

EXERCISES

CHAPTER 3

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1. Reducted

Definition Some rules for reference.

$$\begin{array}{c} \frac{x : \sigma \in \Gamma \quad \Gamma \text{ is a } \lambda 2 \text{ context}}{\Gamma \vdash x : \sigma} \text{ (T-Var)} \qquad \frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M N : \tau} \text{ (T-App)} \\[10pt] \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma. M : \sigma \rightarrow \tau} \text{ (T-Abst)} \qquad \frac{\alpha \in \mathbb{T}_2 \quad \text{FV } \alpha \subseteq \text{dom } \Gamma}{\alpha : * \in \Gamma} \text{ (T-Form)} \\[10pt] \frac{\Gamma \vdash M : \Pi_{\alpha:*.} A \quad \Gamma \vdash B : *}{\Gamma \vdash M B : A [\alpha := B]} \text{ (T2-App)} \qquad \frac{\Gamma, \alpha : * \vdash M : A}{\Gamma \vdash \lambda \alpha : *. M : \Pi_{\alpha:*.} A} \text{ (T2-Abst)} \end{array}$$

In this document, convention is that all type judgements in a proof tree, unless stated otherwise, is derived from a single and unique $\lambda 2$ context per tree. Multiple conclusions might be drawn on a single line from usage of the same inference rule for compactness. Eg:

ex 1. $\alpha, \beta : *$ **T-Form**

Is shorthand for

ex 1. $\Gamma \vdash \alpha : *$ **T-Form**

ex 2. $\Gamma \vdash \beta : *$ **T-Form**

Problem

(3.1) How many $\lambda 2$ contexts consisting of four and only four declarations

- (1) $\Gamma \vdash \alpha : *$ (2) $\Gamma \vdash \beta : *$
 (3) $\Gamma \vdash f : \alpha \rightarrow \beta$ (4) $\Gamma \vdash x : \alpha$

Solution. The last two declarations depend on the first two. Therefore this is an easy combinatorics problem: $2! \times 2! = 4$ contexts:

- 1 – 2 – 3 – 4 1 – 2 – 4 – 3
 2 – 1 – 3 – 4 2 – 1 – 4 – 3

Problem

(3.2) Give a full derivation in $\lambda 2$ to show the following type term is legal:

$$M \equiv \lambda \alpha, \beta, \gamma : *. \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g (f x)$$

Solution.

| | | |
|-----|---|--------------------|
| 1. | $\alpha : *$ | Bound |
| 2. | $\beta : *$ | Bound |
| 3. | $\gamma : *$ | Bound |
| 4. | $f : \alpha \rightarrow \beta$ | Bound |
| 5. | $g : \beta \rightarrow \gamma$ | Bound |
| 6. | $x : \alpha$ | Bound |
| 7. | $\alpha, \beta, \gamma : *$ | T-Form |
| 8. | $\alpha \rightarrow \beta, \beta \rightarrow \gamma : *$ | T-Form |
| 9. | $f : \alpha \rightarrow \beta, x : \alpha$ | T-Var |
| 10. | $f x : \beta$ | 8,8 T-App |
| 11. | $g : \beta \rightarrow \gamma$ | T-Var |
| 12. | $g (f x) : \gamma$ | 11,10 T-App |
| 13. | $\lambda x : \alpha. g (f x) : \alpha \rightarrow \gamma$ | 12 T-Abst |
| 14. | $\lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g (f x) : (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$ | 13 T-Abst |
| 15. | $\lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g (f x) : (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$ | 14 T-Abst |

| | | |
|-----|--|-------------------|
| 16. | $\frac{\lambda\gamma : *. \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g (f x) : \Pi\gamma : *. (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma}{\lambda\beta, \gamma : *. \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g (f x) : \Pi\beta, \gamma : *. (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma}}$ | 15 T2-Abst |
| 17. | | 16 T2-Abst |
| 18. | | |
| | $\lambda\alpha, \beta, \gamma : *. \lambda f : \alpha \rightarrow \beta. \lambda g : \beta \rightarrow \gamma. \lambda x : \alpha. g (f x) : \Pi\alpha, \beta, \gamma : *. (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$ | 17 T2-Abst |

Problem

(3.3 a) Given M in 3.2, and a context Γ such that

$$\Gamma \vdash \text{nat} : *$$

$$\Gamma \vdash \text{bool} : *$$

$$\Gamma \vdash \text{succ} : \text{nat} \rightarrow \text{nat}$$

$$\Gamma \vdash \text{even} : \text{nat} \rightarrow \text{bool}$$

Prove $M \text{ nat nat bool succ even}$ is legal.

Solution. Proof by deriving the term's type.

Proof.

| | | |
|----|--|-------------------|
| 1. | $M : \Pi\alpha, \beta, \gamma : *. (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$ | T-Var |
| 2. | $\text{nat}, \text{bool} : *$ | T-Form |
| 3. | $M \text{ nat} : \Pi\beta, \gamma : *. (\text{nat} \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \text{nat} \rightarrow \gamma$ | 2,1 T2-App |
| 4. | $M \text{ nat nat} : \Pi\gamma : *. (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \gamma) \rightarrow \text{nat} \rightarrow \gamma$ | 2,3 T2-App |
| 5. | | |
| | $M \text{ nat nat bool} : (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \text{bool}) \rightarrow \text{nat} \rightarrow \text{bool}$ | 2,3 T2-App |
| 6. | $\text{succ} : \text{nat} \rightarrow \text{nat}, \text{even} : \text{nat} \rightarrow \text{bool}$ | T-Var |
| 7. | $M \text{ nat nat bool succ} : (\text{nat} \rightarrow \text{bool}) \rightarrow \text{nat} \rightarrow \text{bool}$ | 6,5 T-App |
| 8. | $M \text{ nat nat bool succ even} : \text{nat} \rightarrow \text{bool}$ | 6,7 T-App |

■

Problem

(3.3 b.i) Prove $\lambda x : \text{nat}. \text{even} (\text{succ } x)$ via 3.3 a.

Solution. The result of beta reduction on the term in 3.3 a is what we are proving.

Proof.

$$\begin{aligned}
 & M \text{ nat nat bool succ even} \\
 & \equiv (\lambda \alpha, \beta, \gamma, f, g. \lambda x : \alpha. g (f x)) \text{ nat nat bool succ even} \\
 & \xrightarrow[\beta]{\rightarrow} (\lambda f : \text{nat} \rightarrow \text{nat}. \lambda g : \text{nat} \rightarrow \text{bool}. \lambda x : \text{nat}. g (f (x))) \text{ succ even} \\
 & \xrightarrow[\beta]{\rightarrow} (\lambda x : \text{nat}. \text{even} (\text{succ } x))
 \end{aligned}$$

By the subject reduction lemma, $\lambda x : \text{nat}. \text{even} (\text{succ } x) : \text{nat} \rightarrow \text{bool}$, thus is legal. ■

Problem

(3.3 b.ii) Prove $\lambda x : \text{nat}. \text{even} (\text{succ } x)$ via derivation in the context provided in 3.3 a.

Solution.

Proof.

| | | |
|----|---|------------------|
| 1. | $\text{nat}, \text{bool} : *$ | T-Form |
| 2. | $x : \text{nat}$ | Bound |
| 3. | $\text{succ} : \text{nat} \rightarrow \text{nat}$ | T-Var |
| 4. | $x : \text{nat}$ | T-Var |
| 5. | $\text{succ } x : \text{nat}$ | 3,4 T-App |
| 6. | $\text{even} : \text{nat} \rightarrow \text{bool}$ | T-Var |
| 7. | $\text{even} (\text{succ } x) : \text{bool}$ | 6,5 T-App |
| 8. | $\lambda x : \text{nat}. \text{even} (\text{succ } x) : \text{nat} \rightarrow \text{bool}$ | 7 T-Abst |

■

Problem

(3.4) Give a shorthand (omit T-Var and T-Form) derivation in $\lambda 2$ to show the following term is valid in $\Gamma \equiv \text{nat} : *, \text{bool} : *$

$$(\lambda \alpha, \beta : *. \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g (f (f x))) \text{ nat bool}$$

Solution.

Proof.

| | | |
|-----|---|--------------------|
| 1. | $\alpha, \beta : *$ | Bound |
| 2. | $f : \alpha \rightarrow \alpha$ | Bound |
| 3. | $g : \alpha \rightarrow \beta$ | Bound |
| 4. | $x : \alpha$ | Bound |
| 5. | $f x : \alpha$ | *, T-App |
| 6. | $f (f x) : \alpha$ | *,5 T-App |
| 7. | $g (f (f x)) : \beta$ | *,6 T-App |
| 8. | $\lambda x : \alpha. g (f (f x)) : \alpha \rightarrow \beta$ | 7 T-Abst |
| 9. | $\lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g (f (f x)) : (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$ | 8 T-Abst |
| 10. | $\lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g (f (f x)) : (\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$ | 9 T-Abst |
| 11. | | |
| | $\lambda \alpha, \beta : *. \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g (f (f x)) : \Pi \alpha, \beta : *. (\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \beta$ | 10 T2-Abst |
| 12. | | |
| | $(\lambda \alpha, \beta : *. \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g (f (f x))) \text{ nat} : \Pi \beta : *. (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta$ | *,11 T2-App |
| 13. | | |
| | $(\lambda \alpha, \beta : *. \lambda f : \alpha \rightarrow \alpha. \lambda g : \alpha \rightarrow \beta. \lambda x : \alpha. g (f (f x))) \text{ nat bool} : \Pi \beta : *. (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \text{bool}) \rightarrow \text{nat} \rightarrow \text{bool}$ | *,12 T2-App |

■

Problem

(3.5 a) Let $\perp \equiv \Pi \alpha : *. \alpha$. Prove \perp is legal.

Solution. Here a notion called kind checking is introduced. This has not yet been discussed in this book (?)

Proof.

1. $\alpha : *$ **Bound**
2. $\frac{}{\alpha : *}$ **T-Form**
3. $\Pi \alpha : *. \alpha : * \rightarrow *$ **T2-Abst**

■

Problem

(3.5 b) Consider the context $\Gamma \equiv \beta : *, x : \perp$. Find an inhabitant of type β under Γ .

Solution. $x \beta$ is. Because x is of second-order type, it must be parametric to a type, thus x is of form $\lambda \alpha : *. M$ where $\Gamma, \alpha : * \vdash M : \alpha$.

Proof.

1. $x : \Pi \alpha : *. \alpha$ **T-Var**
2. $\beta : *$ **T-Form**
3. $x \beta : \beta$ **1,2 T2-App**

■

Problem

(3.5 c) Give three inhabitants of $\beta \rightarrow \beta$ in β -nf under Γ in 3.5 b.

Solution.

1. $\lambda y : \beta. y$.

Proof.

1. $y : \beta$ **Bound**
2. $\frac{}{y : \beta}$ **T-Var**
3. $\lambda y : \beta. y : \beta \rightarrow \beta$ **2 T-Abst**

■

2. $\lambda y : \beta. x \beta$.

Proof.

- | | | |
|----|--|-------------------|
| 1. | $y : \beta$ | Bound |
| 2. | $x : \Pi \alpha : * . \alpha$ | T-Var |
| 3. | $\beta : *$ | T-Form |
| 4. | $x \beta : \beta$ | 2,3 T2-App |
| 5. | $\lambda y : \beta. x \beta : \beta \rightarrow \beta$ | 4 T-Abst |

■

3. $\lambda y : \beta. x (\beta \rightarrow \beta) y.$

Proof.

- | | | |
|----|--|-------------------|
| 1. | $y : \beta$ | Bound |
| 2. | $x : \Pi \alpha : * . \alpha$ | T-Var |
| 3. | $\beta \rightarrow \beta : *$ | T-Form |
| 4. | $x (\beta \rightarrow \beta) : \beta \rightarrow \beta$ | 2,3 T2-App |
| 5. | $y : \beta$ | T-Var |
| 6. | $x (\beta \rightarrow \beta) y : \beta$ | 4,5 T-App |
| 7. | $\lambda y : \beta. x (\beta \rightarrow \beta) y : \beta \rightarrow \beta$ | 5 T-Abst |

■

Problem

(3.5 d) Prove that the following terms inhabit the same type in Γ :

$$\lambda f : \beta \rightarrow \beta \rightarrow \beta. f (x \beta)(x \beta)$$

$$x ((\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta)$$

Solution. We simply derive the types.

First Term.

- | | | |
|----|---|-------------------|
| 1. | $f : \beta \rightarrow \beta \rightarrow \beta$ | Bound |
| 2. | $f : \beta \rightarrow \beta \rightarrow \beta$ | T-Var |
| 3. | $x : \Pi \alpha : * . \alpha$ | T-Var |
| 4. | $\beta : *$ | T-Form |
| 5. | $x \beta : \beta$ | 3,4 T2-App |
| 6. | $f (x \beta) : \beta \rightarrow \beta$ | 2,5 T-App |

7. $\frac{}{f(x\ \beta)(x\ \beta) : \beta}$ **6,5 T-App**
8. $\lambda f : \beta \rightarrow \beta \rightarrow \beta. f(x\ \beta)(x\ \beta) : (\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$ **6 T-Abst**

■

Second Term.

1. $(\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta : *$ **T-Form**
2. $x : \Pi \alpha : *. \alpha$ **T-Var**
3. $x((\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta) : (\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$ **2,1 T2-App**

■

The two terms were shown to both inhabit $(\beta \rightarrow \beta \rightarrow \beta) \rightarrow \beta$.

Problem

(3.6 a) Find inhabitant of type

$$\Pi \alpha, \beta : *. (\text{nat} \rightarrow \alpha) \rightarrow (\alpha \rightarrow \text{nat} \rightarrow \beta) \rightarrow \text{nat} \rightarrow \beta$$

In context $\Gamma \equiv \text{nat} : *$.

Solution.

$$\lambda \alpha, \beta : *. \lambda x : \text{nat} \rightarrow \alpha. \lambda y : (\alpha \rightarrow \text{nat} \rightarrow \beta). \lambda z : \text{nat}. y(x\ z)\ z$$

Proof.

1. $\alpha, \beta : *$ **Bound**
2. $\text{nat} \rightarrow \alpha : *$ **T-Form**
3. $x : \text{nat} \rightarrow \alpha$ **Bound**
4. $\alpha \rightarrow \text{nat} \rightarrow \beta : *$ **T-Form**
5. $y : \alpha \rightarrow \text{nat} \rightarrow \beta$ **Bound**
6. $\text{nat} : *$ **Bound**
7. $z : \text{nat}$ **Bound**
8. $y : \alpha \rightarrow \text{nat} \rightarrow \beta$ **T-Var**
9. $x : \text{nat} \rightarrow \alpha$ **T-Var**
10. $z : \text{nat}$ **T-Var**
11. $x\ z : \alpha$ **9,10 T-App**
12. $y(x\ z) : \text{nat} \rightarrow \beta$ **8,11 T-App**
13. $y(x\ z)\ z : \beta$ **12,10 T-App**

| | | |
|-----|---|-------------------|
| 14. | $\lambda z : \mathbf{nat}. y (x z) z : \mathbf{nat} \rightarrow \beta$ | 13 T-Abst |
| 15. | $\lambda y : \alpha \rightarrow \mathbf{nat} \rightarrow \beta. \lambda z : \mathbf{nat}. y (x z) z$ $: (\alpha \rightarrow \mathbf{nat} \rightarrow \beta) \rightarrow \mathbf{nat} \rightarrow \beta$ | 14 T-Abst |
| 16. | $\lambda x : \mathbf{nat} \rightarrow \alpha. y : \alpha \rightarrow \mathbf{nat} \rightarrow \beta. \lambda z : \mathbf{nat}. y (x z) z$ $: (\mathbf{nat} \rightarrow \alpha) \rightarrow (\alpha \rightarrow \mathbf{nat} \rightarrow \beta) \rightarrow \mathbf{nat} \rightarrow \beta$ | 15 T2-Abst |
| 17. | | |
| | $\lambda \alpha, \beta : *. x : \mathbf{nat} \rightarrow \alpha. y : \alpha \rightarrow \mathbf{nat} \rightarrow \beta. \lambda z : \mathbf{nat}. y (x z) z$ $: \Pi \alpha, \beta : *. (\mathbf{nat} \rightarrow \alpha) \rightarrow (\alpha \rightarrow \mathbf{nat} \rightarrow \beta) \rightarrow \mathbf{nat} \rightarrow \beta$ | 16 T2-Abst |

■

Problem

(3.6 b) Find inhabitant of type

$$\Pi \delta : *. ((\alpha \rightarrow \gamma) \rightarrow \delta) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta$$

In context $\Gamma \equiv \alpha : *, \beta : *, \gamma : *$

Solution.

$$\lambda \delta : *. \lambda x : (\alpha \rightarrow \gamma) \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x (\lambda u : \alpha. z (y u))$$

A derivation in shorthand will be given (omitting T-Form / T-Var)

Proof.

| | | |
|-----|---|-----------------|
| 1. | $\delta : *$ | Bound |
| 2. | $x : (\alpha \rightarrow \gamma) \rightarrow \delta$ | Bound |
| 3. | $y : \alpha \rightarrow \beta$ | Bound |
| 4. | $z : \beta \rightarrow \gamma$ | Bound |
| 5. | $u : \alpha$ | Bound |
| 6. | $y u : \beta$ | *, T-App |
| 7. | $z (y u) : \gamma$ | |
| | *,6 T-App | |
| 8. | $\lambda u : \alpha. z (y u) : \alpha \rightarrow \gamma$ | 7 T-Abst |
| 9. | $x (\lambda u : \alpha. z (y u)) : \delta$ | 8 T-Abst |
| 10. | $\lambda z : \beta \rightarrow \gamma. x (\lambda u : \alpha. z (y u)) : (\beta \rightarrow \gamma) \rightarrow \delta$ | 9 T-Abst |

$$11. \quad \left| \begin{array}{l} \lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x (\lambda u : \alpha. z (y u)) \\ : (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta \end{array} \right.$$

10 T-Abst

$$12. \quad \left| \begin{array}{l} \lambda x : (\alpha \rightarrow \gamma) \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x (\lambda u : \alpha. z (y u)) \\ : ((\alpha \rightarrow \gamma) \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta \end{array} \right.$$

11 T-Abst

13.

$$\lambda \delta : *. \lambda x : (\alpha \rightarrow \gamma) \rightarrow \delta. \lambda y : \alpha \rightarrow \beta. \lambda z : \beta \rightarrow \gamma. x (\lambda u : \alpha. z (y u)) \\ : \Pi \delta : *. ((\alpha \rightarrow \gamma) \rightarrow \delta) \rightarrow (\alpha \rightarrow \beta) \rightarrow (\beta \rightarrow \gamma) \rightarrow \delta$$

12 T2-Abst

■

Problem

(3.6 c) Find inhabitant of type

$$\Pi \alpha, \beta, \gamma : *. (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$$

In the empty context

Solution.

$$\lambda \alpha, \beta, \gamma : *. \lambda f : (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma). \lambda x : \alpha. f x (\lambda u : \beta. x)$$

Proof.

| | | |
|-----|--|-------------------|
| 1. | α, β, γ | Bound |
| 2. | $f : \alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma$ | Bound |
| 3. | $x : \alpha$ | Bound |
| 4. | $f x : (\beta \rightarrow \alpha) \rightarrow \gamma$ | *, * T-App |
| 5. | $u : \beta$ | Bound |
| 6. | $x : \alpha$ | T-Var |
| 7. | $\lambda u : \beta. x : \beta \rightarrow \alpha$ | 6 T-Abst |
| 8. | $f x (\lambda u : \beta. x) : \gamma$ | 4,7 T-App |
| 9. | $\lambda x : \alpha. f x (\lambda u : \beta. x) : \alpha \rightarrow \gamma$ | 8 T-Abst |
| | $\lambda f : \alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma. \lambda x : \alpha. f x (\lambda u : \beta. x)$ | |
| 10. | $: (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$ | 9 T-Abst |

11.

$$\lambda\alpha, \beta, \gamma : *. \lambda f : \alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma. \lambda x : \alpha. f x (\lambda u : \beta. x)$$

$$: \Pi\alpha, \beta, \gamma : *. (\alpha \rightarrow (\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \alpha \rightarrow \gamma$$

10 T2-Abst

■

Problem

(3.7) Let $\perp \equiv \Pi\alpha : *. \alpha$ and context $\Gamma \equiv \alpha : *, \beta : *, x : \alpha \rightarrow \perp, f : (\alpha \rightarrow \alpha) \rightarrow \alpha$. Give a derivation that successively calculate an inhabitant of α and β , both in context Γ .

Solution. Have $M : \alpha := f (\lambda n : \alpha. n)$. Then $\Gamma \vdash x M \beta : \beta$.

Typing M.

1. $f : (\alpha \rightarrow \alpha) \rightarrow \alpha$ **T-Var**
2. $n : \alpha$ **Bound**
3. $\frac{}{\lambda n : \alpha. n : \alpha \rightarrow \alpha}$ **T-Var**
4. $\lambda n : \alpha. n : \alpha \rightarrow \alpha$ **3 T-Abst**
5. $f (\lambda n : \alpha. n) : \alpha$ **1,4 T-App**

■

Typing $x M \beta$.

1. $M : \alpha$ **T-Var**
2. $x : \alpha \rightarrow \Pi\alpha : *. \alpha$ **T-Var**
3. $M x : \Pi\alpha : *. \alpha$ **2,1 T-App**
4. $M x \beta : \beta$ **3,* T2-App**

■

Problem

(3.8) Recall $K \equiv \lambda x y . x \in \Lambda$ from untyped lambda calculus. Consider the following types

$$T_1 \equiv \Pi\alpha, \beta : * . \alpha \rightarrow \beta \rightarrow \alpha \quad T_2 \equiv \Pi\alpha : * . \alpha \rightarrow (\Pi\beta : * . \beta \rightarrow \alpha)$$

Find inhabitants of both type $t_1 : T_1$ and $t_2 : T_2$ under the empty context, which may be considered the closed $\lambda 2$ form of $K \in \Lambda_{T_2}$.

Solution.

$$\lambda\alpha, \beta : * . \lambda x : \alpha . \lambda y : \beta . x$$

$$\lambda\alpha : * . \lambda x : \alpha . \lambda\beta : * . \lambda y : \beta . x$$

First Form.

| | | |
|----|--|------------------|
| 1. | $\alpha, \beta : *$ | Bound |
| 2. | $x : \alpha$ | Bound |
| 3. | $y : \beta$ | Bound |
| 4. | $x : \alpha$ | T-Var |
| 5. | $\lambda y : \beta . x : \beta \rightarrow \alpha$ | 4 T-Abst |
| 6. | $\lambda x : \alpha . \lambda y : \beta . x : \alpha \rightarrow \beta \rightarrow \alpha$ | 5 T-Abst |
| 7. | $\lambda\alpha, \beta : * . \lambda x : \alpha . \lambda y : \beta . x : \Pi\alpha, \beta : * . \alpha \rightarrow \beta \rightarrow \alpha$ | 5 T2-Abst |

■

Second Form.

| | | |
|----|--|------------------|
| 1. | $\alpha : *$ | Bound |
| 2. | $x : \alpha$ | Bound |
| 3. | $\beta : *$ | Bound |
| 4. | $y : \beta$ | Bound |
| 5. | $x : \alpha$ | T-Var |
| 6. | $\lambda y : \beta . x : \beta \rightarrow \alpha$ | 5 T-Abst |
| 7. | $\lambda\beta : * . \lambda y : \beta . x : \Pi\beta : * . \beta \rightarrow \alpha$ | 6 T2-Abst |
| 8. | $\lambda x : \alpha . \lambda\beta : * . \lambda y : \beta . x : \alpha \rightarrow (\Pi\beta : * . \beta \rightarrow \alpha)$ | 7 T-Abst |
| 9. | $\lambda\alpha : * . \lambda x : \alpha . \lambda\beta : * . \lambda y : \beta . x : \Pi\alpha : * . \alpha \rightarrow (\Pi\beta : * . \beta \rightarrow \alpha)$ | 8 T2-Abst |

■

Problem

(3.9) Pretype the combinator

$$S \equiv \lambda x y z . x z (y z)$$

In closed form (typable in an empty context) in $\Lambda_{\mathbb{T}2}$.

Solution.

$$S \equiv \lambda \alpha, \beta, \gamma : * . \lambda x : \alpha \rightarrow \beta \rightarrow \gamma . \lambda y : \alpha \rightarrow \beta . \lambda z : \alpha . x z (y z)$$

Proof.

| | | |
|-----|--|-------------------|
| 1. | $\alpha, \beta, \gamma : *$ | Bound |
| 2. | $x : \alpha \rightarrow \beta \rightarrow \gamma$ | Bound |
| 3. | $y : \alpha \rightarrow \beta$ | Bound |
| 4. | $z : \alpha$ | Bound |
| 5. | $x z : \beta \rightarrow \gamma$ | *, T-App |
| 6. | $y x : \beta$ | *, T-App |
| 7. | $x z (y x) : \gamma$ | 5,6 T-App |
| 8. | $\lambda z : \alpha . x z (y x) : \alpha \rightarrow \gamma$ | 7 T-Abst |
| 9. | $\lambda y : \alpha \rightarrow \beta . \lambda z : \alpha . x z (y x) : (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$ | 8 T-Abst |
| 10. | $\lambda x : \alpha \rightarrow \beta \rightarrow \gamma . \lambda y : \alpha \rightarrow \beta . \lambda z : \alpha . x z (y x) : (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$ | 9 T-Abst |
| 11. | | |
| | $\lambda \alpha, \beta, \gamma : * . \lambda x : \alpha \rightarrow \beta \rightarrow \gamma . \lambda y : \alpha \rightarrow \beta . \lambda z : \alpha . x z (y x) : \Pi \alpha, \beta, \gamma : * . (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \beta) \rightarrow \alpha \rightarrow \gamma$ | 10 T2-Abst |

■

Problem

(3.10 a) Consider the term

$$M \equiv \lambda x : \Pi \alpha : * . \alpha \rightarrow \alpha . x (\sigma \rightarrow \sigma)(x \sigma)$$

Prove that M is legal.

Solution. For a term to be legal there must exist a context so that the term could be typed. Here, a witness context is $\Gamma \equiv \sigma : *$.

Proof.

- | | | |
|----|--|------------------|
| 1. | $x : \Pi\alpha : * . \alpha \rightarrow \alpha$ | Bound |
| 2. | $x (\sigma \rightarrow \sigma) : (\sigma \rightarrow \sigma) \rightarrow (\sigma \rightarrow \sigma)$ $x \sigma : \sigma \rightarrow \sigma$ $x (\sigma \rightarrow \sigma)(x \sigma) : \sigma \rightarrow \sigma$ | *, T2-App |
| 3. | | *, T2-App |
| 4. | | 2,3 T-App |
| 5. | | |
- $\lambda x : \Pi\alpha : * . \alpha \rightarrow \alpha. x (\sigma \rightarrow \sigma)(x \sigma)$
 $: (\Pi\alpha : * . \alpha \rightarrow \alpha) \rightarrow \sigma \rightarrow \sigma$
- 4 T-Abst**

■

Problem

(3.10 b) Find a term N such that $M N$ is legal and may be considered to be a way to add type information to $(\lambda x . x x)(\lambda y . y)$

Solution.

$$M \sigma N \equiv (\lambda x : \Pi\alpha : * . \alpha \rightarrow \alpha. x (\sigma \rightarrow \sigma)(x \sigma))\sigma(\lambda y : \sigma. y)$$

Is the same as $(\lambda x . x x)(\lambda y . y)$ modulo type annotations.

Proof.

- | | | |
|----|---|-------------------|
| 1. | $M : (\Pi\alpha : * . \alpha \rightarrow \alpha) \rightarrow \sigma \rightarrow \sigma$ | T-Var |
| 2. | $M \sigma : (\sigma \rightarrow \sigma) \rightarrow \sigma \rightarrow \sigma$ | 1*, T2-App |
| 3. | $y : \sigma$ | Bound |
| 4. | $y : \sigma$ | T-Var |
| 5. | | 4 T-Abst |
| 6. | $M \sigma N : \sigma \rightarrow \sigma$ | 2,5 T-Abst |

■

Problem

(3.11) Recall $\perp \equiv \Pi\alpha : * . \alpha$ from 3.5. Type and prove the following term legal:

$$\lambda x : \perp. x (\perp \rightarrow \perp \rightarrow \perp)(x (\perp \rightarrow \perp) x)(x (\perp \rightarrow \perp \rightarrow \perp) x x)$$

Solution.

Proof. The type $\perp \rightarrow \perp$ is closed and well formed. Therefore, the term is legal.

| | | |
|-----|---|------------------|
| 1. | $\perp : * \equiv \Pi \alpha : * . \alpha$ | T-Form |
| 2. | $x : \perp$ | Bound |
| 3. | $x (\perp \rightarrow \perp \rightarrow \perp) : \perp \rightarrow \perp \rightarrow \perp$ | *, T2-App |
| 4. | $x (\perp \rightarrow \perp) : \perp \rightarrow \perp$ | *, T2-App |
| 5. | $x (\perp \rightarrow \perp) x : \perp$ | 4, T-App |
| 6. | $x (\perp \rightarrow \perp \rightarrow \perp) (x (\perp \rightarrow \perp) x) : \perp \rightarrow \perp$ | 3,5 T-App |
| 7. | $x (\perp \rightarrow \perp \rightarrow \perp) x : \perp \rightarrow \perp$ | 3, T-App |
| 8. | $x (\perp \rightarrow \perp \rightarrow \perp) x x : \perp$ | 7, T-App |
| 9. | $x (\perp \rightarrow \perp \rightarrow \perp) (x (\perp \rightarrow \perp) x) (x (\perp \rightarrow \perp \rightarrow \perp) x x) : \perp$ | 6,8 T-App |
| 10. | | |
| | $\lambda x : \perp. x (\perp \rightarrow \perp \rightarrow \perp) (x (\perp \rightarrow \perp) x)$ | |
| | $(x (\perp \rightarrow \perp \rightarrow \perp) x x) : \perp \rightarrow \perp$ | 9 T-Abst |

■

Problem

(3.12) Given the Polymorphic Church Numerals:

$$\mathbf{nat} \in \mathbb{T}_2 := \Pi \alpha : * . (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$$

$$\bar{0} \equiv \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . x : \mathbf{nat}$$

$$\bar{1} \equiv \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . f x : \mathbf{nat}$$

$$\bar{2} \equiv \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . f (f x) : \mathbf{nat}$$

$$\mathbf{succ} \equiv \lambda n : \mathbf{nat} . \lambda \beta : * . \lambda f : \beta \rightarrow \beta . \lambda x : \beta . f (n \beta f x)$$

Prove that

$$\mathbf{succ} \bar{0} \underset{\beta}{=} \bar{1}$$

$$\mathbf{succ} \bar{1} \underset{\beta}{=} \bar{2}$$

Solution.

$$\begin{aligned}
& \text{succ } \bar{0} \\
& \equiv (\lambda n : \mathbf{nat}. \lambda \beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f (n \beta f x)) (\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. x) \\
& \xrightarrow[\beta]{} (\lambda \beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f ((\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. x) \beta f x)) \\
& \xrightarrow[\beta_{T_2}]{} (\lambda \beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f ((\lambda f : \beta \rightarrow \beta. \lambda x : \beta. x) f x)) \\
& \xrightarrow[\beta]{} (\lambda \beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f x) \stackrel{\beta \rightarrow \alpha}{\alpha_{T_2}} \equiv \bar{1} \\
& \text{succ } \bar{1} \\
& \equiv (\lambda n : \mathbf{nat}. \lambda \beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f (n \beta f x)) (\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f x) \\
& \xrightarrow[\beta]{} (\lambda \beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f ((\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f x) \beta f x)) \\
& \xrightarrow[\beta_{T_2}]{} (\lambda \beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f ((\lambda f : \beta \rightarrow \beta. \lambda x : \beta. f x) f x)) \\
& \xrightarrow[\beta]{} (\lambda \beta : *. \lambda f : \beta \rightarrow \beta. \lambda x : \beta. f (f x)) \stackrel{\beta \rightarrow \alpha}{\alpha_{T_2}} \equiv \bar{2}
\end{aligned}$$

Problem

(3.13 a) We define addition in Polymorphic Church Numerals as

$$\text{add} \equiv \lambda m, n : \mathbf{nat}. \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. m \alpha f (n \alpha f x)$$

Show that

$$\text{add } \bar{1} \ \bar{1} \stackrel{\beta}{=} \bar{2}$$

Solution.

$$\begin{aligned}
& \text{add } \bar{1} \ \bar{1} \\
& \equiv (\lambda m, n : \mathbf{nat}. \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. m \alpha f (n \alpha f x)) \bar{1} \ \bar{1} \\
& \xrightarrow[\beta]{} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. \bar{1} \alpha f (\bar{1} \alpha f x) \\
& \equiv \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. \bar{1} \alpha f ((\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. f x) \alpha f x) \\
& \xrightarrow[\beta]{} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. \bar{1} \alpha f (f x) \\
& \equiv \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. (\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. f x) \alpha f (f x) \\
& \xrightarrow[\beta]{} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \mathbf{nat}. f (f x) \stackrel{\alpha}{=} \bar{2}
\end{aligned}$$

Problem

(3.13 b) Find a term mul simulates multiplication on nat .

Solution.

$$\text{mul} \equiv \lambda m, n : \text{nat}. \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m \alpha (n \alpha f) x$$

Proof. We derive the type first to prove a legal term.

| | | |
|-----|--|-------------------|
| 1. | $m, n : \text{nat}$ | Bound |
| 2. | $\alpha : *$ | Bound |
| 3. | $f : \alpha \rightarrow \alpha$ | Bound |
| 4. | $x : \alpha$ | Bound |
| 5. | $m \alpha : (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$ | *, T2-App |
| 6. | $n \alpha : (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$ | *, T2-App |
| 7. | $n \alpha f : \alpha \rightarrow \alpha$ | 6, T-App |
| 8. | $m \alpha (n \alpha f) : \alpha \rightarrow \alpha$ | 5,7 T-App |
| 9. | $m \alpha (n \alpha f) x : \alpha$ | 8, T-App |
| 10. | $\lambda x : \alpha. m \alpha (n \alpha f) x : \alpha \rightarrow \alpha$ | 9 T-Abst |
| 11. | $\lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m \alpha (n \alpha f) x : (\alpha \rightarrow \alpha) \rightarrow \alpha \rightarrow \alpha$ | 10 T-Abst |
| 12. | $\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m \alpha (n \alpha f) x : \text{nat}$ | 11 T2-Abst |
| 13. | | |

$$\lambda m, n : \text{nat}. \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m \alpha (n \alpha f) x$$

$$: \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$$

12 T-Abst

This proves that the term does indeed produce a natural number from two. Next let's prove that

$$\forall \bar{n}, \bar{m} : \text{nat} \quad \text{mul } \bar{n} \ \bar{m} = \overline{n \times m}$$

It could be proven by induction that

$$\forall \bar{a} : \text{nat} \quad \bar{a} \equiv \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f^a x$$

$$\begin{aligned} \text{mul } \bar{n} \ \bar{m} &\equiv (\lambda m, n : \text{nat}. \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. m \alpha (n \alpha f) x) (\bar{n}) (\bar{m}) \\ &\xrightarrow[\beta]{} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. \bar{n} \alpha (\bar{m} \alpha f) x \\ &\xrightarrow[\beta]{} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. \bar{n} \alpha (\lambda u : \alpha. f^m u) x \\ &\xrightarrow[\beta]{} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. (\lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. f^n x) (\lambda u : \alpha. f^m u) x \\ &\xrightarrow[\beta]{} \lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. (f^m u)^n x \end{aligned}$$

By induction this can be further beta-reduced to

$$\xrightarrow[\beta]{\rightarrow} \lambda \alpha : * . \lambda f : \alpha \rightarrow \alpha . \lambda x : \alpha . f^{mn} x \equiv \overline{m} \overline{n}$$

■

Problem

(3.14) We present the Church-Encoded Boolean:

$$\mathbf{bool} \in \mathbb{T}_2 := \Pi \alpha : * . \alpha \rightarrow \alpha \rightarrow \alpha$$

$$\mathbf{true} \equiv \lambda \alpha : * . \lambda x, y : \alpha . x$$

$$\mathbf{false} \equiv \lambda \alpha : * . \lambda x, y : \alpha . y$$

Construct a $\lambda 2$ term \mathbf{neg} such that $\mathbf{neg} \mathbf{true} \equiv_{\beta} \mathbf{false}$ and $\mathbf{neg} \mathbf{false} \equiv_{\beta} \mathbf{true}$.

Solution.

$$\mathbf{neg} \equiv \lambda b : \mathbf{bool} . \lambda \alpha : * . b \alpha (\mathbf{false} \alpha) (\mathbf{true} \alpha)$$

Neg True.

$$\mathbf{neg} \mathbf{true} \equiv (\lambda b : \mathbf{bool} . \lambda \alpha : * . b \alpha (\mathbf{false} \alpha) (\mathbf{true} \alpha)) \mathbf{true}$$

$$\xrightarrow[\beta]{\rightarrow} \lambda \alpha : * . (\lambda x, y : \alpha . x) (\mathbf{false} \alpha) (\mathbf{true} \alpha)$$

$$\xrightarrow[\beta]{\rightarrow} \lambda \alpha : * . \mathbf{false} \alpha \xrightarrow[\eta]{\rightarrow} \mathbf{false}$$

■

Neg False.

$$\mathbf{neg} \mathbf{false} \equiv (\lambda b : \mathbf{bool} . \lambda \alpha : * . b \alpha (\mathbf{false} \alpha) (\mathbf{true} \alpha)) \mathbf{false}$$

$$\xrightarrow[\beta]{\rightarrow} \lambda \alpha : * . (\lambda x, y : \alpha . y) (\mathbf{false} \alpha) (\mathbf{true} \alpha)$$

$$\xrightarrow[\beta]{\rightarrow} \lambda \alpha : * . \mathbf{true} \alpha \xrightarrow[\eta]{\rightarrow} \mathbf{true}$$

■

Problem

(3.15) Define

$$M \equiv \lambda u, v : \mathbf{bool} . \lambda \beta : * . \lambda x, y : \beta . u \beta (v \beta x y) (v \beta y y)$$

And reduce $M \mathbf{true} \mathbf{true}$, $M \mathbf{true} \mathbf{false}$, $M \mathbf{false} \mathbf{true}$, $M \mathbf{false} \mathbf{false}$, and decide which logical operator is represented by M .

Solution.

$$\begin{aligned}
M \text{ true true} &\equiv (\lambda u, v : \text{bool}. \lambda \beta : *. \lambda x, y : \beta. u \beta (v \beta x y) (v \beta y y)) \text{ true true} \\
&\xrightarrow[\beta]{} \lambda \beta : *. \lambda x, y : \beta. \text{true } \beta (\text{true } \beta x y) (\text{true } \beta y y) \\
&\xrightarrow[\beta]{} \lambda \beta : *. \lambda x, y : \beta. (\text{true } \beta x y) \\
&\xrightarrow[\beta]{} \lambda \beta : *. \lambda x, y : \beta. x \equiv_{\alpha} \text{true}
\end{aligned}$$

$$\begin{aligned}
M \text{ true false} &\equiv (\lambda u, v : \text{bool}. \lambda \beta : *. \lambda x, y : \beta. u \beta (v \beta x y) (v \beta y y)) \text{ true false} \\
&\xrightarrow[\beta]{} \lambda \beta : *. \lambda x, y : \beta. \text{true } \beta (\text{false } \beta x y) (\text{false } \beta y y) \\
&\xrightarrow[\beta]{} \lambda \beta : *. \lambda x, y : \beta. (\text{false } \beta x y) \\
&\xrightarrow[\beta]{} \lambda \beta : *. \lambda x, y : \beta. y \equiv_{\alpha} \text{false}
\end{aligned}$$

$$\begin{aligned}
M \text{ false true} &\equiv (\lambda u, v : \text{bool}. \lambda \beta : *. \lambda x, y : \beta. u \beta (v \beta x y) (v \beta y y)) \text{ false true} \\
&\xrightarrow[\beta]{} \lambda \beta : *. \lambda x, y : \beta. \text{false } \beta (\text{true } \beta x y) (\text{true } \beta y y) \\
&\xrightarrow[\beta]{} \lambda \beta : *. \lambda x, y : \beta. (\text{true } \beta y y) \\
&\xrightarrow[\beta]{} \lambda \beta : *. \lambda x, y : \beta. y \equiv_{\alpha} \text{true}
\end{aligned}$$

$$\begin{aligned}
M \text{ false false} &\equiv (\lambda u, v : \text{bool}. \lambda \beta : *. \lambda x, y : \beta. u \beta (v \beta x y) (v \beta y y)) \text{ false false} \\
&\xrightarrow[\beta]{} \lambda \beta : *. \lambda x, y : \beta. \text{false } \beta (\text{false } \beta x y) (\text{false } \beta y y) \\
&\xrightarrow[\beta]{} \lambda \beta : *. \lambda x, y : \beta. (\text{false } \beta y y) \\
&\xrightarrow[\beta]{} \lambda \beta : *. \lambda x, y : \beta. y \equiv_{\alpha} \text{false}
\end{aligned}$$

Therefore M is equivalent to logical AND.

Problem

(3.16) Find $\lambda 2$ term representing the logical OR, XOR, IMP.

Solution.

$$\begin{aligned}
\text{OR} &\equiv \lambda u, v : \text{bool}. \lambda \beta : *. \lambda x, y : \beta. u \beta x (v \beta x y) \\
\text{XOR} &\equiv \lambda u, v : \text{bool}. \lambda \beta : *. \lambda x, y : \beta. u \beta (v \beta y x) (v \beta x y) \\
\text{IMP} &\equiv \lambda u, v : \text{bool}. \lambda \beta : *. \lambda x, y : \beta. u \beta (v \beta x y) x
\end{aligned}$$

All of them could be checked by finite enumeration over $\text{bool} \times \text{bool}$.

Problem

(3.17) Find $\text{isZero} : \text{nat} \rightarrow \text{bool}$ such that $\forall n : \text{nat}, \text{isZero } n \stackrel{\beta}{=} \text{false}$ except when $n \equiv \bar{0}$.

Solution.

$$\text{isZero} \equiv \lambda n : \text{nat}. n \text{ bool } (\lambda u : \text{bool}. \text{false}) \text{ true}$$

Proof.

$$\begin{aligned} \text{isZero } \bar{0} &\equiv (\lambda n : \text{nat}. n \text{ bool } (\lambda u : \text{bool}. \text{false}) \text{ true}) \bar{0} \\ &\xrightarrow[\beta]{} (\lambda \alpha : *. \lambda f : \alpha \rightarrow \alpha. \lambda x : \alpha. x) \text{ bool } (\lambda u : \text{bool}. \text{false}) \text{ true} \\ &\xrightarrow[\beta]{} (\lambda f : \text{bool} \rightarrow \text{bool}. \lambda x : \text{bool}. x) (\lambda u : \text{bool}. \text{false}) \text{ true} \\ &\xrightarrow[\beta]{} \text{true} \end{aligned}$$

■

By induction it could be proven that any other natural numbers must be applied $\lambda u : \text{bool}. \text{false}$ to the body, making the result false, except for $\bar{0}$, where the function $f : \alpha \rightarrow \alpha$ never got applied.

Problem

(3.18 a) Define type

$$\text{tree} \equiv \Pi \alpha : *. (\text{bool} \rightarrow \alpha) \rightarrow (\text{bool} \rightarrow \alpha \rightarrow \alpha \rightarrow \alpha) \rightarrow \alpha$$

Then we construct an inhabitant

$$\lambda \alpha. *. \lambda u : \text{bool} \rightarrow \alpha. \lambda v : \text{bool} \rightarrow \alpha \rightarrow \alpha \rightarrow \alpha. M$$

Is a node of a binary tree. Sketch graphs of trees where M is

$$\begin{aligned} &u \text{ false} \\ &v \text{ true } (u \text{ false}) (u \text{ true}) \\ &v \text{ true } (u \text{ true}) (v \text{ false } (u \text{ true}) (u \text{ false})) \end{aligned}$$

Solution. A binary tree is usually defined as this:

```
inductive Tree (a : Type) where
| leaf (value : a) : Tree a
| node (left right : Tree a) : Tree a
```

With two constructors: a leaf or a node. Here α is the type of the payload at each node. There are two constructors: u is the left constructor, taking a `bool` value for the direction of the node. The v term is the node constructor, taking a `bool` as the direction, two α -typed terms as it's children.



Problem

(3.18 b) Give a $\lambda 2$ term, which, given input a polymorphic boolean p and two trees s and t , delivers the combined tree with p on top, left subtree s and right subtree t .

Solution.

$$\begin{aligned}
 \text{leaf} &:= \lambda p : \text{bool}. \lambda s, t : \text{tree}. \\
 &\quad \lambda \alpha. \lambda u. \text{bool} \rightarrow \alpha. \lambda v : \text{bool} \rightarrow \alpha \rightarrow \alpha \rightarrow \alpha. \\
 &\quad v \ p \ (s \ \alpha \ u \ v) (t \ \alpha \ u \ v)
 \end{aligned}$$

Proof. We suppose

$$\begin{aligned}
 s &\equiv \lambda \beta : *. \lambda u_s : \text{bool} \rightarrow \beta. \lambda v_s : \text{bool} \rightarrow \beta \rightarrow \beta \rightarrow \beta. S \\
 t &\equiv \lambda \gamma : *. \lambda u_t : \text{bool} \rightarrow \gamma. \lambda v_t : \text{bool} \rightarrow \gamma \rightarrow \gamma \rightarrow \gamma. T
 \end{aligned}$$

We want

$$\text{leaf } p \ s \ t \stackrel{\beta}{=} \lambda \alpha : *. \lambda u : \text{bool} \rightarrow \alpha. \lambda v : \text{bool} \rightarrow \alpha \rightarrow \alpha \rightarrow \alpha. v \ p \ S \ T$$

For compactness we denote

$$\begin{aligned}
 \alpha : * \vdash \tau_{\text{mkleaf}} &\equiv \text{bool} \rightarrow \alpha \\
 \alpha : * \vdash \tau_{\text{mknode}} &\equiv \text{bool} \rightarrow \alpha \rightarrow \alpha \rightarrow \alpha
 \end{aligned}$$

By beta reduction we have

$$\begin{aligned}
\text{leaf } p \ s \ t &\equiv (\lambda p : \text{bool} . \lambda s, t : \text{tree} \dots) \ p \ s \ t \\
&\xrightarrow[\beta]{} (\lambda \alpha : * . \lambda u . \tau_{\text{mkleaf}} . \lambda v : \tau_{\text{mknode}} . v \ p \ (s \ \alpha \ u \ v) (t \ \alpha \ u \ v)) \\
&\xrightarrow[\beta]{} (\lambda \alpha : * . \lambda u . \tau_{\text{mkleaf}} . \lambda v : \tau_{\text{mknode}} . v \ p \ (s \ \alpha \ u \ v) (t \ \alpha \ u \ v)) \\
&\xrightarrow[\beta]{} \lambda \alpha : * . \lambda u . \tau_{\text{mkleaf}} . \lambda v : \tau_{\text{mknode}} . v \ p \ S \ T
\end{aligned}$$

■

Problem

(3.19) If $\Gamma \vdash L : \sigma$, then Γ is a valid λ_2 context.

Solution. The definition of “valid” here would be taken as relative to a judgement as the fact to be able to derive the judgement. Thus, it meant a complete inference path could be made using only statements and judgements derived from the context. Now proof by induction on inference rule that deducted $L : \sigma$.

Case 1 : T-Var. The premise is that Γ is a λ_2 context. ■

Case 2 : T-App. Therefore $L \equiv M \ N$ for some $M, N \in \Lambda_{\mathbb{T}_2}$. Therefore,

$$\Gamma \vdash M : \tau \rightarrow \sigma \quad \Gamma \vdash N : \tau$$

for some $\tau \in \mathbb{T}_2$. By the inductive hypothesis on any premise Γ is a valid λ_2 context. ■

Case 3 : T-Abst. Therefore $L \equiv \lambda x : \alpha . N$ such that

$$\Gamma, x : \alpha \vdash N : \beta$$

for some $\alpha, \beta \in \mathbb{T}_2$ such that $\sigma \equiv \alpha \rightarrow \beta$. By the inductive hypothesis, $\Gamma, x : \alpha$. By the recursive definition of λ_2 contexts, for some valid context Δ , $\forall n \in \text{dom } \Delta$, n could not depend on statement declared after n in the context. Therefore, no statement in Γ could depend on $x : \alpha$. Therefore, Γ is a valid context. ■

Case 4 : T-Form. The premise is that Γ is a valid λ_2 context. ■

Case 5 : T2-App. Therefore $L \equiv N \ B$ for some $N, B \in \mathbb{V}_2$ such that

$$\Gamma \vdash N : \Pi \alpha : * . \sigma \quad \Gamma \vdash B : *$$

By the inductive hypothesis on the any premise Γ is a valid λ_2 context. ■

Case 6 : T2-Abst. Therefore $L \equiv \lambda \alpha : * . M$ for some $M \in \Lambda_{\mathbb{T}_2}$ such that

$$\Gamma, \alpha : * \vdash M : \beta$$

and $\sigma \equiv \Pi\alpha : * . \beta$. By reasoning in *Case 3* no statement in the context could depend on any statement before the latter's declaration. Therefore no statement in Γ could depend on $\alpha : *$, making it a valid context. ■

Problem

(3.20) Prove the free variable lemma for $\lambda 2$.

$$\Gamma \vdash L : \sigma \Rightarrow \text{FV } L \subseteq \text{dom } \Gamma$$

Solution. Proof by induction on inference rules that deduced $L : \sigma$. The only rule not considered is T-Form since all terms apparent in \mathbb{T}_2 .

Case 1 : T-Var. Therefore L is the only free variable in L . By the generation lemma $L : \sigma \in \Gamma$, so $\{L\} \subseteq \text{dom } \Gamma$ ■

Case 2 : T-App. Therefore by the generation lemma $L \equiv M N$ for some $M, N \in \Lambda_{\mathbb{T}_2}$ such that

$$\Gamma \vdash M : \tau \rightarrow \sigma \quad \Gamma \vdash N : \tau$$

For some type τ . By the inductive hypothesis $\text{FV } M \subseteq \Gamma$ and $\text{FV } N \subseteq \text{dom } \Gamma$. Therefore $\text{FV } L = (\text{FV } M) \cup (\text{FV } N) \subseteq \text{dom } \Gamma$. ■

Case 3 : T-Abst. Therefore by the generation lemma $L \equiv \lambda x : \alpha. M$ for some $M \in \Lambda_{\mathbb{T}_2}$ such that

$$\Gamma, x : \alpha \vdash M : \beta$$

and $\sigma \equiv \alpha \rightarrow \beta$. By the inductive hypothesis $\text{FV } M \subseteq \text{dom } \Gamma \cup \{x\}$. Therefore

$$\text{FV } L = \text{FV } M \setminus \{x\} \subseteq (\text{dom } \Gamma \cup \{x\}) \setminus \{x\} = \text{dom } \Gamma$$

Case 4 : T2-App. Therefore $L \equiv B$ for some $N, B \in \mathbb{V}_2$ such that

$$\Gamma \vdash N : \Pi\alpha : * . \sigma \quad \Gamma \vdash B : *$$

By the inductive hypothesis $\text{FV } N \subseteq \text{dom } \Gamma$. Since $B \in \mathbb{T}_2$ then $\text{FV } B = \emptyset$. Therefore $\text{FV } L = \text{FV } N \subseteq \text{dom } \Gamma$. ■

Case 5 : T2-Abst. Therefore $L \equiv \lambda\alpha : * . M$ for some $M \in \Lambda_{\mathbb{T}_2}$ such that

$$\Gamma, \alpha : * \vdash M : \beta$$

and $\sigma \equiv \Pi\alpha : * . \beta$. Because $\alpha \in \mathbb{T}_2$ so $\alpha \notin \text{FV } M$ and $\text{FV } L = \text{FV } M$, thus $\text{FV } M = \text{FV } L \subseteq \Gamma$. ■

Problem

(3.21) Give a recursive definition for $\text{FTV} : \mathbb{T}_2 \cup \Lambda_{\mathbb{T}_2} \rightarrow \mathbb{V}_2$

Solution. Here $\alpha \in \mathbb{V}_2$, $A, B \in \mathbb{T}_2$, $x \in V$ and $M \in \Lambda_{\mathbb{T}_2}$.

| Form | Value |
|--|--------------------------------------|
| $\text{FTV } \alpha$ | $\{\alpha\}$ |
| $\text{FTV } x$ | \emptyset |
| $\text{FTV } (A \rightarrow B)$ | $\text{FTV}(A) \cup \text{FTV}(B)$ |
| $\text{FTV } (\Pi \alpha : * . A)$ | $\text{FTV}(A) \setminus \{\alpha\}$ |
| $\text{FTV } (M N)$ | $\text{FTV}(M) \cup \text{FTV}(N)$ |
| $\text{FTV } (\lambda x : A . M)$ | $\text{FTV}(A) \cup \text{FTV}(M)$ |
| $\text{FTV } (M B)$ | $\text{FV}(M) \cup \text{FTV}(B)$ |
| $\text{FTV } (\lambda \alpha : * . M)$ | $\text{FTV}(M) \setminus \{\alpha\}$ |

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Completed Dec 16 10:11 pm.