EXCERCISES

CHAPTER 5

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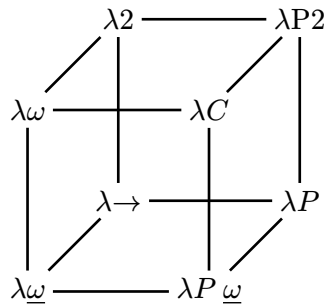
1. Reducted

Reference - Calculus of Constructions

$$\frac{}{\emptyset \vdash * : \square}\text{Sort} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}\text{Var} \qquad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}\text{Weak}$$

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi\, x : A \,.\, B : s_2}\text{Form}$$

$$\frac{\Gamma \vdash M : \Pi\, x : A \,.\, B \quad \Gamma \vdash N : A}{\Gamma \vdash M\, N : B\,[x := N]}\text{App}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi\, x : A \,.\, B : s}{\Gamma \vdash \lambda x : A \,.\, M : \Pi\, x : A \,.\, B}\text{Abst}$$

$$\frac{\Gamma \vdash A : B \quad B \overset{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'}\text{Conv}$$

The λ-Cube

$\boldsymbol{\lambda\to}$	$(*, *)$		
$\boldsymbol{\lambda 2}$	$(*, *)$	$(\square, *)$	
$\boldsymbol{\lambda\underline{\omega}}$	$(*, *)$		$(\square, \square)$
$\boldsymbol{\lambda P}$	$(*, *)$			$(*, \square)$
$\boldsymbol{\lambda\omega}$	$(*, *)$	$(\square, *)$	$(\square, \square)$
$\boldsymbol{\lambda 2}$	$(*, *)$	$(\square, *)$		$(*, \square)$
$\boldsymbol{\lambda\underline{\omega}}$	$(*, *)$		$(\square, \square)$	$(*, \square)$
$\boldsymbol{\lambda P}$	$(*, *)$	$(\square, *)$	$(\square, \square)$	$(*, \square)$

(6.1 a) Give a complete derivation in tree format showing that

$$\bot \equiv \Pi \alpha : *.\alpha$$

is legal in λC.

Solution. Here we will show that there exists $s \in$ sort and Γ such that $\Gamma \vdash \bot : s$.

Proof.

$$\cfrac{\vdash * : \Box \qquad \cfrac{\cfrac{\vdash * : \Box}{\alpha : * \vdash \alpha : *}\text{Var}}{}}{\vdash \Pi \alpha : *.\alpha : *}\text{Form}$$

∎

(6.1 a) Give a complete derivation in tree format showing that $\bot \to \bot$ is legal in λC where

$$\bot \equiv \Pi \alpha : *.\alpha$$

Solution. Here we will show that there exists $s \in$ sort and Γ such that $\Gamma \vdash \bot \to \bot : s$.

Proof.

$$\cfrac{(6.1\ a)\cfrac{}{\vdash \bot : *} \qquad \cfrac{(6.1\ a)\cfrac{}{\vdash \bot : *} \qquad \cfrac{(6.1\ a)\cfrac{}{\vdash \bot : *}}{x : \bot \vdash \bot : *}\text{Weak}}{}}{\vdash \Pi\, x : \bot.\bot : *}\text{Form}$$

∎

(6.1 c) To which systems of the λ-cube does $\bot$ belong? And $\bot \to \bot$?

Solution. The set of (s_1, s_2) pairs in formation rules of the derivation of $\bot$ is $\{(\Box, *)\}$. The minimal system corresponding is $\lambda 2$. The same for $\bot \to \bot$. Therefore $\bot$ and $\bot \to \bot$ belongs to $\lambda 2$, $\lambda \omega$, λP and λC.

(6.2) Given context $\Gamma \equiv S : *, P : S \to *, A : *$. Prove by means of a flag derivation that the following expression is inhabited in λC with respect to Γ:

$$(\Pi\, x : S\,.(A \to P\, x)) \to A \to \Pi\, y : S\,.\,P\, y$$

Solution. The inhabitant is

$$M \equiv \lambda u : (\Pi\, x : S\,.(A \to P\, x)).\lambda v : A\,.\lambda y : S\,.\,u\, y\, v$$

Proof.

1.	$S : *, P : S \to *, A : *$	
2.	$u : \Pi\, x : S\,.(A \to P\, x)$	
3.	$v : A$	
4.	$y : S$	
5.	$u\, y : A \to P\, y$	**2,4 App**
6.	$u\, y\, v : P\, y$	**5,3 App**
7.	$\lambda y : S\,.\,u\, y\, v : \Pi\, y : S\,.\,P\, y$	**6 Abst**
8.	$\lambda v : A\,.\lambda y : S\,.\,u\, y\, v : A \to \Pi\, y : S\,.\,P\, y$	**7 Abst**
	$\lambda u : \Pi\, x : S\,.(A \to P\, x).\lambda v : A\,.\lambda y : S\,.\,u\, y\, v$	
9.	$: \Pi\, x : S\,.(A \to P\, x) \to A \to \Pi\, y : S\,.\,P\, y$	**8 Abst**

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(6.3 a) Let $\mathcal{J}$ be a judgement

$$\mathcal{J} \equiv S : *, P : S \to * \vdash \lambda x : S\,.(P\, x \to \bot) : S \to *$$

Derive $\mathcal{J}$ in λC with shorthand flag notation.

Solution.

1.	$S : *, P : S \to *$	
2.	$x : S$	
3.	$P\, x : *$	**1,2 App**
4.	$\bot : *$	**Weak from 6.1 a**
5.	$P\, x \to \bot : *$	**3,4 Form**

6. $\quad$ $\overline{\vphantom{|}}$ $\lambda x : S \,.\, P\, x \to \bot : S \to *$ $\quad$ **5 Abst**

(6.3 b) Determnmine the (s_1, s_2) pairs corresponding to all Π abstractions occuring in $\mathcal{J}$.

Solution.

Abstraction	**Line Number**	(s_1, s_2)
$P : S \to *$	1	$(*, \square)$
$\bot \equiv \Pi \alpha : * . \alpha$	4	$(\square, *)$
$P\, x \to \bot$	5	$(\square, *)$
$\lambda x : S \,.\, P\, x \to \bot : S \to *$	6	$(*, \square)$

(6.3 c) What is the 'smallest' system in the λ-cube to which $\mathcal{J}$ belongs?

Solution. There are $(*, *) - \lambda{\to}$ pairs, $(*, \square) - \lambda P$ pairs, and $(\square, *) - \lambda 2$. Therefore the minimal system $\mathcal{J}$ belongs to is $\lambda P2$.

(6.4 a) Let $\Gamma \equiv S : *, Q : S \to S \to *$ and

$$M \equiv (\Pi\, x, y : S \,.(Q\, x\, y \to Q\, y\, x \to \bot)) \to \Pi\, z : S \,.(Q\, z\, z \to \bot)$$

Derive $\Gamma \vdash M : *$ and determine the smallest subsystemm to which this judgement belongs.

Solution.

1. $S : *, Q : S \to S \to *$

2. $x : S$

3. $y : S$

4. $Q\,x : S \to *$ **1,2 App**

5. $Q\,x\,y : *$ **4,3 App**

6. $z : Q\,x\,y$

7. $Q\,y : S \to *$ **1,3 App**

8. $Q\,y\,x : *$ **7,2 App**

9. $t : Q\,y\,x$

10. $\bot : *$ **Weak from 6.1 a**

11. $Q\,y\,x \to \bot : *$ **8,10 Form**

12. $Q\,x\,y \to Q\,y\,x \to \bot : *$ **5,11 Form**

13. $\Pi\,y : S \,.\, Q\,x\,y \to Q\,y\,x \to \bot : *$ **1,12 Form**

14. $\Pi\,x, y : S \,.\, Q\,x\,y \to Q\,y\,x \to \bot : *$ **1,13 Form**

15. $a : (\Pi\,x, y : S \,.(Q\,x\,y \to Q\,y\,x \to \bot))$

16. $z : S$

17. $Q\,z : S \to *$ **1,16 App**

18. $Q\,z\,z : *$ **17,16 App**

19. $b : Q\,z\,z$

20. $\bot : *$ **Weak from 6.1 a**

21. $Q\,z\,z \to \bot : *$ **18,20 Form**

22. $\Pi\,z : S \,.\, Q\,z\,z \to \bot : *$ **1,21 Form**

          $\Pi\,x, y : S \,.(Q\,x\,y \to Q\,y\,x \to \bot)$

23. $\to \Pi\,z : S \,.\, Q\,z\,z \to \bot : *$ **14,22 Form**

Here's a table of all Πs that appeared

Abstraction	Line Number	(s_1, s_2)
$S \to *$	1 / 4 / 7 / 17	$(*, \Box)$
$S \to S \to *$	1	$(*, \Box)$
$\bot$	10 / 11 / 12 / 13 / 14 / 15 / 20 / 21 / 22 / 23	$(\Box, *)$
$Q\,y\,x \to \bot$	11 / 12 / 13 / 14 / 15 / 23	$(*, *)$
$Q\,x\,y \to Q\,y\,x \to \bot$	12 / 13 / 14 / 15 / 23	$(*, *)$
$\Pi\,y : S \,.\, Q\,x\,y \to Q\,y\,x \to \bot$	13 / 14 / 23	$(*, *)$

$\Pi\,x,y:S\,.\,Q\,x\,y \to Q\,y\,x \to \bot$	14 / 23	$(*,*)$
$Q\,z\,z \to \bot$	21 / 22 / 23	$(*,*)$
$\Pi\,z:S\,.\,Q\,z\,z \to \bot$	22 / 23	$(*,*)$
$\Pi\,x,y:S\,.\,Q\,x\,y \to Q\,y\,x \to \bot \to$ $\quad \Pi\,z:S\,.\,Q\,z\,z \to \bot$	23	$(*,*)$

There are $(*,*) - \lambda{\to}$ pairs, $(*,\square) - \lambda P$ pairs, and $(\square,*) - \lambda 2$ pairs. Therefore the mimimal system available is $\lambda P2$.

<div style="background:teal;color:white;padding:4px 12px;font-weight:bold">Problem</div>

(6.4 b) Prove in λC that M is inhabited in context Γ.

Solution. A shorthand derivation is given below:

Proof.

1. $S:*, Q:S\to S\to *$
2. $\quad h:\Pi\,x,y:S\,.(Q\,x\,y\to Q\,y\,x\to\bot)$
3. $\quad\quad z:S$
4. $\quad\quad\quad a:Q\,z\,z$
5. $\quad\quad\quad\quad \alpha:*$
6. $\quad\quad\quad\quad\quad h\,z:\Pi\,y:S\,.(Q\,z\,y\to Q\,y\,z\to\bot)$ **2,3 App**
7. $\quad\quad\quad\quad\quad h\,z\,z:Q\,z\,z\to Q\,z\,z\to\bot$ **6,3 App**
8. $\quad\quad\quad\quad\quad h\,z\,z\,a:Q\,z\,z\to\bot$ **7,4 App**
9. $\quad\quad\quad\quad\quad h\,z\,z\,a\,a:\Pi\alpha:*.\alpha$ **8,4 App**
10. $\quad\quad\quad\quad\quad h\,z\,z\,a\,a\,\alpha:\alpha$ **9,5 App**
11. $\quad\quad\quad\quad \lambda\alpha:*.\,h\,z\,z\,a\,a\,\alpha:\Pi\alpha:*.\alpha$ **10 Abst**
12. $\quad\quad\quad \lambda a:Q\,z\,z\,\lambda\alpha:*.\,h\,z\,z\,a\,a\,\alpha:Q\,z\,z\to\bot$ **11 Abst**
13. $\quad\quad \lambda z:S\,.\lambda a:Q\,z\,z\,\lambda\alpha:*.\,h\,z\,z\,a\,a\,\alpha:\Pi\,z:S\,.\,Q\,z\,z\to\bot$ **12 Abst**

$\quad\quad \lambda h:\Pi\,x,y:S\,.(Q\,x\,y\to Q\,y\,x\to\bot)$

$\quad\quad\quad \lambda z:S\,.\lambda a:Q\,z\,z\,\lambda\alpha:*.\,h\,z\,z\,a\,a\,\alpha$

14. $\quad\quad :\Pi\,x,y:S\,.(Q\,x\,y\to Q\,y\,x\to\bot)\to\Pi\,z:S\,.\,Q\,z\,z\to\bot$ **13 Abst**

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(6.4 c) We may consier Q to be a relation on set S. Moreover by PAT we may see $A \to \bot$ as the negation $\neg A$ of prop A. How can M then be interpreted by the PAT paradigm?

Solution. By a direct type-to-proposition translation we have

$$M \equiv \forall x, y \in S, (Q(x, y) \Rightarrow \neg Q(y, x)) \Rightarrow \forall z \in S, (\neg Q(z, z))$$

It expresses the fact if Q is asymmetric then it is irreflective.