

# EXERCISES

## CHAPTER 5

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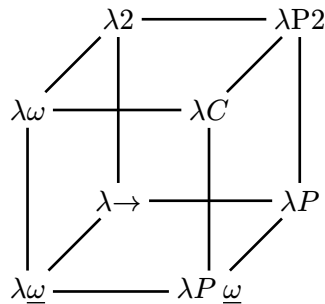
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#### Reference - Calculus of Constructions

$$\begin{array}{c}
 \frac{}{\emptyset \vdash * : \square} \text{Sort} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \text{Var} \qquad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \text{Weak} \\
 \\
 \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A . B : s_2} \text{Form} \\
 \\
 \frac{\Gamma \vdash M : \Pi x : A . B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B [x := N]} \text{App} \\
 \\
 \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A . B : s}{\Gamma \vdash \lambda x : A . M : \Pi x : A . B} \text{Abst} \\
 \\
 \frac{\Gamma \vdash A : B \quad B \stackrel{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \text{Conv}
 \end{array}$$

#### The $\lambda$ -Cube



$\lambda \rightarrow$	$(*, *)$			
$\lambda 2$	$(*, *)$	$(\square, *)$		
$\lambda \omega$	$(*, *)$		$(\square, \square)$	
$\lambda P$	$(*, *)$			$(*, \square)$
$\lambda \omega$	$(*, *)$	$(\square, *)$	$(\square, \square)$	
$\lambda 2P$	$(*, *)$	$(\square, *)$		$(*, \square)$
$\lambda P \omega$	$(*, *)$		$(\square, \square)$	$(*, \square)$
$\lambda C$	$(*, *)$	$(\square, *)$	$(\square, \square)$	$(*, \square)$

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### Problem

(6.1 a) Give a complete derivation in tree format showing that

$$\perp \equiv \Pi \alpha : * . \alpha$$

is legal in  $\lambda C$ .

*Solution.* Here we will show that there exists  $s \in \text{sort}$  and  $\Gamma$  such that  $\Gamma \vdash \perp : s$ .

*Proof.*

$$\frac{\frac{\vdash * : \square}{\alpha : * \vdash \alpha : *} \text{Var}}{\vdash \Pi \alpha : * . \alpha : *} \text{Form}$$

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### Problem

(6.1 a) Give a complete derivation in tree format showing that  $\perp \rightarrow \perp$  is legal in  $\lambda C$  where

$$\perp \equiv \Pi \alpha : * . \alpha$$

*Solution.* Here we will show that there exists  $s \in \text{sort}$  and  $\Gamma$  such that  $\Gamma \vdash \perp \rightarrow \perp : s$ .

*Proof.*

$$\frac{(6.1 \text{ a}) \frac{\vdash \perp : *}{\vdash \perp : *} \quad (6.1 \text{ a}) \frac{\vdash \perp : *}{\vdash \perp : *} \text{Weak}}{\vdash \Pi x : \perp . \perp : *} \text{Form}$$

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### Problem

(6.1 c) To which systems of the  $\lambda$ -cube does  $\perp$  belong? And  $\perp \rightarrow \perp$ ?

*Solution.* The set of  $(s_1, s_2)$  pairs in formation rules of the derivation of  $\perp$  is  $\{(\square, *)\}$ . The minimal system corresponding is  $\lambda 2$ . The same for  $\perp \rightarrow \perp$ . Therefore  $\perp$  and  $\perp \rightarrow \perp$  belongs to  $\lambda 2$ ,  $\lambda \omega$ ,  $\lambda P$  and  $\lambda C$ .

### Problem

(6.2) Given context  $\Gamma \equiv S : *, P : S \rightarrow *, A : *$ . Prove by means of a flag derivation that the following expression is inhabited in  $\lambda C$  with respect to  $\Gamma$ :

$$(\Pi x : S . (A \rightarrow P x)) \rightarrow A \rightarrow \Pi y : S . P y$$

*Solution.* The inhabitant is

$$M \equiv \lambda u : (\Pi x : S . (A \rightarrow P x)) . \lambda v : A . \lambda y : S . u y v$$

*Proof.*

1.	$S : *, P : S \rightarrow *, A : *$	
2.	$u : \Pi x : S . (A \rightarrow P x)$	
3.	$v : A$	
4.	$y : S$	
5.	$u y : A \rightarrow P y$	<b>2,4 App</b>
6.	$u y v : P y$	<b>5,3 App</b>
7.	$\lambda y : S . u y v : \Pi y : S . P y$	<b>6 Abst</b>
8.	$\lambda v : A . \lambda y : S . u y v : A \rightarrow \Pi y : S . P y$	<b>7 Abst</b>
9.	$\lambda u : \Pi x : S . (A \rightarrow P x) . \lambda v : A . \lambda y : S . u y v : \Pi x : S . (A \rightarrow P x) \rightarrow A \rightarrow \Pi y : S . P y$	<b>8 Abst</b>

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### Problem

(6.3 a) Let  $\mathcal{J}$  be a judgement

$$\mathcal{J} \equiv S : *, P : S \rightarrow * \vdash \lambda x : S . (P x \rightarrow \perp) : S \rightarrow *$$

Derive  $\mathcal{J}$  in  $\lambda C$  with shorthand flag notation.

*Solution.*

1.	$S : *, P : S \rightarrow *$	
2.	$x : S$	
3.	$P x : *$	<b>1,2 App</b>
4.	$\perp : *$	<b>Weak from 6.1 a</b>
5.	$P x \rightarrow \perp : *$	<b>3,4 Form</b>

6.  $\lambda x : S . P x \rightarrow \perp : S \rightarrow *$  **5 Abst**

### Problem

(6.3 b) Determine the  $(s_1, s_2)$  pairs corresponding to all  $\Pi$  abstractions occurring in  $\mathcal{J}$ .

*Solution.*

Abstraction	Line Number	$(s_1, s_2)$
$P : S \rightarrow *$	1	$(*, \square)$
$\perp \equiv \Pi \alpha : * . \alpha$	4	$(\square, *)$
$P x \rightarrow \perp$	5	$(\square, *)$
$\lambda x : S . P x \rightarrow \perp : S \rightarrow *$	6	$(*, \square)$

### Problem

(6.3 c) What is the ‘smallest’ system in the  $\lambda$ -cube to which  $\mathcal{J}$  belongs?

*Solution.* There are  $(*, *) - \lambda \rightarrow$  pairs,  $(*, \square) - \lambda P$  pairs, and  $(\square, *) - \lambda 2$ . Therefore the minimal system  $\mathcal{J}$  belongs to is  $\lambda P 2$ .

### Problem

(6.4 a) Let  $\Gamma \equiv S : *, Q : S \rightarrow S \rightarrow *$  and

$$M \equiv (\Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp)) \rightarrow \Pi z : S . (Q z z \rightarrow \perp)$$

Derive  $\Gamma \vdash M : *$  and determine the smallest subsystem to which this judgement belongs.

*Solution.*

1.	$S : *, Q : S \rightarrow S \rightarrow *$	
2.	$x : S$	
3.	$y : S$	
4.	$Q x : S \rightarrow *$	<b>1,2 App</b>
5.	$Q x y : *$	<b>4,3 App</b>
6.	$z : Q x y$	
7.	$Q y : S \rightarrow *$	<b>1,3 App</b>
8.	$Q y x : *$	<b>7,2 App</b>
9.	$t : Q y x$	
10.	$\perp : *$	<b>Weak from 6.1 a</b>
11.	$Q y x \rightarrow \perp : *$	<b>8,10 Form</b>
12.	$Q x y \rightarrow Q y x \rightarrow \perp : *$	<b>5,11 Form</b>
13.	$\Pi y : S . Q x y \rightarrow Q y x \rightarrow \perp : *$	<b>1,12 Form</b>
14.	$\Pi x, y : S . Q x y \rightarrow Q y x \rightarrow \perp : *$	<b>1,13 Form</b>
15.	$a : (\Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp))$	
16.	$z : S$	
17.	$Q z : S \rightarrow *$	<b>1,16 App</b>
18.	$Q z z : *$	<b>17,16 App</b>
19.	$b : Q z z$	
20.	$\perp : *$	<b>Weak from 6.1 a</b>
21.	$Q z z \rightarrow \perp : *$	<b>18,20 Form</b>
22.	$\Pi z : S . Q z z \rightarrow \perp : *$	<b>1,21 Form</b>
23.	$\Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp) \rightarrow \Pi z : S . Q z z \rightarrow \perp : *$	<b>14,22 Form</b>

Here's a table of all  $\Pi$ s that appeared

Abstraction	Line Number	$(s_1, s_2)$
$S \rightarrow *$	1 / 4 / 7 / 17	$(*, \square)$
$S \rightarrow S \rightarrow *$	1	$(*, \square)$
$\perp$	10 / 11 / 12 / 13 / 14 / 15 / 20 / 21 / 22 / 23	$(\square, *)$
$Q y x \rightarrow \perp$	11 / 12 / 13 / 14 / 15 / 23	$(*, *)$
$Q x y \rightarrow Q y x \rightarrow \perp$	12 / 13 / 14 / 15 / 23	$(*, *)$
$\Pi y : S . Q x y \rightarrow Q y x \rightarrow \perp$	13 / 14 / 23	$(*, *)$

$\Pi x, y : S . Q x y \rightarrow Q y x \rightarrow \perp$	14 / 23	$(*, *)$
$Q z z \rightarrow \perp$	21 / 22 / 23	$(*, *)$
$\Pi z : S . Q z z \rightarrow \perp$	22 / 23	$(*, *)$
$\Pi x, y : S . Q x y \rightarrow Q y x \rightarrow \perp \rightarrow$ $\Pi z : S . Q z z \rightarrow \perp$	23	$(*, *)$

There are  $(*, *) - \lambda \rightarrow$  pairs,  $(*, \square) - \lambda P$  pairs, and  $(\square, *) - \lambda 2$  pairs. Therefore the minimal system available is  $\lambda P 2$ .

### Problem

(6.4 b) Prove in  $\lambda C$  that  $M$  is inhabited in context  $\Gamma$ .

*Solution.* A shorthand derivation is given below:

*Proof.*

1.  $S : *, Q : S \rightarrow S \rightarrow *$
2.  $h : \Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp)$
3.  $z : S$
4.  $a : Q z z$
5.  $\alpha : *$
6.  $h z : \Pi y : S . (Q z y \rightarrow Q y z \rightarrow \perp)$  2,3 App
7.  $h z z : Q z z \rightarrow Q z z \rightarrow \perp$  6,3 App
8.  $h z z a : Q z z \rightarrow \perp$  7,4 App
9.  $h z z a a : \Pi \alpha : * . \alpha$  8,4 App
10.  $h z z a a \alpha : \alpha$  9,5 App
11.  $\lambda \alpha : * . h z z a a \alpha : \Pi \alpha : * . \alpha$  10 Abst
12.  $\lambda a : Q z z \lambda \alpha : * . h z z a a \alpha : Q z z \rightarrow \perp$  11 Abst
13.  $\lambda z : S . \lambda a : Q z z \lambda \alpha : * . h z z a a \alpha : \Pi z : S . Q z z \rightarrow \perp$  12 Abst
14.  $\lambda h : \Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp)$   
 $\lambda z : S . \lambda a : Q z z \lambda \alpha : * . h z z a a \alpha$   
 $: \Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp) \rightarrow \Pi z : S . Q z z \rightarrow \perp$  13 Abst

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### Problem

(6.4 c) We may consider  $Q$  to be a relation on set  $S$ . Moreover by PAT we may see  $A \rightarrow \perp$  as the negation  $\neg A$  of prop  $A$ . How can  $M$  then be interpreted by the PAT paradigm?

*Solution.* By a direct type-to-proposition translation we have

$$M \equiv \forall x, y \in S, (Q(x, y) \Rightarrow \neg Q(y, x)) \Rightarrow \forall z \in S, (\neg Q(z, z))$$

It expresses the fact if  $Q$  is asymmetric then it is irreflexive.

### Problem

(5.6 a) Let

$$\mathcal{J} \equiv S : * \vdash \lambda Q : S \rightarrow S \rightarrow *. \lambda x : S. Q x x : (S \rightarrow S \rightarrow *) \rightarrow S \rightarrow *$$

Give a shorthand derivation of  $\mathcal{J}$  and determine the smallest subsystem to which  $\mathcal{J}$  belongs.

*Solution.*

1.	$S : *$	
2.	$Q : S \rightarrow S \rightarrow *$	
3.	$x : S$	
4.	$Q x : S \rightarrow *$	2,3 App
5.	$Q x x : *$	4,3 App
6.	$\lambda x : S. Q x x : S \rightarrow *$	5 Abst
7.	$\lambda Q : S \rightarrow S \rightarrow *. \lambda x : S. Q x x : (S \rightarrow S \rightarrow *) \rightarrow S \rightarrow *$	6 Abst

Abstraction	Line Number	$(s_1, s_2)$
$S \rightarrow *$	2 / 4 / 6 / 7	$(*, \square)$
$S \rightarrow S \rightarrow *$	2 / 7	$(*, \square)$
$(S \rightarrow S \rightarrow *) \rightarrow (S \rightarrow *)$	7	$(\square, \square)$

The judgement contains  $(*, \square) - \lambda P$  pairs and  $(\square, \square) - \lambda \omega$  pairs. Therefore the minimal system  $\mathcal{J}$  belongs to is  $\lambda P \underline{\omega}$ .

### Problem

(6.5 b) In  $\mathcal{J}$  of 6.5 a, we may consider the variable  $Q$  as expressing a relation on set  $S$ . How could you describe the subexpression  $\lambda x : S . Q\ x\ x$  in this settings? And what is then the interpretation of the judgement  $\mathcal{J}$ ?

*Solution.* By a informal translation, the term meant “Given a relation  $Q$  over set  $S$  and an arbitrary element of  $S$ , return whether if  $Q\ (x, x)$  holds”.