

# EXERCISES

## CHAPTER 1

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### 1. Reduced

#### Problem

(1.1) Simplify notation of the following terms

(a)  $(\lambda x . ((x z) y)(x x))$

(b)  $((\lambda x . (\lambda y . (\lambda z . (z ((x y) z)))))(\lambda u . u))$

*Solution.*

(a)  $\lambda x . (x z y)(x x)$

(b)  $(\lambda x y z . x (x y z))(\lambda u . u)$

#### Problem

(1.2) Find the alpha equivalent terms to

$$\lambda x . x (\lambda x . x)$$

In

(a)  $\lambda y . y (\lambda x . x)$

(b)  $\lambda y . y (\lambda x . y)$

(c)  $\lambda y . y (\lambda x . y)$

*Solution.* Only (a).

### Problem

(1.3) Prove

$$\lambda x . x (\lambda z . y) \underset{\alpha}{=} \lambda z . z (\lambda z . y)$$

*Solution.*

*Proof.* By definition of alpha equivalence

$$M \underset{\alpha}{=} N \iff \exists \varphi, M^\varphi \underset{\alpha}{\rightarrow} N \wedge \text{FR } M = \text{FR } N$$

The witness of  $\varphi$  is substituting bound variable  $x$  with  $z$ , and  $z$  is not a free variable in the term, thus the two terms are alpha equivalent.

$$\lambda x . x (\lambda z . y) \xrightarrow[\alpha]{x \rightarrow z} \lambda z . z (\lambda z . y)$$

■

### Problem

(1.4) Consider the following term:

$$U := (\lambda z . z x z)((\lambda y . x y) x)$$

1. Find Sub  $U$
2. Draw tree rep of  $U$
3. Find FV  $U$
4. Find alpha equivalent terms to  $U$  from below and point out which of those follows the Barendregt convention:

(a)  $(\lambda y . y x y)((\lambda z . x z) x)$

(b)  $(\lambda x . x y x)((\lambda z . y z) y)$

(c)  $(\lambda y . y x y)((\lambda y . x y) x)$

(d)  $(\lambda v . (v x) v)((\lambda u . u v) x)$

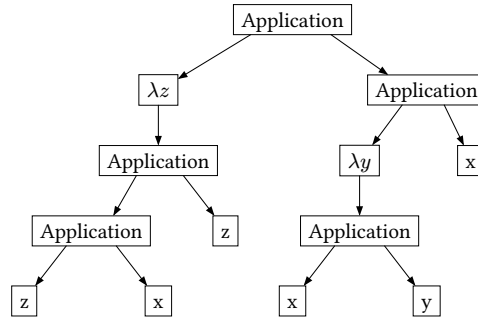
1. Find Sub  $U$ .

*Solution.*

$$\begin{aligned}
\text{Sub } U &= \\
&\{(\lambda z . z x z)((\lambda y . x y) x), (\lambda z . z x z), ((\lambda y . x y) x)\} \cup \\
&\{(\lambda y . x y), (y), (\lambda z . x z), (x)\} \cup \\
&\{(\lambda y . x), (y)\} \cup \{(\lambda z . x), (z)\} \cup \{(y), (x)\} \\
&= \{(\lambda z . z x z)(\lambda y . x y) x, (\lambda z . z x z), (\lambda y . x y) x, \\
&\quad (\lambda y . x y), (\lambda z . x z), (\lambda y . x), (\lambda z . x), y, x\}
\end{aligned}$$

2. Draw a tree rep of  $U$ .

*Solution.*



3. Find  $\text{FV } U$

*Solution.*

$$\begin{aligned}
\text{FV } U &= \text{FV } (\lambda y . y x y) \cup \text{FV } (\lambda z . x z) x \\
&= (\text{FV } y x y) \setminus \{y\} \cup (\text{FV } \lambda z . x z) \cup \{x\} \\
&= (\text{FV } y x) \setminus \{y\} \cup (\text{FV } x z) \setminus \{z\} \cup \{x\} \\
&= \{x\}
\end{aligned}$$

4. Find an alpha-equivalent term.

*Solution.*

$$(a) \stackrel{=}{\alpha} (c) \stackrel{=}{\alpha} U$$

Only (a) follows the Barendregt convention.

### Problem

(1.5) Give the results of the following substitutions

- (a)  $(\lambda x . y (\lambda y . x y))[y := \lambda z . z x]$
- (b)  $((x y z)[x := y])[y := z]$
- (c)  $((\lambda x . x y z)[x := y])[y := z]$
- (d)  $(\lambda y . y y x)[x := y z]$

*Solution.*

- (a)  $(\lambda v . (\lambda z . z x)(\lambda u . v u))$
- (b)  $(y y z)[y := z] = z z z$
- (c)  $(\lambda x . x y z)[y := z] = (\lambda x . x z z)$
- (d)  $(\lambda u . u u (y z))$

### Problem

(1.6)

$$\neg \left( \forall M L N \in \Lambda, M [x := N, y := L] \equiv_{\alpha} M [x := N][y := L] \right)$$

*Solution.*

*Proof.* Because  $\text{RHS} = M [x := N][y := L] = M [x := N [y := L]][y := L]$ , if  $y \in \text{FV } N$ , then what  $x$  gets substituted with will have  $y$  substituted for  $L$ , which is completely different with LHS. ■

### Problem

(1.7) Find all available redexes in

$$U := (\lambda z . z x z)((\lambda y . x y) x)$$

And all reduction pathes to the  $\beta$ -normal form.

*Solution.* The first redex is the term as an application itself; another the second term in the application.



### Problem

(1.8) Show that

$$(\lambda x . x x) y \not\equiv_{\beta} (\lambda x y . y x) x x$$

*Solution.* By Corollary 1.9.9, it suffices to prove the hypothesis with a proof of a common normal reduced form from LHS and RHS not existing.

*Contradiction.* By definition of  $\equiv_{\beta}$ , there exists The set of all terms attainable from  $\beta$ -reduction on  $(\lambda x . x x) y$  and  $(\lambda x y . y x) x x$  do not intersect. Therefore,

$$\neg \left( \exists L \in \Lambda, (\lambda x . x x) y \rightarrow_{\beta} L \wedge (\lambda x y . y x) x x \rightarrow_{\beta} L \right) \implies \neg \left( (\lambda x . x x) y \equiv_{\beta} (\lambda x y . y x) x x \right)$$

■

### Problem

(1.9) Define the combinators

$$K := \lambda x y . x$$

$$S := \lambda x y z . x z (y z)$$

Prove that

$$\forall P Q \in \Lambda, K P Q \rightarrow_{\beta} P$$

$$\forall P Q R \in \Lambda, S P Q R \rightarrow_{\beta} P R (Q R)$$

*Solution.*

*Proof.*

$$K P Q = (\lambda x y . x) P Q \rightarrow_{\beta} (\lambda y . x)[x := P] Q \rightarrow_{\beta} P [y := Q] = P$$

$$S P Q R = (\lambda x y z . x z (y z)) \rightarrow_{\beta} (x z (y z))[x := P][y := Q][z := R] = P R (Q R)$$

■

## Problem

(1.10) We define the church numerals

$$\begin{aligned} \text{zero} &:= \lambda f x . x \\ \text{one} &:= \lambda f x . f x \\ \text{two} &:= \lambda f x . f f x \\ &\dots \\ \text{num}_n &:= \lambda f x . f^n x \end{aligned}$$

And operations

$$\begin{aligned} \text{add} &:= \lambda n m f x . m f (n f x) \\ \text{mul} &:= \lambda n m f x . m (n f) x \end{aligned}$$

Show

- (a)  $\text{add one one} \xrightarrow[\beta]{} \text{two}$
- (b)  $\text{add one one} \not\xrightarrow[\beta]{} \text{mul one zero}$

*Solution.*

$$\begin{aligned} \text{(a) } \text{add one one} &= (\lambda n m f x . m f (n f x))(\lambda f x . f x)(\lambda f x . f x) \\ &\xrightarrow[\beta]{} (\lambda f x . (\lambda f x . f x) f ((\lambda f x . f x) f x)) \\ &\xrightarrow[\beta]{} (\lambda f x . (\lambda x . f x) f x) \\ &\xrightarrow[\beta]{} (\lambda f x . f f x) = \text{two} \end{aligned}$$

$$\begin{aligned} \text{(b) } \text{mul one one} &= (\lambda n m f x . m (n f) x)(\lambda f x . f x)(\lambda f x . f x) \\ &\xrightarrow[\beta]{} \lambda f x . (\lambda f x . f x)((\lambda f x . f x) f) x \\ &\xrightarrow[\beta]{} \lambda f x . f x = \text{one} \end{aligned}$$

Because no intermediate form in the beta reduction process of the two terms are  $\alpha$ -equivalent, by corollary 1.9.9 the two terms are not  $\beta$ -equivalent.

### Problem

(1.11) We define

$$\text{succ} := \lambda m f x . f (m f x) \text{ s.t. } \forall \text{num}_n, \text{succ num}_n = \text{num}_{n+1}$$

Prove

$$\text{succ zero} \stackrel{\beta}{=} \text{one}$$

$$\text{succ one} \stackrel{\beta}{=} \text{two}$$

*Solution.* It suffices to provide a witness of a reduction chain from one side to the other to prove  $\beta$ -equivalence.

*Proof.*

$$\begin{aligned} \text{succ zero} &= (\lambda m f x . f (m f x))(\lambda f x . x) \\ &\rightarrow_{\beta} (\lambda f x . f ((\lambda f x . x) f x)) \\ &\rightarrow_{\beta} (\lambda f x . f x) = \text{one} \end{aligned}$$

The path  $\text{succ zero} \rightarrow_{\beta}^* \text{one}$  derived above is the witness of a reduction chain from LHS to RHS.

$$\begin{aligned} \text{succ one} &= (\lambda m f x . f (m f x))(\lambda f x . f x) \\ &\rightarrow_{\beta} (\lambda f x . f ((\lambda f x . f x) f x)) \\ &\rightarrow_{\beta} (\lambda f x . f (f x)) = \text{two} \end{aligned}$$

The path  $\text{succ one} \rightarrow_{\beta}^* \text{two}$  derived above is the witness of a reduction chain from LHS to RHS. ■

### Problem

(1.12) We define the  $\lambda$ -terms  $\top_{\lambda}$  (true) and  $\perp_{\lambda}$  (false) and  $\neg_{\lambda}$  (not) by:

$$\begin{aligned} \top_{\lambda} &:= \lambda x y . x & \perp_{\lambda} &:= \lambda x y . y \\ \neg_{\lambda} &:= \lambda a . a \perp_{\lambda} \top_{\lambda} \end{aligned}$$

Show that

$$\neg_{\lambda}(\neg_{\lambda} \top_{\lambda}) \stackrel{\beta}{=} \top_{\lambda}$$

$$\neg_{\lambda}(\neg_{\lambda} \perp_{\lambda}) \stackrel{\beta}{=} \perp_{\lambda}$$

*Solution.* It suffices to provide a witness of a reduction chain from one side to the other to prove  $\beta$ -equivalence.

*Proof.*

$$\begin{aligned}
\neg_\lambda(\neg_\lambda \top_\lambda) &= \neg_\lambda((\lambda a . a \perp_\lambda \top_\lambda)(\lambda x y . x)) \\
&\xrightarrow[\beta]{\rightarrow} \neg_\lambda((\lambda x y . x) \perp_\lambda \top_\lambda) \\
&\xrightarrow[\beta]{\rightarrow} (\lambda a . a \perp_\lambda \top_\lambda) \perp_\lambda \\
&\xrightarrow[\beta]{\rightarrow} (\lambda x y . y) \perp_\lambda \top_\lambda \\
&\xrightarrow[\beta]{\rightarrow} \top_\lambda
\end{aligned}$$

$$\begin{aligned}
\neg_\lambda(\neg_\lambda \perp_\lambda) &= \neg_\lambda((\lambda a . a \perp_\lambda \top_\lambda)(\lambda x y . y)) \\
&\xrightarrow[\beta]{\rightarrow} \neg_\lambda((\lambda x y . x) \perp_\lambda \top_\lambda) \\
&\xrightarrow[\beta]{\rightarrow} (\lambda a . a \perp_\lambda \top_\lambda) \top_\lambda \\
&\xrightarrow[\beta]{\rightarrow} (\lambda x y . x) \perp_\lambda \top_\lambda \\
&\xrightarrow[\beta]{\rightarrow} \perp_\lambda
\end{aligned}$$

■

### Problem

(1.13) Define

$$\text{iszero} := \lambda m . m (\lambda x . \perp_\lambda) \top_\lambda$$

Prove

$$\begin{aligned}
&\text{iszero zero} \xrightarrow[\beta]{\rightarrow} \top_\lambda \\
&\forall n \in \mathbb{N}^+, \text{iszero num}_n \xrightarrow[\beta]{\rightarrow} \perp_\lambda
\end{aligned}$$

*Solution.*

$$\begin{aligned}
\text{iszero zero} &= (\lambda m . m (\lambda x . \perp_\lambda) \top_\lambda)(\lambda f x . x) \\
&\xrightarrow[\beta]{\rightarrow} (\lambda f x . x)(\lambda x . \perp_\lambda) \top_\lambda \\
&\xrightarrow[\beta]{\rightarrow} \top_\lambda
\end{aligned}$$



$$\begin{aligned}
\text{iszero num}_n &= (\lambda m . m (\lambda x . \perp_\lambda) \top_\lambda) (\lambda f x . f^n x) \\
&\xrightarrow[\beta]{} (\lambda f x . f^n x) (\lambda x . \perp_\lambda) \top_\lambda \\
&\xrightarrow[\beta]{} (\lambda x . \perp_\lambda) ((\lambda x . \perp_\lambda)^{n-1} \top_\lambda) \xrightarrow[\beta]{} \perp_\lambda
\end{aligned}$$

### Problem

(1.14) If-else can be modeled as

$$\text{ifelse} = \lambda x t f . x t f$$

Where when  $x$ , then  $t$ , else  $f$ . Prove correctness by applying  $\top_\lambda$  and  $\perp_\lambda$  on ifelse.

*Solution.*

$$\begin{aligned}
\text{ifelse } \top_\lambda &= (\lambda x t f . x t f) \top_\lambda \\
&\xrightarrow[\beta]{} (\lambda t f . (\lambda x y . x) t f) \xrightarrow[\beta]{} (\lambda t f . t) \\
\text{ifelse } \perp_\lambda &= (\lambda x t f . x t f) \perp_\lambda \\
&\xrightarrow[\beta]{} (\lambda t f . (\lambda x y . y) t f) \xrightarrow[\beta]{} (\lambda t f . f)
\end{aligned}$$

By applying the results to any two values, the correct corresponding value returns, ex, for ifelse  $\top_\lambda$ ,  $t$  is always returned.

### Problem

(1.15) Prove that  $\Omega := (\lambda x . x x)(\lambda x . x x)$  does not have a  $\beta$ -nf.

*Solution.* Firstly let's prove  $\Omega$ .

*Proof.* Induction on  $\Omega$ 's only reduction path proves that every  $\Omega \xrightarrow[\beta]{} \Omega_i = \Omega$ . For the base case because  $\Omega$  has one and only one redux, it could only reduce to  $\Omega_1$  which is equivalent to itself. For the inductive step,  $\Omega_i = \Omega$ , therefore  $\Omega_i \xrightarrow[\beta]{} \Omega_{i+1}$  is still  $\Omega$ .

By definition, a term having a  $\beta$ -nf requires the existence of a form in  $\beta$ -nf such that the term can reduce to. By induction,  $\Omega$  only reduces to  $\Omega$ , and  $\Omega$  is not in  $\beta$ -nf because it contains  $\beta$ -redex. Therefore,  $\Omega$  can never reduce to a  $\beta$ -nf, thus it does not have a  $\beta$ -nf. ■

### Problem

(1.16) Let  $M$  be a  $\lambda$ -term with the following properties:

- $M$  has a  $\beta$ -nf.
- There exists an infinite reduction path  $M \equiv M_0 \xrightarrow{\beta} M_1 \xrightarrow{\beta} \dots$  on  $M$ .

Prove that every  $M_i$  has a  $\beta$ -nf, and give an example of  $M$ .

*Solution.* An example would be  $(\lambda x y . y)\Omega$ . Reduction can go on infinitely by reducing on  $\Omega$ , but the  $\beta$ -nf of the term is  $\lambda y . y$

*Proof.* Denote  $\beta$ -nf of  $M$  as  $M'$ . For any form in the reduction path,  $M \xrightarrow{\beta} M_i$ . In conjunction with  $M \xrightarrow{\beta} M'$ , by the Church-Rosser theorem, there exists  $L$  such that  $M_i \xrightarrow{\beta} L$  and  $M' \xrightarrow{\beta} L$ . Because  $M'$  is in  $\beta$ -nf,  $L$  can only be  $M'$ , thus  $M_i \xrightarrow{\beta} M'$ , so  $M_i$  is capable of reducing to  $M'$ , a  $\beta$ -nf. Therefore, any form in the reduction path has a  $\beta$ -nf. ■

### Problem

(1.17) If  $M N$  is strongly normalizing, then both  $M$  and  $N$  are strongly normalizing.

*Solution.*

*Proof.* If  $M$  is not strongly normalizing, then there exists a reduction path  $M_0 \xrightarrow{\beta} M_1 \xrightarrow{\beta} \dots$ . Therefore,  $M N$  would have had a reduction path  $M N \xrightarrow{\beta} M_1 N \xrightarrow{\beta} \dots$  that is infinite, which contradicts with  $M N$  being strongly normalizing. Vice versa for  $N$ . ■

### Problem

(1.18) Let  $L, M, N \in \Lambda$  such that  $L \equiv_{\beta} M$  and  $L \xrightarrow{\beta} N$ . Moreover,  $N$  is in  $\beta$ -nf. Prove that  $M \xrightarrow{\beta} N$ .

*Solution.* Corollary 1.9.9.

### Problem

(1.19) Define

$$U := \lambda z x . x (z z x) \quad \text{and} \quad Z := U U$$

Prove  $Z$  is a fixed point combinator.

*Solution.* Proving  $\forall L \in \Lambda, L (Z L) \xrightarrow[\beta]{} Z L$ .

*Proof.*

$$\begin{aligned} Z L &= (\lambda z x . x (z z x)) (\lambda z x . x (z z x)) L \\ &\xrightarrow[\beta]{} L ((\lambda z x . x (z z x)) (\lambda z x . x (z z x)) L) \xrightarrow[\beta]{} L (Z L) \end{aligned}$$

■

### Problem

(1.20) Solve for  $M \in \Lambda$  in each equation:

$$\begin{aligned} M &\equiv_{\beta} \lambda x y . x M y \\ M x y z &\equiv_{\beta} x y z M \end{aligned}$$

*Solution.* By the property of the  $Y$  combinator:

$$f (Y f) = Y f$$

The first equation can be remodeled as

$$M \equiv_{\beta} L M \text{ where } L = \lambda m x y . x m y$$

Solving for fixed point of  $L$  :

$$\begin{aligned} M &\equiv Y L \equiv L (Y L) \\ &= (\lambda x . L (x x)) (\lambda x . L (x x)) \\ &= (\lambda x . (\lambda m u v . u m v) (x x)) (\lambda x . (\lambda m u v . u m v) (x x)) \\ &\xrightarrow[\beta]{} (\lambda x . (\lambda u v . u (x x) v)) (\lambda x . (\lambda u v . u (x x) v)) \end{aligned}$$

The second equation can be  $\eta$ -reduced on both sides:

$$\begin{aligned} M x y z &\equiv_{\beta} x y z M \\ M &\equiv_{\beta} \lambda x y z . x y z M \end{aligned}$$

Remodeling equation:

$$M = N M \text{ where } M = \lambda m x y z . x y z m$$

then

$$\begin{aligned} M &\equiv Y N \equiv N (Y N) \\ &= (\lambda x . N (x x)) (\lambda x . N (x x)) \\ &= (\lambda x . (\lambda m u y z . u y z m) (x x)) (\lambda x . (\lambda m u y z . u y z m) (x x)) \\ &\xrightarrow[\beta]{} (\lambda x . (\lambda u y z . u y z (x x))) (\lambda x . (\lambda u y z . u y z (x x))) \end{aligned}$$