

EXERCISES

CHAPTER 5

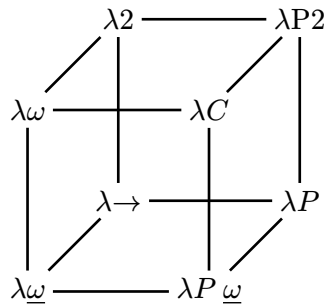
SEAN LI ¹

1. Reducted

Reference - Calculus of Constructions

$$\begin{array}{c}
 \frac{}{\emptyset \vdash * : \square} \text{Sort} \qquad \frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \text{Var} \qquad \frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \text{Weak} \\
 \\
 \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A . B : s_2} \text{Form} \\
 \\
 \frac{\Gamma \vdash M : \Pi x : A . B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B [x := N]} \text{App} \\
 \\
 \frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A . B : s}{\Gamma \vdash \lambda x : A . M : \Pi x : A . B} \text{Abst} \\
 \\
 \frac{\Gamma \vdash A : B \quad B \stackrel{\beta}{=} B' \quad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \text{Conv}
 \end{array}$$

The λ -Cube



$\lambda \rightarrow$	$(*, *)$			
$\lambda 2$	$(*, *)$	$(\square, *)$		
$\lambda \omega$	$(*, *)$		(\square, \square)	
λP	$(*, *)$			$(*, \square)$
$\lambda \omega$	$(*, *)$	$(\square, *)$	(\square, \square)	
$\lambda 2$	$(*, *)$	$(\square, *)$		$(*, \square)$
$\lambda \omega$	$(*, *)$		(\square, \square)	$(*, \square)$
λP	$(*, *)$	$(\square, *)$	(\square, \square)	$(*, \square)$

Problem

(6.1 a) Give a complete derivation in tree format showing that

$$\perp \equiv \Pi \alpha : * . \alpha$$

is legal in λC .

Solution. Here we will show that there exists $s \in \text{sort}$ and Γ such that $\Gamma \vdash \perp : s$.

Proof.

$$\frac{\frac{\vdash * : \square}{\alpha : * \vdash \alpha : *} \text{Var}}{\vdash \Pi \alpha : * . \alpha : *} \text{Form}$$

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Problem

(6.1 a) Give a complete derivation in tree format showing that $\perp \rightarrow \perp$ is legal in λC where

$$\perp \equiv \Pi \alpha : * . \alpha$$

Solution. Here we will show that there exists $s \in \text{sort}$ and Γ such that $\Gamma \vdash \perp \rightarrow \perp : s$.

Proof.

$$\frac{(6.1 \text{ a}) \frac{\vdash \perp : *}{\vdash \perp : *} \quad \frac{(6.1 \text{ a}) \frac{\vdash \perp : *}{\vdash \perp : *} \text{Weak}}{x : \perp \vdash \perp : *} \text{Form}}{\vdash \Pi x : \perp . \perp : *} \text{Form}$$

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Problem

(6.1 c) To which systems of the λ -cube does \perp belong? And $\perp \rightarrow \perp$?

Solution. The set of (s_1, s_2) pairs in formation rules of the derivation of \perp is $\{(\square, *)\}$. The minimal system corresponding is $\lambda 2$. The same for $\perp \rightarrow \perp$. Therefore \perp and $\perp \rightarrow \perp$ belongs to $\lambda 2$, $\lambda \omega$, λP and λC .

Problem

(6.2) Given context $\Gamma \equiv S : *, P : S \rightarrow *, A : *$. Prove by means of a flag derivation that the following expression is inhabited in λC with respect to Γ :

$$(\Pi x : S . (A \rightarrow P x)) \rightarrow A \rightarrow \Pi y : S . P y$$

Solution. The inhabitant is

$$M \equiv \lambda u : (\Pi x : S . (A \rightarrow P x)) . \lambda v : A . \lambda y : S . u y v$$

Proof.

1.	$S : *, P : S \rightarrow *, A : *$	
2.	$u : \Pi x : S . (A \rightarrow P x)$	
3.	$v : A$	
4.	$y : S$	
5.	$u y : A \rightarrow P y$	2,4 App
6.	$u y v : P y$	5,3 App
7.	$\lambda y : S . u y v : \Pi y : S . P y$	6 Abst
8.	$\lambda v : A . \lambda y : S . u y v : A \rightarrow \Pi y : S . P y$	7 Abst
9.	$\lambda u : \Pi x : S . (A \rightarrow P x) . \lambda v : A . \lambda y : S . u y v : \Pi x : S . (A \rightarrow P x) \rightarrow A \rightarrow \Pi y : S . P y$	8 Abst

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Problem

(6.3 a) Let \mathcal{J} be a judgement

$$\mathcal{J} \equiv S : *, P : S \rightarrow * \vdash \lambda x : S . (P x \rightarrow \perp) : S \rightarrow *$$

Derive \mathcal{J} in λC with shorthand flag notation.

Solution.

1.	$S : *, P : S \rightarrow *$	
2.	$x : S$	
3.	$P x : *$	1,2 App
4.	$\perp : *$	Weak from 6.1 a
5.	$P x \rightarrow \perp : *$	3,4 Form

6. $\boxed{\lambda x : S . P x \rightarrow \perp : S \rightarrow *}$ **5 Abst**

Problem

(6.3 b) Determine the (s_1, s_2) pairs corresponding to all Π abstractions occurring in \mathcal{J} .

Solution.

Abstraction	Line Number	(s_1, s_2)
$P : S \rightarrow *$	1	$(*, \square)$
$\perp \equiv \Pi \alpha : * . \alpha$	4	$(\square, *)$
$P x \rightarrow \perp$	5	$(\square, *)$
$\lambda x : S . P x \rightarrow \perp : S \rightarrow *$	6	$(*, \square)$

Problem

(6.3 c) What is the ‘smallest’ system in the λ -cube to which \mathcal{J} belongs?

Solution. There are $(*, *) - \lambda \rightarrow$ pairs, $(*, \square) - \lambda P$ pairs, and $(\square, *) - \lambda 2$. Therefore the minimal system \mathcal{J} belongs to is $\lambda P 2$.

Problem

(6.4 a) Let $\Gamma \equiv S : *, Q : S \rightarrow S \rightarrow *$ and

$$M \equiv (\Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp)) \rightarrow \Pi z : S . (Q z z \rightarrow \perp)$$

Derive $\Gamma \vdash M : *$ and determine the smallest subsystem to which this judgement belongs.

Solution.

1.	$S : *, Q : S \rightarrow S \rightarrow *$	
2.	$x : S$	
3.	$y : S$	
4.	$Q x : S \rightarrow *$	1,2 App
5.	$Q x y : *$	4,3 App
6.	$z : Q x y$	
7.	$Q y : S \rightarrow *$	1,3 App
8.	$Q y x : *$	7,2 App
9.	$t : Q y x$	
10.	$\perp : *$	Weak from 6.1 a
11.	$Q y x \rightarrow \perp : *$	8,10 Form
12.	$Q x y \rightarrow Q y x \rightarrow \perp : *$	5,11 Form
13.	$\Pi y : S . Q x y \rightarrow Q y x \rightarrow \perp : *$	1,12 Form
14.	$\Pi x, y : S . Q x y \rightarrow Q y x \rightarrow \perp : *$	1,13 Form
15.	$a : (\Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp))$	
16.	$z : S$	
17.	$Q z : S \rightarrow *$	1,16 App
18.	$Q z z : *$	17,16 App
19.	$b : Q z z$	
20.	$\perp : *$	Weak from 6.1 a
21.	$Q z z \rightarrow \perp : *$	18,20 Form
22.	$\Pi z : S . Q z z \rightarrow \perp : *$	1,21 Form
23.	$\Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp) \rightarrow \Pi z : S . Q z z \rightarrow \perp : *$	14,22 Form

Here's a table of all Π s that appeared

Abstraction	Line Number	(s_1, s_2)
$S \rightarrow *$	1 / 4 / 7 / 17	$(*, \square)$
$S \rightarrow S \rightarrow *$	1	$(*, \square)$
\perp	10 / 11 / 12 / 13 / 14 / 15 / 20 / 21 / 22 / 23	$(\square, *)$
$Q y x \rightarrow \perp$	11 / 12 / 13 / 14 / 15 / 23	$(*, *)$
$Q x y \rightarrow Q y x \rightarrow \perp$	12 / 13 / 14 / 15 / 23	$(*, *)$
$\Pi y : S . Q x y \rightarrow Q y x \rightarrow \perp$	13 / 14 / 23	$(*, *)$

$\Pi x, y : S . Q x y \rightarrow Q y x \rightarrow \perp$	14 / 23	$(*, *)$
$Q z z \rightarrow \perp$	21 / 22 / 23	$(*, *)$
$\Pi z : S . Q z z \rightarrow \perp$	22 / 23	$(*, *)$
$\Pi x, y : S . Q x y \rightarrow Q y x \rightarrow \perp \rightarrow$ $\Pi z : S . Q z z \rightarrow \perp$	23	$(*, *)$

There are $(*, *) - \lambda \rightarrow$ pairs, $(*, \square) - \lambda P$ pairs, and $(\square, *) - \lambda 2$ pairs. Therefore the minimal system available is $\lambda P 2$.

Problem

(6.4 b) Prove in λC that M is inhabited in context Γ .

Solution. A shorthand derivation is given below:

Proof.

1. $S : *, Q : S \rightarrow S \rightarrow *$
2. $h : \Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp)$
3. $z : S$
4. $a : Q z z$
5. $\alpha : *$
6. $h z : \Pi y : S . (Q z y \rightarrow Q y z \rightarrow \perp)$ 2,3 App
7. $h z z : Q z z \rightarrow Q z z \rightarrow \perp$ 6,3 App
8. $h z z a : Q z z \rightarrow \perp$ 7,4 App
9. $h z z a a : \Pi \alpha : * . \alpha$ 8,4 App
10. $h z z a a \alpha : \alpha$ 9,5 App
11. $\lambda \alpha : * . h z z a a \alpha : \Pi \alpha : * . \alpha$ 10 Abst
12. $\lambda a : Q z z \lambda \alpha : * . h z z a a \alpha : Q z z \rightarrow \perp$ 11 Abst
13. $\lambda z : S . \lambda a : Q z z \lambda \alpha : * . h z z a a \alpha : \Pi z : S . Q z z \rightarrow \perp$ 12 Abst
14. $\lambda h : \Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp)$
 $\lambda z : S . \lambda a : Q z z \lambda \alpha : * . h z z a a \alpha$
 $: \Pi x, y : S . (Q x y \rightarrow Q y x \rightarrow \perp) \rightarrow \Pi z : S . Q z z \rightarrow \perp$ 13 Abst

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Problem

(6.4 c) We may consider Q to be a relation on set S . Moreover by PAT we may see $A \rightarrow \perp$ as the negation $\neg A$ of prop A . How can M then be interpreted by the PAT paradigm?

Solution. By a direct type-to-proposition translation we have

$$M \equiv \forall x, y \in S, (Q(x, y) \Rightarrow \neg Q(y, x)) \Rightarrow \forall z \in S, (\neg Q(z, z))$$

It expresses the fact if Q is asymmetric then it is irreflexive.