Excercises

Chapter 4

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1. Reducted

Definition Some rules for reference.

$$\frac{}{\emptyset \vdash * : \Box}\text{Sort} \qquad \frac{\Gamma \vdash A : s \quad x \notin \text{dom}\,\Gamma}{\Gamma, x : A \vdash x : A}\text{Var}$$

$$\frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s \quad x \notin \text{dom}\,\Gamma}{\Gamma, x : C \vdash A : B}\text{Weak} \qquad \frac{\Gamma \vdash A : s \quad \Gamma \vdash B : s}{\Gamma \vdash A \to B : s}\text{Form}$$

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B}\text{App}$$

$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash A \to B : s}{\Gamma \vdash \lambda x : A \,.\, M : A \to B}\text{Abst}$$

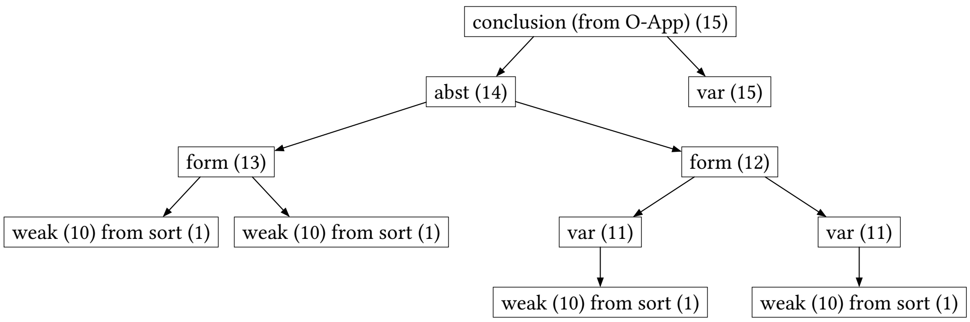
$$\frac{\Gamma \vdash A : B \quad \Gamma \vdash B' : s \quad B \overset{\beta}{=} B'}{\Gamma \vdash A : B'}\text{Conv}$$

Previously an alternative version of the flag derivation was used, only putting up a flag for a local premise (abstraction unwrapping) to save horizontal space. Currently, the standard flag derivation format will be used since now single lines will not be as long.

Problem

(4.1) Give a complete tree diagram of the derivation in section 4.5 (95)

Solution.



Problem

(4.2 a) Give a complete $\lambda\underline{\omega}$ derivation in flag format of

$$\emptyset \vdash (* \to *) \to * : \square$$

Solution.

1. $* : \square$ **Sort**
2. $* \to * : \square$ **1,1 Form**
3. $(* \to *) \to * : \square$ **2,1 Form**

Problem

(4.2 b) Give a complete $\lambda\underline{\omega}$ derivation in flag format of

$$\alpha : *, \beta : * \vdash (\alpha \to \beta) \to \alpha : *$$

Solution.

1.	$\emptyset \vdash * : \square$	**Sort**
2.	$\alpha : *$	
3.	$\quad \alpha : *$	**1 Var**
4.	$\quad * : \square$	**1,1 Weak**
5.	$\quad \beta : *$	
6.	$\quad\quad \alpha : *$	**3,4 Weak**
7.	$\quad\quad \beta : *$	**4 Var**
8.	$\quad\quad \alpha \to \beta : *$	**6,7 Form**
9.	$\quad\quad (\alpha \to \beta) \to \alpha : *$	**8,6 Form**

Problem

(4.3 a) Give a complete $\lambda\underline{\omega}$ derivation in flag format of

$$\alpha, \beta : *, x : \alpha, y : \alpha \to \beta \vdash y\, x : \beta$$

Solution.

1.	$* : \square$	**Sort**
2.	$\alpha : *$	
3.	$\quad \alpha : *$	**1 Var**
4.	$\quad * : \square$	**1,1 Weak**
5.	$\quad \beta : *$	
6.	$\quad\quad \beta : *$	**4 Var**
7.	$\quad\quad \alpha : *$	**3,4 Weak**
8.	$\quad\quad * : \square$	**4,4 Weak**
9.	$\quad\quad x : \alpha$	
10.	$\quad\quad\quad x : \alpha$	**7 Var**
11.	$\quad\quad\quad \alpha : *$	**7,7 Weak**
12.	$\quad\quad\quad \beta : *$	**6,7 Weak**
13.	$\quad\quad\quad \alpha \to \beta : *$	**11,12 Form**
14.	$\quad\quad\quad y : \alpha \to \beta$	
15.	$\quad\quad\quad\quad y : \alpha \to \beta$	**13 Var**
16.	$\quad\quad\quad\quad x : \alpha$	**10,13 Weak**
17.	$\quad\quad\quad\quad y\, x : \beta$	**15,16 App**

Problem

(4.3 b) Give a shortened $\lambda\underline{\omega}$ derivation in flag format of

$$\alpha, \beta : *, x : \alpha, y : \alpha \to \beta, z : \beta \to \alpha \vdash z\,(y\,x) : \alpha$$

Solution.

1. $\alpha : *$
2. $\quad \beta : *$
3. $\quad\quad x : \alpha$
4. $\quad\quad\quad y : \alpha \to \beta$
5. $\quad\quad\quad\quad x : \alpha$ **3 Weak**
6. $\quad\quad\quad\quad z : \beta \to \alpha$
7. $\quad\quad\quad\quad\quad x : \alpha$ **5 Weak**
8. $\quad\quad\quad\quad\quad y : \alpha \to \beta$ **4 Weak**
9. $\quad\quad\quad\quad\quad y\,x : \beta$ **8,7 App**
10. $\quad\quad\quad\quad\quad z\,(y\,x) : \alpha$ **6,9 App**

Problem

(4.4 a) Give a shortened $\lambda\underline{\omega}$ derivation in flag format of

$$\alpha : *, \beta : * \to * \vdash \beta(\beta\alpha) : *$$

Solution.

1. $\alpha : *$
2. $\quad \beta : * \to *$
3. $\quad\quad \beta\alpha : *$ **2,1 App**
4. $\quad\quad \beta(\beta\alpha) : *$ **2,4 App**

Problem

(4.4 b) Give a shortened $\lambda\underline{\omega}$ derivation in flag format of

$$\alpha : *, \beta : * \to *, x : \beta(\beta\alpha) \vdash \lambda y : \alpha.\, x : \alpha \to \beta(\beta\alpha)$$

Solution.

1. $\alpha : *$
2. $\beta : * \to *$
3. $x : \beta(\beta\alpha)$
4. $y : \alpha$
5. $x : \beta(\beta\alpha)$ **3 Var**
6. $\lambda y : \alpha .\, x : \alpha \to \beta(\beta\alpha)$ **5 Abst**

Problem

(4.4 c) Give a shortened $\lambda\underline{\omega}$ derivation in flag format of

$$\emptyset \vdash \lambda\alpha : * .\lambda\beta : * \to * .\beta(\beta\alpha) : * \to (* \to *) \to *$$

Solution.

1. $\alpha : *$
2. $\beta : * \to *$
3. $\beta\alpha : *$ **2,1 App**
4. $\beta(\beta\alpha) : *$ **2,3 App**
5. $\lambda\beta : * \to * .\beta(\beta\alpha) : (* \to *) \to *$ **4 Abst**
6. $\lambda\alpha : * .\lambda\beta : * \to * .\beta(\beta\alpha) : * \to (* \to *) \to *$ **5 Abst**

Problem

(4.4 d) Give a shortened $\lambda\underline{\omega}$ derivation in flag format of

$$\texttt{nat} : * \vdash (\lambda\alpha : * .\lambda\beta : * \to * .\beta(\beta\alpha))\ \texttt{nat}\ (\lambda\gamma : * .\gamma) : *$$

Solution.

1. $\texttt{nat} : *$
2. $\alpha : *$
3. $\beta : * \to *$
4. $\beta\alpha : *$ **3,2 App**
5. $\beta(\beta\alpha) : *$ **3,4 App**
6. $\lambda\beta : * \to * .\beta(\beta\alpha) : (* \to *) \to *$ **5 Abst**

7.	$\lambda\alpha : * .\lambda\beta : * \to * .\beta(\beta\alpha) : * \to (* \to *) \to *$	**6 Abst**
8.	$(\lambda\alpha : * .\lambda\beta : * \to * .\beta(\beta\alpha))\ \mathtt{nat} : (* \to *) \to *$	**7,1 App**
9.	$\gamma : *$	
10.	$\gamma : *$	**9 Var**
11.	$\lambda\gamma : * .\gamma : * \to *$	**10 Abst**
12.	$(\lambda\alpha : * .\lambda\beta : * \to * .\beta(\beta\alpha))\ \mathtt{nat}\ (\lambda\gamma : * .\gamma) : *$	**8,11 App**

Problem

(4.5) Give a shortened $\lambda\underline{\omega}$ derivation in flag format of

$$\alpha : * .\ x : \alpha \vdash \lambda y : \alpha.\ x : (\lambda\beta : * .\beta \to \beta)\alpha$$

Solution.

1.	$\alpha : *$	
2.	$x : \alpha$	
3.	$y : \alpha$	
4.	$x : \alpha$	**2 Weak**
5.	$\lambda y : \alpha.\ x : \alpha \to \alpha$	**4 Abst**
6.	$\beta : *$	
7.	$\beta \to \beta : *$	**6,6 Form**
8.	$\lambda\beta : * .\beta \to \beta : * \to *$	**7 Abst**
9.	$(\lambda\beta : * .\beta \to \beta)\alpha : *$	**8,1 App**
10.	$\lambda y : \alpha.\ x : (\lambda\beta : * .\beta \to \beta)\alpha$	**5,9 Conv**

Problem

(4.6 a) Show that no such context Γ and term M in $\lambda\underline{\omega}$ such that

$$\Gamma \vdash \Box : M$$

is derivable.

Solution. Proof by induction on inference rules. Rules like Sort, Var, Form, Abst, App has syntatically or semantic different conclusions than $\Box : M$.

Case 1 : Rule Weak. Let $\Gamma', C : s \equiv \Gamma$. Therefore this derivation requries a premise $\Gamma' \vdash \Box : M$. By the inductive hypothesis this is impossible. ■

Case 2 : Rule Conv. This derivation requires a premise $\Gamma \vdash \square : M'$ such that $M \underset{\beta}{=} M'$. By the inductive hypothesis this is impossible. ■

By the principle of induction this proves that there's no derivation that could give $\Gamma \vdash \square : M$.

Problem

(4.6 b) Prove there are no such context Γ and terms M and N in $\lambda\underline{\omega}$ such that

$$\Gamma \vdash M \to \square : N$$

Solution. Proof by induction on inference rules. Rules like Sort, Var, Abst, App has syntatically or semantically conclusions than $M \to \square : N$.

Case 1 : Rule Weak. Let $\Gamma', C : s \equiv \Gamma$. The derivation requires a premise $\Gamma' \vdash M \to \square : N$. By the inductive hypothesis this is impossible. ■

Case 2 : Rule Form. This requires a derivation with premise $\Gamma \vdash \square : N$, which by 4.6 a is not possible. ■

Case 3 : Rule Conv. This requires a premise $\Gamma \vdash M \to \square : N'$ such that $N \equiv N'$. By the inductive hypothesis this is impossible. ■

By the principle of induction this proves there's no derivation that could give $\Gamma \vdash M \to \square : N$.

Problem

(4.7 a) Give $\lambda\underline{\omega}$ definition of the notion legal term, $\lambda\underline{\omega}$ context and domain.

Solution.

Definition *Legal Term* are typable terms. That is, a term M is legal iff there exists a context Γ and a legal higher-sorted term α under Γ such that $\Gamma \vdash M : \alpha$.

Definition $\lambda\underline{\omega}$ *Context.*

1. $\emptyset$ is a $\lambda\underline{\omega}$ context.
2. When Γ is a valid $\lambda\underline{\omega}$ context, α is valid under Γ, and type of x is α, and $x \notin \operatorname{dom} \Gamma$, then the context $\Gamma, x : \alpha$ is valid in $\lambda\underline{\omega}$.

Definition *Domain*

1. $\mathrm{dom}\ \emptyset = \{\}$
2. $\mathrm{dom}\ \Gamma, x : s = \mathrm{dom}\ \Gamma \cup \{x\}$

Problem

(4.7 b) Formulate the Free Variables Lemma, Thinning Lemma, and Subtitution Lemma for $\lambda\underline{\omega}$.

Solution.

Lemma 1. **FV Lemma ($\lambda\underline{\omega}$).** For any legal term M under Γ, $\mathrm{FV}\ M \subseteq \mathrm{dom}\ \Gamma$.

More specifically,

$$\begin{aligned} \forall\ M, \alpha \in \Lambda_{\underline{\omega}}, \Gamma \vdash \alpha : s, \Gamma \vdash M : \alpha &\implies \mathrm{FV}\ M \quad\ \ \subseteq \mathrm{dom}\ \Gamma \\ M \equiv \square &\implies \mathrm{FV}\ M \equiv \emptyset \subseteq \mathrm{dom}\ \Gamma \end{aligned}$$

Lemma 2. **Thinning Lemma ($\lambda\underline{\omega}$).** For any legal term M in Γ' and $\Gamma' \subseteq \Gamma$, M is legal under Γ.

Lemma 3. **Substitution Lemma ($\lambda\underline{\omega}$).** Assume term $\kappa : s$ under context Γ'. Under another context Γ'' given a term $\Gamma'' \vdash N : \kappa$ and another context Γ such that $\Gamma, x : \kappa, \Gamma' \vdash M : A$ for some type $A : s$ under Γ. Then

$$\Gamma, \Gamma', \Gamma'' \vdash M\,[x := N] : A$$

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Completed Dec 20 1:23 am.