Excercises

Chapter 5

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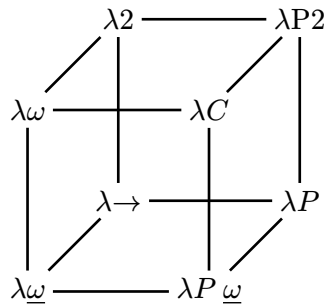
1. Reducted

Reference - Calculus of Constructions

$$\frac{}{\emptyset \vdash * : \square}\ \text{Sort} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}\ \text{Var} \qquad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}\ \text{Weak}$$

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi\, x : A\,.\,B : s_2}\ \text{Form}$$

$$\frac{\Gamma \vdash M : \Pi\, x : A\,.\,B \quad \Gamma \vdash N : A}{\Gamma \vdash M\, N : B\,[x := N]}\ \text{App}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi\, x : A\,.\,B : s}{\Gamma \vdash \lambda x : A\,.\,M : \Pi\, x : A\,.\,B}\ \text{Abst}$$

$$\frac{\Gamma \vdash A : B \quad B \overset{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'}\ \text{Conv}$$

The λ-Cube

$\lambda\!\to$	$(*, *)$			
$\lambda 2$	$(*, *)$	$(\square, *)$		
$\lambda\underline{\omega}$	$(*, *)$		$(\square, \square)$	
λP	$(*, *)$			$(*, \square)$
$\lambda\omega$	$(*, *)$	$(\square, *)$	$(\square, \square)$	
$\lambda 2P$	$(*, *)$	$(\square, *)$		$(*, \square)$
$\lambda P\,\underline{\omega}$	$(*, *)$		$(\square, \square)$	$(*, \square)$
λC	$(*, *)$	$(\square, *)$	$(\square, \square)$	$(*, \square)$

Problem

(6.1 a) Give a complete derivation in tree format showing that

$$\bot \equiv \Pi\alpha : *.\alpha$$

is legal in λC.

Solution. Here we will show that there exists $s \in$ sort and Γ such that $\Gamma \vdash \bot : s$.

Proof.

$$\cfrac{\vdash * : \Box \qquad \cfrac{\cfrac{\vdash * : \Box}{\alpha : * \vdash \alpha : *}\text{Var}}{}}{\vdash \Pi\alpha : *.\alpha : *}\text{Form}$$

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Problem

(6.1 a) Give a complete derivation in tree format showing that $\bot \rightarrow \bot$ is legal in λC where

$$\bot \equiv \Pi\alpha : *.\alpha$$

Solution. Here we will show that there exists $s \in$ sort and Γ such that $\Gamma \vdash \bot \rightarrow \bot : s$.

Proof.

$$\cfrac{(6.1\ a)\cfrac{}{\vdash \bot : *} \qquad \cfrac{(6.1\ a)\cfrac{}{\vdash \bot : *} \qquad \cfrac{(6.1\ a)\cfrac{}{\vdash \bot : *}}{x : \bot \vdash \bot : *}\text{Weak}}{}}{\vdash \Pi\, x : \bot.\bot : *}\text{Form}$$

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Problem

(6.1 c) To which systems of the λ-cube does $\bot$ belong? And $\bot \rightarrow \bot$?

Solution. The set of (s_1, s_2) pairs in formation rules of the derivation of $\bot$ is $\{(\Box, *)\}$. The minimal system corresponding is $\lambda 2$. The same for $\bot \rightarrow \bot$. Therefore $\bot$ and $\bot \rightarrow \bot$ belongs to $\lambda 2$, $\lambda\omega$, λP and λC.

Problem

(6.2) Given context $\Gamma \equiv S : *, P : S \to *, A : *$. Prove by means of a flag derivation that the following expression is inhabited in λC with respect to Γ:

$$(\Pi\, x : S\,.(A \to P\, x)) \to A \to \Pi\, y : S\,.\, P\, y$$

Solution. The inhabitant is

$$M \equiv \lambda u : (\Pi\, x : S\,.(A \to P\, x)).\lambda v : A\,.\lambda y : S\,.\, u\, y\, v$$

Proof.

1.	$S : *, P : S \to *, A : *$	
2.	$u : \Pi\, x : S\,.(A \to P\, x)$	
3.	$v : A$	
4.	$y : S$	
5.	$u\, y : A \to P\, y$	**2,4 App**
6.	$u\, y\, v : P\, y$	**5,3 App**
7.	$\lambda y : S\,.\, u\, y\, v : \Pi\, y : S\,.\, P\, y$	**6 Abst**
8.	$\lambda v : A\,.\lambda y : S\,.\, u\, y\, v : A \to \Pi\, y : S\,.\, P\, y$	**7 Abst**
9.	$\lambda u : \Pi\, x : S\,.(A \to P\, x).\lambda v : A\,.\lambda y : S\,.\, u\, y\, v$ $: \Pi\, x : S\,.(A \to P\, x) \to A \to \Pi\, y : S\,.\, P\, y$	**8 Abst**

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Problem

(6.3 a) Let $\mathcal{J}$ be a judgement

$$\mathcal{J} \equiv S : *, P : S \to * \vdash \lambda x : S\,.(P\, x \to \bot) : S \to *$$

Derive $\mathcal{J}$ in λC with shorthand flag notation.

Solution.

1.	$S : *, P : S \to *$	
2.	$x : S$	
3.	$P\, x : *$	**1,2 App**
4.	$\bot : *$	**Weak from 6.1 a**
5.	$P\, x \to \bot : *$	**3,4 Form**

6. $\lambda x : S \,.\, P\, x \to \bot : S \to *$ **5 Abst**

(6.3 b) Deternmine the (s_1, s_2) pairs corresponding to all Π abstractions occuring in $\mathcal{J}$.

Solution.

Abstraction	Line Number	(s_1, s_2)
$P : S \to *$	1	$(*, \square)$
$\bot \equiv \Pi \alpha : * \,.\, \alpha$	4	$(\square, *)$
$P\, x \to \bot$	5	$(\square, *)$
$\lambda x : S \,.\, P\, x \to \bot : S \to *$	6	$(*, \square)$

(6.3 c) What is the 'smallest' system in the λ-cube to which $\mathcal{J}$ belongs?

Solution. There are $(*, *) - \lambda{\to}$ pairs, $(*, \square) - \lambda P$ pairs, and $(\square, *) - \lambda 2$. Therefore the minimal system $\mathcal{J}$ belongs to is $\lambda P2$.

(6.4 a) Let $\Gamma \equiv S : *, Q : S \to S \to *$ and

$$M \equiv (\Pi\, x, y : S \,.(Q\, x\, y \to Q\, y\, x \to \bot)) \to \Pi\, z : S \,.(Q\, z\, z \to \bot)$$

Derive $\Gamma \vdash M : *$ and determine the smallest subsystemm to which this judgement belongs.

Solution.

1. $S : *, Q : S \to S \to *$
2. $x : S$
3. $y : S$
4. $Q\,x : S \to *$ **1,2 App**
5. $Q\,x\,y : *$ **4,3 App**
6. $z : Q\,x\,y$
7. $Q\,y : S \to *$ **1,3 App**
8. $Q\,y\,x : *$ **7,2 App**
9. $t : Q\,y\,x$
10. $\perp : *$ **Weak from 6.1 a**
11. $Q\,y\,x \to \perp : *$ **8,10 Form**
12. $Q\,x\,y \to Q\,y\,x \to \perp : *$ **5,11 Form**
13. $\Pi\,y : S\,.\,Q\,x\,y \to Q\,y\,x \to \perp : *$ **1,12 Form**
14. $\Pi\,x, y : S\,.\,Q\,x\,y \to Q\,y\,x \to \perp : *$ **1,13 Form**
15. $a : (\Pi\,x, y : S\,.(Q\,x\,y \to Q\,y\,x \to \perp))$
16. $z : S$
17. $Q\,z : S \to *$ **1,16 App**
18. $Q\,z\,z : *$ **17,16 App**
19. $b : Q\,z\,z$
20. $\perp : *$ **Weak from 6.1 a**
21. $Q\,z\,z \to \perp : *$ **18,20 Form**
22. $\Pi\,z : S\,.\,Q\,z\,z \to \perp : *$ **1,21 Form**
       $\Pi\,x, y : S\,.(Q\,x\,y \to Q\,y\,x \to \perp)$
23. $\to \Pi\,z : S\,.\,Q\,z\,z \to \perp : *$ **14,22 Form**

Here's a table of all Πs that appeared

Abstraction	Line Number	(s_1, s_2)
$S \to *$	1 / 4 / 7 / 17	$(*, \square)$
$S \to S \to *$	1	$(*, \square)$
$\perp$	10 / 11 / 12 / 13 / 14 / 15 / 20 / 21 / 22 / 23	$(\square, *)$
$Q\,y\,x \to \perp$	11 / 12 / 13 / 14 / 15 / 23	$(*, *)$
$Q\,x\,y \to Q\,y\,x \to \perp$	12 / 13 / 14 / 15 / 23	$(*, *)$
$\Pi\,y : S\,.\,Q\,x\,y \to Q\,y\,x \to \perp$	13 / 14 / 23	$(*, *)$

$\Pi\,x,y:S\,.\,Q\,x\,y \to Q\,y\,x \to \bot$	14 / 23	$(*,*)$
$Q\,z\,z \to \bot$	21 / 22 / 23	$(*,*)$
$\Pi\,z:S\,.\,Q\,z\,z \to \bot$	22 / 23	$(*,*)$
$\Pi\,x,y:S\,.\,Q\,x\,y \to Q\,y\,x \to \bot \to$ $\Pi\,z:S\,.\,Q\,z\,z \to \bot$	23	$(*,*)$

There are $(*,*) - \lambda{\to}$ pairs, $(*,\square) - \lambda P$ pairs, and $(\square,*) - \lambda 2$ pairs. Therefore the mimimal system available is $\lambda P2$.

(6.4 b) Prove in λC that M is inhabited in context Γ.

Solution. A shorthand derivation is given below:

Proof.

1. $S:*,Q:S \to S \to *$

2. $h:\Pi\,x,y:S\,.(Q\,x\,y \to Q\,y\,x \to \bot)$

3. $z:S$

4. $a:Q\,z\,z$

5. $\alpha:*$

6. $h\,z:\Pi\,y:S\,.(Q\,z\,y \to Q\,y\,z \to \bot)$ **2,3 App**

7. $h\,z\,z:Q\,z\,z \to Q\,z\,z \to \bot$ **6,3 App**

8. $h\,z\,z\,a:Q\,z\,z \to \bot$ **7,4 App**

9. $h\,z\,z\,a\,a:\Pi\alpha:*.\alpha$ **8,4 App**

10. $h\,z\,z\,a\,a\,\alpha:\alpha$ **9,5 App**

11. $\lambda\alpha:*\,.\,h\,z\,z\,a\,a\,\alpha:\Pi\alpha:*.\alpha$ **10 Abst**

12. $\lambda a:Q\,z\,z\,\lambda\alpha:*\,.\,h\,z\,z\,a\,a\,\alpha:Q\,z\,z \to \bot$ **11 Abst**

13. $\lambda z:S\,.\lambda a:Q\,z\,z\,\lambda\alpha:*\,.\,h\,z\,z\,a\,a\,\alpha:\Pi\,z:S\,.\,Q\,z\,z \to \bot$ **12 Abst**

$\lambda h:\Pi\,x,y:S\,.(Q\,x\,y \to Q\,y\,x \to \bot)$

$\lambda z:S\,.\lambda a:Q\,z\,z\,\lambda\alpha:*\,.\,h\,z\,z\,a\,a\,\alpha$

14. $:\Pi\,x,y:S\,.(Q\,x\,y \to Q\,y\,x \to \bot) \to \Pi\,z:S\,.\,Q\,z\,z \to \bot$ **13 Abst**

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(6.4 c) We may consier Q to be a relation on set S. Moreover by PAT we may see $A \to \bot$ as the negation $\neg A$ of prop A. How can M then be interpreted by the PAT paradigm?

Solution. By a direct type-to-proposition translation we have

$$M \equiv \forall x, y \in S, (Q\,(x,y) \Rightarrow \neg\,Q\,(y,x)) \Rightarrow \forall z \in S, (\neg\,Q\,(z,z))$$

It expresses the fact if Q is asymmetric then it is irreflective.

Problem

(5.6 a) Let

$$\mathcal{J} \equiv S : * \vdash \lambda Q : S \to S \to * . \lambda x : S . Q\,x\,x : (S \to S \to *) \to S \to *$$

Give a shorthand derivation of $\mathcal{J}$ and determine the smallest subsystem to which $\mathcal{J}$ belongs.

Solution.

1. $S : *$
2. $Q : S \to S \to *$
3. $x : S$
4. $Q\,x : S \to *$ **2,3 App**
5. $Q\,x\,x : *$ **4,3 App**
6. $\lambda x : S . Q\,x\,x : S \to *$ **5 Abst**
7. $\lambda Q : S \to S \to * . \lambda x : S . Q\,x\,x : (S \to S \to *) \to S \to *$ **6 Abst**

Abstraction	Line Number	(s_1, s_2)
$S \to *$	2 / 4 / 6 / 7	$(*, \square)$
$S \to S \to *$	2 / 7	$(*, \square)$
$(S \to S \to *) \to (S \to *)$	7	$(\square, \square)$

The judgement contains $(*, \square) - \lambda P$ pairs and $(\square, \square) - \lambda \underline{\omega}$ pairs. Therefore the minimal system $\mathcal{J}$ belongs to is $\lambda \mathrm{P}\ \underline{\omega}$.

(6.5 b) In $\mathcal{J}$ of 6.5 a, we may consider the variable Q as expressing a relation on set S. How could you describe the subexpression $\lambda x : S . Q\, x\, x$ in this settings? And what is then the interpretation of the judgement $\mathcal{J}$?

Solution. By a informal translation, the term meant "Given a relation Q over set S and an arbitrary element of S, return whether if $Q\,(x, x)$ holds".

(6.6 a & b) Let

$$M \equiv \lambda S : * .\lambda P : S \to * .\lambda x : S .(P\, x \to \bot)$$

Prove M legal and determine its type. What is the smallest system in which the λ-cube in which M may occur?

Solution.

1.	$* : \square$	**Sort**
2.	$S : *$	
3.	$a : S$	
4.	$* : \square$	
5.	$S \to * : \square$	**2,3 Form**
6.	$P : S \to *$	
7.	$x : S$	
8.	$P\, x : *$	**6,7 App**
9.	$a : P\, x$	
10.	$\bot : *$	**Weak from 6.1 a**
11.	$P\, x \to \bot : *$	**8,10 Form**
12.	$\lambda x : S . P\, x \to \bot : S \to *$	**11,5 Abst**
13.	$S \to * : \square$	**5,5 Weak**
14.	$(S \to *) \to S \to * : \square$	**5,13 Form**
15.	$\lambda P : S \to * .\lambda x : S . P\, x \to \bot : (S \to *) \to S \to *$	**12,5 Abst**
16.	$\Pi\, S : * .(S \to *) \to S \to * : \square$	**1,14 Form**

17.

$\lambda S : * . \lambda P : S \to * . \lambda x : S . P x \to \bot$

  $: \Pi S : * . (S \to *) \to S \to *$ **15,16 Abst**

Abstraction	Line Number	(s_1, s_2)
$S \to *$	5 / 12 / 13 / 14 / 15	$(*, \square)$
$\bot \equiv \Pi \alpha : * . \alpha$	10 / 11	$(\square, *)$
$P x \to \bot$	11 / 12	$(*, *)$
$(S \to *) \to S \to *$	14 / 15	$(\square, *)$
$\Pi S : * . (S \to *) \to S \to *$	16 / 17	$(\square, *)$

The derivation contains $(*, *) - \lambda\to$ pairs, $(*, \square) - \lambda P$ pairs, and $(\square, *) - \lambda 2$ pairs. Therefore the minimal system in which M is legal is $\lambda P2$.

> **Problem**
>
> *(6.6 c)* How could you interpret the constructor M from 6.6 a, if $A \to \bot$ encodes $\neg A$?

Solution. Converting into mathematical function notation,

$$M(S, P, x) = \neg P(x) \text{ where } S \in \text{set}, P \subseteq S, x \in S$$

M constructs the negation of a predicate P over a set S applied to x, an element of S. An inhabitant of M would prove the negation.