

EXERCISES

CHAPTER 1

SEAN LI ¹

1. Reducted

Problem

Simplify notation of the following terms

- (a) $(\lambda x.(((xz)y)(xx)))$
- (b) $((\lambda x.(\lambda y.(\lambda z.(z((xy)z)))))(\lambda u.u))$

Solution.

- (a) $\lambda x.(xzy)(xx)$
- (b) $(\lambda xyz.x(xyz))(\lambda u.u)$

Problem

Find the alpha equivalent terms to

$$\lambda x.x(\lambda x.x)$$

In

- (a) $\lambda y.y(\lambda x.x)$
- (b) $\lambda y.y(\lambda x.y)$
- (c) $\lambda y.y(\lambda x.y)$

Solution. Only (a).

Problem

Prove

$$\lambda x.x(\lambda z.y) \underset{\alpha}{=} \lambda z.z(\lambda z.y)$$

Solution.

Proof. By definition of alpha equivalence

$$M \underset{\alpha}{=} N \iff \exists \varphi, M^\varphi \xrightarrow{\alpha} N \wedge \text{FR } M = \text{FR } N$$

The witness of φ is substituting bound variable x with z , and z is not a free variable in the term, thus the two terms are alpha equivalent.

$$\lambda x.x(\lambda z.y)^{x \rightarrow z} \xrightarrow{\alpha} \lambda z.z(\lambda z.y)$$

■

Problem

Consider the following term:

$$U := (\lambda z.zxz)((\lambda y.xy)x)$$

1. Find Sub U
2. Draw tree rep of U
3. Find FV U
4. Find alpha equivalent terms to U from below and point out which of those follows the Barendregt convention:

- (a) $(\lambda y.yxy)((\lambda z.xz)x)$
- (b) $(\lambda x.xyx)((\lambda z.yz)y)$
- (c) $(\lambda y.yxy)((\lambda y.xy)x)$
- (d) $(\lambda v.(vx)v)((\lambda u.uv)x)$

1. Find Sub U .

Solution.

Sub $U = \{$

$$(\lambda z.zxz)((\lambda y.xy)x), (\lambda z.zxz), ((\lambda y.xy)x),$$

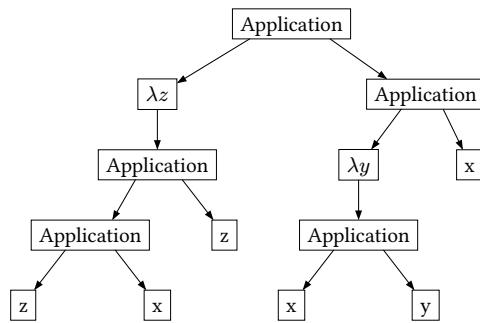
$$(\lambda y.yx), (y), (\lambda z.xz), (x),$$

$$(\lambda y.y), (x), (\lambda z.x), (z), (y), (x)$$

$\}$

2. Draw a tree rep of U .

Solution.



3. Find FV U

Solution.

$$\begin{aligned} \text{FV } U &= \text{FV } (\lambda y.yxy) \cup \text{FV } (\lambda z.xz)x \\ &= (\text{FV } yxy) \setminus \{y\} \cup (\text{FV } \lambda z.xz) \cup \{x\} \\ &= (\text{FV } yx) \setminus \{y\} \cup (\text{FV } xz) \setminus \{z\} \cup \{x\} \\ &= \{x\} \end{aligned}$$

4. Find an alpha-equivalent term.

Solution.

$$(a) \underset{\alpha}{=} (c) \underset{\alpha}{=} U$$

(a), (b), and (c) all follow the Barendregt convention. In (d) the free variable v was bound in the first abstraction.

Problem

Give the results of the following substitutions

- (a) $(\lambda x.y(\lambda y.xy))[y := \lambda z.zx]$
- (b) $((xyz)[x := y])[y := z]$
- (c) $((\lambda x.xyz)[x := y])[y := z]$
- (d) $(\lambda y.yyx)[x := yz]$

Solution.

- (a) $(\lambda v.(\lambda z.zx)(\lambda u.vu))$
- (b) $(yyz)[y := z] = zzz$
- (c) $(\lambda x.xyz)[y := z] = (\lambda x.xzz)$
- (d) $(\lambda u.uu(yz))$

Problem

$$\neg \left(\forall M L N \in \Lambda, M[x := N, y := L] \underset{\alpha}{\equiv} M[x := N][y := L] \right)$$

Solution.

Proof. Because $\text{RHS} = M[x := N][y := L] = M[x := N[y := L]] [y := L]$, if $y \in \text{FV } N$, then what x gets substituted with will have y substituted for L , which is completely different with LHS. ■

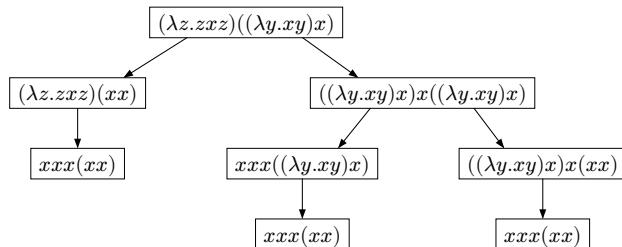
Problem

Find all available redexes in

$$U := (\lambda z.zxz)((\lambda y.xy)x)$$

And all reduction pathes to the β -normal form.

Solution. The first redex is the term as an application itself; another the second term in the application.



Problem

Show that

$$(\lambda x.xx)y \underset{\beta}{\neq} (\lambda xy.yx)xx$$

Solution. By Collorary 1.9.9, it suffices to prove the hypothesis with a proof of a common normal reducted form from LHS and RHS not existing.

Contradiction. By definition of $\overline{\beta}$, there exists The set of all terms attainable from β -reduction on $(\lambda x.xx)y$ and $(\lambda xy.yx)xx$ do not intersect. Therefore,

$$\neg \left(\exists L \in \Lambda, (\lambda x.xx)y \xrightarrow[\beta]{} L \wedge (\lambda xy.yx)xx \xrightarrow[\beta]{} L \right) \Rightarrow \neg \left((\lambda x.xx)y \overline{\beta} (\lambda xy.yx)xx \right)$$

■

Problem

Define the combinators

$$K := \lambda xy.x \\ S := \lambda xyz.xz(yz)$$

Prove that

$$\forall PQ \in \Lambda, KPQ \xrightarrow[\beta]{} P$$

$$\forall PQR \in \Lambda, SPQR \xrightarrow[\beta]{} PR(QR)$$

Solution.

Proof.

$$KPQ = (\lambda xy.x)PQ \xrightarrow[\beta]{} (\lambda y.x)[x := P]Q \xrightarrow[\beta]{} P[y := Q] = P$$

$$SPQR = (\lambda xyz.xz(yz)) \xrightarrow[\beta]{} (xz(yz))[x := P][y := Q][z := R] = PR(QR)$$

■

Problem

We define the church numerals

$$\begin{aligned}\text{zero} &:= \lambda f x. x \\ \text{one} &:= \lambda f x. f x \\ \text{two} &:= \lambda f x. f f x \\ &\dots \\ \text{num}_n &:= \lambda f x. f^n x\end{aligned}$$

And operations

$$\begin{aligned}\text{add} &:= \lambda n m f x. m f(n f x) \\ \text{mul} &:= \lambda n m f x. m(n f) x\end{aligned}$$

Show

- (a) $\text{add one one} \xrightarrow[\beta]{} \text{two}$
- (b) $\text{add one one} \neq \text{mul one zero} \xrightarrow[\beta]$

Solution.

$$\begin{aligned}\text{(a)} \quad \text{add one one} &= (\lambda n m f x. m f(n f x))(\lambda f x. f x)(\lambda f x. f x) \\ &\xrightarrow[\beta]{} (\lambda f x. (\lambda f x. f x) f ((\lambda f x. f x) f x)) \\ &\xrightarrow[\beta]{} (\lambda f x. (\lambda x. f x) f x) \\ &\xrightarrow[\beta]{} (\lambda f x. f f x) = \text{two}\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad \text{mul one one} &= (\lambda n m f x. m(n f) x)(\lambda f x. f x)(\lambda f x. f x) \\ &\xrightarrow[\beta]{} \lambda f x. (\lambda f x. f x)((\lambda f x. f x) f) x \\ &\xrightarrow[\beta]{} \lambda f x. f x = \text{one}\end{aligned}$$

Because no intermediate form in the beta reduction process of the two terms are α -equivalent, by corollary 1.9.9 the two terms are not β -equivalent.

Problem

We define

$$\text{succ} := \lambda mfx.f(mfx) \text{ s.t. } \forall \text{num}_n, \text{succ num}_n = \text{num}_{n+1}$$

Prove

$$\text{succ zero} \xrightarrow{\beta} \text{one}$$

$$\text{succ one} \xrightarrow{\beta} \text{two}$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β -equivalence.

Proof.

$$\begin{aligned} \text{succ zero} &= (\lambda mfx.f(mfx))(\lambda fx.x) \\ &\xrightarrow{\beta} (\lambda fx.f((\lambda fx.x)fx)) \\ &\xrightarrow{\beta} (\lambda fx.fx) = \text{one} \end{aligned}$$

The path $\text{succ zero} \xrightarrow{\beta} \text{one}$ derived above is the witness of a reduction chain from LHS to RHS.

$$\begin{aligned} \text{succ one} &= (\lambda mfx.f(mfx))(\lambda fx.fx) \\ &\xrightarrow{\beta} (\lambda fx.f((\lambda fx.fx)fx)) \\ &\xrightarrow{\beta} (\lambda fx.f(fx)) = \text{two} \end{aligned}$$

The path $\text{succ one} \xrightarrow{\beta} \text{two}$ derived above is the witness of a reduction chain from LHS to RHS. ■

Problem

We define the λ -terms \top_λ (true) and \perp_λ (false) and \neg_λ (not) by:

$$\begin{aligned} \top_\lambda &:= \lambda xy.x & \perp_\lambda &:= \lambda xy.y \\ \neg_\lambda &:= \lambda a.a\perp_\lambda\top_\lambda \end{aligned}$$

Show that

$$\begin{aligned} \neg_\lambda(\neg_\lambda\top_\lambda) &\xrightarrow{\beta} \top_\lambda \\ \neg_\lambda(\neg_\lambda\perp_\lambda) &\xrightarrow{\beta} \perp_\lambda \end{aligned}$$

Solution. It suffices to provide a witness of a reduction chain from one side to the other to prove β -equivalence.

Proof.

$$\begin{aligned}\neg_\lambda(\neg_\lambda \top_\lambda) &= \neg_\lambda((\lambda a.a \perp_\lambda \top_\lambda)(\lambda xy.x)) \\ &\xrightarrow[\beta]{\gg} \neg_\lambda((\lambda xy.x) \perp_\lambda \top_\lambda) \\ &\xrightarrow[\beta]{\gg} (\lambda a.a \perp_\lambda \top_\lambda) \perp_\lambda \\ &\xrightarrow[\beta]{\gg} (\lambda xy.y) \perp_\lambda \top_\lambda \\ &\xrightarrow[\beta]{\gg} \top_\lambda\end{aligned}$$

$$\begin{aligned}\neg_\lambda(\neg_\lambda \perp_\lambda) &= \neg_\lambda((\lambda a.a \perp_\lambda \top_\lambda)(\lambda xy.y)) \\ &\xrightarrow[\beta]{\gg} \neg_\lambda((\lambda xy.x) \perp_\lambda \top_\lambda) \\ &\xrightarrow[\beta]{\gg} (\lambda a.a \perp_\lambda \top_\lambda) \top_\lambda \\ &\xrightarrow[\beta]{\gg} (\lambda xy.x) \perp_\lambda \top_\lambda \\ &\xrightarrow[\beta]{\gg} \perp_\lambda\end{aligned}$$

■

Problem

Define

$$\text{iszzero} := \lambda m.m(\lambda x.\perp_\lambda)\top_\lambda$$

Prove

$$\begin{aligned}\text{iszzero zero} &\xrightarrow[\beta]{\gg} \top_\lambda \\ \forall n \in \mathbb{N}^+, \text{iszzero num}_n &\xrightarrow[\beta]{\gg} \perp_\lambda\end{aligned}$$

Solution.

$$\begin{aligned}\text{iszzero zero} &= (\lambda m.m(\lambda x.\perp_\lambda)\top_\lambda)(\lambda fx.x) \\ &\xrightarrow[\beta]{\gg} (\lambda fx.x)(\lambda x.\perp_\lambda)\top_\lambda \\ &\xrightarrow[\beta]{\gg} \top_\lambda\end{aligned}$$

$$\begin{aligned}
\text{iszzero num}_n &= (\lambda m.m(\lambda x.\perp_\lambda)\top_\lambda)(\lambda f x.f^n x) \\
&\xrightarrow[\beta]{\gg} (\lambda f x.f^n x)(\lambda x.\perp_\lambda)\top_\lambda \\
&\xrightarrow[\beta]{\gg} (\lambda x.\perp_\lambda)((\lambda x.\perp_\lambda)^{n-1}\top_\lambda) \xrightarrow[\beta]{\gg} \perp_\lambda
\end{aligned}$$

Problem

If-else can be modeled as

$$\text{ifelse} = \lambda x t f. x t f$$

Where when x , then t , else f . Prove correctness by applying \top_λ and \perp_λ on ifelse.

Solution.

$$\begin{aligned}
\text{ifelse } \top_\lambda &= (\lambda x t f. x t f)\top_\lambda \\
&\xrightarrow[\beta]{\gg} (\lambda t f.(\lambda x y.x) t f) \xrightarrow[\beta]{\gg} (\lambda t f.t) \\
\text{ifelse } \perp_\lambda &= (\lambda x t f. x t f)\perp_\lambda \\
&\xrightarrow[\beta]{\gg} (\lambda t f.(\lambda x y.y) t f) \xrightarrow[\beta]{\gg} (\lambda t f.f)
\end{aligned}$$

By applying the results to any two values, the correct corresponding value returns, ex, for ifelse \top_λ , t is always returned.

Problem

Prove that $\Omega := (\lambda x.x x)(\lambda x.x x)$.

Solution. Firstly let's prove Ω .

Proof. Induction on Ω 's only reduction path proves that every $\Omega \xrightarrow[\beta]{\gg} \Omega_i = \Omega$. For the base case because Ω has one and only one redux, it could only reduce to Ω_1 which is equivalent to itself. For the inductive step, $\Omega_i = \Omega$, therefore $\Omega_i \xrightarrow[\beta]{\rightarrow} \Omega_{i+1}$ is still Ω .

By definition, a term having a β -nf requires the existence of a form in β -nf such that the term can reduce to. By induction, Ω only reduces to Ω , and Ω is not in β -nf because it contains β -redex. Therefore, Ω can never reduce to a β -nf, thus it does not have a β -nf. ■

Problem

Let M be a λ -term with the following properties:

- M has a β -nf.
- There exists an infinite reduction path $M \equiv M_0 \xrightarrow{\beta} M_1 \xrightarrow{\beta} \dots$ on M .

Prove that every M_i has a β -nf, and give an example of M .

Solution. An example would be $(\lambda xy.y)\Omega$. Reduction can go on infinitely by reducing on Ω , but the β -nf of the term is $\lambda y.y$

Proof. Denote β -nf of M as M' . For any form in the reduction path, $M \xrightarrow[\beta]{} M_i$. In conjunction with $M \xrightarrow[\beta]{} M'$, by the Church-Rosser theorem, there exists L such that $M_i \xrightarrow[\beta]{} L$ and $M' \xrightarrow[\beta]{} L$. Because M' is in β -nf, L can only be M' , thus $M_i \xrightarrow[\beta]{} M'$, so M_i is capable of reducing to M' , a β -nf. Therefore, any form in the reduction path has a β -nf. ■

Problem

If MN is strongly normalizing, then both M and N are strongly normalizing.

Solution.

Proof. If M is not strongly normalizing, then there exists a reduction path $M_0 \xrightarrow[\beta]{} M_1 \xrightarrow[\beta]{} \dots$. Therefore, MN would have had a reduction path $MN \xrightarrow[\beta]{} M_1 N \xrightarrow[\beta]{} \dots$ that is infinite, which contradicts with MN being strongly normalizing. Vice versa for N . ■

Problem

Let $L, M, N \in \Lambda$ such that $L \equiv_{\beta} M$ and $L \xrightarrow[\beta]{} N$. Moreover, N is in β -nf. Prove that $M \xrightarrow[\beta]{} N$.

Solution. Collorary 1.9.9.

Problem

Define

$$U := \lambda zx.x(zzx) \quad \text{and} \quad Z := UU$$

Prove Z is a fixed point combinator.

Solution. Proving $\forall L \in \Lambda, L(ZL) \xrightarrow{\beta} ZL$.

Proof.

$$\begin{aligned} ZL &= (\lambda zx.x(zzx))(\lambda zx.x(zzx))L \\ &\xrightarrow{\beta} L((\lambda zx.x(zzx))(\lambda zx.x(zzx))L) \xrightarrow{\beta} L(ZL) \end{aligned}$$

■

Problem

Solve for $M \in \Lambda$ in each equation:

$$\begin{aligned} M &\equiv \lambda xy.xMy \\ Mxyz &\xrightarrow{\beta} xyzM \end{aligned}$$

Solution. By the property of the Y combinator:

$$f(Yf) = Yf$$

The first equation can be remodeled as

$$M \xrightarrow{\beta} LM \text{ where } L = \lambda mxy.xmy$$

Solving for fixed point of L :

$$\begin{aligned} M &\equiv YL \equiv L(YL) \\ &= (\lambda x.L(xx))(\lambda x.L(xx)) \\ &= (\lambda x.(\lambda muv.umv)(xx))(\lambda x.(\lambda muv.umv)(xx)) \\ &\xrightarrow{\beta} (\lambda x.(\lambda uv.u(xx)v))(\lambda x.(\lambda uv.u(xx)v)) \end{aligned}$$

The second equation can be η -reduced on both sides:

$$\begin{aligned} Mxyz &\xrightarrow{\beta} xyzM \\ M &\equiv \lambda zyx.xyzM \end{aligned}$$

Remodeling equation:

$$M = NM \text{ where } M = \lambda m z y x . x y z m$$

then

$$\begin{aligned} M &\equiv YN \equiv N(YN) \\ &= (\lambda x.N(xx))(\lambda x.N(xx)) \\ &= (\lambda x.(\lambda m z y u . u y z m)(xx))(\lambda x.(\lambda m z y u . u y z m)(xx)) \\ &\xrightarrow{\beta} (\lambda x.(\lambda z y u . u y z (x x))) (\lambda x.(\lambda z y u . u y z (x x))) \end{aligned}$$