi x, y : S . ((P x \rightarrow P y) \rightarrow P (f x))$	
9.	$v (f x) : \Pi y : S . ((P (f x) \rightarrow P y) \rightarrow P (f (f x)))$	8,3 App
10.	$v (f x)(g x) : (P (f x) \rightarrow P (g x)) \rightarrow P (f (f x))$	9,5 App
11.	$v (f x)(g x)(u x) : P (f (f x))$	10,7 App
12.	$\lambda x : S . v (f x)(g x)(u x) : S \rightarrow P (f (f x))$	11 Abst

Problem

(5.11) Let S be a set, with Q and R relations on $S \times S$, and let f and g be functions from S to S . Assume

$$\forall x, y \in S (Q(x, f(y)) \Rightarrow Q(g(x), y))$$

$$\forall x, y \in S (Q(x, f(y)) \Rightarrow R(x, y))$$

$$\forall x \in S (Q(x, f(f(x))))$$

Prove that

$$\forall x \in S, R(g(g(x)), g(x))$$

By giving a context Γ and finding a term M such that

$$\Gamma \vdash M : \Pi x : S . R(g(g(x)))(g(x))$$

Solution. Context Γ is as follows:

$$\Gamma \equiv S : *, f : S \rightarrow S, g : S \rightarrow S$$

$$Q : S \rightarrow S \rightarrow *, R : S \rightarrow S \rightarrow *$$

$$A : \Pi x, y : S . (Q(x, f(y)) \rightarrow Q(g(x), y)),$$

$$B : \Pi x, y : S . (Q(x, f(y)) \rightarrow R(x, y))$$

$$C : \Pi x : S . Q(x, f(f(x)))$$

Derivation.

1.	$S : *, f : S \rightarrow S, g : S \rightarrow S$	
2.	$Q : S \rightarrow S \rightarrow *, R : S \rightarrow S \rightarrow *$	
3.	$A : \Pi x, y : S . (Q(x, f(y)) \rightarrow Q(g(x), y))$	
4.	$B : \Pi x, y : S . (Q(x, f(y)) \rightarrow R(x, y))$	
5.	$C : \Pi x : S . Q(x, f(f(x)))$	
6.	$x : S$	
7.	$g(x) : S$	1,6 App
8.	$C(g(x)) : Q(g(x))(f(f(g(x))))$	5,7 App
9.	$f(g(x)) : S$	1,7 App
10.	$A(g(x)) : \Pi y : S . (Q(g(x), f(y)) \rightarrow (Q(g(g(x))), y))$	3,7 App
	$A(g(x))(f(g(x)))$	
	$: (Q(g(x))(f(f(g(x))))$	
11.	$\rightarrow (Q(g(g(x)))(f(g(x))))$	
	10,9 App	

$$12. \quad \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \end{array} \quad \begin{array}{l} A(gx)(f(gx))(C(gx)) \\ : (Q(g(gx))(f(gx))) \end{array}$$

11,8 App

$$13. \quad \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \end{array} \quad \begin{array}{l} g(gx) : S \\ B(g(gx)) \\ : \Pi y : S. (Q(g(gx))(f(y)) \\ \rightarrow R(g(gx))y) \end{array}$$

1,7 App

4,13 App

$$15. \quad \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \end{array} \quad \begin{array}{l} B(g(gx))(gx) \\ : (Q(g(gx))(f(gx))) \rightarrow (R(g(gx))(gx)) \end{array}$$

14,7 App

$$16. \quad \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \end{array} \quad \begin{array}{l} B(g(gx))(gx)(A(gx)(f(gx))(C(gx))) \\ : (R(g(gx))(gx)) \end{array}$$

15,12 App

$$17. \quad \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} \quad \begin{array}{l} \lambda x : S. B(g(gx))(gx)(A(gx)(f(gx))(C(gx))) \\ : \Pi x : S. (R(g(gx))(gx)) \end{array}$$

17 Abst

■