

# EXERCISES

## CHAPTER 9

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1. Redacted

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**Definition** Extended Rules for  $\lambda D_0$

$$\frac{\Delta; \Gamma \vdash K : L \quad \Delta; \bar{a} : \bar{M} \vdash M : N}{\Delta, (\bar{x} : \bar{A} \triangleright a(\bar{x}) := M : N); \Gamma \vdash K : L} \text{def}$$
$$\frac{\Delta, \mathcal{D}; \Gamma \vdash * : \square \quad \Delta, \mathcal{D}; \Gamma \vdash \bar{U} : A \ [\bar{x} := \bar{U}] \quad \mathcal{D} \equiv \bar{x} : \bar{A} \triangleright a(\bar{x}) := M : N}{\Delta, \mathcal{D}; \Gamma \vdash a(\bar{U}) : N \ [\bar{x} := \bar{U}]} \text{inst}$$
$$\frac{\Delta; \Gamma \vdash x : A \quad \Delta; \Gamma \vdash A : s \quad A \stackrel{\Delta, \beta}{=} B}{\Delta; \Gamma \vdash x : B} \text{conv}$$

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*Lemma 1.* Given  $\mathcal{D} \equiv \bar{x} : \bar{A} \triangleright a(\bar{x}) := M : N$  and  $a \notin \Delta$

$$\frac{\Delta; \bar{x} : \bar{A} \vdash M : N}{\Delta, \mathcal{D}; \bar{x} : \bar{A} \vdash a(\bar{x}) : N} \text{par}$$

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### Problem

(9.1) Given

$$(\mathcal{D}_1) \quad x : \mathbb{Z}, y : \mathbb{Z} \triangleright a(x, y) := x^2 + y^2 : \mathbb{Z}$$

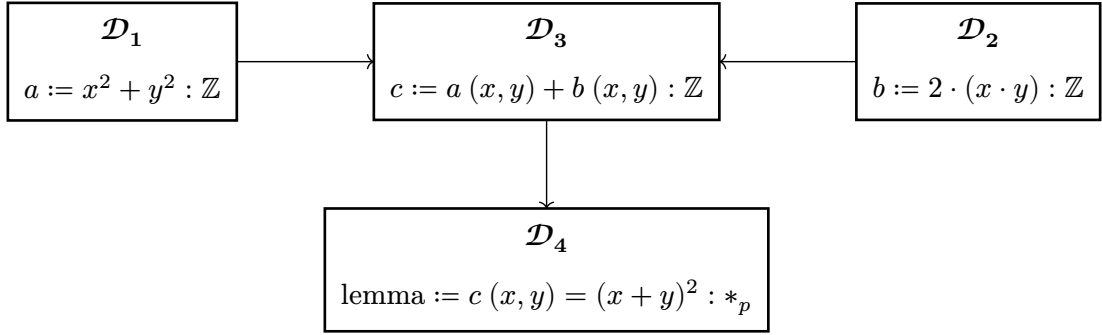
$$(\mathcal{D}_2) \quad x : \mathbb{Z}, y : \mathbb{Z} \triangleright b(x, y) := 2 \cdot (x \cdot y) : \mathbb{Z}$$

$$(\mathcal{D}_3) \quad x : \mathbb{Z}, y : \mathbb{Z} \triangleright c(x, y) := a(x, y) + b(x, y) : \mathbb{Z}$$

$$(\mathcal{D}_4) \quad x : \mathbb{Z}, y : \mathbb{Z} \triangleright \text{lemma}(x, y) := c(x, y) = (x + y)^2 : *_p$$

Consider  $\Delta \equiv \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ . Describe the dependencies between the four definitions and give all possible linearizations of the corresponding partial order.

*Solution.* Hasse diagram given below



The only incomparable pair is  $(\mathcal{D}_1, \mathcal{D}_2)$ . Therefore there are two possible linearizations:

$$(1) \quad \mathcal{D}_1 \leq \mathcal{D}_2 \leq \mathcal{D}_3 \leq \mathcal{D}_4$$

$$(2) \quad \mathcal{D}_2 \leq \mathcal{D}_1 \leq \mathcal{D}_3 \leq \mathcal{D}_4$$

### Problem

(9.2) Consider

$$\mathcal{D}_i \equiv \bar{x} : \bar{A} \triangleright a(\bar{x}) := K : L$$

$$\mathcal{D}_j \equiv \bar{y} : \bar{B} \triangleright b(\bar{y}) := M : N$$

Let  $\Delta; \Gamma \vdash U : V$  and assume  $\mathcal{D}_i$  and  $\mathcal{D}_j$  are elements of list  $\Delta$ , where  $\mathcal{D}_i$  precedes  $\mathcal{D}_j$ . Describe precisely where constant  $a$  may occur in  $\mathcal{D}_i$  and  $\mathcal{D}_j$  and where constant  $b$  may occur in  $\Delta$ .

*Solution.* In order for  $\mathcal{D}_i$  to be a valid definition,  $\bar{x} : \bar{A} \vdash K : L$  must be legal. Therefore by the free variable lemma any free variables in  $K$  and  $L$  must be in  $\bar{x} : \bar{A}$ , which by the time, does not yet contain  $a$ 's definition. Therefore,  $a$  could only appear in  $\mathcal{D}_j$ .

By similar reasoning  $b$  could only have appeared in definitions after  $\mathcal{D}_j$ . Assuming the list sorted by the suffix, then  $b$  could only have been in any  $\mathcal{D}_k$  where  $k > j$ .