Exercises

Chapter 9

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1. Redacted

Definition Extended Rules for λD_0

$$\frac{\Delta; \Gamma \vdash K : L \qquad \Delta; \overline{a} : \overline{M} \vdash M : N}{\Delta, \left(\overline{x} : \overline{A} \rhd a\left(\overline{x}\right) := M : N\right); \Gamma \vdash K : L}\,\text{def}$$

$$\frac{\Delta, \mathcal{D}; \Gamma \vdash * : \square \qquad \Delta, \mathcal{D}; \Gamma \vdash \overline{U} : \overline{A\left[\overline{x} := \overline{U}\right]} \qquad \mathcal{D} \equiv \overline{x} : \overline{A} \rhd a\left(\overline{x}\right) := M : N}{\Delta, \mathcal{D}; \Gamma \vdash a\left(\overline{U}\right) : N\left[\overline{x} := \overline{U}\right]}\,\text{inst}$$

$$\frac{\Delta; \Gamma \vdash x : A \qquad \Delta; \Gamma \vdash A : s \qquad A \overset{\Delta,\beta}{=} B}{\Delta; \Gamma \vdash x : B}\,\text{conv}$$

Lemma 1. Given $\mathcal{D} \equiv \overline{x} : \overline{A} \rhd a\left(\overline{x}\right) := M : N$ and $a \notin \Delta$

$$\frac{\Delta; \overline{x} : \overline{A} \vdash M : N}{\Delta, \mathcal{D}; \overline{x} : \overline{A} \vdash a\left(\overline{x}\right) : N}\,\text{par}$$

Problem

(9.1) Given

$$
\begin{array}{llcll}
(\mathcal{D}_1) & x:\mathbb{Z}, y:\mathbb{Z} & \rhd & a\,(x,y) & := x^2+y^2 \quad : \mathbb{Z} \\
(\mathcal{D}_2) & x:\mathbb{Z}, y:\mathbb{Z} & \rhd & b\,(x,y) & := 2\cdot(x\cdot y) \quad : \mathbb{Z} \\
(\mathcal{D}_3) & x:\mathbb{Z}, y:\mathbb{Z} & \rhd & c\,(x,y) & := a\,(x,y)+b\,(x,y) \quad : \mathbb{Z} \\
(\mathcal{D}_4) & x:\mathbb{Z}, y:\mathbb{Z} & \rhd & \text{lemma}(x,y) & := c\,(x,y)=(x+y)^2 : *_p
\end{array}
$$

Consider $\Delta \equiv \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$. Describe the dependencies between the four definitions and give all possible linearizations of the corresponding partial order.

Solution. Hasse diagram given below

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\boldsymbol{\mathcal{D}_1} \\ a := x^2+y^2 : \mathbb{Z}\end{array}} \arrow[r] & \boxed{\begin{array}{c}\boldsymbol{\mathcal{D}_3} \\ c := a\,(x,y)+b\,(x,y) : \mathbb{Z}\end{array}} \arrow[d] & \boxed{\begin{array}{c}\boldsymbol{\mathcal{D}_2} \\ b := 2\cdot(x\cdot y) : \mathbb{Z}\end{array}} \arrow[l] \\
& \boxed{\begin{array}{c}\boldsymbol{\mathcal{D}_4} \\ \text{lemma} := c\,(x,y)=(x+y)^2 : *_p\end{array}} &
\end{tikzcd}
$$

The only incomparable pair is $(\mathcal{D}_1, \mathcal{D}_2)$. Therefore there are two possible linearizations:

$$
\begin{array}{ll}
(1) & \mathcal{D}_1 \leq \mathcal{D}_2 \leq \mathcal{D}_3 \leq \mathcal{D}_4 \\
(2) & \mathcal{D}_2 \leq \mathcal{D}_1 \leq \mathcal{D}_3 \leq \mathcal{D}_4
\end{array}
$$

Problem

(9.2) Consider

$$\mathcal{D}_i \equiv \overline{x} : \overline{A} \rhd a\,(\overline{x}) := K : L$$
$$\mathcal{D}_j \equiv \overline{y} : \overline{B} \rhd b\,(\overline{y}) := M : N$$

Let $\Delta; \Gamma \vdash U : V$ and assume $\mathcal{D}_i$ and $\mathcal{D}_j$ are elements of list Δ, where $\mathcal{D}_i$ precedes $\mathcal{D}_j$. Describe precisely where constant a may occur in $\mathcal{D}_i$ and $\mathcal{D}_j$ and where constant b may occur in Δ.

Solution. In order for $\mathcal{D}_i$ to be a valid definition, $\overline{x} : \overline{A} \vdash K : L$ must be legal. Therefore by the free variable lemma any free variables in K and L must be in $\overline{x} : \overline{A}$, which by the time, does not yet contain a's definition. Therefore, a could only appear in $\mathcal{D}_j$.

By similar reasoning b could only have appeared in definitions after $\mathcal{D}_j$. Assuming the list sorted by the suffix, then b could only have been in any $\mathcal{D}_k$ where $k > j$.