Excercises

Chapter 3

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1. Redacted

Definition Some rules for reference.

$$\frac{x : \sigma \in \Gamma \qquad \Gamma \text{ is a } \lambda 2 \text{ context}}{\Gamma \vdash x : \sigma}\,(\text{T-Var}) \qquad \frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash M\ N : \tau}\,(\text{T-App})$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x : \sigma .\ M : \sigma \to \tau}\,(\text{T-Abst}) \qquad \frac{\alpha \in \mathbb{T}_2 \qquad \text{FV } \alpha \subseteq \text{dom } \Gamma}{\alpha : * \in \Gamma}\,(\text{T-Form})$$

$$\frac{\Gamma \vdash M : \Pi_{\alpha:*}.\ A \qquad \Gamma \vdash B : *}{\Gamma \vdash M\ B : A\,[\alpha \coloneqq B]}\,(\text{T2-App}) \qquad \frac{\Gamma, \alpha : * \vdash M : A}{\Gamma \vdash \lambda\alpha : *\ .\ M : \Pi_{\alpha:*} : A}\,(\text{T2-Abst})$$

In this document, convention is that all type judgements in a proof tree, unless stated otherwise, is derived from a single and unique $\lambda 2$ context per tree. Multiple conclusions might be drawn on a single line from usage of the same inference rule for compactness. Eg:

ex 1. $\alpha, \beta : *$ **T-Form**

Is shorthand for

ex 1. $\Gamma \vdash \alpha : *$ **T-Form**

ex 2. $\Gamma \vdash \beta : *$ **T-Form**

Problem

(3.1) How many $\lambda 2$ contexts consisting of four and only four declarations

$$(1)\ \Gamma \vdash \alpha : * \qquad (2)\ \Gamma \vdash \beta : *$$
$$(3)\ \Gamma \vdash f : \alpha \to \beta \quad (4)\ \Gamma \vdash x : \alpha$$

Solution. The last two declarations depende on the first two. Therefore this is an easy combinatorics problem: $2! \times 2! = 4$ contexts:

$$\begin{array}{cc} 1-2-3-4 & 1-2-4-3 \\ 2-1-3-4 & 2-1-4-3 \end{array}$$

Problem

(3.2) Give a full derivation in $\lambda 2$ to show the following type term is legal:

$$M \equiv \lambda \alpha, \beta, \gamma : * .\lambda f : \alpha \to \beta . \lambda g : \beta \to \gamma . \lambda x : \alpha .\, g\,(f\ x)$$

Solution.

1.	$\alpha : *$	**Bound**
2.	$\beta : *$	**Bound**
3.	$\gamma : *$	**Bound**
4.	$f : \alpha \to \beta$	**Bound**
5.	$g : \beta \to \gamma$	**Bound**
6.	$x : \alpha$	**Bound**
7.	$\alpha, \beta, \gamma : *$	**T-Form**
8.	$\alpha \to \beta, \beta \to \gamma : *$	**T-Form**
9.	$f : \alpha \to \beta, x : \alpha$	**T-Var**
10.	$f\ x : \beta$	**8,8 T-App**
11.	$g : \beta \to \gamma$	**T-Var**
12.	$g\,(f\ x) : \gamma$	**11,10 T-App**
13.	$\lambda x : \alpha .\, g\,(f\ x) : \alpha \to \gamma$	**12 T-Abst**
14.	$\lambda g : \beta \to \gamma . \lambda x : \alpha .\, g\,(f\ x) : (\beta \to \gamma) \to \alpha \to \gamma$	**13 T-Abst**
15.	$\lambda f : \alpha \to \beta . \lambda g : \beta \to \gamma . \lambda x : \alpha .\, g\,(f\ x)$ $: (\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$	**14 T-Abst**

16. $\lambda\gamma : * .\lambda f : \alpha \to \beta.\lambda g : \beta \to \gamma.\lambda x : \alpha.\, g\,(f\,x)$
$: \Pi\gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **15 T2-Abst**

17. $\lambda\beta, \gamma : * .\lambda f : \alpha \to \beta.\lambda g : \beta \to \gamma.\lambda x : \alpha.\, g\,(f\,x)$
$: \Pi\beta, \gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **16 T2-Abst**

18.

$\lambda\alpha, \beta, \gamma : * .\lambda f : \alpha \to \beta.\lambda g : \beta \to \gamma.\lambda x : \alpha.\, g\,(f\,x)$
$: \Pi\alpha, \beta, \gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **17 T2-Abst**

Problem

(3.3 a) Given M in 3.2, and a context Γ such that

$$\Gamma \vdash \texttt{nat} : *$$

$$\Gamma \vdash \texttt{bool} : *$$

$$\Gamma \vdash \text{succ} : \texttt{nat} \to \texttt{nat}$$

$$\Gamma \vdash \text{even} : \texttt{nat} \to \texttt{bool}$$

Prove $M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \text{succ}\ \text{even}$ is legal.

Solution. Proof by deriving the term's type.

Proof.

1. $M : \Pi\alpha, \beta, \gamma : * .(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$ **T-Var**
2. $\texttt{nat}, \texttt{bool} : *$ **T-Form**
3. $M\ \texttt{nat} : \Pi\beta, \gamma : * .(\texttt{nat} \to \beta) \to (\beta \to \gamma) \to \texttt{nat} \to \gamma$ **2,1 T2-App**
4. $M\ \texttt{nat}\ \texttt{nat} : \Pi\gamma : * .(\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \gamma) \to \texttt{nat} \to \gamma$ **2,3 T2-App**
5.

$M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool} : (\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \texttt{bool}) \to \texttt{nat} \to \texttt{bool}$ **2,3 T2-App**

6. $\text{succ} : \texttt{nat} \to \texttt{nat}, \text{even} : \texttt{nat} \to \texttt{bool}$ **T-Var**
7. $M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \text{succ} : (\texttt{nat} \to \texttt{bool}) \to \texttt{nat} \to \texttt{bool}$ **6,5 T-App**
8. $M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \text{succ}\ \text{even} : \texttt{nat} \to \texttt{bool}$ **6,7 T-App**

■

Problem

(3.3 b.i) Prove $\lambda x : \texttt{nat}.\ \text{even}\ (\text{succ}\ x)$ via 3.3 a.

Solution. The result of beta reduction on the term in 3.3 a is what we are proving.

Proof.

$$\begin{aligned}
& M\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \text{succ}\ \text{even} \\
\equiv\ & (\lambda \alpha, \beta, \gamma, f, g\ .\lambda x : \alpha.\ g\ (f\ x))\ \texttt{nat}\ \texttt{nat}\ \texttt{bool}\ \text{succ}\ \text{even} \\
\twoheadrightarrow_{\beta}\ & (\lambda f : \texttt{nat} \to \texttt{nat}.\lambda g : \texttt{nat} \to \texttt{bool}.\lambda x : \texttt{nat}.\ g\ (f\ (x)))\ \text{succ}\ \text{even} \\
\to_{\beta}\ & (\lambda x : \texttt{nat}.\ \text{even}\ (\text{succ}\ x))
\end{aligned}$$

By the subject reduction lemma, $\lambda x : \texttt{nat}.\ \text{even}\ (\text{succ}\ x)) : \texttt{nat} \to \texttt{bool}$, thus is legal. ■

Problem

(3.3 b.ii) Prove $\lambda x : \texttt{nat}.\ \text{even}\ (\text{succ}\ x)$ via derivation in the context provided in 3.3 a.

Solution.

Proof.

1.	$\texttt{nat}, \texttt{bool} : *$	**T-Form**
2.	$x : \texttt{nat}$	**Bound**
3.	$\quad$ | $\text{succ} : \texttt{nat} \to \texttt{nat}$	**T-Var**
4.	$\quad$ | $x : \texttt{nat}$	**T-Var**
5.	$\quad$ | $\text{succ}\ x : \texttt{nat}$	**3,4 T-App**
6.	$\quad$ | $\text{even} : \texttt{nat} \to \texttt{bool}$	**T-Var**
7.	$\quad$ | $\text{even}\ (\text{succ}\ x) : \texttt{bool}$	**6,5 T-App**
8.	$\lambda x : \texttt{nat}.\ \text{even}\ (\text{succ}\ x) : \texttt{nat} \to \texttt{bool}$	**7 T-Abst**

■

Problem

(3.4) Give a shorthanded (omit T-Var and T-Form) derivation in $\lambda 2$ to show the following term is valid in $\Gamma \equiv \texttt{nat} : *, \texttt{bool} : *$

$$(\lambda \alpha, \beta : *\ .\lambda f : \alpha \to \alpha.\lambda g : \alpha \to \beta.\lambda x : \alpha.\ g\ (f\ (f\ x)))\ \texttt{nat}\ \texttt{bool}$$

Solution.

Proof.

1.	$\alpha, \beta : *$	**Bound**
2.	$f : \alpha \to \alpha$	**Bound**
3.	$g : \alpha \to \beta$	**Bound**
4.	$x : \alpha$	**Bound**
5.	$f\ x : \alpha$	***,* T-App**
6.	$f\ (f\ x) : \alpha$	***,5 T-App**
7.	$g\ (f\ (f\ x)) : \beta$	***,6 T-App**
8.	$\lambda x : \alpha.\, g\ (f\ (f\ x)) : \alpha \to \beta$	**7 T-Abst**
9.	$\lambda g : \alpha \to \beta.\lambda x : \alpha.\, g\ (f\ (f\ x)) : (\alpha \to \beta) \to \alpha \to \beta$	**8 T-Abst**
10.	$\lambda f : \alpha \to \alpha.\lambda g : \alpha \to \beta.\lambda x : \alpha.\, g\ (f\ (f\ x))$ $: (\alpha \to \alpha) \to (\alpha \to \beta) \to \alpha \to \beta$	**9 T-Abst**

11.

$$\lambda \alpha, \beta : * .\lambda f : \alpha \to \alpha.\lambda g : \alpha \to \beta.\lambda x : \alpha.\, g\ (f\ (f\ x)) \\ : \Pi \alpha, \beta : * .(\alpha \to \alpha) \to (\alpha \to \beta) \to \alpha \to \beta$$ **10 T2-Abst**

12.

$$(\lambda \alpha, \beta : * .\lambda f : \alpha \to \alpha.\lambda g : \alpha \to \beta.\lambda x : \alpha.\, g\ (f\ (f\ x)))\ \texttt{nat} \\ : \Pi \beta : * .(\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \beta) \to \texttt{nat} \to \beta$$ ***,11 T2-App**

13.

$$(\lambda \alpha, \beta : * .\lambda f : \alpha \to \alpha.\lambda g : \alpha \to \beta.\lambda x : \alpha.\, g\ (f\ (f\ x)))\ \texttt{nat}\ \texttt{bool} \\ : \Pi \beta : * .(\texttt{nat} \to \texttt{nat}) \to (\texttt{nat} \to \texttt{bool}) \to \texttt{nat} \to \texttt{bool}$$ ***,12 T2-App**

■

Problem

(3.5 a) Let $\perp \equiv \Pi \alpha : * .\alpha$. Prove $\perp$ is legal.

Solution. Here a notion called kind checking is introduced. This has not yet been discussed in this book (?

Proof.

1.	$\alpha : *$	**Bound**
2.	$\alpha : *$	**T-Form**

3. $\Pi\alpha : * . \alpha : * \to *$ **T2-Abst**

■

Problem

(3.5 b) Consider the context $\Gamma \equiv \beta : *, x : \bot$. Find an inhabitant of type β under Γ.

Solution. $x\,\beta$ is. Because x is of second-order type, it must be parametric to a type, thus x is of form $\lambda\alpha : * \,.\, M$ where $\Gamma, \alpha : * \vdash M : \alpha$.

Proof.

1. $x : \Pi\alpha : * . \alpha$ **T-Var**
2. $\beta : *$ **T-Form**
3. $x\,\beta : \beta$ **1,2 T2-App**

■

Problem

(3.5 c) Give three inhabitants of $\beta \to \beta$ in β-nf under Γ in 3.5 b.

Solution.
1. $\lambda y : \beta.\, y$.

Proof.

1. $y : \beta$ **Bound**
2. $\quad y : \beta$ **T-Var**
3. $\lambda y : \beta.\, y : \beta \to \beta$ **2 T-Abst**

■

2. $\lambda y : \beta.\, x\,\beta$.

Proof.

1. $y : \beta$ **Bound**
2. $\quad x : \Pi\alpha : * . \alpha$ **T-Var**
3. $\quad \beta : *$ **T-Form**
4. $\quad x\,\beta : \beta$ **2,3 T2-App**

5. $\lambda y : \beta .\, x\, \beta : \beta \to \beta$ **4 T-Abst**

$\blacksquare$

3. $\lambda y : \beta .\, x\, (\beta \to \beta)\, y$.

Proof.

1.	$y : \beta$	**Bound**
2.	$\quad x : \Pi\alpha : * .\alpha$	**T-Var**
3.	$\quad \beta \to \beta : *$	**T-Form**
4.	$\quad x\, (\beta \to \beta) : \beta \to \beta$	**2,3 T2-App**
5.	$\quad y : \beta$	**T-Var**
6.	$\quad x\, (\beta \to \beta)\, y : \beta$	**4,5 T-App**
7.	$\lambda y : \beta .\, x\, (\beta \to \beta)\, y : \beta \to \beta$	**5 T-Abst**

$\blacksquare$

Problem

(3.5 d) Prove that the following terms inhabit the same type in Γ:

$$\lambda f : \beta \to \beta \to \beta .\, f\, (x\, \beta)(x\, \beta)$$

$$x\, ((\beta \to \beta \to \beta) \to \beta)$$

Solution. We simply derive the types.

First Term.

1.	$f : \beta \to \beta \to \beta$	**Bound**
2.	$\quad f : \beta \to \beta \to \beta$	**T-Var**
3.	$\quad x : \Pi\alpha : * .\alpha$	**T-Var**
4.	$\quad \beta : *$	**T-Form**
5.	$\quad x\, \beta : \beta$	**3,4 T2-App**
6.	$\quad f\, (x\, \beta) : \beta \to \beta$	**2,5 T-App**
7.	$\quad f\, (x\, \beta)(x\, \beta) : \beta$	**6,5 T-App**
8.	$\lambda f : \beta \to \beta \to \beta .\, f\, (x\, \beta)(x\, \beta) : (\beta \to \beta \to \beta) \to \beta$	**6 T-Abst**

$\blacksquare$

Second Term.

1.	$(\beta \to \beta \to \beta) \to \beta : *$	**T-Form**
2.	$x : \Pi\alpha : * .\alpha$	**T-Var**
3.	$x\,((\beta \to \beta \to \beta) \to \beta) : (\beta \to \beta \to \beta) \to \beta$	**2,1 T2-App**

■

The two terms were shown to both inhabit $(\beta \to \beta \to \beta) \to \beta$.

Problem

(3.6 a) Find inhabitant of type

$$\Pi\alpha, \beta : * .(\mathtt{nat} \to \alpha) \to (\alpha \to \mathtt{nat} \to \beta) \to \mathtt{nat} \to \beta$$

In context $\Gamma \equiv \mathtt{nat} : *$.

Solution.

$$\lambda\alpha, \beta : * .\lambda x : \mathtt{nat} \to \alpha.\lambda y : (\alpha \to \mathtt{nat} \to \beta).\lambda z : \mathtt{nat}.\, y\,(x\,z)\,z$$

Proof.

1.	$\alpha, \beta : *$	**Bound**
2.	$\mathtt{nat} \to \alpha : *$	**T-Form**
3.	$x : \mathtt{nat} \to \alpha$	**Bound**
4.	$\alpha \to \mathtt{nat} \to \beta : *$	**T-Form**
5.	$y : \alpha \to \mathtt{nat} \to \beta$	**Bound**
6.	$\mathtt{nat} : *$	**Bound**
7.	$z : \mathtt{nat}$	**Bound**
8.	$y : \alpha \to \mathtt{nat} \to \beta$	**T-Var**
9.	$x : \mathtt{nat} \to \alpha$	**T-Var**
10.	$z : \mathtt{nat}$	**T-Var**
11.	$x\,z : \alpha$	**9,10 T-App**
12.	$y\,(x\,z) : \mathtt{nat} \to \beta$	**8,11 T-App**
13.	$y\,(x\,z)\,z : \beta$	**12,10 T-App**
14.	$\lambda z : \mathtt{nat}.\, y\,(x\,z)\,z : \mathtt{nat} \to \beta$	**13 T-Abst**
15.	$\lambda y : \alpha \to \mathtt{nat} \to \beta.\lambda z : \mathtt{nat}.\, y\,(x\,z)\,z$ $: (\alpha \to \mathtt{nat} \to \beta) \to \mathtt{nat} \to \beta$	**14 T-Abst**
16.	$\lambda x : \mathtt{nat} \to \alpha.\, y : \alpha \to \mathtt{nat} \to \beta.\lambda z : \mathtt{nat}.\, y\,(x\,z)\,z$ $: (\mathtt{nat} \to \alpha) \to (\alpha \to \mathtt{nat} \to \beta) \to \mathtt{nat} \to \beta$	**15 T2-Abst**

17.

$\lambda\alpha, \beta : * \, . \, x : \mathtt{nat} \to \alpha . \, y : \alpha \to \mathtt{nat} \to \beta . \lambda z : \mathtt{nat} . \, y \, (x \, z) \, z$

$: \Pi\alpha, \beta : * \, .(\mathtt{nat} \to \alpha) \to (\alpha \to \mathtt{nat} \to \beta) \to \mathtt{nat} \to \beta$ **16 T2-Abst**

■

Problem

(3.6 b) Find inhabitant of type

$$\Pi\delta : * \, .((\alpha \to \gamma) \to \delta) \to (\alpha \to \beta) \to (\beta \to \gamma) \to \delta$$

In context $\Gamma \equiv \alpha : *, \beta : *, \gamma : *$

Solution.

$$\lambda\delta : * \, .\lambda x : (\alpha \to \gamma) \to \delta . \lambda y : \alpha \to \beta . \lambda z : \beta \to \gamma . \, x \, (\lambda u : \alpha . \, z \, (y \, u))$$

A derivation in shorthand will be given (omitting T-Form / T-Var)

Proof.

1.	$\delta : *$	**Bound**
2.	$x : (\alpha \to \gamma) \to \delta$	**Bound**
3.	$y : \alpha \to \beta$	**Bound**
4.	$z : \beta \to \gamma$	**Bound**
5.	$u : \alpha$	**Bound**
6.	$y \, u : \beta$	***,* T-App**
7.	$z \, (y \, u) : \gamma$	***,6 T-App**
8.	$\lambda u : \alpha . \, z \, (y \, u) : \alpha \to \gamma$	**7 T-Abst**
9.	$x \, (\lambda u : \alpha . \, z \, (y \, u)) : \delta$	**8 T-Abst**
10.	$\lambda z : \beta \to \gamma . \, x \, (\lambda u : \alpha . \, z \, (y \, u)) : (\beta \to \gamma) \to \delta$	**9 T-Abst**
11.	$\lambda y : \alpha \to \beta . \lambda z : \beta \to \gamma . \, x \, (\lambda u : \alpha . \, z \, (y \, u))$ $: (\alpha \to \beta) \to (\beta \to \gamma) \to \delta$	**10 T-Abst**
12.	$\lambda x : (\alpha \to \gamma) \to \delta . \lambda y : \alpha \to \beta . \lambda z : \beta \to \gamma . \, x \, (\lambda u : \alpha . \, z \, (y \, u))$ $: ((\alpha \to \gamma) \to \beta) \to (\alpha \to \beta) \to (\beta \to \gamma) \to \delta$	**11 T-Abst**
13.		

$\lambda\delta : * \, .\lambda x : (\alpha \to \gamma) \to \delta . \lambda y : \alpha \to \beta . \lambda z : \beta \to \gamma . \, x \, (\lambda u : \alpha . \, z \, (y \, u))$

$: \Pi\delta : * \, .((\alpha \to \gamma) \to \delta) \to (\alpha \to \beta) \to (\beta \to \gamma) \to \delta$ **12 T2-Abst**

∎

Problem

(3.6 c) Find inhabitant of type

$$\Pi\alpha, \beta, \gamma : * .(\alpha \to (\beta \to \alpha) \to \gamma) \to \alpha \to \gamma$$

In the empty context

Solution.

$$\lambda\alpha, \beta, \gamma : * .\lambda f : (\alpha \to (\beta \to \alpha) \to \gamma).\lambda x : \alpha.\ f\ x\ (\lambda u : \beta.\ x)$$

Proof.

1.	α, β, γ	**Bound**
2.	$\quad f : \alpha \to (\beta \to \alpha) \to \gamma$	**Bound**
3.	$\quad\quad x : \alpha$	**Bound**
4.	$\quad\quad\quad f\ x : (\beta \to \alpha) \to \gamma$	***,* T-App**
5.	$\quad\quad\quad u : \beta$	**Bound**
6.	$\quad\quad\quad\quad x : \alpha$	**T-Var**
7.	$\quad\quad\quad \lambda u : \beta.\ x : \beta \to \alpha$	**6 T-Abst**
8.	$\quad\quad\quad f\ x\ (\lambda u : \beta.\ x) : \gamma$	**4,7 T-App**
9.	$\quad\quad \lambda x : \alpha.\ f\ x\ (\lambda u : \beta.\ x) : \alpha \to \gamma$	**8 T-Abst**
10.	$\quad \lambda f : \alpha \to (\beta \to \alpha) \to \gamma.\lambda x : \alpha.\ f\ x\ (\lambda u : \beta.\ x)$ $: (\alpha \to (\beta \to \alpha) \to \gamma) \to \alpha \to \gamma$	**9 T-Abst**
11.	$\lambda\alpha, \beta, \gamma : * .\lambda f : \alpha \to (\beta \to \alpha) \to \gamma.\lambda x : \alpha.\ f\ x\ (\lambda u : \beta.\ x)$ $: \Pi\alpha, \beta, \gamma : * .(\alpha \to (\beta \to \alpha) \to \gamma) \to \alpha \to \gamma$	**10 T2-Abst**

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Problem

(3.7) Let $\bot \equiv \Pi\alpha : * .\alpha$ and context $\Gamma \equiv \alpha : *, \beta : *, x : \alpha \to \bot, f : (\alpha \to \alpha) \to \alpha$. Give a derivation that succesively calculate an inhabitant of α and β, both in context Γ.

Solution. Have $M : \alpha := f\ (\lambda n : \alpha.\ n)$. Then $\Gamma \vdash x\ M\ \beta : \beta$.

Typing M.

1. $f : (\alpha \to \alpha) \to \alpha$ **T-Var**
2. $n : \alpha$ **Bound**
3. $\quad n : \alpha$ **T-Var**
4. $\lambda n : \alpha.\, n : \alpha \to \alpha$ **3 T-Abst**
5. $f\ (\lambda n : \alpha.\, n) : \alpha$ **1,4 T-App**

■

Typing $x\ M\ \beta$.

1. $M : \alpha$ **T-Var**
2. $x : \alpha \to \Pi\alpha : * .\alpha$ **T-Var**
3. $M\ x : \Pi\alpha : * .\alpha$ **2,1 T-App**
4. $M\ x\ \beta : \beta$ **3,* T2-App**

■

Problem

(3.8) Recall $K \equiv \lambda x\ y\ .\ x \in \Lambda$ from untyped lambda calculus. Consider the following types

$$T_1 \equiv \Pi\alpha, \beta : * .\alpha \to \beta \to \alpha \quad T_2 \equiv \Pi\alpha : * .\alpha \to (\Pi\beta : * .\beta \to \alpha)$$

Find inhabitants of both type $t_1 : T_1$ and $t_2 : T_2$ under the empty context, which may be considered the closed $\lambda 2$ form of $K \in \Lambda_{\mathbb{T}2}$.

Solution.

$$\lambda\alpha, \beta : * .\lambda x : \alpha.\lambda y : \beta.\, x$$

$$\lambda\alpha : * .\lambda x : \alpha.\lambda\beta : * .\lambda y : \beta.\, x$$

First Form.

1. $\alpha, \beta : *$ **Bound**
2. $\quad x : \alpha$ **Bound**
3. $\quad\quad y : \beta$ **Bound**
4. $\quad\quad\quad x : \alpha$ **T-Var**
5. $\quad\quad \lambda y : \beta.\, x : \beta \to \alpha$ **4 T-Abst**
6. $\quad \lambda x : \alpha.\lambda y : \beta.\, x : \alpha \to \beta \to \alpha$ **5 T-Abst**

7.	$\lambda \alpha, \beta : * . \lambda x : \alpha . \lambda y : \beta .\, x : \Pi \alpha, \beta : * . \alpha \to \beta \to \alpha$	**5 T2-Abst**

∎

Second Form.

1.	$\alpha : *$	**Bound**
2.	$\quad x : \alpha$	**Bound**
3.	$\quad\quad \beta : *$	**Bound**
4.	$\quad\quad\quad y : \beta$	**Bound**
5.	$\quad\quad\quad\quad x : \alpha$	**T-Var**
6.	$\quad\quad\quad \lambda y : \beta .\, x : \beta \to \alpha$	**5 T-Abst**
7.	$\quad\quad \lambda \beta : * . \lambda y : \beta .\, x : \Pi \beta : * . \beta \to \alpha$	**6 T2-Abst**
8.	$\quad \lambda x : \alpha . \lambda \beta : * . \lambda y : \beta .\, x : \alpha \to (\Pi \beta : * . \beta \to \alpha)$	**7 T-Abst**
9.	$\lambda \alpha : * . \lambda x : \alpha . \lambda \beta : * . \lambda y : \beta .\, x : \Pi \alpha : * . \alpha \to (\Pi \beta : * . \beta \to \alpha)$	**8 T2-Abst**

∎

Problem

(3.9) Pretype the combinator

$$S \equiv \lambda x\, y\, z \,.\, x\, z\, (y\, z)$$

In closed form (typable in an empty context) in $\Lambda_{\mathbb{T}2}$.

Solution.

$$S \equiv \lambda \alpha, \beta, \gamma : * . \lambda x : \alpha \to \beta \to \gamma . \lambda y : \alpha \to \beta . \lambda z : \alpha .\, x\, z\, (y\, z)$$

Proof.

1.	$\alpha, \beta, \gamma : *$	**Bound**
2.	$\quad x : \alpha \to \beta \to \gamma$	**Bound**
3.	$\quad\quad y : \alpha \to \beta$	**Bound**
4.	$\quad\quad\quad z : \alpha$	**Bound**
5.	$\quad\quad\quad\quad x\, z : \beta \to \gamma$	***,* T-App**
6.	$\quad\quad\quad\quad y\, x : \beta$	***,* T-App**
7.	$\quad\quad\quad\quad x\, z\, (y\, x) : \gamma$	**5,6 T-App**
8.	$\quad\quad\quad \lambda z : \alpha .\, x\, z\, (y\, x) : \alpha \to \gamma$	**7 T-Abst**
9.	$\quad\quad \lambda y : \alpha \to \beta . \lambda z : \alpha .\, x\, z\, (y\, x) : (\alpha \to \beta) \to \alpha \to \gamma$	**8 T-Abst**

10. $\left| \begin{array}{l} \lambda x : \alpha \to \beta \to \gamma . \lambda y : \alpha \to \beta . \lambda z : \alpha .\, x\, z\, (y\, x) \\ \quad : (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma \end{array} \right.$ **9 T-Abst**

11.

$\lambda \alpha, \beta, \gamma : * . \lambda x : \alpha \to \beta \to \gamma . \lambda y : \alpha \to \beta . \lambda z : \alpha .\, x\, z\, (y\, x)$

$\quad : \Pi \alpha, \beta, \gamma : * . (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$ **10 T2-Abst**

■

Problem

(3.10 a) Consider the term

$$M \equiv \lambda x : \Pi \alpha : * . \alpha \to \alpha .\, x\, (\sigma \to \sigma)(x\, \sigma)$$

Prove that M is legal.

Solution. For a term to be legal there must exist a context so that the term could be typed. Here, a witness context is $\Gamma \equiv \sigma : *$.

Proof.

1. $x : \Pi \alpha : * . \alpha \to \alpha$ **Bound**
2. $\quad x\, (\sigma \to \sigma) : (\sigma \to \sigma) \to (\sigma \to \sigma)$ *,* **T2-App**
3. $\quad x\, \sigma : \sigma \to \sigma$ *,* **T2-App**
4. $\quad x\, (\sigma \to \sigma)(x\, \sigma) : \sigma \to \sigma$ **2,3 T-App**
5.

$\lambda x : \Pi \alpha : * . \alpha \to \alpha .\, x\, (\sigma \to \sigma)(x\, \sigma)$

$\quad : (\Pi \alpha : * . \alpha \to \alpha) \to \sigma \to \sigma$ **4 T-Abst**

■

Problem

(3.10 b) Find a term N such that $M\, N$ is legal and may be considered to be a way to add type information to $(\lambda x \,.\, x\, x)(\lambda y \,.\, y)$

Solution.

$$M\, \sigma\, N \equiv (\lambda x : \Pi \alpha : * . \alpha \to \alpha .\, x\, (\sigma \to \sigma)(x\, \sigma))\sigma(\lambda y : \sigma .\, y)$$

Is the same as $(\lambda x\,.\,x\,x)(\lambda y\,.\,y)$ modulo type annotations.

Proof.

1.	$M : (\Pi\alpha : *\,.\alpha \to \alpha) \to \sigma \to \sigma$	**T-Var**
2.	$M\,\sigma : (\sigma \to \sigma) \to \sigma \to \sigma$	**1,* T2-App**
3.	$y : \sigma$	**Bound**
4.	$\quad y : \sigma$	**T-Var**
5.	have $N := \lambda y : \sigma.\, y : \sigma \to \sigma$	**4 T-Abst**
6.	$M\,\sigma\,N : \sigma \to \sigma$	**2,5 T-Abst**

■

Problem

(3.11) Recall $\bot \equiv \Pi\alpha : *\,.\alpha$ from 3.5. Type and prove the following term legal:

$$\lambda x : \bot.\, x\,(\bot \to \bot \to \bot)(x\,(\bot \to \bot)\,x)(x\,(\bot \to \bot \to \bot)\,x\,x)$$

Solution.

Proof. The type $\bot \to \bot$ is closed and well formed. Therefore, the term is legal.

1.	$\bot : * \equiv \Pi\alpha : *\,.\alpha$	**T-Form**
2.	$x : \bot$	**Bound**
3.	$\quad x\,(\bot \to \bot \to \bot) : \bot \to \bot \to \bot$	***,* T2-App**
4.	$\quad x\,(\bot \to \bot) : \bot \to \bot$	***,* T2-App**
5.	$\quad x\,(\bot \to \bot)\,x : \bot$	**4,* T-App**
6.	$\quad x\,(\bot \to \bot \to \bot)(x\,(\bot \to \bot)\,x) : \bot \to \bot$	**3,5 T-App**
7.	$\quad x\,(\bot \to \bot \to \bot)\,x : \bot \to \bot$	**3,* T-App**
8.	$\quad x\,(\bot \to \bot \to \bot)\,x\,x : \bot$	**7,* T-App**
9.	$\quad x\,(\bot \to \bot \to \bot)(x\,(\bot \to \bot)\,x)(x\,(\bot \to \bot \to \bot)\,x\,x) : \bot$	**6,8 T-App**
10.	$\lambda x : \bot.\, x\,(\bot \to \bot \to \bot)(x\,(\bot \to \bot)\,x)$ $(x\,(\bot \to \bot \to \bot)\,x\,x) : \bot \to \bot$	**9 T-Abst**

■

Problem

(3.12) Given the Polymorphic Church Numerals:

$$\begin{aligned}
\texttt{nat} \in \mathbb{T}_2 &\coloneqq \Pi\alpha : * .(\alpha \to \alpha) \to \alpha \to \alpha \\
\overline{0} &\equiv \lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ x : \texttt{nat} \\
\overline{1} &\equiv \lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ f\ x : \texttt{nat} \\
\overline{2} &\equiv \lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ f\ (f\ x) : \texttt{nat} \\
\text{succ} &\equiv \lambda n : \texttt{nat}.\lambda\beta : * .\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ (n\ \beta\ f\ x)
\end{aligned}$$

Prove that

$$\text{succ}\ \overline{0} =_\beta \overline{1}$$

$$\text{succ}\ \overline{1} =_\beta \overline{2}$$

Solution.

$$\begin{aligned}
&\text{succ}\ \overline{0} \\
\equiv\ &(\lambda n : \texttt{nat}.\lambda\beta : * .\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ (n\ \beta\ f\ x))(\lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ x) \\
\underset{\beta}{\to}\ &(\lambda\beta : * .\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ ((\lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ x)\beta\ f\ x)) \\
\underset{\beta_{\mathbb{T}2}}{\to}\ &(\lambda\beta : * .\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ ((\lambda f : \beta \to \beta.\lambda x : \beta.\ x)\ f\ x)) \\
\underset{\beta}{\twoheadrightarrow}\ &(\lambda\beta : * .\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ x) \overset{\beta \to \alpha}{\underset{\alpha_{\mathbb{T}2}}{\equiv}} \overline{1}
\end{aligned}$$

$$\begin{aligned}
&\text{succ}\ \overline{1} \\
\equiv\ &(\lambda n : \texttt{nat}.\lambda\beta : * .\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ (n\ \beta\ f\ x))(\lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ f\ x) \\
\underset{\beta}{\to}\ &(\lambda\beta : * .\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ ((\lambda\alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ f\ x)\beta\ f\ x)) \\
\underset{\beta_{\mathbb{T}2}}{\to}\ &(\lambda\beta : * .\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ ((\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ x)\ f\ x)) \\
\underset{\beta}{\twoheadrightarrow}\ &(\lambda\beta : * .\lambda f : \beta \to \beta.\lambda x : \beta.\ f\ (f\ x)) \overset{\beta \to \alpha}{\underset{\alpha_{\mathbb{T}2}}{\equiv}} \overline{2}
\end{aligned}$$

Problem

(3.13 a) We define addition in Polymorphic Church Numerals as

$$\text{add} \equiv \lambda m, n : \texttt{nat}.\lambda \alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \texttt{nat}.\, m\, \alpha\, f\, (n\, \alpha\, f\, x)$$

Show that

$$\text{add}\ \overline{1}\quad \overline{1} \underset{\beta}{=} \overline{2}$$

Solution.

$$\begin{aligned}
&\text{add}\ \overline{1}\quad \overline{1} \\
\equiv\ & (\lambda m, n : \texttt{nat}.\lambda \alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\, m\, \alpha\, f\, (n\, \alpha\, f\, x))\overline{1}\quad \overline{1} \\
\underset{\beta}{\twoheadrightarrow}\ & \lambda \alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\overline{1}\alpha\, f\, \left(\overline{1}\alpha\, f\, x\right) \\
\equiv\ & \lambda \alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\overline{1}\alpha\, f\, ((\lambda \alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\, f\, x)\alpha\, f\, x) \\
\underset{\beta}{\twoheadrightarrow}\ & \lambda \alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\overline{1}\alpha\, f\, (f\, x) \\
\equiv\ & \lambda \alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.(\lambda \alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\, f\, x)\alpha\, f\, (f\, x) \\
\underset{\beta}{\twoheadrightarrow}\ & \lambda \alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\, f\, (f\, x) \underset{\alpha}{\equiv} \overline{2}
\end{aligned}$$

Problem

(3.13 b) Find a term mul simulates multiplication on $\texttt{nat}$.

Solution.

$$\text{mul} \equiv \lambda m, n : \texttt{nat}.\lambda \alpha : *.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\, m\, \alpha (n\, \alpha\, f)\, x$$

Proof. We derive the type first to prove a legal term.

1.	$m, n : \texttt{nat}$	**Bound**
2.	$\quad \alpha : *$	**Bound**
3.	$\quad\quad f : \alpha \to \alpha$	**Bound**
4.	$\quad\quad\quad x : \alpha$	**Bound**
5.	$\quad\quad\quad\quad m\, \alpha : (\alpha \to \alpha) \to \alpha \to \alpha$	***,* T2-App**
6.	$\quad\quad\quad\quad n\, \alpha : (\alpha \to \alpha) \to \alpha \to \alpha$	***,* T2-App**
7.	$\quad\quad\quad\quad n\, \alpha\, f : \alpha \to \alpha$	**6,* T-App**
8.	$\quad\quad\quad\quad m\, \alpha (n\, \alpha\, f) : \alpha \to \alpha$	**5,7 T-App**

9. $m\,\alpha(n\,\alpha\,f)\,x : \alpha$ **8,* T-App**

10. $\lambda x : \alpha.\, m\,\alpha(n\,\alpha\,f)\,x : \alpha \to \alpha$ **9 T-Abst**

11. $\lambda f : \alpha \to \alpha.\lambda x : \alpha.\, m\,\alpha(n\,\alpha\,f)\,x : (\alpha \to \alpha) \to \alpha \to \alpha$ **10 T-Abst**

12. $\lambda \alpha : *\,.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\, m\,\alpha(n\,\alpha\,f)\,x : \texttt{nat}$ **11 T2-Abst**

13.

$\lambda m, n : \texttt{nat}.\lambda \alpha : *\,.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\, m\,\alpha(n\,\alpha\,f)\,x$
$: \texttt{nat} \to \texttt{nat} \to \texttt{nat}$ **12 T-Abst**

This proves that the term does indeed produce a natural number from two. Next let's prove that

$$\forall\, \overline{n}, \overline{m} : \texttt{nat} \quad \mathrm{mul}\,\overline{n} \quad \overline{m} = \overline{n \times m}$$

It could be proven by induction that

$$\forall\, \overline{a} : \texttt{nat} \quad \overline{a} \equiv \lambda \alpha : *\,.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\, f^{a}\,x$$

$$\begin{aligned}
\mathrm{mul}\,\overline{n} \quad \overline{m} &\equiv (\lambda m, n : \texttt{nat}.\lambda \alpha : *\,.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\, m\,\alpha(n\,\alpha\,f)\,x)(\overline{n})(\overline{m}) \\
&\twoheadrightarrow_{\beta} \lambda \alpha : *\,.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\, \overline{n}\,\alpha(\overline{m}\,\alpha\,f)\,x \\
&\twoheadrightarrow_{\beta} \lambda \alpha : *\,.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\, \overline{n}\,\alpha(\lambda u : \alpha.\, f^{m}\,u)\,x \\
&\twoheadrightarrow_{\beta} \lambda \alpha : *\,.\lambda f : \alpha \to \alpha.\lambda x : \alpha.(\lambda f : \alpha \to \alpha.\lambda x : \alpha.\, f^{n}\,x)(\lambda u : \alpha.\, f^{m}\,u)\,x \\
&\twoheadrightarrow_{\beta} \lambda \alpha : *\,.\lambda f : \alpha \to \alpha.\lambda x : \alpha.(\lambda u : \alpha.\, f^{m}\,u)^{n}\,x
\end{aligned}$$

By induction this can be further beta-reduced to

$$\twoheadrightarrow_{\beta} \lambda \alpha : *\,.\lambda f : \alpha \to \alpha.\lambda x : \alpha.\, f^{mn}\,x \equiv \overline{m\,n}$$

■

Problem

(3.14) We present the Church-Encoded Boolean:

$$\begin{aligned}
\texttt{bool} \in \mathbb{T}_2 &:= \Pi \alpha : *\,.\alpha \to \alpha \to \alpha \\
\mathrm{true} &\equiv \lambda \alpha : *\,.\lambda x, y : \alpha.\, x \\
\mathrm{false} &\equiv \lambda \alpha : *\,.\lambda x, y : \alpha.\, y
\end{aligned}$$

Construct a $\lambda 2$ term neg such that neg true $=_{\beta}$ false and neg false $=_{\beta}$ true.

Solution.

$$\mathrm{neg} \equiv \lambda b : \texttt{bool}.\lambda \alpha : *\,.\, b\,\alpha(\mathrm{false}\,\alpha)(\mathrm{true}\,\alpha)$$

Neg True.

$$\begin{aligned}\text{neg true} &\equiv (\lambda b : \texttt{bool}.\lambda\alpha : * .\, b\,\alpha(\text{false}\ \alpha)(\text{true}\ \alpha))\ \text{true} \\ &\underset{\beta}{\twoheadrightarrow} \lambda\alpha : * .(\lambda x, y : \alpha.\, x)(\text{false}\ \alpha)(\text{true}\ \alpha) \\ &\underset{\beta}{\rightarrow} \lambda\alpha : * .\ \text{false}\ \alpha \underset{\eta}{\rightarrow} \text{false}\end{aligned}$$

■

Neg False.

$$\begin{aligned}\text{neg false} &\equiv (\lambda b : \texttt{bool}.\lambda\alpha : * .\, b\,\alpha(\text{false}\ \alpha)(\text{true}\ \alpha))\ \text{false} \\ &\underset{\beta}{\twoheadrightarrow} \lambda\alpha : * .(\lambda x, y : \alpha.\, y)(\text{false}\ \alpha)(\text{true}\ \alpha) \\ &\underset{\beta}{\rightarrow} \lambda\alpha : * .\ \text{true}\ \alpha \underset{\eta}{\rightarrow} \text{true}\end{aligned}$$

■

Problem

(3.15) Define

$$M \equiv \lambda u, v : \texttt{bool}.\lambda\beta : * .\lambda x, y : \beta.\, u\,\beta(v\,\beta\,x\,y)(v\,\beta\,y\,y)$$

And reduce M true true, M true false, M false true, M false false, and decide which logical operator is represented by M.

Solution.

$$\begin{aligned}M\ \text{true true} &\equiv (\lambda u, v : \texttt{bool}.\lambda\beta : * .\lambda x, y : \beta.\, u\,\beta(v\,\beta\,x\,y)(v\,\beta\,y\,y))\ \text{true true} \\ &\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.\ \text{true}\ \beta(\text{true}\ \beta\,x\,y)(\text{true}\ \beta\,y\,y) \\ &\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.(\text{true}\ \beta\,x\,y) \\ &\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.\, x \underset{\alpha}{\equiv} \text{true}\end{aligned}$$

$$\begin{aligned}M\ \text{true false} &\equiv (\lambda u, v : \texttt{bool}.\lambda\beta : * .\lambda x, y : \beta.\, u\,\beta(v\,\beta\,x\,y)(v\,\beta\,y\,y))\ \text{true false} \\ &\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.\ \text{true}\ \beta(\text{false}\ \beta\,x\,y)(\text{false}\ \beta\,y\,y) \\ &\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.(\text{false}\ \beta\,x\,y) \\ &\underset{\beta}{\twoheadrightarrow} \lambda\beta : * .\lambda x, y : \beta.\, y \underset{\alpha}{\equiv} \text{false}\end{aligned}$$

$$
\begin{aligned}
M \text{ false true} \equiv{}& (\lambda u, v : \texttt{bool}.\lambda\beta : *.\lambda x, y : \beta.\, u\,\beta(v\,\beta\,x\,y)(v\,\beta\,y\,y)) \text{ false true} \\
&\twoheadrightarrow_\beta \lambda\beta : *.\lambda x, y : \beta.\ \text{false}\ \beta(\text{true}\ \beta\,x\,y)(\text{true}\ \beta\,y\,y) \\
&\twoheadrightarrow_\beta \lambda\beta : *.\lambda x, y : \beta.(\text{true}\ \beta\,y\,y) \\
&\twoheadrightarrow_\beta \lambda\beta : *.\lambda x, y : \beta.\, y \equiv_\alpha \text{false}
\end{aligned}
$$

$$
\begin{aligned}
M \text{ false false} \equiv{}& (\lambda u, v : \texttt{bool}.\lambda\beta : *.\lambda x, y : \beta.\, u\,\beta(v\,\beta\,x\,y)(v\,\beta\,y\,y)) \text{ false false} \\
&\twoheadrightarrow_\beta \lambda\beta : *.\lambda x, y : \beta.\ \text{false}\ \beta(\text{false}\ \beta\,x\,y)(\text{false}\ \beta\,y\,y) \\
&\twoheadrightarrow_\beta \lambda\beta : *.\lambda x, y : \beta.(\text{false}\ \beta\,y\,y) \\
&\twoheadrightarrow_\beta \lambda\beta : *.\lambda x, y : \beta.\, y \equiv_\alpha \text{false}
\end{aligned}
$$

Therefore M is equivalent to logical AND.

Problem

(3.16) Find $\lambda 2$ term representing the logical OR, XOR, IMP.

Solution.

$$
\begin{aligned}
\text{OR} &\equiv \lambda u, v : \texttt{bool}.\lambda\beta : *.\lambda x, y : \beta.\, u\,\beta\,x\,(v\,\beta\,x\,y) \\
\text{XOR} &\equiv \lambda u, v : \texttt{bool}.\lambda\beta : *.\lambda x, y : \beta.\, u\,\beta(v\,\beta\,y\,x)(v\,\beta\,x\,y) \\
\text{IMP} &\equiv \lambda u, v : \texttt{bool}.\lambda\beta : *.\lambda x, y : \beta.\, u\,\beta(v\,\beta\,x\,y)\,x
\end{aligned}
$$

All of them could be checked by finite enumeration over $\texttt{bool} \times \texttt{bool}$.

Problem

(3.17) Find $\text{isZero} : \texttt{nat} \to \texttt{bool}$ such that $\forall\, n : \texttt{nat}, \text{isZero}\ n =_\beta \text{false}$ except when $n \equiv \overline{0}$.

Solution.

$$
\text{isZero} \equiv \lambda n : \texttt{nat}.\, n\,\texttt{bool}\ (\lambda u : \texttt{bool}.\ \text{false})\ \text{true}
$$

Proof.

$$
\begin{aligned}
\text{isZero } \overline{0} &\equiv (\lambda n : \texttt{nat}.\, n\, \texttt{bool}\ (\lambda u : \texttt{bool}.\ \text{false})\ \text{true})\overline{0} \\
&\twoheadrightarrow_\beta (\lambda \alpha : * .\lambda f : \alpha \to \alpha.\lambda x : \alpha.\ x)\ \texttt{bool}\ (\lambda u : \texttt{bool}.\ \text{false})\ \text{true} \\
&\twoheadrightarrow_\beta (\lambda f : \texttt{bool} \to \texttt{bool}.\lambda x : \texttt{bool}.\ x)(\lambda u : \texttt{bool}.\ \text{false})\ \text{true} \\
&\twoheadrightarrow_\beta \text{true}
\end{aligned}
$$

■

By induction it could be proven that any other natural numbers must be applied $\lambda u : \texttt{bool}.\ \text{false}$ to the body, making the result false, except for $\overline{0}$, where the function $f : \alpha \to \alpha$ never got applied.

Problem

(3.18 a) Define type

$$\texttt{tree} \equiv \Pi\alpha : * .(\texttt{bool} \to \alpha) \to (\texttt{bool} \to \alpha \to \alpha \to \alpha) \to \alpha$$

Then we construct an inhabitant

$$\lambda\alpha. * .\lambda u : \texttt{bool} \to \alpha.\lambda v : \texttt{bool} \to \alpha \to \alpha \to \alpha.\ M$$

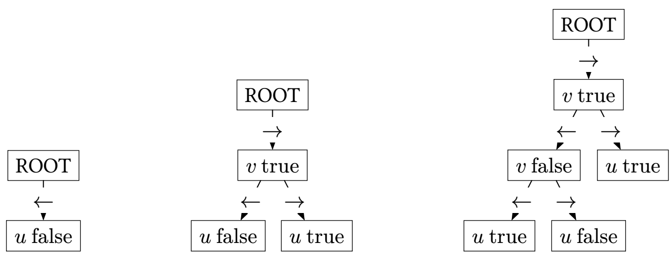
Is a node of a binary tree. Sketch graphs of trees where M is

$$u\ \text{false}$$
$$v\ \text{true}\ \ (u\ \text{false})(u\ \text{true})$$
$$v\ \text{true}\ \ (u\ \text{true})(v\ \text{false}\ \ (u\ \text{true})(u\ \text{false}))$$

Solution. A binary tree is usually defined as this:

```
inductive Tree (α : Type) where
  | leaf (value : α) : Tree α
  | node (left right : Tree α) : Tree α
```

With two constructors: a leaf or a node. Here α is the type of the payload at each node. There are two constructors: u is the left constructor, taking a `bool` value for the direction of the node. The v term is the node constructor, taking a `bool` as the direction, two α-typed terms as it's children.



Problem

(3.18 b) Give a $\lambda 2$ term, which, given input a polymorphic boolean p and two trees s and t, delivers the combined tree with p on top, left subtree s and right subtree t.

Solution.

$$\begin{aligned}\text{leaf} := \lambda p : \texttt{bool}.\lambda s, t : \texttt{tree}.\\ \lambda\alpha.\lambda u\,.\,\texttt{bool} \to \alpha.\lambda v : \texttt{bool} \to \alpha \to \alpha \to \alpha.\\ v\,p\,(s\,\alpha\,u\,v)(t\,\alpha\,u\,v)\end{aligned}$$

Proof. We suppose

$$s \equiv \lambda\beta : *\,.\lambda u_s : \texttt{bool} \to \beta.\lambda v_s : \texttt{bool} \to \beta \to \beta \to \beta.\,S$$
$$t \equiv \lambda\gamma : *\,.\lambda u_t : \texttt{bool} \to \gamma.\lambda v_t : \texttt{bool} \to \gamma \to \gamma \to \gamma.\,T$$

We want

$$\text{leaf}\,p\,s\,t \underset{\beta}{=} \lambda\alpha : *\,.\lambda u : \texttt{bool} \to \alpha.\lambda v : \texttt{bool} \to \alpha \to \alpha \to \alpha.\,v\,p\,S\,T$$

For compactness we denote

$$\alpha : * \vdash \tau_{\text{mkleaf}} \equiv \texttt{bool} \to \alpha$$
$$\alpha : * \vdash \tau_{\text{mknode}} \equiv \texttt{bool} \to \alpha \to \alpha \to \alpha$$

By beta reduction we have

$$\begin{aligned}\text{leaf}\,p\,s\,t &\equiv (\lambda p : \texttt{bool}.\lambda s, t : \texttt{tree}....)\,p\,s\,t\\ &\underset{\beta}{\twoheadrightarrow} (\lambda\alpha : *\,.\lambda u\,.\tau_{\text{mkleaf}}.\lambda v : \tau_{\text{mknode}}.\,v\,p\,(s\,\alpha\,u\,v)(t\,\alpha\,u\,v))\\ &\underset{\beta}{\twoheadrightarrow} (\lambda\alpha : *\,.\lambda u\,.\tau_{\text{mkleaf}}.\lambda v : \tau_{\text{mknode}}.\,v\,p\,(s\,\alpha\,u\,v)(t\,\alpha\,u\,v))\\ &\underset{\beta}{\twoheadrightarrow} \lambda\alpha : *\,.\lambda u\,.\tau_{\text{mkleaf}}.\lambda v : \tau_{\text{mknode}}.\,v\,p\,S\,T\end{aligned}$$

■

Problem

(3.19) If $\Gamma \vdash L : \sigma$, then Γ is a valid $\lambda 2$ context.

Solution. The definition of “valid” here would be taken as relative to a judgement as the fact to be able to derive the judgement. Thus, it meant a complete inference path could be made using only statements and judgements derived from the context. Now proof by induction on inference rule that deducted $L : \sigma$.

Case 1 : T-Var. The premise is that Γ is a $\lambda 2$ context. ■

Case 2 : T-App. Therefore $L \equiv M\ N$ for some $M, N \in \Lambda_{\mathbb{T}2}$. Therefore,

$$\Gamma \vdash M : \tau \to \sigma \qquad \Gamma \vdash N : \tau$$

for some $\tau \in \mathbb{T}_2$. By the inductive hypothesis on any premise Γ is a valid $\lambda 2$ context. ■

Case 3 : T-Abst. Therefore $L \equiv \lambda x : \alpha.\ N$ such that

$$\Gamma, x : \alpha \vdash N : \beta$$

for some $\alpha, \beta \in \mathbb{T}_2$ such that $\sigma \equiv \alpha \to \beta$. By the inductive hypothesis, $\Gamma, x : \alpha$. By the recursive definition of $\lambda 2$ contexts, for some valid context Δ, $\forall\ n \in \text{dom}\ \Delta$, n could not depende on statement declared after n in the context. Therefore, no statement in Γ could depend on $x : \alpha$. Therefore, Γ is a valid context. ■

Case 4 : T-Form. The premise is that Γ is a valid $\lambda 2$ context. ■

Case 5 : T2-App. Therefore $L \equiv N\ B$ for some $N, B \in \mathbb{V}_2$ such that

$$\Gamma \vdash N : \Pi\alpha : * .\sigma \qquad \Gamma \vdash B : *$$

By the inductive hypothesis on the any premise Γ is a valid $\lambda 2$ context. ■

Case 6 : T2-Abst. Therefore $L \equiv \lambda\alpha : *\ .\ M$ for some $M \in \Lambda_{\mathbb{T}2}$ such that

$$\Gamma, \alpha : * \vdash M : \beta$$

and $\sigma \equiv \Pi\alpha : * .\beta$. By reasoning in *Case 3* no statement in the context could depend on any statement before the latter's declaration. Therefore no statement in Γ could depend on $\alpha : *$, making it a valid context. ■

Problem

(3.20) Prove the free variable lemma for $\lambda 2$.

$$\Gamma \vdash L : \sigma \Rightarrow \text{FV}\ L \subseteq \text{dom}\ \Gamma$$

Solution. Proof by induction on inference rules that deduced $L : \sigma$. The only rule not considered is T-Form since all terms apparent in $\mathbb{T}_2$.

Case 1 : T-Var. Therefore L is the only free variable in L. By the generation lemma $L : \sigma \in \Gamma$, so $\{L\} \subseteq \text{dom}\ \Gamma$ ■

Case 2 : T-App. Therefore by the generation lemma $L \equiv M\ N$ for some $M, N \in \Lambda_{\mathbb{T}2}$ such that

$$\Gamma \vdash M : \tau \to \sigma \qquad \Gamma \vdash N : \tau$$

For some type τ. By the inductive hypothesis FV $M \subseteq \Gamma$ and FV $N \subseteq \text{dom}\ \Gamma$. Therefore FV $L = (\text{FV}\ M) \cup (\text{FV}\ N) \subseteq \text{dom}\ \Gamma$. ■

Case 3 : T-Abst. Therefore by the generation lemma $L \equiv \lambda x : \alpha .\ M$ for some $M \in \Lambda_{\mathbb{T}2}$ such that

$$\Gamma, x : \alpha \vdash M : \beta$$

and $\sigma \equiv \alpha \to \beta$. By the inductive hypothesis FV $M \subseteq \text{dom } \Gamma \cup \{x\}$. Therefore

$$\text{FV } L = \text{FV } M \setminus \{x\} \subseteq (\text{dom } \Gamma \cup \{x\}) \setminus \{x\} = \text{dom } \Gamma$$

■

Case 4 : T2-App. Therefore $L \equiv B$ for some $N, B \in \mathbb{V}_2$ such that

$$\Gamma \vdash N : \Pi\alpha : * . \sigma \qquad \Gamma \vdash B : *$$

By the inductive hypothesis FV $N \subseteq \text{dom } \Gamma$. Since $B \in \mathbb{T}_2$ then FV $B = \emptyset$. Therefore FV $L =$ FV $N \subseteq \text{dom } \Gamma$. ■

Case 5 : T2-Abst. Therefore $L \equiv \lambda\alpha : * .\ M$ for some $M \in \Lambda_{\mathbb{T}2}$ such that

$$\Gamma, \alpha : * \vdash M : \beta$$

and $\sigma \equiv \Pi\alpha : * . \beta$. Because $\alpha \in \mathbb{T}_2$ so $\alpha \notin$ FV M and FV $L =$ FV M, thus FV $M =$ FV $L \subseteq \Gamma$. ■

Problem

(3.21) Give a recursive definition for FTV : $\mathbb{T}_2 \cup \Lambda_{\mathbb{T}2} \to \mathbb{V}_2$

Solution. Here $\alpha \in \mathbb{V}_2$, $A, B \in \mathbb{T}_2$, $x \in V$ and $M \in \Lambda_{\mathbb{T}2}$.

Form	**Value**
FTV α	$\{\alpha\}$
FTV x	$\emptyset$
FTV $(A \to B)$	FTV$(A) \cup$ FTV(B)
FTV $(\Pi\alpha : * . A)$	FTV$(A) \setminus \{\alpha\}$
FTV $(M\ N)$	FTV$(M) \cup$ FTV(N)
FTV $(\lambda x : A . M)$	FTV$(A) \cup$ FTV(M)
FTV $(M\ B)$	FV$(M) \cup$ FTV(B)
FTV $(\lambda\alpha : * . M)$	FTV$(M) \setminus \{\alpha\}$

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Completed Dec 16 10:11 pm.