Excercises

Chapter 5

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1. Reducted

Definition Some rules for reference.

λP Calculus Rules

$$\frac{}{\emptyset \vdash * : \square}\,\text{Sort} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}\,\text{Var} \qquad \frac{\Gamma \vdash A : B \qquad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}\,\text{Weak}$$

$$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi\, x : A \,.\, B : s}\,\text{Form} \qquad \frac{\Gamma \vdash M : \Pi\, x : A \,.\, B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B\,[x := N]}\,\text{App}$$

$$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash \Pi\, x : A \,.\, B : s}{\Gamma \vdash \lambda x : A \,.\, M : \Pi\, x : A \,.\, B}\,\text{Abst}$$

$$\frac{\Gamma \vdash A : B \qquad B \overset{\beta}{=} B' \qquad \Gamma \vdash B' : s}{\Gamma \vdash A : B'}\,\text{Conv}$$

Predicate Logic

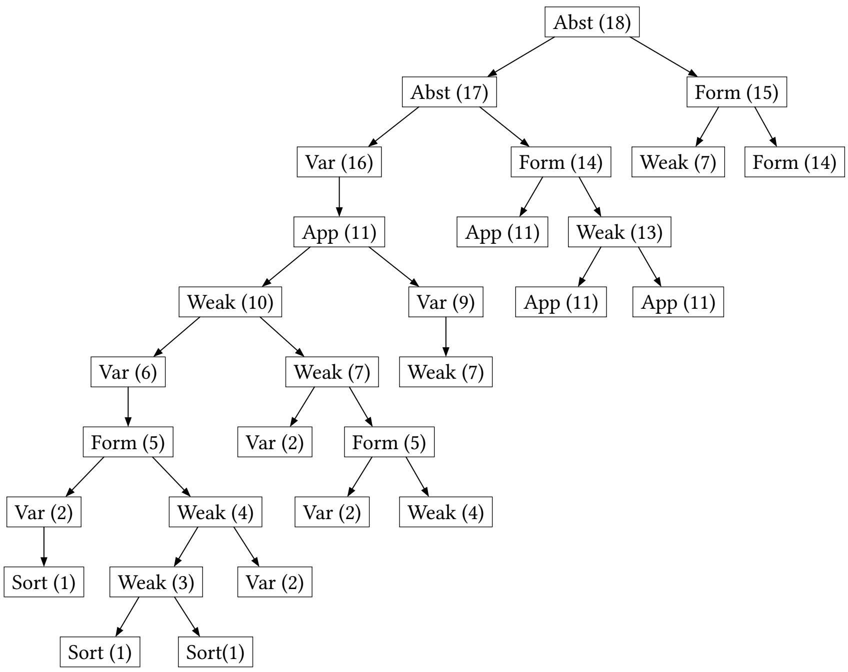
$$\frac{\begin{array}{ll} 1. & \text{Assume } A \\ 2. & \quad \dots \\ 3. & \quad B \end{array}}{A \Rightarrow B}\,{\Rightarrow}\text{I} \qquad \frac{A \Rightarrow B \qquad A}{B}\,{\Rightarrow}\text{E} \qquad \frac{\begin{array}{ll} 1. & \text{Let } a \in S \\ 2. & \quad \dots \\ 3. & \quad P\,(a) \end{array}}{\forall a \in S, P\,(a)}\,\forall\text{I}$$

$$\frac{\forall a \in S \qquad N \in S}{P\,(N)}\,\forall\text{E}$$

Problem

(5.1) Give a diagram of the tree corresponding to the complete tree derivation of line 18 of Section 5.3 (P 107)

Solution.



Problem

(5.2) Give a complete λP derivation of

$$S : * \vdash S \to S \to * : \square$$

In tree format and flag format.

Solution.

Tree Derivation.

$$(7)\,\dfrac{(3) S : * \vdash S : * \qquad (6)\,\dfrac{(4)\,\dfrac{\vdash * : \square \qquad \vdash * : \square}{S : * \vdash * : \square}\,\text{Weak} \qquad (3) S : * \vdash S : *}{S : *, x : S \vdash * : \square}\,\text{Weak}}{S : * \vdash S \to * : \square}$$

$$\dfrac{(3)\,\dfrac{\vdash * : \square}{S : * \vdash S : *}\,\text{Var} \qquad (9)\,\dfrac{(7) S : * \vdash S \to * : \square \qquad (3) S : * \vdash S : *}{S : *, x : S \vdash S \to * : \square}\,\text{Weak}}{S : * \vdash S \to S \to * : \square}\,\text{Form}$$

■

Flag Derivation.

1.	$* : \square$	**Sort**
2.	$S : *$	
3.	$\quad S : *$	**1 Var**
4.	$\quad * : \square$	**1,1 Weak**
5.	$\quad x : S$	
6.	$\quad\quad * : \square$	**4,3 Weak**
7.	$\quad S \to * : \square$	**3,6 Form**
8.	$\quad x : S$	
9.	$\quad\quad S \to * : \square$	**7,3 Weak**
10.	$\quad S \to S \to * : \square$	**3,9 Form**

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Problem

(5.3) Derive

$$S : *, Q : S \to S \to * \vdash \Pi\, x : S\, .\Pi\, y : S\, .\, Q\, x\, y : *$$

Solution.

1.	$* : \square$	**Sort**
2.	$S : *$	
3.	$\quad S : *$	**1 Var**
4.	$\quad * : \square$	**1,1 Weak**
5.	$\quad x : S$	
6.	$\quad\quad * : \square$	**4,3 Weak**
7.	$\quad S \to * : \square$	**3,6 Form**
8.	$\quad x : S$	
9.	$\quad\quad S \to * : \square$	**7,3 Weak**
10.	$\quad S \to S \to * : \square$	**3,9 Form**
11.	$\quad Q : S \to S \to *$	

12.	$Q : S \to S \to *$	**10 Var**
13.	$S : *$	**3,10 Weak**
14.	$* : \square$	**4,10 Weak**
15.	$x : S$	
16.	$* : \square$	**14,13 Weak**
17.	$S : *$	**13,13 Weak**
18.	$x : S$	**13 Var**
19.	$Q : S \to S \to *$	**12,13 Weak**
20.	$y : S$	
21.	$y : S$	**17 Var**
22.	$Q : S \to S \to *$	**19,17 Weak**
23.	$x : S$	**18,17 Weak**
24.	$Q\,x : S \to *$	**22,23 App**
25.	$Q\,x\,y : *$	**24,21 App**
26.	$\Pi\, y : S \,.\, Q\,x\,y : *$	**17,25 Form**
27.	$\Pi\, x : S \,.\Pi\, y : S \,.\, Q\,x\,y : *$	**13,26 Form**

Problem

(5.4) Prove that $*$ is the only valid kind in λP.

Solution.

Proof. The only possible way to construct a new kind is through the Form rule and the Sort axiom. Because we are trying to construct a kind, s here stands for $\square$.

$$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : \square}{\Gamma \vdash \Pi\, x : A \,.\, B : \square}\ \text{Form}$$

One could only construct new kinds with kinds, which requires $A : \square$ and $B : \square$. This contradicts with $A : *$. ∎

Problem

(5.5) Prove that $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ is a tautology by given a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash A \to (A \to B) \to B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1. $A : *$
2. $\quad B : *$
3. $\quad\quad x : A$
4. $\quad\quad\quad y : A \to B$
5. $\quad\quad\quad\quad y\,x : B$ — **4,3 App**
6. $\quad\quad\quad \lambda y : A \to B\,.\,y\,x : (A \to B) \to B$ — **5 Abst**
7. $\quad\quad \lambda x : A\,.\lambda y : A \to B\,.\,y\,x : A \to (A \to B) \to B$ — **5 Abst**

∎

Problem

(5.6 a) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ a tautology using natural deduction.

Solution.

Proof.

1. Assume $A \Rightarrow (A \Rightarrow B)$
2. $\quad A \Rightarrow (A \Rightarrow B)$
3. $\quad$ Assume A
4. $\quad\quad A$
5. $\quad\quad A \Rightarrow B$ — **2,4 $\Rightarrow$E**
6. $\quad\quad B$ — **5,4 $\Rightarrow$E**
7. $\quad A \Rightarrow B$ — **3,6 $\Rightarrow$I**
8. $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ — **1,7 $\Rightarrow$I**

∎

Problem

(5.6 b) Prove $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ using a shorthand λP derivation

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash (A \to A \to B) \to A \to B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$x : A \to A \to B$	
4.	$y : A$	
5.	$x\, y : A \to B$	**3,4 App**
6.	$x\, y\, y : B$	**5,4 App**
7.	$\lambda y : A \,.\, x\, y\, y : A \to B$	**6 Abst**
8.	$\lambda x : A \to A \to B \,.\lambda y : A \,.\, x\, y\, y : (A \to A \to B) \to A \to B$	**7 Abst**

■

Problem

(5.7 a) Proof $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \to B) \to (B \to C) \to A \to C$$

And finding an inhabitant in a context only with definition of A, B, and C is equivalent to a proof of tautologousness.

Proof.

1.	$A : *$	
2.	$B : *$	
3.	$C : *$	
4.	$x : A \to B$	
5.	$y : B \to C$	
6.	$a : A$	
7.	$x\, a : B$	**4,6 App**
8.	$y\,(x\, a) : C$	**5,7 App**

9. $\lambda a : A \,.\, y\,(x\,z) \colon A \to C$ **8 Abst**

10. $\lambda y : B \to C \,.\lambda a : A \,.\, y\,(x\,z) \colon (B \to C) \to A \to C$ **9 Abst**

11. $\lambda x : A \to B \,.\lambda y : B \to C \,.\lambda a : A \,.\, y\,(x\,z)$
$\colon (A \to B) \to (B \to C) \to A \to C$ **10 Abst**

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Problem

(5.7 b) Proof $((A \Rightarrow B) \Rightarrow A) \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B : * \vdash ((A \to B) \to A) \to (A \to B) \to B$$

And finding an inhabitant in a context only with definition of A and B is equivalent to a proof of tautologousness.

Proof.

1. $A : *$
2. $B : *$
3. $x : (A \to B) \to A$
4. $y : A \to B$
5. $x\,y : A$ **3,4 App**
6. $y\,(x\,y) \colon B$ **4,5 App**
7. $\lambda y : A \to B \,.\, y\,(x\,y) \colon (A \to B) \to B$ **6 Abst**
8. $\lambda x : (A \to B) \to A \,.\lambda y : A \to B \,.\, y\,(x\,y)$
$\colon ((A \to B) \to A) \to (A \to B) \to B$ **7 Abst**

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Problem

(5.7 c) Proof $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ using a shorthand λP derivation.

Solution. By the principle of PAT, this proposition is equivalent to the type

$$A, B, C : * \vdash (A \to B \to C) \to (A \to B) \to A \to C$$

And finding an inhabitant in a context only with definition of A, B, and C is equivalent to a proof of tautologousness.

Proof.

1. $A : *$
2. $B : *$
3. $C : *$
4. $x : A \to B \to C$
5. $y : A \to B$
6. $a : A$
7. $x\, a : B \to C$ **4,6 App**
8. $y\, a : B$ **5,6 App**
9. $x\, a\, (y\, a) : C$ **7,8 App**
10. $\lambda a : A \,.\, x\, a\, (y\, a) : A \to C$ **9 Abst**
11. $\lambda y : A \to B \,.\lambda a : A \,.\, x\, a\, (y\, a) : (A \to B) \to A \to C$ **10 Abst**
12. $\lambda x : A \to B \to C \,.\lambda y : A \to B \,.\lambda a : A \,.\, x\, a\, (y\, a)$
 $: (A \to B \to C) \to (A \to B) \to A \to C$ **10 Abst**

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Problem

(5.8 a) Let $\Gamma \equiv S : *, P : S \to *, Q : S \to *$, find an inhabitant of

$$\Pi\, x : S \,.\, P\, x \to Q\, x \to P\, x$$

with respect to Γ and give a shorthand λP derivation

Solution.

1. $S : *$
2. $P : S \to *$
3. $Q : S \to *$
4. $x : S$
5. $a : P\, x$
6. $b : Q\, x$
7. $a : P\, x$ **2,4 App**

8. $\lambda b : Q\, x\, .\, S : Q\, x \to P\, x$ **7 Abst**

9. $\lambda a : P\, x\, .\lambda b : Q\, x\, .\, a : P\, x \to Q\, x \to P\, x$ **8 Abst**

$\lambda x : S\, .\lambda a : P\, x\, .\lambda b : Q\, x\, .\, a$

10. $: \Pi\, x : S\, .\, P\, x \to Q\, x \to P\, x$ **9 Abst**

Problem

(5.8 b) Let $\Gamma \equiv S : *, P : S \to *, Q : S \to *$, find an inhabitant of

$$\Pi\, x : S\, .\, P\, x \to Q\, x \to P\, x$$

By proving the corresponding proposition in natural deduction.

Solution. The corresponding proposition and premises are

$$\frac{S \in \text{Set} \qquad P : S \to \text{Prop} \qquad Q : S \to \text{Prop}}{\forall a \in S, P\,(a) \Rightarrow (Q\,(a) \Rightarrow P\,(a))}$$

Proof.

1. Let $a \in S$
2. Assume $P\,(a)$
3. Assume $Q\,(a)$
4. $P\,(a)$
5. $Q\,(a) \Rightarrow P\,(a)$ **3,4 ⇒I**
6. $P\,(a) \Rightarrow (Q\,(a) \Rightarrow P\,(a))$ **2,5 ⇒I**
7. $\forall a \in S, P\,(a) \Rightarrow (Q\,(a) \Rightarrow P\,(a))$ **1,6 ∀I**

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Problem

(5.9 a) Give proof for

$$(\forall x \in S, Q\,(x)) \Rightarrow (\forall y \in S, P\,(y) \Rightarrow Q\,(y))$$

by natural deduction and a λP derivation.

Solution.

Natural Deduction.

1. Assume $\forall x \in S, Q\ (x)$
2. Let $y \in S$
3. Assume $P\ (y)$
4. $Q\ (y)$ **1,2 $\forall$E**
5. $P\ (y) \Rightarrow Q\ (y)$ **3,4 $\Rightarrow$I**
6. $\forall y \in S, P\ (y) \Rightarrow Q\ (y)$ **2,5 $\forall$I**
7. $(\forall x \in S, Q\ (x) \Rightarrow (\forall y \in S, P\ (y) \Rightarrow Q\ (y))$ **1,6 $\Rightarrow$I**

■

λP Derivation. Corresponding type is

$$S : *, P : S \to *, Q : S \to * \vdash (\Pi\, x : S\,.\,Q\ x) \to (\Pi\, y : S\,.\,P\ y \to Q\ y)$$

1. $S : *$
2. $P : S \to *$
3. $Q : S \to *$
4. $a : \Pi\, x : S\,.\,Q\ x$
5. $y : S$
6. $z : P\ y$
7. $a\ y : Q\ y$ **4,5 App**
8. $\lambda z : P\ y\,.\,a\ y : P\ y \to Q\ y$ **7 Abst**
9. $\lambda y : S\,.\lambda z : P\ y\,.\,a\ y : \Pi\, y : S\,.\,P\ y \to Q\ y$ **7 Abst**
10. $\lambda a : \Pi\, x : S\,.\,Q\ x\,.\lambda y : S\,.\lambda z : P\ y\,.\,a\ y$ $: (\Pi\, x : S\,.\,Q\ x) \to (\Pi\, y : S\,.\,P\ y \to Q\ y)$ **7 Abst**

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Problem

(5.9 b) Give proof for

$$\forall x \in S, (P\ (x) \Rightarrow Q\ (x)) \Rightarrow ((\forall y \in S, P\ (y)) \Rightarrow (\forall z \in S, Q\ (z)))$$

by natural deduction and a λP derivation

Solution.

Natural Deduction.

1. Assume $\forall x \in S, (P\ (x) \Rightarrow Q\ (x))$

2. Assume $\forall y \in S, P\,(y)$

3. Let $z \in S$

4. $P\,(z)$ **2,3 $\forall$E**

5. $P\,(z) \Rightarrow Q\,(z)$ **1,3 $\forall$E**

6. $Q\,(z)$ **5,4 $\Rightarrow$E**

7. $\forall z \in S, Q\,(z)$ **3,6 $\forall$I**

8. $\forall y \in S, P\,(y) \Rightarrow (\forall z \in S, Q\,(z))$ **2,7 $\forall$I**

9. $\forall x \in S, (P\,(x) \Rightarrow Q\,(x)) \Rightarrow ((\forall y \in S, P\,(y)) \Rightarrow (\forall z \in S, Q\,(z)))$ **1,8 $\forall$I**

■

λP Derivation. Corresponding type is

$$S : *, P : S \to *, Q : S \to *$$
$$\vdash (\Pi\, x : S\,.\, P\, x \to Q\, x) \to (\Pi\, y : S\,.\, P\, y) \to (\Pi\, z : S\,.\, Q\, z)$$

1. $S : *$

2. $P : S \to *$

3. $Q : S \to *$

4. $a : \Pi\, x : S\,.\, P\, x \to Q\, x$

5. $b : \Pi\, y : S\,.\, P\, y$

6. $z : S$

7. $b\, z : P\, z$ **5,6 App**

8. $a\, z : P\, z \to Q\, z$ **4,6 App**

9. $a\, z\, (b\, z) : Q\, z$ **8,7 App**

10. $\lambda z : S\,.\, a\, z\, (b\, z) : \Pi\, z : S\,.\, Q\, z$ **9 Abst**

11. $\lambda b : (\Pi\, y : S\,.\, P\, y).\lambda z : S\,.\, a\, z\, (b\, z)$ $: (\Pi\, y : S\,.\, P\, y) \to \Pi\, z : S\,.\, Q\, z$ **10 Abst**

12. $\lambda a : (\Pi\, x : S\,.\, P\, x \to Q\, x).\lambda b : (\Pi\, y : S\,.\, P\, y).$ $\lambda z : S\,.\, a\, z\, (b\, z)$ $: (\Pi\, x : S\,.\, P\, x \to Q\, x) \to$ $(\Pi\, y : S\,.\, P\, y) \to \Pi\, z : S\,.\, Q\, z$ **10 Abst**

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