

EXERCISES

CHAPTER 9

SEAN LI ¹

1. Redacted

Definition Extended Rules for λD_0

$$\frac{\Delta; \Gamma \vdash K : L \quad \Delta; \bar{a} : \bar{M} \vdash M : N}{\Delta, (\bar{x} : \bar{A} \triangleright a(\bar{x}) := M : N); \Gamma \vdash K : L} \text{def}$$
$$\frac{\Delta, \mathcal{D}; \Gamma \vdash * : \square \quad \Delta, \mathcal{D}; \Gamma \vdash \bar{U} : A \ [\bar{x} := \bar{U}] \quad \mathcal{D} \equiv \bar{x} : \bar{A} \triangleright a(\bar{x}) := M : N}{\Delta, \mathcal{D}; \Gamma \vdash a(\bar{U}) : N \ [\bar{x} := \bar{U}]} \text{inst}$$
$$\frac{\Delta; \Gamma \vdash x : A \quad \Delta; \Gamma \vdash A : s \quad A \stackrel{\Delta, \beta}{=} B}{\Delta; \Gamma \vdash x : B} \text{conv}$$

Lemma 1. Given $\mathcal{D} \equiv \bar{x} : \bar{A} \triangleright a(\bar{x}) := M : N$ and $a \notin \Delta$

$$\frac{\Delta; \bar{x} : \bar{A} \vdash M : N}{\Delta, \mathcal{D}; \bar{x} : \bar{A} \vdash a(\bar{x}) : N} \text{par}$$

Problem

(9.1) Given

$$(\mathcal{D}_1) \quad x : \mathbb{Z}, y : \mathbb{Z} \triangleright a(x, y) := x^2 + y^2 : \mathbb{Z}$$

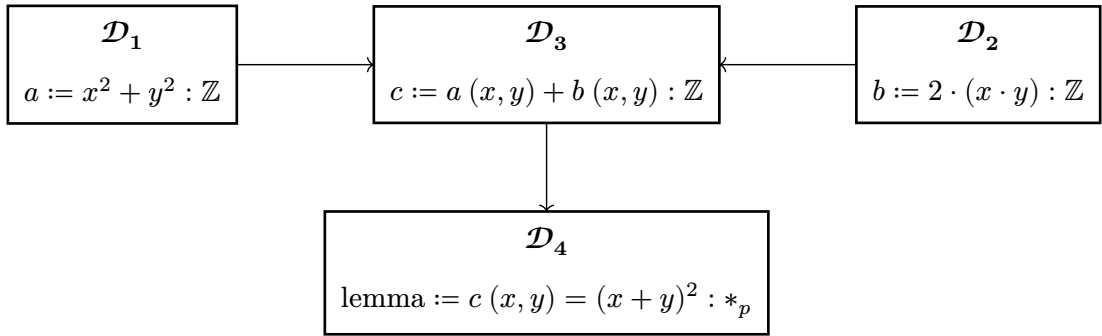
$$(\mathcal{D}_2) \quad x : \mathbb{Z}, y : \mathbb{Z} \triangleright b(x, y) := 2 \cdot (x \cdot y) : \mathbb{Z}$$

$$(\mathcal{D}_3) \quad x : \mathbb{Z}, y : \mathbb{Z} \triangleright c(x, y) := a(x, y) + b(x, y) : \mathbb{Z}$$

$$(\mathcal{D}_4) \quad x : \mathbb{Z}, y : \mathbb{Z} \triangleright \text{lemma}(x, y) := c(x, y) = (x + y)^2 : *_p$$

Consider $\Delta \equiv \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$. Describe the dependencies between the four definitions and give all possible linearizations of the corresponding partial order.

Solution. Hasse diagram given below



The only incomparable pair is $(\mathcal{D}_1, \mathcal{D}_2)$. Therefore there are two possible linearizations:

$$(1) \quad \mathcal{D}_1 \leq \mathcal{D}_2 \leq \mathcal{D}_3 \leq \mathcal{D}_4$$

$$(2) \quad \mathcal{D}_2 \leq \mathcal{D}_1 \leq \mathcal{D}_3 \leq \mathcal{D}_4$$

Problem

(9.2) Consider

$$\mathcal{D}_i \equiv \bar{x} : \bar{A} \triangleright a(\bar{x}) := K : L$$

$$\mathcal{D}_j \equiv \bar{y} : \bar{B} \triangleright b(\bar{y}) := M : N$$

Let $\Delta; \Gamma \vdash U : V$ and assume \mathcal{D}_i and \mathcal{D}_j are elements of list Δ , where \mathcal{D}_i precedes \mathcal{D}_j . Describe precisely where constant a may occur in \mathcal{D}_i and \mathcal{D}_j and where constant b may occur in Δ .

Solution. In order for \mathcal{D}_i to be a valid definition, $\bar{x} : \bar{A} \vdash K : L$ must be legal. Therefore by the free variable lemma any free variables in K and L must be in $\bar{x} : \bar{A}$, which by the time, does not yet contain a 's definition. Therefore, a could only appear in \mathcal{D}_j .

By similar reasoning b could only have appeared in definitions after \mathcal{D}_j . Assuming the list sorted by the suffix, then b could only have been in any \mathcal{D}_k where $k > j$.

Problem

(9.3) Recall Q 8.2

1. $V : *_s$
2. $u : V \subseteq \mathbb{R}$
3. $\text{bounded-from-above}(V, u) := \exists y : \mathbb{R}. \forall x : \mathbb{R}. (x \in V \Rightarrow x \leq y) : *_p$
4. $s : \mathbb{R}$
5. $\text{upper-bound}(V, u, s) := \forall x \in \mathbb{R}. (x \in V \Rightarrow x \leq s) : *_p$
 $\text{least-upper-bound}(V, u, s) := \text{upper-bound}(V, u, s) \wedge$
 $\forall x : \mathbb{R}. (x < s \Rightarrow \neg \text{upper-bound}(V, u, x)) : *_p$
6. $v : V \neq \emptyset$
7. $w : \text{bounded-from-above}(V, u)$
8. $p_4(V, u, w, v) := \text{sorry} : \exists^1 s : \mathbb{R}. \text{least-upper-bound}(V, u, s)$
9. $S := \left\{ x : \mathbb{R} \mid \exists n : \mathbb{R}. \left(n \in \mathbb{N} \wedge x = \frac{n}{n+1} \right) \right\}$
10. $p_6 := \text{sorry} : S \subseteq \mathbb{R}$
11. $p_7 := \text{sorry} : \text{bounded-from-above}(S, p_6)$
12. $p_8 := \text{sorry} : \text{least-upper-bound}(S, p_6, 1)$

Write p_8 out such that all definitions have been unfolded.

Solution.

$$\begin{aligned}
p_8 &:= \text{least-upper-bound}(S, p_6, 1) \\
&\stackrel{=}{=} \text{upper-bound}(S, p_6, 1) \wedge \forall x : \mathbb{R}. (x < 1 \Rightarrow \neg \text{upper-bound}(S, p_6, x)) \\
&\stackrel{=}{=} \forall x : \mathbb{R}. (x \in S \Rightarrow x \leq 1) \wedge \forall x : \mathbb{R}. (x < 1 \Rightarrow \neg(\forall y : \mathbb{R}. (y \in S \Rightarrow y \leq x))) \\
&\stackrel{=}{=} \forall x : \mathbb{R}. \left(x \in \left\{ x : \mathbb{R} \mid \exists n : \mathbb{R}. \left(n \in \mathbb{N} \wedge x = \frac{n}{n+1} \right) \right\} \Rightarrow x \leq 1 \right) \wedge \\
&\quad \forall x : \mathbb{R}. \left(x < 1 \Rightarrow \neg \forall y : \mathbb{R}. \left(y \in \left\{ k : \mathbb{R} \mid \exists n : \mathbb{R}. \left(n \in \mathbb{N} \wedge k = \frac{n}{n+1} \right) \right\} \Rightarrow y \leq x \right) \right)
\end{aligned}$$

Problem

(9.4) Recall $\Delta \equiv \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ from 9.1. Give a complete δ -reduction diagram for $c(a(u, v), b(w, w))$

Solution. Too long to contain. An algorithm for finding the graph is proposed as below:

```

1 Let  $V := \emptyset$  : Set of type  $\mathcal{E}_{\lambda D}$ 
2 Let  $E := \emptyset$  : Set of type  $(\mathcal{E}_{\lambda D} \times \mathcal{E}_{\lambda D})$ 
3 Define procedure  $\text{reduce}(t : \mathcal{E}_{\lambda D}, \Delta : \text{Env})$  do
4   If  $t \in V$  then terminate
5   Else
6     Set  $V := V \cup \{t\}$ 
7     Loop for each redex  $r$  of  $t$  do
8       Let  $r' :=$  outermost one-step  $\delta$ -reduction of  $r$ 
9       Let  $t' := t[r := r']$ 
10      Set  $E := E \cup \{(t, t')\}$ 
11      Execute  $\text{reduce}(t', \Delta)$ 
12    End loop
13  End if
14 End reduce
15 Main
16   Define  $\Delta := \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ 
17   Execute  $\text{reduce}(c(a(u, v), b(w, w)), \Delta)$  and discard result
18   Graph  $(V, E)$ 
19   Terminates
20 End main

```

Problem

(9.5) Check that all instantiations in the 8.2 proof is legal under *inst* rule.

Solution. It is trivial that the first (well-formed context and environment) and third (definition existence) holds for all instantiations.

The instantiation on line 6 ...upper-bound(V, u, s)... requires the following premises after expansion:

$$\begin{aligned}
\Delta; \Gamma \vdash V : (*_s [V := V, u := u, s := s] \text{ which is } *_s) & \quad \checkmark \\
\Delta; \Gamma \vdash u : (V \subseteq \mathbb{R}[V := V, u := u, s := s] \text{ which is } V) & \quad \checkmark \\
\Delta; \Gamma \vdash s : (\mathbb{R}[V := V, u := u, s := s] \text{ which is } \mathbb{R}) & \quad \checkmark
\end{aligned}$$

The instantiation on line 6 ...upper-bound(V, u, x)... requires the following premises after expansion:

$$\begin{aligned}
\Delta; \Gamma \vdash V : (*_s [V := V, u := u, s := x] \text{ which is } *_s) & \quad \checkmark \\
\Delta; \Gamma \vdash u : (V \subseteq \mathbb{R}[V := V, u := u, s := x] \text{ which is } V) & \quad \checkmark \\
\Delta; \Gamma \vdash x : (\mathbb{R}[V := V, u := u, s := x] \text{ which is } \mathbb{R}) & \quad \checkmark
\end{aligned}$$

The instantiation on line 8 ...bounded-from-above(V, u)... requires the following premises after expansion:

$$\begin{aligned}
\Delta; \Gamma \vdash V : (*_s [V := V, u := u] \text{ which is } *_s) & \quad \checkmark \\
\Delta; \Gamma \vdash u : (V \subseteq \mathbb{R}[V := V, u := u] \text{ which is } V \subseteq \mathbb{R}) & \quad \checkmark
\end{aligned}$$

The instantiation on line 9 ...least-upper-bound(V, u, s)... requires the following premises after expansion:

$$\begin{aligned}
\Delta; \Gamma \vdash V : (*_s [V := V, u := u, s := s] \text{ which is } *_s) & \quad \checkmark \\
\Delta; \Gamma \vdash u : ((V \subseteq \mathbb{R})[V := V, u := u, s := s] \text{ which is } V \subseteq \mathbb{R}) & \quad \checkmark \\
\Delta; \Gamma \vdash s : (\mathbb{R}[V := V, u := u, s := s] \text{ which is } \mathbb{R}) & \quad \checkmark
\end{aligned}$$

The instantiation on line 12 ...bounded-from-above(S, p_6)... requires the following premises after expansion:

$$\begin{aligned}
\Delta; \Gamma \vdash S : (*_s [V := S, u := p_6] \text{ which is } *_s) & \quad \checkmark \\
\Delta; \Gamma \vdash p_6 : ((V \subseteq \mathbb{R})[V := S, u := p_6] \text{ which is } S \subseteq \mathbb{R}) & \quad \checkmark
\end{aligned}$$

The instantiation on line 13 ...least-upper-bound($S, p_6, 1$)... requires the following premises after expansion:

$$\begin{aligned}
\Delta; \Gamma \vdash S : (*_s [V := S, u := p_6, s := 1] \text{ which is } *_s) & \quad \checkmark \\
\Delta; \Gamma \vdash p_6 : ((V \subseteq \mathbb{R})[V := S, u := p_6, s := 1] \text{ which is } S \subseteq \mathbb{R}) & \quad \checkmark \\
\Delta; \Gamma \vdash 1 : (\mathbb{R}[V := S, u := p_6, s := 1] \text{ which is } \mathbb{R}) & \quad \checkmark
\end{aligned}$$

Therefore all of the instantiations are valid.