

# EXERCISES

## CHAPTER 3

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1. Reducted

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**Definition** Some rules for reference.

$$\frac{x : \sigma \in \Gamma \quad \Gamma \text{ is a } \lambda 2 \text{ context}}{\Gamma \vdash x : \sigma} \text{ (T-Var)} \quad \frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau} \text{ (T-App)}$$
$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma. M : \sigma \rightarrow \tau} \text{ (T-Abst)} \quad \frac{\alpha \in \mathbb{T}_2 \quad \forall \tau \in \text{FV } \alpha, \Gamma \vdash \tau : *}{\alpha : * \in \Gamma} \text{ (T-Form)}$$
$$\frac{\Gamma \vdash M : \Pi_{\alpha : *} A \quad \Gamma \vdash B : *}{\Gamma \vdash MB : A[\alpha := B]} \text{ (T2-App)} \quad \frac{\Gamma, \alpha : * \vdash M : A}{\Gamma \vdash \lambda \alpha : *. M : \Pi_{\alpha : *} A} \text{ (T2-Abst)}$$

In this document, convention is that all type judgements in a proof tree, unless stated otherwise, is derived from a single and unique  $\lambda 2$  context per tree. Multiple conclusions might be drawn on a single line from usage of the same inference rule for compactness. Eg:

*ex* 1.  $\alpha, \beta : *$  **T-Form**

Is shorthand for

*ex* 1.  $\Gamma \vdash \alpha : *$  **T-Form**

*ex* 2.  $\Gamma \vdash \beta : *$  **T-Form**

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### Problem

(3.1) How many  $\lambda 2$  contexts consisting of four and only four declarations

- (1)  $\Gamma \vdash \alpha : *$
- (2)  $\Gamma \vdash \beta : *$
- (3)  $\Gamma \vdash f : \alpha \rightarrow \beta$
- (4)  $\Gamma \vdash x : \alpha$

*Solution.* The last two declarations depende on the first two. Therefore this is an easy combinatorics problem:  $2! \times 2! = 4$  contexts:

$$\begin{array}{ll} 1 - 2 - 3 - 4 & 1 - 2 - 4 - 3 \\ 2 - 1 - 3 - 4 & 2 - 1 - 4 - 3 \end{array}$$

### Problem

(3.2) Give a full derivation in  $\lambda 2$  to show the following type term is legal:

$$M \equiv \lambda \alpha, \beta, \gamma : * . \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx)$$

*Solution.*

		<b>Bound</b>
1.	$\alpha : *$	
2.	$\beta : *$	<b>Bound</b>
3.	$\gamma : *$	<b>Bound</b>
4.	$f : \alpha \rightarrow \beta$	<b>Bound</b>
5.	$g : \beta \rightarrow \gamma$	<b>Bound</b>
6.	$x : \alpha$	<b>Bound</b>
7.	$\alpha, \beta, \gamma : *$	<b>T-Form</b>
8.	$\alpha \rightarrow \beta, \beta \rightarrow \gamma : *$	<b>T-Form</b>
9.	$f : \alpha \rightarrow \beta, x : \alpha$	<b>T-Var</b>
10.	$fx : \beta$	<b>8,8 T-App</b>
11.	$g : \beta \rightarrow \gamma$	<b>T-Var</b>
12.	$g(fx) : \gamma$	<b>11,10 T-App</b>
13.	$\lambda x : \alpha. g(fx) : \alpha \rightarrow \gamma$	<b>12 T-Abst</b>
14.	$\lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) : (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	<b>13 T-Abst</b>
15.	$\lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) : (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	<b>14 T-Abst</b>

16.	$\begin{array}{l} \lambda\gamma : * . \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) \\ : \Pi\gamma : * . (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma \end{array}$	<b>15 T2-Abst</b>
17.	$\begin{array}{l} \lambda\beta, \gamma : * . \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) \\ : \Pi\beta, \gamma : * . (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma \end{array}$	<b>16 T2-Abst</b>
18.	$\begin{array}{l} \lambda\alpha, \beta, \gamma : * . \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g(fx) \\ : \Pi\alpha, \beta, \gamma : * . (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma \end{array}$	<b>17 T2-Abst</b>

### Problem

(3.3 a) Given  $M$  in 3.2, and a context  $\Gamma$  such that

$$\Gamma \vdash \text{nat} : *$$

$$\Gamma \vdash \text{bool} : *$$

$$\Gamma \vdash \text{succ} : \text{nat} \rightarrow \text{nat}$$

$$\Gamma \vdash \text{even} : \text{nat} \rightarrow \text{bool}$$

Prove  $M \text{ nat } \text{nat } \text{bool } \text{succ } \text{even}$  is legal.

*Solution.* Proof by deriving the term's type.

*Proof.*

1.	$M : \Pi\alpha, \beta, \gamma : * . (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	<b>T-Var</b>
2.	$\text{nat}, \text{bool} : *$	<b>T-Form</b>
3.	$M \text{ nat} : \Pi\beta, \gamma : * . (\text{nat} \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \text{nat} \rightarrow \gamma$	<b>2,1 T2-App</b>
4.	$M \text{ nat } \text{nat} : \Pi\gamma : * . (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \gamma) \rightarrow \text{nat} \rightarrow \gamma$	<b>2,3 T2-App</b>
5.	$M \text{ nat } \text{nat } \text{bool} : (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \text{bool}) \rightarrow \text{nat} \rightarrow \text{bool}$	<b>2,3 T2-App</b>
6.	$\text{succ} : \text{nat} \rightarrow \text{nat}, \text{even} : \text{nat} \rightarrow \text{bool}$	<b>T-Var</b>
7.	$M \text{ nat } \text{nat } \text{bool } \text{succ} : (\text{nat} \rightarrow \text{bool}) \rightarrow \text{nat} \rightarrow \text{bool}$	<b>6,5 T-App</b>
8.	$M \text{ nat } \text{nat } \text{bool } \text{succ } \text{even} : \text{nat} \rightarrow \text{bool}$	<b>6,7 T-App</b>

■

### Problem

(3.3 b.i) Prove  $\lambda x : \text{nat}. \text{even}(\text{succ } x)$  via 3.3 a.

*Solution.* The result of beta reduction on the term in 3.3 a is what we are proving.

*Proof.*

$$\begin{aligned} M & \text{ nat nat bool succ even} \\ & \equiv (\lambda \alpha, \beta, \gamma, f, g. \lambda x : \alpha. g(fx)) \text{ nat nat bool succ even} \\ & \xrightarrow[\beta]{\beta} (\lambda f : \text{nat} \rightarrow \text{nat}. \lambda g : \text{nat} \rightarrow \text{bool}. \lambda x : \text{nat}. g(f(x))) \text{ succ even} \\ & \xrightarrow[\beta]{} (\lambda x : \text{nat}. \text{even}(\text{succ } x)) \end{aligned}$$

By the subject reduction lemma,  $\lambda x : \text{nat}. \text{even}(\text{succ } x) : \text{nat} \rightarrow \text{bool}$ , thus is legal. ■

### Problem

(3.3 b.ii) Prove  $\lambda x : \text{nat}. \text{even}(\text{succ } x)$  via derivation in the context provided in 3.3 a.

*Solution.*

*Proof.*

1.	$\text{nat}, \text{bool} : *$	<b>T-Form</b>
2.	$x : \text{nat}$	<b>Bound</b>
3.	$\text{succ} : \text{nat} \rightarrow \text{nat}$	<b>T-Var</b>
4.	$x : \text{nat}$	<b>T-Var</b>
5.	$\text{succ } x : \text{nat}$	<b>3,4 T-App</b>
6.	$\text{even} : \text{nat} \rightarrow \text{bool}$	<b>T-Var</b>
7.	$\text{even}(\text{succ } x) : \text{bool}$	<b>6,5 T-App</b>
8.	$\lambda x : \text{nat}. \text{even}(\text{succ } x) : \text{nat} \rightarrow \text{bool}$	<b>7 T-Abst</b>

### Problem

(3.4) Give a shorthanded (omit T-Var and T-Form) derivation in  $\lambda 2$  to show the following term is valid in  $\Gamma \equiv \text{nat} : *, \text{bool} : *$

$$(\lambda\alpha, \beta : * . \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx))) \text{ nat bool}$$

*Solution.*

*Proof.*

1.	$\alpha, \beta : *$	<b>Bound</b>
2.	$f : \alpha \rightarrow \alpha$	<b>Bound</b>
3.	$g : \alpha \rightarrow \beta$	<b>Bound</b>
4.	$x : \alpha$	<b>Bound</b>
5.	$fx : \alpha$	$^{*,*} \text{T-App}$
6.	$f(fx) : \alpha$	$^{*,5} \text{T-App}$
7.	$g(f(fx)) : \beta$	$^{*,6} \text{T-App}$
8.	$\lambda x : \alpha. g(f(fx)) : \alpha \rightarrow \beta$	<b>7 T-Abst</b>
9.	$\lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx)) : (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$	<b>8 T-Abst</b>
10.	$\lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx))$	
	$: (\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$	<b>9 T-Abst</b>
11.		
12.	$\lambda\alpha, \beta : * . \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx))$	
	$: \Pi\alpha, \beta : * . (\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$	<b>10 T2-Abst</b>
13.	$(\lambda\alpha, \beta : * . \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx))) \text{ nat}$	
	$: \Pi\beta : * . (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta$	$^{*,11} \text{T2-App}$
	$(\lambda\alpha, \beta : * . \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g(f(fx))) \text{ nat bool}$	
	$: \Pi\beta : * . (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \text{bool}) \rightarrow \text{nat} \rightarrow \text{bool}$	$^{*,12} \text{T2-App}$

■

### Problem

(3.5 a) Let  $\perp \equiv \Pi\alpha : * . \alpha$ . Prove  $\perp$  is legal.

*Solution.* Here a notion called kind checking is introduced. This has not yet been discussed in this book (?)

*Proof.*

- |  |               |
|--|---------------|
| 1. $\alpha : *$                              | <b>Bound</b>  |
| 2. $\boxed{\alpha : *}$                      | <b>T-Form</b> |
| 3. $\Pi\alpha : *. \alpha : * \rightarrow *$ | <b>T-Form</b> |

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### Problem

(3.5 b) Consider the context  $\Gamma \equiv \beta : *, x : \perp$ . Find an inhabitant of type  $\beta$  under  $\Gamma$ .

*Solution.*  $x\beta$  is. Because  $x$  is of second-order type, it must be parametric to a type, thus  $x$  is of form  $\lambda\alpha : *. M$  where  $\Gamma, \alpha : * \vdash M : \alpha$ .

*Proof.*

- |                                |                   |
|--------------------------------|-------------------|
| 1. $x : \Pi\alpha : *. \alpha$ | <b>T-Var</b>      |
| 2. $\beta : *$                 | <b>T-Form</b>     |
| 3. $x\beta : \beta$            | <b>1,2 T2-App</b> |

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### Problem

(3.5 c) Give three inhabitants of  $\beta \rightarrow \beta$  in  $\beta$ -nf under  $\Gamma$  in 3.5 b.

*Solution.*

1.  $\lambda y : \beta. y$ .

*Proof.*

- |   |                 |
|---|-----------------|
| 1. $y : \beta$                                      | <b>Bound</b>    |
| 2. $\boxed{y : \beta}$                              | <b>T-Var</b>    |
| 3. $\lambda y : \beta. y : \beta \rightarrow \beta$ | <b>2 T-Abst</b> |

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2.  $\lambda y : \beta. x\beta$ .

*Proof.*

1.	$y : \beta$	<b>Bound</b>
2.	$x : \Pi\alpha : * . \alpha$	<b>T-Var</b>
3.	$\beta : *$	<b>T-Form</b>
4.	$x\beta : \beta$	<b>2,3 T2-App</b>
5.	$\lambda y : \beta. x\beta : \beta \rightarrow \beta$	<b>4 T-Abst</b>

■

$$3. \lambda y : \beta. x(\beta \rightarrow \beta)y.$$

*Proof.*

1.	$y : \beta$	<b>Bound</b>
2.	$x : \Pi\alpha : * . \alpha$	<b>T-Var</b>
3.	$\beta \rightarrow \beta : *$	<b>T-Form</b>
4.	$x(\beta \rightarrow \beta) : \beta \rightarrow \beta$	<b>2,3 T2-App</b>
5.	$y : \beta$	<b>T-Var</b>
6.	$x(\beta \rightarrow \beta)y : \beta$	<b>4,5 T-App</b>
7.	$\lambda y : \beta. x(\beta \rightarrow \beta)y : \beta \rightarrow \beta$	<b>5 T-Abst</b>

■

### Problem

(3.5 d) Prove that the following terms inhabit the same type in  $\Gamma$ :

$$\lambda f : \beta \rightarrow \beta \rightarrow \beta. f(x\beta)(x\beta)$$

$$x((\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta)$$

*Solution.* We simply derive the types.

*First Term.*

1.	$f : \beta \rightarrow \beta \rightarrow \beta$	<b>Bound</b>
2.	$f : \beta \rightarrow \beta \rightarrow \beta$	<b>T-Var</b>
3.	$x : \Pi\alpha : * . \alpha$	<b>T-Var</b>
4.	$\beta : *$	<b>T-Form</b>
5.	$x\beta : \beta$	<b>3,4 T2-App</b>
6.	$f(x\beta) : \beta \rightarrow \beta$	<b>2,5 T-App</b>

7.  $f(x\beta)(x\beta) : \beta$  **6,5 T-App**  
8.  $\lambda f : \beta \rightarrow \beta \rightarrow \beta. f(x\beta)(x\beta) : (\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$  **6 T-Abst**

■

*Second Term.*

1.  $(\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta : *$  **T-Form**  
2.  $x : \Pi \alpha : * . \alpha$  **T-Var**  
3.  $x((\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta) : (\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$  **2,1 T2-App**

■

The two terms were shown to both inhabit  $(\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$ .

### Problem

(3.6 a) Find inhabitant of type

$$\Pi \alpha, \beta : * . (\text{nat} \rightarrow \alpha) \rightarrow (\alpha \rightarrow \text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta$$

In context  $\Gamma \equiv \text{nat} : *$ .

*Solution.*

$$\lambda \alpha, \beta : * . \lambda x : \text{nat} \rightarrow \alpha. \lambda y : (\alpha \rightarrow \text{nat} \rightarrow \beta). \lambda z : \text{nat}. y(xz)z$$

*Proof.*

- |  |                    |
|--|--------------------|
| 1. $\alpha, \beta : *$                                   | <b>Bound</b>       |
| 2. $\text{nat} \rightarrow \alpha : *$                   | <b>T-Form</b>      |
| 3. $x : \text{nat} \rightarrow \alpha$                   | <b>Bound</b>       |
| 4. $\alpha \rightarrow \text{nat} \rightarrow \beta : *$ | <b>T-Form</b>      |
| 5. $y : \alpha \rightarrow \text{nat} \rightarrow \beta$ | <b>Bound</b>       |
| 6. $\text{nat} : *$                                      | <b>Bound</b>       |
| 7. $z : \text{nat}$                                      | <b>Bound</b>       |
| 8. $y : \alpha \rightarrow \text{nat} \rightarrow \beta$ | <b>T-Var</b>       |
| 9. $x : \text{nat} \rightarrow \alpha$                   | <b>T-Var</b>       |
| 10. $z : \text{nat}$                                     | <b>T-Var</b>       |
| 11. $xz : \alpha$  | <b>9,10 T-App</b>  |
| 12. $y(xz) : \text{nat} \rightarrow \beta$               | <b>8,11 T-App</b>  |
| 13. $y(xz)z : \beta$                                     | <b>12,10 T-App</b> |

14.	$\boxed{\lambda z : \text{nat}.y(xz)z : \text{nat} \rightarrow \beta}$ $\lambda y : \alpha \rightarrow \text{nat} \rightarrow \beta. \lambda z : \text{nat}.y(xz)z$	<b>13 T-Abst</b>
15.	$\boxed{:(\alpha \rightarrow \text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta}$	<b>14 T-Abst</b>
16.	$\boxed{\lambda x : \text{nat} \rightarrow \alpha.y : \alpha \rightarrow \text{nat} \rightarrow \beta. \lambda z : \text{nat}.y(xz)z}$ $:(\text{nat} \rightarrow \alpha) \rightarrow (\alpha \rightarrow \text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta$	<b>15 T-Abst</b>
17.		
	$\lambda \alpha, \beta : * . x : \text{nat} \rightarrow \alpha.y : \alpha \rightarrow \text{nat} \rightarrow \beta. \lambda z : \text{nat}.y(xz)z$ $:\Pi \alpha, \beta : *. (\text{nat} \rightarrow \alpha) \rightarrow (\alpha \rightarrow \text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta$	<b>16 T2-Abst</b>

■

### Problem

(3.6 b) Find inhabitant of type

$$\Pi \delta : * . ((\alpha \rightarrow \gamma) \rightarrow \delta) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta$$

In context  $\Gamma \equiv \alpha : *, \beta : *, \gamma : *$

*Solution.*

$$\lambda \delta : * . \lambda x : (\alpha \rightarrow \gamma) \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x(\lambda u : \alpha.z(yu))$$

A derivation in shorthand will be given (omitting T-Form / T-Var)

*Proof.*

1.	$\delta : *$	<b>Bound</b>
2.	$x : (\alpha \rightarrow \gamma) \rightarrow \delta$	<b>Bound</b>
3.	$y : \alpha \rightarrow \beta$	<b>Bound</b>
4.	$z : \beta \rightarrow \gamma$	<b>Bound</b>
5.	$u : \alpha$	<b>Bound</b>
6.	$yu : \beta$	<b>*, T-App</b>
7.	$z(yu) : \gamma$	<b>*,6 T-App</b>
8.	$\lambda u : \alpha.z(yu) : \alpha \rightarrow \gamma$	<b>7 T-Abst</b>
9.	$x(\lambda u : \alpha.z(yu)) : \delta$	<b>8 T-Abst</b>
10.	$\lambda z : \beta \rightarrow \gamma. x(\lambda u : \alpha.z(yu)) : (\beta \rightarrow \gamma) \rightarrow \delta$	<b>9 T-Abst</b>
11.	$\lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x(\lambda u : \alpha.z(yu))$	
	$: (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta$	<b>10 T-Abst</b>

12. $\boxed{\begin{array}{l} \lambda x : (\alpha \rightarrow \gamma) \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x(\lambda u : \alpha. z(yu)) \\ : ((\alpha \rightarrow \gamma) \rightarrow \delta) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta \end{array}}$	<b>11 T-Abst</b>
13. $\begin{array}{l} \lambda \delta : * . \lambda x : (\alpha \rightarrow \gamma) \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x(\lambda u : \alpha. z(yu)) \\ : \Pi \delta : * . ((\alpha \rightarrow \gamma) \rightarrow \delta) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta \end{array}$	<b>12 T-Abst</b>

■

### Problem

(3.6 c) Find inhabitant of type

$$\Pi \alpha, \beta, \gamma : * . (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$$

In the empty context

*Solution.*

$$\lambda \alpha, \beta, \gamma : * . \lambda f : (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) . \lambda x : \alpha. f x (\lambda u : \beta. x)$$

*Proof.*

1. $\alpha, \beta, \gamma$	<b>Bound</b>
2. $f : \alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma$	<b>Bound</b>
3. $x : \alpha$	<b>Bound</b>
4. $fx : (\beta \rightarrow \alpha) \rightarrow \gamma$	$^{**}$ <b>T-App</b>
5. $u : \beta$	<b>Bound</b>
6. $x : \alpha$	<b>T-Var</b>
7. $\lambda u : \beta. x : \beta \rightarrow \alpha$	<b>6 T-Abst</b>
8. $fx(\lambda u : \beta. x) : \gamma$	<b>4,7 T-App</b>
9. $\lambda x : \alpha. fx(\lambda u : \beta. x) : \alpha \rightarrow \gamma$	<b>8 T-Abst</b>
10. $\lambda f : \alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma. \lambda x : \alpha. fx(\lambda u : \beta. x)$ $: (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	<b>9 T-Abst</b>
11. $\lambda \alpha, \beta, \gamma : * . \lambda f : \alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma. \lambda x : \alpha. fx(\lambda u : \beta. x)$ $: \Pi \alpha, \beta, \gamma : * . (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$	<b>10 T2-Abst</b>

■

### Problem

(3.7) Let  $\perp \equiv \Pi\alpha : * . \alpha$  and context  $\Gamma \equiv \alpha : *, \beta : *, x : \alpha \rightarrow \perp, f : (\alpha \rightarrow \alpha) \rightarrow \alpha$ . Give a derivation that successively calculate an inhabitant of  $\alpha$  and  $\beta$ , both in context  $\Gamma$ .

*Solution.* Have  $M : \alpha := f(\lambda n : \alpha.n)$ . Then  $\Gamma \vdash xM\beta : \beta$ .

*Typing M.*

1.  $f : (\alpha \rightarrow \alpha) \rightarrow \alpha$  **T-Var**
2.  $n : \alpha$  **Bound**
3.  $\boxed{n : \alpha}$  **T-Var**
4.  $\lambda n : \alpha. n : \alpha \rightarrow \alpha$  **3 T-Abst**
5.  $f(\lambda n : \alpha. n) : \alpha$  **1,4 T-App**

■

*Typing  $xM\beta$ .*

1.  $M : \alpha$  **T-Var**
2.  $x : \alpha \rightarrow \Pi\alpha : * . \alpha$  **T-Var**
3.  $Mx : \Pi\alpha : * . \alpha$  **2,1 T-App**
4.  $Mx\beta : \beta$  **3, \* T2-App**

■

### Problem

(3.8) Recall  $K \equiv \lambda xy.x \in \Lambda$  from untyped lambda calculus. Consider the following types

$$T_1 \equiv \Pi\alpha, \beta : * . \alpha \rightarrow \beta \rightarrow \alpha \quad T_2 \equiv \Pi\alpha : * . \alpha \rightarrow (\Pi\beta : * . \beta \rightarrow \alpha)$$

Find inhabitants of both type  $t_1 : T_1$  and  $t_2 : T_2$  under the empty context, which may be considered the closed  $\lambda 2$  form of  $K \in \Lambda_{T2}$ .

*Solution.*

$$\lambda\alpha, \beta : * . \lambda x : \alpha. \lambda y : \beta. x$$

$$\lambda\alpha : * . \lambda x : \alpha. \lambda\beta : * . \lambda y : \beta. x$$

*First Form.*

1.	$\alpha, \beta : *$	<b>Bound</b>
2.	$x : \alpha$	<b>Bound</b>
3.	$y : \beta$	<b>Bound</b>
4.	$x : \alpha$	<b>T-Var</b>
5.	$\lambda y : \beta. x : \beta \rightarrow \alpha$	<b>4 T-Abst</b>
6.	$\lambda x : \alpha. \lambda y : \beta. x : \alpha \rightarrow \beta \rightarrow \alpha$	<b>5 T-Abst</b>
7.	$\lambda \alpha, \beta : * . \lambda x : \alpha. \lambda y : \beta. x : \Pi \alpha, \beta : * . \alpha \rightarrow \beta \rightarrow \alpha$	<b>5 T-Abst</b>

■

*Second Form.*

1.	$\alpha : *$	<b>Bound</b>
2.	$x : \alpha$	<b>Bound</b>
3.	$\beta : *$	<b>Bound</b>
4.	$y : \beta$	<b>Bound</b>
5.	$x : \alpha$	<b>T-Var</b>
6.	$\lambda y : \beta. x : \beta \rightarrow \alpha$	<b>5 T-Abst</b>
7.	$\lambda \beta : * . \lambda y : \beta. x : \Pi \beta : * . \beta \rightarrow \alpha$	<b>6 T-Abst</b>
8.	$\lambda x : \alpha. \lambda \beta : * . \lambda y : \beta. x : \alpha \rightarrow (\Pi \beta : * . \beta \rightarrow \alpha)$	<b>7 T-Abst</b>
9.	$\lambda \alpha : * . \lambda x : \alpha. \lambda \beta : * . \lambda y : \beta. x : \Pi \alpha : * . \alpha \rightarrow (\Pi \beta : * . \beta \rightarrow \alpha)$	<b>8 T-Abst</b>

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