Excercises

Chapter 3

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1. Reducted

Definition Some rules for reference.

$$\frac{x : \sigma \in \Gamma \qquad \Gamma \text{ is a } \lambda 2 \text{ context}}{\Gamma \vdash x : \sigma}\text{(T-Var)} \qquad \frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}\text{(T-App)}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma . M : \sigma \to \tau}\text{(T-Abst)} \qquad \frac{\alpha \in \mathbb{T}_2 \qquad \forall \tau \in \text{FV } \alpha, \Gamma \vdash \tau : *}{\alpha : * \in \Gamma}\text{(T-Form)}$$

$$\frac{\Gamma \vdash M : \Pi_{\alpha:*}.A \qquad \Gamma \vdash B : *}{\Gamma \vdash MB : A[\alpha := B]}\text{(T2-App)} \qquad \frac{\Gamma, \alpha : * \vdash M : A}{\Gamma \vdash \lambda\alpha : * . M : \Pi_{\alpha:*} : A}\text{(T2-Abst)}$$

In this document, convention is that all type judgements in a proof tree, unless stated otherwise, is derived from a single and unique $\lambda 2$ context per tree. Multiple conclusions might be drawn on a single line from usage of the same inference rule for compactness. Eg:

ex 1. $\alpha, \beta : *$ **T-Form**

Is shorthand for

ex 1. $\Gamma \vdash \alpha : *$ **T-Form**
ex 2. $\Gamma \vdash \beta : *$ **T-Form**

Problem

(3.1) How many $\lambda 2$ contexts consisting of four and only four declarations

$$(1)\ \Gamma \vdash \alpha : * \qquad (2)\ \Gamma \vdash \beta : *$$
$$(3)\ \Gamma \vdash f : \alpha \to \beta \qquad (4)\ \Gamma \vdash x : \alpha$$

Solution. The last two declarations depende on the first two. Therefore this is an easy combinatorics problem: $2! \times 2! = 4$ contexts:

$$\begin{array}{cc} 1-2-3-4 & 1-2-4-3 \\ 2-1-3-4 & 2-1-4-3 \end{array}$$

Problem

(3.2) Give a full derivation in $\lambda 2$ to show the following type term is legal:

$$M \equiv \lambda \alpha, \beta, \gamma : * . \lambda f : \alpha \to \beta . \lambda g : \beta \to \gamma . \lambda x : \alpha . g(fx)$$

Solution.

1.	$\alpha : *$	**Bound**
2.	$\beta : *$	**Bound**
3.	$\gamma : *$	**Bound**
4.	$f : \alpha \to \beta$	**Bound**
5.	$g : \beta \to \gamma$	**Bound**
6.	$x : \alpha$	**Bound**
7.	$\alpha, \beta, \gamma : *$	**T-Form**
8.	$\alpha \to \beta, \beta \to \gamma : *$	**T-Form**
9.	$f : \alpha \to \beta, x : \alpha$	**T-Var**
10.	$fx : \beta$	**8,8 T-App**
11.	$g : \beta \to \gamma$	**T-Var**
12.	$g(fx) : \gamma$	**11,10 T-App**
13.	$\lambda x : \alpha . g(fx) : \alpha \to \gamma$	**12 T-Abst**
14.	$\lambda g : \beta \to \gamma . \lambda x : \alpha . g(fx) : (\beta \to \gamma) \to \alpha \to \gamma$	**13 T-Abst**
15.	$\lambda f : \alpha \to \beta . \lambda g : \beta \to \gamma . \lambda x : \alpha . g(fx)$ $: (\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$	**14 T-Abst**

16. $\lambda\gamma : * .\lambda f : \alpha \to \beta.\lambda g : \beta \to \gamma.\lambda x : \alpha.g(fx)$
$\quad : \Pi\gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **15 T2-Abst**

17. $\lambda\beta, \gamma : * .\lambda f : \alpha \to \beta.\lambda g : \beta \to \gamma.\lambda x : \alpha.g(fx)$
$\quad : \Pi\beta, \gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **16 T2-Abst**

18.

$\lambda\alpha, \beta, \gamma : * .\lambda f : \alpha \to \beta.\lambda g : \beta \to \gamma.\lambda x : \alpha.g(fx)$
$\quad : \Pi\alpha, \beta, \gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **17 T2-Abst**

Problem

(3.3 a) Given M in 3.2, and a context Γ such that

$$\Gamma \vdash \texttt{nat} : *$$

$$\Gamma \vdash \texttt{bool} : *$$

$$\Gamma \vdash \text{succ} : \texttt{nat} \to \texttt{nat}$$

$$\Gamma \vdash \text{even} : \texttt{nat} \to \texttt{bool}$$

Prove M $\texttt{nat nat bool}$ succ even is legal.

Solution. Proof by deriving the term's type.

Proof.

1. $M : \Pi\alpha, \beta, \gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **T-Var**
2. $\texttt{nat}, \texttt{bool} : *$ **T-Form**
3. $M\ \texttt{nat} : \Pi\beta, \gamma : * .(\texttt{nat} \to \beta) \to (\beta \to \gamma) \to \texttt{nat} \to \gamma$ **2,1 T2-App**
4. $M\ \texttt{nat}\ \texttt{nat} : \Pi\gamma : * .(\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \gamma) \to \texttt{nat} \to \gamma$ **2,3 T2-App**
5.

$M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool} : (\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \texttt{bool}) \to \texttt{nat} \to \texttt{bool}$ **2,3 T2-App**

6. $\text{succ} : \texttt{nat} \to \texttt{nat}, \text{even} : \texttt{nat} \to \texttt{bool}$ **T-Var**
7. $M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \text{succ} : (\texttt{nat} \to \texttt{bool}) \to \texttt{nat} \to \texttt{bool}$ **6,5 T-App**
8. $M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \text{succ}\ \text{even} : \texttt{nat} \to \texttt{bool}$ **6,7 T-App**

∎

Problem

(3.3 b.i) Prove $\lambda x : \texttt{nat}.\ \text{even}\ (\text{succ}\ x)$ via 3.3 a.

Solution. The result of beta reduction on the term in 3.3 a is what we are proving.

Proof.

$$
\begin{aligned}
& M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \text{succ even} \\
\equiv\ & (\lambda \alpha, \beta, \gamma, f, g.\lambda x : \alpha.g(fx))\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \text{succ even} \\
\twoheadrightarrow_\beta\ & (\lambda f : \texttt{nat} \to \texttt{nat}.\lambda g : \texttt{nat} \to \texttt{bool}.\lambda x : \texttt{nat}.g(f(x)))\ \text{succ even} \\
\to_\beta\ & (\lambda x : \texttt{nat}.\ \text{even}\ (\text{succ}\ x))
\end{aligned}
$$

By the subject reduction lemma, $\lambda x : \texttt{nat}.\ \text{even}\ (\text{succ}\ x)) : \texttt{nat} \to \texttt{bool}$, thus is legal. ■

Problem

(3.3 b.ii) Prove $\lambda x : \texttt{nat}.\ \text{even}\ (\text{succ}\ x)$ via derivation in the context provided in 3.3 a.

Solution.

Proof.

1.	$\texttt{nat}, \texttt{bool} : *$	**T-Form**
2.	$x : \texttt{nat}$	**Bound**
3.	│ $\text{succ} : \texttt{nat} \to \texttt{nat}$	**T-Var**
4.	│ $x : \texttt{nat}$	**T-Var**
5.	│ $\text{succ}\ x : \texttt{nat}$	**3,4 T-App**
6.	│ $\text{even} : \texttt{nat} \to \texttt{bool}$	**T-Var**
7.	│ $\text{even}\ (\text{succ}\ x) : \texttt{bool}$	**6,5 T-App**
8.	$\lambda x : \texttt{nat}.\ \text{even}\ (\text{succ}\ x) : \texttt{nat} \to \texttt{bool}$	**7 T-Abst**

■

Problem

(3.4) Give a shorthanded (omit T-Var and T-Form) derivation in $\lambda 2$ to show the following term is valid in $\Gamma \equiv \texttt{nat} : *, \texttt{bool} : *$

$$(\lambda\alpha, \beta : * .\lambda f : \alpha \to \alpha.\lambda g : \alpha \to \beta.\lambda x : \alpha.g(f(fx)))\ \texttt{nat}\ \texttt{bool}$$

Solution.

Proof.

1.	$\alpha, \beta : *$	**Bound**
2.	$f : \alpha \to \alpha$	**Bound**
3.	$g : \alpha \to \beta$	**Bound**
4.	$x : \alpha$	**Bound**
5.	$fx : \alpha$	***,* T-App**
6.	$f(fx) : \alpha$	***,5 T-App**
7.	$g(f(fx)) : \beta$	***,6 T-App**
8.	$\lambda x : \alpha.g(f(fx)) : \alpha \to \beta$	**7 T-Abst**
9.	$\lambda g : \alpha \to \beta.\lambda x : \alpha.g(f(fx)) : (\alpha \to \beta) \to \alpha \to \beta$	**8 T-Abst**
10.	$\lambda f : \alpha \to \alpha.\lambda g : \alpha \to \beta.\lambda x : \alpha.g(f(fx))$ $: (\alpha \to \alpha) \to (\alpha \to \beta) \to \alpha \to \beta$	**9 T-Abst**

11.

$\lambda\alpha, \beta : * .\lambda f : \alpha \to \alpha.\lambda g : \alpha \to \beta.\lambda x : \alpha.g(f(fx))$
$\quad : \Pi\alpha, \beta : * .(\alpha \to \alpha) \to (\alpha \to \beta) \to \alpha \to \beta$ **10 T2-Abst**

12.

$(\lambda\alpha, \beta : * .\lambda f : \alpha \to \alpha.\lambda g : \alpha \to \beta.\lambda x : \alpha.g(f(fx)))\ \texttt{nat}$
$\quad : \Pi\beta : * .(\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \beta) \to \texttt{nat} \to \beta$ ***,11 T2-App**

13.

$(\lambda\alpha, \beta : * .\lambda f : \alpha \to \alpha.\lambda g : \alpha \to \beta.\lambda x : \alpha.g(f(fx)))\ \texttt{nat}\ \texttt{bool}$
$\quad : \Pi\beta : * .(\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \texttt{bool}) \to \texttt{nat} \to \texttt{bool}$ ***,12 T2-App**

■

Problem

(3.5 a) Let $\bot \equiv \Pi\alpha : * .\alpha$. Prove $\bot$ is legal.

Solution. Here a notion called kind checking is introduced. This has not yet been discussed in this book (?

Proof.

1.	$\alpha : *$	**Bound**
2.	$\quad \alpha : *$	**T-Form**
3.	$\Pi\alpha : * .\alpha : * \to *$	**T-Form**

■

Problem

(3.5 b) Consider the context $\Gamma \equiv \beta : *, x : \bot$. Find an inhabitant of type β under Γ.

Solution. $x\beta$ is. Because x is of second-order type, it must be parametric to a type, thus x is of form $\lambda\alpha : * .M$ where $\Gamma, \alpha : * \vdash M : \alpha$.

Proof.

1.	$x : \Pi\alpha : * .\alpha$	**T-Var**
2.	$\beta : *$	**T-Form**
3.	$x\beta : \beta$	**1,2 T2-App**

■

Problem

(3.5 c) Give three inhabitants of $\beta \to \beta$ in β-nf under Γ in 3.5 b.

Solution.
1. $\lambda y : \beta.y$.

Proof.

1.	$y : \beta$	**Bound**
2.	$\quad y : \beta$	**T-Var**
3.	$\lambda y : \beta.y : \beta \to \beta$	**2 T-Abst**

■

2. $\lambda y : \beta.x\beta$.

Proof.

1.	$y : \beta$	**Bound**
2.	$\quad x : \Pi\alpha : *.\alpha$	**T-Var**
3.	$\quad \beta : *$	**T-Form**
4.	$\quad x\beta : \beta$	**2,3 T2-App**
5.	$\lambda y : \beta.x\beta : \beta \to \beta$	**4 T-Abst**

■

3. $\lambda y : \beta.x(\beta \to \beta)y$.

Proof.

1.	$y : \beta$	**Bound**
2.	$\quad x : \Pi\alpha : *.\alpha$	**T-Var**
3.	$\quad \beta \to \beta : *$	**T-Form**
4.	$\quad x(\beta \to \beta) : \beta \to \beta$	**2,3 T2-App**
5.	$\quad y : \beta$	**T-Var**
6.	$\quad x(\beta \to \beta)y : \beta$	**4,5 T-App**
7.	$\lambda y : \beta.x(\beta \to \beta)y : \beta \to \beta$	**5 T-Abst**

■

Problem

(3.5 d) Prove that the following terms inhabit the same type in Γ:

$$\lambda f : \beta \to \beta \to \beta.f(x\beta)(x\beta)$$

$$x((\beta \to \beta \to \beta) \to \beta)$$

Solution. We simply derive the types.

First Term.

1.	$f : \beta \to \beta \to \beta$	**Bound**
2.	$\quad f : \beta \to \beta \to \beta$	**T-Var**
3.	$\quad x : \Pi\alpha : *.\alpha$	**T-Var**
4.	$\quad \beta : *$	**T-Form**
5.	$\quad x\beta : \beta$	**3,4 T2-App**
6.	$\quad f(x\beta) : \beta \to \beta$	**2,5 T-App**

7. $\quad f(x\beta)(x\beta) : \beta$ **6,5 T-App**

8. $\lambda f : \beta \to \beta \to \beta . f(x\beta)(x\beta) : (\beta \to \beta \to \beta) \to \beta$ **6 T-Abst**

■

Second Term.

1. $(\beta \to \beta \to \beta) \to \beta : *$ **T-Form**
2. $x : \Pi\alpha : * . \alpha$ **T-Var**
3. $x((\beta \to \beta \to \beta) \to \beta) : (\beta \to \beta \to \beta) \to \beta$ **2,1 T2-App**

■

The two terms were shown to both inhabit $(\beta \to \beta \to \beta) \to \beta$.

Problem

(3.6 a) Find inhabitant of type

$$\Pi\alpha, \beta : * . (\texttt{nat} \to \alpha) \to (\alpha \to \texttt{nat} \to \beta) \to \texttt{nat} \to \beta$$

In context $\Gamma \equiv \texttt{nat} : *$.

Solution.

$$\lambda\alpha, \beta : * . \lambda x : \texttt{nat} \to \alpha . \lambda y : (\alpha \to \texttt{nat} \to \beta) . \lambda z : \texttt{nat} . y(xz)z$$

Proof.

1. $\alpha, \beta : *$ **Bound**
2. $\texttt{nat} \to \alpha : *$ **T-Form**
3. $\quad x : \texttt{nat} \to \alpha$ **Bound**
4. $\quad \alpha \to \texttt{nat} \to \beta : *$ **T-Form**
5. $\quad\quad y : \alpha \to \texttt{nat} \to \beta$ **Bound**
6. $\quad\quad \texttt{nat} : *$ **Bound**
7. $\quad\quad\quad z : \texttt{nat}$ **Bound**
8. $\quad\quad\quad\quad y : \alpha \to \texttt{nat} \to \beta$ **T-Var**
9. $\quad\quad\quad\quad x : \texttt{nat} \to \alpha$ **T-Var**
10. $\quad\quad\quad\quad z : \texttt{nat}$ **T-Var**
11. $\quad\quad\quad\quad xz : \alpha$ **9,10 T-App**
12. $\quad\quad\quad\quad y(xz) : \texttt{nat} \to \beta$ **8,11 T-App**
13. $\quad\quad\quad\quad y(xz)z : \beta$ **12,10 T-App**

14. $\lambda z : \mathtt{nat}.y(xz)z : \mathtt{nat} \to \beta$ **13 T-Abst**

15. $\lambda y : \alpha \to \mathtt{nat} \to \beta.\lambda z : \mathtt{nat}.y(xz)z$
$: (\alpha \to \mathtt{nat} \to \beta) \to \mathtt{nat} \to \beta$ **14 T-Abst**

16. $\lambda x : \mathtt{nat} \to \alpha.y : \alpha \to \mathtt{nat} \to \beta.\lambda z : \mathtt{nat}.y(xz)z$
$: (\mathtt{nat} \to \alpha) \to (\alpha \to \mathtt{nat} \to \beta) \to \mathtt{nat} \to \beta$ **15 T2-Abst**

17.

$\lambda\alpha, \beta : *.x : \mathtt{nat} \to \alpha.y : \alpha \to \mathtt{nat} \to \beta.\lambda z : \mathtt{nat}.y(xz)z$
$: \Pi\alpha, \beta : *.(\mathtt{nat} \to \alpha) \to (\alpha \to \mathtt{nat} \to \beta) \to \mathtt{nat} \to \beta$ **16 T2-Abst**

■

Problem

(3.6 b) Find inhabitant of type

$$\Pi\delta : *.((\alpha \to \gamma) \to \delta) \to (\alpha \to \beta) \to (\beta \to \gamma) \to \delta$$

In context $\Gamma \equiv \alpha : *, \beta : *, \gamma : *$

Solution.

$$\lambda\delta : *.\lambda x : (\alpha \to \gamma) \to \delta.\lambda y : \alpha \to \beta.\lambda z : \beta \to \gamma.x(\lambda u : \alpha.z(yu))$$

A derivation in shorthand will be given (omitting T-Form / T-Var)

Proof.

1. $\delta : *$ **Bound**
2. $x : (\alpha \to \gamma) \to \delta$ **Bound**
3. $y : \alpha \to \beta$ **Bound**
4. $z : \beta \to \gamma$ **Bound**
5. $u : \alpha$ **Bound**
6. $yu : \beta$ ***,* T-App**
7. $z(yu) : \gamma$ ***,6 T-App**
8. $\lambda u : \alpha.z(yu) : \alpha \to \gamma$ **7 T-Abst**
9. $x(\lambda u : \alpha.z(yu)) : \delta$ **8 T-Abst**
10. $\lambda z : \beta \to \gamma.x(\lambda u : \alpha.z(yu)) : (\beta \to \gamma) \to \delta$ **9 T-Abst**
11. $\lambda y : \alpha \to \beta.\lambda z : \beta \to \gamma.x(\lambda u : \alpha.z(yu))$
$: (\alpha \to \beta) \to (\beta \to \gamma) \to \delta$ **10 T-Abst**

12. $\lambda x : (\alpha \to \gamma) \to \delta.\lambda y : \alpha \to \beta.\lambda z : \beta \to \gamma.x(\lambda u : \alpha.z(yu))$ $: ((\alpha \to \gamma) \to \beta) \to (\alpha \to \beta) \to (\beta \to \gamma) \to \delta$ **11 T-Abst**

13.

$$\lambda\delta : * .\lambda x : (\alpha \to \gamma) \to \delta.\lambda y : \alpha \to \beta.\lambda z : \beta \to \gamma.x(\lambda u : \alpha.z(yu))$$
$$: \Pi\delta : * .((\alpha \to \gamma) \to \delta) \to (\alpha \to \beta) \to (\beta \to \gamma) \to \delta$$ **12 T2-Abst**

■

Problem

(3.6 c) Find inhabitant of type

$$\Pi\alpha, \beta, \gamma : * .(\alpha \to (\beta \to \alpha) \to \gamma) \to \alpha \to \gamma$$

In the empty context

Solution.

$$\lambda\alpha, \beta, \gamma : * .\lambda f : (\alpha \to (\beta \to \alpha) \to \gamma).\lambda x : \alpha.fx(\lambda u : \beta.x)$$

Proof.

1. α, β, γ **Bound**
2. $f : \alpha \to (\beta \to \alpha) \to \gamma$ **Bound**
3. $x : \alpha$ **Bound**
4. $fx : (\beta \to \alpha) \to \gamma$ ***,* T-App**
5. $u : \beta$ **Bound**
6. $x : \alpha$ **T-Var**
7. $\lambda u : \beta.x : \beta \to \alpha$ **6 T-Abst**
8. $fx(\lambda u : \beta.x) : \gamma$ **4,7 T-App**
9. $\lambda x : \alpha.fx(\lambda u : \beta.x) : \alpha \to \gamma$ **8 T-Abst**
10. $\lambda f : \alpha \to (\beta \to \alpha) \to \gamma.\lambda x : \alpha.fx(\lambda u : \beta.x)$ $: (\alpha \to (\beta \to \alpha) \to \gamma) \to \alpha \to \gamma$ **9 T-Abst**
11.

$$\lambda\alpha, \beta, \gamma : * .\lambda f : \alpha \to (\beta \to \alpha) \to \gamma.\lambda x : \alpha.fx(\lambda u : \beta.x)$$
$$: \Pi\alpha, \beta, \gamma : * .(\alpha \to (\beta \to \alpha) \to \gamma) \to \alpha \to \gamma$$ **10 T2-Abst**

■

Problem

(3.7) Let $\perp \equiv \Pi\alpha : * .\alpha$ and context $\Gamma \equiv \alpha : *, \beta : *, x : \alpha \to \perp, f : (\alpha \to \alpha) \to \alpha$. Give a derivation that succesively calculate an inhabitant of α and β, both in context Γ.

Solution. Have $M : \alpha := f(\lambda n : \alpha.n)$. Then $\Gamma \vdash xM\beta : \beta$.

Typing M.

1.	$f : (\alpha \to \alpha) \to \alpha$	**T-Var**
2.	$n : \alpha$	**Bound**
3.	$\quad n : \alpha$	**T-Var**
4.	$\lambda n : \alpha.n : \alpha \to \alpha$	**3 T-Abst**
5.	$f(\lambda n : \alpha.n) : \alpha$	**1,4 T-App**

∎

Typing $xM\beta$.

1.	$M : \alpha$	**T-Var**
2.	$x : \alpha \to \Pi\alpha : * .\alpha$	**T-Var**
3.	$Mx : \Pi\alpha : * .\alpha$	**2,1 T-App**
4.	$Mx\beta : \beta$	**3,* T2-App**

∎

Problem

(3.8) Recall $K \equiv \lambda xy.x \in \Lambda$ from untyped lambda calculus. Consider the following types

$$T_1 \equiv \Pi\alpha, \beta : * .\alpha \to \beta \to \alpha \quad T_2 \equiv \Pi\alpha : * .\alpha \to (\Pi\beta : * .\beta \to \alpha)$$

Find inhabitants of both type $t_1 : T_1$ and $t_2 : T_2$ under the empty context, which may be considered the closed $\lambda 2$ form of $K \in \Lambda_{\mathbb{T}2}$.

Solution.

$$\lambda\alpha, \beta : * .\lambda x : \alpha.\lambda y : \beta.x$$

$$\lambda\alpha : * .\lambda x : \alpha.\lambda\beta : * .\lambda y : \beta.x$$

First Form.

1.	$\alpha, \beta : *$	**Bound**
2.	$\quad x : \alpha$	**Bound**
3.	$\quad\quad y : \beta$	**Bound**
4.	$\quad\quad\quad x : \alpha$	**T-Var**
5.	$\quad\quad \lambda y : \beta . x : \beta \to \alpha$	**4 T-Abst**
6.	$\quad \lambda x : \alpha . \lambda y : \beta . x : \alpha \to \beta \to \alpha$	**5 T-Abst**
7.	$\lambda \alpha, \beta : * . \lambda x : \alpha . \lambda y : \beta . x : \Pi \alpha, \beta : * . \alpha \to \beta \to \alpha$	**5 T2-Abst**

■

Second Form.

1.	$\alpha : *$	**Bound**
2.	$\quad x : \alpha$	**Bound**
3.	$\quad\quad \beta : *$	**Bound**
4.	$\quad\quad\quad y : \beta$	**Bound**
5.	$\quad\quad\quad\quad x : \alpha$	**T-Var**
6.	$\quad\quad\quad \lambda y : \beta . x : \beta \to \alpha$	**5 T-Abst**
7.	$\quad\quad \lambda \beta : * . \lambda y : \beta . x : \Pi \beta : * . \beta \to \alpha$	**6 T2-Abst**
8.	$\quad \lambda x : \alpha . \lambda \beta : * . \lambda y : \beta . x : \alpha \to (\Pi \beta : * . \beta \to \alpha)$	**7 T-Abst**
9.	$\lambda \alpha : * . \lambda x : \alpha . \lambda \beta : * . \lambda y : \beta . x : \Pi \alpha : * . \alpha \to (\Pi \beta : * . \beta \to \alpha)$	**8 T2-Abst**

■

Problem

(3.9) Pretype the combinator

$$S \equiv \lambda xyz.xz(yz)$$

In closed form (typable in an empty context) in $\Lambda_{\mathbb{T}2}$.

Solution.

$$S \equiv \lambda \alpha, \beta, \gamma : * . \lambda x : \alpha \to \beta \to \gamma . \lambda y : \alpha \to \beta . \lambda z : \alpha . xz(yz)$$

Proof.

1.	$\alpha, \beta, \gamma : *$	**Bound**
2.	$\quad x : \alpha \to \beta \to \gamma$	**Bound**
3.	$\quad\quad y : \alpha \to \beta$	**Bound**

4.	$z : \alpha$	**Bound**
5.	$xz : \beta \to \gamma$	***,* T-App**
6.	$yx : \beta$	***,* T-App**
7.	$xz(yx) : \gamma$	**5,6 T-App**
8.	$\lambda z : \alpha.xz(yx) : \alpha \to \gamma$	**7 T-Abst**
9.	$\lambda y : \alpha \to \beta.\lambda z : \alpha.xz(yx) : (\alpha \to \beta) \to \alpha \to \gamma$	**8 T-Abst**
10.	$\lambda x : \alpha \to \beta \to \gamma.\lambda y : \alpha \to \beta.\lambda z : \alpha.xz(yx)$ $: (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$	**9 T-Abst**
11.		

$$\lambda\alpha, \beta, \gamma : * .\lambda x : \alpha \to \beta \to \gamma.\lambda y : \alpha \to \beta.\lambda z : \alpha.xz(yx)$$
$$: \Pi\alpha, \beta, \gamma : * .(\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma \qquad \textbf{10 T2-Abst}$$

■

Problem

(3.10 a) Consider the term

$$M \equiv \lambda x : \Pi\alpha : * .\alpha \to \alpha.x(\sigma \to \sigma)(x\sigma)$$

Prove that M is legal.

Solution. For a term to be legal there must exist a context so that the term could be typed. Here, a witness context is $\Gamma \equiv \sigma : *$.

Proof.

1.	$x : \Pi\alpha : * .\alpha \to \alpha$	**Bound**
2.	$x(\sigma \to \sigma) : (\sigma \to \sigma) \to (\sigma \to \sigma)$	***,* T2-App**
3.	$x\sigma : \sigma \to \sigma$	***,* T2-App**
4.	$x(\sigma \to \sigma)(x\sigma) : \sigma \to \sigma$	**2,3 T-App**
5.	$\lambda x : \Pi\alpha : * .\alpha \to \alpha.x(\sigma \to \sigma)(x\sigma) : (\Pi\alpha : * .\alpha \to \alpha) \to \sigma \to \sigma$	**4 T-Abst**

■

Problem

(3.10 b) Find a term N such that MN is legal and may be considered to be a way to add type information to $(\lambda x.xx)(\lambda y.y)$

Solution.

$$M\sigma N \equiv (\lambda x : \Pi\alpha : * .\alpha \to \alpha.x(\sigma \to \sigma)(x\sigma))\sigma(\lambda y : \sigma.y)$$

Is the same as $(\lambda x.xx)(\lambda y.y)$ modulo type annotations.

Proof.

1.	$M : (\Pi\alpha : * .\alpha \to \alpha) \to \sigma \to \sigma$	**T-Var**
2.	$M\sigma : (\sigma \to \sigma) \to \sigma \to \sigma$	**1,* T2-App**
3.	$y : \sigma$	**Bound**
4.	$\quad y : \sigma$	**T-Var**
5.	have $N := \lambda y : \sigma.y : \sigma \to \sigma$	**4 T-Abst**
6.	$M\sigma N : \sigma \to \sigma$	**2,5 T-Abst**

■

Problem

(3.11) Recall $\perp \equiv \Pi\alpha : * .\alpha$ from 3.5. Type and prove the following term legal:

$$\lambda x : \perp.x(\perp \to \perp \to \perp)(x(\perp \to \perp)x)(x(\perp \to \perp \to \perp)xx)$$

Solution.

Proof. The type $\perp \to \perp$ is closed and well formed. Therefore, the term is legal.

1.	$\perp : * \equiv \Pi\alpha : * .\alpha$	**T-Form**
2.	$x : \perp$	**Bound**
3.	$\quad x(\perp \to \perp \to \perp) : \perp \to \perp \to \perp$	***,* T2-App**
4.	$\quad x(\perp \to \perp) : \perp \to \perp$	***,* T2-App**
5.	$\quad x(\perp \to \perp)x : \perp$	**4,* T-App**
6.	$\quad x(\perp \to \perp \to \perp)(x(\perp \to \perp)x) : \perp \to \perp$	**3,5 T-App**
7.	$\quad x(\perp \to \perp \to \perp)x : \perp \to \perp$	**3,* T-App**
8.	$\quad x(\perp \to \perp \to \perp)xx : \perp$	**7,* T-App**
9.	$\quad x(\perp \to \perp \to \perp)(x(\perp \to \perp)x)(x(\perp \to \perp \to \perp)xx) : \perp$	**6,8 T-App**
10.		

$\lambda x : \perp.x(\perp \to \perp \to \perp)(x(\perp \to \perp)x)$
$\quad(x(\perp \to \perp \to \perp)xx) : \perp \to \perp$ **9 T-Abst**

■