Excercises

Chapter 1

Sean Li [1]

1. Reducted

Problem

(1.1) Simplify notation of the following terms

$$\text{(a)}\ (\lambda x.(((xz)y)(xx)))$$
$$\text{(b)}\ ((\lambda x.(\lambda y.(\lambda z.(z((xy)z)))))(\lambda u.u))$$

Solution.

$$\text{(a)}\ \lambda x.(xzy)(xx)$$
$$\text{(b)}\ (\lambda xyz.x(xyz))(\lambda u.u)$$

Problem

(1.2) Find the alpha equivalent terms to

$$\lambda x.x(\lambda x.x)$$

In

$$\text{(a)}\ \lambda y.y(\lambda x.x)$$
$$\text{(b)}\ \lambda y.y(\lambda x.y)$$
$$\text{(c)}\ \lambda y.y(\lambda x.y)$$

Solution. Only (a).

Problem

(1.3) Prove

$$\lambda x.x(\lambda z.y) \underset{\alpha}{=} \lambda z.z(\lambda z.y)$$

Solution.

Proof. By definition of alpha equivalence

$$M \underset{\alpha}{=} N \iff \exists \varphi, M^{\varphi} \underset{\alpha}{\to} N \wedge \mathrm{FR}\ M = \mathrm{FR}\ N$$

The witness of φ is substituting bound variable x with z, and z is not a free variable in the term, thus the two terms are alpha equivalent.

$$\lambda x.x(\lambda z.y)^{x \to z} \underset{\alpha}{\to} \lambda z.z(\lambda z.y)$$

■

Problem

(1.4) Consider the following term:

$$U := (\lambda z.zxz)((\lambda y.xy)x)$$

1. Find Sub U
2. Draw tree rep of U
3. Find FV U
4. Find alpha equivalent terms to U from below and point out which of those follows the Barendregt convention:

$$\text{(a)}\ (\lambda y.yxy)((\lambda z.xz)x)$$
$$\text{(b)}\ (\lambda x.xyx)((\lambda z.yz)y)$$
$$\text{(c)}\ (\lambda y.yxy)((\lambda y.xy)x)$$
$$\text{(d)}\ (\lambda v.(vx)v)((\lambda u.uv)x)$$

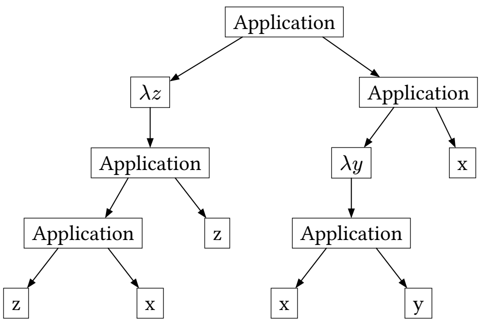
1. Find Sub U.

Solution.

$$
\begin{aligned}
\text{Sub } U = \\
&\{(\lambda z.zxz)((\lambda y.xy)x), (\lambda z.zxz), ((\lambda y.xy)x)\} \cup \\
&\{(\lambda y.xy), (y), (\lambda z.xz), (x)\} \cup \\
&\{(\lambda y.x), (y)\} \cup \{(\lambda z.x), (z)\} \cup \{(y), (x)\} \\
= &\{(\lambda z.zxz)(\lambda y.xy)x, (\lambda z.zxz), (\lambda y.xy)x, \\
&(\lambda y.xy), (\lambda z.xz), (\lambda y.x), (\lambda z.x), y, x\}
\end{aligned}
$$

2. Draw a tree rep of U.

Solution.



3. Find FV U

Solution.

$$
\begin{aligned}
\text{FV } U &= \text{FV } (\lambda y.yxy) \cup \text{FV } (\lambda z.xz)x \\
&= (\text{FV } yxy) \setminus \{y\} \cup (\text{FV } \lambda z.xz) \cup \{x\} \\
&= (\text{FV } yx) \setminus \{y\} \cup (\text{FV } xz) \setminus \{z\} \cup \{x\} \\
&= \{x\}
\end{aligned}
$$

4. Find an alpha-equivalent term.

Solution.

$$(a) \underset{\alpha}{=} (c) \underset{\alpha}{=} U$$

Only (a) follows the Barendregt convention.

Problem

(1.5) Give the results of the following substitutions

$$
\begin{aligned}
&\text{(a)}\ (\lambda x.y(\lambda y.xy))[y := \lambda z.zx] \\
&\text{(b)}\ ((xyz)[x := y])[y := z] \\
&\text{(c)}\ ((\lambda x.xyz)[x := y])[y := z] \\
&\text{(d)}\ (\lambda y.yyx)[x := yz]
\end{aligned}
$$

Solution.

(a) $(\lambda v.(\lambda z.zx)(\lambda u.vu))$

(b) $(yyz)[y := z] = zzz$

(c) $(\lambda x.xyz)[y := z] = (\lambda x.xzz)$

(d) $(\lambda u.uu(yz))$

Problem

(1.6)

$$\neg\Big(\forall MLN \in \Lambda, M[x := N, y := L] \underset{\alpha}{\equiv} M[x := N][y := L]\Big)$$

Solution.

Proof. Because RHS $= M[x := N][y := L] = M[x := N[y := L]][y := L]$, if $y \in$ FV N, then what x gets substituted with will have y substituted for L, which is completely different with LHS. ∎

Problem

(1.7) Find all available redexes in

$$U := (\lambda z.zxz)((\lambda y.xy)x)$$

And all reduction pathes to the β-normal form.

Solution. The first redex is the term as an application itself; another the second term in the application.

$$\begin{tikzcd}
 & \boxed{(\lambda z.zxz)((\lambda y.xy)x)} \arrow[dl] \arrow[dr] & & \\
\boxed{(\lambda z.zxz)(xx)} \arrow[d] & & \boxed{((\lambda y.xy)x)x((\lambda y.xy)x)} \arrow[dl] \arrow[dr] & \\
\boxed{xxx(xx)} & \boxed{xxx((\lambda y.xy)x)} \arrow[d] & & \boxed{((\lambda y.xy)x)x(xx)} \arrow[d] \\
 & \boxed{xxx(xx)} & & \boxed{xxx(xx)}
\end{tikzcd}$$

Problem

(1.8) Show that

$$(\lambda x.xx)y \underset{\beta}{\neq} (\lambda xy.yx)xx$$

Solution. By Collorary 1.9.9, it suffices to prove the hypothesis with a proof of a common normal reducted form from LHS and RHS not existing.

Contradiction. By definition of $\underset{\beta}{=}$, there exists The set of all terms attainable from β -reduction on $(\lambda x.xx)y$ and $(\lambda xy.yx)xx$ do not intersect. Therefore,

$$\neg\left(\exists L \in \Lambda, (\lambda x.xx)y \underset{\beta}{\twoheadrightarrow} L \land (\lambda xy.yx)xx \underset{\beta}{\twoheadrightarrow} L\right) \implies \neg\left((\lambda x.xx)y \underset{\beta}{=} (\lambda xy.yx)xx\right)$$

■

Problem

(1.9) Define the combinators

$$K := \lambda xy.x$$
$$S := \lambda xyz.xz(yz)$$

Prove that

$$\forall PQ \in \Lambda, KPQ \underset{\beta}{\twoheadrightarrow} P$$

$$\forall PQR \in \Lambda, SPQR \underset{\beta}{\twoheadrightarrow} PR(QR)$$

Solution.

Proof.

$$KPQ = (\lambda xy.x)PQ \underset{\beta}{\to} (\lambda y.x)[x := P]Q \underset{\beta}{\to} P[y := Q] = P$$

$$SPQR = (\lambda xyz.xz(yz)) \underset{\beta}{\twoheadrightarrow} (xz(yz))[x := P][y := Q][z := R] = PR(QR)$$

■

Problem

(1.10) We define the church numerals

$$\begin{aligned} \text{zero} &:= \lambda f x . x \\ \text{one} &:= \lambda f x . f x \\ \text{two} &:= \lambda f x . f f x \\ &\dots \\ \text{num}_n &:= \lambda f x . f^n x \end{aligned}$$

And operations

$$\begin{aligned} \text{add} &:= \lambda n m f x . m f (n f x) \\ \text{mul} &:= \lambda n m f x . m (n f) x \end{aligned}$$

Show

$$\text{(a) add one one} \underset{\beta}{\twoheadrightarrow} \text{two}$$

$$\text{(b) add one one} \underset{\beta}{\neq} \text{mul one zero}$$

Solution.

$$\begin{aligned} \text{(a)} \quad \text{add one one} &= (\lambda n m f x . m f (n f x))(\lambda f x . f x)(\lambda f x . f x) \\ &\underset{\beta}{\twoheadrightarrow} (\lambda f x . (\lambda f x . f x) f ((\lambda f x . f x) f x)) \\ &\underset{\beta}{\twoheadrightarrow} (\lambda f x . (\lambda x . f x) f x) \\ &\underset{\beta}{\twoheadrightarrow} (\lambda f x . f f x) = \text{two} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \text{mul one one} &= (\lambda n m f x . m (n f) x)(\lambda f x . f x)(\lambda f x . f x) \\ &\underset{\beta}{\twoheadrightarrow} \lambda f x . (\lambda f x . f x)((\lambda f x . f x) f) x \\ &\underset{\beta}{\twoheadrightarrow} \lambda f x . f x = \text{one} \end{aligned}$$

Because no intermediate form in the beta reduction process of the two terms are α-equivalent, by collorary 1.9.9 the two terms are not β-equivalent.

Problem

(1.11) We define

$$\text{succ} := \lambda mfx.f(mfx) \ \text{ s.t. } \forall \text{num}_n, \text{succ num}_n = \text{num}_{n+1}$$

Prove

$$\text{succ zero} \underset{\beta}{=} \text{one}$$
$$\text{succ one} \underset{\beta}{=} \text{two}$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$\begin{aligned}\text{succ zero} &= (\lambda mfx.f(mfx))(\lambda fx.x)\\ &\underset{\beta}{\to} (\lambda fx.f((\lambda fx.x)fx))\\ &\underset{\beta}{\twoheadrightarrow} (\lambda fx.fx) = \text{one}\end{aligned}$$

The path succ zero $\underset{\beta}{\twoheadrightarrow}$ one derived above is the witness of a reduction chain from LHS to RHS.

$$\begin{aligned}\text{succ one} &= (\lambda mfx.f(mfx))(\lambda fx.fx)\\ &\underset{\beta}{\to} (\lambda fx.f((\lambda fx.fx)fx))\\ &\underset{\beta}{\twoheadrightarrow} (\lambda fx.f(fx)) = \text{two}\end{aligned}$$

The path succ one $\underset{\beta}{\twoheadrightarrow}$ two derived above is the witness of a reduction chain from LHS to RHS. ■

Problem

(1.12) We define the λ-terms $\top_\lambda$(true) and $\bot_\lambda$(false) and $\neg_\lambda$(not) by:

$$\top_\lambda := \lambda xy.x \quad \bot_\lambda := \lambda xy.y$$
$$\neg_\lambda := \lambda a.a\bot_\lambda\top_\lambda$$

Show that

$$\neg_\lambda(\neg_\lambda\top_\lambda) \underset{\beta}{=} \top_\lambda$$
$$\neg_\lambda(\neg_\lambda\bot_\lambda) \underset{\beta}{=} \bot_\lambda$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β-equivalence.

Proof.

$$\begin{aligned}
\neg_\lambda(\neg_\lambda \top_\lambda) &= \neg_\lambda((\lambda a.a\bot_\lambda\top_\lambda)(\lambda xy.x)) \\
&\underset{\beta}{\twoheadrightarrow} \neg_\lambda((\lambda xy.x)\bot_\lambda\top_\lambda) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda a.a\bot_\lambda\top_\lambda)\bot_\lambda \\
&\underset{\beta}{\twoheadrightarrow} (\lambda xy.y)\bot_\lambda\top_\lambda \\
&\underset{\beta}{\twoheadrightarrow} \top_\lambda
\end{aligned}$$

$$\begin{aligned}
\neg_\lambda(\neg_\lambda \bot_\lambda) &= \neg_\lambda((\lambda a.a\bot_\lambda\top_\lambda)(\lambda xy.y)) \\
&\underset{\beta}{\twoheadrightarrow} \neg_\lambda((\lambda xy.x)\bot_\lambda\top_\lambda) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda a.a\bot_\lambda\top_\lambda)\top_\lambda \\
&\underset{\beta}{\twoheadrightarrow} (\lambda xy.x)\bot_\lambda\top_\lambda \\
&\underset{\beta}{\twoheadrightarrow} \bot_\lambda
\end{aligned}$$

■

Problem

(1.13) Define

$$\text{iszero} := \lambda m.m(\lambda x.\bot_\lambda)\top_\lambda$$

Prove

$$\text{iszero zero} \underset{\beta}{\twoheadrightarrow} \top_\lambda$$

$$\forall n \in \mathbb{N}^+, \text{iszero num}_n \underset{\beta}{\twoheadrightarrow} \bot_\lambda$$

Solution.

$$\begin{aligned}
\text{iszero zero} &= (\lambda m.m(\lambda x.\bot_\lambda)\top_\lambda)(\lambda fx.x) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda fx.x)(\lambda x.\bot_\lambda)\top_\lambda \\
&\underset{\beta}{\twoheadrightarrow} \top_\lambda
\end{aligned}$$

$$
\begin{aligned}
\text{iszero num}_n &= (\lambda m.m(\lambda x.\bot_\lambda)\top_\lambda)(\lambda fx.f^n x) \\
&\underset{\beta}{\twoheadrightarrow} (\lambda fx.f^n x)(\lambda x.\bot_\lambda)\top_\lambda \\
&\underset{\beta}{\twoheadrightarrow} (\lambda x.\bot_\lambda)\big((\lambda x.\bot_\lambda)^{n-1}\top_\lambda\big) \underset{\beta}{\twoheadrightarrow} \bot_\lambda
\end{aligned}
$$

Problem

(1.14) If-else can be modeled as

$$\text{ifelse} = \lambda xtf.xtf$$

Where when x, then t, else f. Prove correctness by applying $\top_\lambda$ and $\bot_\lambda$ on ifelse.

Solution.

$$
\begin{aligned}
\text{ifelse } \top_\lambda &= (\lambda xtf.xtf)\top_\lambda \\
&\underset{\beta}{\twoheadrightarrow} (\lambda tf.(\lambda xy.x)tf) \underset{\beta}{\twoheadrightarrow} (\lambda tf.t)
\end{aligned}
$$

$$
\begin{aligned}
\text{ifelse } \top_\lambda &= (\lambda xtf.xtf)\bot_\lambda \\
&\underset{\beta}{\twoheadrightarrow} (\lambda tf.(\lambda xy.y)tf) \underset{\beta}{\twoheadrightarrow} (\lambda tf.f)
\end{aligned}
$$

By applying the results to any two values, the correct corresponding value returns, ex, for ifelse $\top_\lambda$, t is always returned.

Problem

(1.15) Prove that $\Omega := (\lambda x.xx)(\lambda x.xx)$ does not have a β-nf.

Solution. Firstly let's prove Ω.

Proof. Induction on Ω's only reduction path proves that every $\Omega \underset{\beta}{\twoheadrightarrow} \Omega_i = \Omega$. For the base case because Ω has one and only one redux, it could only reduce to Ω_1 which is equivalent to itself. For the inductive step, $\Omega_i = \Omega$, therefore $\Omega_i \underset{\beta}{\to} \Omega_{i+1}$ is still Ω.

By definition, a term having a β-nf requires the existence of a form in β-nf such that the term can reduce to. By induction, Ω only reduces to Ω, and Ω is not in β-nf because it contains β-redex. Therefore, Ω can never reduce to a β-nf, thus it does not have a β-nf. ■

Problem

(1.16) Let M be a λ-term with the following properties:

- M has a β-nf.
- There exists an infinite reduction path $M \equiv M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \dots$ on M.

Prove that every M_i has a β-nf, and give an example of M.

Solution. An example would be $(\lambda xy.y)\Omega$. Reduction can go on infinitely by reducing on Ω, but the β-nf of the term is $\lambda y.y$

Proof. Denote β-nf of M as M'. For any form in the reduction path, $M \underset{\beta}{\twoheadrightarrow} M_i$. In conjunction with $M \underset{\beta}{\twoheadrightarrow} M'$, by the Church-Rosser theorem, there exists L such that $M_i \underset{\beta}{\twoheadrightarrow} L$ and $M' \underset{\beta}{\twoheadrightarrow} L$. Because M' is in β-nf, L can only be M', thus $M_i \underset{\beta}{\twoheadrightarrow} M'$, so M_i is capable of reducing to M', a β-nf. Therefore, any form in the reduction path has a β-nf. ■

Problem

(1.17) If MN is strongly normalizing, then both M and N are strongly normalizing.

Solution.

Proof. If M is not strongly normalizing, then there exists a reduction path $M_0 \underset{\beta}{\to} M_1 \underset{\beta}{\to} \dots$ Therefore, MN would have had a reduction path $MN \underset{\beta}{\to} M_1 N \underset{\beta}{\to} \dots$ that is infinite, which contradicts with MN being strongly normalizing. Vice versa for N. ■

Problem

(1.18) Let $L, M, N \in \Lambda$ such that $L \underset{\beta}{=} M$ and $L \underset{\beta}{\twoheadrightarrow} N$. Moreover, N is in β-nf. Prove that $M \underset{\beta}{\twoheadrightarrow} N$.

Solution. Collorary 1.9.9.

Problem

(1.19) Define

$$U := \lambda zx.x(zzx) \quad \text{and} \quad Z := UU$$

Prove Z is a fixed point combinator.

Solution. Proving $\forall L \in \Lambda, L(ZL) \underset{\beta}{\twoheadrightarrow} ZL$.

Proof.

$$\begin{aligned} ZL &= (\lambda zx.x(zzx))(\lambda zx.x(zzx))L \\ &\underset{\beta}{\twoheadrightarrow} L((\lambda zx.x(zzx))(\lambda zx.x(zzx))L) \underset{\beta}{\twoheadrightarrow} L(ZL) \end{aligned}$$

■

Problem

(1.20) Solve for $M \in \Lambda$ in each equation:

$$\begin{aligned} M &\underset{\beta}{=} \lambda xy.xMy \\ Mxyz &\underset{\beta}{=} xyzM \end{aligned}$$

Solution. By the property of the Y combinator:

$$f(Yf) = Yf$$

The first equation can be remodeled as

$$M \underset{\beta}{=} LM \text{ where } L = \lambda mxy.xmy$$

Solving for fixed point of L :

$$\begin{aligned} M &\equiv YL \equiv L(YL) \\ &= (\lambda x.L(xx))(\lambda x.L(xx)) \\ &= (\lambda x.(\lambda muv.umv)(xx))(\lambda x.(\lambda muv.umv)(xx)) \\ &\underset{\beta}{\twoheadrightarrow} (\lambda x.(\lambda uv.u(xx)v))(\lambda x.(\lambda uv.u(xx)v)) \end{aligned}$$

The second equation can be η-reduced on both sides:

$$\begin{aligned} Mxyz &\underset{\beta}{=} xyzM \\ M &\underset{\beta}{=} \lambda xyz.xyzM \end{aligned}$$

Remodeling equation:

$$M = NM \text{ where } M = \lambda mxyz.xyzm$$

then

$$\begin{aligned} M &\equiv YN \equiv N(YN) \\ &= (\lambda x.N(xx))(\lambda x.N(xx)) \\ &= (\lambda x.(\lambda muyz.uyzm)(xx))(\lambda x.(\lambda muyz.uyzm)(xx)) \\ &\twoheadrightarrow_{\beta} (\lambda x.(\lambda uyz.uyz(xx)))(\lambda x.(\lambda uyz.uyz(xx))) \end{aligned}$$