

# EXERCISES

## CHAPTER 5

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1. Redacted

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**Definition** Some rules for reference.

### $\lambda P$ Calculus Rules

$$\begin{array}{c} \frac{}{\emptyset \vdash * : \square} \text{Sort} \quad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \text{Var} \quad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \text{Weak} \\[10pt] \frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi x : A . B : s} \text{Form} \quad \frac{\Gamma \vdash M : \Pi x : A . B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B [x := N]} \text{App} \\[10pt] \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A . B : s}{\Gamma \vdash \lambda x : A . M : \Pi x : A . B} \text{Abst} \\[10pt] \frac{\Gamma \vdash A : B \quad B \stackrel{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \text{Conv} \end{array}$$

### Predicate Logic

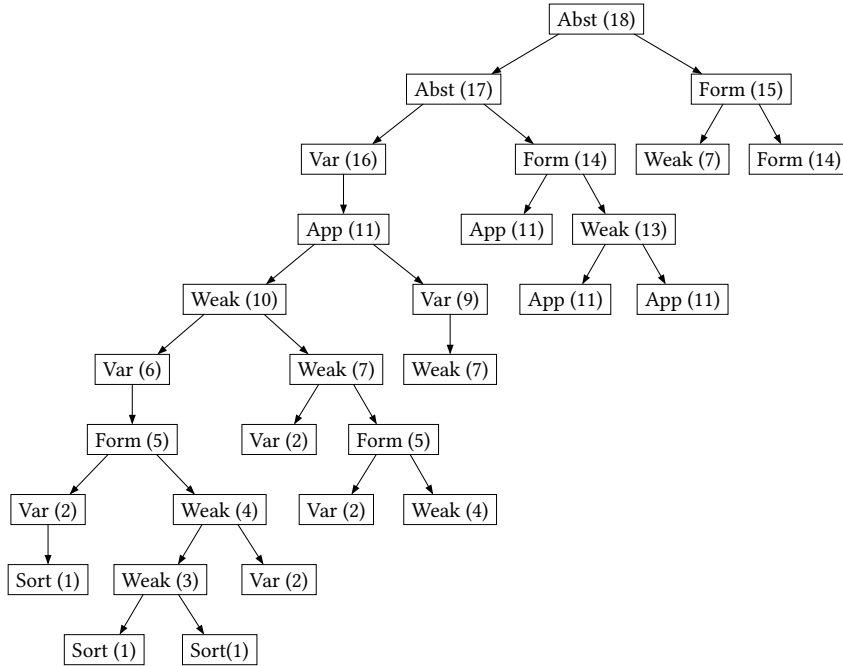
$$\begin{array}{c} \begin{array}{l} 1. \text{ Assume } A \\ 2. \quad \left| \begin{array}{l} \dots \\ 3. \quad B \end{array} \right. \\ \hline A \Rightarrow B \end{array} \Rightarrow I \quad \frac{A \Rightarrow B \quad A}{B} \Rightarrow E \quad \begin{array}{l} 1. \text{ Let } a \in S \\ 2. \quad \left| \begin{array}{l} \dots \\ 3. \quad P(a) \end{array} \right. \\ \hline \forall a \in S, P(a) \end{array} \forall I \\[10pt] \frac{\forall a \in S \quad N \in S}{P(N)} \forall E \end{array}$$

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### Problem

(5.1) Give a diagram of the tree corresponding to the complete tree derivation of line 18 of Section 5.3 (P 107)

*Solution.*



### Problem

(5.2) Give a complete  $\lambda P$  derivation of

$$S : * \vdash S \rightarrow S \rightarrow * : \square$$

In tree format and flag format.

*Solution.*

*Tree Derivation.*

$$\begin{array}{c}
 (4) \frac{\vdash * : \square \quad \vdash * : \square}{S : * \vdash * : \square} \text{Weak} \quad (3) S : * \vdash S : * \\
 (6) \frac{S : * \vdash * : \square \quad S : * \vdash S : *}{S : *, x : S \vdash * : \square} \text{Weak} \\
 (7) \frac{(3) S : * \vdash S : * \quad S : *, x : S \vdash * : \square}{S : * \vdash S \rightarrow * : \square}
 \end{array}$$

$$\begin{array}{c}
 (3) \frac{\vdash * : \square}{S : * \vdash S : *} \text{Var} \quad (9) \frac{(7) S : * \vdash S \rightarrow * : \square \quad (3) S : * \vdash S : *}{S : *, x : S \vdash S \rightarrow * : \square} \text{Weak} \\
 \hline
 S : * \vdash S \rightarrow S \rightarrow * : \square \text{Form}
 \end{array}$$

■

*Flag Derivation.*

1.	$* : \square$	<b>Sort</b>
2.	$S : *$	
3.	$S : *$	<b>1 Var</b>
4.	$* : \square$	<b>1,1 Weak</b>
5.	$x : S$	
6.	$\begin{array}{ l} * : \square \end{array}$	<b>4,3 Weak</b>
7.	$S \rightarrow * : \square$	<b>3,6 Form</b>
8.	$x : S$	
9.	$\begin{array}{ l} S \rightarrow * : \square \end{array}$	<b>7,3 Weak</b>
10.	$S \rightarrow S \rightarrow * : \square$	<b>3,9 Form</b>

■

### Problem

(5.3) Derive

$$S : *, Q : S \rightarrow S \rightarrow * \vdash \Pi x : S . \Pi y : S . Q x y : *$$

*Solution.*

1.	$* : \square$	<b>Sort</b>
2.	$S : *$	
3.	$S : *$	<b>1 Var</b>
4.	$* : \square$	<b>1,1 Weak</b>
5.	$x : S$	
6.	$\begin{array}{ l} * : \square \end{array}$	<b>4,3 Weak</b>
7.	$S \rightarrow * : \square$	<b>3,6 Form</b>
8.	$x : S$	
9.	$\begin{array}{ l} S \rightarrow * : \square \end{array}$	<b>7,3 Weak</b>
10.	$S \rightarrow S \rightarrow * : \square$	<b>3,9 Form</b>
11.	$Q : S \rightarrow S \rightarrow *$	

12.	$Q : S \rightarrow S \rightarrow *$	<b>10 Var</b>
13.	$S : *$	<b>3,10 Weak</b>
14.	$* : \square$	<b>4,10 Weak</b>
15.	$x : S$	
16.	$* : \square$	<b>14,13 Weak</b>
17.	$S : *$	<b>13,13 Weak</b>
18.	$x : S$	<b>13 Var</b>
19.	$Q : S \rightarrow S \rightarrow *$	<b>12,13 Weak</b>
20.	$y : S$	
21.	$y : S$	<b>17 Var</b>
22.	$Q : S \rightarrow S \rightarrow *$	<b>19,17 Weak</b>
23.	$x : S$	<b>18,17 Weak</b>
24.	$Q x : S \rightarrow *$	<b>22,23 App</b>
25.	$Q x y : *$	<b>24,21 App</b>
26.	$\Pi y : S . Q x y : *$	<b>17,25 Form</b>
27.	$\Pi x : S . \Pi y : S . Q x y : *$	<b>13,26 Form</b>

### Problem

(5.4) Prove that  $*$  is the only valid kind in  $\lambda P$ .

*Solution.*

*Proof.* The only possible way to construct a new kind is through the Form rule and the Sort axiom. Because we are trying to construct a kind,  $s$  here stands for  $\square$ .

$$\frac{\Gamma \vdash A : * \quad \Gamma, x : A \vdash B : \square}{\Gamma \vdash \Pi x : A . B : \square} \text{Form}$$

One could only construct new kinds with kinds, which requires  $A : \square$  and  $B : \square$ . This contradicts with  $A : *$ . ■

### Problem

(5.5) Prove that  $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$  is a tautology by given a shorthand  $\lambda P$  derivation.

*Solution.* By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash A \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of  $A$  and  $B$  is equivalent to a proof of tautologousness.

*Proof.*

1.	$A : *$	
2.	$B : *$	
3.	$x : A$	
4.	$y : A \rightarrow B$	
5.	$y x : B$	<b>4,3 App</b>
6.	$\lambda y : A \rightarrow B . y x : (A \rightarrow B) \rightarrow B$	<b>5 Abst</b>
7.	$\lambda x : A . \lambda y : A \rightarrow B . y x : A \rightarrow (A \rightarrow B) \rightarrow B$	<b>5 Abst</b>

■

### Problem

(5.6 a) Prove  $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$  a tautology using natural deduction.

*Solution.*

*Proof.*

1.	Assume $A \Rightarrow (A \Rightarrow B)$	
2.	$A \Rightarrow (A \Rightarrow B)$	
3.	Assume $A$	
4.	$A$	
5.	$A \Rightarrow B$	<b>2,4 <math>\Rightarrow</math>E</b>
6.	$B$	<b>5,4 <math>\Rightarrow</math>E</b>
7.	$A \Rightarrow B$	<b>3,6 <math>\Rightarrow</math>I</b>
8.	$(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$	<b>1,7 <math>\Rightarrow</math>I</b>

■

### Problem

(5.6 b) Prove  $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$  using a shorthand  $\lambda P$  derivation

*Solution.* By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$$

And finding an inhabitant in a context only with definition of  $A$  and  $B$  is equivalent to a proof of tautologousness.

*Proof.*

1.	$A : *$	
2.	$B : *$	
3.	$x : A \rightarrow A \rightarrow B$	
4.	$y : A$	
5.	$x y : A \rightarrow B$	3,4 App
6.	$x y y : B$	5,4 App
7.	$\lambda y : A . x y y : A \rightarrow B$	6 Abst
8.	$\lambda x : A \rightarrow A \rightarrow B . \lambda y : A . x y y : (A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$	7 Abst

■

### Problem

(5.7 a) Proof  $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$  using a shorthand  $\lambda P$  derivation.

*Solution.* By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of  $A$ ,  $B$ , and  $C$  is equivalent to a proof of tautologousness.

*Proof.*

1.	$A : *$	
2.	$B : *$	
3.	$C : *$	
4.	$x : A \rightarrow B$	
5.	$y : B \rightarrow C$	
6.	$a : A$	
7.	$x a : B$	4,6 App
8.	$y (x a) : C$	5,7 App
9.	$\lambda a : A . y (x a) : A \rightarrow C$	8 Abst
10.	$\lambda y : B \rightarrow C . \lambda a : A . y (x a) : (B \rightarrow C) \rightarrow A \rightarrow C$	9 Abst

$$\begin{array}{c}
11. \quad \left| \begin{array}{l} \lambda x : A \rightarrow B . \lambda y : B \rightarrow C . \lambda a : A . y (x z) \\ : (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow A \rightarrow C \end{array} \right. \quad \mathbf{10 \text{ Abst}}
\end{array}$$

■

### Problem

(5.7 b) Proof  $((A \Rightarrow B) \Rightarrow A) \Rightarrow ((A \Rightarrow B) \Rightarrow B)$  using a shorthand  $\lambda P$  derivation.

*Solution.* By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B$$

And finding an inhabitant in a context only with definition of  $A$  and  $B$  is equivalent to a proof of tautologousness.

*Proof.*

$$\begin{array}{c}
1. \quad A : * \\
2. \quad \left| B : * \right. \\
3. \quad \left| \left| x : (A \rightarrow B) \rightarrow A \right. \right. \\
4. \quad \left| \left| \left| y : A \rightarrow B \right. \right. \right. \\
5. \quad \left| \left| \left| \left| x y : A \right. \right. \right. \quad \mathbf{3,4 \text{ App}} \\
6. \quad \left| \left| \left| \left| y (x y) : B \right. \right. \right. \quad \mathbf{4,5 \text{ App}} \\
7. \quad \left| \left| \left| \lambda y : A \rightarrow B . y (x y) : (A \rightarrow B) \rightarrow B \right. \right. \quad \mathbf{6 \text{ Abst}} \\
8. \quad \left| \left| \lambda x : (A \rightarrow B) \rightarrow A . \lambda y : A \rightarrow B . y (x y) \right. \right. \\
\quad \left| : ((A \rightarrow B) \rightarrow A) \rightarrow (A \rightarrow B) \rightarrow B \quad \mathbf{7 \text{ Abst}} \right.
\end{array}$$

■

### Problem

(5.7 c) Proof  $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$  using a shorthand  $\lambda P$  derivation.

*Solution.* By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$$

And finding an inhabitant in a context only with definition of  $A$ ,  $B$ , and  $C$  is equivalent to a proof of tautologousness.

*Proof.*

1.	$A : *$		
2.	$B : *$		
3.	$C : *$		
4.	$x : A \rightarrow B \rightarrow C$		
5.	$y : A \rightarrow B$		
6.	$a : A$		
7.	$x a : B \rightarrow C$	4,6 App	
8.	$y a : B$	5,6 App	
9.	$x a (y a) : C$	7,8 App	
10.	$\lambda a : A . x a (y a) : A \rightarrow C$	9 Abst	
11.	$\lambda y : A \rightarrow B . \lambda a : A . x a (y a) : (A \rightarrow B) \rightarrow A \rightarrow C$	10 Abst	
12.	$\lambda x : A \rightarrow B \rightarrow C . \lambda y : A \rightarrow B . \lambda a : A . x a (y a) : (A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C$	10 Abst	

■

### Problem

(5.8 a) Let  $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *$ , find an inhabitant of

$$\Pi x : S . P x \rightarrow Q x \rightarrow P x$$

with respect to  $\Gamma$  and give a shorthand  $\lambda P$  derivation

*Solution.*

1.	$S : *$		
2.	$P : S \rightarrow *$		
3.	$Q : S \rightarrow *$		
4.	$x : S$		
5.	$a : P x$		
6.	$b : Q x$		
7.	$a : P x$	2,4 App	
8.	$\lambda b : Q x . S : Q x \rightarrow P x$	7 Abst	
9.	$\lambda a : P x . \lambda b : Q x . a : P x \rightarrow Q x \rightarrow P x$	8 Abst	



$$10. \quad \left| \left| \begin{array}{l} \lambda x : S . \lambda a : P \ x . \lambda b : Q \ x . a \\ : \Pi x : S . P \ x \rightarrow Q \ x \rightarrow P \ x \end{array} \right. \right.$$

**9 Abst**

### Problem

(5.8 b) Let  $\Gamma \equiv S : *, P : S \rightarrow *, Q : S \rightarrow *$ , find an inhabitant of

$$\Pi x : S . P \ x \rightarrow Q \ x \rightarrow P \ x$$

By proving the corresponding proposition in natural deduction.

*Solution.* The corresponding proposition and premises are

$$\frac{S \in \text{Set} \quad P : S \rightarrow \text{Prop} \quad Q : S \rightarrow \text{Prop}}{\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a))}$$

*Proof.*

1. Let  $a \in S$
2.     | Assume  $P(a)$
3.     |     | Assume  $Q(a)$
4.     |     |     |  $P(a)$
5.     |     |     |  $Q(a) \Rightarrow P(a)$      **3,4  $\Rightarrow$ I**
6.     |     |  $P(a) \Rightarrow (Q(a) \Rightarrow P(a))$      **2,5  $\Rightarrow$ I**
7.  $\forall a \in S, P(a) \Rightarrow (Q(a) \Rightarrow P(a))$      **1,6  $\forall$ I**

■

### Problem

(5.9 a) Give proof for

$$(\forall x \in S, Q(x)) \Rightarrow (\forall y \in S, P(y) \Rightarrow Q(y))$$

by natural deduction and a  $\lambda P$  derivation.

*Solution.*

*Natural Deduction.*

1. Assume  $\forall x \in S, Q(x)$
2.     | Let  $y \in S$

3.		Assume $P(y)$	
4.		$Q(y)$	<b>1,2 <math>\forall E</math></b>
5.		$P(y) \Rightarrow Q(y)$	<b>3,4 <math>\Rightarrow I</math></b>
6.		$\forall y \in S, P(y) \Rightarrow Q(y)$	<b>2,5 <math>\forall I</math></b>
7.		$(\forall x \in S, Q(x) \Rightarrow (\forall y \in S, P(y) \Rightarrow Q(y)))$	<b>1,6 <math>\Rightarrow I</math></b>

■

*$\lambda P$  Derivation.* Corresponding type is

$$S : *, P : S \rightarrow *, Q : S \rightarrow * \vdash (\Pi x : S . Q x) \rightarrow (\Pi y : S . P y \rightarrow Q y)$$

1.	$S : *$	
2.	$P : S \rightarrow *$	
3.	$Q : S \rightarrow *$	
4.	$a : \Pi x : S . Q x$	
5.	$y : S$	
6.	$z : P y$	
7.	$a y : Q y$	<b>4,5 App</b>
8.	$\lambda z : P y . a y : P y \rightarrow Q y$	<b>7 Abst</b>
9.	$\lambda y : S . \lambda z : P y . a y : \Pi y : S . P y \rightarrow Q y$	<b>7 Abst</b>
	$\lambda a : \Pi x : S . Q x . \lambda y : S . \lambda z : P y . a y$	
10.	$: (\Pi x : S . Q x) \rightarrow (\Pi y : S . P y \rightarrow Q y)$	<b>7 Abst</b>

■

## Problem

(5.9 b) Give proof for

$$\forall x \in S, (P(x) \Rightarrow Q(x)) \Rightarrow ((\forall y \in S, P(y)) \Rightarrow (\forall z \in S, Q(z)))$$

by natural deduction and a  $\lambda P$  derivation

*Solution.*

*Natural Deduction.*

1.	Assume $\forall x \in S, (P(x) \Rightarrow Q(x))$
2.	Assume $\forall y \in S, P(y)$
3.	Let $z \in S$

4.	$P(z)$	<b>2,3 <math>\forall E</math></b>
5.	$P(z) \Rightarrow Q(z)$	<b>1,3 <math>\forall E</math></b>
6.	$Q(z)$	<b>5,4 <math>\Rightarrow E</math></b>
7.	$\forall z \in S, Q(z)$	<b>3,6 <math>\forall I</math></b>
8.	$\forall y \in S, P(y) \Rightarrow (\forall z \in S, Q(z))$	<b>2,7 <math>\forall I</math></b>
9.	$\forall x \in S, (P(x) \Rightarrow Q(x)) \Rightarrow ((\forall y \in S, P(y)) \Rightarrow (\forall z \in S, Q(z)))$	<b>1,8 <math>\forall I</math></b>

■

*$\lambda P$  Derivation.* Corresponding type is

$$S : *, P : S \rightarrow *, Q : S \rightarrow *$$

$$\vdash (\Pi x : S . P x \rightarrow Q x) \rightarrow (\Pi y : S . P y) \rightarrow (\Pi z : S . Q z)$$

1.	$S : *$	
2.	$P : S \rightarrow *$	
3.	$Q : S \rightarrow *$	
4.	$a : \Pi x : S . P x \rightarrow Q x$	
5.	$b : \Pi y : S . P y$	
6.	$z : S$	
7.	$b z : P z$	<b>5,6 App</b>
8.	$a z : P z \rightarrow Q z$	<b>4,6 App</b>
9.	$a z (b z) : Q z$	<b>8,7 App</b>
10.	$\lambda z : S . a z (b z) : \Pi z : S . Q z$	<b>9 Abst</b>
11.	$\lambda b : (\Pi y : S . P y) . \lambda z : S . a z (b z)$ $: (\Pi y : S . P y) \rightarrow \Pi z : S . Q z$	<b>10 Abst</b>
12.	$\lambda a : (\Pi x : S . P x \rightarrow Q x) . \lambda b : (\Pi y : S . P y) .$ $\lambda z : S . a z (b z)$ $: (\Pi x : S . P x \rightarrow Q x) \rightarrow$ $(\Pi y : S . P y) \rightarrow \Pi z : S . Q z$	<b>10 Abst</b>

■

## Problem

(5.10) Given a context

$$\begin{aligned}\Gamma &\equiv S : *, P : S \rightarrow *, f : S \rightarrow S, g : S \rightarrow S, \\ u &: \Pi x : S . (P (f x) \rightarrow P (g x)), \\ v &: \Pi x, y : S . ((P x \rightarrow P y) \rightarrow P (f x))\end{aligned}$$

Let

$$M \equiv \lambda x : S . v (f x)(g x)(u x)$$

Type  $M$  under  $\Gamma$ .

*Solution.*

1.	$x : S$	
2.	$f : S \rightarrow S$	
3.	$f x : S$	<b>2,1 App</b>
4.	$g : S \rightarrow S$	
5.	$g x : S$	<b>4,1 App</b>
6.	$u : \Pi x : S . (P (f x) \rightarrow P (g x))$	
7.	$u x : P (f x) \rightarrow P (g x)$	<b>6,1 App</b>
8.	$v : \Pi x, y : S . ((P x \rightarrow P y) \rightarrow P (f x))$	
9.	$v (f x) : \Pi y : S . ((P (f x) \rightarrow P y) \rightarrow P (f (f x)))$	<b>8,3 App</b>
10.	$v (f x)(g x) : (P (f x) \rightarrow P (g x)) \rightarrow P (f (f x))$	<b>9,5 App</b>
11.	$v (f x)(g x)(u x) : P (f (f x))$	<b>10,7 App</b>
12.	$\lambda x : S . v (f x)(g x)(u x) : S \rightarrow P (f (f x))$	<b>11 Abst</b>

### Problem

(5.11) Let  $S$  be a set, with  $Q$  and  $R$  relations on  $S \times S$ , and let  $f$  and  $g$  be functions from  $S$  to  $S$ . Assume

$$\forall x, y \in S (Q(x, f(y)) \Rightarrow Q(g(x), y))$$

$$\forall x, y \in S (Q(x, f(y)) \Rightarrow R(x, y))$$

$$\forall x \in S (Q(x, f(f(x))))$$

Prove that

$$\forall x \in S, R(g(g(x)), g(x))$$

By giving a context  $\Gamma$  and finding a term  $M$  such that

$$\Gamma \vdash M : \Pi x : S. R(g(g(x)))(g(x))$$

*Solution.* Context  $\Gamma$  is as follows:

$$\Gamma \equiv S : *, f : S \rightarrow S, g : S \rightarrow S$$

$$Q : S \rightarrow S \rightarrow *, R : S \rightarrow S \rightarrow *$$

$$A : \Pi x, y : S. (Q(x, f(y)) \rightarrow Q(g(x), y)),$$

$$B : \Pi x, y : S. (Q(x, f(y)) \rightarrow R(x, y))$$

$$C : \Pi x : S. Q(x, f(f(x)))$$

*Derivation.*

- |     |  |                 |
|-----|--|-----------------|
| 1.  | $S : *, f : S \rightarrow S, g : S \rightarrow S$                      |                 |
| 2.  | $Q : S \rightarrow S \rightarrow *, R : S \rightarrow S \rightarrow *$ |                 |
| 3.  | $A : \Pi x, y : S. (Q(x, f(y)) \rightarrow Q(g(x), y))$                |                 |
| 4.  | $B : \Pi x, y : S. (Q(x, f(y)) \rightarrow R(x, y))$                   |                 |
| 5.  | $C : \Pi x : S. Q(x, f(f(x)))$   |                 |
| 6.  | $x : S$  |                 |
| 7.  | $g(x) : S$   | <b>1,6 App</b>  |
| 8.  | $C(g(x)) : Q(g(x))(f(f(g(x))))$  | <b>5,7 App</b>  |
| 9.  | $f(g(x)) : S$  | <b>1,7 App</b>  |
| 10. | $A(g(x)) : \Pi y : S. (Q(g(x))(f(y)) \rightarrow (Q(g(g(x))) y))$      | <b>3,7 App</b>  |
|     | $A(g(x))(f(g(x)))$   |                 |
|     | $: (Q(g(x))(f(f(g(x))))$   |                 |
| 11. | $\rightarrow (Q(g(g(x)))(f(g(x))))$                                    | <b>10,9 App</b> |

12.	$A(g\ x)(f(g\ x))(C(g\ x))$ $: (Q(g(g\ x))(f(g\ x)))$	<b>11,8 App</b>
13.	$g(g\ x) : S$ $B(g(g\ x))$ $: \Pi y : S . (Q(g(g\ x))(f\ y))$	<b>1,7 App</b>
14.	$\rightarrow R(g(g\ x))\ y)$	<b>4,13 App</b>
15.	$B(g(g\ x))(g\ x)$ $: (Q(g(g\ x))(f(g\ x))) \rightarrow (R(g(g\ x))(g\ x))$	<b>14,7 App</b>
16.	$B(g(g\ x))(g\ x)(A(g\ x)(f(g\ x))(C(g\ x)))$ $: (R(g(g\ x))(g\ x))$	<b>15,12 App</b>
17.	$\lambda x : S . B(g(g\ x))(g\ x)(A(g\ x)(f(g\ x))(C(g\ x)))$ $: \Pi x : S . (R(g(g\ x))(g\ x))$	<b>17 Abst</b>

■

### Problem

(5.12 a) In  $\lambda P$ , consider the context

$$\begin{aligned}\Gamma &\equiv S : *, R : S \rightarrow S \rightarrow *, \\ u &: \Pi x, y : S . R\ x\ y \rightarrow R\ y\ x \\ v &: \Pi x, y, z : S . R\ x\ y \rightarrow R\ x\ z \rightarrow R\ y\ z\end{aligned}$$

Show that  $R$  is reflexive over  $S \times S$ . That is, construct  $M$  such that

$$\Gamma \vdash M : \Pi x, y : S . R\ x\ y \rightarrow R\ x\ x$$

*Solution.*

*Proof.*

1.  $S : *, R : S \rightarrow S \rightarrow *$
2.  $A : \Pi u, v : S . R\ u\ v \rightarrow R\ v\ u$
3.  $B : \Pi u, v, w : S . R\ u\ v \rightarrow R\ u\ w \rightarrow R\ v\ w$
4.  $\left| \begin{array}{l} x : S \\ y : S \\ h : R\ x\ y \end{array} \right|$
- 5.
- 6.

7.	$B x : \Pi v, w : S . R x v \rightarrow R v w \rightarrow R x w$	3,4 App
8.	$B x y : \Pi w : S . R x y \rightarrow R y w \rightarrow R x w$	7,5 App
9.	$B x y x : R x y \rightarrow R y x \rightarrow R x x$	7,5 App
10.	$A x : \Pi v : R x v \rightarrow R v x$	2,4 App
11.	$A x y : R x y \rightarrow R y x$	10,5 App
12.	$A x y h : R y x$	11,6 App
13.	$B x y x h : R y x \rightarrow R x x$	9,6 App
14.	$B x y x h (A x y h) : R x x$	13,12 App
15.	$\lambda h : R x y . B x y x h (A x y h) : R x y \rightarrow R x x$	14 Abst
16.	$\lambda y : S . \lambda h : R x y . B x y x h (A x y h)$ $: \Pi y : S . R x y \rightarrow R x x$	15 Abst
17.	$\lambda x, y : S . \lambda h : R x y . B x y x h (A x y h)$ $: \Pi x, y : S . R x y \rightarrow R x x$	16 Abst

■

### Problem

(5.12 b) Given the context  $\Gamma$  in 5.12 a, prove transitivity of  $R$  by constructing  $M$  such that

$$\Gamma \vdash M : \Pi x, y, z : S . R x y \rightarrow R y z \rightarrow R x z$$

*Solution.*

*Proof.*

1.	$S : *, R : S \rightarrow S \rightarrow *$	
2.	$A : \Pi u, v : S . R u v \rightarrow R v u$	
3.	$B : \Pi u, v, w : S . R u v \rightarrow R u w \rightarrow R v w$	
4.	$x : S$	
5.	$y : S$	
6.	$z : S$	
7.	$h : R x y$	
8.	$r : R y z$	
9.	$A x : \Pi v : S . R x v \rightarrow R v x$	2,4 App
10.	$A x y : R x y \rightarrow R y x$	9,5 App
11.	$A x y h : R y x$	10,7 App

12.	$B y : \Pi v, w : S . R y v \rightarrow R y w \rightarrow R v w$	<b>3,5 App</b>
13.	$B y x : \Pi w : S . R y x \rightarrow R y w \rightarrow R x w$	<b>12,4 App</b>
14.	$B y x z : R y x \rightarrow R y z \rightarrow R x z$	<b>12,4 App</b>
15.	$B y x z (A x y h) : R y z \rightarrow R x z$	<b>14,11 App</b>
16.	$B y x z (A x y h) r : R x z$	<b>15,8 App</b>
17.	$\lambda r : R y z . B y x z (A x y h) r : R y z \rightarrow R x z$	<b>16 Abst</b>
18.	$\lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$ $: R x y \rightarrow R y z \rightarrow R x z$	<b>17 Abst</b>
19.	$\lambda z : S . \lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$ $: \Pi z : S . R x y \rightarrow R y z \rightarrow R x z$	<b>18 Abst</b>
20.	$\lambda y, z : S . \lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$ $: \Pi y, z : S . R x y \rightarrow R y z \rightarrow R x z$	<b>19 Abst</b>
21.	$\lambda x, y, z : S . \lambda h : R x y . \lambda r : R y z . B y x z (A x y h) r$ $: \Pi x, y, z : S . R x y \rightarrow R y z \rightarrow R x z$	<b>20 Abst</b>

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Completed Dec 22 6:51 pm.