

EXERCISES

CHAPTER 1

SEAN LI ¹

1. Reduced

Problem

Simplify notation of the following terms

(a) $(\lambda x.((xz)y)(xx))$

(b) $((\lambda x.(\lambda y.(\lambda z.(z((xy)z)))))(\lambda u.u))$

Solution.

(a) $\lambda x.(xyz)(xx)$

(b) $(\lambda xyz.x(xyz))(\lambda u.u)$

Problem

Find the alpha equivalent terms to

$$\lambda x.x(\lambda x.x)$$

In

(a) $\lambda y.y(\lambda x.x)$

(b) $\lambda y.y(\lambda x.y)$

(c) $\lambda y.y(\lambda x.y)$

Solution. Only (a).

Problem

Prove

$$\lambda x.x(\lambda z.y) \underset{\alpha}{=} \lambda z.z(\lambda z.y)$$

Solution.

Proof. By definition of alpha equivalence

$$M \underset{\alpha}{=} N \iff \exists \varphi, M^\varphi \underset{\alpha}{\rightarrow} N \wedge \text{FR } M = \text{FR } N$$

The witness of φ is substituting bound variable x with z , and z is not a free variable in the term, thus the two terms are alpha equivalent.

$$\lambda x.x(\lambda z.y) \xrightarrow[\alpha]{x \rightarrow z} \lambda z.z(\lambda z.y)$$

■

Problem

Consider the following term:

$$U := (\lambda z.zxz)((\lambda y.xy)x)$$

1. Find Sub U
2. Draw tree rep of U
3. Find FV U
4. Find alpha equivalent terms to U from below and point out which of those follows the Barendregt convention:

(a) $(\lambda y.yxy)((\lambda z.xz)x)$

(b) $(\lambda x.xyx)((\lambda z.yz)y)$

(c) $(\lambda y.yxy)((\lambda y.xy)x)$

(d) $(\lambda v.(vx)v)((\lambda u.uv)x)$

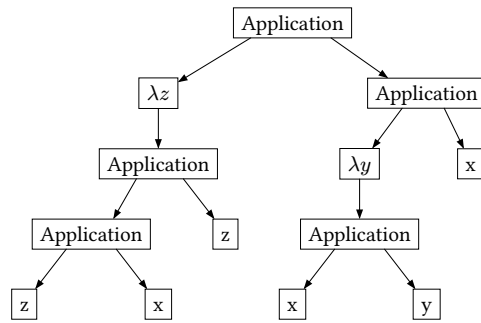
1. Find Sub U .

Solution.

$$\text{Sub } U = \{ \\
(\lambda z.zxz)((\lambda y.xy)x), (\lambda z.zxz), ((\lambda y.xy)x), \\
(\lambda y.yx), (y), (\lambda z.xz), (x), \\
(\lambda y.y), (x), (\lambda z.x), (z), (y), (x) \\
\}$$

2. Draw a tree rep of U .

Solution.



3. Find $\text{FV } U$

Solution.

$$\begin{aligned} \text{FV } U &= \text{FV } (\lambda y.yxy) \cup \text{FV } (\lambda z.xz)x \\ &= (\text{FV } yxy) \setminus \{y\} \cup (\text{FV } \lambda z.xz) \cup \{x\} \\ &= (\text{FV } yx) \setminus \{y\} \cup (\text{FV } xz) \setminus \{z\} \cup \{x\} \\ &= \{x\} \end{aligned}$$

4. Find an alpha-equivalent term.

Solution.

$$(a) \stackrel{\alpha}{=} (c) \stackrel{\alpha}{=} U$$

(a), (b), and (c) all follow the Barendregt convention. In (d) the free variable v was bound in the first abstraction.

Problem

Give the results of the following substitutions

- (a) $(\lambda x.y(\lambda y.xy))[y := \lambda z.zx]$
- (b) $((xyz)[x := y])[y := z]$
- (c) $((\lambda x.xyz)[x := y])[y := z]$
- (d) $(\lambda y.yyx)[x := yz]$

Solution.

- (a) $(\lambda v.(\lambda z.zx)(\lambda u.vu))$
- (b) $(yyz)[y := z] = zzz$
- (c) $(\lambda x.xyz)[y := z] = (\lambda x.xzz)$
- (d) $(\lambda u.uu(yz))$

Problem

$$\neg \left(\forall MLN \in \Lambda, M[x := N, y := L] \equiv_{\alpha} M[x := N][y := L] \right)$$

Solution.

Proof. Because $\text{RHS} = M[x := N][y := L] = M[x := N[y := L]][y := L]$, if $y \in \text{FV } N$, then what x gets substituted with will have y substituted for L , which is completely different with LHS. ■

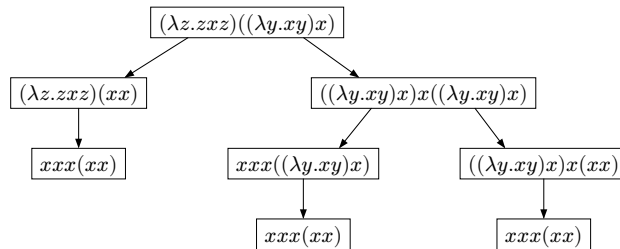
Problem

Find all available redexes in

$$U := (\lambda z.zxz)((\lambda y.xy)x)$$

And all reduction paths to the β -normal form.

Solution. The first redex is the term as an application itself; another the second term in the application.



Problem

Show that

$$(\lambda x.xx)y \not\equiv_{\beta} (\lambda xy.yx)xx$$

Solution. By Collorary 1.9.9, it suffices to prove the hypothesis with a proof of a common normal redacted form from LHS and RHS not existing.

Contradiction. By definition of $\overset{=}{\beta}$, there exists The set of all terms attainable from β -reduction on $(\lambda x.xx)y$ and $(\lambda xy.yx)xx$ do not intersect. Therefore,

$$\neg \left(\exists L \in \Lambda, (\lambda x.xx)y \overset{\rightarrow}{\beta} L \wedge (\lambda xy.yx)xx \overset{\rightarrow}{\beta} L \right) \implies \neg \left((\lambda x.xx)y \overset{=}{\beta} (\lambda xy.yx)xx \right)$$

■

Problem

Define the combinators

$$K := \lambda xy.x$$

$$S := \lambda xyz.xz(yz)$$

Prove that

$$\forall PQ \in \Lambda, K PQ \overset{\rightarrow}{\beta} P$$

$$\forall PQR \in \Lambda, SPQR \overset{\rightarrow}{\beta} PR(QR)$$

Solution.

Proof.

$$K PQ = (\lambda xy.x)PQ \overset{\rightarrow}{\beta} (\lambda y.x)[x := P]Q \overset{\rightarrow}{\beta} P[y := Q] = P$$

$$SPQR = (\lambda xyz.xz(yz)) \overset{\rightarrow}{\beta} (xz(yz))[x := P][y := Q][z := R] = PR(QR)$$

■

Problem

We define the church numerals

$$\begin{aligned}\text{zero} &:= \lambda f x. x \\ \text{one} &:= \lambda f x. f x \\ \text{two} &:= \lambda f x. f f x \\ &\dots \\ \text{num}_n &:= \lambda f x. f^n x\end{aligned}$$

And operations

$$\begin{aligned}\text{add} &:= \lambda n m f x. m f (n f x) \\ \text{mul} &:= \lambda n m f x. m (n f) x\end{aligned}$$

Show

- (a) $\text{add one one} \xrightarrow[\beta]{} \text{two}$
- (b) $\text{add one one} \not\xrightarrow[\beta]{} \text{mul one zero}$

Solution.

$$\begin{aligned}\text{(a) } \text{add one one} &= (\lambda n m f x. m f (n f x)) (\lambda f x. f x) (\lambda f x. f x) \\ &\xrightarrow[\beta]{} (\lambda f x. (\lambda f x. f x) f ((\lambda f x. f x) f x)) \\ &\xrightarrow[\beta]{} (\lambda f x. (\lambda x. f x) f x) \\ &\xrightarrow[\beta]{} (\lambda f x. f f x) = \text{two} \\ \text{(b) } \text{mul one one} &= (\lambda n m f x. m (n f) x) (\lambda f x. f x) (\lambda f x. f x) \\ &\xrightarrow[\beta]{} \lambda f x. (\lambda f x. f x) ((\lambda f x. f x) f) x \\ &\xrightarrow[\beta]{} \lambda f x. f x = \text{one}\end{aligned}$$

Because no intermediate form in the beta reduction process of the two terms are α -equivalent, by collorary 1.9.9 the two terms are not β -equivalent.

Problem

We define

$$\text{succ} := \lambda m f x. f(m f x) \text{ s.t. } \forall \text{num}_n, \text{succ num}_n = \text{num}_{n+1}$$

Prove

$$\text{succ zero} \stackrel{\beta}{=} \text{one}$$

$$\text{succ one} \stackrel{\beta}{=} \text{two}$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β -equivalence.

Proof.

$$\begin{aligned} \text{succ zero} &= (\lambda m f x. f(m f x))(\lambda f x. x) \\ &\rightarrow_{\beta} (\lambda f x. f((\lambda f x. x) f x)) \\ &\rightarrow_{\beta} (\lambda f x. f x) = \text{one} \end{aligned}$$

The path $\text{succ zero} \rightarrow_{\beta} \text{one}$ derived above is the witness of a reduction chain from LHS to RHS.

$$\begin{aligned} \text{succ one} &= (\lambda m f x. f(m f x))(\lambda f x. f x) \\ &\rightarrow_{\beta} (\lambda f x. f((\lambda f x. f x) f x)) \\ &\rightarrow_{\beta} (\lambda f x. f(f x)) = \text{two} \end{aligned}$$

The path $\text{succ one} \rightarrow_{\beta} \text{two}$ derived above is the witness of a reduction chain from LHS to RHS. ■

Problem

We define the λ -terms \top_{λ} (true) and \perp_{λ} (false) and \neg_{λ} (not) by:

$$\begin{aligned} \top_{\lambda} &:= \lambda x y. x & \perp_{\lambda} &:= \lambda x y. y \\ \neg_{\lambda} &:= \lambda a. a \perp_{\lambda} \top_{\lambda} \end{aligned}$$

Show that

$$\neg_{\lambda}(\neg_{\lambda} \top_{\lambda}) \stackrel{\beta}{=} \top_{\lambda}$$

$$\neg_{\lambda}(\neg_{\lambda} \perp_{\lambda}) \stackrel{\beta}{=} \perp_{\lambda}$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β -equivalence.

Proof.

$$\begin{aligned}
\neg_\lambda(\neg_\lambda \top_\lambda) &= \neg_\lambda((\lambda a. a \perp_\lambda \top_\lambda)(\lambda xy. x)) \\
&\xrightarrow[\beta]{\rightarrow} \neg_\lambda((\lambda xy. x) \perp_\lambda \top_\lambda) \\
&\xrightarrow[\beta]{\rightarrow} (\lambda a. a \perp_\lambda \top_\lambda) \perp_\lambda \\
&\xrightarrow[\beta]{\rightarrow} (\lambda xy. y) \perp_\lambda \top_\lambda \\
&\xrightarrow[\beta]{\rightarrow} \top_\lambda
\end{aligned}$$

$$\begin{aligned}
\neg_\lambda(\neg_\lambda \perp_\lambda) &= \neg_\lambda((\lambda a. a \perp_\lambda \top_\lambda)(\lambda xy. y)) \\
&\xrightarrow[\beta]{\rightarrow} \neg_\lambda((\lambda xy. x) \perp_\lambda \top_\lambda) \\
&\xrightarrow[\beta]{\rightarrow} (\lambda a. a \perp_\lambda \top_\lambda) \top_\lambda \\
&\xrightarrow[\beta]{\rightarrow} (\lambda xy. x) \perp_\lambda \top_\lambda \\
&\xrightarrow[\beta]{\rightarrow} \perp_\lambda
\end{aligned}$$

■

Problem

Define

$$\text{iszero} := \lambda m. m(\lambda x. \perp_\lambda) \top_\lambda$$

Prove

$$\text{iszero zero} \xrightarrow[\beta]{\rightarrow} \top_\lambda$$

$$\forall n \in \mathbb{N}^+, \text{iszero num}_n \xrightarrow[\beta]{\rightarrow} \perp_\lambda$$

Solution.

$$\begin{aligned}
\text{iszero zero} &= (\lambda m. m(\lambda x. \perp_\lambda) \top_\lambda)(\lambda fx. x) \\
&\xrightarrow[\beta]{\rightarrow} (\lambda fx. x)(\lambda x. \perp_\lambda) \top_\lambda \\
&\xrightarrow[\beta]{\rightarrow} \top_\lambda
\end{aligned}$$

$$\begin{aligned}
\text{iszero num}_n &= (\lambda m.m(\lambda x.\perp_\lambda)\top_\lambda)(\lambda f x.f^n x) \\
&\xrightarrow[\beta]{\rightarrow} (\lambda f x.f^n x)(\lambda x.\perp_\lambda)\top_\lambda \\
&\xrightarrow[\beta]{\rightarrow} (\lambda x.\perp_\lambda)((\lambda x.\perp_\lambda)^{n-1}\top_\lambda) \xrightarrow[\beta]{\rightarrow} \perp_\lambda
\end{aligned}$$

Problem

If-else can be modeled as

$$\text{ifelse} = \lambda x t f.x t f$$

Where when x , then t , else f . Prove correctness by applying \top_λ and \perp_λ on ifelse.

Solution.

$$\begin{aligned}
\text{ifelse } \top_\lambda &= (\lambda x t f.x t f)\top_\lambda \\
&\xrightarrow[\beta]{\rightarrow} (\lambda t f.(\lambda x y.x) t f) \xrightarrow[\beta]{\rightarrow} (\lambda t f.t) \\
\text{ifelse } \perp_\lambda &= (\lambda x t f.x t f)\perp_\lambda \\
&\xrightarrow[\beta]{\rightarrow} (\lambda t f.(\lambda x y.y) t f) \xrightarrow[\beta]{\rightarrow} (\lambda t f.f)
\end{aligned}$$

By applying the results to any two values, the correct corresponding value returns, ex, for ifelse \top_λ , t is always returned.

Problem

Prove that $\Omega := (\lambda x.xx)(\lambda x.xx)$ does not have a β -nf.

Solution. Firstly let's prove Ω .

Proof. Induction on Ω 's only reduction path proves that every $\Omega \xrightarrow[\beta]{\rightarrow} \Omega_i = \Omega$. For the base case because Ω has one and only one redux, it could only reduce to Ω_1 which is equivalent to itself. For the inductive step, $\Omega_i = \Omega$, therefore $\Omega_i \xrightarrow[\beta]{\rightarrow} \Omega_{i+1}$ is still Ω .

By definition, a term having a β -nf requires the existence of a form in β -nf such that the term can reduce to. By induction, Ω only reduces to Ω , and Ω is not in β -nf because it contains β -redex. Therefore, Ω can never reduce to a β -nf, thus it does not have a β -nf. ■

Problem

Let M be a λ -term with the following properties:

- M has a β -nf.
- There exists an infinite reduction path $M \equiv M_0 \xrightarrow{\beta} M_1 \xrightarrow{\beta} \dots$ on M .

Prove that every M_i has a β -nf, and give an example of M .

Solution. An example would be $(\lambda xy.y)\Omega$. Reduction can go on infinitely by reducing on Ω , but the β -nf of the term is $\lambda y.y$

Proof. Denote β -nf of M as M' . For any form in the reduction path, $M \xrightarrow{\beta} M_i$. In conjunction with $M \xrightarrow{\beta} M'$, by the Church-Rosser theorem, there exists L such that $M_i \xrightarrow{\beta} L$ and $M' \xrightarrow{\beta} L$. Because M' is in β -nf, L can only be M' , thus $M_i \xrightarrow{\beta} M'$, so M_i is capable of reducing to M' , a β -nf. Therefore, any form in the reduction path has a β -nf. ■

Problem

If MN is strongly normalizing, then both M and N are strongly normalizing.

Solution.

Proof. If M is not strongly normalizing, then there exists a reduction path $M_0 \xrightarrow{\beta} M_1 \xrightarrow{\beta} \dots$. Therefore, MN would have had a reduction path $MN \xrightarrow{\beta} M_1N \xrightarrow{\beta} \dots$ that is infinite, which contradicts with MN being strongly normalizing. Vice versa for N . ■

Problem

Let $L, M, N \in \Lambda$ such that $L \equiv_{\beta} M$ and $L \xrightarrow{\beta} N$. Moreover, N is in β -nf. Prove that $M \xrightarrow{\beta} N$.

Solution. Corollary 1.9.9.

Problem

Define

$$U := \lambda zx.x(zzx) \quad \text{and} \quad Z := UU$$

Prove Z is a fixed point combinator.

Solution. Proving $\forall L \in \Lambda, L(ZL) \xrightarrow[\beta]{} ZL$.

Proof.

$$\begin{aligned} ZL &= (\lambda zx.x(zzx))(\lambda zx.x(zzx))L \\ &\xrightarrow[\beta]{} L((\lambda zx.x(zzx))(\lambda zx.x(zzx))L) \xrightarrow[\beta]{} L(ZL) \end{aligned}$$

■

Problem

Solve for $M \in \Lambda$ in each equation:

$$\begin{aligned} M &\stackrel{=}{\beta} \lambda xy.xMy \\ Mxyz &\stackrel{=}{\beta} xyzM \end{aligned}$$

Solution. By the property of the Y combinator:

$$f(Yf) = Yf$$

The first equation can be remodeled as

$$M \stackrel{=}{\beta} LM \quad \text{where} \quad L = \lambda mxy.xmy$$

Solving for fixed point of L :

$$\begin{aligned} M &\equiv YL \equiv L(YL) \\ &= (\lambda x.L(xx))(\lambda x.L(xx)) \\ &= (\lambda x.(\lambda muv.umv)(xx))(\lambda x.(\lambda muv.umv)(xx)) \\ &\xrightarrow[\beta]{} (\lambda x.(\lambda uv.u(xx)v))(\lambda x.(\lambda uv.u(xx)v)) \end{aligned}$$

The second equation can be η -reduced on both sides:

$$\begin{aligned} Mxyz &\stackrel{=}{\beta} xyzM \\ M &\stackrel{=}{\beta} \lambda xyz.xyzM \end{aligned}$$

Remodeling equation:

$$M = NM \text{ where } M = \lambda mxyz.xyzm$$

then

$$\begin{aligned} M &\equiv YN \equiv N(YN) \\ &= (\lambda x.N(xx))(\lambda x.N(xx)) \\ &= (\lambda x.(\lambda mxyz.xyzm)(xx))(\lambda x.(\lambda mxyz.xyzm)(xx)) \\ &\xrightarrow[\beta]{} (\lambda x.(\lambda yz.yz(xx)))(\lambda x.(\lambda yz.yz(xx))) \end{aligned}$$