

# EXERCISES

## CHAPTER 4

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1. Redacted

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**Definition** Some rules for reference.

$$\frac{}{\emptyset \vdash * : \square} \text{Sort} \qquad \frac{\Gamma \vdash A : s \quad x \notin \text{dom } \Gamma}{\Gamma, x : A \vdash x : A} \text{Var}$$

$$\frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s \quad x \notin \text{dom } \Gamma}{\Gamma, x : C \vdash A : B} \text{Weak} \qquad \frac{\Gamma \vdash A : s \quad \Gamma \vdash B : s}{\Gamma \vdash A \rightarrow B : s} \text{Form}$$

$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B} \text{App} \qquad \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash A \rightarrow B : s}{\Gamma \vdash \lambda x : A . M : A \rightarrow B} \text{Abst}$$

$$\frac{\Gamma \vdash A : B \quad \Gamma \vdash B' : s \quad B \stackrel{\beta}{=} B'}{\Gamma \vdash A : B'} \text{Conv}$$

Previously an alternative version of the flag derivation was used, only putting up a flag for a local premise (abstraction unwrapping) to save horizontal space. Currently, the standard flag derivation format will be used since now single lines will not be as long.

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### Problem

(4.1) Give a complete tree diagram of the derivation in section 4.5 (95)

*Solution.*



### Problem

(4.2 a) Give a complete  $\lambda\omega$  derivation in flag format of

$$\emptyset \vdash (* \rightarrow *) \rightarrow * : \square$$

*Solution.*

- |  |                 |
|--|-----------------|
| 1. $* : \square$                               | <b>Sort</b>     |
| 2. $* \rightarrow * : \square$                 | <b>1,1 Form</b> |
| 3. $(* \rightarrow *) \rightarrow * : \square$ | <b>2,1 Form</b> |

### Problem

(4.2 b) Give a complete  $\lambda\omega$  derivation in flag format of

$$\alpha : *, \beta : * \vdash (\alpha \rightarrow \beta) \rightarrow \alpha : *$$

*Solution.*

1.	$\emptyset \vdash * : \square$	<b>Sort</b>
2.	$\alpha : *$	
3.	$\alpha : *$	<b>1 Var</b>
4.	$* : \square$	<b>1,1 Weak</b>
5.	$\beta : *$	
6.	$\alpha : *$	<b>3,4 Weak</b>
7.	$\beta : *$	<b>4 Var</b>
8.	$\alpha \rightarrow \beta : *$	<b>6,7 Form</b>
9.	$(\alpha \rightarrow \beta) \rightarrow \alpha : *$	<b>8,6 Form</b>

### Problem

(4.3 a) Give a complete  $\lambda\omega$  derivation in flag format of

$$\alpha, \beta : *, x : \alpha, y : \alpha \rightarrow \beta \vdash y x : \beta$$

*Solution.*

1.	$* : \square$	<b>Sort</b>
2.	$\alpha : *$	
3.	$\alpha : *$	<b>1 Var</b>
4.	$* : \square$	<b>1,1 Weak</b>
5.	$\beta : *$	
6.	$\beta : *$	<b>4 Var</b>
7.	$\alpha : *$	<b>3,4 Weak</b>
8.	$* : \square$	<b>4,4 Weak</b>
9.	$x : \alpha$	
10.	$x : \alpha$	<b>7 Var</b>
11.	$\alpha : *$	<b>7,7 Weak</b>
12.	$\beta : *$	<b>6,7 Weak</b>
13.	$\alpha \rightarrow \beta : *$	<b>11,12 Form</b>
14.	$y : \alpha \rightarrow \beta$	
15.	$y : \alpha \rightarrow \beta$	<b>13 Var</b>
16.	$x : \alpha$	<b>10,13 Weak</b>
17.	$y x : \beta$	<b>15,16 App</b>

**Problem**

(4.3 b) Give a shortened  $\lambda\omega$  derivation in flag format of

$$\alpha, \beta : *, x : \alpha, y : \alpha \rightarrow \beta, z : \beta \rightarrow \alpha \vdash z(y\ x) : \alpha$$

(4.3 b) Give a shortened  $\lambda\underline{\omega}$  derivation in flag format of

$$\alpha, \beta : *, x : \alpha, y : \alpha \rightarrow \beta, z : \beta \rightarrow \alpha \vdash z(y\ x) : \alpha$$

$$\alpha, \beta : *, x : \alpha, y : \alpha \rightarrow \beta, z : \beta \rightarrow \alpha \vdash z (y x) : \alpha$$

*Solution.*

1.	$\alpha : *$				
2.	$\beta : *$				
3.	$x : \alpha$				
4.	$y : \alpha \rightarrow \beta$				
5.	$x : \alpha$			<b>3 Weak</b>	
6.	$z : \beta \rightarrow \alpha$				
7.	$x : \alpha$			<b>5 Weak</b>	
8.	$y : \alpha \rightarrow \beta$			<b>4 Weak</b>	
9.	$y\ x : \beta$			<b>8,7 App</b>	
10.	$z\ (y\ x) : \alpha$			<b>6,9 App</b>	

**Problem**

(4.4 a) Give a shortened  $\lambda_{\omega}$  derivation in flag format of

$$\alpha : *, \beta : * \rightarrow * \vdash \beta(\beta\alpha) : *$$

(4.4 a) Give a shortened  $\lambda_{\underline{\omega}}$  derivation in flag format of

$$\alpha : *, \beta : * \rightarrow * \vdash \beta(\beta\alpha) : *$$

$$\alpha : *, \beta : * \rightarrow * \vdash \beta(\beta\alpha) : *$$

*Solution.*

$$\begin{array}{ll}
1. & \alpha : * \\
2. & \left| \beta : * \rightarrow * \right. \\
3. & \left| \beta\alpha : * \right. \quad \mathbf{2,1 \text{ App}} \\
4. & \left| \beta(\beta\alpha) : * \right. \quad \mathbf{2,4 \text{ App}}
\end{array}$$

**Problem**

(4.4 b) Give a shortened  $\lambda\omega$  derivation in flag format of

$$\alpha : *, \beta : * \rightarrow *, x : \beta(\beta\alpha) \vdash \lambda y : \alpha. x : \alpha \rightarrow \beta(\beta\alpha)$$

(4.4 b) Give a shortened  $\lambda\omega$  derivation in flag format of

$$\alpha : *, \beta : * \rightarrow *, x : \beta(\beta\alpha) \vdash \lambda y : \alpha. x : \alpha \rightarrow \beta(\beta\alpha)$$

$$\alpha : *, \beta : * \rightarrow *, x : \beta(\beta\alpha) \vdash \lambda y : \alpha. x : \alpha \rightarrow \beta(\beta\alpha)$$

*Solution.*

1.	$\alpha : *$	
2.	$\beta : * \rightarrow *$	
3.	$x : \beta(\beta\alpha)$	
4.	$y : \alpha$	
5.	$x : \beta(\beta\alpha)$	<b>3 Var</b>
6.	$\lambda y : \alpha. x : \alpha \rightarrow \beta(\beta\alpha)$	<b>5 Abst</b>

### Problem

(4.4 c) Give a shortened  $\lambda\omega$  derivation in flag format of

$$\emptyset \vdash \lambda\alpha : *. \lambda\beta : * \rightarrow *. \beta(\beta\alpha) : * \rightarrow (* \rightarrow *) \rightarrow *$$

*Solution.*

1.	$\alpha : *$	
2.	$\beta : * \rightarrow *$	
3.	$\beta\alpha : *$	<b>2,1 App</b>
4.	$\beta(\beta\alpha) : *$	<b>2,3 App</b>
5.	$\lambda\beta : * \rightarrow *. \beta(\beta\alpha) : (* \rightarrow *) \rightarrow *$	<b>4 Abst</b>
6.	$\lambda\alpha : *. \lambda\beta : * \rightarrow *. \beta(\beta\alpha) : * \rightarrow (* \rightarrow *) \rightarrow *$	<b>5 Abst</b>

### Problem

(4.4 d) Give a shortened  $\lambda\omega$  derivation in flag format of

$$\text{nat} : * \vdash (\lambda\alpha : *. \lambda\beta : * \rightarrow *. \beta(\beta\alpha)) \text{ nat } (\lambda\gamma : *. \gamma) : *$$

*Solution.*

1.	$\text{nat} : *$	
2.	$\alpha : *$	
3.	$\beta : * \rightarrow *$	
4.	$\beta\alpha : *$	<b>3,2 App</b>
5.	$\beta(\beta\alpha) : *$	<b>3,4 App</b>
6.	$\lambda\beta : * \rightarrow *. \beta(\beta\alpha) : (* \rightarrow *) \rightarrow *$	<b>5 Abst</b>

7.	$\lambda\alpha : *. \lambda\beta : * \rightarrow *. \beta(\beta\alpha) : * \rightarrow (* \rightarrow *) \rightarrow *$	<b>6 Abst</b>
8.	$(\lambda\alpha : *. \lambda\beta : * \rightarrow *. \beta(\beta\alpha)) \text{ nat} : (* \rightarrow *) \rightarrow *$	<b>7,1 App</b>
9.	$\gamma : *$	
10.	$\boxed{\gamma : *}$	<b>9 Var</b>
11.	$\lambda\gamma : *. \gamma : * \rightarrow *$	<b>10 Abst</b>
12.	$(\lambda\alpha : *. \lambda\beta : * \rightarrow *. \beta(\beta\alpha)) \text{ nat} (\lambda\gamma : *. \gamma) : *$	<b>8,11 App</b>

### Problem

(4.5) Give a shortened  $\lambda\omega$  derivation in flag format of

$$\alpha : *. x : \alpha \vdash \lambda y : \alpha. x : (\lambda\beta : *. \beta \rightarrow \beta)\alpha$$

*Solution.*

1.	$\alpha : *$	
2.	$x : \alpha$	
3.	$y : \alpha$	
4.	$\boxed{x : \alpha}$	<b>2 Weak</b>
5.	$\lambda y : \alpha. x : \alpha \rightarrow \alpha$	<b>4 Abst</b>
6.	$\beta : *$	
7.	$\boxed{\beta \rightarrow \beta : *}$	<b>6,6 Form</b>
8.	$\lambda\beta : *. \beta \rightarrow \beta : * \rightarrow *$	<b>7 Abst</b>
9.	$(\lambda\beta : *. \beta \rightarrow \beta)\alpha : *$	<b>8,1 App</b>
10.	$\boxed{\lambda y : \alpha. x : (\lambda\beta : *. \beta \rightarrow \beta)\alpha}$	<b>5,9 Conv</b>

### Problem

(4.6 a) Show that no such context  $\Gamma$  and term  $M$  in  $\lambda\omega$  such that

$$\Gamma \vdash \Box : M$$

is derivable.

*Solution.* Proof by induction on inference rules. Rules like Sort, Var, Form, Abst, App has syntactically or semantic different conclusions than  $\Box : M$ .

*Case 1 : Rule Weak.* Let  $\Gamma', C : s \equiv \Gamma$ . Therefore this derivation requires a premise  $\Gamma' \vdash \Box : M$ . By the inductive hypothesis this is impossible. ■

*Case 2 : Rule Conv.* This derivation requires a premise  $\Gamma \vdash \square : M'$  such that  $M \equiv_{\beta} M'$ . By the inductive hypothesis this is impossible. ■

By the principle of induction this proves that there's no derivation that could give  $\Gamma \vdash \square : M$ .

### Problem

(4.6 b) Prove there are no such context  $\Gamma$  and terms  $M$  and  $N$  in  $\lambda\omega$  such that

$$\Gamma \vdash M \rightarrow \square : N$$

*Solution.* Proof by induction on inference rules. Rules like Sort, Var, Abst, App has syntactically or semantically conclusions than  $M \rightarrow \square : N$ .

*Case 1 : Rule Weak.* Let  $\Gamma', C : s \equiv \Gamma$ . The derivation requires a premise  $\Gamma' \vdash M \rightarrow \square : N$ . By the inductive hypothesis this is impossible. ■

*Case 2 : Rule Form.* This requires a derivation with premise  $\Gamma \vdash \square : N$ , which by 4.6 a is not possible. ■

*Case 3 : Rule Conv.* This requires a premise  $\Gamma \vdash M \rightarrow \square : N'$  such that  $N \equiv N'$ . By the inductive hypothesis this is impossible. ■

By the principle of induction this proves there's no derivation that could give  $\Gamma \vdash M \rightarrow \square : N$ .

### Problem

(4.7 a) Give  $\lambda\omega$  definition of the notion legal term,  $\lambda\omega$  context and domain.

*Solution.*

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**Definition** *Legal Term* are typable terms. That is, a term  $M$  is legal iff there exists a context  $\Gamma$  and a legal higher-sorted term  $\alpha$  under  $\Gamma$  such that  $\Gamma \vdash M : \alpha$ .

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**Definition**  $\lambda\omega$  Context.

1.  $\emptyset$  is a  $\lambda\omega$  context.
  2. When  $\Gamma$  is a valid  $\lambda\omega$  context,  $\alpha$  is valid under  $\Gamma$ , and type of  $x$  is  $\alpha$ , and  $x \notin \text{dom } \Gamma$ , then the context  $\Gamma, x : \alpha$  is valid in  $\lambda\omega$ .
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**Definition** *Domain*

1.  $\text{dom } \emptyset = \{\}$
  2.  $\text{dom } \Gamma, x : s = \text{dom } \Gamma \cup \{x\}$
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**Problem**

(4.7 b) Formulate the Free Variables Lemma, Thinning Lemma, and Substitution Lemma for  $\lambda\omega$ .

*Solution.*

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*Lemma 1.*     **FV Lemma ( $\lambda\omega$ ).** For any legal term  $M$  under  $\Gamma$ ,  $\text{FV } M \subseteq \text{dom } \Gamma$ .

More specifically,

$$\begin{aligned} \forall M, \alpha \in \Lambda_{\omega}, \Gamma \vdash \alpha : s, \Gamma \vdash M : \alpha &\implies \text{FV } M \subseteq \text{dom } \Gamma \\ M \equiv \square &\implies \text{FV } M \equiv \emptyset \subseteq \text{dom } \Gamma \end{aligned}$$

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*Lemma 2.*     **Thinning Lemma ( $\lambda\omega$ ).** For any legal term  $M$  in  $\Gamma'$  and  $\Gamma' \subseteq \Gamma$ ,  $M$  is legal under  $\Gamma$ .

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*Lemma 3.*     **Substitution Lemma ( $\lambda\omega$ ).** Assume term  $\kappa : s$  under context  $\Gamma'$ . Under another context  $\Gamma''$  given a term  $\Gamma'' \vdash N : \kappa$  and another context  $\Gamma$  such that  $\Gamma, x : \kappa, \Gamma' \vdash M : A$  for some type  $A : s$  under  $\Gamma$ . Then

$$\Gamma, \Gamma', \Gamma'' \vdash M [x := N] : A$$

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Completed Dec 20 1:23 am.